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THE POISSON PROBLEM WITH MIXED BOUNDARY CONDITIONS IN SOBOLEV AND BESOV SPACES IN NON-SMOOTH DOMAINS

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Abstract. We introduce certain Sobolev-Besov spaces which are particularly well adapted for measuring the smoothness of data and solutions of mixed boundary value problems in Lipschitz domains. In particular, these are used to obtain sharp well-posedness results for the Poisson problem for the Laplacian with mixed boundary conditions on bounded Lipschitz domains which satisfy a suitable geometric condition introduced by R. Brown in (1994). In this context, we obtain results which generalize those by D. Jerison and C. Kenig (1995) as well as E. Fabes, O. Mendez and M. Mitrea (1998). Applications to Hodge theory and the regularity of Green operators are also presented.

1. Introduction

Given a domain $\Omega \subseteq \mathbb{R}^n$, consider the Poisson boundary value problems with Dirichlet and Neumann boundary conditions

$$(1.1)\qquad (Dir)\begin{cases}\Delta u = f & \text{in } \Omega,\\ u|_{\partial\Omega} = g & \text{on } \partial\Omega,\end{cases}\qquad (Neu)\begin{cases}\Delta u = f & \text{in } \Omega,\\ \partial_\nu u = g & \text{on } \partial\Omega,\end{cases}$$

where ν stands for the outward unit normal vector to $\partial\Omega$ and ∂_ν is the normal derivative.

The optimal range of solvability for (Dir) in (1.1) on scales of Sobolev-Besov spaces on arbitrary Lipschitz domains has been identified by D. Jerison and C. Kenig in [25]. Using delicate estimates for the harmonic measure associated with Ω they establish the following result. For each $\varepsilon \in (0, 1/2]$ let $\mathcal{H}_\varepsilon$ stand for the interior of the hexagonal region in the plane with vertices

$$(1.2)\qquad (0,0),\quad (\varepsilon,0),\quad \Big(1,\frac{1}{2}-\varepsilon\Big),\quad (1,1),\quad (1-\varepsilon,1),\quad \Big(0,\frac{1}{2}+\varepsilon\Big).$$

Then, for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the problem (Dir) in (1.1) has a unique solution $u \in L^p_{s+1/p}(\Omega)$ for any $f \in L^p_{s+(1/p)-2}(\Omega)$ and $g \in B^{p,p}_s(\partial\Omega)$ whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$. Here $L^p_\alpha(\Omega)$

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and $B^{p,q}_s(\partial\Omega)$ stand for the scales of Sobolev and Besov spaces on Ω and $\partial\Omega$, respectively; we refer the reader to the body of the paper for complete definitions.

An alternative approach to both types of boundary conditions in (1.1) and which emphasizes the functional analytical properties of boundary layer potentials on scales of Sobolev-Besov spaces has been developed by E. Fabes, O. Mendez and M. Mitrea in [16]. This, in principle, does not differentiate between Dirichlet and Neumann type boundary conditions and allows for a unified treatment of (1.1). Indeed, the authors just mentioned recovered the main (positive) results in [25] and, in addition, proved the following. For any bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$ with connected boundary there exists $\varepsilon=\varepsilon(\Omega)>0$ such that the problem (Neu) in (1.1), with $f\in L^{p'}_{(1/p')-s-1,0}(\Omega)$ and $g\in B^{p',p'}_{-s}(\partial\Omega)$, subject to the (necessary) compatibility condition $\langle f,1\rangle=\langle g,1\rangle$, has a solution (which is unique modulo additive constants) $u\in L^{p'}_{1-s+(1/p')}(\Omega)$ whenever $(s,1/p)\in\mathcal{H}_\varepsilon$, with $1/p+1/p'=1$.

Both (Dir) and (Neu) in (1.1) are special cases of the more general case when *mixed boundary conditions* are considered. This is also known as the *Zaremba problem* and reads

$$\Delta u=F\quad\text{in }\Omega,\quad \partial_\nu u|_N=f,\quad u\,|_D=g, \tag{1.3}$$

where D and N are disjoint open subsets of $\partial\Omega$ which share a common boundary, i.e., $\partial D=\partial N$.

The Poisson problem with mixed Dirichlet-Neumann boundary conditions arises naturally in connection to a series of important problems in mathematical physics and engineering dealing with conductivity, heat transfer, wave phenomena, electrostatics, metallurgical melting, and stamp problems in elasticity and hydrodynamics. Specific references can be found in [1], [12], [13], [17], [22], [23], [27], [33], [36], [42], [43], [46], [47], [48], [49], [50], [57]. Other interesting, physically relevant mixed boundary value problems are those associated with the Maxwell equations and the Dirac operator.

In each case, the physical properties of the bounding surface dictate the type of boundary conditions one must impose. One example which is intuitively simple is that of an iceberg Ω floating while partially submerged in water. In this scenario, $u(x)$ is the temperature at the point $x\in\Omega$. On D, the portion of $\partial\Omega$ lying below the waterline, Ω behaves like a thermostat and, hence, one has to impose a (homogeneous) Dirichlet boundary condition. On the remaining portion of the boundary $N:=\partial\Omega\setminus\bar{D}$, lying in the air, Ω behaves like a perfect insulator and, thus, one must impose a (homogeneous) Neumann boundary condition on N.

Mixed boundary problems are also important in numerical analysis. For instance, in [32] the authors devise an algorithm for solving the Cauchy problem for elliptic equations via an iterative procedure which requires solving mixed boundary value problems for the original equation at each step. See also [2], [14], [41] and the references therein.

For Lipschitz domains, the issue of the regularity of the solution of (1.3) has been raised by C. Kenig on p. 120 of [31]. To answer this question, R. Brown has introduced in [5] the class of the so-called *creased domains* which, roughly speaking, means that D and N are separated by a Lipschitz interface ("crease" or "collision submanifold") and the angle between D and N is $<\pi$. In this class, he proved that

$$\|\nabla u\|_{L^2(\partial\Omega)}\leq C\Big(\|\nabla_{tan}u\|_{L^2(D)}+\|\partial_\nu u\|_{L^2(N)}\Big) \tag{1.4}$$

uniformly for u harmonic in Ω. In a subsequent paper, [53], R. Brown and J. Sykes succeeded in proving an L^p-version of (1.4), valid for all $p \in (1,2]$. Let us also mention the recent paper [7] by R. Brown, L. Capogna and L. Lanzani in which the authors study the L^p Zaremba problem for Laplace's equation in two-dimensional Lipschitz graph domains with Lipschitz constant at most 1.

The work in [5], [53] emphasized L^p-data along with nontangential maximal function estimates for the solution, and the next natural step is to consider (1.3) in the context of *Sobolev-Besov spaces*. In this paper we undertake this task and initiate the study of elliptic problems equipped with mixed boundary conditions in nonsmooth domains when the size/smoothness of both the data and the solutions are measured on Sobolev-Besov scales.

The main result of this paper, which answers questions posed to us by B. Schulze and I. Lasiecka, reads as follows (see the body of the paper for precise definitions of all spaces involved and for the way traces are considered).

Theorem. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded creased domain whose boundary is connected and decomposes into two pieces D and N. Then there exists $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ such that for each $(s, 1/p) \in \mathcal{H}_\varepsilon$ the Poisson mixed boundary value problem*

$$(1.5)\qquad \begin{cases} u \in L^p_{s+\frac{1}{p}}(\Omega), \\ \Delta u = F\,|_{\Omega}, \quad F \in \Big(L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*, \\ u\,|_{D} = g \in B^{p,p}_s(D), \\ \partial_\nu u\,|_{N} = f \in B^{p,p}_{s-1}(N), \end{cases}$$

is well-posed whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$ (here, as before, $1/p + 1/p' = 1$). Furthermore, a similar result is valid for

$$(1.6)\qquad \begin{cases} u \in B^{p,p}_{s+\frac{1}{p}}(\Omega), \\ \Delta u = F\,|_{\Omega}, \quad F \in \Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*, \\ u\,|_{D} = g \in B^{p,p}_s(D), \\ \partial_\nu u\,|_{N} = f \in B^{p,p}_{s-1}(N). \end{cases}$$

This theorem (which is sharp in the class of creased domains) unifies and extends the main results in [25] and [16] (which correspond to $N = \emptyset$ and $D = \emptyset$, respectively). Furthermore, the main result in [53] can be viewed as a natural endpoint case of the above theorem (corresponding to $F = 0$ and $s = 1$). Another well-known case corresponds to taking $s = 1/p = 1/2$, in which case the classical Lax-Milgram lemma (cf., e.g., [26] or §7 in [37]) applies. Incidentally, the point $(1/2, 1/2)$ is the center of the hexagonal region $\mathcal{H}_\varepsilon$ and, hence, our main theorem can be regarded as a far-reaching generalization of the variational Lax-Milgram result for the Zaremba problem. The limitation to three or more space dimensions is inherited from [53], and the problem slightly changes its character for domains in $\mathbb{R}^2$. This latter case will be treated separately in [39].

The fact that we work in the class of creased domains means that all smooth domains fall outside the scope of our main theorem. Yet, even when $\partial\Omega \in C^\infty$ we do not expect the problems (1.5)-(1.6) to be well-posed for all $(s, 1/p) \in (0,1) \times (0,1)$, given the rough nature of the boundary conditions. Indeed, (1.5)-(1.6) fit the model

$$\left\{\begin{array}{l} \Delta u = F \quad \text{in } \Omega, \\ a\,u \,+\, b\,\partial_\nu u \text{ prescribed on } \partial\Omega, \end{array}\right. \tag{1.7}$$

where the (variable) coefficients $a := \chi_D$ and $b := \chi_N$ satisfy the ellipticity condition $a^2 + b^2 \neq 0$ on $\partial\Omega$ but otherwise are very rough. For example, when $\partial\Omega \in C^\infty$ and $s = 1 - 1/p$, the natural limitations on p are $4/3 < p < 4$ (cf. [40]). That this is in the nature of best possible can be seen by analyzing the standard pathological example, offered by (a suitable truncation of) the harmonic function $u(x,y) := \mathrm{Im}\,(x+iy)^{1/2}$, $(x,y) \in \mathbb{R}^2_+$, which satisfies homogeneous Dirichlet (Neumann) boundary conditions on the positive (negative) real semiaxis. Cf. also [49] and the example discussed in §4 of [33].

