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H^∞-CALCULUS FOR THE SUM OF NON-COMMUTING OPERATORS

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Abstract. A recent result of Kalton and Weis is extended to the case of non-commuting operators, employing the commutator condition of Labbas and Terreni, or of Da Prato and Grisvard. Under appropriate assumptions it is shown that the sum of two non-commuting operators admits an $\mathcal{H}^\infty$-calculus. The main results are then applied to a parabolic problem on a wedge domain.

1. Introduction

In recent years the method of operator sums has become an important tool for proving optimal regularity results for partial differential and integro-differential equations, as well as for abstract evolutionary problems; see for instance [8, 9, 10, 19, 20, 21]. This method was introduced in the fundamental paper of da Prato and Grisvard [5] and has been developed further in the case of two commuting operators, A and B, by Dore and Venni [7], Prüss and Sohr [23], and more recently by Kalton and Weis [13]. Since in these results the sum $A+B$ with natural domain $D(A+B) = D(A) \cap D(B)$ has similar properties as A and B, one obtains the important feature that the method can be iterated, and hence, complicated operators can be built up from simpler ones.

If the operators are non-commuting, matters are, naturally, much more involved. However, it is known that the Da Prato-Grisvard theorem remains valid if A and B satisfy certain commutator estimates. Such conditions were already introduced by Da Prato and Grisvard [5], and later on, Labbas and Terreni [15] proposed another, more flexible one. In Monniaux and Prüss [16], the Dore-Venni theorem was extended to the non-commuting case, employing the Labbas-Terreni condition.

An extension of the Kalton-Weis theorem to the non-commutative case for the Labbas-Terreni condition was obtained by Strkalj [26], provided the underlying Banach space is B-convex. However, no such results are known for the Da Prato-Grisvard condition, and it is also not known whether the result of Monniaux and Prüss or Strkalj can be iterated. It is the purpose of this paper to present a non-commutative version of the Kalton-Weis theorem, employing the commutator condition of Labbas and Terreni, as well as that of Da Prato and Grisvard, without any assumption on the Banach space. Under stronger hypotheses we show that the sum $A + B$ admits an $\mathcal{H}^\infty$-calculus, so that the sum method can also be iterated in the non-commuting case.

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The plan for this paper is as follows. In section 2 we introduce the necessary notation and the concepts and results relevant for this paper. Our main theorem is formulated in section 3 and proved in section 4. We conclude the paper with some applications to partial differential operators on domains with wedges or corners. We are obtaining a new, purely operator-theoretic proof of a recent result due to Nazarov [17] and Solonnikov [25]. The main theorem of this paper will be instrumental for the study of the Navier-Stokes equations in a wedge domain, as well as for some free boundary problems with moving contact lines and prescribed contact angles; see [22] for some results in this direction.

2. Summary of results for the commuting case

In the following, $X = (X, |\cdot|)$ always denotes a Banach space with norm $|\cdot|$, and $\mathcal{B}(X)$ stands for the space of all bounded linear operators on X, where we will again use the notation $|\cdot|$ for the norm in $\mathcal{B}(X)$. If A is a linear operator on X, then $D(A)$, $R(A)$, $N(A)$ denote the domain, the range, and the kernel of A, whereas $\rho(A)$ and $\sigma(A)$ stand for the resolvent set and the spectrum of A, respectively. An operator A is called *sectorial* if

- $D(A)$ and $R(A)$ are dense in X,
- $(-\infty, 0) \subset \rho(A)$ and $|t(t+A)^{-1}| \leq M$ for $t > 0$.

The class of all sectorial operators is denoted by $\mathcal{S}(X)$. If A is sectorial, then it is closed, and it follows from the ergodic theorem that $N(A) = 0$. Moreover, by a Neumann series argument one obtains that $\rho(-A)$ contains a sector

$$\Sigma_\phi := \{z \in \mathbb{C} : \ z \neq 0, \ |\arg(z)| < \phi\}.$$

Consequently, it is meaningful to define the *spectral angle* ϕ_A of A by means of

$$\phi_A := \inf\{\phi > 0 : \rho(-A) \supset \Sigma_{\pi-\phi}, \ M_{\pi-\phi} < \infty\},$$

where $M_\phi := \sup\{|\lambda(\lambda + A)^{-1}| : \ \lambda \in \Sigma_\phi\}$. Obviously we have

$$\pi > \phi_A \geq \arg(\sigma(A)) := \sup\{|\arg(\lambda)| : \ \lambda \neq 0, \ \lambda \in \sigma(A)\}.$$

Given two linear operators A and B we define

$$(A + B)x := Ax + Bx, \quad x \in D(A + B) := D(A) \cap D(B).$$

A and B are said to *commute* if there are numbers $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ such that

$$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}.$$

In this case, the commutativity relation holds for all $\lambda \in \rho(A)$ and $\mu \in \rho(B)$.

In their seminal paper [5], Da Prato and Grisvard proved the following result: suppose $A, B \in \mathcal{S}(X)$ commute and the *parabolicity condition* $\phi_A + \phi_B < \pi$ holds true. Then $A + B$ is closable and its closure $L := \overline{A + B}$ is again sectorial with spectral angle $\phi_L \leq \max\{\phi_A, \phi_B\}$.

The natural question in this context then is whether or not $A + B$ is already closed, i.e. if *maximal regularity* holds. Da Prato and Grisvard were able to answer this question in the affirmative for some special cases when X is a Hilbert space, and in real interpolation spaces associated with A and B. In general, however, maximal regularity does not hold, not even in Hilbert spaces, as was pointed out by Baillon and Clément [1].

An important step forward was made by Dore and Venni [7]. To describe their result, recall that a Banach space X is said to belong to the class $\mathcal{HT}$ if the *Hilbert transform*, defined by

$$(Hf)(t) := \lim_{\varepsilon\to 0}\int_{|s|\geq\varepsilon} f(t-s)\frac{ds}{\pi s}, \quad t\in\mathbb{R},\ f\in C_0^\infty(\mathbb{R};X),$$

extends to a bounded linear operator on $L_2(\mathbb{R};X)$. If A is sectorial, then the complex powers A^z of A are well defined, and they give rise to closed, densely defined operators on X, which satisfy the group property $A^uA^v = A^{u+v}$ in an appropriate sense; see for instance [6].

A is said to admit *bounded imaginary powers* if the set $\{A^{is} : |s|\leq 1\}\subset\mathcal{B}(X)$ is uniformly bounded. The class of such operators is denoted by $\mathcal{BIP}(X)$. If A admits bounded imaginary powers, then it is not difficult to show that $\{A^{is}\}_{s\in\mathbb{R}}$ forms a C_0-group of bounded linear operators. The type θ_A of this group is called the *power angle* of A, i.e. we have

$$\theta_A := \overline{\lim}_{|s|\to\infty}|s|^{-1}\log|A^{is}|.$$

The Dore-Venni theorem in the extended version given by Prüss and Sohr [23] states that $A+B$ is closed, provided $X\in\mathcal{HT}$, $A,B\in\mathcal{BIP}(X)$, A,B commute, and the *strong parabolicity condition* $\theta_A+\theta_B<\pi$ is satisfied. Moreover, in that paper it is proved that $A+B$ is not only sectorial, but admits bounded imaginary powers as well, with power angle $\theta_{A+B}\leq\max\{\theta_A,\theta_B\}$. This shows that the Dore-Venni theorem can be iterated.

