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# THE ARITHMETIC AND COMBINATORICS OF BUILDINGS FOR $Sp_n$

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Abstract. In this paper, we investigate both arithmetic and combinatorial aspects of buildings and associated Hecke operators for $Sp_n(K)$ with $K$ a local field. We characterize the action of the affine Weyl group in terms of a symplectic basis for an apartment, characterize the special vertices as those which are self-dual with respect to the induced inner product, and establish a one-to-one correspondence between the special vertices in an apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

We then give a natural representation of the local Hecke algebra over $K$ acting on the special vertices of the Bruhat-Tits building for $Sp_n(K)$. Finally, we give an application of the Hecke operators defined on the building by characterizing minimal walks on the building for $Sp_n$.

## 1. Introduction

Buildings play a large role in the study of classical groups [9] and, in particular, in the study of Hecke algebras associated to these groups [4]. In [7], Serre defined Hecke operators acting on trees (the building associated to $SL_2$ over a local field), and this work was generalized to $SL_n$ in [2]. In this paper, we investigate both arithmetic and combinatorial aspects of buildings and associated Hecke operators for $Sp_n(K)$ with $K$ a local field.

Compared to the theory of buildings for the special linear group, the theory for the symplectic group is far less developed, so the first part of this paper is devoted to giving more concrete characterizations of the vertices in an apartment with particular attention to the so-called special vertices. We note that in the case of $SL_n$ all vertices in the building are special. We characterize the action of the affine Weyl group in terms of a symplectic basis for an apartment, characterize the special vertices as those which are self-dual with respect to the induced inner product, and establish a one-to-one correspondence between the special vertices in an apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

We next establish connections between the symplectic elementary divisor theory of lattices over the ring of integers $\mathcal{O}$ of $K$ and double cosets of the group $\Gamma = Sp_n(\mathcal{O})$. Using this correspondence, we define Hecke operators on the building which act as generalized adjacency operators on the underlying graph. We then give a natural (essentially faithful) representation of the local Hecke algebra over $K$ acting on the special vertices of the Bruhat-Tits building for $Sp_n(K)$. Finally,

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we give an application of the Hecke operators by characterizing minimal walks on the 1-subcomplex of the building for $Sp_n$ generated by the special vertices.

## 2. The building for $Sp_n$

In this section, we consider the building for $Sp_n$ over a local field and give intrinsic characterizations of its apartments and special vertices. In particular, we give a concrete characterization of the action of the affine Weyl group, $\widetilde{C}_n$, in terms of a symplectic basis for an apartment. Moreover, after associating each vertex with a homothety class of a lattice in the symplectic space, we show that special vertices are precisely those which are self-dual with respect to the induced inner product. For our application to walks on the building, we further establish a one-to-one correspondence between the special vertices in an apartment and the elements of $\mathbb{Z}^{n+1}/\mathbb{Z}(2,1,\dots 1)$, and interpret the induced group structure in terms of walks on the building.

Throughout, let $K$ be a local field, $\mathcal{O}$ its ring of integers, $\pi \in \mathcal{O}$ a uniformizing parameter, $k = \mathcal{O}/\pi\mathcal{O}$ the residue field, and $(V, \langle *, * \rangle)$ a symplectic (non-degenerate alternating) space of dimension $2n$ over $K$. For an integer $n \geq 1$, let $I_n$ be the $n \times n$ identity matrix and $J_n$ the $2n \times 2n$ matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Define the group of symplectic similitudes of $K$ as $GSp_n(K) = \{M \in M_{2n}(K) \mid M^t J_n M = r(M) J_n\}$ where $r(M) \in K^\times$. Note that $Sp_n(K)$ consists of those elements $M \in GSp_n(K)$ with $r(M) = 1$. Now let $S = K^\times/\mathcal{O}^\times$; for convenience, we take $S = \{\pi^\nu \mid \nu \in \mathbb{Z}\}$. We will denote by $GSp_n^S(K) = \{M \in GSp_n(K) \mid r(M) \in S\}$. Finally, let $\Gamma = Sp_n(\mathcal{O})$.

The Bruhat-Tits building for $Sp_n(K)$ is an $n$-dimensional simplicial complex, $\Delta_n$, whose vertices are homothety classes of lattices in $V$. One defines an incidence relation on the vertices, and the resulting flag complex is the building. In general, our focus will be on an apartment in the building, and we will need a careful understanding of how the vertices are indexed by classes of lattices. Some of the basic material can be found in Chapter 20 of [5], which we supplement where germane.

**Definition 2.1.** An $\mathcal{O}$-lattice $\Lambda \subset V$ is a free $\mathcal{O}$-module of rank $2n$, and is called primitive if $\langle \Lambda, \Lambda \rangle \subseteq \mathcal{O}$ and $\langle *, * \rangle$ induces a non-degenerate form on the alternating space $\Lambda/\pi\Lambda$ over $k$.

Following [5], we first give a general description of the building. We describe an apartment system for the building as follows (see [5]). A frame is an unordered $n$-tuple $\{\lambda_1^1, \lambda_1^2\}, \dots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines $\{\lambda_i^1, \lambda_i^2\}$ such that $V = \sum_1^n (\lambda_i^1 + \lambda_i^2)$, $(\lambda_i^1 + \lambda_i^2)$ is orthogonal to $(\lambda_j^1 + \lambda_j^2)$ for $i \neq j$, and each $(\lambda_i^1 + \lambda_i^2)$ is a hyperbolic plane. We say that the frame determines the apartment $\Sigma$. Vertices in $\Sigma$ are homothety classes of lattices, denoted $[\Lambda]$. A vertex $[\Lambda]$ lies in $\Sigma$ (determined by the above frame) if there are free $\mathcal{O}$-modules $M_i^j \subset \lambda_i^j$ such that $\Lambda = \bigoplus_{i,j} M_i^j$ for some (and hence every) representative $\Lambda$ of the homothety class. More concretely, vertices of the building are homothety classes of lattices $[\Lambda]$ which possess a representative $\Lambda$ such that: there exists a lattice $\Lambda_0$ with $\pi^{-1}\Lambda_0$ primitive, $\Lambda_0 \subseteq \Lambda \subseteq \pi^{-1}\Lambda_0$, and $\langle \Lambda, \Lambda \rangle \subseteq \pi\mathcal{O}$; equivalently, $\Lambda/\Lambda_0$ is a totally isotropic $k$-subspace of the non-degenerate alternating space $\pi^{-1}\Lambda_0/\Lambda_0$. To define the building, we start with the set of vertices and define an incidence relation on them as follows: For vertices $t, t'$, we say $t \sim t'$ if there are lattices $\Lambda_t \in t$ and $\Lambda_{t'} \in t'$ and a lattice $\Lambda_0$ such that $\pi^{-1}\Lambda_0$ is primitive, $\Lambda_0 \subseteq \Lambda_t, \Lambda_{t'} \subseteq \pi^{-1}\Lambda_0$, and either $\Lambda_t \subset \Lambda_{t'}$ or $\Lambda_{t'} \subset \Lambda_t$. The associated flag complex yields the building. The maximal simplices (chambers) are unordered $(n+1)$-tuples $[\Lambda_0], [\Lambda_1], \ldots, [\Lambda_n]$ of homothety classes of lattices with representatives $\Lambda_i$ satisfying: $\pi^{-1}\Lambda_0$ is primitive, $\Lambda_0 \subseteq \Lambda_i \subseteq \pi^{-1}\Lambda_0$, and $\Lambda_1/\Lambda_0 \subset \Lambda_2/\Lambda_0 \subset \cdots \subset \Lambda_n/\Lambda_0$ is a maximal isotropic flag of $k$-subspaces in $\pi^{-1}\Lambda_0/\Lambda_0$.