The organization of the paper is as follows. In Section 2 we collect notation and basic definitions, while in Section 3 we review the Hardy, Besov and Sobolev spaces on $\mathbb{R}^n$, on a domain $\Omega \subset \mathbb{R}^n$ and on its boundary $\partial\Omega$. In Section 4 we introduce certain function spaces which are well adapted to mixed problems on Lipschitz domains and discuss their functional analytic properties (such as interpolation, characterizations of dual spaces and density results). This analysis is further developed in Section 5 where traces and extension operators are studied in detail. In Section 6 we briefly recall some abstract tools from functional analysis which are useful in the present context, such as Banach envelopes of non-locally-convex Hardy and Besov spaces and the stability of certain properties of linear operators acting on complex interpolation scales of quasi-Banach spaces. The Neumann-to-Dirichlet operator is introduced in Section 7, and we derive optimal invertibility results for this operator on diagonal Besov space on creased domains. In Section 8 we present our main result, dealing with the well-posedness of the Poisson problem for the Laplacian with mixed boundary conditions on Sobolev-Besov spaces on creased domains. Here we also establish a singular integral operator representation of the solution. Finally, in Section 9, we discuss several applications of this result to Helmholtz-type decompositions (extending work from [16]) and the regularity of Green operators (generalizing B. Dahlberg's $L^p - L^q$ estimates from [9]).

2. Lipschitz and creased domains

Given a metric space $\mathcal{X}$, call a function $f : \mathcal{X} \to \mathbb{C}$ *Lipschitz* provided there exists $M > 0$ such that $|f(x) - f(y)| \leq M\,\mathrm{dist}\,(x,y)$ for each $x, y \in \mathcal{X}$. The class of Lipschitz functions on $\mathcal{X}$ is denoted by $\mathrm{Lip}\,(\mathcal{X})$. We also write $\mathrm{Lip}_c(\mathcal{X})$ for the collection of all Lipschitz functions defined on $\mathcal{X}$ which have compact support.

When $\mathcal{X}$ is a Lipschitz hypersurface in $\mathbb{R}^n$, we shall occasionally refer to $\mathrm{Lip}\,(\mathcal{X})$ as *test* functions on $\mathcal{X}$ and to $\Big(\mathrm{Lip}\,(\mathcal{X})\Big)^*$ as the set of *distributions* on $\mathcal{X}$ (hereafter, we let the superscript asterisk denote the topological dual).

Definition 2.1. A bounded domain $\Omega \subset \mathbb{R}^n$ is called Lipschitz if for any $x_0 \in \partial\Omega$ there exist $r, h > 0$ and a coordinate system $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^n$ (isometric to the canonical one) with origin at x_0 along with a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ which is Lipschitz and so that the following holds. If $C(r,h)$ denotes the cylinder $\{(x_1,\dots,x_n) : |x_j| < r \text{ all } j\} \times (0,h) \subset \mathbb{R}^n$, then

(2.1)

$$\Omega \cap C(r,h) = \{X = (x_1,\dots,x_n) : |x_j| < r \text{ all } j \text{ and } x_n > \varphi(x_1,\dots,x_{n-1})\},$$

$$\partial\Omega \cap C(r,h) = \{X = (x_1,\dots,x_n) : |x_j| < r \text{ all } j \text{ and } x_n = \varphi(x_1,\dots,x_{n-1})\}.$$

For each point $x = (x_1,x_2,...,x_n)$ in $\mathbb{R}^n$, we set $x' := (x_1,x_2,...,x_{n-1})$ and $x'' := (x_2,...,x_{n-1})$. In particular, $x' = (x_1,x'')$ and $x = (x',x_n)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. An open set $\Sigma \subseteq \partial\Omega$ is called an *admissible patch* if for every $x_o \in \partial\Sigma$ there exists a new system of orthogonal axes, obtained from the original one via a rigid motion such that x_o is the origin in this system of coordinates and such that the following holds. There exists a cube $Q = Q_1 \times Q_2 \times \cdots \times Q_n \subset \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ centered at 0 and two Lipschitz functions

$$\begin{aligned} &\varphi : Q' := Q_1 \times \cdots \times Q_{n-1} \longrightarrow Q_n, \qquad \varphi(0) = 0,\\ &\psi : Q'' := Q_2 \times \cdots \times Q_{n-1} \longrightarrow Q_1, \qquad \psi(0) = 0, \end{aligned} \tag{2.2}$$

satisfying

$$\begin{aligned} &\Sigma \cap Q = \big\{(x',\varphi(x')) : x' \in Q' \text{ and } \psi(x'') < x_1\big\},\\ &\Big(\partial\Omega \setminus \bar\Sigma\Big) \cap Q = \big\{(x',\varphi(x')) : x' \in Q' \text{ and } \psi(x'') > x_1\big\},\\ &\partial\Sigma \cap Q = \Big\{\big(\psi(x''),x'',\varphi(\psi(x''),x'')\big) : x'' \in Q''\Big\}. \end{aligned} \tag{2.3}$$

It follows that $\partial\Omega \setminus \bar\Sigma$ is an admissible patch whenever Σ is.

Next, recall that the non-tangential maximal operator acting on an arbitrary function $u : \Omega \to \mathbb{R}$ is given at each boundary point x by

$$u^*(x) := \sup\{|u(y)| : y \in \gamma(x)\}. \tag{2.4}$$

Here, for a fixed, sufficiently large constant $\kappa > 1$, the non-tangential approach region corresponding to $x \in \partial\Omega$ is defined by

$$\gamma(x) := \{y \in \Omega : |x-y| < \kappa\,\mathrm{dist}\,(y,\partial\Omega)\}. \tag{2.5}$$

Definition 2.2. Let Ω be a special Lipschitz domain $\mathbb{R}^n$ and suppose that $D, N \subset \partial\Omega$ are two non-empty, disjoint admissible patches satisfying $\bar D \cap \bar N = \partial D = \partial N$ and $\bar D \cup \bar N = \partial\Omega$. The domain Ω is called creased provided that the following hold:

(i) There exists a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with the property that $\Omega = \{(x',x_n) \in \mathbb{R}^n : x_n > \phi(x')\}$.

(ii) There exists a Lipschitz function $\psi : \mathbb{R}^{n-2} \to \mathbb{R}$ such that

$$N = \{(x_1,x'',x_n) \in \mathbb{R}^n : x_1 > \psi(x'')\} \cap \partial\Omega$$

and

$$D = \{(x_1,x'',x_n) : x_1 < \psi(x'')\} \cap \partial\Omega.$$

(iii) There exist $\delta_D, \delta_N \ge 0$ with $\delta_D + \delta_N > 0$ such that

$$\frac{\partial\phi}{\partial x_1} \ge \delta_N \quad \text{almost everywhere on} \quad \{(x_1,x'',x_n) \in \mathbb{R}^n : x_1 > \psi(x'')\}$$

and

$$\frac{\partial\phi}{\partial x_1} \le -\delta_D \quad \text{almost everywhere on} \quad \{(x_1,x'',x_n) \in \mathbb{R}^n : x_1 < \psi(x'')\}.$$

Definition 2.3. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with connected boundary and suppose that $D, N \subset \partial\Omega$ are two non-empty, disjoint admissible patches satisfying $\bar{D} \cap \bar{N} = \partial D = \partial N$ and $\bar{D} \cup \bar{N} = \partial\Omega$. The domain Ω is called creased provided the following hold:

(i) There exist $P_i \in \partial\Omega$, $i = 1, \ldots, M$ and $r > 0$ such that $\partial\Omega \subset \bigcup_{i=1}^{M} B_r(P_i)$.

(ii) For each $i = 1, \ldots, M$ there exist a coordinate system $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^n$ with origin at P_i and a Lipschitz function $\phi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that the set $\Omega_i := \{(x', x_n) \in \mathbb{R}^n : x_n > \phi_i(x')\}$, with boundary decomposition $\partial\Omega_i = N_i \cup D_i$, is a special creased Lipschitz domain in the sense of Definition 2.2, and

$$
\begin{aligned}
\Omega \cup B_{2r}(P_i) &= \Omega_i \cup B_{2r}(P_i), \\
D \cup B_{2r}(P_i) &= D_i \cup B_{2r}(P_i), \\
N \cup B_{2r}(P_i) &= N_i \cup B_{2r}(P_i).
\end{aligned} \tag{2.6}
$$

3. Review of function spaces on Lipschitz domains

In this section we introduce the function spaces relevant for the exposition. We start by recalling the Lebesgue, Sobolev, Besov and atomic Hardy spaces of $\mathbb{R}^n$, Ω and $\partial\Omega$.

For $1 < p < \infty$, we denote by $L^p(\mathbb{R}^n)$ the space of p-th power integrable functions in $\mathbb{R}^n$ and for each $1 < p < \infty$ and $\alpha \in \mathbb{R}$ denote by $L^p_\alpha(\mathbb{R}^n)$ the classical L^p-based Sobolev spaces with smoothness α in $\mathbb{R}^n$. That is,

$$
L^p_\alpha(\mathbb{R}^n) = (I - \Delta)^{-\alpha/2} L^p(\mathbb{R}^n). \tag{3.1}
$$

Going further, we let $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$ stand, respectively, for the classical Besov and Triebel-Lizorkin spaces in $\mathbb{R}^n$ (see, for example, [52], [44]).

