

MODULI OF ROOTS OF LINE BUNDLES ON CURVES

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ABSTRACT. We treat the problem of completing the moduli space for roots of line bundles on curves. Special attention is devoted to higher spin curves within the universal Picard scheme. Two new different constructions, both using line bundles on nodal curves as boundary points, are carried out and compared with pre-existing ones.

1. INTRODUCTION

The problem of compactifying the moduli space for roots of line bundles on curves is the object of this paper.

More precisely, consider a family of nodal curves $f: \mathcal{C} \rightarrow B$, with nonsingular fibers over an open subset U of B ; let \mathcal{N} be a line bundle on \mathcal{C} , viewed as a family of line bundles on the fibers of f . Assume that the relative degree of \mathcal{N} is divisible by some positive integer r . Then the family of r -th roots¹ of the restriction of \mathcal{N} to the nonsingular fibers is an étale covering of U .

The issue is how to compactify over B such a covering, in a modular way. We define and explore two different approaches, and compare our resulting moduli spaces between themselves and with pre-existing ones.

One of the first cases in which such a question was considered is the one of 2-torsion points in the Jacobian, that is, the case $N = \mathcal{O}$, $r = 2$. This was solved by A. Beauville [4] by means of admissible coverings.

Another remarkable, well-known instance is the one of the so-called “higher spin curves”, which corresponds to $\mathcal{N} = \omega_f$ and, more generally, to $\mathcal{N} = \omega_f^{\otimes l}$.

The case of genuine “spin curves”, that is, $r = 2$ (and $l = 1$), was solved by Cornalba in [8], where a geometrically meaningful compactification was constructed over the moduli space \overline{M}_g of Deligne-Mumford stable curves. The boundary points are certain line bundles on nodal curves.

Later T. J. Jarvis (in [14] and [15]) used rank 1 torsion-free sheaves to approach the problem (over stable curves) for all r and l . In particular, he constructed two stack compactifications $\overline{\mathfrak{S}}$ and ROOT, the first of which is the normalization of the second, and turns out to be smooth. For $r = 2$ his compactifications are all isomorphic to the one of [8].

Other methods have recently been used by D. Abramovich and T. J. Jarvis [1] and by A. Chiodo [7]. Both employ (differently) “twisted curves”, which, roughly speaking, are nodal curves with a stack structure at some nodes. The resulting compactifications are isomorphic to Jarvis’s $\overline{\mathfrak{S}}$.

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¹If L and N are line bundles on X such that $L^{\otimes r} = N$, we call L an r -th root of N .

Over the last decade, interest in higher spin curves has been revived by the generalized Witten conjecture, which predicts that the intersection theory on their moduli spaces is governed by the Gelfand-Dikiĭ (also known as higher KdV) hierarchy. This conjecture is open;² see [18] for more details and some recent progress.

The first part of our paper presents a solution of the general problem in the same spirit as [8]; we complete the r -th root functor (and its moduli scheme) for any family and any line bundle \mathcal{N} as above, by means of line bundles on nodal curves. For clarity (and to distinguish it from other approaches) we name the resulting space the moduli space for *limit roots*. Some of the results in this part can already be found, in one form or another, in the existing literature; however, it seemed worthwhile to present a coherent account of the subject based on the ideas of [8]. In order to limit the length of the paper, we work in the context of schemes and coarse moduli spaces, although it is quite clear that our methods lend themselves to the construction of stacks which are fine moduli spaces for the problem at hand. We plan to return to the question in a future paper.

As in the case $r = 2$ (see [6]), the combinatorial aspects of our construction play a significant role, and enable us to offer a fairly explicit description of our compactification.

We prove in Theorem 4.2.3 that, in the special case of higher spin curves, our moduli space is isomorphic to the coarse space underlying Jarvis's ROOT.

The last section is devoted to our second approach, whose goal is to obtain a completion within the compactified Picard scheme, where, obviously, r -th roots on smooth curves naturally live. Because of the relative newness of this method, we only apply it to higher spin curves over \overline{M}_g . A modular compactification of the universal Picard scheme was constructed in [5] by Caporaso, by means of line bundles on semistable curves. This space was later given a description in terms of rank 1 torsion-free sheaves by R. Pandharipande [20], as a special case of a more general result, valid for all ranks.

The question on whether the existing compactifications of higher spin curves embed in the universal Picard scheme seemed, at least to us, irresistible. All the more so because of the strong similarities between the boundary points used by the various authors: (analogous) line bundles in [8] and [5] on the one hand, and rank 1 torsion-free sheaves in [14] and [20] on the other.

Therefore we construct and study a new modular compactification of higher spin curves, $\widehat{S}_g^{r,l}$, sitting inside the universal Picard scheme, and whose main new feature is that it is not finite over \overline{M}_g , as soon as $r \geq 3$.

We then prove that, for $r \geq 3$, the previously constructed moduli spaces of limit roots, which we denote by $\overline{S}_g^{r,l}$, as well as those constructed in [14], [15], [1] and [7], do not embed in the universal Picard scheme. More precisely, neither is the natural birational map $\chi: \overline{S}_g^{r,l} \dashrightarrow \widehat{S}_g^{r,l}$ regular, nor is its inverse (Theorem 5.4.2 and its corollary). The case $r = 2$, treated by C. Fontanari in [12], is an exception, as χ is regular and bijective.

For higher rank, in the same spirit as ours, a compactification of the moduli space of vector bundles over curves using, instead of torsion-free sheaves, vector bundles over semistable curves, has recently been constructed by A. Schmitt [22],

²*Added in proof.* The generalized Witten conjecture has recently been proved by C. Faber, S. Shadrin and D. Zvonkine in Faber, Shadrin, Zvonkine: *Tautological relations and the r -spin Witten conjecture*. Preprint math.AG/0612510.

continuing upon the work of C. S. Seshadri and others (see [23] and references therein).

1.1. Notation. Throughout the paper we work over the field of complex numbers. We shall use the words “line bundle” as a synonym of “invertible sheaf”.

For any scheme Z , we denote by $\nu = \nu_Z: Z^\nu \rightarrow Z$ the normalization morphism.

C will always be a connected, nodal, projective curve of (arithmetic) genus g . We denote by g^ν the genus of its normalization.

To C we associate a *dual graph* Γ_C , as follows. The set of *vertices* of Γ_C is the set of components of C^ν . The set of *half-edges* is the set of all points of C^ν mapping to a node of C . Two half-edges form an *edge* of Γ_C when the two corresponding points map to the same node of C . Recall that $g = g^\nu + b_1(\Gamma_C)$, where $b_1(\Gamma_C)$ is the first Betti number. Let Δ be a set of nodes of C , i.e., of edges of Γ_C . We denote by $\tilde{\Delta}$ the subgraph of Γ_C consisting of Δ plus the abutting vertices.

We say that a nodal curve X is obtained from C by *blowing-up* Δ if there exists $\pi: X \rightarrow C$ such that $\pi^{-1}(n_i) = E_i \simeq \mathbb{P}^1$ for any $n_i \in \Delta$, and $\pi: X \setminus \bigcup_i E_i \rightarrow C \setminus \Delta$ is an isomorphism. We call $\pi: X \rightarrow C$ a *blow-up* of C and E_i an *exceptional component*. Set $\tilde{X} = X \setminus \bigcup_i E_i$. Then $\tilde{\pi}: \tilde{X} \rightarrow C$ is the normalization of C at Δ . For any $n_i \in \Delta$, set $\{p_i, q_i\} = E_i \cap \tilde{X} = \tilde{\pi}^{-1}(n_i)$; the points p_i and q_i are called *exceptional nodes*.

For families of curves, we use the same letters but different styles to denote the total space and a special fiber, and a star superscript to indicate restriction to the complement of the special fiber. For instance, $\mathcal{C} \rightarrow B$ will denote a family of curves with special fiber C , while \mathcal{C}^* will stand for $\mathcal{C} \setminus C$.

Let $\mathcal{C} \rightarrow B$ be a family of nodal curves. We denote by C_b the fiber over $b \in B$. We say that a family of nodal curves $\mathcal{X} \rightarrow B$, endowed with a B -morphism $\pi: \mathcal{X} \rightarrow \mathcal{C}$, is a *family of blow-ups* of \mathcal{C} , if for any $b \in B$ the restriction $\pi|_{X_b}: X_b \rightarrow C_b$ is a blow-up of C_b . If $\mathcal{N} \in \text{Pic } \mathcal{C}$, we set $N_b := \mathcal{N}|_{C_b}$ and, as above, $\mathcal{N}^* = \mathcal{N}|_{\mathcal{C}^*}$.

We denote by S a smooth, connected 1-dimensional scheme, not necessarily complete, with a fixed point $s_0 \in S$, and we set $S^* = S \setminus \{s_0\}$.

We shall attach to X a second graph Σ_X , whose vertices are the connected components of \tilde{X} and whose edges are the exceptional components of X . Note that Σ_X is obtained from Γ_C by contracting all edges corresponding to nodes which are not blown up in X .

For any graph Γ and commutative group G , we denote by $\mathcal{C}^0(\Gamma, G)$ and $\mathcal{C}^1(\Gamma, G)$ the groups of formal linear combinations, respectively, of vertices and edges of Γ with coefficients in G . When we fix an orientation for Γ , the boundary and coboundary operators $\partial: \mathcal{C}^1(\Gamma, G) \rightarrow \mathcal{C}^0(\Gamma, G)$ and $\delta: \mathcal{C}^0(\Gamma, G) \rightarrow \mathcal{C}^1(\Gamma, G)$ are defined in the usual way. Then $H_1(\Gamma, G) := \ker \partial \subset \mathcal{C}^1(\Gamma, G)$.

For any positive integer r , μ_r is group of r -th roots of unity.

2. LIMIT ROOTS OF LINE BUNDLES

Let C be a nodal curve, r a positive integer, and N a line bundle on C whose degree is divisible by r .

2.1. The main definition.

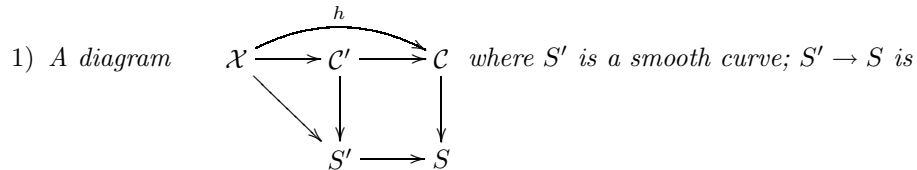
Definition 2.1.1. Consider a triple (X, L, α) , where X is a blow-up of C , L is a line bundle on X , and α is a homomorphism $\alpha: L^{\otimes r} \rightarrow \pi^*(N)$ ($\pi: X \rightarrow C$ is the

natural map). We say that (X, L, α) is a *limit r -th root* of (C, N) if the following properties are satisfied:

- (i) the restriction of L to every exceptional component of X has degree 1;
- (ii) the map α is an isomorphism at all points of X not belonging to an exceptional component;
- (iii) for every exceptional component E_i of X , the orders of vanishing of α at p_i and q_i add up to r .

The following is an analogue, for roots of line bundles, of the stable reduction theorem.

Proposition 2.1.2. *Let $\mathcal{C} \rightarrow S$ be a family of nodal curves, smooth over S^* . Consider line bundles $\mathcal{N} \in \text{Pic } \mathcal{C}$ and $\mathcal{L} \in \text{Pic } \mathcal{C}^*$ with an isomorphism $\alpha: \mathcal{L}^{\otimes r} \xrightarrow{\sim} \mathcal{N}|_{\mathcal{C}^*}$. Then there exist:*



- a finite morphism of degree r , étale over S^* ; $\mathcal{C}' := \mathcal{C} \times_S S'$; $\mathcal{X} \rightarrow \mathcal{C}'$ is an isomorphism outside the central fiber; \mathcal{X} is a blow-up of \mathcal{C}' .
- 2) A line bundle $\mathcal{L}' \in \text{Pic } \mathcal{X}$ such that $\mathcal{L}'|_{\mathcal{X}^*} = h^*(\mathcal{L})$.
- 3) An isomorphism $\alpha': (\mathcal{L}')^{\otimes r} \xrightarrow{\sim} h^*(\mathcal{N})(-D)$, where D is an effective Cartier divisor supported on the exceptional components of the central fiber, such that $\alpha'|_{\mathcal{X}^*} = h^*(\alpha)$ and $(\mathcal{X}, \mathcal{L}'|_{\mathcal{X}}, \alpha'|_{\mathcal{X}})$ is a limit r -th root of $(\mathcal{C}, \mathcal{N}|_{\mathcal{C}})$.

Moreover, for any $(\mathcal{X}_1 \rightarrow \mathcal{C}', \mathcal{L}'_1, \alpha'_1, D_1)$ and $(\mathcal{X}_2 \rightarrow \mathcal{C}', \mathcal{L}'_2, \alpha'_2, D_2)$ satisfying the above conditions, there exists an isomorphism $\sigma: \mathcal{X}_1 \xrightarrow{\sim} \mathcal{X}_2$ over \mathcal{C}' such that $\sigma^*(D_2) = D_1$, $\sigma^*(\mathcal{L}'_2) \simeq \mathcal{L}'_1$, and this last isomorphism is compatible with α'_1 and α'_2 .

The proof is in [14], section §4.2.2. For more details on the work of Jarvis on the subject, see section 4.2.

The notion of a limit root generalizes to families in a natural way. Let $\mathcal{C} \rightarrow T$ be a family of nodal curves and $\mathcal{N} \in \text{Pic } \mathcal{C}$. A *limit r -th root* $(\mathcal{X}, \mathcal{L}, \alpha)$ of $(\mathcal{C}, \mathcal{N})$ is the datum of a family $\pi: \mathcal{X} \rightarrow \mathcal{C}$ of blow-ups of \mathcal{C} , a line bundle $\mathcal{L} \in \text{Pic } \mathcal{X}$ and a homomorphism $\alpha: \mathcal{L}^{\otimes r} \rightarrow \pi^*(\mathcal{N})$, such that for all $t \in T$, (X_t, L_t, α_t) is a limit r -th root of (C_t, N_t) .

We can restate Proposition 2.1.2 by saying that if $\mathcal{C} \rightarrow S$ is a family of nodal curves, $\mathcal{N} \in \text{Pic } \mathcal{C}$ and $(\mathcal{C}^*, \mathcal{L}, \alpha)$ is an r -th root of $\mathcal{N}|_{\mathcal{C}^*}$, then for some finite covering $S' \rightarrow S$ of degree r there exists a limit r -th root $(\mathcal{X}, \mathcal{L}', \alpha')$ of the pull-back of \mathcal{N} to the base-changed family $\mathcal{C}' \rightarrow S'$ which extends over all of S' the pull-back of $(\mathcal{C}^* \rightarrow S^*, \mathcal{L}, \alpha)$.

2.2. Weighted graphs. To a limit r -th root (X, L, α) we shall attach a *weighted subgraph* of the dual graph Γ_C of C . The underlying graph is the subgraph $\overline{\Delta}$ corresponding to the set Δ of nodes which are blown up in $\pi: X \rightarrow C$. The weight function w is defined on the exceptional nodes (or, equivalently, on the half-edges of $\overline{\Delta}$), takes values in $\{1, \dots, r - 1\}$, and is given by $w(p_i) = u_i$, $w(q_i) = v_i$, where

u_i, v_i are the orders of vanishing of α at the points p_i, q_i respectively. We denote this weighted graph by Δ^w . Clearly, the following conditions are satisfied:

- (C1) $u_i + v_i = r$ for any i ;
- (C2) for every irreducible component C_j of C , the sum of all weights assigned to the vertex corresponding C_j is congruent to $\deg_{C_j} N$ modulo r .

Conversely, for any assignment of a weighted subgraph Δ^w of Γ_C satisfying conditions (C1) and (C2), there is a limit r -th root of N whose graph is Δ^w . In fact, let $\tilde{\pi}: \tilde{X} \rightarrow C$ be the partial normalization of C at Δ . By (C2), the line bundle $\tilde{\pi}^*(N)(-\sum_{n_i \in \Delta} (u_i p_i + v_i q_i))$ on \tilde{X} admits an r -th root $\tilde{L} \in \text{Pic } \tilde{X}$. Let X be the blow-up of C at Δ . We obtain a limit root (X, L, α) of N by gluing to \tilde{L} (however we wish) the degree one line bundle on each exceptional component; the map $\alpha: L^{\otimes r} \rightarrow \pi^*(N)$ is defined to agree with $(\tilde{L})^{\otimes r} = \tilde{\pi}^*(N)(-\sum_i (u_i p_i + v_i q_i)) \hookrightarrow \tilde{\pi}^*(N)$ on \tilde{X} , and to be zero on all exceptional components.

Remark 2.2.1. The *multidegree* $\underline{d} = \underline{\deg} N$ of N is the set of the degrees $\deg_{C_j} N$ for all irreducible components C_j of C . Its class modulo r can be viewed as an element $[\underline{d}]_r$ of $\mathcal{C}^0(\Gamma_C, \mathbb{Z}/r)$ defined naturally as

$$[\underline{d}]_r := \sum_j \left(\deg_{C_j} N \pmod{r} \right) [C_j].$$

Of course, condition (C2) above only depends on $[\underline{d}]_r$.

Now fix an orientation for Γ_C ; then there is a natural bijection between $\mathcal{C}^1(\Gamma_C, \mathbb{Z}/r)$ and the set of weighted subgraphs of Γ_C satisfying (C1): to a weighted subgraph Δ^w we associate the 1-chain $\sum_i u_i [n_i]$ with the convention that the edge n_i is oriented entering the vertex corresponding to u_i .

In this set-up, the boundary operator $\partial: \mathcal{C}^1(\Gamma_C, \mathbb{Z}/r) \rightarrow \mathcal{C}^0(\Gamma_C, \mathbb{Z}/r)$ can be viewed as a map from weighted subgraphs of Γ_C satisfying (C1) to multidegrees reduced modulo r . Then Δ^w satisfies (C2) with respect to $[\underline{d}]_r$ if and only if $\partial(\Delta^w) = [\underline{d}]_r$. Since the cardinality of $\partial^{-1}([\underline{d}]_r)$ equals that of $\ker \partial = H_1(\Gamma_C, \mathbb{Z}/r)$, we see that:

- for every N as above, there exist $r^{b_1(\Gamma_C)}$ weighted subgraphs of Γ_C satisfying (C1) and (C2) with respect to N .