Now we establish a more concrete realization of the apartment. Fix a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of $V$ ($\langle u_i, w_j\rangle = \delta_{ij}$ (Kronecker delta), $\langle u_i, u_j\rangle = \langle w_i, w_j\rangle = 0$), and let $\Lambda$ be the $\mathcal{O}$-lattice $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$. We note that $\langle \Lambda, \Lambda\rangle \subseteq \mathcal{O}$ if and only if $\langle \pi^{a_i}u_i, \pi^{b_i}w_i\rangle = \pi^{a_i+b_i} \in \mathcal{O}$, which is true if and only if $a_i + b_i \geq 0$. Given $a_i + b_i \geq 0$, the induced alternating form on $\Lambda/\pi\Lambda$ is non-degenerate over $k = \mathcal{O}/\pi\mathcal{O}$ if and only if $a_i + b_i = 0$ for all $i$. Since the basis and uniformizing parameter will remain fixed throughout this paper, we shall lighten the notation, and for a lattice $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$, we will denote it and its class via the exponents as $\Lambda = (a_1, \ldots, a_n; b_1, \ldots, b_n)$, and $[\Lambda] = [a_1, \ldots, a_n; b_1, \ldots, b_n]$, respectively.

**Example 2.2.** With the fixed symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ for $V$, put $\lambda_i^1 = Ku_i$ and $\lambda_i^2 = Kw_i$. The frame $\{\lambda_i^1, \lambda_i^2\}$ determines an apartment $\Sigma$. Let $\Lambda_0 = \pi(\oplus\mathcal{O}u_i \oplus \mathcal{O}w_i)$. Then $\pi^{-1}\Lambda_0$ is primitive. The following ascending chains of lattices determine two chambers in $\Sigma$ containing the vertex $[\Lambda_0]$:

$\Lambda_0 = (1, \ldots, 1; 1, \ldots, 1) \subsetneq \Lambda_1 = (0, 1, \ldots, 1; 1, \ldots, 1) \subsetneq \Lambda_2 = (0, 0, 1, \ldots, 1; 1, \ldots, 1) \subsetneq \cdots \subsetneq \Lambda_n = (0, 0, \ldots, 0; 1, \ldots, 1) \subsetneq \pi^{-1}\Lambda_0$,

$\Lambda_0 = (1, \ldots, 1; 1, \ldots, 1) \subsetneq \Lambda_1 = (1, \ldots, 1; 0, 1, \ldots, 1) \subsetneq \Lambda_2 = (1, \ldots, 1; 0, 0, 1 \ldots, 1) \subsetneq \cdots \subsetneq \Lambda_n = (1, \ldots, 1; 0, 0, \ldots, 0) \subsetneq \pi^{-1}\Lambda_0$.

To see how the rest of the apartment is laid out, one must understand the action of the reflections which generate the Weyl group associated to the building on the lattices. The affine Weyl group is of type $\widetilde{C}_n$ which has a Coxeter diagram:

$$\bullet_1 = \bullet_2 - \bullet_3 \cdots\cdots \bullet_{n-1} - \bullet_n = \bullet_{n+1}$$

with $(n+1)$ vertices; the two endpoints are *special* vertices in the sense of [6], i.e., deleting either of them produces a Coxeter diagram for the spherical Weyl group $C_n$. Associated to each vertex $i$ in the Coxeter diagram is a reflection $s_i$, and the collection of reflections satisfy the standard rules $s_i^2 = 1$ and $s_i s_j$ has order $m_{ij}$, indicated by the Coxeter diagram ($m_{12} = m_{n(n+1)} = 4$, $m_{i(i+1)} = 3$, for $i \neq 1, n$, and $m_{ij} = 2$ otherwise).

Acting on the symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$, define the reflections (any basis vector not specified is fixed):

$s_1$ : Interchange $u_n$ and $w_n$;

(2.1) $s_j (2 \leq j \leq n)$ : Interchange $u_{n-j+1} \leftrightarrow u_{n-j+2}$ and $w_{n-j+1} \leftrightarrow w_{n-j+2}$;

$s_{n+1} : u_1 \mapsto \pi w_1, w_1 \mapsto \pi^{-1}u_1$.

That is, acting on a vertex $[a_1, \ldots, a_n; b_1, \ldots, b_n]$,

$s_1$ takes $[a_1, \ldots, a_n; b_1, \ldots, b_n]$ to $[a_1, \ldots, a_{n-1}, b_n; b_1, \ldots, b_{n-1}, a_n]$;

$s_j (2 \leq j \leq n)$ takes $[a_1, \ldots, a_n; b_1, \ldots, b_n]$ to

(2.2) $[a_1, \ldots, a_{n-j}, a_{n-j+2}, a_{n-j+1}, \ldots, a_n; b_1, \ldots, b_{n-j}, b_{n-j+2}, b_{n-j+1}, \ldots, b_n]$;

$s_{n+1}$ takes $[a_1, \ldots, a_n; b_1, \ldots, b_n]$ to $[b_1 - 1, a_2, \ldots, a_n; a_1 + 1, b_2, \ldots, b_n]$.

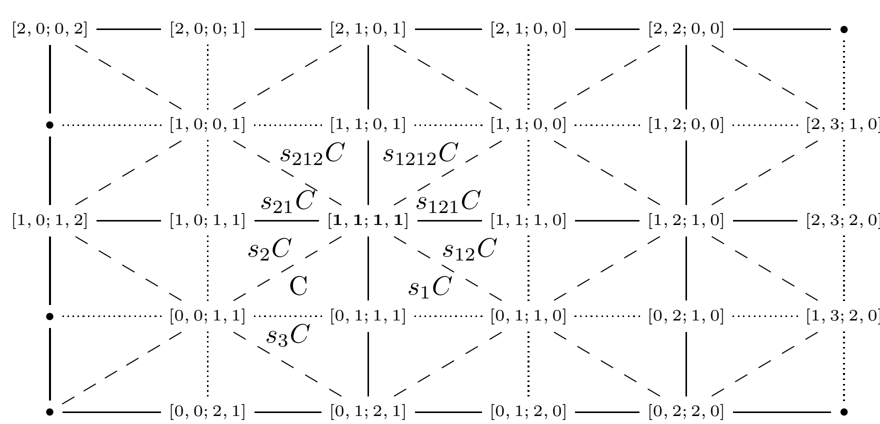
The group $\widetilde{C}_n$ is generated by $s_1, \ldots, s_{n+1}$; one easily checks that the $s_i$ satisfy the prescribed Coxeter data. To label the apartment $\Sigma$, we first note that each chamber contains two special vertices: one fixed by the reflections $s_1, \ldots, s_n$ and the other by $s_2, \ldots, s_{n+1}$. It is easily seen that the vertex $v = [a_1, \ldots, a_n; b_1, \ldots, b_n]$ is fixed by the first $n$ reflections if and only if $a_k = b_l$ for all $1 \leq k, l \leq n$. It is fixed by the last $n$ reflections if and only if for all $2 \leq j \leq n$, $a_{n-j+1} = a_{n-j+2}$, $b_{n-j+1} = b_{n-j+2}$ and $b_1 = a_1 + 1$; in other words, if and only if there is an integer $m$ such that for all $1 \leq k \leq n$, $a_k = m = b_k - 1$.

Since the group $\widetilde{C}_n$ acts transitively on the chambers in the apartment, we facilitate a labeling of the vertices of $\Sigma$ by fixing a chamber $C$, which we will call the fundamental chamber, and letting the group act on it. The chamber we choose is determined by the first chain of lattices in Example 2.2: $\Lambda_0 = (1, \ldots, 1; 1, \ldots, 1) \subsetneq \Lambda_1 = (0, 1, \ldots, 1; 1, \ldots, 1) \subsetneq \Lambda_2 = (0, 0, 1, \ldots, 1; 1, \ldots, 1) \subsetneq \cdots \subsetneq \Lambda_n = (0, 0, \ldots, 0; 1, \ldots, 1) \subsetneq \pi^{-1}\Lambda_0$. Note that the special vertices of this fundamental chamber are $[\Lambda_0]$ and $[\Lambda_n]$. The lattices $\Lambda_i$ defined here will be used in subsequent sections.

The codimension-one faces of this fundamental chamber may be labeled by the reflections $s_i$ so that their action on $C$ generates the rest of the chambers in the apartment. We illustrate this in the example below with $n = 2$.