To state the third, more recent result in this line, the Kalton-Weis theorem [13], we have to introduce some further notation. If A is sectorial, the functional calculus of Dunford given by

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda-A)^{-1}d\lambda$$

is a well-defined algebra homomorphism $\Phi_A : \mathcal{H}_0(\Sigma_\phi)\to\mathcal{B}(X)$, where $\mathcal{H}_0(\Sigma_\phi)$ denotes the set of all functions $f:\Sigma_\phi\to\mathbb{C}$ that are holomorphic and that satisfy the condition

$$\sup_{\lambda\in\Sigma_\phi}(|\lambda^{-\varepsilon}f(\lambda)|+|\lambda^\varepsilon f(\lambda)|)<\infty \text{ for some } \varepsilon>0 \text{ and some } \phi>\phi_A.$$

Here Γ denotes a contour $\Gamma = e^{i\theta}(\infty,0]\cup e^{-i\theta}[0,\infty)$ with $\theta\in(\phi_A,\phi)$. A is said to admit an $\mathcal{H}^\infty$-*calculus* if there are numbers $\phi>\phi_A$ and $M>0$ such that the estimate

$$|f(A)|\leq M|f|_{\mathcal{H}^\infty(\Sigma_\phi)}, \quad f\in\mathcal{H}_0(\Sigma_\phi), \tag{2.1}$$

is valid. In this case, the Dunford calculus extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$; see for instance [6] for more details. We denote the class of sectorial operators which admit an $\mathcal{H}^\infty$-calculus by $\mathcal{H}^\infty(X)$. The infimum ϕ_A^∞ of all angles ϕ such that (2.1) holds for some constant $C>0$ is called the $\mathcal{H}^\infty$-*angle* of A. Since the functions $f_s(z)=z^{is}$ belong to $\mathcal{H}^\infty(\Sigma_\phi)$ for any $s\in\mathbb{R}$ and $\phi\in(0,\pi)$, we have the inclusions

$$\mathcal{H}^\infty(X)\subset\mathcal{BIP}(X)\subset\mathcal{S}(X).$$

Moreover, we have the following relation between the angles introduced so far:

$$\phi_A^\infty\geq\theta_A\geq\phi_A.$$

The first relation is obvious by the choice $f(z) = z^{is}$, and the second one has been proved by Prüss and Sohr [23].

Let $\mathcal{T} \subset \mathcal{B}(X)$ be an arbitrary set of bounded linear operators on X. Then $\mathcal{T}$ is called *$\mathcal{R}$-bounded* if there is a constant $M > 0$ such that the inequality

$$\mathbb{E}(|\sum_{i=1}^{N} \varepsilon_i T_i x_i|) \leq M\mathbb{E}(|\sum_{i=1}^{N} \varepsilon_i x_i|) \tag{2.2}$$

is valid for every $N \in \mathbb{N}$, $T_i \in \mathcal{T}$, $x_i \in X$, and all independent symmetric $\{\pm 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{A}, P)$ with expectation $\mathbb{E}$. The smallest constant M in (2.2) is called the *$\mathcal{R}$-bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. A sectorial operator A is called *$\mathcal{R}$-sectorial* if the set

$$\{\lambda(\lambda + A)^{-1} : \ \lambda \in \Sigma_{\pi-\phi}\} \text{ is } \mathcal{R}\text{-bounded for some } \phi \in (0, \pi).$$

The infimum ϕ_A^R of such angles ϕ is called the *$\mathcal{R}$-angle* of A. We denote the class of $\mathcal{R}$-sectorial operators by $\mathcal{RS}(X)$. The relation $\phi_A^R \geq \phi_A$ is clear. If X is a space of class $\mathcal{HT}$ and $A \in \mathcal{BIP}(X)$, then it has been shown by Clément and Prüss [3] that $A \in \mathcal{RS}(X)$ with $\phi_A^R \leq \theta_A$.

Finally, an operator $A \in \mathcal{H}^\infty(X)$ is said to admit an *$\mathcal{R}$-bounded* $\mathcal{H}^\infty$-calculus if the set

$$\{f(A) : \ f \in \mathcal{H}^\infty(\Sigma_\phi), |f|_{\mathcal{H}^\infty(\Sigma_\phi)} \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi \in (0, \pi)$. Again, the infimum $\phi_A^{R\infty}$ of such ϕ is called the $\mathcal{RH}^\infty$-angle of A, and the class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

The Kalton-Weis theorem [13] implies the following: suppose that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$, A, B commute, and $\phi_A^\infty + \phi_B^R < \pi$. Then $A+B$ is closed. It further implies that $A + B$ admits an $\mathcal{H}^\infty$-calculus as well, provided we have, in addition, $B \in \mathcal{RH}^\infty(X)$ and $\phi_A^\infty + \phi_B^{R\infty} < \pi$. Consequently, the Kalton-Weis theorem may be iterated as well. Note that in contrast to the Dore-Venni theorem, no condition on the geometry of the underlying Banach space X is needed.

We refer to the monograph of Denk, Hieber, and Prüss [6] as well as to [3, 4, 11, 14, 20, 27] for further information and background material.

Remark 2.1. If X enjoys the so-called *property* α (see [2]), then every operator $A \in \mathcal{H}^\infty(X)$ already has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, that is,

$$\mathcal{H}^\infty(X) = \mathcal{RH}^\infty(X) \quad \text{and} \quad \phi_A^{R\infty} = \phi_A^\infty;$$

see Kalton and Weis [13]. In particular, the L_p-spaces with $1 < p < \infty$ have property α; see [2].

3. The non-commuting case. Main result

In this section we formulate our main result for non-commuting operators. We first recall the commutator condition introduced by Da Prato and Grisvard [5]. Suppose that A and B are sectorial operators, defined on a Banach space X, and suppose that

$$\begin{cases} 0 \in \rho(A). \text{ There are constants } c > 0,\ \alpha, \beta > 0,\ \beta < 1,\ \alpha + \beta > 1, \\ \psi_A > \phi_A,\ \psi_B > \phi_B,\ \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \\ (\lambda + A)^{-1} D(B) \subset D(B) \text{ and} \\ |[B(\lambda + A)^{-1} - (\lambda + A)^{-1} B](\mu + B)^{-1}| \leq c/(1 + |\lambda|)^\alpha |\mu|^\beta. \end{cases} \tag{3.1}$$

Then it was shown in [5] that the closure $L = \overline{A+B}$ is invertible, sectorial and $\phi_L \le \max\{\psi_A, \psi_B\}$ holds, provided the constant c in (3.1) is sufficiently small.