As is well known, $F^{p,2}_\alpha(\mathbb{R}^n) = L^p_\alpha(\mathbb{R}^n)$ whenever $1 < p < \infty$, $\alpha \in \mathbb{R}$, and $F^{p,2}_0(\mathbb{R}^n) = h^p(\mathbb{R}^n)$, the local Hardy space discussed in [21], if $0 < p \leq 1$. We shall also use a special notation, namely $h^{1,p}(\mathbb{R}^n)$, for the Triebel-Lizorkin space $F^{p,2}_1(\mathbb{R}^n)$ whenever $0 < p \leq 1$, and view this as a Hardy-based Sobolev space of smoothness one.

Then, for each open set Ω in $\mathbb{R}^n$, set

$$
\begin{aligned}
& L^p_\alpha(\Omega) := \{f|_\Omega : f \in L^p_\alpha(\mathbb{R}^n)\}, \quad 1 < p < \infty,\ \alpha \in \mathbb{R}, \\
& B^{p,q}_\alpha(\Omega) := \{f|_\Omega : f \in B^{p,q}_\alpha(\mathbb{R}^n)\}, \quad 1 \leq p, q \leq \infty,\ \alpha \in \mathbb{R}, \\
& h^p(\Omega) := \{f|_\Omega : f \in F^{p,2}_0(\mathbb{R}^n)\}, \quad 0 < p \leq 1, \\
& h^{1,p}(\Omega) := \{f|_\Omega : f \in F^{p,2}_1(\mathbb{R}^n)\}, \quad 0 < p \leq 1.
\end{aligned} \tag{3.2}
$$

It is proved in [45] that there exists a linear, continuous operator E_Ω mapping distributions in Ω into tempered distributions in $\mathbb{R}^n$ such that

$$
\begin{aligned}
& E_\Omega : L^p_\alpha(\Omega) \longrightarrow L^p_\alpha(\mathbb{R}^n), \quad 1 < p < \infty,\ \alpha \in \mathbb{R}, \\
& E_\Omega : B^{p,q}_\alpha(\Omega) \longrightarrow B^{p,q}_\alpha(\mathbb{R}^n), \quad 1 \leq p, q \leq \infty,\ \alpha \in \mathbb{R}, \\
& E_\Omega : h^p(\Omega) \longrightarrow F^{p,2}_0(\mathbb{R}^n), \quad 0 < p \leq 1, \\
& E_\Omega : h^{1,p}(\Omega) \longrightarrow F^{p,2}_1(\mathbb{R}^n), \quad 0 < p \leq 1,
\end{aligned} \tag{3.3}
$$

boundedly and which satisfies $R_\Omega \circ E_\Omega = I$ in each case (here and elsewhere R_Ω is the restriction to Ω and I denotes the identity operator). This result builds on the earlier work of many people, including A.P. Calderón [8], E.M. Stein [52], G.A. Kalyabin [30], among others.

Assume that $1 < p < \infty$ and $0 \leq s \leq 1$. Then the Sobolev spaces $L^p_s(\mathbb{R}^{n-1})$ are invariant under pointwise multiplication by Lipschitz maps and are stable under composition by Lipschitz diffeomorphisms. This allows us to lift this scale on the boundaries of Lipschitz domains as follows. First, whenever Ω is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ we define

$$f \in L^p_s(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot)) \in L^p_s(\mathbb{R}^{n-1}). \tag{3.4}$$

Regarding Sobolev spaces with a negative amount of smoothness, we agree that

$$f \in L^p_{-s}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in L^p_{-s}(\mathbb{R}^{n-1}), \tag{3.5}$$

whenever $1 < p < \infty$, $0 \leq s \leq 1$.

These definitions then readily extend to the case of *bounded* Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. In this scenario, it is not difficult to check that

$$L^p_1(\partial\Omega) = \{f \in L^p(\partial\Omega) : \nabla_{tan} f \in L^p(\partial\Omega)\}, \qquad 1 < p < \infty, \tag{3.6}$$

where ∇_{tan} is the tangential gradient on $\partial\Omega$, and

$$L^p_{-s}(\partial\Omega) = \Big(L^{p'}_s(\partial\Omega)\Big)^*, \qquad 1 < p < \infty,\ 0 \leq s \leq 1, \tag{3.7}$$

where $1/p + 1/p' = 1$ and, given a Banach space $\mathcal{X}$, $\mathcal{X}^*$ denotes its dual.

The case of Besov spaces is handled similarly. More specifically, the analogues of (3.4)-(3.5) are

$$f \in B^{p,q}_s(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot)) \in B^{p,q}_s(\mathbb{R}^{n-1}), \qquad 1 \leq p, q \leq \infty,\ \ 0 < s < 1, \tag{3.8}$$

and, for $1 \leq p, q \leq \infty$, $0 < s < 1$,

$$f \in B^{p,q}_{-s}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1 + |\nabla\varphi(\cdot)|^2} \in B^{p,q}_{-s}(\mathbb{R}^{n-1}). \tag{3.9}$$

Once again, these definitions are adapted to the case of a bounded Lipschitz domain via a standard patching argument, involving a smooth partition of unity.

Note that $B^{\infty,\infty}_s(\partial\Omega)$ coincides with $C^s(\partial\Omega)$, the space of Hölder continuous functions of order s on $\partial\Omega$ and that

$$B^{p,q}_{-s}(\partial\Omega) = \Big(B^{p',q'}_s(\partial\Omega)\Big)^*, \tag{3.10}$$

for each $0 < s < 1$, $1 < p, q \leq \infty$ (again, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$).

Throughout the paper, we let $[\cdot\,,\,\cdot]_\theta$ and $(\cdot\,,\,\cdot)_{\theta,q}$ stand for the standard *complex* and *real* interpolation brackets. For definitions and basic properties we refer the reader to [4], [54], [44], [3]. Let Ω be an arbitrary, bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $1 < p_0, p_1 < \infty$, $0 \leq s_0, s_1 \leq 1$. Then

$$\begin{aligned} &[L^{p_0}_{s_0}(\partial\Omega), L^{p_1}_{s_1}(\partial\Omega)]_\theta = L^p_s(\partial\Omega), \\ &[L^{p_0}_{-s_0}(\partial\Omega), L^{p_1}_{-s_1}(\partial\Omega)]_\theta = L^p_{-s}(\partial\Omega), \end{aligned} \tag{3.11}$$

where $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$ and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$. Furthermore, if $s_0\neq s_1$ and $1<q\leq\infty$, then

$$\begin{aligned}&(L^p_{s_0}(\partial\Omega),L^p_{s_1}(\partial\Omega))_{\theta,q}=B^{p,q}_s(\partial\Omega),\\&(L^p_{-s_0}(\partial\Omega),L^p_{-s_1}(\partial\Omega))_{\theta,q}=B^{p,q}_{-s}(\partial\Omega).\end{aligned}\tag{3.12}$$

Next, assume that $1\leq p,q_0,q_1,q\leq\infty$ and $0<s_0\neq s_1<1$. Then, with $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$,

$$\begin{aligned}&(B^{p,q_0}_{s_0}(\partial\Omega),B^{p,q_1}_{s_1}(\partial\Omega))_{\theta,q}=B^{p,q}_s(\partial\Omega),\\&(B^{p,q_0}_{-s_0}(\partial\Omega),B^{p,q_1}_{-s_1}(\partial\Omega))_{\theta,q}=B^{p,q}_{-s}(\partial\Omega).\end{aligned}\tag{3.13}$$

Furthermore, if $1\leq p_0,q_0,p_1,q_1\leq\infty$ and $0<s_0\neq s_1<1$, then

$$\begin{aligned}&[B^{p_0,q_0}_{s_0}(\partial\Omega),B^{p_1,q_1}_{s_1}(\partial\Omega)]_{\theta}=B^{p,q}_s(\partial\Omega),\\&[B^{p_0,q_0}_{-s_0}(\partial\Omega),B^{p_1,q_1}_{-s_1}(\partial\Omega)]_{\theta}=B^{p,q}_{-s}(\partial\Omega),\end{aligned}\tag{3.14}$$

where $0<\theta<1$, $s:=(1-\theta)s_0+\theta s_1$, $\frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $\frac{1}{q}:=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$.

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p,s satisfy $1\leq p,q\leq\infty$ and $0<s<1$. Then the restriction to the boundary extends to a linear bounded operator

$$\mathrm{Tr}:B^{p,q}_{s+\frac{1}{p}}(\Omega)\longrightarrow B^{p,q}_s(\partial\Omega).\tag{3.15}$$

For this range of indices, Tr is onto and has a bounded right inverse

$$\mathrm{Ext}:B^{p,q}_s(\partial\Omega)\longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega).\tag{3.16}$$

Similar considerations hold for

$$\mathrm{Tr}:L^p_{s+\frac{1}{p}}(\Omega)\longrightarrow B^{p,p}_s(\partial\Omega)\tag{3.17}$$

whenever $1<p<\infty$, $0<s<1$. In this situation, there exists a linear, bounded right inverse

$$\mathrm{Ext}:B^{p,p}_s(\partial\Omega)\longrightarrow L^p_{s+\frac{1}{p}}(\Omega).\tag{3.18}$$

For the case of smooth domains see [18] and [54] (where one can also find references to the literature dealing with the upper half-space). Adaptations to Lipschitz domains are in [28]; see also the discussion in [24]. A new proof and an extension of these trace results has been recently worked out in [35].