**Example 2.3.** In labeling the chambers, it is natural to first establish the *residue* of the special vertex $[\Lambda_0]$ in $\Sigma$; that is, the set of chambers in $\Sigma$ containing it. The residue is naturally associated with the *link* of the vertex (see [3]). The residue is simply obtained by letting the spherical Weyl group $C_n = \langle s_1, \ldots, s_n \rangle \subsetneq \widetilde{C}_n$ act on the fundamental chamber. The Weyl group $C_n$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ (the signed permutation group) and has order $2^n n!$, so for $n = 2$ we expect 8 chambers in the apartment containing the given special vertex $[\Lambda_0]$. Thus, we start with the fundamental chamber $C$ given by the chain $\Lambda_0 = (1, 1; 1, 1) \subset (0, 1; 1, 1) \subset (0, 0; 1, 1) \subset (0, 0; 0, 0) = \pi^{-1}\Lambda_0$, and act on this chain with the group $C_2 = \langle s_1, s_2 \rangle = \{1, s_1, s_2, s_2s_1, s_1s_2, s_1s_2s_1, s_2s_1s_2, s_2s_1s_2s_1 = s_1s_2s_1s_2\}$. For ease in labeling, we abbreviate the composition $s_{i_1}s_{i_2}\cdots s_{i_k}$ as $s_{i_1 i_2 \cdots i_k}$.

We also indicate the action of the generators of $\widetilde{C}_2$. Continuing to apply the reflections in this way admits a labeling of the vertices of $\Sigma$ by classes of lattices:



## 3. Special vertices and the associated 1-complex

We consider the subcomplex of the building $\Delta_n$ obtained by restricting our attention to the special vertices. We give a lattice-theoretic characterization of the special vertices, show that there is a natural group structure on the set of special vertices, and then investigate the properties of the 1-complex one obtains from the building by restricting to special vertices. In particular, we show that the 1-complex is connected.

We retain the notation of the previous section with our fixed basis of the symplectic space $V$, apartment $\Sigma$, and fundamental chamber $C$. Since the building $\Delta_n$ arises from a $BN$-pair, we know (see Theorem p. 112 of [3]) that the action of $Sp_n(K)$ on $\Delta_n$ is type-preserving and strongly transitive. We briefly explain these terms (see [3] for a detailed explanation). The vertices of a building can be labeled in exactly the same sense that the vertices of a graph can be colored — with no two vertices connected by an edge having the same color. That the action is type-preserving means that it preserves the label (color) of the vertices. The labels for the vertices in $\Delta_n$ can be taken from the set $\{1, 2, \ldots, n+1\}$, corresponding to the $n+1$ vertices in a chamber. The action is strongly transitive in the sense (see [3]) that $Sp_n(K)$ acts transitively on the chambers of $\Delta_n$ and that the stabilizer of a given chamber $C$ acts transitively on the set of apartments containing $C$. Equivalently, we can say that $Sp_n(K)$ acts transitively on the apartments of $\Delta_n$ and the stabilizer of a given apartment $\Sigma$ acts transitively on the chambers in $\Sigma$.

In particular, because this action is type-preserving, special vertices are mapped to special vertices. Moreover, in the fixed chamber $C$ in $\Sigma$, the special vertex $[\Lambda_0] = [1, \ldots, 1; 1, \ldots, 1]$ is mapped to the other special vertex $[\Lambda_n] = [0, \ldots, 0; 1, \ldots, 1]$ by means of the matrix $\mathrm{diag}(1, \ldots, 1, \pi, \ldots, \pi) \in GSp_n^S(K)$. Because the action of $Sp_n(K)$ is also transitive on the chambers of $\Delta_n$, it is clear that every special vertex in the building is the image of $[\Lambda_0]$ under the action of $GSp_n^S(K)$. The converse is also true; to see this, we give an alternate characterization of special vertices as those which are *self-dual*.

Using our shorthand notation for lattices (relative to our fixed basis) in $V$, let $\Lambda$ be the lattice $\Lambda = (a_1, \ldots, a_n; b_1, \ldots, b_n)$. The dual lattice $\Lambda^\sharp$ is defined to be $\{v \in V \mid \langle v, \Lambda\rangle \subseteq \mathcal{O}\}$. It too is a lattice, and it is easily seen from the bilinearity of the alternating form that $\Lambda^\sharp = (-b_1, \ldots, -b_n; -a_1, \ldots, -a_n)$. It is also clear that $(\pi^\nu\Lambda)^\sharp = \pi^{-\nu}\Lambda^\sharp$, so $[\Lambda^\sharp]$ depends only on $[\Lambda]$, and in particular $[\Lambda] = [\Lambda^\sharp]$ if and only if $\pi^\mu\Lambda^\sharp = \Lambda$ for some $\mu \in \mathbb{Z}$.

**Proposition 3.1.** *Let $\Lambda = (a_1, \ldots, a_n; b_1, \ldots, b_n)$. Then $[\Lambda] = [\Lambda^\sharp]$ if and only if there exists an integer $\mu$, such that for all $i$, $a_i + b_i = \mu$. In this case we call the vertex self-dual.*

*Proof.* Using our explicit characterization of the dual lattice, $[\Lambda] = [\Lambda^\sharp]$ if and only if there exists an integer $\mu$ such that $\pi^\mu\Lambda^\sharp = \Lambda$; that is, if and only if $\mu - b_i = a_i$ and $\mu - a_i = b_i$, which is true if and only if $\mu = a_i + b_i$ for all $i$. □

**Proposition 3.2.** *If $\Lambda = (a_1, \ldots, a_n; b_1, \ldots, b_n)$ and $[\Lambda]$ is self-dual, then its images under the affine Weyl group $\widetilde{C}_n$ are again self-dual vertices. Moreover, the image of any non-self dual vertex is again not self-dual.*

*Proof.* We need only check this for the generators $s_i$ of the affine Weyl group, and all of these assertions are obvious from the above definitions. □

**Proposition 3.3.** *The group $GSp_n^S(K)$ acts transitively on the special vertices in the building $\Delta_n$.*

*Proof.* We have already observed that every special vertex in the building is the image of $[\Lambda_0]$ under the action of $GSp_n^S(K)$. We need only observe that the action of $GSp_n^S(K)$ on $[\Lambda_0]$ is always a special vertex. To see this, recall that $Sp_n(K)$ acts in a type-preserving manner on the vertices of $\Delta_n$; in particular, it takes special vertices to special vertices. Since $\Gamma = Sp_n(\mathcal{O}) \subset Sp_n(K)$, we know that any element $\xi \in GSp_n^S(K)$ will act (on vertex type) in the same way as any element of $\Gamma\xi\Gamma$. Thus by Lemma 4.1 (see below), we may assume that $\xi = \operatorname{diag}(\pi^{a_1}, \ldots, \pi^{a_n}, \pi^{b_1}, \ldots, \pi^{b_n})$ with $a_i + b_i$ constant. It is clear that the action of this $\xi$ on $[\Lambda_0]$ produces a self-dual vertex $v_0$ in $\Sigma$. We need to show that this vertex is special. If $v_0$ is not special, then via $\widetilde{C}_n$, we can translate $v_0$ back to a non-special vertex in the fundamental chamber $C$. By examining the chain of lattices which define $C$, we see that only two vertices are self-dual, and they are the special vertices. This means that $v_0$ is not self-dual, contradicting Proposition 3.2. $\square$

**Corollary 3.4.** *A vertex in the building $\Delta_n$ is special if and only if it is self-dual relative to any apartment in which it is viewed.*

*Proof.* Let $v_0$ be a special vertex and $\Sigma$ an apartment containing it. Let $C'$ be any chamber in $\Sigma$ containing $v_0$. For convenience of notation, we assume the same basis as before and fix a fundamental chamber $C$. Since there is an element of the Weyl group $\widetilde{C}_n \subset Sp_n(K)$ that maps $C'$ to $C$, $v_0$ is mapped to one of the two special vertices of $C$, which are the only self-dual vertices in $C$. By Proposition 3.2, $v_0$ is self-dual. The converse follows from the proof of Proposition 3.3. $\square$