A different, more flexible condition was later introduced by Labbas and Terreni [15]. It reads as follows:

$$(3.2)\quad \begin{cases} 0 \in \rho(A). \text{ There are constants } c > 0,\ \ 0 \le \alpha < \beta < 1, \\ \psi_A > \phi_A,\ \psi_B > \phi_B,\ \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \\ |A(\lambda+A)^{-1}[A^{-1}(\mu+B)^{-1} - (\mu+B)^{-1}A^{-1}]| \le c/(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{cases}$$

In Monniaux and Prüss [16], the Labbas-Terreni condition was employed to extend the Dore-Venni theorem to the non-commuting case. In particular, in that paper it was proved that $A+B$ with natural domain is closed and sectorial with spectral angle $\phi_{A+B} \le \max\{\psi_A, \psi_B\}$ provided $X \in \mathcal{HT}$, $A, B \in \mathcal{BIP}(X)$, and (3.2) holds with a sufficiently small constant $c > 0$. The Kalton-Weis theorem has been extended to the non-commuting case by Strkalj [26], provided the Labbas-Terreni conditions hold with sufficiently small $c > 0$ and X is B-convex.

We are now in a position to state our main results.

Theorem 3.1. *Suppose $A \in \mathcal{H}^\infty(X)$, $B \in \mathcal{RS}(X)$ and suppose that* (3.1) *or* (3.2) *holds for some angles $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^R$ such that $\psi_A + \psi_B < \pi$. Then there is a constant $c_0 > 0$ such that $A+B$ is invertible and sectorial with*

$$\phi_{A+B} \le \max\{\psi_A, \psi_B\}$$

whenever $c < c_0$. Moreover, if in addition $B \in \mathcal{RH}^\infty(X)$ and $\psi_B > \phi_B^{R\infty}$, then $A+B \in \mathcal{H}^\infty(X)$ and $\phi_{A+B}^\infty \le \max\{\psi_A, \psi_B\}$.

As for the smallness of the constant c in the commutator condition (3.2), we remark that (3.2) and also (3.1) are invariant under shifts $\nu + A$, resp. $\nu + B$. Thus by enlarging α and decreasing β slightly in (3.1)–(3.2), we obtain smallness of c at the expense of a shift. This remark leads to the following corollary of Theorem 3.1.

Corollary 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Then there is $\nu \ge 0$ such that $\nu + A + B$ is sectorial with spectral angle not larger than* $\max\{\psi_A, \psi_B\}$. *If $B \in \mathcal{RH}^\infty(X)$ and $\psi_B > \phi_B^{R\infty}$, we have $\nu + A + B \in \mathcal{H}^\infty(X)$ as well and $\phi_{\nu+A+B}^\infty \le \max\{\psi_A, \psi_B\}$.*

4. The proof of the main result

(i) The proof is based on the Da Prato-Grisvard formulae

$$(4.1)\qquad S_\lambda = \frac{1}{2\pi i}\int_{\Gamma_r^\theta} (z+\lambda+A)^{-1}(z-B)^{-1}dz$$

and

$$(4.2)\qquad T_\lambda = \frac{1}{2\pi i}\int_{\Gamma_r^\theta} (z-B)^{-1}(z+\lambda+A)^{-1}dz,$$

where $|\arg(\lambda)| \le \pi - \psi$ for some $\psi > \max\{\psi_A, \psi_B\}$. Here Γ_r^θ means the contour

$$\Gamma_r^\theta = (\infty, r)e^{i\theta} \cup re^{i[\theta, \pi-\theta]} \cup [r, \infty)e^{-i\theta},$$

with $\psi_B < \theta < \min\{\psi, \pi - \psi_A\}$ and $0 < r \le \max\{\varepsilon_0, |\lambda|\sin(\psi - \psi_A)\}$, where ε_0 is sufficiently small. Here we recall that A is invertible by assumption. The integrals

defining S_λ and T_λ are absolutely convergent and, by the resolvent estimates of A and B, we obtain the estimate

$$|S_\lambda|,\ |T_\lambda| \le \frac{C}{1+|\lambda|}\,, \qquad \lambda \in \bar{\Sigma}_{\pi-\psi}, \tag{4.3}$$

with a constant $C > 0$ that is independent of λ. By Cauchy's theorem it is easy to deduce the identities

$$(\lambda + A)S_\lambda x + S_\lambda Bx = x, \quad x \in D(B), \quad \lambda \in \bar{\Sigma}_{\pi-\psi}, \tag{4.4}$$

and

$$T_\lambda(\lambda + A)x + BT_\lambda x = x, \quad x \in D(A), \quad \lambda \in \bar{\Sigma}_{\pi-\psi}. \tag{4.5}$$

Therefore, AS_λ and $S_\lambda B$ are bounded or unbounded simultaneously, as are $T_\lambda A$ and BT_λ. On the other hand we have the identities

$$\begin{aligned} S_\lambda Bx - BT_\lambda x &= \frac{1}{2\pi i}\int_{\Gamma_0^\theta} z[(z+\lambda+A)^{-1},(z-B)^{-1}]x\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_0^\theta} z(z-B)^{-1}[(z+\lambda+A)^{-1},B](z-B)^{-1}x\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_0^\theta} zA(z+\lambda+A)^{-1}[A^{-1},(z-B)^{-1}]A(z+\lambda+A)^{-1}x\,dz \end{aligned}$$

for $x \in D(B)$, where as usual $[S,T] = ST - TS$ denotes the commutator of the bounded linear operators S and T on X. Conditions (3.1) or (3.2) show that

$$|S_\lambda Bx - BT_\lambda x| \le \frac{M}{(1+|\lambda|)^\eta}\,|x|, \quad \lambda \in \bar{\Sigma}_{\pi-\psi},$$

where $\eta = \alpha+\beta-1$ in the case of (3.1), and $\eta = \beta-\alpha$ in the case of (3.2). Therefore, $S_\lambda B - BT_\lambda$ is in $\mathcal{B}(X)$, and $S_\lambda B - BT_\lambda$ is uniformly bounded in $\lambda \in \bar{\Sigma}_{\pi-\psi}$. Thus the operators AS_λ, $S_\lambda B$, BT_λ, $T_\lambda A$ are bounded or unbounded simultaneously. In the first case, all the operators are bounded uniformly in λ.

(ii) We will now assume that AS_λ (or equivalently, $S_\lambda B$, BT_λ, $T_\lambda A$) is bounded in $\mathcal{B}(X)$, uniformly in $\lambda \in \bar{\Sigma}_{\pi-\psi}$. This assumption will be justified in (vi).