In closing, we note that Hardy and Hardy-Sobolev spaces on the boundary of a Lipschitz domain can also be introduced using the same technology as before. More specifically, if $\frac{n-1}{n}<p\leq 1$ and Ω is the unbounded region in $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$, we define the Hardy space $h^p(\partial\Omega)$ by requiring

$$f\in h^p(\partial\Omega)\Longleftrightarrow f(\cdot,\varphi(\cdot))\sqrt{1+|\nabla\varphi(\cdot)|^2}\in h^p(\mathbb{R}^{n-1}),\tag{3.19}$$

where $h^p(\mathbb{R}^{n-1})$ is the local Hardy space in the $(n-1)$-dimensional space introduced by D. Goldberg in [21]. As is well known, $h^p(\mathbb{R}^{n-1})=F^{p,2}_0(\mathbb{R}^{n-1})$. Accordingly, this definition can be adapted to the case of bounded Lipschitz domains in $\mathbb{R}^n$ by proceeding as before. In a similar fashion, the Hardy-Sobolev space (of order one) is defined by

$$f\in h^{1,p}(\partial\Omega)\Longleftrightarrow f(\cdot,\varphi(\cdot))\in F^{p,2}_1(\mathbb{R}^{n-1}).\tag{3.20}$$

4. Function spaces adapted to mixed problems: I

Here we discuss the versions of the spaces of distributions on the boundaries of Lipschitz domains, defined in the previous section, which are going to be relevant in the case of mixed boundary value problems.

We debut with a few notational conventions. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider an open subset Σ of $\partial\Omega$. If $\mathcal{X}(\partial\Omega)$ is a space of functions defined on $\partial\Omega$ we let

$$\begin{aligned} &\mathcal{X}(\Sigma) := \{f|_\Sigma : f \in \mathcal{X}(\partial\Omega)\}, \\ &\mathcal{X}_0(\Sigma) := \left\{f \in \mathcal{X}(\partial\Omega) : \operatorname{supp} f \subseteq \bar{\Sigma}\right\}. \end{aligned} \tag{4.1}$$

In our exposition the role of $\mathcal{X}(\partial\Omega)$ will be played by the boundary function spaces introduced in (3.4), (3.5), (3.19), (3.20). In particular, this defines $B^{p,q}_s(\Sigma)$, $B^{p,q}_{s,0}(\Sigma)$, $L^p_s(\Sigma)$, $L^p_{s,0}(\Sigma)$, $h^p(\Sigma)$, $h^p_0(\Sigma)$, $h^{1,p}(\Sigma)$ and $h^{1,p}_0(\Sigma)$.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $\Sigma \subset \partial\Omega$ is an admissible patch. Then there exists a linear operator* Ext_Σ *which acts in each of the following cases:*

$$\mathrm{Ext}_\Sigma : L^p_s(\Sigma) \longrightarrow L^p_s(\partial\Omega), \quad 1 < p < \infty,\ |s| \leq 1, \tag{4.2}$$

$$\mathrm{Ext}_\Sigma : B^{p,q}_s(\Sigma) \longrightarrow B^{p,q}_s(\partial\Omega), \quad 1 \leq p, q \leq \infty,\ 0 < |s| < 1, \tag{4.3}$$

$$\mathrm{Ext}_\Sigma : h^p(\Sigma) \longrightarrow h^p(\partial\Omega), \quad \frac{n-1}{n} < p \leq 1, \tag{4.4}$$

$$\mathrm{Ext}_\Sigma : h^{1,p}(\Sigma) \longrightarrow h^{1,p}(\partial\Omega), \quad \frac{n-1}{n} < p \leq 1, \tag{4.5}$$

in a bounded fashion and such that $R_\Sigma \circ \mathrm{Ext}_\Sigma = \mathrm{I}$, the identity operator, where R_Σ is the operator of restriction to Σ.

Proof. The problem localizes, so it suffices to work in a small neighborhood of a point $x_o \in \partial\Sigma$. Based on the definitions of the admissible patch and of the smoothness spaces involved, matters can be further reduced to the case when $\partial\Omega$ is flat and Σ is a (piece of a) Lipschitz graph. In this latter scenario, a Rychkov-type extension operator (much as the one displayed on the second line of (3.3)) does the job. The operator Ext_Σ we are after is ultimately obtained by pulling back to $\partial\Omega$ and patching together such local extension operators (via a smooth partition of unity). $\square$

Recall that a family of spaces $\mathcal{X} = \{X_\omega\}_{\omega\in\mathcal{O}}$, where $\mathcal{O}$ is an open, convex subset of a linear space, is called a complex interpolation scale if for any $\theta \in (0,1)$ and $\omega_0, \omega_1 \in \mathcal{O}$,

$$[X_{\omega_0}, X_{\omega_1}]_\theta = X_\omega, \qquad \omega := (1-\theta)\omega_0 + \theta\omega_1. \tag{4.6}$$

Real interpolation scales are introduced similarly. Simple examples of complex interpolation scales are constructed starting with a compatible couple of spaces X_0, X_1 and then defining $\mathcal{X} := \{X_\theta\}_{\theta\in(0,1)}$, where $X_\theta := [X_0, X_1]_\theta$ for each $\theta \in (0,1)$.

Next, suppose that X_i, Y_i, $i = 0, 1$, are two compatible couple of spaces for which there exist morphisms $\mathcal{I} : X_i \to Y_i$ and $\mathcal{P} : Y_i \to X_i$ such that $\mathcal{P} \circ \mathcal{I}$ is the identity on the scale X_i for $i = 0, 1$. Then

$$\mathcal{P}([Y_0, Y_1]_\theta) = [X_0, X_1]_\theta, \qquad \forall\, \theta \in (0,1), \tag{4.7}$$

and we shall call $\mathcal{X} := \{[X_0, X_1]_\theta\}_{\theta\in(0,1)}$ a retract of $\mathcal{Y} := \{[Y_0, Y_1]_\theta\}_{\theta\in(0,1)}$. Similar considerations apply in connection with the real method of interpolation.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $\Sigma \subset \partial\Omega$ is an admissible patch. Then for any $1 < p_0, p_1 < \infty$ we have*

$$[L^{p_0}_{s_0}(\Sigma), L^{p_1}_{s_1}(\Sigma)]_\theta = L^p_s(\Sigma), \quad [L^{p_0}_{s_0,0}(\Sigma), L^{p_1}_{s_1,0}(\Sigma)]_\theta = L^p_{s,0}(\Sigma), \tag{4.8}$$

where $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$, and either $0 \le s_0, s_1 \le 1$, or $-1 \le s_0, s_1 \le 0$. Moreover,

$$(L^p_{s_0}(\Sigma), L^p_{s_1}(\Sigma))_{\theta,q} = B^{p,q}_s(\Sigma), \quad (L^p_{s_0,0}(\Sigma), L^p_{s_1,0}(\Sigma))_{\theta,q} = B^{p,q}_{s,0}(\Sigma), \tag{4.9}$$

if, in addition, $s_0 \neq s_1$ and $1 < q \le \infty$.

Similar results hold for Besov spaces as well. More concretely, assume that $1 \le p, q_0, q_1, q \le \infty$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. Then

$$(B^{p,q_0}_{s_0}(\Sigma), B^{p,q_1}_{s_1}(\Sigma))_{\theta,q} = B^{p,q}_s(\Sigma), \quad (B^{p,q_0}_{s_0,0}(\Sigma), B^{p,q_1}_{s_1,0}(\Sigma))_{\theta,q} = B^{p,q}_{s,0}(\Sigma) \tag{4.10}$$

if either $0 < s_0 \neq s_1 < 1$ or $-1 < s_0 \neq s_1 < 0$. Also, if $1 \le p_0, q_0, p_1, q_1 \le \infty$, $0 < \theta < 1$, and $s := (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then

$$[B^{p_0,q_0}_{s_0}(\Sigma), B^{p_1,q_1}_{s_1}(\Sigma)]_\theta = B^{p,q}_s(\Sigma), \quad [B^{p_0,q_0}_{s_0,0}(\Sigma), B^{p_1,q_1}_{s_1,0}(\Sigma)]_\theta = B^{p,q}_{s,0}(\Sigma), \tag{4.11}$$

provided either $0 < s_0 \neq s_1 < 1$ or $-1 < s_0 \neq s_1 < 0$.

Proof. The idea is to show that the scales $L^p_s(\Sigma)$ and $L^p_{s,0}(\Sigma)$ are retracts of $L^p_s(\partial\Omega)$, and that $B^{p,q}_s(\Sigma)$ and $B^{p,q}_{s,0}(\Sigma)$, are retracts of $B^{p,q}_s(\partial\Omega)$. Then the desired conclusions follow from (3.13)-(3.14) and the discussion preceding the statement of the proposition.