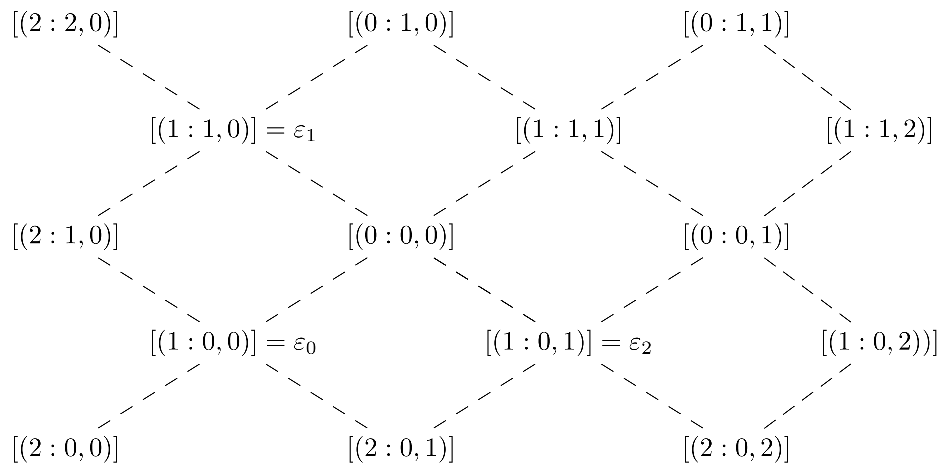
For our application to walks on the building, it is convenient here to make one further characterization of the special vertices in an apartment. As above, we work in the fixed apartment $\Sigma$ with symplectic basis $\{u_i, w_i\}$. From the above discussion, we saw that given a lattice $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$, the vertex $v_0 = [\Lambda] = [a_1, \ldots, a_n; b_1, \ldots, b_n]$ is special (self-dual) if and only if $a_i + b_i = \mu$ is constant. Moreover, the lattice $\Lambda$ is completely characterized by the data $(\mu, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$. For two special vertices $v_0$ and $v_0' = [a_1', \ldots, a_n'; b_1', \ldots, b_n']$, we have that $v_0 = v_0'$ if and only if $a_i' = a_i + k$ and $b_i' = b_i + k$ for all $i$ and some $k \in \mathbb{Z}$. To try to avoid confusion between the two different notations characterizing classes of lattices, we denote $v_0$ by $[(\mu : a_1, \ldots, a_n)]$. Then $[(\mu : a_1, \ldots, a_n)] = [(\mu' : a_1', \ldots, a_n')]$ if and only if $a_i' = a_i + k$ and $\mu' = \mu + 2k$. Thus there is a one-to-one correspondence between the special vertices in the apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

Indeed the natural group operation defined on $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$ induces one on the special vertices of an apartment, and there is a natural geometric interpretation of this group operation as well. We show that the subcomplex of $\Delta_n$ obtained by restriction only to special vertices is a connected 1-complex, and that the group operation corresponds to certain walks on this graph. A characterization of minimal walks is given in the final section of this paper.

Consider the special vertices in $\Sigma$ in the residue of a fundamental chamber containing $[(0 : 0, \ldots, 0)]$. Using the reflections defined in equation (2.2), it is easy to see that the collection of special vertices in this residue (excluding $[(0 : 0, \ldots, 0)]$ itself) consists of all vertices of the form $[(1 : \varepsilon_1, \ldots, \varepsilon_n)]$, where $\varepsilon_i \in \{0, 1\}$. That is, the special vertex $[\Lambda_n] = [0, \dots 0; 1, \dots, 1]$ in the fundamental chamber $C$ can be mapped to $[\delta_1, \dots, \delta_n; 1 - \delta_1, \dots, 1 - \delta_n]$ ($\delta_i \in \{0, 1\}$) by applying the reflections $s_1, \dots, s_n$ which generate the spherical Weyl group. For example, if $j$ is the smallest index such that $\delta_j = 1$, then the composition $s_{n-j+1} \cdots s_2 s_1$ maps $[\Lambda_n]$ to $[\gamma_1, \dots, \gamma_n; 1 - \gamma_1, \dots, 1 - \gamma_n]$ with $\gamma_j = 1$ and $\gamma_i = 0$ for $i \neq j$. Iterating in the obvious manner produces the desired special vertex.

Of these special vertices we now specify certain ones which offer us geometric insight into the group operation induced on the special vertices. For $1 \leq k \leq n$, denote by $\varepsilon_k$ the special vertex in the residue of $[\Lambda_0]$ having the form $\varepsilon_k = [(1 : \delta_1, \dots, \delta_n)]$ with $\delta_k = 1$, and $\delta_i = 0$ (for $i \neq k$), and let $\varepsilon_0 = [(1 : 0, \dots, 0)]$. Let $[(\mu : a_1, \dots, a_n)]$ be an element of $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \dots, 1)$. It is clear that as elements of the group, $[(\mu : a_1, \dots, a_n)] = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_n\varepsilon_n + (\mu - \sum_1^n a_i)\varepsilon_0$. The geometric interpretation we shall establish is that the $\varepsilon_k$ represent directions to walk from $[\Lambda_0]$ in the apartment $\Sigma$. As the vertices $\varepsilon_k$ are adjacent to $[\Lambda_0]$, these are walks of length one. We will show that any special vertex in the building is the endpoint of a walk, and in particular, that the subcomplex of the building generated by the special vertices is connected.

**Example 3.5.** For $Sp_2(K)$ we have the following (partial) labeling of the special vertices in an apartment by elements of $\mathbb{Z}^3/\mathbb{Z}(2, 1, 1)$. Note that in considering the 1-subcomplex of the apartment, we have removed all non-special vertices and the corresponding edges. Compare with Example 2.3.




The special vertices in the residue of $[(0 : 0, 0)]$ are labeled clockwise $[(1 : 1, 1)]$, $\varepsilon_2 = [(1 : 0, 1)]$, $\varepsilon_0 = [(1 : 0, 0)]$, and $\varepsilon_1 = [(1 : 1, 0)]$, and they define directions in which to move (relative to $[(0 : 0, 0)]$) within the apartment consistent with the group law: For example, $\varepsilon_2 - 2\varepsilon_0$ corresponds to a walk from $[(0 : 0, 0)]$ moving one unit in the direction indicated by $\varepsilon_2$ and then two units in the opposite direction indicated by $\varepsilon_0$, bringing us to $[(-1 : 0, 1)] = [(1 : 1, 2)]$. Thus, we can think of a vertex $[(\mu : a_1, \dots, a_n)]$ as the endpoint of a walk along the 1-subcomplex of the apartment (consisting of only the special vertices and associated edges) which is given by moving a certain number of units in the above mentioned directions.

**Proposition 3.6.** *The subcomplex generated by restricting to special vertices in $\Delta_n$ is a connected 1-complex.*

*Proof.* Clearly, given any two special vertices, we may assume they lie in a given apartment, so we use our fixed apartment $\Sigma$. It is clear that as elements of the group $\mathbb{Z}^{n+1}/\mathbb{Z}(2,1,\ldots,1)$, the element $[(\mu : a_1,\ldots,a_n)] = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_n\varepsilon_n + (\mu - \sum_1^n a_i)\varepsilon_0$, so it suffices to show for any special vertex $v_0 = [(\mu : a_1,\ldots,a_n)]$, that $v_0 \pm \varepsilon_k$ $(0 \le k \le n)$ is a special vertex incident to $v_0$ in $\Delta_n$ (in fact, in $\Sigma$). We treat the case of $v_0 + \varepsilon_k$; the case of $v_0 - \varepsilon_k$ is analogous (if $\varepsilon_k = [a_i; b_i]$, then $v_0 - \varepsilon_k = v_0 + [b_i; a_i]$).

To establish this, we return to the definition of the incidence relation defined in section 2. Given a special vertex $v_0 = [(\mu : a_1,\ldots,a_n)]$, we may reduce modulo $[(2:1,\ldots,1)]$ and so assume $\mu = 0$ or $1$.

If $\mu = 0$, let $L$ be the lattice $(a_1,\ldots,a_n; -a_1,\ldots,-a_n)$. Note that $v_0 = [L]$ and that $L$ is a primitive lattice (see section 2). Being somewhat sloppy, we want to define $L_k$ as a lattice representing $v_0 + \varepsilon_k$. More precisely, let $L_0 = (a_1,\ldots,a_n; 1-a_1,\ldots,1-a_n)$ and for $1 \le k \le n$, let $L_k = (a_1,\ldots,a_{k-1}, 1+a_k, a_{k+1},\ldots,a_n; 1-a_1,\ldots,1-a_{k-1}, -a_k, 1-a_{k+1},\ldots,1-a_n)$. In terms of the group, $v_0 + \varepsilon_k = [L_k]$. We note that $\pi L \subset L_k \subset L$ and that $L$ is primitive which means $[L] = v_0$ and $[L_k] = v_0 + \varepsilon_k$ are incident special vertices.