Then in the case of condition (3.2) we may proceed as in Monniaux and Prüss [16] to obtain the inverse of $\lambda + A + B$. We do not repeat the details here, but observe that we then have

$$(\lambda + A + B)^{-1} = S_\lambda - (\lambda + A)^{-1}Q_\lambda(I + Q_\lambda)^{-1}(\lambda + A)S_\lambda = S_\lambda + R_\lambda. \tag{4.6}$$

Here due to (3.2)

$$Q_\lambda = \frac{1}{2\pi i}\int_{\Gamma_0^\theta} zA(z+\lambda+A)^{-1}[A^{-1},(z-B)^{-1}]A(\lambda+A)^{-1}dz$$

is defined by an absolutely convergent integral, and we have the estimate

$$|Q_\lambda| \le c\frac{C}{(1+|\lambda|)^{\beta-\alpha}}\,, \qquad \lambda \in \bar{\Sigma}_{\pi-\psi},$$

where C is a constant that does not depend on λ. This shows that $I + Q_\lambda$ is invertible with say $|(I + Q_\lambda)^{-1}| \le 2$, provided the constant $c > 0$ from (3.2) is

sufficiently small. Therefore in this case the remainder term in (4.6) satisfies

$$|R_\lambda| \le \frac{C}{(1+|\lambda|)^{1+\varepsilon}}\,, \qquad \lambda \in \bar{\Sigma}_{\pi-\psi}, \tag{4.7}$$

where $\varepsilon = \beta - \alpha > 0$.

Let us now consider the Da Prato-Grisvard condition (3.1). In this case we can write

$$(\lambda + A + B)^{-1} = S_\lambda - S_\lambda Q_\lambda (I + Q_\lambda)^{-1} = S_\lambda + R_\lambda \tag{4.8}$$

with

$$Q_\lambda = [B, S_\lambda] = \frac{1}{2\pi i}\int_{\Gamma_0^\theta} [B, (\lambda + z + A)^{-1}](z - B)^{-1} dz,$$

and we arrive again at estimate (4.7), with $\varepsilon = \alpha + \beta - 1 > 0$ this time, thanks to the commutator estimate (3.1). Strictly speaking, $S_\lambda(I + Q_\lambda)^{-1}$ gives a right inverse to $\lambda + A + B$, and we still have to show that $\lambda + A + B$ is injective. To do so, let us assume that $(\lambda + Au + B)u = 0$ for some $u \in D(A) \cap D(B)$. Then applying T_λ to this equation and using (4.5) we obtain

$$0 = T_\lambda(\lambda + A + B)u = u - [B, T_\lambda]u.$$

This yields

$$\begin{aligned}(1+B)^\gamma u &= (1+B)^\gamma [B, T_\lambda]u \\ &= \frac{1}{2\pi i}\int_{\Gamma_r^\theta} (1+B)^\gamma (z-B)^{-1}[B, (z+\lambda+A)^{-1}](1+B)^{-\gamma}(1+B)^\gamma u\, dz\end{aligned}$$

with $\gamma \in (0,1)$ to be chosen later. Using the functional calculus for B with an appropriate contour Γ_B we get

$$\begin{aligned}(1+B)^\gamma u = \frac{1}{(2\pi i)^2}\int_{\Gamma_0^\theta}\int_{\Gamma_B} &(1+B)^\gamma (z-B)^{-1}[B, (z+\lambda+A)^{-1}] \\ &\times (1+\zeta)^{-\gamma}(\zeta - B)^{-1}(1+B)^\gamma u\, d\zeta\, dz.\end{aligned}$$

Using (3.1) then yields

$$|(1+B)^\gamma u| \le cC \int_{\Gamma_0^\theta}\int_{\Gamma_B} \frac{|(1+B)^\gamma u|\ |d\zeta|\ |dz|}{|1+z|^{(1-\gamma)}(1+|\lambda+z|)^\alpha |\zeta|^\beta |1+\zeta|^\gamma}\,,$$

and we conclude that

$$|(1+B)^\gamma u| \le cC|(1+B)^\gamma u|,$$

for some constant $C > 0$, provided $1-\beta < \gamma < \alpha$. Thus if $c > 0$ is sufficiently small, then $(1+B)^\gamma u = 0$, and hence $u = 0$, which completes the proof of uniqueness. In summary, (4.6) and (4.8) imply that $\lambda + A + B$ is invertible for all $\lambda \in \bar{\Sigma}_{\pi-\psi}$.

We will now show that

$$\sup_{\lambda \in \Sigma_{\pi-\psi}} |\lambda(\lambda + A + B)^{-1}| < \infty\,. \tag{4.9}$$

It follows from (4.4) that

$$\lambda S_\lambda x = (\lambda + A)S_\lambda x - AS_\lambda x = x - S_\lambda Bx - AS_\lambda x. \tag{4.10}$$

This relation certainly holds true for every $x \in D(B)$. According to our assumption, the operators $S_\lambda B$ and AS_λ admit unique extensions in $\mathcal{B}(X)$, again denoted by the same symbol, and these extensions are uniformly bounded in λ. The assertion in (4.9) is now a consequence of (4.10) and (4.6)–(4.8).

It remains to prove that the domain $D(A) \cap D(B)$ is dense in X. This is easy to show in the case of condition (3.1), since $(1+\delta A)^{-1}(1+\delta B)^{-1}u$ belongs to $D(A) \cap D(B)$ and converges to u as $\delta \to 0$. For the Labbas-Terreni condition, density of $D(A) \cap D(B)$ would be obvious if X was reflexive, since every operator L with a resolvent estimate of the form $\sup_{t>0} |t(t+L)^{-1}| < \infty$ has dense domain. However, since the underlying Banach space X is arbitrary, we cannot use this argument. Instead we will prove that

$$\lambda(\lambda + A + B)^{-1} \to I \quad \text{strongly in } X \text{ as } \lambda \to \infty. \tag{4.11}$$

We conclude from (4.10) (and our assumptions) that

$$\lambda S_\lambda x = x - S_\lambda Bx - S_\lambda Ax - [A, S_\lambda]x$$

for every $x \in D(A)$. We will now consider each of the terms in the equation above separately. Clearly, $S_\lambda Bx \to 0$ for all $x \in D(B)$, and hence $S_\lambda B \to 0$ strongly by its boundedness. In particular, $S_\lambda Bx \to 0$ for every $x \in D(A)$. Clearly, $S_\lambda Ax \to 0$ for any $x \in D(A)$. A short calculation shows that

$$[A, S_\lambda]x = \frac{1}{2\pi i} \int_{\Gamma_r^\theta} A(z + \lambda + A)^{-1}[(z-B)^{-1}, A^{-1}]Ax\, dz$$

for $x \in D(A)$. Here we choose $r = |\lambda| \sin(\psi - \psi_A)$. It follows from the commutator condition (3.2) that

$$|[A, S_\lambda]x| \le C\, \frac{1}{(1+|\lambda|)^{1+\beta-\alpha}}\, |Ax|.$$

Therefore, $[A, S_\lambda]x \to 0$ for every $x \in D(A)$. Hence $\lambda S_\lambda x \to x$ for every $x \in D(A)$. We have already observed in (4.10) that λS_λ admits a bounded extension in $\mathcal{B}(X)$, and therefore $\lambda S_\lambda \to I$ strongly in X. Equations (4.6)–(4.7) now yield the assertion in (4.11), and this shows that $D(A) \cap D(B)$ is dense in X.

In summary, we have proved the first assertion of Theorem 3.1, provided we can justify the assumption made at the beginning of (ii).