Indeed, for the spaces of distributions on Σ, this follows directly from Proposition 4.1. As for the remaining spaces, we consider the operator

$$P_\Sigma := I - \Big(\mathrm{Ext}_{\partial\Omega\setminus\bar\Sigma}\Big) \circ \Big(R_{\partial\Omega\setminus\bar\Sigma}\Big), \tag{4.12}$$

where $\mathrm{Ext}_{\partial\Omega\setminus\bar\Sigma}$ is the extension operator from Proposition 4.1 (for $\partial\Omega \setminus \bar\Sigma$ in place of Σ) and $R_{\partial\Omega\setminus\bar\Sigma}$ is the operator of restriction to $\partial\Omega\setminus\bar\Sigma$. It is straightforward to check that, for every $1 \le p, q \le \infty$ and $0 < s < 1$,

$$P_\Sigma\,(L^p_s(\partial\Omega)) = L^p_{s,0}(\Sigma) \quad \text{and} \quad P_\Sigma \circ \iota = I \quad \text{on} \quad L^p_{s,0}(\Sigma), \tag{4.13}$$

where ι denotes the inclusion of $L^p_{s,0}(\Sigma)$ into $L^p_s(\partial\Omega)$. Consequently, the scale $L^p_{s,0}(\Sigma)$ is a retract of $L^p_s(\partial\Omega)$. A similar reasoning works for $B^{p,q}_{s,0}(\Sigma)$, and this concludes the proof of the proposition. □

Proposition 4.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Sigma \subseteq \partial\Omega$ be an admissible patch. Then for each $\frac{n-1}{n} < p < 1$,*

$$(h^p_0(\Sigma))^* \quad \textit{is isomorphic to} \quad B^{\infty,\infty}_s(\Sigma), \quad \textit{where} \quad s = (n-1)\Big(\frac{1}{p} - 1\Big). \tag{4.14}$$

Proof. Consider $\Lambda \in (h^p_0(\Sigma))^*$ and recall the linear and continuous operator P_Σ from (4.12). Then, since $P_\Sigma : h^p(\partial\Omega) \to h^p_0(\Sigma)$, it follows that $\Lambda \circ P_\Sigma \in (h^p(\partial\Omega))^* = B^{\infty,\infty}_s(\partial\Omega)$. Thus, there exists a unique element $\phi_\Lambda \in B^{\infty,\infty}_s(\partial\Omega)$ such that

$$\Lambda\left(P_\Sigma(\psi)\right)) = \langle \phi_\Lambda, \psi\rangle, \quad \forall\, \psi \in h^p(\partial\Omega). \tag{4.15}$$

Throughout the paper, $\langle\cdot,\cdot\rangle$ will denote various duality pairings, and the nature of the spaces in question should be clear from the context. For example, in (4.15), $\langle\cdot,\cdot\rangle$ stands for the pairing of elements from $h^p(\partial\Omega)$ with elements from its dual space, $B^{\infty,\infty}_s(\partial\Omega)$. Next, define

$$\Phi : (h^p_0(\Sigma))^* \longrightarrow B^{\infty,\infty}_s(\Sigma), \quad \Phi(\Lambda) := \phi_\Lambda\Big|_\Sigma, \tag{4.16}$$

where ϕ_Λ is as in (4.15), and consider

$$\Psi : B^{\infty,\infty}_s(\Sigma) \longrightarrow (h^p_0(\Sigma))^*, \quad (\Psi(f))\,(\xi) := \langle F, \xi\rangle, \quad \forall\, \xi \in h^p_0(\Sigma), \tag{4.17}$$

where $F \in B^{\infty,\infty}_s(\partial\Omega)$ is such that $F\Big|_\Sigma = f$.

We now remark that

$$\{\psi \in \mathrm{Lip}(\partial\Omega) : \operatorname{supp}\psi \subset \Sigma\} \hookrightarrow h^p_0(\Sigma) \quad \text{densely, for} \quad \frac{n-1}{n} < p \le 1. \tag{4.18}$$

The proof proceeds via a localization argument, and pulling back to the Euclidean model. Based on this, it follows that the mapping Ψ is well-defined, i.e., $\Psi(f)$ does not depend on the particular choice of an extension of $f \in B^{\infty,\infty}_s(\Sigma)$ to an element F in $B^{\infty,\infty}_s(\partial\Omega)$. Indeed, if $F_1, F_2 \in B^{\infty,\infty}_s(\partial\Omega)$ satisfy $F_1\Big|_\Sigma = F_2\Big|_\Sigma = f$, then $\psi_j \to \xi$ in $h^p_0(\Sigma)$ as $j \to \infty$ and $\operatorname{supp}(F_1 - F_2) \subseteq \partial\Omega\setminus\bar{\Sigma}$. Thus, given $\xi \in h^p_0(\Sigma)$ and choosing $\psi_j \in \mathrm{Lip}(\partial\Omega)$ with $\operatorname{supp}\psi_j \subset \Sigma$, we may write $\langle F_1 - F_2, \xi\rangle = \lim_{j\to\infty}\langle F_1 - F_2, \psi_j\rangle = 0$, due to simple support considerations. This finishes the proof of the fact that Ψ is well-defined.

The proof of (4.14) is finished as soon as we show that Φ and Ψ from (4.16)-(4.17) are inverse to each other. To this end, let $f \in B^{\infty,\infty}_s(\Sigma)$ and, according to (4.15), note that we have

$$\Phi(\Psi(f)) = \phi_{\Psi(f)}\Big|_\Sigma, \quad \text{where} \quad \langle \phi_{\Psi(f)}, \psi\rangle = (\Psi(f))\left(P_\Sigma(\psi)\right), \quad \forall\, \psi \in h^p(\partial\Omega). \tag{4.19}$$

On the other hand, using (4.17) we have that $(\Psi(f))\,(P_\Sigma(\psi)) = \langle \mathrm{Ext}_\Sigma f, P_\Sigma(\psi)\rangle$. In particular, when $\psi \in h^p_0(\Sigma)$ and, therefore, $P_\Sigma(\psi) = \psi$ it follows based on (4.19) that $\langle \phi_{\Psi(f)}, \psi\rangle = \langle \mathrm{Ext}_\Sigma f, \psi\rangle$. Note that the "test" functions in Σ, i.e., functions $\varphi \in \mathrm{Lip}\,(\Sigma)$ with compact support, satisfy $\tilde{\varphi} \in h^p_0(\Sigma)$, where tilde denotes the extension by zero to the whole $\partial\Omega$. Consequently, in the sense of distributions, $\phi_{\Psi(f)}\Big|_\Sigma = \mathrm{Ext}_\Sigma f\Big|_\Sigma = f$ and this shows that $\Phi \circ \Psi = I$ on $B^{\infty,\infty}_s(\Sigma)$.

It remains to consider $\Psi \circ \Phi$, for which we fix an arbitrary $\Lambda \in (h^p_0(\Sigma))^*$. Then, if $\xi \in h^p_0(\Sigma)$, we have $(\Psi \circ \Phi(\Lambda))\,(\xi) = \langle F, \xi\rangle$, where the sole requirement on the function $F \in B^{\infty,\infty}_s(\partial\Omega)$ is that $F\Big|_\Sigma = \Phi(\Lambda)$. In particular, we may choose $F = \phi_\Lambda$ (defined as in (4.15)) since, by design, $\phi_\Lambda\Big|_\Sigma = \Phi(\Lambda)$. Then

$$(\Psi \circ \Phi(\Lambda))\,(\xi) = \langle \phi_\Lambda, \xi\rangle = \Lambda(P_\Sigma(\xi)) = \Lambda(\xi), \tag{4.20}$$

where the second equality follows from the fact that $\xi \in h^p_0(\Sigma)$ and, hence, $P_\Sigma(\xi) = \xi$. This shows that $\Psi \circ \Phi = I$ on $(h^p_0(\Sigma))^*$ and finishes the proof. $\square$

For the applications that we have in mind the following variant of Proposition 4.3 is better suited.

Proposition 4.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Sigma \subseteq \partial\Omega$ be an admissible patch. Then for each $\frac{n-1}{n} < p < 1$ and $s = (n-1)\Big(\frac{1}{p} - 1\Big)$,*

$$\big(h^p_0(\Sigma) \cap \{1\}^\perp\big)^* \quad \textit{is isomorphic to} \quad B^{\infty,\infty}_s(\Sigma)\Big/\mathbb{R}. \tag{4.21}$$

Proof. This largely parallels the proof of Proposition 4.3 so we choose to emphasize only the novel points. Fix

$$\eta \in \mathrm{Lip}\,(\partial\Omega) \quad \text{with} \quad \mathrm{supp}\,\eta \subset \Sigma \quad \text{and} \quad \langle 1, \eta\rangle = 1 \tag{4.22}$$

and introduce the projection $\pi(f) := f - \langle f, 1\rangle\eta$.

If $\Lambda \in \big(h^p_0(\Sigma) \cap \{1\}^\perp\big)^*$ it follows that $\Lambda \circ \pi \circ P_\Sigma \in (h^p(\partial\Omega))^* = B^{\infty,\infty}_s(\partial\Omega)$ and, consequently, there exists a unique $\phi_\Lambda \in B^{\infty,\infty}_s(\partial\Omega)$ such that

$$\Lambda\left(\pi(P_\Sigma(\psi))\right) = \langle \phi_\Lambda, \psi\rangle, \quad \forall\, \psi \in h^p(\partial\Omega). \tag{4.23}$$

The analogue of (4.16) in this situation becomes

$$\Phi : \big(h^p_0(\Sigma) \cap \{1\}^\perp\big)^* \to B^{\infty,\infty}_s(\Sigma)\Big/\mathbb{R}, \quad \Phi(\Lambda) := \Big[\phi_\Lambda\Big|_\Sigma\Big], \tag{4.24}$$

where $[f]$ stands for "f modulo constants". Next, in place of (4.17) we take

$$\Psi : B^{\infty,\infty}_s(\Sigma)\Big/\mathbb{R} \longrightarrow \big(h^p_0(\Sigma) \cap \{1\}^\perp\big)^*, \;\; \Psi([f])(\xi) := \langle F, \xi\rangle, \tag{4.25}$$

for all $\xi \in h^p_0(\Sigma) \cap \{1\}^\perp$, where $F \in B^{\infty,\infty}_s(\partial\Omega)$ is such that $F\Big|_\Sigma$ and f differ by a constant.