If $\mu = 1$, the roles of $L$ and $L_k$ reverse as follows. Let $L = (a_1,\ldots,a_n; 1-a_1,\ldots,1-a_n)$. Note that $v_0 = [L]$. Let $L_k$ be defined as above, and note that $v_0 + \varepsilon_k = [L_k]$ and that $L_k \subset L \subset \pi^{-1}L_k$ with $\pi^{-1}L_k$ primitive; hence $v_0 = [L]$ and $v_0 + \varepsilon_k = [L_k]$ are incident special vertices. $\square$

We shall return to this 1-subcomplex in the final section of the paper where we make use of Hecke operators to characterize the endpoints of minimal walks on this graph.

## 4. A representation of the local Hecke algebra

To produce operators acting on the building, we define an essentially faithful representation of a local Hecke algebra acting on the special vertices of the building for $Sp_n$. This representation is quite natural, generalizing both the notion of adjacency operators on a graph and Serre's action of the Hecke algebra on trees (see [7] for the case of $SL_2$ and [2] for higher rank generalizations). To start, we need to discuss how the lattices which define the special vertices of the building are connected to elementary divisors and how the elementary divisors are connected to double cosets of the Hecke algebra.

4.1. **Symplectic lattices and elementary divisors.** We begin with a short discussion about lattices and elementary divisors in the symplectic setting. Retaining the notation of earlier sections, $K$ is a local field, $\mathcal{O}$ its ring of integers, $\pi$ a fixed uniformizing parameter, and $(V, \langle *, * \rangle)$ a $2n$-dimensional symplectic space over $K$. Let $S = K^\times/\mathcal{O}^\times$; for a convenient set of representatives we fix $S = \{\pi^\nu \mid \nu \in \mathbb{Z}\}$. As before, we denote by $GSp_n^S(K) = \{M \in GSp_n(K) \mid r(M) \in S\}$. We again note that $Sp_n(K) \subset GSp_n^S(K)$, and put $\Gamma = Sp_n(\mathcal{O})$.

Fix a symplectic basis $\{u_1,\ldots,u_n,w_1,\ldots,w_n\}$ of $V$ satisfying $\langle u_i, w_j\rangle = \delta_{ij}$ (Kronecker delta), $\langle u_i, u_j\rangle = \langle w_i, w_j\rangle = 0$. With obvious modification to the proof, the following is Lemma 3.6 of [1].

**Lemma 4.1.** *Let $\xi \in GSp_n^S(K)$. Then every double coset $\Gamma\xi\Gamma$ has a unique representative of the form $sd(\xi) = \operatorname{diag}(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)$, where $\alpha_i, \beta_i \in S$ satisfy $\alpha_i \mid \alpha_{i+1}$, $\alpha_n \mid \beta_n$, $\beta_{i+1} \mid \beta_i$, and $\alpha_i\beta_i = r(\xi)$.*

*Remark* 4.2. Classically, the diagonal representative of the double coset is called the Smith normal form of the matrix $\xi$, while in more modern terms, this process reflects the $p$-adic Cartan decomposition of the group.

We call a lattice symplectic if it has an $\mathcal{O}$-basis which is a symplectic basis for $V$ with respect to the alternating bilinear form on $V$. The following proposition is easily established. Note that in its statement and proof, we follow [8] and use a right action on lattices to facilitate a cosmetically more pleasing result in Lemma 4.6.

**Proposition 4.3.** *Let $\mathcal{L}$ be a symplectic lattice. Then $\Gamma = Sp_n(\mathcal{O})$ can be identified with $\{A \in GSp_n^S(K) \mid \mathcal{L}A = \mathcal{L}\}$, where $A$ acts on $\mathcal{L}$ as the matrix of a linear transformation with respect to a fixed basis of $\mathcal{L}$.*

To set up the correct analog of elementary divisor theory, we need to fuss a bit more than in the general linear case. To begin, fix a symplectic lattice $\mathcal{L}$ and put $\mathcal{R} = \mathcal{R}_{\mathcal{L}} = \{\mathcal{L}A \mid A \in GSp_n^S(K)\}$. Note that in the general linear case, $GSp_n$ would be replaced by $GL_{2n}$, and $\mathcal{R}$ would be the set of all lattices of full rank in $V$, and so $\mathcal{R}$ would not need to be defined at all.

**Lemma 4.4.** *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. Then there exists a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of $V$ and elements $\alpha_i, \beta_i \in S$ satisfying $\beta_1\mathcal{O} \subset \cdots \subset \beta_n\mathcal{O} \subset \alpha_n\mathcal{O} \subset \cdots \subset \alpha_1\mathcal{O}$ and $\beta_i\alpha_i = r \in S$ such that $\mathcal{M} = \bigoplus_{i=1}^n \mathcal{O}u_i \oplus \bigoplus_{i=1}^n \mathcal{O}w_i$ and $\mathcal{N} = \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i$.*

*Remark* 4.5. The ideals $\alpha_i\mathcal{O}$ and $\beta_j\mathcal{O}$ are called the symplectic divisors of $\mathcal{N}$ in $\mathcal{M}$ and coincide with the standard elementary divisors $\{\mathcal{M} : \mathcal{N}\}$ since $\Gamma \subset SL_{2n}(\mathcal{O})$. That is, if we choose two lattices from $\mathcal{R}$ and consider their elementary divisors in the traditional sense, they are in fact symplectic elementary divisors with the above-stated additional properties. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are as in the lemma, we will write $\{\mathcal{M} : \mathcal{N}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ to mean there exist bases of $\mathcal{M}$ and $\mathcal{N}$ as in the lemma.

*Proof.* Since $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{R}$, there exists an $A \in GSp_n^S(K)$ with $\mathcal{N} = \mathcal{M}A$. Identify $\Gamma$ with the stabilizer of $\mathcal{M}$. By Lemma 4.1, $sd(A) = \mathrm{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \gamma_1 A \gamma_2$ for some $\gamma_i \in \Gamma$, where $sd(A)$ is the *symplectic divisor* matrix of $A$. Finally, since $\mathcal{M}\gamma_i = \mathcal{M}$, it is clear that $\{\mathcal{M} : \mathcal{N}\} = \{\mathcal{M}\gamma_1 : \mathcal{M}\gamma_1 A\} = \{\mathcal{M}\gamma_1\gamma_2 : \mathcal{M}\gamma_1 A\gamma_2\} = \{\mathcal{M} : \mathcal{M}sd(A)\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, from which the lemma follows. □

**Lemma 4.6.** *For $A$ and $B$ in $GSp_n^S(K)$, $\Gamma A = \Gamma B$ if and only if $\mathcal{L}A = \mathcal{L}B$.*

*Proof.* Note that $\Gamma A = \Gamma B$ if and only if $AB^{-1} \in \Gamma$, which by Proposition 4.3 is true if and only if $\mathcal{L} = \mathcal{L}AB^{-1}$. □

**Lemma 4.7.** *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. The elementary divisors of $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{L}$ satisfy $\{\mathcal{L} : \mathcal{M}\} = \{\mathcal{L} : \mathcal{N}\}$ if and only if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$.*

*Proof.* The result is clear if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$. To prove the converse, we note that by definition of the symplectic elementary divisors, there exist elements $\alpha_i,\ \beta_i \in S$ satisfying $\beta_1\mathcal{O} \subset \cdots \subset \beta_n\mathcal{O} \subset \alpha_n\mathcal{O} \subset \cdots \subset \alpha_1\mathcal{O}$ and $\beta_i\alpha_i = r \in S$ and symplectic $\mathcal{O}$-bases

$$\{u_1^{(j)}, \ldots, u_n^{(j)}, w_1^{(j)}, \ldots, w_n^{(j)}\} \qquad (j = 1, 2)$$

of $\mathcal{L}$ such that

$$\mathcal{L} = \bigoplus_{i=1}^n \mathcal{O}u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}w_i^{(1)}, \qquad \mathcal{M} = \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i^{(1)},$$
$$\mathcal{L} = \bigoplus_{i=1}^n \mathcal{O}u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}w_i^{(2)}, \qquad \mathcal{N} = \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i^{(2)}.$$