(iii) To prove that $A+B$ admits an $\mathcal{H}^\infty$-calculus, we fix a function $f \in \mathcal{H}_0^\infty(\Sigma_\psi)$ and choose $\theta \in (\max\{\psi_A, \psi_B\}, \psi)$. Then by definition

$$\begin{aligned} f(A+B) &= \frac{1}{2\pi i} \int_{\Gamma_0^\theta} f(\lambda)(\lambda - (A+B))^{-1} d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma_0^\theta} f(\lambda) S_{-\lambda} d\lambda - \frac{1}{2\pi i} \int_{\Gamma_0^\theta} f(\lambda) R_{-\lambda} d\lambda \\ &= T^1 f + T^2 f. \end{aligned}$$

By (4.7) we easily obtain

$$|T^2 f|_{\mathcal{B}(X)} \le C|f|_{\mathcal{H}^\infty} \int_{\Gamma_0^\theta} \frac{|d\lambda|}{(1+|\lambda|)^{1+\varepsilon}} \le C_1 |f|_{\mathcal{H}^\infty}.$$

Hence in order to prove the $\mathcal{H}^\infty$-estimate, it is enough to consider $T^1 f$.

By Cauchy's theorem we may write

$$S_\lambda = \frac{1}{2\pi i}\int_\Gamma (z-A)^{-1}(\lambda+z+B)^{-1}dz$$

with an appropriate contour Γ. Then we have

$$\begin{aligned} T^1 f &= -\frac{1}{2\pi i}\int_{\Gamma_0^\theta} f(\lambda)S_{-\lambda}d\lambda \\ &= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma_0^\theta} f(\lambda)(z-A)^{-1}(\lambda-z-B)^{-1}d\lambda dz \\ &= \frac{1}{2\pi i}\int_\Gamma (z-A)^{-1}f(z+B)dz\,. \end{aligned}$$

To prove boundedness of T^1 we symmetrize as follows. Cauchy's theorem implies

$$\int_\Gamma (z-A)^{-1}f(z+B)dz = \int_\Gamma A^\gamma(z-A)^{-1}f(z+B)dz/z^\gamma,$$

where $\gamma\in(0,1)$. Setting $F(z)=f(z+B)$ we obtain the decomposition $T^1f = T_1^1f - T_2^1f$, where

$$\begin{aligned} T_1^1 f &= \frac{1}{2\pi i}\int_\Gamma g_1(z,A)F(z)g_2(z,A)dz/z^\gamma, \\ T_2^1 f &= \frac{1}{2\pi i}\int_{\Gamma_0^\theta} g_1(z,A)[F(z),g_2(z,A)]dz/z^\gamma, \end{aligned}$$

with $g_j(z,\zeta)=[\zeta^\gamma/(z-\zeta)]^{\delta_j}$, $j=1,2$, and $\delta_1+\delta_2=1$.

(iv) We next prove boundedness of T_2^1f and provide the choices of $\gamma,\delta_j\in(0,1)$. Once more this involves the commutator conditions. Consider first condition (3.2). We have

$$[F(z),g_2(z,A)] = \frac{1}{(2\pi i)^2}\int_{\Gamma_0^\theta}\int_{\Gamma_A} f(\mu)g_2(z,\lambda)[(\mu-z-B)^{-1},(\lambda-A)^{-1}]d\lambda\, d\mu,$$

where Γ_A denotes an appropriate contour. Then (3.2) implies

$$\begin{aligned} |[F(z),g_2(z,A)]| &\le C|f|_{\mathcal{H}^\infty}\int_{\Gamma_0^\theta}\int_{\Gamma_A}\Big(\frac{|\lambda|^\gamma}{|z-\lambda|}\Big)^{\delta_2}(1+|\lambda|)^{\alpha-1}|\mu-z|^{-(1+\beta)}|d\lambda||d\mu| \\ &\le C|f|_{\mathcal{H}^\infty}|z|^{-(\beta-\alpha+\delta_2(1-\gamma))}, \end{aligned}$$

by a scaling argument, provided we choose $\delta_2,\gamma\in(0,1)$ in such a way that $\alpha<\delta_2(1-\gamma)$. Using this estimate in the definition of T_2^1f we get

$$\begin{aligned} |T_2^1 f|_{\mathcal{B}(X)} &\le C|f|_{\mathcal{H}^\infty}\int_\Gamma \frac{|dz|}{(1+|z|)^{\delta_1(1-\gamma)}|z|^{(\beta-\alpha+\delta_2(1-\gamma))}|z|^\gamma} \\ &\le C|f|_{\mathcal{H}^\infty}, \end{aligned}$$

provided $\beta-\alpha<\delta_1(1-\gamma)$. The choice $\gamma=(1-\beta)/2$ and $\delta_2=(2\alpha+(1-\beta)/2)/(1+\beta)$ meets these requirements.

In the case of the Da Prato-Grisvard condition (3.1) we obtain a similar estimate. This time a feasible choice is $\gamma=(1-\beta)/2$ and $\delta_2=2(1-\alpha-\beta/2)/(1-\beta)+1/2$.

(v) To estimate T_1^1f we use the technique introduced by Kalton and Weis [13].

We begin with the following lemma from that paper. For the sake of completeness a proof is incuded here.

Lemma 4.1. *Suppose $A \in \mathcal{H}^\infty(X)$, $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$, $\phi > \phi_A^\infty$. Then there is a constant $C > 0$ such that*

$$|\sum_{k\in\mathbb{Z}} \alpha_k h(2^k tA)|_{\mathcal{B}(X)} \le C \sup_{k\in\mathbb{Z}} |\alpha_k|$$

for all $\alpha_k \in \mathbb{C}$ and $t > 0$.

Proof. Let $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ be given. Then we have

$$|h(z)| \le c\frac{|z|^\beta}{1+|z|^{2\beta}}, \quad z \in \Sigma_\phi,$$

for some numbers $\beta > 0$ and $c > 0$. Set $f(z) = \sum_{k\in\mathbb{Z}} \alpha_k h(2^k tz)$. This series is absolutely convergent as can be seen from the estimate

$$|f(z)| \le |\alpha|_\infty \sum_k |h(2^k tz)| \le C|\alpha|_\infty,$$

since

$$\sum_k |h(2^k tz)| \le c\sum_k \frac{(r2^k)^\beta}{1+(r2^k)^{2\beta}} \le \frac{2c}{1-2^{-\beta}}, \quad r = t|z|.$$

Therefore $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and so by $A \in \mathcal{H}^\infty(X)$, $\phi > \phi_A^\infty$ we obtain

$$|\sum_{k\in\mathbb{Z}} \alpha_k h(2^k tA)|_{\mathcal{B}(X)} = |f(A)|_{\mathcal{B}(X)} \le C_A |f|_{\mathcal{H}^\infty} \le C|\alpha|_\infty. \qquad \square$$