2.3. Automorphisms of limit roots.

Definition 2.3.1. An *isomorphism* of limit r -th roots $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ and $(\mathcal{X}' \rightarrow T, \mathcal{L}', \alpha')$ of (C, \mathcal{N}) is the datum of:

- (a) an isomorphism $\sigma: \mathcal{X} \rightarrow \mathcal{X}'$ over \mathcal{C} ,
- (b) an isomorphism $\tau: \sigma^* \mathcal{L}' \rightarrow \mathcal{L}$ that makes the following diagram commute:

$$\begin{array}{ccc} \sigma^*(\mathcal{L}')^{\otimes r} & \xrightarrow{\tau^{\otimes r}} & \mathcal{L}^{\otimes r} \\ \sigma^*(\alpha') \downarrow & & \downarrow \alpha \\ \sigma^*(\pi')^*(\mathcal{N}) & \xrightarrow{\sim} & \pi^*(\mathcal{N}) \end{array}$$

This notion of isomorphism agrees with the one of [9], rather than with the one of [8]. At the level of automorphisms, for instance, the difference is that here fiber multiplication in \mathcal{L} by a nontrivial root of unity is viewed as a nontrivial automorphism of the limit root. To make contact with the terminology of [8] and

[9], an isomorphism between \mathcal{X} and \mathcal{X}' over \mathcal{C} will sometimes be referred to as an *inessential* isomorphism. Likewise, automorphisms of \mathcal{X} over \mathcal{C} will sometimes be called inessential automorphisms of \mathcal{X} . As customary, the group of these automorphisms will be denoted $\text{Aut}_{\mathcal{C}}(\mathcal{X})$. Given a limit root $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$, we denote by $\text{Aut}(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ the group of its automorphisms. Clearly, $\text{Aut}(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ maps to $\text{Aut}_{\mathcal{C}}(\mathcal{X})$; the kernel is naturally isomorphic to a product of copies of μ_r , one for each connected component of T .

We now state and prove some of the basic properties of automorphisms of limit roots. Let X be a blow-up of C , and let E_1, \dots, E_m be its exceptional components. View each E_i as a copy of the Riemann sphere, with the origin placed at q_i and the point at infinity placed at p_i . Then any inessential automorphism of X acts on each E_i as multiplication by a nonzero constant l_i , and any assignment of the l_i corresponds to an inessential automorphism. This proves the first part of the following result.

Lemma 2.3.2. *Let (X, L, α) be a limit root of (C, N) , and fix an orientation on the graph Σ_X . Then:*

- (i) *There is a natural identification $\text{Aut}_{\mathcal{C}}(X) \simeq \mathcal{C}^1(\Sigma_X, \mathbb{C}^*)$.*
- (ii) *There is a natural identification $\text{Aut}(X, L, \alpha) \simeq \mathcal{C}^0(\Sigma_X, \mu_r)$, and the homomorphism $\text{Aut}(X, L, \alpha) \rightarrow \text{Aut}_{\mathcal{C}}(X)$ corresponds to the composition of the coboundary map $\delta: \mathcal{C}^0(\Sigma_X, \mu_r) \rightarrow \mathcal{C}^1(\Sigma_X, \mu_r)$ with the inclusion $\mathcal{C}^1(\Sigma_X, \mu_r) \rightarrow \mathcal{C}^1(\Sigma_X, \mathbb{C}^*)$.*

Proof. We have already proved (i). To prove (ii), it is convenient to work with the geometric bundles $\mathbb{V}(L)$ and $\mathbb{V}(L^{\otimes r})$ rather than with L and $L^{\otimes r}$. In terms of these, an automorphism of (X, L, α) is a pair (σ, τ) , where σ is an inessential automorphism of X and τ is an automorphism of $\mathbb{V}(L)$ which is compatible with σ , is linear on the fibers of $\mathbb{V}(L) \rightarrow X$, and is such that $\alpha \circ \tau^{\otimes r} = \alpha$. This last condition means that, above each connected component X_j of \tilde{X} , $\tau^{\otimes r}$ is the identity, and hence τ must be the fiberwise multiplication by an r -th root of unity l_j . If E_i is an exceptional component connecting X_j and X_k , the restrictions of τ and σ to $\mathbb{V}(L)|_{E_i} \rightarrow E_i$ can thus be viewed as an automorphism pair of $\mathbb{V}(\mathcal{O}(1)) \rightarrow \mathbb{P}^1$ which is the multiplication by l_j on the fiber above the origin and the multiplication by l_k on the fiber above the point at infinity. These pairs are easy to describe. View \mathbb{P}^1 as the projective space $\mathbb{P}V$ constructed on a two-dimensional vector space V , and its origin and point at infinity as $[v_0]$ and $[v_1]$, where v_0, v_1 is a basis of V . Then $\mathbb{V}(\mathcal{O}(1))$ is the set of all pairs (ℓ, φ) , where ℓ is a one-dimensional subspace of V and φ an element of its dual. An automorphism pair as above then acts on $\mathbb{V}(\mathcal{O}(1))$ via $(\ell, \varphi) \mapsto (f(\ell), \varphi \circ f^{-1})$, where f is the automorphism of V such that $f(v_0) = l_j^{-1}v_0$, $f(v_1) = l_k^{-1}v_0$, and on $\mathbb{P}V$ via $\ell \mapsto f(\ell)$. Thus σ acts on E_i as multiplication by l_j/l_k . This means that giving an automorphism of (X, L, α) is the same as giving an r -th root of unity for each connected component of \tilde{X} , that is, an element of $\mathcal{C}^0(\Sigma_X, \mu_r)$, and that the map $\text{Aut}(X, L, \alpha) \rightarrow \text{Aut}_{\mathcal{C}}(X)$ is as claimed. \square

It follows from Lemma 2.3.2 that $\text{Aut}(X, L, \alpha)$ has cardinality r^γ , where γ is the number of connected components of \tilde{X} .

Fix (C, N) and $\pi : X \rightarrow C$ as usual. We need to further investigate the action of $\text{Aut}_{\mathcal{C}}(X)$ on certain line bundles on X . Consider pairs (M, β) , where M is a line bundle on X , $\beta : M \rightarrow \pi^*(N)$ is a homomorphism which is an isomorphism on

the complement of the exceptional components of X , and M is assumed to have degree r on each exceptional component. This degree condition forces β to vanish identically on each exceptional component

An example of this situation is provided by $\alpha: L^{\otimes r} \rightarrow \pi^*(N)$, where (X, L, α) is a limit r -th root.

Lemma 2.3.3. *Let (M, β) and (M', β') be two pairs as above. Then:*

- (i) *There exist $\sigma \in \text{Aut}_C(X)$ and an isomorphism $\vartheta: M \rightarrow \sigma^*(M')$ such that $\beta = \sigma^*(\beta') \circ \vartheta$ if and only if, for any exceptional node q , the orders of vanishing of β and β' at q are the same.*
- (ii) *Under the identification (i) of Lemma 2.3.2, the set of those $\sigma \in \text{Aut}_C(X)$ such that there is an isomorphism $\vartheta: M \rightarrow \sigma^*(M)$ with the property that $\beta = \sigma^*(\beta) \circ \vartheta$ corresponds to $\mathcal{C}^1(\Sigma_X, \mu_r)$.*

Proof. The proof is similar to the one of Lemma 2.3.2. The “only if” part of (i) is clear. To prove the converse, denote by u_i (resp., v_i) the order of vanishing of β and β' at p_i (resp., q_i). The homomorphisms β and β' determine isomorphisms $M|_{\tilde{X}} \simeq \pi^*(N)(-\sum(u_i p_i + v_i q_i)) \simeq M'|_{\tilde{X}}$, hence there is a unique isomorphism $\vartheta': M|_{\tilde{X}} \rightarrow M'|_{\tilde{X}}$ such that $\beta|_{\tilde{X}} = \beta'|_{\tilde{X}} \circ \vartheta'$. Clearly, ϑ must restrict to ϑ' on \tilde{X} . It remains to construct σ and ϑ on the exceptional components. If E_i is one of these, we identify the restrictions to E_i of $\mathbb{V}(M) \rightarrow X$ and $\mathbb{V}(M') \rightarrow X$ to $\mathbb{V}(\mathcal{O}(r)) \rightarrow \mathbb{P}(V)$, where V is a two-dimensional vector space. We also think of $\mathbb{P}(V)$ as being attached to the rest of X at $[v_0]$ and $[v_1]$, where v_0, v_1 is a basis of V . The isomorphisms ϑ and σ , if they exist, restrict on $\mathbb{V}(\mathcal{O}(r)) \rightarrow \mathbb{P}(V)$ to an automorphism pair fixing $[v_0]$ and $[v_1]$. Now, $\mathbb{V}(\mathcal{O}(r))$ is the set of all pairs $(\ell, \varphi^{\otimes r})$, where ℓ is a one-dimensional subspace of V and φ an element of its dual. Any automorphism pair fixing $[v_0]$ and $[v_1]$ then acts on $\mathbb{V}(\mathcal{O}(r))$ via $(\ell, \varphi^{\otimes r}) \mapsto (f(\ell), (\varphi \circ f^{-1})^{\otimes r})$, where f is an automorphism of V such that $f(v_0) = b_0 v_0, f(v_1) = b_1 v_0, b_0, b_1 \in \mathbb{C}^*$, and on $\mathbb{P}V$ via $\ell \mapsto f(\ell)$. The action of such an automorphism pair on the fiber above $[v_i], i = 0, 1$, corresponds to multiplication by b_i^{-r} , hence we can choose b_0 and b_1 so that it matches the one of ϑ' ; the choices are unique up to multiplication by r -th roots of unity. This proves the existence of σ and ϑ . It also proves (ii). In fact, σ acts on E_i as multiplication by b_1/b_0 . In the setup of (ii), both b_0 and b_1 are (arbitrary) roots of unity. The conclusion is that σ acts on each exceptional component as multiplication by an r -th root of unity, and conversely that any element of $\mathcal{C}^1(\Sigma_X, \mu_r)$ comes from an automorphism σ as in (ii). □

2.4. The general moduli problem. Let B be a scheme, $f: \mathcal{C} \rightarrow B$ a family of nodal curves, and $\mathcal{N} \in \text{Pic } \mathcal{C}$ a line bundle of relative degree divisible by r . We shall use the following notation: for any B -scheme T with structure map $p: T \rightarrow B$ we set

$$\begin{array}{ccc} \mathcal{C}_T := \mathcal{C} \times_B T & \xrightarrow{\bar{p}} & \mathcal{C} \\ \downarrow & & \downarrow f \\ T & \xrightarrow{p} & B \end{array}$$

where $\bar{p}: \mathcal{C}_T \rightarrow \mathcal{C}$ is the projection. The pull-back of \mathcal{N} to \mathcal{C}_T is denoted by

$$\mathcal{N}_T := \bar{p}^* \mathcal{N}.$$

Having fixed $B, f: \mathcal{C} \rightarrow B$ and \mathcal{N} as above, we introduce a contravariant functor

$$\overline{\mathcal{S}}_f^r(\mathcal{N}): \{B\text{-schemes}\} \longrightarrow \{\text{sets}\},$$

encoding the moduli problem for limit r -th roots of \mathcal{N} , as follows. For every B -scheme T , the set $\overline{\mathcal{S}}_f^r(\mathcal{N})(T)$ is the set of all limit r -th roots of \mathcal{N}_T (which is, with the above notation, a line bundle on the total space of the family $\mathcal{C}_T \rightarrow T$), modulo isomorphisms of limit roots.

For every morphism of B -schemes $\rho: T \rightarrow T'$, the corresponding map

$$\overline{\mathcal{S}}_f^r(\mathcal{N})(\rho): \overline{\mathcal{S}}_f^r(\mathcal{N})(T') \longrightarrow \overline{\mathcal{S}}_f^r(\mathcal{N})(T)$$

sends a limit r -th root $(\mathcal{X}', \mathcal{L}', \alpha')$ of $\mathcal{N}_{T'}$ to the limit root $(\mathcal{X}, \mathcal{L}, \alpha)$ of \mathcal{N}_T defined by $\mathcal{X} := \mathcal{X}' \times_{T'} T$ and $\mathcal{L} := \overline{p}^* \mathcal{L}'$, where $\overline{p}: \mathcal{X} \rightarrow \mathcal{X}'$ is the projection. Finally, the morphism α' is the natural pull-back of α .

We shall prove

Theorem 2.4.1. *Let $f: \mathcal{C} \rightarrow B$ be a family of nodal curves, with B a quasi-projective scheme. Let $\mathcal{N} \in \text{Pic } \mathcal{C}$ be a line bundle on \mathcal{C} of relative degree divisible by r .*

Then the functor $\overline{\mathcal{S}}_f^r(\mathcal{N})$ is coarsely represented by a quasiprojective scheme $\overline{\mathcal{S}}_f^r(\mathcal{N})$, finite over B . If B is projective, then $\overline{\mathcal{S}}_f^r(\mathcal{N})$ is projective.

We shall denote by $S_f^r(\mathcal{N})$ the open subscheme of $\overline{\mathcal{S}}_f^r(\mathcal{N})$ parametrizing (closed) points whose underlying curve has no exceptional components.

Proposition 2.4.2. *Let $\xi = (X, L, \alpha) \in \overline{\mathcal{S}}_f^r(\mathcal{N})$. The morphism $p: \overline{\mathcal{S}}_f^r(\mathcal{N}) \rightarrow B$ is smooth at ξ if and only if $b_1(\Sigma_X) = 0$.*

In particular, if B is smooth, $\overline{\mathcal{S}}_f^r(\mathcal{N})$ is smooth at all points ξ such that X has no exceptional components, or is of compact type.

The proofs of these results are in section 3.4.

Remark 2.4.3. A purely formal consequence of Theorem 2.4.1 is the following useful fact. Let B' be a B -scheme, and denote by

$$f': \mathcal{C}' = \mathcal{C} \times_B B' \longrightarrow B'$$

the family obtained by base change, and by \mathcal{N}' the pull-back of \mathcal{N} to \mathcal{C}' . Then we naturally have

$$\overline{\mathcal{S}}_{f'}^r(\mathcal{N}') \simeq \overline{\mathcal{S}}_f^r(\mathcal{N}) \times_B B'.$$

Remark 2.4.4. Let $f: \mathcal{C} \rightarrow B$ and $\mathcal{N} \in \text{Pic } \mathcal{C}$ be as above. For any $\mathcal{L}_0 \in \text{Pic } \mathcal{C}$, the functors $\overline{\mathcal{S}}_f^r(\mathcal{N})$ and $\overline{\mathcal{S}}_f^r(\mathcal{N} \otimes \mathcal{L}_0^{\otimes r})$ are isomorphic, so the same holds for the corresponding moduli spaces $\overline{\mathcal{S}}_f^r(\mathcal{N})$ and $\overline{\mathcal{S}}_f^r(\mathcal{N} \otimes \mathcal{L}_0^{\otimes r})$. This is because for any limit r -th root $(\mathcal{X}, \mathcal{L})$ of \mathcal{N} , $(\mathcal{X}, \mathcal{L} \otimes \pi^*(\mathcal{L}_0))$ is a limit r -th root of $\mathcal{N} \otimes \mathcal{L}_0^{\otimes r}$.

Remark 2.4.5. Let $f: \mathcal{C} \rightarrow B$ and $\mathcal{N} \in \text{Pic } \mathcal{C}$ be as above. For any positive integer s , there is a natural injective morphism

$$J: \overline{\mathcal{S}}_f^r(\mathcal{N}) \longrightarrow \overline{\mathcal{S}}_f^{rs}(\mathcal{N}^{\otimes s})$$

over B . Indeed for any limit r -th root $(\mathcal{X}, \mathcal{L}, \alpha)$ of \mathcal{N} , $(\mathcal{X}, \mathcal{L}, \alpha^{\otimes s})$ is a limit (rs) -th root of $\mathcal{N}^{\otimes s}$. The image of J is a union of irreducible components of $\overline{\mathcal{S}}_f^{rs}(\mathcal{N}^{\otimes s})$.

3. PROOF OF THE EXISTENCE THEOREM

We shall now construct $\overline{S}_f^r(\mathcal{N})$, and prove Theorem 2.4.1. First of all, working in the analytic category, we construct the universal deformation of a limit root. Then we glue together the bases of the universal deformations, in order to give a complex algebraic structure to $\overline{S}_f^r(\mathcal{N})$.

We will make use of the following simple results. The first one is well known, while the second is Lemma 1.1 of [8].

Remark 3.0.6. Let $\mathcal{Y} \rightarrow T$ be a family of nodal curves, t_0 a point of T , and let Y be the fiber over t_0 . Consider two line bundles $\mathcal{P} \in \text{Pic } \mathcal{Y}$ and $L \in \text{Pic } Y$ such that there exists an isomorphism $\iota_0: L^{\otimes r} \rightarrow \mathcal{P}|_Y$. Then, up to shrinking T , there exist a line bundle $\mathcal{L} \in \text{Pic } \mathcal{Y}$ extending L and an isomorphism $\iota: \mathcal{L}^{\otimes r} \rightarrow \mathcal{P}$ extending ι_0 . Moreover, this extension is unique up to isomorphism, meaning that if (\mathcal{L}', ι') is another extension, then there is an isomorphism $\chi: \mathcal{L} \rightarrow \mathcal{L}'$ on a neighbourhood of Y which restricts to the identity on L and is such that $\iota = \iota' \circ \chi^{\otimes r}$.

Remark 3.0.7. Let $\mathcal{Y} \rightarrow T$ be a family of nodal curves, and let E be a smooth rational component of one of its fibers. If \mathcal{L} is a line bundle on \mathcal{Y} whose restriction to E has zero degree, then \mathcal{L} is trivial on a neighbourhood of E in \mathcal{Y} .

3.1. Local structure near exceptional curves. Let $\mathcal{C} \rightarrow T$ be a family of nodal curves, $\mathcal{N} \in \text{Pic } \mathcal{C}$ and $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ a limit r -th root of $(\mathcal{C}, \mathcal{N})$. Fix an exceptional component $E \subset \mathcal{X}$ lying over a point $t_0 \in T$, let u, v be the corresponding weights, and denote by n the point of \mathcal{C} to which E contracts. As usual, $\pi: \mathcal{X} \rightarrow \mathcal{C}$ is the natural map; recall that $\pi^*\mathcal{N}$ is trivial on a neighbourhood of E . We shall describe the geometry of $\mathcal{X} \rightarrow T$ near E .

The prototype is the following family: for the base, consider the affine plane curve $C_{u,v} \subset \mathbb{A}_{w,z}^2$ ³ given by the equation $w^u = z^v$. The total space is the surface

$$S_{u,v} = \{xs_0 = ws_1, ys_1 = zs_0, w^u = z^v\} \subset \mathbb{A}_{x,y,w,z}^4 \times \mathbb{P}_{s_0:s_1}^1.$$

The projection to $\mathbb{A}_{w,z}^2$ yields a family of open curves

$$f_{u,v}: S_{u,v} \rightarrow C_{u,v}$$

whose only singular fiber lies over the point $(0, 0) \in C_{u,v}$ and contains an exceptional component, i.e., the locus of all points of $S_{u,v}$ for which $x = y = w = z = 0$.

Note that $S_{u,v}$ has $d = (u, v)$ irreducible components, each one isomorphic to $S_{u/d, v/d}$. For instance, when $d = 1$, $S_{u,v}$ is smooth if and only if $u = v = 1$ (in which case it is a resolution of an A_1 -singularity); it is normal if and only if $u = 1$ or $v = 1$ (in which case it has an isolated A_{v-1} or A_{u-1} singularity in one exceptional node). If $u > 1$ and $v > 1$, $S_{u,v}$ is singular along the whole special fiber of $f_{u,v}$ (the curve $w = z = 0$).