Let $A$ be the matrix of the linear transformation (with respect to either basis) taking $u_i^{(1)} \mapsto u_i^{(2)}$, and $w_i^{(1)} \mapsto w_i^{(2)}$. Clearly $A \in Sp_n(K) \subset GSp_n^S(K)$ as it maps one symplectic basis to another. Since $\mathcal{L}A = \mathcal{L}$, $A \in \Gamma$ by Proposition 4.3. Since $A$ obviously maps $\mathcal{M}$ to $\mathcal{N}$, the proof is complete. □

**Proposition 4.8.** *Let $\mathcal{L} \in \mathcal{R}$, $\Gamma$ the stabilizer of $\mathcal{L}$, $A \in GSp_n^S(K)$, and*

$$\Gamma A\Gamma = \Gamma sd(A)\Gamma = \Gamma \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\Gamma.$$

*Then $\Gamma\xi \mapsto \mathcal{L}\xi$ gives a one-to-one correspondence between the cosets $\Gamma\xi$ in $\Gamma A\Gamma$ and lattices $M \in \mathcal{R}$ with $\{\mathcal{L} : \mathcal{M}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$.*

*Proof.* We may assume that $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. If $\Gamma\xi = \Gamma A\delta$ with $\delta \in \Gamma$, then $\mathcal{L}\xi \in \mathcal{R}$ and we have $\{\mathcal{L} : \mathcal{L}\xi\} = \{\mathcal{L} : \mathcal{L}A\delta\} = \{\mathcal{L} : \mathcal{L}A\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n\}$. Conversely, if $\mathcal{M} \in \mathcal{R}$ and $\{\mathcal{L} : \mathcal{M}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, then by Lemma 4.7, there exists an element $B \in \Gamma$ such that $\mathcal{M} = \mathcal{L}AB$. Clearly, $\Gamma AB \subset \Gamma A\Gamma$. The correspondence is one-to-one since by Lemma 4.6, $\Gamma\xi = \Gamma\zeta$ if and only if $\mathcal{L}\xi = \mathcal{L}\zeta$. □

4.2. **The representation.** We now give a natural representation of the local Hecke algebra in which the Hecke operators act on the special vertices of the building for $Sp_n(K)$. In addition, we shall show how the operators in this representation space correspond to adjacency operators on the associated 1-subcomplex of the building. In the next section, we use these operators to characterize minimal walks on this subset of the building.

Often in the context of buildings, especially as so much of the theory is related to the representation theory of classical $p$-adic groups, one considers the local Hecke algebra as a convolution algebra of compactly supported $\Gamma$-bi-invariant functions $GSp_n(K) \to \mathbb{C}$ [4]. In this setting, the classical double cosets considered in the previous section are viewed as characteristic functions associated to the double cosets. In large part, the purpose is then to obtain a natural action of the Hecke algebra on the set of compactly supported functions which act on the vertices of the building in question. Usually this occurs by identifying the set of vertices in the building with a quotient such as $GSp_n(K)/\Gamma$. While this perspective affords a rather general context in which to view many similar problems, we have not chosen this perspective, as it would move the exposition a good deal farther from the concrete characterizations of special vertices in terms of lattices. Indeed, given the explicit lattice-theoretic characterization of special vertices, Lemma 4.7 and Proposition 4.8 provide a transparent connection between the right cosets comprising a given double coset and sublattices of a given lattice with a prescribed set of elementary divisors. An operator which sums over right cosets of a given double coset is a classically defined Hecke operator in the sense of [8], while the notion of summing over (classes of) lattices is the immediate analog of Serre's original work on trees [7], as well as its generalizations [2].

To define the representation, let $E$ be any field of characteristic zero, and consider the local Hecke algebra $\mathcal{H}$ generated as a vector space over $E$ by the double cosets $\Gamma\xi\Gamma$ with $\xi \in GSp_n^S(K)$. By Lemma 4.1, we may assume all $\xi$ have the form $\xi = \text{diag}(\pi^{a_1}, \dots, \pi^{a_n}, \pi^{b_1}, \dots, \pi^{b_n})$, where $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$. To introduce the algebra structure on $\mathcal{H}$, we give its multiplication law (e.g., see section 3.1 of [8]):

Let $\xi_1 = \text{diag}(\pi^{a_1}, \dots, \pi^{a_n}, \pi^{b_1}, \dots, \pi^{b_n})$ and $\xi_2 = \text{diag}(\pi^{c_1}, \dots, \pi^{c_n}, \pi^{d_1}, \dots, \pi^{d_n})$ be elements of $GSp_n^S(K)$, and write $\Gamma\xi_1\Gamma$ as the disjoint union $\bigcup \Gamma\alpha_i$ and $\Gamma\xi_2\Gamma$ as the disjoint union $\bigcup \Gamma\beta_j$. Then

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \Gamma\xi_1\Gamma\xi_2\Gamma = \sum_{\Gamma\xi\Gamma} c(\xi)\Gamma\xi\Gamma,$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subset \Gamma\xi_1\Gamma\xi_2\Gamma$ and $c(\xi)$ is the number of pairs $(i, j)$ for which $\Gamma\alpha_i\beta_j = \Gamma\xi$.

We have previously noted that the vertices of the building, $\text{Vert}(\Delta_n)$, are in one-to-one correspondence with homothety classes of certain lattices in our fixed vector space $V$; however, our action will only be on the special vertices. So we let $\mathcal{B}$ be the vector space over $E$ with a basis consisting of the special vertices in $\text{Vert}(\Delta_n)$.

Let $L$ be a lattice in $V$ with $[L]$ a special vertex in $\Delta_n$, and identify $\Gamma = Sp_n(\mathcal{O})$ with the stabilizer of $L$ in $GSp_n^S(K)$. Let $\xi = \text{diag}(\pi^{a_1}, \dots, \pi^{a_n}, \pi^{b_1}, \dots, \pi^{b_n}) \in GSp_n^S(K)$. By Proposition 4.8, we know that the double coset $\Gamma\xi\Gamma$ determines a collection of right cosets $\{\Gamma\xi_\nu\}$ which are in one-to-one correspondence with the collection of lattices $\{M\}$ with $\{L : M\} = \{\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n}\}$. Note that all of these lattices $M$ are contained in $\mathcal{R} = \{LA \mid A \in GSp_n^S(K)\}$, and hence by the discussion above, their classes are all special vertices.

In the context of Hecke operators acting on modular forms, the natural action of a double coset on the modular form is to sum the actions on the form by the right cosets comprising the double coset. Using the notation above, it is then natural to define the operator $T_{\mathcal{B}}(\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n}) \in \text{End}(\mathcal{B})$ induced by

$$T_{\mathcal{B}}(\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n})([L]) = \sum_{\{L:M\} = \{\pi^{a_1}, \dots, \pi^{a_n}, \pi^{b_1}, \dots, \pi^{b_n}\}} [M],$$

where the sum is over all (special) vertices in the building with prescribed elementary divisors. For brevity, we shall write $T_{\mathcal{B}}(\xi)([L]) = \sum_{\{L:M\}=\xi} [M]$. The map is clearly well-defined and (by definition) linear.