Since the integral defining $T_1^1 f$ is absolutely convergent, we have

$$T_1^1 f = \lim_{N\to\infty} \frac{1}{2\pi i}\int_{\Gamma_N} g_1(\lambda, A)F(\lambda)g_2(\lambda, A)d\lambda/\lambda^\gamma = \lim_{N\to\infty} T_N f,$$

where $\Gamma_N = \{\lambda \in \Gamma : 2^{-N} \le |\lambda| \le 2^N\}$. We write $T_N = T_N^+ + T_N^-$, with

$$\begin{aligned}
T_N^\pm f &= \frac{e^{\pm i\theta(1-\gamma)}}{2\pi i}\int_{2^{-N}}^{2^N} g_1(re^{\pm i\theta}, A)F(re^{\pm i\theta})g_2(re^{\pm i\theta}, A)dr/r^\gamma \\
&= \frac{e^{\pm i\theta(1-\gamma)}}{2\pi i}\sum_{k=-N}^{N-1}\int_{2^k}^{2^{k+1}} h_{1,\pm}(A/r)F(re^{\pm i\theta})h_{2,\pm}(A/r)dr/r \\
&= \frac{e^{\pm i\theta(1-\gamma)}}{2\pi i}\sum_{k=-N}^{N-1}\int_1^2 h_{1,\pm}(A/t2^k)F(t2^k e^{\pm i\theta})h_{2,\pm}(A/t2^k)dt/t \\
&= \frac{e^{\pm i\theta(1-\gamma)}}{2\pi i}\int_1^2 R_N^\pm(t)f dt/t,
\end{aligned}$$

where $h_{j,\pm}(z) = [z^\gamma/(e^{\pm i\theta} - z)]^{\delta_j}$ belongs to $\mathcal{H}_0^\infty(\Sigma_\phi)$, for $\phi \in (\phi_A^\infty, \theta)$. Let $(\Omega, \mathcal{A}, \mu)$

be a probability space and let ε_k be independent symmetric $\{\pm 1\}$-valued random variables on this probability space. We randomize and estimate as follows:

$$\begin{aligned}
|\langle [R_N^{\pm}(t)f]x, x^*\rangle| &= |\sum_{k=-N}^{N-1} \langle h_{1,\pm}(A/t2^k)F(t2^k e^{\pm i\theta})h_{2,\pm}(A/t2^k)x, x^*\rangle| \\
&= |\int_\Omega \sum_{k=-N}^{N-1} \varepsilon_k^2 \langle h_{1,\pm}(A/t2^k)F(t2^k e^{\pm i\theta})h_{2,\pm}(A/t2^k)x, x^*\rangle d\mu| \\
&= |\int_\Omega \langle \sum_{k=-N}^{N-1} \varepsilon_k F(t2^k e^{\pm i\theta})h_{2,\pm}(A/t2^k)x, \sum_{k=-N}^{N-1} \varepsilon_k h_{1,\pm}(A^*/t2^k)x^*\rangle d\mu| \\
&\le |\sum_{k=-N}^{N-1} \varepsilon_k F(t2^k e^{\pm i\theta})h_{2,\pm}(A/t2^k)x|_{L_2(\Omega;X)} |\sum_{k=-N}^{N-1} \varepsilon_k h_{1,\pm}(A^*/t2^k)x^*|_{L_2(\Omega;X^*)} \\
&\le \mathcal{R}(F(\Sigma_\phi))|\sum_{k=-N}^{N-1} \varepsilon_k h_{2,\pm}(A/t2^k)x|_{L_2(\Omega;X)} |\sum_{k=-N}^{N-1} \varepsilon_k h_{1,\pm}(A^*/t2^k)x^*|_{L_2(\Omega;X^*)} \\
&\le C\mathcal{R}(F(\Sigma_\phi))|x||x^*|,
\end{aligned}$$

by Lemma 4.1. Here we have employed $\mathcal{R}$-boundedness of $F(\Sigma_\phi) = f(\Sigma_\phi + B)$, that is, the assumption that B has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus. This shows that $R_N^{\pm}(t)f$ is uniformly bounded in $t \in [1,2]$ and in $N \in \mathbb{N}$, hence so is $T_1^1 f$, with

$$|T_1^1 f|_{\mathcal{B}(X)} \le C\mathcal{R}(f(\Sigma_\phi + B)) \le C|f|_{\mathcal{H}^\infty}.$$

From (iii), (iv) and the last estimate we obtain that $A + B$ again admits an $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle smaller than or equal to $\max\{\phi_A^\infty, \phi_B^{R\infty}\}$. Note that $A^* \in \mathcal{H}(X^*)$ with $\phi_A^\infty = \phi_{A^*}^\infty$ in case $D(A^*)$ is dense in X. If $D(A^*)$ is not dense in X^*, then we may use the sun-dual $A^\odot$ on $X^\odot$ instead; see Hille-Phillips [12, Section 14.2].

(vi) We now show that, say, $S_\lambda B$ is uniformly bounded in $\mathcal{B}(X)$ for all $\lambda \in \Sigma_{\pi-\psi}$. In order to prove this we introduce the Yosida approximation $B_\delta = B(1+\delta B)^{-1}$ of B, where $\delta > 0$, and we recall that $B_\delta x \to Bx$ for $x \in D(B)$ as $\delta \to 0$. We use the methods from (iv) and (v) to write

$$\begin{aligned}
S_\lambda B_\delta &= \frac{1}{2\pi i}\int_\Gamma (z-A)^{-1}B(\lambda+z+B)^{-1}(1+\delta B)^{-1}dz \\
&= \frac{1}{2\pi i}\int_\Gamma A^\gamma(z-A)^{-1}B(\lambda+z+B)^{-1}(1+\delta B)^{-1}dz/z^\gamma \\
&= \frac{1}{2\pi i}\int_\Gamma g_1(z,A)B(\lambda+z+B)^{-1}(1+\delta B)^{-1}g_2(z,A)dz/z^\gamma \\
&\quad -\frac{1}{2\pi i}\int_\Gamma g_1(z,A)[B(\lambda+z+B)^{-1}(1+\delta B)^{-1}, g_2(z,A)]dz/z^\gamma \\
&= T_{\lambda,\delta}^1 + T_{\lambda,\delta}^2.
\end{aligned}$$

Then the arguments given in (v) yield the estimate

$$\begin{aligned}
|T_{\lambda,\delta}^1| &\le C\mathcal{R}(B(1+\delta B)^{-1}(\lambda+\Sigma_\phi+B)^{-1}) \\
&\le C\mathcal{R}(\{(1+\delta B)^{-1} : \delta > 0\})\mathcal{R}(B(\Sigma_\phi+B)^{-1}) < \infty,
\end{aligned}$$

where $C > 0$ is independent of δ and λ, provided B is $\mathcal{R}$-sectorial and $\phi < \pi - \phi_B^R$. Similarly, the estimates from (iv) yield $|T^2_{\lambda,\delta}| \le C$ with $C > 0$ independent of δ and λ. Passing to the limit $\delta \to 0$ we obtain boundedness of $S_\lambda B$, uniformly in λ. This completes the proof of Theorem 3.1. □

5. Parabolic equations on wedges and cones

In this section we consider an application of our main results to the diffusion equation on a domain of wedge or cone type, that is, on the domain $G = \mathbb{R}^m \times C_\Omega$, where $\Omega \subset S^{n-1}$ is open with smooth boundary $\partial\Omega \neq \emptyset$, and C_Ω denotes the cone

$$C_\Omega = \{x \in \mathbb{R}^n : \ x \neq 0, \ x/|x| \in \Omega\}.$$

We then consider the problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } G \times (0,T), \\ u = 0 & \text{on } \partial G \times (0,T), \\ u|_{t=0} = 0 & \text{on } G. \end{cases} \tag{5.1}$$

Here $m \in \mathbb{N}_0$ and $2 \le n \in \mathbb{N}$. The function f is given in a weighted L_p-space, i.e.

$$f \in L_p(J \times \mathbb{R}^m; L_p(C_\Omega; |x|^\gamma dx)),$$

where $\gamma \in \mathbb{R}$ will be chosen appropriately, and $J = [0,T]$.