Much as before, one can check that the mapping Ψ is well-defined and the goal is to show that the maps (4.24)-(4.25) are inverse to each other. Let us first prove that $\Phi \circ \Psi = I$ on $B^{\infty,\infty}_s(\Sigma)\Big/\mathbb{R}$. Since for each $f \in B^{\infty,\infty}_s(\Sigma)$ we have

$$\Phi(\Psi([f])) = \Big[\phi_{\Psi([f])}\Big|_\Sigma\Big] \text{ with } \langle \phi_{\Psi([f])}, \psi\rangle = \Psi([f])\left(\pi(P_\Sigma(\psi))\right), \; \forall\, \psi \in h^p(\partial\Omega), \tag{4.26}$$

the desired conclusion follows as soon as we prove that $\Big[\phi_{\Psi([f])}\Big|_\Sigma\Big] = [f]$, i.e. that $\phi_{\Psi([f])}\Big|_\Sigma = f + c$ for some constant $c \in \mathbb{R}$. Nonetheless, a straightforward calculation based on unraveling definitions shows that $c := -\langle f, \eta|_\Sigma\rangle$ will do.

The proof of the fact that $\Psi \circ \Phi = I$ on $\big(h^p_0(\Sigma) \cap \{1\}^\perp\big)^*$ closely mirrors that of the corresponding case in the proof of Proposition 4.3 and we omit it. $\square$

Proposition 4.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Sigma \subseteq \partial\Omega$ be an admissible patch. Then, for each $\frac{n-1}{n} < p < 1$ and $s = (n-1)\Big(\frac{1}{p} - 1\Big)$,*

$$\big(h^{1,p}(\Sigma)\big)^* \quad \textit{is isomorphic to} \quad B^{\infty,\infty}_{s-1,0}(\Sigma) \tag{4.27}$$

and

$$\Big(h^{1,p}(\Sigma)\Big/\mathbb{R}\Big)^* \quad \textit{is isomorphic to} \quad B^{\infty,\infty}_{s-1,0}(\Sigma) \cap \{1\}^\perp. \tag{4.28}$$

Proof. Consider (4.27) and, for starters, introduce the mapping

$$\tilde{\Phi} : \big(h^{1,p}(\Sigma)\big)^* \to B^{\infty,\infty}_{s-1,0}(\Sigma), \quad \tilde{\Phi}(\Lambda) := \Lambda \circ R_\Sigma, \quad \forall\, \Lambda \in \big(h^{1,p}(\Sigma)\big)^*. \tag{4.29}$$

To see that this is well-defined we note that $\big(h^{1,p}(\partial\Omega)\big)^* = B^{\infty,\infty}_{s-1}(\partial\Omega)$; cf. Section 6 for a discussion. Consequently, $\Lambda \circ R_\Sigma \in B^{\infty,\infty}_{s-1}(\partial\Omega)$ and $(\Lambda \circ R_\Sigma)\Big|_{\partial\Omega\setminus\bar{\Sigma}} = 0$, which are the attributes of distributions in $B^{\infty,\infty}_{s-1,0}(\Sigma)$. Going further, we also introduce

$$\tilde{\Psi} : B^{\infty,\infty}_{s-1,0}(\Sigma) \to \big(h^{1,p}(\Sigma)\big)^*, \quad \Big(\tilde{\Psi}(\xi)\Big)(f) := \langle \xi, F\rangle, \tag{4.30}$$

where $\xi \in B^{\infty,\infty}_{s-1,0}(\Sigma)$, $f \in h^{1,p}(\Sigma)$, and $F \in h^{1,p}(\partial\Omega)$ is chosen such that $F\Big|_\Sigma = f$. To show that the mapping (4.30) is well-defined let $F_1, F_2 \in h^{1,p}(\partial\Omega)$ be two extensions of $f \in h^{1,p}(\Sigma)$. Then $(F_1 - F_2)\Big|_\Sigma = 0$ so that $F_1 - F_2 \in h^{1,p}_0(\partial\Omega \setminus \bar{\Sigma})$. Much as before,

$$\{\psi \in \mathrm{Lip}(\partial\Omega) : \operatorname{supp}\psi \subset \partial\Omega \setminus \bar{\Sigma}\} \hookrightarrow h^{1,p}_0(\partial\Omega \setminus \bar{\Sigma}) \quad \text{densely, for} \quad \frac{n-1}{n} < p \leq 1, \tag{4.31}$$

and, therefore, $F_1 - F_2$ can be approximated in $h^{1,p}(\partial\Omega)$ by functions from $\mathrm{Lip}(\partial\Omega)$ supported in $\partial\Omega \setminus \bar{\Sigma}$. Since $\xi\Big|_{\partial\Omega\setminus\bar{\Sigma}} = 0$, it follows that $\langle F_1 - F_2, \xi\rangle = 0$ and, hence, $\tilde{\Psi}$ is well-defined.

Next, we aim to show that the mappings (4.29)-(4.30) are inverses to each other. Fix $\xi \in B^{\infty,\infty}_{s-1,0}(\Sigma)$ and $\psi \in B^{\infty,\infty}_{1-s}(\Sigma)$. Using (4.29)-(4.30) we may write

$$\langle \Big(\tilde{\Phi} \circ \tilde{\Psi}\Big)(\xi), \psi\rangle = \tilde{\Psi}(\xi) \circ R_\Sigma(\psi) = \langle \xi, \psi\rangle. \tag{4.32}$$

This shows that $\tilde{\Phi} \circ \tilde{\Psi} = I$ on $B^{\infty,\infty}_{s-1,0}(\Sigma)$. Next, for arbitrary $\Lambda \in \big(h^{1,p}(\Sigma)\big)^*$ and $\psi \in h^{1,p}(\Sigma)$ we have $\langle \Big(\tilde{\Psi} \circ \tilde{\Phi}\Big)(\Lambda), \psi\rangle = \langle \tilde{\Phi}(\Lambda), \mathrm{Ext}_\Sigma(\psi)\rangle = \langle \Lambda, \mathrm{Ext}_\Sigma(\psi)\Big|_\Sigma\rangle = \langle \Lambda, \psi\rangle$, since the operator Ext_Σ (introduced in (4.3)) satisfies $\mathrm{Ext}_\Sigma(\psi)|_\Sigma = \psi$. Thus, $\tilde{\Psi} \circ \tilde{\Phi} = I$ on $\big(h^{1,p}(\Sigma)\big)^*$, finishing the proof of (4.27).

The proof of (4.28) is a minor modification of the above argument. The main alterations consist of taking

$$\tilde{\Phi} : \Big(h^{1,p}(\Sigma)\Big/\mathbb{R}\Big)^* \to B^{\infty,\infty}_{s-1,0}(\Sigma) \cap \{1\}^\perp,\ \tilde{\Phi}(\Lambda) := \Lambda \circ [\cdot] \circ R_\Sigma,\ \forall\, \Lambda \in \Big(h^{1,p}(\Sigma)\Big/\mathbb{R}\Big)^* \tag{4.33}$$

in place of (4.29) where, as before, the bracket $[\cdot]$ stands for the operation of "modding out" constants and, in place of (4.30),

$$\tilde{\Psi} : B^{\infty,\infty}_{s-1,0}(\Sigma) \cap \{1\}^\perp \to \Big(h^{1,p}(\Sigma)\Big/\mathbb{R}\Big)^*, \quad \tilde{\Psi}(\xi)([f]) := \langle \xi, F\rangle, \tag{4.34}$$

where $\xi \in B^{\infty,\infty}_{s-1,0}(\Sigma)$, $f \in h^{1,p}(\Sigma)$, and $F \in h^{1,p}(\partial\Omega)$ is chosen such that $F\Big|_\Sigma$ differs from f by a constant. It is then straightforward to check that these mappings are well-defined and inverse to each other. $\square$

We conclude with a couple of observations which are going to be of importance later on.

Remark I. There are natural versions of the mappings (4.24)-(4.25) acting as isomorphisms of

$$
\begin{aligned}
&\Phi : \left(L^p_0(\Sigma)\cap\{1\}^\perp\right)^* \longrightarrow L^{p'}(\Sigma)\Big/\mathbb{R},\\
&\Psi : L^{p'}(\Sigma)\Big/\mathbb{R} \longrightarrow \left(L^p_0(\Sigma)\cap\{1\}^\perp\right)^*,
\end{aligned}
\tag{4.35}
$$

for each $1<p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$. These are defined in a similar manner to (4.24)-(4.25) and continue to be inverse to each other.

Furthermore, similar conclusions hold in the case of the mappings (4.29)-(4.30) acting, for each $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, as isomorphisms of

$$
\begin{aligned}
&\tilde{\Phi} : \left(L^p_1(\Sigma)\Big/\mathbb{R}\right)^* \longrightarrow L^{p'}_{-1,0}(\Sigma)\cap\{1\}^\perp,\\
&\tilde{\Psi} : L^{p'}_{-1,0}(\Sigma)\cap\{1\}^\perp \longrightarrow \left(L^p_1(\Sigma)\Big/\mathbb{R}\right)^*.
\end{aligned}
\tag{4.36}
$$

Remark II. There is an analogue of Proposition 4.3 valid for Besov spaces. More specifically, under the same geometric assumptions, a similar proof yields the following duality results:

$$
\begin{aligned}
&(B^{p,q}_s(\Sigma))^* \quad \text{is isomorphic to} \quad B^{p',q'}_{-s,0}(\Sigma),\\
&\left(B^{p,q}_{s,0}(\Sigma)\right)^* \quad \text{is isomorphic to} \quad B^{p',q'}_{-s}(\Sigma),
\end{aligned}
\tag{4.37}
$$

for each $1<p,q<\infty$ and $0<s<1$, where $1/p+1/p'=1$, $1/q+1/q'=1$.