**Theorem 4.9.** *The correspondence $\Gamma\xi\Gamma \mapsto T_{\mathcal{B}}(\xi)$ induces a representation $\Psi : \mathcal{H} \to \text{End}(\mathcal{B})$, whose kernel consists of double cosets of the form $\Gamma\xi\Gamma$ with $\xi = \pi^\mu I_{2n}$, $\mu \in \mathbb{Z}$.*

*Proof.* We first verify that $\Psi$ is a ring homomorphism. Using the notation above, we have

$$\begin{aligned} T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) &= T_{\mathcal{B}}(\xi_1)\Big(\sum_{\{L:M\}=\xi_2} [M]\Big) \\ &= \sum_{\{L:M\}=\xi_2} \sum_{\{M:N\}=\xi_1} [N]. \end{aligned}$$

By Proposition 4.8, each lattice $M$ for which $\{L : M\} = \xi_2$ is of the form $M = L\beta_j$. Now

$$\{M : N\} = \xi_1 \iff \{L\beta_j : N\} = \xi_1 \iff \{L : N\beta_j^{-1}\} = \xi_1.$$

Let $P$ be such that $\{L : P\} = \xi_1$. Then, again by Proposition 4.8, $P = L\alpha_i$ for some $i$. But then $P = N\beta_j^{-1}$, so $N = P\beta_j = L\alpha_i\beta_j$. Thus,

$$T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) = \sum_{\{L:M\}=\xi_2}\sum_{\{M:N\}=\xi_1}[N] = \sum_{i,j}[L\alpha_i\beta_j].$$

From the discussion preceding the theorem (and once again Proposition 4.8), this last sum is exactly $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L])$, which is the image of $(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma)$.

To compute the kernel of $\Psi$, suppose $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)$ is the trivial map. Then

$$\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L]) = \sum_{\Gamma\xi\Gamma} c(\xi)\sum_{\{L:M\}=\xi}[M] = [L]$$

for all special vertices $[L] \in \mathrm{Vert}(\Delta_n)$. But since the special vertices $[M] \in \mathrm{Vert}(\Delta_n)$ are a basis for $\mathcal{B}$, we have only one $\xi$, and for that $\xi$, $c(\xi) = 1$. Thus, $\sum_{\{L:M\}=\xi}[M] = [L]$ for all $[L]$. Now, if $\Gamma\xi\Gamma = \bigcup \Gamma\xi_\nu$, then by Proposition 4.8, $\sum_{\{L:M\}=\xi}[M] = \sum_\nu [L\xi_\nu] = [L]$, so there can be only one right coset: $\Gamma\xi\Gamma = \Gamma\xi$, and $[L\xi] = [L]$. Since $\{L : L\xi\} = \xi$, we must have $\xi = \pi^\mu I_{2n}$ for some integer $\mu$. $\square$

We have suggested that the operator $T_{\mathcal{B}}$ can be interpreted as an adjacency operator. To give a flavor of things, we begin with an example. The reader should refer to Example 2.3 for the labeling of the vertices.

**Example 4.10.** For $Sp_2$, there are three generators of the algebra $\mathcal{H}$, $T(\pi) = \Gamma\,\mathrm{diag}(1, 1, \pi, \pi)\Gamma$, $T_1^2(\pi^2) = \Gamma\,\mathrm{diag}(1, \pi, \pi^2, \pi)\Gamma$ and $T_2^2(\pi^2) = \Gamma\,\mathrm{diag}(\pi, \pi, \pi, \pi)\Gamma$ whose images under the representation are respectively $T_{\mathcal{B}}(1, 1, \pi, \pi)$, $T_{\mathcal{B}}(1, \pi, \pi^2, \pi)$, and $T_{\mathcal{B}}(\pi, \pi, \pi, \pi)$. The last operator acts trivially, but the first two are of real interest. Restricted to the fundamental apartment (see Example 2.3), $T_{\mathcal{B}}(1, 1, \pi, \pi)$ sums the four special vertices closest to $[1, 1; 1, 1]$, namely $[0, 0; 1, 1] + [0, 1; 1, 0] + [1, 1; 0, 0] + [1, 0; 0, 1]$, while $T_{\mathcal{B}}(1, \pi, \pi^2, \pi)$ sums the four special vertices "next closest" to $[1, 1; 1, 1]$, namely $[0, 1; 2, 1] + [1, 2; 1, 0] + [2, 1; 0, 1] + [1, 0; 1, 2]$. Thus, both operators act as adjacency operators on the underlying 1-complex.

We amplify this example with some general considerations. An adjacency operator is often defined as a sum of vertices at a fixed distance from a given vertex, where the definition of distance can vary. In the context of a building, there are many notions of distance, some abstract, and others tied to the characterization of the building as a chamber complex. We focus on two: length of edge path (since we are looking at the underlying 1-complex of special vertices) and gallery length (the number of codimension-one faces crossed in moving from one chamber to another). The first adjacency operator in the example corresponds to identifying special vertices which are edge distance one (gallery distance zero) from the given vertex, while the second corresponds to special vertices edge distance two (gallery distance one) from the given vertex. We examine this in some detail, reducing the considerations to our fixed apartment $\Sigma$.

Recall that we defined our fundamental chamber $C$ by means of the lattices: $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n \subset \pi^{-1}\Lambda_0$, where $[\Lambda_0]$ and $[\Lambda_n]$ are the two special vertices in $C$. In the affine Weyl group $\widetilde{C}_n = \langle s_1, \ldots, s_{n+1}\rangle$, the subgroup $C_n = \langle s_1, \ldots, s_n\rangle$ is the stabilizer of the vertex $[\Lambda_0]$, and the image of $C$ under this subgroup generates the residue (in $\Sigma$) of that vertex. More explicitly, the chambers in the residue of $[\Lambda_0]$ in $\Sigma$ have the form $\gamma C$, corresponding to the chain of lattices $\Lambda_0 \subset \gamma\Lambda_1 \subset \cdots \subset \gamma\Lambda_n \subset \pi^{-1}\Lambda_0$.

To analyze the situation further, consider the subgroup $W$ of $C_n$ that fixes $\Lambda_n$. Thus, $W$ is the stabilizer in $\widetilde{C}_n$ of the edge with vertices $[\Lambda_0]$ and $[\Lambda_n]$. Using the standard poset isomorphism between faces of the fundamental chamber and *special* subgroups of $C_n$ (generated by subsets of $\{s_1, \ldots, s_n\}$), we see that $W = \langle s_2, \ldots, s_n \rangle$. The group $W$ is a spherical Weyl group of type $A_{n-1}$ and hence is isomorphic to the symmetric group on $n$ letters. Thus, within the apartment, the special vertices in the residue of $[\Lambda_0]$ have the form $[\gamma\Lambda_n]$, where $\gamma \in C_n/W$, producing $2^n$ such vertices. One can, in fact, be completely explicit.

In equation (2.2), we describe the action of the reflections $s_i$ on the vertices $[a_1, \ldots, a_n; b_1, \ldots, b_n]$. Identifying $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ with $(1, 2, \ldots, 2n)$ in the obvious manner, we rewrite the $s_i$ $(1 \leq i \leq n)$ as elements of the symmetric group on $2n$ letters written as the product of disjoint transpositions:

$$\begin{aligned} &s_1 : (n\ 2n); \\ &s_j(2 \leq j \leq n) : ((n-j+1)\ (n-j+2))\ \ ((2n-j+1)\ (2n-j+2)). \end{aligned} \tag{4.1}$$

It is easy to see that $s_2, \ldots, s_n$ act as the transpositions $(n-1\ n), (n-2\ n-1), \ldots, (1\ 2)$ on the first $n$ entries of the lattice (with mirrored action on the last $n$ entries), so they clearly generate the symmetric group on $n$ letters. Acting on $\Lambda_n = (0, \ldots, 0; 1, \ldots, 1)$, we see that $s_1$ takes $\Lambda_n$ to $(0, \ldots, 0, 1; 1, \ldots, 1, 0)$, and then acting repeatedly by elements of $W$ and $s_1$ produces the $2^n$ representatives of the form $\gamma\Lambda_n = (a_1, \ldots, a_n; b_1, \ldots, b_n)$ with $a_i, b_i \in \{0, 1\}$ and $a_i + b_i = 1$ for all $i$.