We now show that the morphism $f_{u,v}: S_{u,v} \rightarrow C_{u,v}$ is a model for $\mathcal{X} \rightarrow T$ near E . To set up, recall that a neighbourhood V of n in \mathcal{C} is of the form $\{xy = \ell\}$, where ℓ is a regular function on a neighbourhood T_0 of t_0 . In other terms, there is

³The notation $\mathbb{A}_{w,z}^2$ means, here and similarly below, that w, z are the coordinate functions on \mathbb{A}^2 .

a fiber product diagram

$$(1) \quad \begin{array}{ccc} \mathcal{C} \supset V & \longrightarrow & V' \\ \downarrow & & \downarrow e \\ T_0 & \xrightarrow{\ell} & \mathbb{A}_s^1 \end{array}$$

where V' is a neighbourhood of the origin in $\mathbb{A}_{x,y}^2$ and e is given by $s = xy$.

The curve $C_{u,v}$ maps to \mathbb{A}_s^1 via $(w, z) \mapsto wz$, and $S_{u,v}$ to $\mathbb{A}_{x,y}^2$ via projection. We will show that ℓ lifts to a morphism $h: T_0 \rightarrow C_{u,v}$. More precisely, we shall prove the following result.

Lemma 3.1.1. *Possibly after shrinking T_0 , diagram (1) lifts to a fiber product diagram*

$$\begin{array}{ccc} \mathcal{X} \supset U & \longrightarrow & U' \subset S_{u,v} \\ \downarrow & & \downarrow f_{u,v} \\ T_0 & \xrightarrow{h} & C_{u,v} \end{array}$$

where U is a neighbourhood of E in \mathcal{X} and U' is a neighbourhood of the exceptional curve in $S_{u,v}$. Moreover, the lifting can be chosen so that the following holds. Let $\zeta_{\mathcal{N}}$ be any generator of $\pi^*\mathcal{N}$ over U , and denote by U_0 (resp., U_1) the complement in U of the inverse image of the locus $s_1 = 0$ (resp., $s_0 = 0$). Then there are generators ζ_0 and ζ_1 of $\mathcal{L}^{\otimes r}$ over U_0 and U_1 such that

$$s_0^r \zeta_0 = s_1^r \zeta_1 \text{ on } U_0 \cap U_1, \quad \alpha(\zeta_0) = x^u \zeta_{\mathcal{N}} \text{ on } U_0, \quad \alpha(\zeta_1) = y^v \zeta_{\mathcal{N}} \text{ on } U_1.^4$$

Proof. A weaker lifting property than the one claimed obviously holds. Consider the threefold

$$Z = \{xs_0 = ws_1, ys_1 = zs_0\} \subset \mathbb{A}_{x,y,w,z}^4 \times \mathbb{P}_{s_0:s_1}^1.$$

The projection to $\mathbb{A}_{w,z}^2$ yields a family of open curves $\phi: Z \rightarrow \mathbb{A}_{w,z}^2$, whose restriction to $C_{u,v}$ is clearly just $S_{u,v} \rightarrow C_{u,v}$. Then, after suitable shrinkings, diagram (1) lifts (nonuniquely) to

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & & \downarrow \phi \\ T_0 & \xrightarrow{k} & \mathbb{A}_{w,z}^2 \end{array}$$

where $W \subset Z$ is a neighbourhood of the exceptional curve in the central fiber of ϕ . In concrete terms, this means that U has equations

$$x\eta_0 = k_0 \text{ in } U_0, \quad y\eta_1 = k_1 \text{ in } U_1,$$

where $\eta_0\eta_1 = 1$ on $U_0 \cap U_1$, x and y vanish on E , η_0 and η_1 vanish on $\tilde{X} \cap U$, and k_0 and k_1 are regular functions on T_0 which vanish at t_0 . Clearly, η_0 is the pull-back of the quotient s_0/s_1 , and η_1 is its inverse. What must be shown is that k can be chosen in such a way that $k_0^u = k_1^v$. This is where the presence of a limit root comes into play.

Consider the line bundle \mathcal{M} on U whose transition function, relative to the cover $\{U_0, U_1\}$, is $\eta_1^r \in \mathcal{O}^*(U_0 \cap U_1)$. Clearly, $\deg \mathcal{M}|_E = \deg \mathcal{L}|_E^{\otimes r} = r$. By Remark

⁴For simplicity, we continue to use x and y to denote the pull-backs of these functions to U .

3.0.7, we may assume that $\mathcal{L}|_U^{\otimes r} \simeq \mathcal{M}$, possibly after shrinking U . Thus there are generators ζ'_0, ζ'_1 of $\mathcal{L}^{\otimes r}$, over U_0 and U_1 , respectively, such that $\zeta'_0 = \eta_1^r \zeta'_1$ on $U_0 \cap U_1$.

Let $\zeta_{\mathcal{N}}$ be a generator of $\pi^*(\mathcal{N})$ over U . We have $\alpha(\zeta'_i) = c_i \zeta_{\mathcal{N}}$ on U_i , $i = 0, 1$, where c_0 and c_1 are regular functions on U_0 and U_1 , respectively. Consider the expressions of c_0 and c_1 as power series

$$(2) \quad c_0 = \sum_{i \geq 0} a_i x^i + \sum_{i < 0} a_i \eta_0^{-i} \text{ in } U_0, \quad c_1 = \sum_{i \geq 0} b_i y^i + \sum_{i < 0} b_i \eta_1^{-i} \text{ in } U_1,$$

where the a_i and the b_i are regular functions on T_0 . Since $\pi^*(\mathcal{N})$ has degree zero on E while $\mathcal{L}^{\otimes r}$ has degree r , α is identically zero on E , which gives $a_i(t_0) = b_i(t_0) = 0$ for all $i \leq 0$. On the other hand, by definition, the order of vanishing of $(c_0)|_{X \cap U_0}$ at p is u and that of $(c_1)|_{X \cap U_1}$ at q is v , so $a_u(t_0) \neq 0$ and $b_v(t_0) \neq 0$. Up to shrinking T_0 , we may thus assume that a_u and b_v are units.

On $U_0 \cap U_1$ we have $\alpha(\zeta'_0) = \eta_1^r \alpha(\zeta'_1)$, so $c_0 = \eta_1^r c_1$. We also have $\eta_0 = \eta_1^{-1}$, $x = k_0 \eta_1$, $y = k_1 \eta_1^{-1}$. Substituting, we get

$$(3) \quad \sum_{i \geq 0} a_i k_0^i \eta_1^i + \sum_{i < 0} a_i \eta_1^i = \sum_{i \geq 0} b_i k_1^i \eta_1^{r-i} + \sum_{i < 0} b_i \eta_1^{r-i} \text{ in } U_0 \cap U_1.$$

Comparing terms of degree u in η_1 , we get $a_u k_0^u = b_v k_1^v$. This is not exactly what we are asking for, but we may proceed as follows. Let δ be an r -th root of b_v/a_u (a unit), and define a morphism $h: T_0 \rightarrow \mathbb{A}_{w,z}^2$ by setting $h_0 = \delta^{-1} k_0$, $h_1 = \delta k_1$. Clearly, $h_0^u = h_1^v$, and h is a lifting of ℓ , since $h_0 h_1 = k_0 k_1 = \ell$. A compatible lifting of $V \rightarrow \mathbb{A}_{x,y}^2$ to a morphism $U \rightarrow S_{u,v}$ is given by

$$\begin{aligned} (x, \eta_0, t) &\mapsto (x, \eta_0 k_1(t), h_0(t), h_1(t), [\delta^{-1} \eta_0 : 1]) \text{ on } U_0, \\ (y, \eta_1, t) &\mapsto (\eta_1 k_0(t), y, h_0(t), h_1(t), [1 : \delta \eta_1]) \text{ on } U_1, \end{aligned}$$

where t stands for a variable point in T_0 .

It remains to find the local generators ζ_0 and ζ_1 . What we have done so far amounts to showing that, replacing η_0 with $\delta^{-1} \eta_0$ and η_1 with $\delta \eta_1$, we may assume that U is defined by equations $x \eta_0 = h_0$ on U_0 and $y \eta_1 = h_1$ on U_1 , where $\eta_0 \eta_1 = 1$ on $U_0 \cap U_1$ and $h_0^u = h_1^v$. Moreover, replacing ζ'_0 with $\delta^u \zeta'_0$ and ζ'_1 with $\delta^{-v} \zeta'_1$, we may also assume that the coefficients a_u and b_v in the power series developments (2) are equal, while conserving the property that $\zeta'_0 = \eta_1^r \zeta'_1$. Now, comparing terms of all degrees in identity (3), we conclude that

$$c_0 = x^u \rho, \quad c_1 = y^v \rho,$$

where

$$\rho = a_u + \sum_{i > 0} a_{u+i} x^i + \sum_{i > 0} b_{v+i} y^i = b_v + \sum_{i > 0} a_{u+i} x^i + \sum_{i > 0} b_{v+i} y^i.$$

Since ρ is a unit on a neighbourhood of E , possibly after shrinking U it makes sense to set

$$\zeta_i = \zeta'_i / \rho, \quad i = 0, 1,$$

and these local generators of $\mathcal{L}^{\otimes r}$ have all the required properties. □

Remark 3.1.2. Set $d := (u, v)$. Consider the surface

$$\tilde{S}_{u,v} = \{x s_0 = \tau^{v/d} s_1, y s_1 = \tau^{u/d} s_0\} \subset \mathbb{A}_{x,y,\tau}^3 \times \mathbb{P}_{s_0:s_1}^1$$

and the family of curves $\tilde{f}_{u,v}: \tilde{S}_{u,v} \rightarrow \mathbb{A}_\tau^1$ induced by the projection. Note that if $d = 1$, this is the normalization of $f_{u,v}: S_{u,v} \rightarrow C_{u,v}$.

If the base T is normal, then \mathcal{X} has the following stronger property:

- (P) There exist neighbourhoods T_0 of t_0 in T and U of E in \mathcal{X} , and a regular map $h: T_0 \rightarrow \mathbb{A}_\tau^1$, such that $h(t_0) = 0$ and U is isomorphic to a neighbourhood of the exceptional curve in the base change of the surface $\tilde{S}_{u/d, v/d}$ via h .

3.2. The model family of curves. Let C be a nodal curve and let Δ^w be a weighted subgraph of Γ_C satisfying condition (C1) of section 2.2. Let X be the curve obtained from C by blowing up the nodes in Δ . Let $\underline{\mathcal{C}} \rightarrow \underline{D}$ be a semi-universal deformation of the curve C , where \underline{D} is the unit polydisc in \mathbb{C}^M with coordinates t_1, \dots, t_M . Observe that this deformation is universal if and only if C is stable. If this is the case, M equals $3g - 3$; otherwise it is strictly larger than $3g - 3$.

We shall construct a finite cover $D_{\Delta^w} \rightarrow \underline{D}$ and a family of blow-ups $\pi: \mathcal{X}_{\Delta^w} \rightarrow C_{\Delta^w} \rightarrow D_{\Delta^w}$ (where $C_{\Delta^w} = \underline{\mathcal{C}} \times_{\underline{D}} D_{\Delta^w}$) with central fiber X :

$$\begin{array}{ccccc} \mathcal{X}_{\Delta^w} & \longrightarrow & C_{\Delta^w} & \longrightarrow & D_{\Delta^w} \\ & & \downarrow & & \downarrow \\ & & \underline{\mathcal{C}} & \longrightarrow & \underline{D} \end{array}$$

The family $\mathcal{X}_{\Delta^w} \rightarrow D_{\Delta^w}$ will depend on C and on the weighted graph Δ^w ; it will be used to construct the universal deformation of any limit r -th root (X, L, α) with weighted graph Δ^w .

Denote by E_1, \dots, E_m the exceptional components of X and by n_1, \dots, n_m the corresponding nodes in C . For any $i = 1, \dots, m$ let u_i, v_i be the weights associated to n_i and let $\{t_i = 0\}$ be the locus in \underline{D} where the node n_i persists. We write $\underline{D} = D \times D'$, where D is the unit polydisc with coordinates t_1, \dots, t_m and D' corresponds to the remaining t_i 's. Consider

$$C_{u_1, v_1} \times \dots \times C_{u_m, v_m} = \{w_1^{u_1} = z_1^{v_1}, \dots, w_m^{u_m} = z_m^{v_m}\} \subset \mathbb{A}_{w_1, z_1, \dots, w_m, z_m}^{2m}$$

and set

$$D_{\Delta^w} := \prod_{i=1}^m C_{u_i, v_i} \times D'.$$

Consider the morphism $\prod_i C_{u_i, v_i} \rightarrow D$ defined by

$$t_i = w_i z_i \quad \text{for all } i = 1, \dots, m$$

and the induced morphism $D_{\Delta^w} \rightarrow \underline{D}$. Let $C_{\Delta^w} \rightarrow D_{\Delta^w}$ be the pull-back of the family $\underline{\mathcal{C}} \rightarrow \underline{D}$ to D_{Δ^w} .

A neighbourhood V_i of n_i in C_{Δ^w} can be thought of as

$$V_i = \{xy = w_i z_i, w_1^{u_1} = z_1^{v_1}, \dots, w_m^{u_m} = z_m^{v_m}\} \subset \mathbb{A}_{x, y, w_1, z_1, \dots, w_m, z_m, t_{m+1}, \dots, t_M}^{M+m+2}.$$

Now let $\pi: \mathcal{X}_{\Delta^w} \rightarrow C_{\Delta^w}$ be the blow-up locally described as follows:

$$U_i = \pi^{-1}(V_i) = \{xs_0 = w_i s_1, ys_1 = z_i s_0, w_1^{u_1} = z_1^{v_1}, \dots, w_m^{u_m} = z_m^{v_m}\}$$

in $\mathbb{A}^{M+m+2} \times \mathbb{P}_{s_0: s_1}^1$. Note that $\pi|_{U_i}: U_i \rightarrow V_i$ is the blow-up of the ideal (x, w_i) , or equivalently of (y, z_i) . Now $\mathcal{X}_{\Delta^w} \rightarrow D_{\Delta^w}$ is a family of blow-ups of $C_{\Delta^w} \rightarrow D_{\Delta^w}$, and has X as central fiber.

Cover each U_i by the two affine open subsets $U_{i0} = \{s_1 \neq 0\}$ and $U_{i1} = \{s_0 \neq 0\}$. Set $\eta_0 := s_0/s_1$ and $\eta_1 := s_1/s_0$. Then the total space \mathcal{X}_{Δ^w} has equations

$$(4) \quad w_1^{u_1} = z_1^{v_1}, \dots, w_m^{u_m} = z_m^{v_m}, \quad \begin{cases} x\eta_0 = w_i & \text{in } U_{i0}, \\ y\eta_1 = z_i & \text{in } U_{i1}, \end{cases}$$

and $\eta_0\eta_1 = 1$ on $U_{i0} \cap U_{i1}$. Let \mathcal{E} be the Cartier divisor given by

$$(5) \quad x^{u_i} \text{ on } U_{i0} \text{ and } y^{v_i} \text{ on } U_{i1}, \text{ for } i = 1, \dots, m.$$

On $U_{i0} \cap U_{i1}$ we have $x^{u_i}/y^{v_i} = \eta_1^r \in \mathcal{O}_{\mathcal{X}_{\Delta^w}}^*(U_{i0} \cap U_{i1})$. Note that:

- \mathcal{E} is effective; its support in each U_i is the inverse image via π of the locus of the i -th node $\{x = y = w_i = z_i\}$ in $V_i \subset \mathcal{C}_{\Delta^w}$;
- $\mathcal{E}|_{\bar{X}} = \sum_{i=1}^m (u_i p_i + v_i q_i)$;
- $\mathcal{O}_{\mathcal{X}_{\Delta^w}}(-\mathcal{E})|_{E_i}$ has degree r for any $i = 1, \dots, m$.

The following result is a global version of Lemma 3.1.1.

Lemma 3.2.1. *Let $\mathcal{C} \rightarrow T$ be a family of nodal curves, $\mathcal{N} \in \text{Pic } \mathcal{C}$ and $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ a limit r -th root of $(\mathcal{C}, \mathcal{N})$. Let $t_0 \in T$, and let Δ^w be the weighted graph of the fiber over t_0 .*

Possibly after shrinking T , there exist a fiber product diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\pi} & \mathcal{C} & \longrightarrow & T \\ \downarrow k & & \downarrow & & \downarrow h \\ \mathcal{X}_{\Delta^w} & \longrightarrow & \mathcal{C}_{\Delta^w} & \longrightarrow & D_{\Delta^w} \end{array}$$

where $h(t_0) = (0, \dots, 0)$, and an isomorphism $\vartheta: \mathcal{L}^{\otimes r} \rightarrow \pi^*(\mathcal{N}) \otimes k^*(\mathcal{O}(-\mathcal{E}))$ whose composition with the natural map $\pi^*(\mathcal{N}) \otimes k^*(\mathcal{O}(-\mathcal{E})) \rightarrow \pi^*(\mathcal{N})$ is α .

Proof. Call \mathcal{C} and \mathcal{X} the fibers of $\mathcal{C} \rightarrow T$ and $\mathcal{X} \rightarrow T$ over t_0 . Since $\mathcal{C} \rightarrow T$ is a deformation of \mathcal{C} , up to shrinking T there is a morphism $\ell: T \rightarrow \underline{D}$ such that $\ell(t_0) = (0, \dots, 0)$ and $\mathcal{C} \simeq \underline{\mathcal{C}} \times_{\underline{D}} T$. By Lemma 3.1.1, ℓ lifts to $h: T \rightarrow D_{\Delta^w}$ and $\mathcal{X} \simeq \mathcal{X}_{\Delta^w} \times_{D_{\Delta^w}} T$. We define ϑ to agree with α away from the exceptional components, as $k^*(\mathcal{O}(-\mathcal{E}))$ is canonically isomorphic to $\mathcal{O}_{\mathcal{X}}$ in this region. What must be seen is that ϑ can be extended to all of \mathcal{X} . Let E_i be an exceptional component of \mathcal{X} , and let $\zeta_{\mathcal{N}}$ be a generator of $\pi^*(\mathcal{N})$ on a neighbourhood of E_i . Near E_i , the space \mathcal{X} is obtained from (4) by base change via h . Let U_0 and U_1 be the inverse images of U_{i0} and U_{i1} . As in Lemma 3.1.1, we write x and y also to indicate the pull-backs of these functions to \mathcal{X} . We know from Lemma 3.1.1 that there are generators ζ_0 and ζ_1 for $\mathcal{L}^{\otimes r}$, over U_0 and U_1 , respectively, with the property that

$$(6) \quad \zeta_0 = \eta_1^r \zeta_1 \text{ on } U_0 \cap U_1, \quad \alpha(\zeta_0) = x^u \zeta_{\mathcal{N}}, \quad \alpha(\zeta_1) = y^v \zeta_{\mathcal{N}},$$

while $\pi^*(\mathcal{N}) \otimes k^*(\mathcal{O}(-\mathcal{E}))$ is generated on U_0 and U_1 by $\zeta'_0 = \zeta_{\mathcal{N}} \otimes k^*(x^u)$ and $\zeta'_1 = \zeta_{\mathcal{N}} \otimes k^*(y^v)$. Now we can extend ϑ across E_i by sending ζ_0 to ζ'_0 and ζ_1 to ζ'_1 ; since $\zeta'_0 = \eta_1^r \zeta'_1$, the first of the identities (6) says that this is not ambiguous. On the other hand, since ζ'_0 maps to $x^u \zeta_{\mathcal{N}}$ in $\pi^*(\mathcal{N})$, and ζ'_1 to $y^v \zeta_{\mathcal{N}}$, the remaining identities say that the composition of ϑ with $\pi^*(\mathcal{N}) \otimes k^*(\mathcal{O}(-\mathcal{E})) \rightarrow \pi^*(\mathcal{N})$ agrees with α . □

Remark 3.2.2. Since D_{Δ^w} equals $C_{u_1, v_1} \times \dots \times C_{u_m, v_m} \times D'$, it is normal if and only if, for every i , either $u_i = 1$, or $v_i = 1$. This is always true if $r = 2$ or $r = 3$.