It is natural to introduce polar coordinates in the x-variables, i.e. $x = r\zeta$ where $\zeta \in \Omega$ and $r > 0$. Then the diffusion operator $\partial_t - \Delta$ transforms into

$$\partial_t - \Delta_y - [\partial_r^2 + \frac{n-1}{r}\partial_r] - \frac{\Delta_\zeta}{r^2},$$

where y denotes the variable in $\mathbb{R}^m$, Δ_y is the Laplacian in the y-variables and Δ_ζ means the Laplace-Beltrami operator on S^{n-1}. The underlying space for the function f now is

$$f \in L_p(J \times \mathbb{R}^m \times \Omega; L_p(\mathbb{R}_+; r^{\gamma+n} dr/r)),$$

where the measure on Ω is the surface measure. It is also natural to employ the Euler transformation $r = e^x$ where now $x \in \mathbb{R}$. Setting

$$g(t,y,\zeta,x) = r^{2-\beta} f(t,y,\zeta,r), \quad u(t,y,\zeta,r) = r^\beta v(t,y,\zeta,\log r),$$

we arrive at the following problem for the unknown function v:

$$\begin{cases} e^{2x}(\partial_t - \Delta_y)v + P(\partial_x)v - \Delta_\zeta v = g & \text{in } (0,T) \times \mathbb{R}^m \times \Omega \times \mathbb{R}, \\ v = 0 & \text{on } (0,T) \times \mathbb{R}^m \times \partial\Omega \times \mathbb{R}, \\ v|_{t=0} = 0 & \text{on } \mathbb{R}^m \times \Omega \times \mathbb{R}. \end{cases} \tag{5.2}$$

The resulting equations are now defined in a smooth domain. They contain the (non-standard) differential operators $e^{2x}\partial_t$ and $e^{2x}\Delta_y$. We observe that these operators do not commute with $P(\partial_x)$.

Next we note that

$$\int_{\mathbb{R}} |g(t,y,\zeta,x)|^p dx = \int_0^\infty |r^{2-\beta} f(t,y,\zeta,r)|^p dr/r < \infty,$$

in case we choose $p(2-\beta) = \gamma + n$, that is, $\beta = 2 - (\gamma+n)/p$. Making this choice of β, we can remove the weight and work in the unweighted base space

$$X := L_p(J \times \mathbb{R}^m \times \Omega \times \mathbb{R}).$$

The differential operator $P(\partial_x)$ is given by the polynomial

$$P(z) = -[z^2 + (2\beta + n - 2)z + \beta(\beta + n - 2)] = -z^2 + a_1 z + a_0,$$

as a simple computation shows.

In order to derive the unique solvability and maximal regularity of (5.2) we introduce the following basic operators.

(i) Define B in $L_p(\mathbb{R})$ by means of

$$Bu(x) = P(\partial_x)u(x), \quad x \in \mathbb{R}, \quad u \in D(B) = H_p^2(\mathbb{R}).$$

It is well known that the spectrum of B is given by the parabola $P(i\mathbb{R})$, which opens to the right, is symmetric about the real axis, and has its vertex at $a_0 \in \mathbb{R}$. This follows from the Mikhlin multiplier theorem, and the latter also implies that $\omega + B$ belongs to $\mathcal{H}^\infty(L_p(\mathbb{R}))$ for $\omega > -a_0$, with $\phi^\infty_{\omega+B} < \pi/2$. Even more, the $\mathcal{H}^\infty$-calculus is $\mathcal{R}$-bounded with the same angle, as follows from Remark 2.1. The same results are valid for the canonical extension of B to X, which we again denote by B. We also observe that $B - a_0$ is accretive in X, since $-\partial_x^2$ is so on $L_p(\mathbb{R})$, and $\pm\partial_x$ are as well.

(ii) Define L in $L_p(\Omega)$ by means of

$$Lu(\zeta) = -\Delta_\zeta u(\zeta), \quad \zeta \in \Omega, \quad D(L) = \{u \in H_p^2(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

This operator has pure discrete point spectrum $\sigma(L) = \{\lambda_k\}_{k\in\mathbb{N}} \subset (0, \infty)$. It is self adjoint and positive definite in L_2 and has $\mathcal{H}^\infty$-calculus in L_p with $\phi^\infty_L = 0$. These facts are also well known and follow easily from known results on elliptic differential operators by a coordinate transform to a flat domain in $\mathbb{R}^{n-1}$, e.g. by a stereographic projection; see for instance Prüss and Sohr [24]. The canonical extension of L to X enjoys the same properties and will again be denoted by L.

(iii) Let C in $L_p(\mathbb{R}^m)$ be the negative Laplacian

$$Cu(y) = -\Delta_y u(y), \quad y \in \mathbb{R}^m, \quad u \in D(C) = H_p^2(\mathbb{R}^m).$$

C also admits an $\mathcal{H}^\infty$-calculus in $L_p(\mathbb{R}^m)$ with $\phi^\infty_C = 0$. In fact the $\mathcal{H}^\infty$-calculus for C is also $\mathcal{R}$-bounded with the same angle; see again [6]. The spectrum of C is the half-line $\mathbb{R}_+$. The same is true for its canonical extension C to X. Note that C is accretive.

(iv) Next consider G on $L_p(J)$ defined by

$$Gu(t) = \partial_t u(t) \quad t \in J, \quad D(G) = {}_0H_p^1(J) := \{u \in H_p^1(J) : \ u(0) = 0\}.$$

This operator admits a bounded $\mathcal{H}^\infty$-calculus with $\phi^\infty_G = \pi/2$, and the same holds true for its canonical extension G to X. Note that G is invertible and accretive.

(v) The final ingredient we need is the multiplication operator M in $L_p(\mathbb{R})$ given by

$$Mu(x) = e^{2x}u(x), \quad x \in \mathbb{R}, \quad D(M) = \{u \in L_p(\mathbb{R}) : \ Mu \in L_p(\mathbb{R})\}.$$

This operator has an $\mathcal{H}^\infty$-calculus in $L_p(\mathbb{R})$, which is also $\mathcal{R}$-bounded by the contraction principle of Kahane, and $\phi^{R\infty}_M = 0$. The spectrum of M is the half-line $[0, \infty)$. Once more, the canonical extension of M to X has the same properties.