Proposition 4.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $\Sigma\subseteq\partial\Omega$ be an admissible patch. Then the family of spaces defined as $h^p_0(\Sigma)\cap\{1\}^\perp$ whenever $\frac{n-1}{n}<p\le 1$, and $L^p_0(\Sigma)\cap\{1\}^\perp$ whenever $1<p<\infty$, is a complex interpolation scale.*

Furthermore, the family of spaces defined as $h^{1,p}(\Sigma)/\mathbb{R}$ for $\frac{n-1}{n}<p\le 1$, and $L^p_1(\Sigma)/\mathbb{R}$ for $1<p<\infty$, is also a complex interpolation scale.

Proof. If we consider the families of spaces $\{\mathcal{X}_p\}_{\frac{n-1}{n}<p<\infty}$ and $\{\mathcal{Y}_p\}_{\frac{n-1}{n}<p<\infty}$ given by

$$
\mathcal{X}_p := \begin{cases} h^p_0(\Sigma), & p\in(\frac{n-1}{n},1]\\ L^p_0(\Sigma), & 1<p<\infty,\end{cases} \quad \text{and } \mathcal{Y}_p(\Sigma) := \begin{cases} h^{1,p}(\Sigma), & p\in(\frac{n-1}{n},1]\\ L^p_1(\Sigma), & 1<p<\infty,\end{cases}
\tag{4.38}
$$

then the goal is to prove that

$$
\left\{\mathcal{X}_p\cap\{1\}^\perp\right\}_{\frac{n-1}{n}<p<\infty} \quad \text{and} \quad \left\{\mathcal{Y}_p(\Sigma)\Big/\mathbb{R}\right\}_{\frac{n-1}{n}<p<\infty}
\tag{4.39}
$$

are complex interpolation scales. To this end, we first observe that, thanks to Proposition 4.5, $\{\mathcal{Y}_p(\Sigma)\}_p$ is a retract of $\{\mathcal{Y}_p(\partial\Omega)\}_p$, which is known to be a complex interpolation scale (cf. [38]). Consequently, so is $\{\mathcal{Y}_p(\Sigma)\}_p$. Next, it is also known that

$$
\mathcal{Z}_p := h^p(\partial\Omega) \ \text{ if } \ p\in\left(\frac{n-1}{n},1\right], \ \text{ and } \ \mathcal{Z}_p := L^p(\partial\Omega) \ \text{ if } \ 1<p<\infty
\tag{4.40}
$$

is a complex interpolation scale. Since P_Σ (introduced in (4.12)) is a common projection operator from $\mathcal{Z}_p$ onto $\mathcal{X}_p$, it follows that $\{\mathcal{X}_p\}_p$ is a complex interpolation scale as well.

Turning to the scales in (4.39), we note that $\{\mathcal{X}_p \cap \{1\}^\perp\}_p$ is a retract of $\{\mathcal{X}_p\}_p$ (via inclusion and the projection operator π defined in the proof of Proposition 4.4). Hence, $\{\mathcal{X}_p \cap \{1\}^\perp\}_p$ is a complex interpolation scale itself.

Finally, observe that the mapping $f \mapsto f - [\text{measure}\,(\Sigma)]^{-1}\langle f, 1\rangle$ induces an isomorphism of $\mathcal{Y}_p(\Sigma)/\mathbb{R}$ onto $\mathcal{Y}_p(\Sigma) \cap \{1\}^\perp$ and that, as before, the latter family of spaces is a complex interpolation scale. All in all, $\{\mathcal{Y}_p(\Sigma)/\mathbb{R}\}_p$ is a complex interpolation scale, as desired. $\square$

5. Function spaces adapted to mixed problems: II

In this section we focus on spaces which are well-suited for the treatment of mixed boundary value problems consisting of functions in domains of the Euclidean space. To fix ideas, let $\Omega \subset \mathbb{R}^n$, Σ be an open subset of $\partial\Omega$ and introduce

(5.1)
$$L^p_\alpha(\mathbb{R}^n;\Sigma) := \text{ the closure of } C^\infty_c(\mathbb{R}^n \setminus \bar{\Sigma}) \text{ in } L^p_\alpha(\mathbb{R}^n),\ 1 < p < \infty,\ \alpha \in \mathbb{R},$$
$$B^{p,q}_\alpha(\mathbb{R}^n;\Sigma) := \text{ the closure of } C^\infty_c(\mathbb{R}^n \setminus \bar{\Sigma}) \text{ in } B^{p,q}_\alpha(\mathbb{R}^n),\ 1 \le p, q \le \infty,\ \alpha \in \mathbb{R},$$

where the subscript c indicates compact support and

$$\begin{aligned} L^p_{\alpha,z}(\Omega;\Sigma) &:= \{u|_\Omega :\ u \in L^p_\alpha(\mathbb{R}^n;\Sigma)\},\ 1 < p < \infty,\ \alpha \in \mathbb{R}, \\ B^{p,q}_{\alpha,z}(\Omega;\Sigma) &:= \{u|_\Omega :\ u \in B^{p,q}_\alpha(\mathbb{R}^n;\Sigma)\},\ 1 \le p, q \le \infty,\ \alpha \in \mathbb{R}. \end{aligned} \tag{5.2}$$

If $\mathcal{Z}(\Omega)$ is a Banach space of functions on Ω we set

$$\mathcal{Z}(\Omega;\Sigma) := \overline{\mathcal{C}^\infty_c(\mathbb{R}^n \setminus \bar{\Sigma})|_\Omega}^{\mathcal{Z}(\Omega)}, \tag{5.3}$$

where $\bar{\cdot}^{\mathcal{Z}(\Omega)}$ denotes closure in the $\mathcal{Z}(\Omega)$ norm. Hereafter $\mathcal{Z}(\Omega)$ will be one of the spaces (3.2). Note that

$$\mathcal{Z}(\Omega;\emptyset) = \mathcal{Z}(\Omega) \text{ and } \mathcal{Z}(\Omega;\partial\Omega) = \overline{\mathcal{C}^\infty_c(\Omega)}^{\mathcal{Z}(\Omega)}. \tag{5.4}$$

In the next few propositions below we study how these spaces are related to one another.

Proposition 5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that Σ is an admissible patch for $\partial\Omega$. Then*

$$L^p_{\alpha,z}(\Omega;\Sigma) \hookrightarrow L^p_\alpha(\Omega;\Sigma), \qquad 1 < p < \infty,\ \alpha \in \mathbb{R}, \tag{5.5}$$

and

$$B^{p,q}_{\alpha,z}(\Omega;\Sigma) \hookrightarrow B^{p,q}_\alpha(\Omega;\Sigma), \qquad 1 \le p, q \le \infty,\ \alpha \in \mathbb{R}, \tag{5.6}$$

Furthermore,

$$L^p_\alpha(\Omega;\Sigma) = L^p_{\alpha,z}(\Omega;\Sigma) = L^p_\alpha(\Omega), \quad 1 < p < \infty \text{ and } -1 + \frac{1}{p} < \alpha < \frac{1}{p}, \tag{5.7}$$

and, for each $1 \le q < \infty$,

$$B^{p,q}_\alpha(\Omega;\Sigma) = B^{p,q}_{\alpha,z}(\Omega;\Sigma) = B^{p,q}_\alpha(\Omega), \quad 1 \le p < \infty \text{ and } -1 + \frac{1}{p} < \alpha < \frac{1}{p}. \tag{5.8}$$

Proof. Let us first deal with the case of Sobolev spaces. Since by definition the space $C^\infty_c(\mathbb{R}^n \setminus \bar{\Sigma})$ is dense in $L^p_\alpha(\mathbb{R}^n;\Sigma)$ and since the operator of restriction to Ω maps the latter space boundedly onto $L^p_{\alpha,z}(\Omega;\Sigma)$, (5.5) follows.

Turning our attention to (5.7) we first recall that, for the range of indices under discussion, $C^\infty_c(\Omega)$ is dense in $L^p_\alpha(\Omega)$. This readily entails the equality $L^p_\alpha(\Omega;\Sigma) = L^p_\alpha(\Omega)$. Granted (5.5), there remains to prove that $L^p_\alpha(\Omega) \hookrightarrow L^p_{\alpha,z}(\Omega;\Sigma)$. This, in turn, is a consequence of the definitions and the well-known fact that $L^p_\alpha(\Omega) = \{u|_\Omega : \exists u_j \in \widetilde{C^\infty_c(\Omega)} \text{ such that } u_j \to u \text{ in } L^p_\alpha(\mathbb{R}^n)\}$ whenever $1 < p < \infty$ and $-1 + 1/p < \alpha < 1/p$.

Finally, that analogous results hold for Besov spaces is proved in a similar manner. □

Proposition 5.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and suppose that $\Sigma \subset \partial\Omega$ is an arbitrary admissible patch. Then*

$$L^p_\alpha(\Omega;\Sigma) = L^p_{\alpha,z}(\Omega;\Sigma) \tag{5.9}$$

provided

$$1 < p < \infty, \ \alpha > -1 + \frac{1}{p} \quad \text{and} \quad \alpha - \frac{1}{p} \notin \mathbb{N}. \tag{5.10}$$

Moreover,

$$B^{p,q}_\alpha(\Omega;\Sigma) = B^{p,q}_{\alpha,z}(\Omega;\Sigma) \tag{5.11}$$

whenever

$$1 < p,q < \infty, \ \alpha > -1 + \frac{1}{p} \quad \text{and} \quad \alpha - \frac{1}{p} \notin \mathbb{N}. \tag{5.12}$$

Proof. Consider first (5.9) and note that the right-to-left inclusion is contained in (5.5). To see the opposite one, select $k \in \mathbb{N}_o$ such that $k - 1 < \alpha - 1/p < k$. Note that, when $k = 0$, this forces $-1 + 1/p < \alpha < 1/p$ in which case the desired conclusion follows from Proposition 5.1, so we may assume that $k \ge 1$ for the remainder of the proof. Also, there is no loss of generality in assuming that Ω is the domain in $\mathbb{R}^n$ lying above a Lipschitz graph. Tacitly, it will also be assumed that all functions we work with have a compact, suitably localized support. This can be arranged using a smooth partition of unity.