3.3. Construction of universal deformations. We are now in a position to construct the universal deformation of a limit root. We place ourselves in the setup of section 2.4. Let B be a scheme, $f: C \rightarrow B$ a family of nodal curves and $\mathcal{N} \in \text{Pic } C$ a line bundle of relative degree divisible by r .

Fix $b_0 \in B$, set $C = f^{-1}(b_0)$ and let $\underline{C} \rightarrow \underline{D}$ be a semi-universal deformation of C . There exist a neighbourhood B_0 of b_0 in B and a morphism $B_0 \rightarrow \underline{D}$, $b_0 \mapsto (0, \dots, 0)$, such that $\mathcal{C}_{B_0} \simeq \underline{C} \times_{\underline{D}} B_0$.

Let $\xi := (X, L, \alpha)$ be a limit r -th root of (C, N_{b_0}) and let Δ^w be the associated weighted subgraph of Γ_C . Let $\mathcal{X}_{\Delta^w} \rightarrow D_{\Delta^w}$ be the deformation of X constructed above. Set

$$U_\xi := B_0 \times_{\underline{D}} D_{\Delta^w},$$

let $u_0 \in U_\xi$ be the point that maps to $b_0 \in B_0$ and define $\mathcal{C}_\xi, \mathcal{X}_\xi$ in the natural way. We denote by \mathcal{N}_ξ the pull-back of \mathcal{N} under $\mathcal{C}_\xi \rightarrow \mathcal{C}_{B_0}$; observe that in general \mathcal{N}_ξ is not the pull-back of a line bundle on \mathcal{C}_{Δ^w} :

$$\begin{array}{ccccccc}
 & & & \mathcal{N}_\xi & & & \\
 & & & \downarrow & & & \\
 X \hookrightarrow & \mathcal{X}_\xi & \xrightarrow{\pi_\xi} & \mathcal{C}_\xi & \longrightarrow & U_\xi & \longrightarrow & B_0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \mathcal{X}_{\Delta^w} & \longrightarrow & \mathcal{C}_{\Delta^w} & \longrightarrow & D_{\Delta^w} & \longrightarrow & \underline{D}
 \end{array}$$

Our goal is to prove that L extends to a line bundle on \mathcal{X}_ξ , giving a family of limit r -th roots of \mathcal{N}_ξ .

A key step is the next lemma, concerning automorphisms of the above diagram. As usual, when $W \rightarrow Z$ is a morphism of schemes, we write $\text{Aut}_Z(W)$ to denote the group of automorphisms of W over Z . We shall make use of Lemma 2.3.2 by fixing an orientation on the graph Σ_X and hence an isomorphism $\text{Aut}_C(X) \simeq \mathcal{C}^1(\Sigma_X, \mathbb{C}^*)$.

Lemma 3.3.1. *The action of $G = \text{Aut}_{\underline{D}}(D_{\Delta^w})$ on D_{Δ^w} lifts to:*

- a) a natural action on \mathcal{X}_{Δ^w} , which leaves the divisor \mathcal{E} invariant;
- b) natural actions $G \rightarrow \text{Aut}_{B_0}(U_\xi)$, $G \rightarrow \text{Aut}_{B_0}(\mathcal{C}_\xi)$, and $G \rightarrow \text{Aut}_{B_0}(\mathcal{X}_\xi)$, compatible with the projections $\mathcal{X}_\xi \rightarrow \mathcal{C}_\xi \rightarrow U_\xi$.

Moreover, the homomorphism $G \rightarrow \text{Aut}(X)$ induced by $G \rightarrow \text{Aut}_{B_0}(\mathcal{X}_\xi)$ yields an isomorphism $G \xrightarrow{\sim} \mathcal{C}^1(\Sigma_X, \mu_r) \subset \text{Aut}_C(X)$.

Proof. Automorphisms of D_{Δ^w} over \underline{D} are automorphisms of $C_{u_1, v_1} \times \dots \times C_{u_m, v_m}$ over D . Any such automorphism has the form $(w_i, z_i) \mapsto (\zeta_i w_i, \zeta_i^{-1} z_i)$ with ζ_i an r -th root of unity; thus $G \simeq (\mu_r)^m$.

We show that G acts on \mathcal{X}_{Δ^w} and that the action on the central fiber is by inessential automorphisms, giving $G \simeq \mathcal{C}^1(\Sigma_X, \mu_r) \subset \text{Aut}_C(X)$ (cf. Lemma 2.3.2). Fix $(\zeta_1, \dots, \zeta_m) \in (\mu_r)^m$. In each U_i the space \mathcal{X}_{Δ^w} has equations (4) and the action is given by

$$\begin{aligned}
 (7) \quad & (x, \eta_0, \dots, w_i, \dots) \mapsto (x, \zeta_i \eta_0, \dots, \zeta_i w_i, \dots) \text{ in } U_{i0}, \\
 & (y, \eta_1, \dots, z_i, \dots) \mapsto (y, \zeta_i^{-1} \eta_1, \dots, \zeta_i^{-1} z_i, \dots) \text{ in } U_{i1},
 \end{aligned}$$

hence the induced automorphism of X is inessential and corresponds to $(\zeta_1, \dots, \zeta_m) \in \mathcal{C}^1(\Sigma_X, \mu_r)$. It is clear that \mathcal{E} is invariant under this action.

Now, using the universal property of fiber products, it is immediate to see that the action of G lifts to U_ξ , \mathcal{C}_ξ , and \mathcal{X}_ξ with the desired properties. \square

In what follows, if $\sigma \in G$, we shall denote by σ_{U_ξ} , $\sigma_{\mathcal{C}_\xi}$, and $\sigma_{\mathcal{X}_\xi}$ the corresponding automorphisms of U_ξ , \mathcal{C}_ξ , and \mathcal{X}_ξ .

We now address the problem of extending L . Recall that there exists an effective Cartier divisor \mathcal{E} on \mathcal{X}_{Δ^w} such that $\mathcal{E}|_{\widehat{X}} = \sum_{i=1}^m (u_i p_i + v_i q_i)$ and $\mathcal{O}_{\mathcal{X}_{\Delta^w}}(-\mathcal{E})|_{E_i}$ has degree r for any $i = 1, \dots, m$. Denote by \mathcal{G}_ξ the pull-back of $\mathcal{O}_{\mathcal{X}_{\Delta^w}}(-\mathcal{E})$ to \mathcal{X}_ξ . Then part (i) of Lemma 2.3.3 implies that there are an inessential automorphism σ of X and an isomorphism $\vartheta: L^{\otimes r} \rightarrow \sigma^*(\pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi \otimes \mathcal{O}_X)$ such that α is the composition of ϑ with the pull-back of $\pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi \otimes \mathcal{O}_X \rightarrow \pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{O}_X$. Hence, up to modifying via σ the identification of X with the central fiber of \mathcal{X}_ξ , we can assume that $\alpha: L^{\otimes r} \rightarrow \pi_{b_0}^*(N_{b_0})$ agrees with the restriction to X of the natural map $\pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi \rightarrow \pi_\xi^*(\mathcal{N}_\xi)$.

Shrinking U_ξ and B_0 , if necessary, we can extend L to $\mathcal{L}_\xi \in \text{Pic } \mathcal{X}_\xi$ so that

$$\mathcal{L}_\xi^{\otimes r} \simeq \pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi,$$

by Remark 3.0.6. Let α_ξ be the composition of this isomorphism with $\pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi \rightarrow \pi_\xi^*(\mathcal{N}_\xi)$. Now $(\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi)$ is a limit r -th root of \mathcal{N}_ξ . Moreover, there is an isomorphism $\widehat{\psi}$ of limit roots between ξ and the fiber of the family $(\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi)$ over $u_0 \in U_\xi$. The pair $((\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi), \widehat{\psi})$ is a universal deformation for ξ .

Proposition 3.3.2. *Let $p: T \rightarrow B$ be a morphism of schemes, and $(\mathcal{X} \rightarrow T, \mathcal{L}, \beta)$ a limit r -th root of \mathcal{N}_T . Let $t_0 \in T$ be such that $p(t_0) = b_0$, and assume that there is an isomorphism $\widehat{\phi}$ of limit roots between $\xi = (X, L, \alpha)$ and the fiber of $(\mathcal{X} \rightarrow T, \mathcal{L}, \beta)$ over t_0 . Then, possibly after shrinking T , the deformation $((\mathcal{X} \rightarrow T, \mathcal{L}, \beta), \widehat{\phi})$ is isomorphic to the pull-back of $((\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi), \widehat{\psi})$ via a unique morphism of B -schemes $\gamma: T \rightarrow U_\xi$ such that $\gamma(t_0) = u_0$; moreover, the isomorphism is unique.*

Proof of existence. We know by Lemma 3.2.1 that, up to shrinking T , there exists $h: T \rightarrow D_{\Delta^w}$ such that $\mathcal{C}_T \simeq \mathcal{C}_{\Delta^w} \times_{D_{\Delta^w}} T$ and $\mathcal{X} \simeq \mathcal{X}_{\Delta^w} \times_{D_{\Delta^w}} T$. Recall that $U_\xi = B_0 \times_{\underline{D}} D_{\Delta^w}$, and set $\gamma = p \times h$. Then $\gamma: T \rightarrow U_\xi$ has the property that $\gamma(t_0) = u_0$ and that there is a commutative diagram with fiber product squares

$$(8) \quad \begin{array}{ccccc} \mathcal{X} & \xrightarrow{\pi} & \mathcal{C}_T & \longrightarrow & T \\ \downarrow \delta & & \downarrow \kappa & & \downarrow \gamma \\ \mathcal{X}_\xi & \xrightarrow{\pi_\xi} & \mathcal{C}_\xi & \longrightarrow & U_\xi \longrightarrow B_0 \subset B \end{array}$$

Let \mathcal{L}' and β' be the line bundle and the homomorphism obtained by pulling back \mathcal{L}_ξ and α_ξ via δ . Note that $(\mathcal{X}, \mathcal{L}', \beta')$ is a limit r -th root of \mathcal{N}_T , since $\kappa^*(\mathcal{N}_\xi) = \mathcal{N}_T$ by the commutativity of (8). Moreover, it follows from Lemma 3.2.1 and from the definition of \mathcal{L}_ξ that there is an isomorphism $\vartheta: \mathcal{L}^{\otimes r} \rightarrow \mathcal{L}'^{\otimes r}$ such that $\beta' \circ \vartheta = \beta$.

Now let $\phi: X \rightarrow X_{t_0}$ and $\psi: X \rightarrow (X_\xi)_{u_0}$ be the isomorphisms of schemes underlying $\widehat{\phi}$ and $\widehat{\psi}$, and set $L' = \phi^*(\mathcal{L}')$. Then $\sigma := \psi^{-1} \circ \delta \circ \phi$ is an inessential automorphism of X with the property that $\sigma^*(L) = L'$, and ϑ yields an isomorphism $\chi: L^{\otimes r} \rightarrow \sigma^*(L)^{\otimes r}$ such that $\alpha = \sigma^*(\alpha) \circ \chi$. Then part ii) of Lemma 2.3.3 says that σ belongs to $\mathcal{C}^1(\Sigma_X, \mu_r)$ which, by Lemma 3.3.1, can be identified with $G = \text{Aut}_{\underline{D}}(D_{\Delta^w})$. Moreover, Lemma 3.3.1, suitably interpreted, implies that $\psi \circ \sigma =$

$\sigma_{\mathcal{X}_\xi} \circ \psi$. Thus, replacing γ with $\sigma_{U_\xi}^{-1} \circ \gamma$, κ with $\sigma_{C_\xi}^{-1} \circ \kappa$, and δ with $\sigma_{\mathcal{X}_\xi}^{-1} \circ \delta$, we may suppose that $\psi = \delta \circ \phi$ (recall that, always by Lemma 3.3.1, \mathcal{E} is G -invariant). A consequence is that \mathcal{L} and \mathcal{L}' agree at t_0 . On the other hand, $\mathcal{L}^{\otimes r}$ and $\mathcal{L}'^{\otimes r}$ are isomorphic via ϑ , so Remark 3.0.6 implies that there is an isomorphism between \mathcal{L} and \mathcal{L}' whose r -th tensor power is ϑ . At this point we are almost done: $(\mathcal{X}, \mathcal{L}, \beta)$ is certainly the pull-back of $(\mathcal{X}_\xi, \mathcal{L}_\xi, \alpha_\xi)$. Moreover, $\widehat{\phi}$ is the pull-back of $\widehat{\psi}$, except for one little detail, namely that the corresponding isomorphisms between L and the pull-back of \mathcal{L} might not be equal. However, since the r -th tensor powers of these isomorphisms are equal, the ratio between the two is an r -th root of unity. To fix this it suffices to multiply the isomorphism between \mathcal{L} and \mathcal{L}' by a suitable root of unity. \square

Proof of uniqueness. Assume that the morphism $p: T \rightarrow B_0$ has another lifting $\gamma': T \rightarrow U_\xi$ such that $((\mathcal{X}, \mathcal{L}, \beta), \widehat{\phi})$ is isomorphic to the pull-back of $((\mathcal{X}_\xi, \mathcal{L}_\xi, \alpha_\xi), \widehat{\psi})$ via γ' , and let

$$(9) \quad \begin{array}{ccccc} \mathcal{X} & \xrightarrow{\pi} & \mathcal{C}_T & \longrightarrow & T \\ \downarrow \delta' & & \downarrow \kappa' & & \downarrow \gamma' \\ \mathcal{X}_\xi & \xrightarrow{\pi_\xi} & \mathcal{C}_\xi & \longrightarrow & U_\xi \longrightarrow B_0 \subset B \end{array}$$

be the corresponding commutative diagram. Call h and h' the morphisms $T \rightarrow D_{\Delta^w}$ induced, respectively, by γ and γ' . Then h and h' are both liftings of $T \rightarrow \underline{D}$, so there exists $\sigma \in G$ such that $h' = \sigma \circ h$. As p is fixed, this yields that $\gamma' = \sigma_{U_\xi} \circ \gamma$, and hence also that $\kappa' = \sigma_{C_\xi} \circ \kappa$. If we compose the vertical arrows of (9) with the actions of the inverse of σ on the respective target spaces, we get another commutative diagram whose vertical arrows are $\delta'' := \sigma_{\mathcal{X}_\xi}^{-1} \circ \delta'$, κ , and γ . Clearly, δ'' and δ differ at most by an automorphism of \mathcal{X} over \mathcal{C}_T ; in other words, there is $\iota \in \text{Aut}_{\mathcal{C}_T}(\mathcal{X})$ such that $\delta'' = \delta \circ \iota$. Moreover, there is an isomorphism between $\mathcal{L}^{\otimes r}$ and $\iota^*(\mathcal{L}^{\otimes r})$ which is compatible with β . We need an auxiliary lemma.

Lemma 3.3.3. *Let $\iota \in \text{Aut}_{\mathcal{C}_T}(\mathcal{X})$. Suppose that there exists an isomorphism $\vartheta: \mathcal{L}^{\otimes r} \rightarrow \iota^*(\mathcal{L}^{\otimes r})$ such that $\beta = \iota^*(\beta) \circ \vartheta$. Then there exists $\tau \in G$ such that, possibly after shrinking T , $\delta \circ \iota = \tau_{\mathcal{X}_\xi} \circ \delta$.*

Proof of the lemma. This is essentially a version “with parameters” of part (ii) of Lemma 2.3.3. Let E be an exceptional component of X_{t_0} . Recall from Lemma 3.1.1 that there are open sets U_0 and U_1 covering E such that \mathcal{X} is of the form $x\eta_0 = h_0$ in U_0 and of the form $x\eta_1 = h_1$ in U_1 , where $\eta_0\eta_1 = 1$ and h_0, h_1 are functions on a neighbourhood of t_0 such that $h_0^u = h_1^v$. Since ι is an automorphism over \mathcal{C}_T , $\iota^*(x) = x$ and $\iota^*(y) = y$. Write

$$\iota^*(\eta_0) = \sum_{i \geq 0} a_i x^i + \sum_{i < 0} a_i \eta_0^{-i}, \quad \iota^*(\eta_1) = \sum_{i \geq 0} b_i y^i + \sum_{i < 0} b_i \eta_1^{-i}.$$

From $\iota^*(x)\iota^*(\eta_0) = h_0$ we get in particular that $a_i = 0$ for $i \geq 0$. Similarly, $b_i = 0$ for $i \geq 0$. On the other hand, a_{-1} and b_{-1} are both different from zero, since ι restricts to an automorphism of E . Starting from $\iota^*(\eta_0)\iota^*(\eta_1) = 1$, another simple power series computation then yields that $a_i = b_i = 0$ for $i < -1$, and that $a_{-1}b_{-1} = 1$. Thus $\iota^*(\eta_0) = k^{-1}\eta_0$ and $\iota^*(\eta_1) = k\eta_1$, where $k = b_{-1}$ is a unit on a neighbourhood of t_0 . To prove the lemma it suffices to show that k is an r -th root of

unity. Now, Lemma 3.1.1 says that there are generators ζ_i for $\mathcal{L}^{\otimes r}$ on U_i , $i = 0, 1$, and $\zeta_{\mathcal{N}}$ for $\pi^*(\mathcal{N}_T)$ on $U_0 \cup U_1$, with the property that $\zeta_0 = \eta_1^r \zeta_1$, $\beta(\zeta_0) = x^u \zeta_{\mathcal{N}}$, and $\beta(\zeta_1) = y^v \zeta_{\mathcal{N}}$. Write $\vartheta(\zeta_i) = f_i \iota^*(\zeta_i)$, $i = 0, 1$, where

$$f_0 = \sum_{i \geq 0} c_i x^i + \sum_{i < 0} c_i \eta_0^{-i}, \quad f_1 = \sum_{i \geq 0} d_i y^i + \sum_{i < 0} d_i \eta_1^{-i}.$$

From $\beta(\zeta_0) = x^u \zeta_{\mathcal{N}}$ and $\beta = \iota^*(\beta) \circ \vartheta$ we then get that $x^u = x^u f_0$; this yields that $c_i = 0$ for $i > 0$ and that $c_0 = 1$. Similarly, $d_i = 0$ for $i > 0$, and $d_0 = 1$. On the other hand,

$$\eta_1^r f_1 \iota^*(\zeta_1) = \eta_1^r \vartheta(\zeta_1) = \vartheta(\zeta_0) = f_0 \iota^*(\zeta_0) = f_0 \iota^*(\eta_1^r \zeta_1) = k^r \eta_1^r f_0 \iota^*(\zeta_1).$$

This gives that $d_0 \eta_1^r + d_{-1} \eta_1^{r-1} + \dots = k^r (c_0 \eta_1^r + c_{-1} \eta_1^{r+1} + \dots)$, and hence in particular that $k^r = 1$, as desired. \square

We return to the proof of uniqueness in Proposition 3.3.2. Lemma 3.3.3 implies that $\delta' = \sigma_{\mathcal{X}_\xi} \circ \tau_{\mathcal{X}_\xi} \circ \delta$. On the other hand, $\tau_{\mathcal{C}_\xi} \circ \kappa = \kappa$ and $\tau_{U_\xi} \circ \gamma = \gamma$, since $\tau_{\mathcal{X}_\xi}$ corresponds, via δ , to an automorphism of \mathcal{X} over \mathcal{C}_T . Thus, replacing the original σ with $\sigma\tau$, we may in fact suppose that not only $\gamma' = \sigma_{U_\xi} \circ \gamma$ and $\kappa' = \sigma_{\mathcal{C}_\xi} \circ \kappa$, but also $\delta' = \sigma_{\mathcal{X}_\xi} \circ \delta$. In particular, $\sigma_{\mathcal{X}_\xi} \circ \delta \circ \phi = \psi$. However, by the proof of existence, ψ is also equal to $\delta \circ \phi$, so $\sigma_{\mathcal{X}_\xi}$ acts trivially on the fiber of $\mathcal{X}_\xi \rightarrow U_\xi$ at u_0 . Since this action corresponds via ψ to the action of σ on X , it follows that $\sigma = \text{Id}$, $\gamma' = \gamma$, and $\delta' = \delta$. That there is a unique isomorphism $\mathcal{L} \simeq \delta^*(\mathcal{L}_\xi)$ which is compatible with β and $\delta^*(\alpha_\xi)$ and agrees with the given one on the fibers above t_0 follows from the uniqueness part of Remark 3.0.6. This finishes the proof of Proposition 3.3.2. \square

Remark 3.3.4. It follows from the construction of $(\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi)$ that this family is a universal deformation for any one of its fibers.