Now we can build up the differential operator defined by the left-hand side of (5.2). First, since G and C commute and G is invertible, the sum $G + C$ with domain $D(G) \cap D(C)$ is invertible and

$$G + C \in \mathcal{H}^\infty(X) \quad \text{with angle} \quad \phi^\infty_{G+C} \le \pi/2.$$

This follows for instance from Theorem 3.1 since C is in $\mathcal{RH}^\infty(X)$. Next we consider the product $A := (G+C)M$ with natural domain

$$D(A) = \{u \in D(M) : Mu \in D(G) \cap D(C)\}.$$

Since $G+C$ is invertible, A is closed, hence sectorial and in $\mathcal{BIP}(X)$, with $\theta_A \leq \theta_{G+C} + \theta_M \leq \pi/2$ by a result due to Prüss and Sohr [23]. Due to [13, Theorem 4.4] we further see that

$$A \in \mathcal{H}^\infty(X) \quad \text{with angle} \quad \phi_A^\infty \leq \pi/2.$$

Since M is a multiplication with a positive function, it follows from the accretivity of G and C that A is accretive as well.

Next we consider the sum $A+B$ with natural domain. It is here where we need the full strength of Theorem 3.1, since A and B do not commute. Let us compute the commutator (3.2) for A and B. For this purpose we note first that B, C and G commute, however, M and B are non-commuting. We have the important relation

$$MB = e^{2x}P(\partial_x) = P(\partial_x - 2)e^{2x} = B_{-2}M, \tag{5.3}$$

where B_{-2} is defined in the same way as B. This implies

$$M(\mu+\omega+B)^{-1} = (\mu+\omega+B_{-2})^{-1}M,$$

and hence

$$[A, (\mu+\omega+B)^{-1}] = (\mu+\omega+B)^{-1}Q(\mu+\omega+B_{-2})^{-1}A,$$

where $Q = B - B_{-2} = P(\partial_x) - P(\partial_x - 2) = Q(\partial_x)$ is a differential operator of first order. Here and in the following we assume that $\omega > 0$ is fixed such that $\sigma(\omega+B)$ as well as $\sigma(\omega+B_2)$ is contained in $[Re\, z > 0]$. This implies that both operators $\omega+B$ and $\omega+B_2$ are sectorial with angle strictly less than $\pi/2$. Let $\eta > 0$. One readily verifies that

$$\begin{aligned}
&(\eta+A)(\lambda+\eta+A)^{-1}[(\eta+A)^{-1}, (\lambda+\omega+B)^{-1}] \\
&= -(\lambda+\eta+A)^{-1}[A, (\mu+\omega+B)^{-1}](\eta+A)^{-1} \\
&= -(\lambda+\eta+A)^{-1}(\mu+\omega+B)^{-1}Q(\mu+\omega+B_{-2})^{-1}A(\eta+A)^{-1}\,.
\end{aligned}$$

Therefore, we obtain the estimate

$$\begin{aligned}
&|(\eta+A)(\lambda+\eta+A)^{-1}[(\eta+A)^{-1}, (\mu+\omega+B)^{-1}]| \\
&\leq \frac{C_\varepsilon}{\eta+|\lambda|}|(\mu+\omega+B)^{-1}|\,|Q(\mu+\omega+B_{-2})^{-1}||A(\eta+A)^{-1}| \\
&\leq \frac{C}{(1+|\lambda|)\,|\mu|^{3/2}}
\end{aligned}$$

for all $\lambda \in \Sigma_{\pi/2-\varepsilon}$, $\mu \in \Sigma_{\pi/2+2\varepsilon}$, where $\varepsilon > 0$ is small, and $C = C(\varepsilon, \eta)$. Thus (3.2) holds with $\alpha = 0$, $\beta = 1/2$, $\psi_A = \pi/2 + \varepsilon$ and $\psi_B = \pi/2 - 2\varepsilon$. By Corollary 3.2 we may conclude that

$$\nu + A + B \in \mathcal{H}^\infty(X) \quad \text{with angle} \quad \phi_{\nu+A+B}^\infty \leq \pi/2 + \varepsilon,$$

where ν is sufficiently large.

To conclude, observe that $A+B-a_0$ with natural domain $D(A) \cap D(B)$ is again accretive, hence $A+B-a_0+\varepsilon$ is sectorial. Theorem 8.5 of Prüss [18] then implies

that $A+B+L$ is invertible with natural domain, provided $\sigma(A+B)\cap\sigma(-L)=\emptyset$. Since $\sigma(A+B)\subset\{z\in\mathbb{C}:\ \mathrm{Re}z\geq a_0\}$ and $\sigma(L)=\{\lambda_k\}_{k\in\mathbb{N}}$, the latter is satisfied if

$$\lambda_1 > -a_0 = \beta(\beta+n-2) = (2-n/p-\gamma/p)(n-n/p-\gamma/p) \tag{5.4}$$

is valid. This is the condition found by Nazarov in his recent paper [17]. We may now summarize our considerations in the following result.

Corollary 5.1. *Suppose $1<p<\infty$ and suppose that $\gamma\in\mathbb{R}$ is subject to condition* (5.4)*, where $\lambda_1>0$ denotes the first eigenvalue of the Laplace-Beltrami operator on $\Omega\subset S^{n-1}$ with Dirichlet boundary conditions. Then for each $f\in L_p(J\times\mathbb{R}^m\times\Omega\times\mathbb{R})$ there is a unique solution v of* (5.2) *in the regularity class*

$$\begin{aligned}&v\in L_p(J\times\mathbb{R}^m;H^2_p(\Omega\times\mathbb{R})),\\ &e^{2x}v\in H^1_p(J;L_p(\mathbb{R}^m\times\Omega\times\mathbb{R}))\cap L_p(J;H^2_p(\mathbb{R}^m;L_p(\Omega\times\mathbb{R}))).\end{aligned}$$

In particular, the map $[v\mapsto f]$ defines an isomorphism between the corresponding spaces.

We may now transform this result back to the original variables to obtain precisely Nazarov's result for (5.1).

Corollary 5.2. *Suppose $1<p<\infty$ and suppose that $\gamma\in\mathbb{R}$ is subject to condition* (5.4)*, where $\lambda_1>0$ denotes the first eigenvalue of the Laplace-Beltrami operator on $\Omega\subset S^{n-1}$ with Dirichlet boundary conditions. Then for each*

$$f\in L_p(J\times\mathbb{R}^m;L_p(C_\Omega,|x|^\gamma dx))$$

there is a unique solution u of (5.1) *with regularity*

$$u, u/|x|^2, \partial_t u, \nabla^2 u\in L_p(J\times\mathbb{R}^m;L_p(C_\Omega,|x|^\gamma dx)).$$

The solution map $[u\mapsto f]$ defines an isomorphism between the corresponding function spaces.

For simplicity we have chosen the integrability exponent $p\in(1,\infty)$ to be the same for the variables t, x and y. By the arguments given above it also follows that we may choose different exponents for these variables, and we may arrange them in any order.

We also note that the method described above can be applied to other problems on cone and wedge domains, like the Navier-Stokes equations, or free boundary value problems with moving contact lines and prescribed contact angles. These will be topics for our future work.

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