Next, consider the adjacency operator corresponding to special vertices which are gallery distance one from $[\Lambda_0]$. Notice that the reflection $s_{n+1}$ fixes the vertices $[\Lambda_k]$ for $1 \leq k \leq n$, so $s_{n+1}$ takes the fundamental chamber $C$ to $s_{n+1}C$ given by the chain of lattices: $s_{n+1}\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n \subset \pi^{-1}s_{n+1}\Lambda_0$. The chamber $s_{n+1}C$ contains the special vertex $s_{n+1}\Lambda_0 = [0, 1, \ldots, 1; 2, 1, \ldots, 1]$ and shares the codimension-one face generated by the $[\Lambda_k]$ for $1 \leq k \leq n$; hence, the special vertices $[\Lambda_0]$ and $s_{n+1}[\Lambda_0]$ are gallery distance one apart. The translation of $C$ to $\gamma C$ $(\gamma \in C_n)$ produces the set of all special vertices in $\Sigma$ which are gallery distance one from $[\Lambda_0]$, namely $\gamma s_{n+1}[\Lambda_0]$. The distinct vertices correspond to $\gamma \in C_n/(C_n \cap s_{n+1}C_n s_{n+1})$.

## 5. Hecke operators and walks

In Example 3.5, we first suggested a connection between the labeling of special vertices and walks on the 1-subcomplex of the building generated by the special vertices. In this last section, we characterize minimal walks in the building of a prescribed length in terms of the action of the Hecke operators defined in the previous section.

Fix an apartment in the building by specifying a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$. We previously showed that the special vertices in the apartment are in one-to-one correspondence with the elements of $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$. Recall the Hecke operators

$$T_{\mathcal{B}}(\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n})([L]) = \sum_{\{L:M\} = \{\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}\}} [M].$$

Restricted to our given apartment, this sum is fairly easy to characterize. All lattices $M$ in the apartment have the form $[c_1, \dots, c_n; d_1, \dots, d_n]$. For simplicity, normalize $L = [0, \dots, 0; 0, \dots, 0]$. Then $\{L : M\} = \{a_1, \dots, a_n, b_1, \dots, b_n\}$ means that the $c_i$ and $d_i$ are chosen from among the $a_i$ and $b_i$. But the choices are more constrained. For each $i$, $c_i$ is either some $a_{\sigma(i)}$ or some $b_{\sigma(i)}$ for $\sigma \in S_n$. But then $d_i$ is determined by the choice of $c_i$ since $c_i + d_i$ is constant. In particular (assuming the normalization of $L$ as above), the set of lattices $M$ with the prescribed elementary divisors are those obtained by acting on $[a_1, \dots, a_n; b_1, \dots, b_n]$ by all the elements of the spherical Weyl group $C_n$.

The interpretation of $T_{\mathcal{B}}(\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n})$ on the building $\Delta_n$ is a bit more complicated. By a minimal walk between two vertices, we simply mean a walk (a sequence of vertices $\{v_1, \dots, v_m\}$ in which each pair $\{v_i, v_{i+1}\}$ is connected by an edge) between the two vertices which is of minimal length. Again we reiterate that we are considering only the 1-subcomplex of the building spanned by the special vertices. We characterize the endpoints of minimal walks in the building in the following theorem.

**Theorem 5.1.** *Let $v_0 = [L]$ be a special vertex in the Bruhat-Tits building $\Delta_n$ for $Sp_n(K)$. The set of special vertices in the building which are endpoints of minimal walks of length $m$ from $v_0$ are the summands of*

$$\sum_{0 \le a_2 \le \cdots \le a_n \le m/2} T_{\mathcal{B}}(1, \pi^{a_2}, \dots, \pi^{a_n}; \pi^m, \pi^{m-a_2}, \dots, \pi^{m-a_n})([L]).$$

*Proof.* Consider a minimal walk, $\gamma$, between two vertices $v_0$ and $v_m$ in $\Delta_n$. Denote the walk by the sequence of vertices through which it passes: $\gamma = \{v_0, v_1, \dots, v_m\}$. Choose chambers $C_0$ and $C_m$ with $v_0 \in C_0$ and $v_m \in C_m$, and let $\Sigma$ be an apartment containing $C_0$ and $C_m$. Finally, let $\rho = \rho_{\Sigma, C_0}$ be the canonical retraction of $\Delta_n$ onto $\Sigma$ centered at $C_0$. The canonical retraction $\rho$ is the unique chamber map (a simplicial map which preserves dimensions of simplices) from $\Delta_n \to \Sigma$ that fixes $C_0$ pointwise and preserves gallery distances from $C_0$ (see chapter 4 of either [3] or [5]).

Since the retraction $\rho$ is a simplicial map, it takes the walk $\gamma$ to another walk $\rho(\gamma) = \{\rho(v_0), \rho(v_1), \dots, \rho(v_{m-1}), \rho(v_m)\}$ contained in $\Sigma$. But $v_0$ and $v_m$ are both in $\Sigma$, so they are fixed pointwise by $\rho$, making $\rho(\gamma)$ a walk in $\Sigma$ from $v_0$ to $v_m$. Moreover, it is clear that $\rho(\gamma)$ has length at most $m$, since it is a walk defined by $m+1$ (not necessarily distinct) vertices, and hence, by at most $m+1$ distinct vertices. Finally, since $m$ is the length of any minimal walk from $v_0$ to $v_m$, $\rho(\gamma)$ must have length $m$ and hence, is a minimal walk in $\Sigma$ from $v_0$ to $v_m$.

Since our interest is only to count the endpoints of minimal walks of length $m$, we may assume from the argument above that any such walk is wholly contained in an apartment. Thus, we need only characterize the vertices of an apartment which are the endpoints of minimal walks (in that apartment) of length $m$. Let $v = [a_1, \dots, a_n; b_1, \dots, b_n]$ $(a_i + b_i = \mu)$ be such a vertex. The Weyl group acting on the apartment will take any walk in the apartment to another of the same length. Since we will use the Weyl group to count endpoints of minimal walks in the apartment, there is no loss of generality in assuming that $v$ is chosen with $0 \le a_1 \le \cdots \le a_n \le b_n \le \cdots \le b_1$. Moreover, since the vertex is defined by the homothety class of a lattice, we may assume that $a_1 = 0$. Recall that there is a one-to-one correspondence between the vertices of an apartment and elements in $\mathbb{Z}^{n+1}/\mathbb{Z}(2,1,1,\dots,1)$. Our normalized $v$ has the form $v = [(\mu : 0, a_2, \dots, a_n)]$, where $0 \le a_2 \le \cdots \le a_n \le \mu$. In fact all the $a_i \le \mu/2$ since $a_i \le a_n \le b_n$ and $a_n + b_n = \mu$. We claim that $\mu = m$. Define elements of $\mathbb{Z}^{n+1}$: $\delta_0 = (1, 0, \dots, 0)$, $\delta_1 = (1, 0, \dots, 0, 1), \dots, \delta_{n-1} = (1, 0, 1, \dots, 1)$. First note that the directions $[\delta_0]$, $[\delta_1], \dots, [\delta_{n-1}]$ are independent in the sense that $\sum_{k=0}^{n-1} c_k\delta_k \in \mathbb{Z}(2, 1, \dots, 1)$ if and only if $\sum_{k=0}^{n-1} c_k\delta_k = 0$ if and only if $c_k = 0$ for all $k$. Now we return to our vertex $v = [(\mu : 0, a_2, \dots, a_n)]$ as above. If $\mu = 1$, then $0 \le a_2 \le \cdots \le a_n \le 1/2$, so $v = [(1 : 0, \dots, 0)]$ is one of the special vertices in the residue of $[(0 : 0, \dots, 0)]$ and hence, is the endpoint of a walk of length one.

Next consider the case $\mu > 1$. Then

$$v=[(\mu : 0, a_2, \dots, a_n)] = a_2[\delta_{n-1}]+(a_3-a_2)[\delta_{n-2}]+\cdots+(a_n-a_{n-1})[\delta_1]+(\mu-a_n)[\delta_0].$$

Each summand has the form $c[\delta_i]$ and so represents a walk of length $c$ in the direction $[\delta_i]$. By the independence of the $[\delta_i]$, we conclude that the above walk is minimal (and of length $\mu$); hence, $\mu = m$.

For a vertex $v$, denote by $v^{C_n}$ the orbit of $v$ under the action of the spherical Weyl group. Then in a given apartment, the endpoints of minimal walks of length $m$ starting from $[(0 : 0, \dots, 0)]$ are given by the summands of

$$\sum_{0\le a_2\le\cdots\le a_n\le m/2} [(m : 0, a_2, \dots, a_n)]^{C_n}.$$

From this, the theorem follows immediately. □

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