3.4. The moduli scheme of limit roots. We are ready to prove Theorem 2.4.1. Consider the set

$$\overline{S}_f^r(\mathcal{N}) = \coprod_{b \in B} \{\text{limit } r\text{-th roots of } (C_b, N_b)\} / \sim,$$

where \sim is the equivalence relation given by isomorphism of limit roots.

Fix $b_0 \in B$ and a limit r -th root $\xi = (X, L, \alpha)$ of (C_{b_0}, N_{b_0}) . Let $(\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi)$ be the universal deformation of ξ and $B_0 \subset B$ the open subset of B dominated by U_ξ .

Recall that, by Lemmas 2.3.2 and 3.3.1, there are natural identifications and maps:

$$\text{Aut}(\xi) = \mathcal{C}^0(\Sigma_X, \mu_r) \rightarrow \mathcal{C}^1(\Sigma_X, \mu_r) = G.$$

Via this homomorphism, the group $\text{Aut}(\xi)$ acts on U_ξ , \mathcal{C}_ξ , and \mathcal{X}_ξ ; if $\sigma \in \text{Aut}(\xi)$, we will denote by σ_{U_ξ} , $\sigma_{\mathcal{C}_\xi}$, and $\sigma_{\mathcal{X}_\xi}$ the corresponding automorphisms of these spaces. The universality of the family $(\mathcal{X}_\xi, \mathcal{L}_\xi, \alpha_\xi)$ shown in Proposition 3.3.2 implies that, if $\sigma \in \text{Aut}(\xi)$, then $\sigma_{\mathcal{X}_\xi}^*(\mathcal{L}_\xi) \simeq \mathcal{L}_\xi$ and $\sigma_{\mathcal{X}_\xi}^*(\alpha_\xi) \simeq \alpha_\xi$.

Lemma 3.4.1. *Let $b \in B_0$, let $u_1, u_2 \in U_\xi$ be points lying over b , and let (X_1, L_1, α_1) , (X_2, L_2, α_2) be the corresponding limit r -th roots of (C_b, N_b) . The following are equivalent:*

- (i) *there exists $\sigma \in \text{Aut}(\xi)$ such that $\sigma_{U_\xi}(u_1) = u_2$;*
- (ii) *there exists an isomorphism of limit roots between (X_1, L_1, α_1) and (X_2, L_2, α_2) .*

Proof. The implication (i) \Rightarrow (ii) is an immediate consequence of what we observed above: if there exists $\sigma \in \text{Aut}(\xi)$ such that $\sigma_{U_\xi}(u_1) = u_2$, then $(\sigma_{\mathcal{X}_\xi})|_{X_1} : X_1 \rightarrow X_2$ is an inessential isomorphism which induces an isomorphism of limit roots.

Conversely, assume that there is an inessential isomorphism $\phi : X_1 \rightarrow X_2$ that induces an isomorphism between the corresponding limit roots. We are going to show first of all that $\phi = (\sigma_{\mathcal{X}_\xi})|_{X_1}$ for some $\sigma \in G$, and then that σ belongs to the image of $\text{Aut}(\xi)$ in G . Let $d \in \underline{D}$ be the image of b , and $d_1, d_2 \in D_{\Delta^w}$ the images of u_1, u_2 . Recall that G is the group of automorphisms of D_{Δ^w} over \underline{D} , and that $\underline{D} = D_{\Delta^w}/G$. Choose $\tau \in G$ such that $\tau(d_1) = d_2$. Then $\tau_{U_\xi}(u_1) = u_2$ and

$$\rho := \phi^{-1} \circ (\tau_{\mathcal{X}_\xi})|_{X_1} : X_1 \longrightarrow X_1$$

is an inessential automorphism of X_1 . Note that $\rho \in \mathcal{C}^1(\Sigma_{X_1}, \mu_r) \subset \text{Aut}_{C_b}(X_1)$ by construction.

Set $\xi_1 := (X_1, L_1, \alpha_1)$. The universal deformation $(\mathcal{X}_{\xi_1} \rightarrow U_{\xi_1}, \mathcal{L}_{\xi_1}, \alpha_{\xi_1})$ of ξ_1 is naturally an open subfamily of $(\mathcal{X}_\xi \rightarrow U_\xi, \mathcal{L}_\xi, \alpha_\xi)$ (see Remark 3.3.4). Applying Lemma 3.3.1 with respect to ξ_1 , we see that there is a subgroup G_1 of G which is naturally identified with $\mathcal{C}^1(\Sigma_{X_1}, \mu_r) \subset \text{Aut}_{C_b}(X_1)$. If $\{j_1, \dots, j_k\} \subset \{1, \dots, m\}$ are the indices for which the exceptional component E_i of X does not deform to X_1 (namely, in C_b the i -th node is smoothed), then under the isomorphism $G \simeq (\mu_r)^m$, G_1 is the subgroup of $(\zeta_1, \dots, \zeta_m)$ with $\zeta_{j_1} = \dots = \zeta_{j_k} = 1$.

Let $\tau' \in G_1$ be such that $(\tau'_{\mathcal{X}_\xi})|_{X_1} = \rho$, and set $\sigma := \tau \circ (\tau')^{-1}$. Then $(\sigma_{\mathcal{X}_\xi})|_{X_1} = \phi$. Note that $\sigma_{\mathcal{X}_\xi}^*(\mathcal{G}_\xi) \simeq \mathcal{G}_\xi$ and $\sigma_{\mathcal{X}_\xi}^*(\pi_\xi^*(\mathcal{N}_\xi)) \simeq \pi_\xi^*(\mathcal{N}_\xi)$ (this is true for all elements of G). In fact, the first property is easily deduced from (5) and (7); for the second property, it is enough to observe that $\sigma_{\mathcal{C}_\xi}^*(\mathcal{N}_\xi) \simeq \mathcal{N}_\xi$ and that $\pi_\xi \circ \sigma_{\mathcal{X}_\xi} = \sigma_{\mathcal{C}_\xi} \circ \pi_\xi$, so $\sigma_{\mathcal{X}_\xi}^*(\pi_\xi^*(\mathcal{N}_\xi)) \simeq \pi_\xi^*(\sigma_{\mathcal{C}_\xi}^*(\mathcal{N}_\xi)) \simeq \pi_\xi^*(\mathcal{N}_\xi)$.

Now, since $\mathcal{L}_\xi^{\otimes r} \simeq \pi_\xi^*(\mathcal{N}_\xi) \otimes \mathcal{G}_\xi$, we get $\sigma_{\mathcal{X}_\xi}^*(\mathcal{L}_\xi^{\otimes r}) \simeq \mathcal{L}_\xi^{\otimes r}$. By construction

$$\sigma_{\mathcal{X}_\xi}^*(\mathcal{L}_\xi)|_{X_1} = \phi^*(L_2) \simeq L_1 = \mathcal{L}_\xi|_{X_1},$$

so we get $\sigma_{\mathcal{X}_\xi}^*(\mathcal{L}_\xi) \simeq \mathcal{L}_\xi$ by Remark 3.0.6, and σ comes from $\text{Aut}(\xi)$. □

Proof of Theorem 2.4.1. For every r -th root ξ as above we have a well-defined, injective map between B-sets:

$$\beta_\xi : U_\xi / \text{Aut}(\xi) \longrightarrow \overline{S}_f^r(\mathcal{N}),$$

and the image of β_ξ inherits a complex structure from $U_\xi / \text{Aut}(\xi)$. Since $\overline{S}_f^r(\mathcal{N})$ is covered by subsets of the form $\text{Im } \beta_\xi$, in order for these “charts” to define a complex structure on $\overline{S}_f^r(\mathcal{N})$, the following must hold: if ξ_1 and ξ_2 are limit roots such that $\text{Im } \beta_{\xi_1}$ and $\text{Im } \beta_{\xi_2}$ intersect, then $\beta_{\xi_2}^{-1}(\text{Im } \beta_{\xi_1})$ is open and the composition $\beta_{\xi_2}^{-1} \circ \beta_{\xi_1}$ is holomorphic. That this is true follows easily from Proposition 3.3.2, Remark 3.3.4, and Lemma 3.4.1. In fact, choose a limit root η corresponding to a point in the intersection of $\text{Im } \beta_{\xi_1}$ and $\text{Im } \beta_{\xi_2}$. Then there are natural (and canonically determined) open immersions $J_i : U_\eta \hookrightarrow U_{\xi_i}$, compatible with the actions of $\text{Aut}(\eta)$ and $\text{Aut}(\xi_i)$. Thus J_i induces a morphism $\overline{J}_i : U_\eta / \text{Aut}(\eta) \hookrightarrow U_{\xi_i} / \text{Aut}(\xi_i)$, which is an embedding by Lemma 3.4.1. Finally, we have $\beta_\eta = \beta_{\xi_i} \circ \overline{J}_i$.

The analytic morphism $p : \overline{S}_f^r(\mathcal{N}) \rightarrow B$ has finite fibers. Proposition 2.1.2 implies that p is proper, hence it is a finite projective morphism. As B is quasiprojective, so is $\overline{S}_f^r(\mathcal{N})$.

Finally, the scheme $\overline{S}_f^r(\mathcal{N})$ is a coarse moduli space for the functor $\overline{S}_f^r(\mathcal{N})$. In fact, for any B -scheme T , the map $\overline{S}_f^r(\mathcal{N})(T) \rightarrow \mathcal{H}om_B(T, \overline{S}_f^r(\mathcal{N}))$ is naturally defined as follows. To any limit r -th root $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$ of \mathcal{N}_T , we associate the morphism $T \rightarrow \overline{S}_f^r(\mathcal{N})$ which is locally given by Proposition 3.3.2. Clearly, this morphism depends only on the isomorphism class of $(\mathcal{X} \rightarrow T, \mathcal{L}, \alpha)$. \square

Proof of Proposition 2.4.2. The morphism p is smooth at a point ξ if and only if $U_\xi/\text{Aut}(\xi) \simeq B_0$, that is, if and only if $\text{Aut}(\xi) = G$. This is equivalent to $b_1(\Sigma_X) = 0$. \square

4. GEOMETRY OF THE MODULI SPACE OF LIMIT ROOTS

4.1. Limit roots for a fixed curve: examples. Let C be a nodal curve and $N \in \text{Pic } C$. We denote by $\overline{S}_C^r(N)$ the zero-dimensional scheme $\overline{S}_{f_C}^r(N)$, where $f_C: C \rightarrow \{pt\}$ is the trivial family. Note that, by Remark 2.4.3, for any family $f: \mathcal{C} \rightarrow B$ whose fiber over $b_0 \in B$ is C and for any $\mathcal{N} \in \text{Pic } \mathcal{C}$ such that $\mathcal{N}|_C = N$, the scheme-theoretical fiber over b_0 of the finite morphism $\overline{S}_f^r(\mathcal{N}) \rightarrow B$ is isomorphic to $\overline{S}_C^r(N)$.

We have seen in Remark 2.2.1 that there are $r^{b_1(\Gamma_C)}$ weighted subgraphs of Γ_C satisfying conditions (C1) and (C2). Let Δ^w be one of them. Recall that Δ^w carries the weights u_i, v_i on each edge, and that its supporting graph $\overline{\Delta}$ is a subgraph of Γ_C . As usual, denote by g^ν the genus of the normalization of C . The set of nodes Δ determines a partial normalization $\tilde{\pi}: \tilde{X} \rightarrow C$ of C , and the weights determine a line bundle $\tilde{\pi}^*(N)(-\sum_i(u_i p_i + v_i q_i))$ on \tilde{X} . The dual graph of \tilde{X} coincides with $\Gamma_C \setminus \Delta$, so we have $r^{2g^\nu + b_1(\Gamma_C \setminus \Delta)}$ choices for a line bundle $\tilde{L} \in \text{Pic } \tilde{X}$ such that $\tilde{L}^{\otimes r} \simeq \tilde{\pi}^*(N)(-\sum_i(u_i p_i + v_i q_i))$ (r^{2g^ν} choices for the pull-back of \tilde{L} to the normalization of \tilde{X} , and $r^{b_1(\Gamma_C \setminus \Delta)}$ choices for the gluings at the nodes). Each choice of \tilde{L} gives a point (X, L, α) in $\overline{S}_C^r(N)$ with weighted graph Δ^w .

Lemma 4.1.1. *The geometric multiplicity of the connected component of $\overline{S}_C^r(N)$ supported on (X, L, α) is $r^{b_1(\Sigma_X)} = r^{b_1(\Gamma_C) - b_1(\Gamma_C \setminus \Delta)}$.*

Proof. Recall that Σ_X is the graph whose vertices are the connected components of \tilde{X} and whose edges are the exceptional components of X . The connected component of $\overline{S}_C^r(N)$ corresponding to (X, L, α) is isomorphic to the fiber over the origin of the morphism $D_{\Delta^w}/\text{Aut}(X, L, \alpha) \rightarrow \underline{D}$ (see section 3). If X has m exceptional components and γ is the number of connected components of \tilde{X} , then the order of ramification of $D_{\Delta^w}/\text{Aut}(X, L, \alpha) \rightarrow \underline{D}$ over the origin is $r^m/r^{\gamma-1} = r^{b_1(\Sigma_X)}$. Now note that Σ_X is obtained from Γ_C by contracting all edges in $\Gamma_C \setminus \Delta$, thus $b_1(\Sigma_X) = b_1(\Gamma_C) - b_1(\Gamma_C \setminus \Delta)$. \square

In conclusion, using Remark 2.2.1, we have that the length of $\overline{S}_C^r(N)$ is

$$\sum_{\Delta^w \in \partial^{-1}([\underline{d}]_r)} r^{2g^\nu + b_1(\Gamma_C \setminus \Delta)} \cdot r^{b_1(\Gamma_C) - b_1(\Gamma_C \setminus \Delta)} = r^{b_1(\Gamma_C)} \cdot r^{2g^\nu + b_1(\Gamma_C)} = r^{2g},$$

as expected. Here, as usual, $[\underline{d}]_r$ stands for the modulo r multidegree of N .

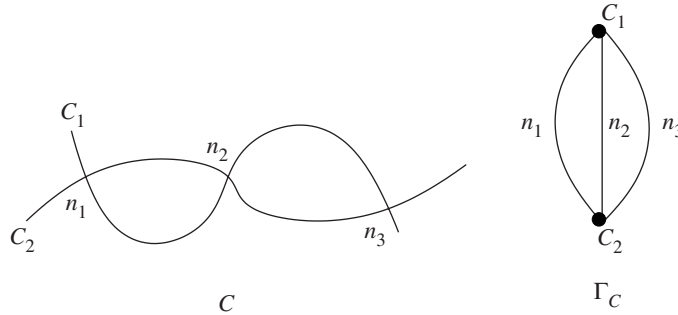
Remark 4.1.2. Consider a limit root (C, L) having C itself as an underlying curve; then the scheme $\overline{S}_C^r(N)$ is reduced at the point (C, L) .

Remark 4.1.3. By construction $D_{\Delta^w} \rightarrow \underline{D}$ depends only on C and on Δ^w , while $\text{Aut}(X, L, \alpha)$ depends only on Σ_X and r , by Lemma 2.3.2. Thus

- the scheme structure on $\overline{\mathcal{S}}_C^r(N)$ depends only on $[\underline{d}]_r$.

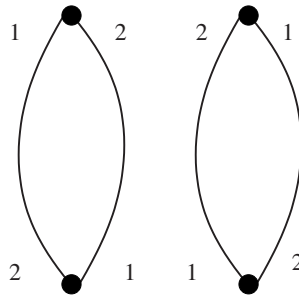
Example 4.1.4 (Compact type). Let $C = \bigcup_j C_j$ be a curve of compact type. Then $b_1(\Gamma_C) = 0$, hence there is a unique weighted subgraph of Γ_C satisfying (C1) and (C2) (which depends on $[\underline{d}]_r$ by the previous remark). So there is a unique blow-up X of C (X is of compact type) with a divisor $D = \sum_i (u_i p_i + v_i q_i)$, such that every limit root in $\overline{\mathcal{S}}_C^r(N)$ is of type (X, L, α) with $L|_X^{\otimes r} \simeq \tilde{\pi}^*(N)(-D)$. In particular, $X = C$ if and only if $[\underline{d}]_r = 0$. Note also that $\overline{\mathcal{S}}_C^r(N)$ is reduced.

Example 4.1.5. Let C be a curve of genus g with two smooth components C_1, C_2 and three nodes n_1, n_2, n_3 :

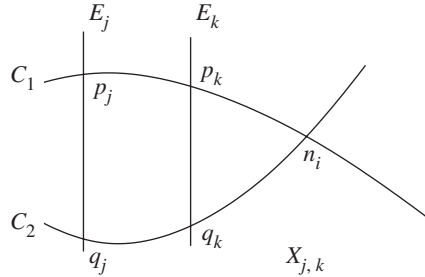


We describe $\overline{\mathcal{S}}_C^3(\mathcal{O}_C)$. Since $b_1(\Gamma_C) = 2$, Γ_C must have $3^2 = 9$ weighted subgraphs satisfying (C1) and (C2). One is \emptyset . This choice corresponds to points of type (C, L) with $L^{\otimes 3} \simeq \mathcal{O}_C$; there are $9 \cdot 3^{2g^\nu}$ of them (with $g^\nu = g - 2$), all reduced points in $\overline{\mathcal{S}}_C^3(\mathcal{O}_C)$.

For each Δ consisting of two edges $\{n_j, n_k\}$ (where $j, k \in \{1, 2, 3\}$), we have two possible choices for the weights:

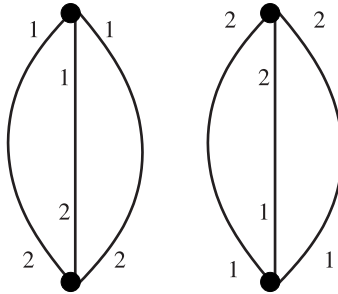


Denote by $X_{j,k}$ the curve obtained by blowing up Δ .

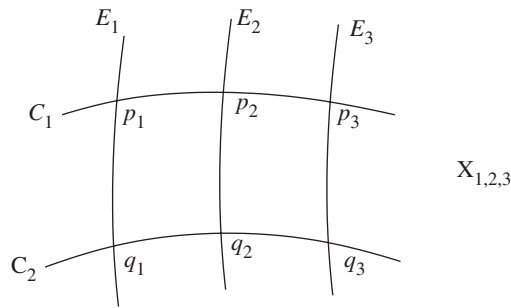


We get points of type $(X_{j,k}, L)$ where $L|_{\widetilde{X}_{j,k}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{j,k}}(-2p_j - q_j - p_k - 2q_k)$ or $L|_{\widetilde{X}_{j,k}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{j,k}}(-p_j - 2q_j - 2p_k - q_k)$, depending on the weights. Since $\Gamma_C \setminus \Delta$ is contractible, we have $b_1(\Gamma_C \setminus \Delta) = 0$, so these points have multiplicity 9 in $\overline{S}_C^3(\mathcal{O}_C)$; there are $6 \cdot 3^{2g^\nu}$ of them.

Finally, when $\Delta = \{n_1, n_2, n_3\}$, we have two possible choices for the weights:



Denote by $X_{1,2,3}$ the corresponding curve.



We get points $(X_{1,2,3}, L)$, where $L|_{\widetilde{X}_{1,2,3}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{1,2,3}}(-p_1 - p_2 - p_3 - 2q_1 - 2q_2 - 2q_3)$ or $L|_{\widetilde{X}_{1,2,3}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{1,2,3}}(-2p_1 - 2p_2 - 2p_3 - q_1 - q_2 - q_3)$. There are $2 \cdot 3^{2g^\nu}$ of these points, each of them with multiplicity 9 in $\overline{S}_C^3(\mathcal{O}_C)$.

4.2. Higher spin curves. Let $\overline{M}_{g,n}$ be the moduli space, and $\overline{\mathcal{M}}_{g,n}$ the moduli stack, of stable n -pointed curves of genus g . Let ω_{u_n} be the relative dualizing sheaf, and $\Sigma_1, \dots, \Sigma_n$ the tautological sections, of the universal curve $u_n: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$.

Fix integers l, m_1, \dots, m_n such that $r \mid l(2g - 2) + m_1 + \dots + m_n$, and set $\underline{m} := \{m_1, \dots, m_n\}$ and

$$\omega_{\underline{m}}^l := \omega_{u_n}^{\otimes l} (m_1 \Sigma_1 + \dots + m_n \Sigma_n) \in \text{Pic } \overline{\mathcal{C}}_{g,n}.$$

We introduce a contravariant functor

$$\overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}}: \{\text{schemes}\} \longrightarrow \{\text{sets}\},$$

encoding the moduli problem for limit r -th roots of $\omega_{\underline{m}}^l$ over $\overline{\mathcal{M}}_{g,n}$, as follows. Look at pairs $(f: \mathcal{C} \rightarrow B, \xi)$ consisting of a family $f: \mathcal{C} \rightarrow B$ of n -pointed genus g curves and a limit root ξ of $(\mathcal{C}, \omega_f^{\otimes l}(\sum m_i \sigma_i))$, where $\sigma_1, \dots, \sigma_n$ are the tautological sections of f and ω_f is the relative dualizing sheaf. Two such pairs $(f: \mathcal{C} \rightarrow B, \xi)$ and $(f': \mathcal{C}' \rightarrow B, \xi')$ are considered equivalent if there is an isomorphism $\kappa: \mathcal{C} \rightarrow \mathcal{C}'$ of families of n -pointed curves and an isomorphism, as limit roots of $(\mathcal{C}, \omega_f^{\otimes l}(\sum m_i \sigma_i))$, between ξ and $\kappa^*(\xi')$ (observe that, if $\sigma'_1, \dots, \sigma'_n$ are the tautological sections of f' , then $\omega_f^{\otimes l}(\sum m_i \sigma_i)$ and $\kappa^*(\omega_{f'}^{\otimes l}(\sum m_i \sigma'_i))$ are canonically isomorphic). Note that this is clearly a coarser equivalence relation than the one given by isomorphism of limit roots over a fixed family of curves.

Then, for every scheme B , we define $\overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}}(B)$ to be the set of all equivalence classes of pairs $(f: \mathcal{C} \rightarrow B, \xi)$ as above. Arguing exactly as in the proof of Theorem 2.4.1 we obtain:

Theorem 4.2.1. *The functor $\overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}}$ is coarsely represented by a projective scheme $\overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}}$, finite over $\overline{\mathcal{M}}_{g,n}$.*

We call $\overline{\varphi}: \overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}} \rightarrow \overline{\mathcal{M}}_{g,n}$ the structure morphism. For any stable n -pointed curve $(C, \sigma_1, \dots, \sigma_n)$, the fiber of $\overline{\varphi}$ over $[C]$ is

$$\overline{\mathcal{S}}_C^r(\omega_C^{\otimes l}(m_1 \sigma_1 + \dots + m_n \sigma_n)) / \text{Aut}(C, \sigma_1, \dots, \sigma_n).$$

When we consider stable curves without markings (i.e., when $n = 0$), we denote by $\overline{\mathcal{S}}_g^{r,l}$ the corresponding moduli space, and by $\overline{\mathcal{S}}_C^{r,l}$ the fiber of $\overline{\varphi}$ over $[C] \in \overline{\mathcal{M}}_g$. The scheme $\overline{\mathcal{S}}_g^{2,1}$ is the moduli space of spin curves, constructed in [8].

Denote by $S_{g,n}^{r,l,\underline{m}}$ the open subscheme of $\overline{\mathcal{S}}_{g,n}^{r,l,\underline{m}}$ parametrizing points (C, L) with C stable, and let $\varphi: S_{g,n}^{r,l,\underline{m}} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the natural morphism.

Pairs (C, L) with C a smooth curve and $L \in \text{Pic } C$ such that $L^{\otimes r} \simeq \omega_C$ are generally called r -spin curves. Their moduli space admits a natural stratification by loci parametrising line bundles L with increasing $h^0(C, L)$, so that the largest locus corresponds to “effective r -spin curves”; we refer to the recent paper of A. Polishchuk [21] for more details and open problems related to this interesting aspect.

Several modular compactifications of the moduli space of r -spin curves, and more generally of $S_{g,n}^{r,l,\underline{m}}$, have been introduced by T. J. Jarvis [14, 15, 16], by means of rank 1, torsion-free sheaves.

The next statement summarizes well-known results relating line bundles and torsion-free sheaves of rank 1.

Proposition 4.2.2. *Let B be an integral scheme and $f: \mathcal{C} \rightarrow B$ a family of nodal curves.*

- (I) Let $\pi: \mathcal{X} \rightarrow \mathcal{C}$ be a family of blow-ups of \mathcal{C} and let $\mathcal{L} \in \text{Pic } \mathcal{X}$ be a line bundle having degree 1 on every exceptional component. Then $\pi_*(\mathcal{L})$ is a relatively torsion-free sheaf of rank 1, flat over B .
- (II) Conversely, suppose that \mathcal{F} is a relatively torsion-free sheaf of rank 1 on \mathcal{C} , flat over B . Then there exist a family $\pi: \mathcal{X} \rightarrow \mathcal{C}$ of blow-ups of \mathcal{C} and a line bundle $\mathcal{L} \in \text{Pic } \mathcal{X}$ having degree 1 on all exceptional components, such that $\mathcal{F} \simeq \pi_*(\mathcal{L})$.
- (III) Let $\pi: \mathcal{X} \rightarrow \mathcal{C}$, $\pi': \mathcal{X}' \rightarrow \mathcal{C}$ be families of blow-ups of \mathcal{C} and $\mathcal{L} \in \text{Pic } \mathcal{X}$, $\mathcal{L}' \in \text{Pic } \mathcal{X}'$ line bundles having degree 1 on every exceptional component. Then $\pi_*(\mathcal{L}) \simeq \pi'_*(\mathcal{L}')$ if and only if there exists an isomorphism $\sigma: \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ over \mathcal{C} such that $\mathcal{L} \simeq \sigma^*(\mathcal{L}')$.

Proof. (I) is a local property, see [14], Proposition 3.1.2.

(II) is a consequence of G. Faltings’ local characterization of torsion-free sheaves [11]; see [14], §3.1 and 3.2, in particular Theorem 3.2.2.

(III) The “if” part is clear. To prove the converse, it suffices to note that there is an isomorphism $\tau: \mathcal{X} \rightarrow \text{Proj}(\bigoplus_{k \geq 0} \text{Sym}^k(\pi_*(\mathcal{L})))$ of spaces over \mathcal{C} , with the property that $\tau^*(\mathcal{O}(1))$ is isomorphic to \mathcal{L} , and similarly for \mathcal{X}' and \mathcal{L}' . \square

Jarvis introduces the notion of an r -th root sheaf of a line bundle $N \in \text{Pic } C$, which is a rank 1, torsion-free sheaf \mathcal{F} on C together with a homomorphism $\beta: \mathcal{F}^{\otimes r} \rightarrow N$ with the following properties (see [16], Definition 2.3):

- $r \deg \mathcal{F} = \deg N$;
- β is an isomorphism on the locus where \mathcal{F} is locally free;
- at any node of C where \mathcal{F} is not locally free, the length of the cokernel of β is $r - 1$.

Such an r -th root sheaf is always of the form $\pi_*(L)$, where X is the curve obtained from C by blowing-up the nodes where \mathcal{F} is not locally free, $\pi: X \rightarrow C$ is the natural morphism and (X, L, β) is a limit r -th root of (C, N) . It is easy to see, using Proposition 4.2.2, that the moduli functor of r -th root sheaves of ω_m^l over $\overline{M}_{g,n}$ is isomorphic to $\overline{S}_{g,n}^{r,l,m}$ (see [14], 3.1 and 4.2.2, and [16], 2.2.2 and page 38). Hence we have the following:

Theorem 4.2.3. *The scheme $\overline{S}_{g,n}^{r,l,m}$ is isomorphic over $\overline{M}_{g,n}$ to the coarse moduli space of the stack $\text{ROOT}_{g,n}^{1/r}(\omega_m^l)$ (in the notation of [15]) of r -th root sheaves of ω_m^l .*

Observe that, exactly as in [15], one could construct different (less singular) compactifications of $S_{g,n}^{r,l,m}$ (and more generally of $S_f^r(\mathcal{N})$) in two ways:

- (a) requiring, in the definition of a family of limit roots, that property (P) of Remark 3.1.2 holds; this amounts to considering only “not too singular” families of limit roots;
- (b) attaching to a limit r -th root the data of a set of limit d -th roots for any positive d dividing r , i.e., considering “coherent nets of roots”. Set-theoretically, this amounts to attaching to a limit r -th root $\xi = (X, L, \alpha)$ the data of the gluings for ξ^m for any integer m dividing r .

5. HIGHER SPIN CURVES IN THE UNIVERSAL PICARD SCHEME

5.1. **Compactifying the Picard functor.** Assume $g \geq 3$ and let $P_{d,g} \rightarrow M_g^0$ be the degree d universal Picard variety⁵ whose fiber over a smooth curve C is $\text{Pic}^d C$, the variety of line bundles of degree d on C . Let $\overline{P}_{d,g} \rightarrow \overline{M}_g$ be the modular compactification constructed in [5]; $\overline{P}_{d,g}$ is an integral, normal projective scheme. Its fiber over $C \in \overline{M}_g^0$ is denoted by \overline{P}_C^d and gives a compactification of (a finite number of copies of) the generalized Jacobian of C . The moduli properties of $\overline{P}_{d,g}$ are described in [5], section 8. Its boundary points are of the same nature as limit roots: they correspond to line bundles on quasistable curves⁶ having degree 1 on exceptional components.

The moduli space of higher spin curves over M_g^0 naturally embeds in $P_{d,g}$ (see Lemma-Definition 5.2.1); its projective closure in $\overline{P}_{d,g}$ is thus a new compactification which we shall now study.

We recall from [5] a few basic properties of $\overline{P}_{d,g}$, which, as \overline{M}_g , is constructed by means of Geometric Invariant Theory (see [13]; in this section we freely use the language of [19]). We begin with some useful conventions and numerical preliminaries. Let d be an integer; we denote by \underline{d} an element of \mathbb{Z}^γ whose entries add up to d , that is, $\underline{d} = \{d_1, \dots, d_\gamma\}$ and $\sum d_i = d$. We say that \underline{d} is divisible by some integer r if all of its entries are; if that is the case, we write $\underline{d} \equiv 0 \pmod{r}$. If q is a rational number, we set $q\underline{d} = \{qd_1, \dots, qd_\gamma\}$, and we say that $q\underline{d}$ is integer if qd_i is integer for every i .

For any nodal curve X , we denote by X_1, \dots, X_γ its irreducible components and set $k_i = \#X_i \cap \overline{X \setminus X_i}$. Let M be a line bundle of degree d on X ; denote by $\underline{d} = \text{deg } M = \{d_1, \dots, d_\gamma\}$ its multidegree, where $d_i := \text{deg}_{X_i} M$. For the dualizing sheaf of X we use the special notation $w_i := \text{deg}_{X_i} \omega_X = 2g(X_i) - 2 + k_i$ and

$$\underline{w}_X := \text{deg } \omega_X = \{w_1, \dots, w_\gamma\}.$$

Similarly, for a subcurve Z of X , we set $k_Z := \#Z \cap \overline{X \setminus Z}$, $w_Z := \text{deg}_Z \omega_X = 2g(Z) - 2 + k_Z$ and $d_Z := \text{deg}_Z M$.

Let X be a quasistable curve. As usual, we denote by E_i its exceptional components and set $\tilde{X} = \overline{X \setminus \bigcup_i E_i}$ (thus \tilde{X} is a partial normalization of C).

Definition 5.1.1. Let X be a quasistable curve and $M \in \text{Pic } X$. We shall say that the multidegree \underline{d} of M is *balanced* if $\text{deg}_E M = 1$ on every exceptional component E of X and if, for every subcurve Z of X , the following inequality (known as Basic Inequality) holds:

$$\left| d_Z - d \frac{w_Z}{2g - 2} \right| \leq \frac{k_Z}{2}.$$

We shall say that \underline{d} is *stably balanced* if it is balanced and if for every subcurve Z of X such that

$$d_Z - d \frac{w_Z}{2g - 2} = -\frac{k_Z}{2},$$

we have that Z contains \tilde{X} .

Similarly, we shall say that the line bundle M is *balanced* (or *stably balanced*) if its multidegree is.

⁵ M_g^0 (and likewise \overline{M}_g^0) stands for the locus of curves with a trivial automorphism group.

⁶A *quasistable curve* X is a blow-up of a stable curve C . We call C the *stable model* of X .

It is easy to see that if \underline{d} is balanced and Z is a connected component of \tilde{X} , then $d_Z - d \frac{w_Z}{2g-2} = -\frac{k_Z}{2}$. Hence if \underline{d} is stably balanced, \tilde{X} is connected.

We shall need the following elementary

Lemma 5.1.2. *Let X be a quasistable curve.*

- (i) *Let $M \in \text{Pic } X$ be such that $M^{\otimes r} \simeq \omega_X^{\otimes l}$ for some integers r and l ; then M is stably balanced.*
- (ii) *$M \in \text{Pic } X$ is balanced (stably balanced) if and only if $M \otimes \omega_X^{\otimes t}$ is balanced (stably balanced) for some integer t .*
- (iii) *Assume that the stable model of X is irreducible and let $M \in \text{Pic } X$ be a line bundle having degree 1 on all exceptional components. Then M is stably balanced.*

The goal of the following definitions is to measure the nonseparatedness of the Picard functor.

Definition 5.1.3. Let X be a nodal curve and $T \in \text{Pic } X$; we say that T is a *twister* if there exists a one-parameter smoothing $\mathcal{X} \rightarrow S$ of X such that

$$T \simeq \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\gamma} a_i X_i\right) \otimes \mathcal{O}_X$$

for some Cartier divisor $\sum_{i=1}^{\gamma} a_i X_i$ on \mathcal{X} (where $a_i \in \mathbb{Z}$). The set of all twistors of X is denoted by $\text{Tw}(X)$.

Definition 5.1.4. Consider two balanced line bundles $M \in \text{Pic } X$ and $M' \in \text{Pic } X'$, where X and X' are quasistable curves.

We say that M and M' are *equivalent* if there exists a semistable curve Y dominating⁷ both X and X' and a twister $T \in \text{Tw}(Y)$ such that, denoting by M_Y and M'_Y the pull-backs of M and M' to Y , we have

$$M'_Y \simeq M_Y \otimes T.$$

In particular, for any automorphism σ of X , $\sigma^*(M)$ is equivalent to M .

To say that M and M' are equivalent is to say that their pull-backs to Y are both limits of the same family of line bundles. More precisely, let $\mathcal{Y} \rightarrow S$ be a one-parameter smoothing of Y such that the twister T of Definition 5.1.4 is $T = \mathcal{O}_{\mathcal{Y}}(D) \otimes \mathcal{O}_{\mathcal{Y}}$, with $\text{Supp } D \subset Y$. Let $\mathcal{M} \in \text{Pic } \mathcal{Y}$ be such that $\mathcal{M}|_Y \simeq M_Y$; then $\mathcal{M} \otimes \mathcal{O}_{\mathcal{Y}}(D) \otimes \mathcal{O}_{\mathcal{Y}} \simeq M'_Y$ so that M_Y and M'_Y are both limits of the family corresponding to \mathcal{M}^* (recall that the “ $*$ ” denotes restriction away from the central fiber). Therefore, the line bundles M_Y and M'_Y (and likewise M and M') must be identified in any proper completion of the Picard scheme over \overline{M}_g .

Remark 5.1.5. Let $M, M' \in \text{Pic } X$ be stably balanced. Then a simple numerical checking shows that they are equivalent if and only if there exists an automorphism σ of X such that $M' \simeq \sigma^*(M)$. If the stable model of X has no nontrivial automorphisms, then M and M' are equivalent if and only if $M|_{\tilde{X}} \simeq M'|_{\tilde{X}}$.

Fix a large d ($d \geq 20(g-1)$ would work), set $s = d - g$ and consider the Hilbert scheme $\text{Hilb}_{\mathbb{P}^s}^{d,g}$ of connected curves of degree d and genus g in \mathbb{P}^s ; the group

⁷We say that a semistable curve Y dominates a nodal curve X if it can be obtained from X by a finite sequence of blow-ups.

$SL(s + 1)$ naturally acts on $\text{Hilb}_{\mathbb{P}^s}^{d,g}$. Recall that there exist linearizations for such an action such that the following facts hold.

Theorem 5.1.6 ([5]). *Let $X \subset \mathbb{P}^s$ be a connected curve of genus g .*

- (1) *The Hilbert point of X is GIT-semistable if and only if X is quasistable and $\mathcal{O}_X(1)$ is balanced.*
- (2) *The Hilbert point of X is GIT-stable if and only if X is quasistable and $\mathcal{O}_X(1)$ is stably balanced.*
- (3) *Assume that the Hilbert points of X and $X' \subset \mathbb{P}^s$ are GIT-semistable. Then they are GIT-equivalent if and only if $\mathcal{O}_X(1)$ and $\mathcal{O}_{X'}(1)$ are equivalent.*

Proof. We have assembled together, for convenience, various results from [5]: for (1) see Propositions 3.1 and 6.1 (the “only if” part is due to D. Gieseker [13]); for (2) see Lemma 6.1. The “only if” of part (3) is Lemma 5.2; the “if” is a direct consequence of the existence of the GIT quotient as a projective scheme (see below). \square

Then let $\overline{P}_{d,g}$ be the GIT-quotient of the locus of all GIT-semistable points in $\text{Hilb}_{\mathbb{P}^s}^{d,g}$; it is a normal, integral scheme, flat over \overline{M}_g^0 (Theorem 6.1 in [5]).

The above result permits a modular interpretation of $\overline{P}_{d,g}$ as a modular compactification of the universal Picard variety. Let X be a quasistable curve and let $M \in \text{Pic } X$ be very ample of degree d . By Theorem 5.1.6, if we embed X in \mathbb{P}^s by M , the Hilbert point of X is GIT-semistable if and only if M is balanced. Thus $\overline{P}_{d,g}$ parametrizes equivalence classes of balanced line bundles on quasistable curves.

Associating to $M \in \text{Pic } X$ the stable model C of X yields the natural morphism $\overline{P}_{d,g} \rightarrow \overline{M}_g$; \overline{P}_C^d denotes its fiber over $[C] \in \overline{M}_g$.

Recall (Theorem 6.1 in [5]) that \overline{P}_C^d is a finite union of g -dimensional irreducible components, one for each stably balanced multidegree (of degree d) on C :

$$\overline{P}_C^d = \bigcup_{\underline{d} \text{ stably balanced}} \overline{P}_C^{\underline{d}}.$$

Moreover, if C has no nontrivial automorphisms, every component of \overline{P}_C^d contains a copy of the generalized Jacobian of C as a dense open subscheme:

$$\text{Pic}^{\underline{d}} C \subset \overline{P}_C^{\underline{d}},$$

where $\text{Pic}^{\underline{d}} C$ is the variety of line bundles on C having multidegree \underline{d} . If C is irreducible or of compact type, then $\overline{P}_C^{\underline{d}}$ is irreducible.

Remark 5.1.7. By [5], Lemma 8.1, for any integer t , $\overline{P}_{d,g}$ and $\overline{P}_{d+t(2g-2),g}$ are isomorphic over \overline{M}_g via the morphism $M \mapsto M \otimes \omega_X^{\otimes t}$ for any $M \in \text{Pic } X$. Hence we can assume that $\overline{P}_{d,g}$ exists for any $d \in \mathbb{Z}$, even if it is constructed only for $d \gg 0$. Moreover, M is balanced, or stably balanced, if and only if $M \otimes \omega_X^{\otimes t}$ is (by part (ii) of Lemma 5.1.2).

5.2. A new compactification of higher spin curves. Let C be a stable curve and $L \in \text{Pic } C$ an r -th root of $\omega_C^{\otimes t}$; then, by part (ii) of Lemma 5.1.2, L is stably

balanced, therefore, by part (2) of 5.1.6, it is identified with a point of $\overline{P}_{l(2g-2)/r, g}$. More precisely, consistently with section 4.2, we denote by $S_g^{r, l}$ the open subscheme of $\overline{S}_g^{r, l}$ corresponding to pairs (C, L) as above. Obviously, $S_g^{r, l}$ is the coarse moduli space for the similarly defined functor $\mathcal{S}_g^{r, l}$. We have:

Lemma-Definition 5.2.1. *There is a natural embedding*

$$\chi: S_g^{r, l} \hookrightarrow \overline{P}_{l(2g-2)/r, g}.$$

We define $\widehat{S}_g^{r, l}$ to be the closure, in the projective variety $\overline{P}_{l(2g-2)/r, g}$, of the image of χ , and

$$\widehat{\varphi}: \widehat{S}_g^{r, l} \longrightarrow \overline{M}_g$$

to be the natural morphism.

Proof. The set-theoretic inclusion has already been defined before the statement. To complete the proof, note that there is an obvious inclusion of functors

$$\mathcal{S}_g^{r, l} \hookrightarrow \overline{\mathcal{P}}_{l(2g-2)/r, g},$$

and hence the desired embedding follows from [5], Proposition 8.1(1). □

This defines a new compactification of the moduli space of higher spin curves, which we shall compare with the previously constructed $\overline{S}_g^{r, l}$. As for $\overline{S}_g^{r, l}$, the boundary points of $\widehat{S}_g^{r, l}$ correspond to line bundles on quasistable curves, having degree 1 on exceptional components. Here is a more precise description.

Proposition 5.2.2. *The points of $\widehat{S}_g^{r, l}$ are in bijective correspondence with equivalence classes of balanced line bundles $M \in \text{Pic } X$ such that X is a quasistable curve of genus g and there exists a twister T on X for which the following relation holds:*

$$M^{\otimes r} \simeq \omega_X^{\otimes l} \otimes T.$$

Proof. Let $\lambda \in \widehat{S}_g^{r, l}$. Since λ belongs to the closure of (smooth) r -spin curves, there exist a representative $M \in \text{Pic } X$ of λ , a one-parameter smoothing $f: \mathcal{X} \rightarrow S$ of X , and a line bundle \mathcal{M} on \mathcal{X} , such that $\mathcal{M} \otimes \mathcal{O}_{\mathcal{X}} \simeq M$ and such that

$$(\mathcal{M}^*)^{\otimes r} \simeq (\omega_f^*)^{\otimes l}.$$

Then we have

$$\mathcal{M}^{\otimes r} \simeq \omega_f^{\otimes l} \otimes \mathcal{O}_{\mathcal{X}}(D),$$

where D is a Cartier divisor supported on the central fiber. Finally, the line bundle $T = \mathcal{O}_{\mathcal{X}}(D) \otimes \mathcal{O}_X$ is a twister of X ; therefore, restricting the above relation to X , we are done.

Now we prove the converse: let X be a quasistable curve of genus g and let $T \in \text{Tw}(X)$ be such that $(\underline{\text{deg}} T + l\underline{w}_X)r^{-1}$ is integer and balanced. Observe that $\omega_X^{\otimes l} \otimes T$ has degree divisible by r on each irreducible component of X , so it admits some r -th root $M \in \text{Pic } X$. Let us show that $M \in \widehat{S}_g^{r, l}$.

M is balanced, so it defines a point in $\overline{P}_{d, g}$. Moreover, by the definition of twister, there exist a 1-parameter smoothing $f: \mathcal{X} \rightarrow S$ of X and a Cartier divisor D , supported on the central fiber, such that

$$\mathcal{O}_{\mathcal{X}}(D) \otimes \mathcal{O}_X = T.$$

Then M is an r -th root of $(\omega_f^{\otimes l} \otimes \mathcal{O}_{\mathcal{X}}(D)) \otimes \mathcal{O}_X$, and by Remark 3.0.6 it has a (unique) extension to $\mathcal{M} \in \text{Pic } \mathcal{X}$ such that

$$\mathcal{M}^{\otimes r} \simeq \omega_f^{\otimes l} \otimes \mathcal{O}_{\mathcal{X}}(D).$$

Hence M is a limit of smooth r -spin curves, and, as such, it lies in $\widehat{S}_g^{r,l}$. □

Remark 5.2.3. The proposition highlights the combinatorial nature of $\widehat{S}_g^{r,l}$ by saying that the points of $\widehat{S}_g^{r,l}$ are determined by all twistors T on X such that $r^{-1} \underline{\deg}(T \otimes \omega_X^{\otimes l})$ is an integer and balanced. It is easy to see that this is equivalent to the following properties of $\underline{\deg} T$:

- (i) $\deg_{E_i} T = r$ for every exceptional component E_i of X ;
- (ii) $\deg_{X_j} T \equiv (-lw_j) \pmod{r}$ for every nonexceptional component X_j of X ;
- (iii) $|\deg_Z T| \leq \frac{k_Z r}{2}$ for every subcurve Z of X .

5.3. Fiberwise description. Consider again the embedding χ defined in Lemma-Definition 5.2.1. As $S_g^{r,l}$ is dense in $\overline{S}_g^{r,l}$ (and in $\widehat{S}_g^{r,l}$ by definition), χ will be viewed as a birational map

$$\chi: \overline{S}_g^{r,l} \dashrightarrow \widehat{S}_g^{r,l}.$$

For any stable curve C of genus g , we denote by $\widehat{S}_C^{r,l}$ the fiber of $\widehat{\varphi}: \widehat{S}_g^{r,l} \rightarrow \overline{M}_g$ over $[C]$. Throughout this section we assume that the automorphism group of C is trivial. If C is smooth, then $\widehat{S}_C^{r,l} = \overline{S}_C^{r,l}$ is the reduced, zero-dimensional scheme parametrizing r -th roots of $\omega_C^{\otimes l}$.

We have a first, easy result relating the two above moduli spaces.

Lemma 5.3.1. *Let $(X, L, \alpha) \in \overline{S}_g^{r,l}$ be a limit root and let C be a stable curve.*

- (i) *The map χ is regular at every point (X, L, α) such that L is balanced.*
- (ii) *If every limit root in $\overline{S}_C^{r,l}$ is balanced (for instance, if C is irreducible), then the restriction $\chi|_{\overline{S}_C^{r,l}}: \overline{S}_C^{r,l} \rightarrow \widehat{S}_C^{r,l}$ is surjective. In particular, $\dim \widehat{S}_C^{r,l} = 0$.*

Proof. The argument for (i) is analogous to the one proving Theorem 5.1.6. The requirement that L be balanced naturally defines a subfunctor of $\overline{S}_g^{r,l}$, which has a natural inclusion in the functor $\overline{\mathcal{P}}_{l(2g-2)/r, g}$. Hence the statement follows from Proposition 8.1 of [5].

Now (ii). We just proved that χ is defined on $\overline{S}_C^{r,l}$ and that, for any $\xi \in \overline{S}_C^{r,l}$ with $\xi = (X, L, \alpha)$, the image $\chi(\xi)$ is the class defined by L in $\widehat{S}_C^{r,l}$. Let $\lambda \in \widehat{S}_C^{r,l}$ and choose a representative $M \in \text{Pic } X$ for λ , where X is some blow-up of C ; pick a one-parameter smoothing of X , $f: \mathcal{X} \rightarrow S$, and $\mathcal{M} \in \text{Pic } \mathcal{X}$ such that

$$(\mathcal{M}^*)^{\otimes r} \simeq (\omega_{f^*})^{\otimes l}$$

(“ $*$ ” means away from the special fiber). Then by Proposition 2.1.2, up to replacing S by a finite (ramified) covering, the family $(f^*: \mathcal{X}^* \rightarrow S^*, \mathcal{M}^*)$ determines a unique limit root $\xi \in \overline{S}_C^{r,l}$. By continuity, $\chi(\xi) = \lambda$, hence surjectivity. The fact that (ii) applies to irreducible curves descends from Lemma 5.1.2, (iii). □

Observe that, if (X, L, α) and (X, L, β) are two balanced limit roots of $\omega_C^{\otimes l}$, then they are identified by χ to the same point in $\widehat{S}_g^{r,l}$, regardless of their being isomorphic or not as limits roots (cf. Lemma 2.3.3, part (ii)).

The main difference between $\widehat{S}_g^{r,l}$ and $\overline{S}_g^{r,l}$ is that, if $r \geq 3$, the morphism $\widehat{\varphi}$ is not finite. First we have

Lemma 5.3.2. *Let Z be an irreducible component of $\widehat{S}_C^{r,l}$. Then*

$$\dim Z \leq b_1(\Gamma_C).$$

In particular, if C is of compact type, then $\dim \widehat{S}_C^{r,l} = 0$.

Proof. Let Z be an irreducible component of $\widehat{S}_C^{r,l}$ and let $\lambda \in Z$. By Proposition 5.2.2, there exists a representative $L \in \text{Pic } X$ for λ such that $L^{\otimes r} \simeq \omega_X^{\otimes l} \otimes T$ for a twister $T \in \text{Tw}(X)$. For every given T there are obviously finitely many $L \in \text{Pic } X$ satisfying this relation. We claim that the twistors T vary in algebraic varieties of dimension at most $b_1(\Gamma_C)$.

Let $\nu: X^\nu \rightarrow X$ be the normalization, T a twister, and set $G := \nu^*T \in \text{Pic } X^\nu$. Then every $T' \in \text{Tw}(X)$ such that $\nu^*T' = G$ has the same multidegree as T , and hence all the r -th roots of $\omega_X^{\otimes l} \otimes T'$ lie in $\widehat{S}_C^{r,l}$ (see Remark 5.2.3). Under the map $\nu^*: \text{Pic } X \rightarrow \text{Pic } X^\nu$ the fiber of G is a $(\mathbb{C}^*)^b$, where

$$b = b_1(\Gamma_X) = b_1(\Gamma_C);$$

now, T' moves in such a fiber, and hence in a variety of dimension at most $b_1(\Gamma_C)$. On the other hand, the varying of G does not contribute to the dimension of Z : it is in fact easy to see that the set $\nu^*(\text{Tw } X)$ (where G ranges) is a discrete subset of $\text{Pic } X^\nu$. Therefore we are done.

Finally, if C is of compact type, then $b_1(\Gamma_C) = 0$, so the last statement is a special case of the first. □

Proposition 5.3.3. *Let $C = C_1 \cup C_2$ with C_i smooth and $\#C_1 \cap C_2 = k \geq 2$. Then $\dim \widehat{S}_C^{r,l} = k - 1$, with the following list of exceptions, where $\dim \widehat{S}_C^{r,l} = 0$:*

- (i) $r = 2$;
- (ii) $k = 2, l \underline{w}_C \equiv 0 \pmod{r}$;
- (iii) $k = 3, r = 4, l \underline{w}_C \equiv (2, 2) \pmod{4}$;
- (iv) $k = 4, r = 3, l \underline{w}_C \equiv 0 \pmod{3}$.

Proof. In the four exceptional cases listed in the statement, it is easy to check that all limit roots in $\overline{S}_C^{r,l}$ are balanced, hence by Lemma 5.3.1, part (ii), we have $\dim \widehat{S}_C^{r,l} = 0$.

Denote by n_1, \dots, n_k the nodes of C , and by $\nu: C^\nu \rightarrow C$ the normalization. We divide the proof into three steps.

Step 1. *Fix a twister T on C . Let $T' \in \text{Pic } C$ be such that $\nu^*(T') = \nu^*(T)$; then T' is a twister.*

Let $\mathcal{C} \rightarrow S$ be such that $T = \mathcal{O}_{\mathcal{C}}(\sum_i a_i C_i) \otimes \mathcal{O}_{\mathcal{C}}$, and suppose that \mathcal{C} has an A_{h_j-1} singularity in n_j . Introduce the universal deformation $\underline{\mathcal{C}} \rightarrow \underline{D}$ of C , where \underline{D} is a smooth polydisc. Assume, as usual, that the locus where the nodes n_1, \dots, n_k are preserved is defined by the vanishing of the first k coordinates on \underline{D} . Consider

the family $\mathcal{C}' \rightarrow S$ obtained by pulling back $\underline{\mathcal{C}}$ via the morphism

$$S \longrightarrow \underline{D},$$

$$t \mapsto (c_1 t^{h_1}, \dots, c_k t^{h_k}, 0, \dots, 0),$$

where t is the local coordinate on S and the $c_j \in \mathbb{C}^*$ are arbitrary. Then \mathcal{C}' , just like \mathcal{C} , has an A_{h_j-1} singularity at n_j ; therefore $T' := \mathcal{O}_{\mathcal{C}'}(\sum_i a_i C_i) \otimes \mathcal{O}_C$ is a line bundle and thus a twister. Finally, it is easy to see that T' has gluing datum c_j over n_j . The conclusion is that, for any gluing assignment of ν^*T over the nodes of C , we obtain a twister, as desired.

From now on we exclude cases (i), \dots , (iv) above.

Step 2. *There exists a nonzero twister T_0 on C such that the r -th roots of $\omega_C^{\otimes l} \otimes T_0$ are stably balanced and, in particular, correspond to points of $\widehat{S}_C^{r,l}$.*

Pick k nonzero integers b_1, \dots, b_k having the same sign. As usual, denote by p_1, \dots, p_k (respectively, q_1, \dots, q_k) the inverse images in C_1 (respectively, in C_2) of the nodes. Define $G \in \text{Pic } C^\nu$ by

$$G|_{C_1} = \mathcal{O}_{C_1}(\sum_{i=1}^k b_i p_i), \quad G|_{C_2} = \mathcal{O}_{C_2}(-\sum_{i=1}^k b_i q_i).$$

Then there exists a twister T on C such that $\nu^*T = G$. In fact, set $h := \text{lcm}(|b_1|, \dots, |b_k|)$ and $h_i := h/|b_i|$. Consider a one-parameter smoothing $\mathcal{C} \rightarrow S$ of C having an A_{h_i-1} singularity in n_i for $i = 1, \dots, k$. Set $D := hC_2$ if the b_i are positive, $D := -hC_2$ if the b_i are negative. Then D is Cartier and $T := \mathcal{O}_C(D) \otimes \mathcal{O}_C$ is a twister such that $T|_{C_1} = \mathcal{O}_{C_1}(b_1 p_1 + \dots + b_k p_k)$ and $T|_{C_2} = \mathcal{O}_{C_2}(-b_1 q_1 - \dots - b_k q_k)$, that is, $\nu^*T = G$.

Now set $s = \sum b_i$; obviously $|s| \geq k$. By Proposition 5.2.2, Step 2 will be proved by solving the following numerical problem.

Find an integer s such that

- (1) $|s| \geq k$,
- (2) $lw_1 + s \equiv 0 \pmod{r}$ and $lw_2 - s \equiv 0 \pmod{r}$,
- (3) $|s| < \frac{kr}{2}$,

where (2) and (3) are parts (ii) and (iii) of Remark 5.2.3, and we ask for strict inequality in (3) because we want the degree to be stably balanced.

Write $s = rj - lw_1$ for some integer j . It finally suffices to find an integer j so that (1) and (3) above are satisfied, that is, $k \leq |rj - lw_1| < \frac{kr}{2}$. In other words, we are reduced to showing that the set

$$\left(-\frac{k}{2} + \frac{lw_1}{r}, -\frac{k}{r} + \frac{lw_1}{r} \right] \cup \left[\frac{k}{r} + \frac{lw_1}{r}, \frac{k}{2} + \frac{lw_1}{r} \right)$$

contains an integer j (provided that cases (i), \dots , (iv) are excluded). The proof of this fact is elementary and easy, therefore we omit it.

Step 3. *Steps 1 and 2 imply the proposition.*

Pick a twister T_0 as in Step 2. By Proposition 5.2.2 and Remark 5.2.3, the set of all such T_0 is characterized by some numerical conditions on their multidegree, hence on the multidegree of their pull-back to the normalization of C . Therefore, by Step 1, if there exists one T_0 , then there exists a $b_1(\Gamma_C)$ -dimensional family \mathcal{F} of them: $\mathcal{F} \simeq (\mathbb{C}^*)^{b_1(\Gamma_C)}$ is the entire fiber of ν^* over $\nu^*(T_0)$. This yields that for every $T \in \mathcal{F}$ the r -th roots of $\omega_C^{\otimes l} \otimes T$ lie in $\widehat{S}_C^{r,l}$, and correspond to different points,

because they are stably balanced. We conclude that $\widehat{S}_C^{r,l}$ contains subvarieties of dimension $b_1(\Gamma_C) = k - 1$; by Lemma 5.3.2 we are done. \square

5.4. Comparison results. We begin with the case $r = 2$, which is essentially already known, as we will explain. Under this condition limit roots are always balanced.

Lemma 5.4.1. *Assume that $r = 2$. Let C be a stable curve and let $(X, L, \alpha) \in \overline{S}_g^{2,l}$ be a limit square root of $(C, \omega_C^{\otimes l})$. Then L is balanced, and the orbit of the corresponding Hilbert point is closed in the GIT-semistable locus.*

Proof. Fix a subcurve Z of X , let k_Z and w_Z be as usual, and let k'_Z be the number of exceptional components of X intersecting Z and not contained in Z . Then we have

$$k'_Z \equiv lk_Z \pmod{2},$$

$k'_Z \leq k_Z$ and $2d_Z = lw_Z - k'_Z$. The Basic Inequality (see Definition 5.1.1) for Z simplifies to $k'_Z \leq k_Z$, which is always satisfied, thus L is balanced and the Hilbert point is GIT-semistable (Theorem 5.1.6). Moreover, if equality holds, every node on $Z \cap \overline{X} \setminus \overline{Z}$ is exceptional. Then Lemma 6.1 of [5] applies, giving that the Hilbert point has closed orbit, as stated. \square

In view of Lemma 5.3.1, one obtains that $\chi: \overline{S}_g^{2,l} \dashrightarrow \widehat{S}_g^{2,l}$ is regular for all l ; moreover, χ is bijective because of the last assertion of the previous lemma (cf. [12]).

When $r \geq 3$ the situation is quite different, for, in general, neither the birational map $\chi: \overline{S}_g^{r,l} \dashrightarrow \widehat{S}_g^{r,l}$ nor its inverse are regular, as we shall presently see.

Theorem 5.4.2. *Let $r \geq 3$.*

- (i) χ is regular in codimension 1.
- (ii) *If r does not divide $2l$, then χ is not regular in codimension 2.*
If r divides $2l$, then χ is not regular in codimension 3, unless $r = 4$ and $l \equiv 2 \pmod{4}$.
If $r = 4$ and $l \equiv 2 \pmod{4}$, then χ is not regular in codimension 4.
- (iii) *If $r \geq 5$ and $g \geq 7$, or $r = 3, 4$ and $g \geq 10$, then χ^{-1} is not regular in codimension 1.*

Proof. First of all, recall that $\overline{\varphi}: \overline{S}_g^{r,l} \rightarrow \overline{M}_g$ is finite. By Lemma 5.3.1, χ is regular over the locus of irreducible curves in \overline{M}_g . Let $U \subset \overline{S}_g^{r,l}$ be the inverse image of the locus of curves of compact type in \overline{M}_g . Then, by Proposition 2.4.2 applied to $B = \overline{M}_g$, U is normal, so χ is regular in codimension 1 on U , hence regular in codimension 1 on $\overline{S}_g^{r,l}$. This settles (i).

For (ii) we use Proposition 5.3.3, which describes boundary strata in \overline{M}_g over which $\widehat{\varphi}: \widehat{S}_g^{r,l} \rightarrow \overline{M}_g$ is not finite. Since $\overline{\varphi}$ is finite, χ , being dominant, cannot be regular over such strata.

In case r does not divide $2l$ we consider the locus of curves $C = C_1 \cup C_2$ with C_i smooth, $\#C_1 \cap C_2 = 2$ and C_1 of genus 1 (hence C_2 of genus $g - 2$). Then $lw_C = (2l, 2l(g - 2)) \not\equiv 0 \pmod{r}$, so by Proposition 5.3.3 $\widehat{\varphi}$ is not finite over this locus.

If r divides $2l$ and $(r, l) \neq (4, 2 + 4j)$, again by Proposition 5.3.3, $\widehat{\varphi}$ is not finite over all codimension 3 loci whose general curve is a union of 2 smooth components meeting at 3 points.

Finally, if $r = 4$ and $l \equiv 2 \pmod{4}$, $\widehat{\varphi}$ is not finite over all codimension 4 loci whose general curve is a union of 2 smooth components meeting at 4 points.

(iii) Consider $\xi = (X, L, \alpha) \in \overline{S}_C^{r,l}$ and a one-parameter smoothing $(f: \mathcal{X} \rightarrow S, \mathcal{L})$ of ξ . Then the moduli morphism $S^* \rightarrow \widehat{S}_g^{r,l}$ completes to a morphism $S \rightarrow \widehat{S}_g^{r,l}$. Denote by $\lambda \in \widehat{S}_C^{r,l}$ the image of $s_0 \in S$. Clearly, if χ^{-1} is defined at λ , then $\chi^{-1}(\lambda) = \xi$. We shall exhibit, for a suitable choice of C , r distinct points $\xi_1, \dots, \xi_r \in \overline{S}_C^{r,l}$ and one-parameter deformations f_1, \dots, f_r as above, such that the associated λ_i all lie in the same irreducible component of $\widehat{S}_C^{r,l}$. This implies that χ^{-1} is not defined at λ_i , as the ξ_i are irreducible components of $\overline{S}_C^{r,l}$.

Let $C = C_1 \cup C_2$ with C_i smooth, $C_1 \simeq \mathbb{P}^1$ and $\#C_1 \cap C_2 = k$ (hence C has genus $g \geq k - 1$ and $w_1 = k - 2$).

We treat the case $r \geq 5$, $g \geq 7$ and $l = 1$, the remaining cases being similar. Let $k = 7$, fix 5 nodes n_1, \dots, n_5 of C and consider the weighted subgraph of Γ_C given by these five nodes with weights $u_1 = \dots = u_5 = 1$ and $v_1 = \dots = v_5 = r - 1$. This weighted graph satisfies conditions (C1) and (C2) of section 2.2 with respect to ω_C .

Let X be the blow-up of C at n_1, \dots, n_5 . Fix $L_i \in \text{Pic } C_i$ such that

$$L_1^{\otimes r} \simeq (\omega_X)_{|C_1} \otimes \mathcal{O}_{C_1}(-\sum_{j=1}^5 p_j) \quad \text{and} \quad L_2^{\otimes r} \simeq (\omega_X)_{|C_2} \otimes \mathcal{O}_{C_2}(-\sum_{j=1}^5 (r-1)q_j).$$

The choices for the gluings of L_1 and L_2 over n_6 and n_7 yield r distinct limit roots $\xi_i = (X, L_i)$ of (C, ω_C) , $i = \{1, \dots, r\}$. Now consider a general one-parameter smoothing $(f_i: \mathcal{X} \rightarrow S, \mathcal{L}_i)$ of (X, L_i) . The surface \mathcal{X} is smooth at $q_1, \dots, q_5, n_6, n_7$ and has A_{r-2} singularities at p_1, \dots, p_5 ; therefore C_2 is a Cartier divisor. The line bundle $\mathcal{M}_i := \mathcal{L}_i \otimes \mathcal{O}_{\mathcal{X}}(-C_2)$ has degree zero on all exceptional components of X , hence it descends to $\mathcal{M}'_i \in \text{Pic } \mathcal{C}$, where $\mathcal{C} \rightarrow S$ is the stable model of $\mathcal{X} \rightarrow S$. Now set $M_i := (\mathcal{M}'_i)_{|C}$. Then M_i is stably balanced and determines a point $\lambda_i \in \widehat{S}_C^{r,1}$. Observe that $M_i^{\otimes r} \simeq \omega_C \otimes T_i$, where T_i is a twister on C and

$$(T_i)_{|C_1} \simeq \mathcal{O}_{C_1}(-\sum_{j=1}^5 p_j), \quad (T_i)_{|C_2} \simeq \mathcal{O}_{C_2}(\sum_{j=1}^5 q_j).$$

Then, as in the proof of Proposition 5.3.3, we see that there is a 6-dimensional irreducible component Z of $\widehat{S}_C^{r,1}$ containing $\lambda_1, \dots, \lambda_r$. Moreover, as we vary the families f_i , the points λ_i cover an open subset of Z . This shows that Z is entirely contained in the subscheme where χ^{-1} is not defined. Since the curve C itself varies in a codimension 7 stratum of \overline{M}_g , we obtain that χ^{-1} is not regular over a codimension 1 subscheme of $\widehat{S}_g^{r,1}$. \square

Our comparison results extend easily to torsion-free sheaves compactifications. More precisely, recall that Pandharipande constructs in [20] a compactification $\overline{U}_g(d, 1)$ of $P_{d,g}$ over \overline{M}_g , by means of rank 1 torsion-free sheaves, and shows that this compactification is isomorphic to $\overline{P}_{d,g}$ under the correspondence given by Proposition 4.2.2. Let $R_g^{1/r}(\omega^{\otimes l})$ be the coarse moduli space of Jarvis's stack

$\text{ROOT}_g^{1/r}(\omega^{\otimes l})$ of r -th root sheaves of $\omega^{\otimes l}$ over \overline{M}_g (see section 4.2). Set $d = l(2g - 2)/r$. Then we have a commutative diagram:

$$\begin{CD} \overline{S}_g^{r,l} @>\sim>> R_g^{1/r}(\omega^{\otimes l}) \\ @V\chi VV @VV\chi' V \\ \overline{P}_{d,g} @>\sim>> \overline{U}_g(d,1) \end{CD}$$

Therefore all results about χ hold equally for χ' . In particular, we have:

Corollary 5.4.3. *The coarse moduli spaces of the stacks $\text{ROOT}_g^{1/r}(\omega^{\otimes l})$ and $\overline{\mathcal{S}}_g^{1/r}(\omega^{\otimes l})$ (in the notation of [15]) do not embed in $\overline{U}_g(d,1)$.*

Proof. For $R_g^{1/r}(\omega^{\otimes l})$, the statement is an immediate consequence of Theorem 4.2.3 and Theorem 5.4.2 (ii). Let $T_g^{1/r}(\omega^{\otimes l})$ be the coarse moduli space of the stack $\overline{\mathcal{S}}_g^{1/r}(\omega^{\otimes l})$. Then $T_g^{1/r}(\omega^{\otimes l})$ is the normalization of $R_g^{1/r}(\omega^{\otimes l})$, so it is finite over \overline{M}_g . The statement follows. \square

The literature on compactified Picard varieties is quite rich and varied, so that it would be worthwhile to study the same issue with respect to other compactifications, such as those of [24], [10], [17], for example. This naturally leads to the problem of comparing among each other various constructions of the compactified jacobian; the general situation is not as clear-cut as it is for the two compactifications ([5] and [20]) used in our paper. Recent results in [2] show that, under suitable hypotheses, the spaces of [5] and [24] coincide; other correlation results are, at least to us, not known.

We conclude with a detailed example.

Example 5.4.4. Let C be as in Example 4.1.5. Assume also, for simplicity, that $C_i \not\cong \mathbb{P}^1$ for $i = 1, 2$. Under this assumption, two limit third roots are isomorphic if and only if the underlying line bundles are isomorphic off the exceptional components. The quasistable curves having stable model C are X_i , obtained blowing up the node n_i for $i = 1, 2, 3$, and $X_{j,k}$ and $X_{1,2,3}$, described in Example 4.1.5. The balanced multidegrees are:

- $\underline{d} = (d_{C_1}, d_{C_2}) = (1, -1), (0, 0)$ and $(-1, 1)$ on C ;
- $\underline{d} = (d_{C_1}, d_{C_2}, d_{E_i}) = (0, -1, 1)$ and $(-1, 0, 1)$ on X_i ;
- $\underline{d} = (d_{C_1}, d_{C_2}, d_{E_j}, d_{E_k}) = (-1, -1, 1, 1)$ on $X_{j,k}$.

All these degrees are stably balanced; no multidegree on $X_{1,2,3}$ is balanced.

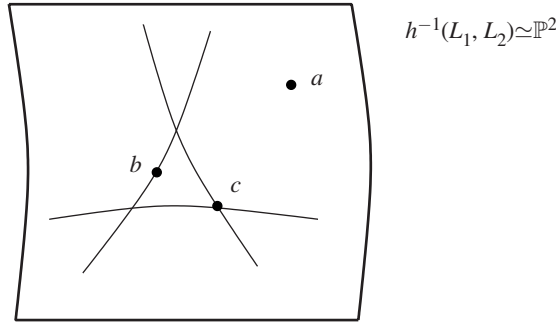
Let \overline{P}_C^0 be the compactified jacobian of C . Recall from [5], §7.3, that \overline{P}_C^0 is the union of strata $V_I^{\underline{d}}$ parametrized by the balanced multidegrees listed above. Each stratum $V_I^{\underline{d}}$ is isomorphic to the variety of line bundles of multidegree \underline{d} on the quasistable curve obtained blowing-up the nodes $\{n_i\}_{i \in I}$ of C (we denote by $V_{\emptyset}^{\underline{d}}$ the strata $V_{\emptyset}^{\underline{d}}$). The three open strata $V^{(1,-1)}$, $V^{(0,0)}$ and $V^{(-1,1)}$ are $(\mathbb{C}^*)^2$ -bundles over $\text{Pic}^1 C_1 \times \text{Pic}^{-1} C_2$, $\text{Pic}^0 C_1 \times \text{Pic}^0 C_2$ and $\text{Pic}^{-1} C_1 \times \text{Pic}^1 C_2$ respectively; their closures are the irreducible components of \overline{P}_C^0 . In $\overline{V}^{(1,-1)}$ and $\overline{V}^{(-1,1)}$, the fibers are compactified by \mathbb{P}^2 ; in $\overline{V}^{(0,0)}$ they are compactified by \mathbb{P}^2 blown-up in three points.

These three irreducible components intersect along the codimension 1 strata; see [5], page 651, for more details.

Recall that $g^\nu = g - 2$; we are going to show that $\widehat{S}_C^{3,0}$ consists of $15 \cdot 3^{2g^\nu}$ points and $2 \cdot 3^{2g^\nu}$ disjoint copies of \mathbb{P}^2 .

Among the points, $9 \cdot 3^{2g^\nu}$ lie in $V^{(0,0)}$, and are the $L \in \text{Pic } C$ with $L^3 \simeq \mathcal{O}_C$; the remaining $6 \cdot 3^{2g^\nu}$ are the $L \in \text{Pic } X_{j,k}$ with $L_{|\widetilde{X}_{j,k}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{j,k}}(-2p_j - q_j - p_k - 2q_k)$ or $L_{|\widetilde{X}_{j,k}}^{\otimes 3} \simeq \mathcal{O}_{\widetilde{X}_{j,k}}(-p_j - 2q_j - 2p_k - q_k)$; they lie in $V_{j,k}^{(-1,-1,1,1)}$. All these points are limit roots and lie also in $\overline{S}_C^{3,0}$; note that the remaining points in $\overline{S}_C^{3,0}$ have unbalanced multidegree.

Each \mathbb{P}^2 in $\widehat{S}_C^{3,0}$ corresponds to a choice of $L_1 \in \text{Pic } C_1$ and $L_2 \in \text{Pic } C_2$ such that either $L_1^{\otimes 3} \simeq \mathcal{O}_{C_1}(p_1 + p_2 + p_3)$ and $L_2^{\otimes 3} \simeq \mathcal{O}_{C_2}(-q_1 - q_2 - q_3)$, or conversely. We describe the first case, the other being symmetric. So, fix such L_1 and L_2 . Note that $(L_1, L_2) \in \text{Pic}^1 C_1 \times \text{Pic}^{-1} C_2$. Then the corresponding \mathbb{P}^2 is the fiber over (L_1, L_2) of the \mathbb{P}^2 -bundle $h: \overline{V}^{(1,-1)} \rightarrow \text{Pic}^1 C_1 \times \text{Pic}^{-1} C_2$. This \mathbb{P}^2 contains three types of points:



The points in $V^{(1,-1)}$ (α in the figure) correspond to the $L \in \text{Pic } C$ with $L_{|C_1} \simeq L_1$ and $L_{|C_2} \simeq L_2$; the open stratum $h^{-1}(L_1, L_2) \cap V^{(1,-1)} = (\mathbb{C}^*)^2 \subset \mathbb{P}^2$ parametrizes the gluing data for L_1 and L_2 over C . These points are obtained as the central fiber of a 1-parameter family $(\mathcal{C} \rightarrow S, \mathcal{L})$, where \mathcal{C} is smooth and $\mathcal{L} \in \text{Pic } \mathcal{C}$ is such that $\mathcal{L}^{\otimes 3} \simeq \mathcal{O}_{\mathcal{C}}(-C_1)$. The line bundle $\mathcal{O}_{\mathcal{C}}(-C_1)|_{\mathcal{C}}$ depends on the family.

The points in $V_1^{(0,-1,1)} \cup V_2^{(0,-1,1)} \cup V_3^{(0,-1,1)}$ (β in the figure) are the $L \in \text{Pic } X_i$ with $L_{|C_1} \simeq L_1 \otimes \mathcal{O}_{C_1}(-p_i)$ and $L_{|C_2} \simeq L_2$. For each i , the open stratum $h^{-1}(L_1, L_2) \cap V_i^{(0,-1,1)} \simeq \mathbb{C}^* \subset \mathbb{P}^2$ parametrizes the gluing data for $L_1 \otimes \mathcal{O}_{C_1}(-p_i)$ and L_2 over \widetilde{X}_i . These points are obtained as the central fiber of a 1-parameter family $(\mathcal{X}_i \rightarrow S, \mathcal{L})$, where \mathcal{X}_i has an A_2 singularity in q_i and $\mathcal{L} \in \text{Pic } \mathcal{X}_i$ is such that $\mathcal{L}^{\otimes 3} \simeq \mathcal{O}_{\mathcal{X}_i}(-C_1 - 3E_i)$.

Finally, the three points in $V_{1,2}^{(-1,-1,1,1)}$, $V_{1,3}^{(-1,-1,1,1)}$ and $V_{2,3}^{(-1,-1,1,1)}$ (γ in the figure) are the $L \in \text{Pic } X_{j,k}$ with $L_{|C_1} \simeq L_1 \otimes \mathcal{O}_{C_1}(-p_j - p_k)$ and $L_{|C_2} \simeq L_2$. They are obtained as the central fiber of a 1-parameter family $(\mathcal{X}_{j,k} \rightarrow S, \mathcal{L})$,

where $\mathcal{X}_{j,k}$ has A_2 singularities in q_j and q_k , and $\mathcal{L} \in \text{Pic } \mathcal{X}_{j,k}$ is such that $\mathcal{L}^{\otimes 3} \simeq \mathcal{O}_{\mathcal{X}_{j,k}}(-C_1 - 3E_j - 3E_k)$.

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