As is well known, in this context there exists an infinite, upright circular cone Γ, with vertex at the origin in $\mathbb{R}^n$ such that $x + \Gamma \subset \Omega$ for every $x \in \bar{\Omega}$. Bring in Calderón's extension operator $\mathcal{E}_k$. This was originally introduced in [8] as

$$\begin{aligned} \mathcal{E}_k u(x) := (-1)^k \int_{S^{n-1}} \int_0^\infty \phi(\rho\omega)\rho^{n-1} \Big\{ \Big(\frac{\partial}{\partial\rho}\Big)^k [u(x-\rho\omega)] \Big\}^{\sim} d\rho\, d\omega \\ - \int_{S^{n-1}} \int_0^\infty \eta(\rho\omega) \widetilde{u(x-\rho\omega)}\, d\rho\, d\omega, \qquad x \in \mathbb{R}^n, \end{aligned}$$

where the large tilde (as well as $\{\cdots\}^{\sim}$) denotes extension by zero outside the domain Ω, $\eta := (\partial/\partial\rho)^k[\rho^{n-1}\phi(\rho\omega)]$ and $\phi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n+k$ for $0<|x|\ll 1$, has bounded support, vanishes outside the cone $-\Gamma$, and is normalized such that

$$\int_{S^{n-1}} \rho^{n-k}\phi(\rho\omega)\,d\omega = \frac{1}{(k-1)!} \quad \text{for} \quad \rho \quad \text{small}. \tag{5.13}$$

Above, $x=\rho\omega$ with $\rho:=|x|$ and $\omega:=x/|x|\in S^{n-1}$, is the polar representation of $x\in\mathbb{R}^n\setminus\{0\}$. Note that $\eta\in C_c^\infty(\mathbb{R}^n)$ and $0\notin\operatorname{supp}\eta$. Although we will not make it explicit, it is further assumed that the operator (5.13) is truncated by a smooth function supported in a neighborhood of $\bar\Omega$ and which is identically one on Ω.

The integral operator just constructed enjoys the following properties. First,

$$u\in C^\infty(\bar\Omega)\Longrightarrow \mathcal{E}_k u\Big|_{\Omega} = u \quad \text{in } \Omega, \tag{5.14}$$

and we recall the classical proof. Fix $x\in\Omega$ and note that $\tilde u(x-\rho\omega)=u(x-\rho\omega)$ whenever $\rho\geq 0$ and $\omega\in-\Gamma$. The idea is then to integrate by parts k times in ρ for each fixed $\omega\in-\Gamma$:

$$\begin{aligned}
&\int_0^\infty \phi(\rho\omega)\rho^{n-1}\Big(\frac{\partial}{\partial\rho}\Big)^k[u(x-\rho\omega)]\,d\rho\\
&\quad=\sum_{j=0}^{k-1}(-1)^j\Big(\frac{\partial}{\partial\rho}\Big)^j[\phi(\rho\omega)\rho^{n-1}]\Big(\frac{\partial}{\partial\rho}\Big)^{k-j-1}[u(x-\rho\omega)]\Big|_{\rho=0}^{\rho=\infty}\\
&\qquad+(-1)^k\int_0^\infty u(x-\rho\omega)\Big(\frac{\partial}{\partial\rho}\Big)^k[\phi(\rho\omega)\rho^{n-1}]\,d\rho.
\end{aligned} \tag{5.15}$$

In the above sum, the terms corresponding to $0\leq j\leq k-2$ vanish due to the fact that $\rho^{n-1}\phi(\rho\omega)$ is homogeneous of degree $k-1$ and has bounded support. Also, the term corresponding to $j=k-1$ yields, after integrating in ω over the unit sphere, $(-1)^k u(x)$. Here the normalization condition (5.13) is used. Finally, the last (double) integral in (5.13) is designed to cancel the contribution from the very last term in (5.15).

Second, we claim that

$$u\in C_c^\infty(\mathbb{R}^n\setminus\bar\Sigma)\Big|_{\Omega}\Longrightarrow \mathcal{E}_k u\in C^\infty(\mathbb{R}^n) \text{ and vanishes near } \Sigma. \tag{5.16}$$

The fact that $\mathcal{E}_k u$ is smooth in $\mathbb{R}^n$ whenever $u\in C^\infty(\bar\Omega)$ is evident from the definition of $\mathcal{E}_k$. Next, $\mathcal{E}_k u(x)=u(x)=0$ if $x\in\Omega$ lies sufficiently close to $\bar\Sigma$, so it remains to analyze the case when x belongs to the complement of $\bar\Omega$.

In this latter scenario, we proceed along the lines of the proof of (5.14) with the following alterations. Fix $u\in C_c^\infty(\mathbb{R}^n\setminus\bar\Sigma)\Big|_{\Omega}$ and let $x\in\mathbb{R}^n\setminus\bar\Omega$ be a point sufficiently close to $\bar\Sigma$. Then, for each $\omega\in S^{n-1}$ there exists a unique $\rho(x,\omega)>0$

such that $x-\rho(x,\omega)\omega\in\partial\Omega$. Furthermore, it can be assumed that $x-\rho(x,\omega)\omega$ is sufficiently close to $\bar{\Sigma}$ so that, in particular, u along with its derivatives vanishes at $x-\rho(x,\omega)\omega$. Then, much as in (5.15),

(5.17)
$$\begin{aligned}
\int_0^\infty \phi(\rho\omega)\rho^{n-1}\Big\{\Big(\frac{\partial}{\partial\rho}\Big)^k[u(x-\rho\omega)]\Big\}^{\widetilde{}}\,d\rho &= \int_{\rho(x,\omega)}^\infty \phi(\rho\omega)\rho^{n-1}\Big(\frac{\partial}{\partial\rho}\Big)^k[u(x-\rho\omega)]\,d\rho\\
&= \sum_{j=0}^{k-1}(-1)^j\Big(\frac{\partial}{\partial\rho}\Big)^j[\phi(\rho\omega)\rho^{n-1}]\Big(\frac{\partial}{\partial\rho}\Big)^{k-j-1}[u(x-\rho\omega)]\Big|_{\rho=\rho(x,\omega)}^{\rho=\infty}\\
&\quad + (-1)^k\int_{\rho(x,\omega)}^\infty u(x-\rho\omega)\Big(\frac{\partial}{\partial\rho}\Big)^k[\phi(\rho\omega)\rho^{n-1}]\,d\rho\\
&= (-1)^k\int_0^\infty \eta(\rho\omega)u\widetilde{(x-\rho\omega)}\,d\rho.
\end{aligned}$$

The reason for which there is no contribution from the sum above is that the expression $(\partial/\partial\rho)^{k-j-1}[u(x-\rho\omega)]\Big|_{\rho=\rho(x,\omega)}$ is zero since u vanishes near Σ. This justifies (5.16).

Going further, as in [8] we observe that

$$\Big(\frac{\partial}{\partial\rho}\Big)^k=\sum_{|\gamma|=k}\frac{k!}{\gamma!}\omega^\gamma\Big(\frac{\partial}{\partial x}\Big)^\gamma,\qquad x=\rho\omega, \tag{5.18}$$

so that by actually carrying out the differentiation with respect to ρ in (5.13) we arrive at

$$\begin{aligned}
\mathcal{E}_k u &= \int_{S^{n-1}}\int_0^\infty \phi(\rho\omega)\Big[\sum_{|\gamma|=k}\frac{k!}{\gamma!}\omega^\gamma\Big\{\Big(\frac{\partial}{\partial x}\Big)^\gamma u(x-\rho\omega)\Big\}^{\widetilde{}}\ \Big]\rho^{n-1}\,d\rho\,d\omega\\
&\quad -\int_{S^{n-1}}\int_0^\infty \eta(\rho\omega)u\widetilde{(x-\rho\omega)}\,d\rho\,d\omega\\
&= \sum_{|\gamma|=k}\varphi_\gamma * u_\gamma+\xi*\tilde{u},
\end{aligned}\tag{5.19}$$

where

$$\varphi_\gamma:=(k!/\gamma!)\omega^\gamma\phi(\rho\omega),\quad u_\gamma:=\widetilde{(\partial/\partial x)^\gamma} u,\quad \xi:=\eta(x)/|x|^{n-1}. \tag{5.20}$$

Note that each $\varphi_\gamma\in C^\infty(\mathbb{R}^n\setminus\{0\})$ has bounded support and is homogeneous of degree $-n+k$ for $0<|x|\ll 1$. Based on this, it follows from the classical Calderón-Zygmund theory that the corresponding convolution operators, i.e. $u\mapsto\varphi_\gamma*u$, for $|\gamma|=k$, are smoothing of order k. Also, $\xi\in C_c^\infty(\mathbb{R}^n)$ so that $\xi*\cdot$ is smoothing of any order. Consequently, for each $u \in C^\infty(\bar{\Omega})$ of bounded support,

$$\begin{aligned}\|\mathcal{E}_k u\|_{L^p_\alpha(\mathbb{R}^n)} &\leq C \sum_{|\gamma|=k} \|\varphi_\gamma * u_\gamma\|_{L^p_\alpha(\mathbb{R}^n)} + C\|\xi * \tilde{u}\|_{L^p_\alpha(\mathbb{R}^n)} \\ &\leq C \sum_{|\gamma|=k} \|\widetilde{\partial^\gamma u}\|_{L^p_{\alpha-k}(\mathbb{R}^n)} + C\|\tilde{u}\|_{L^p(\mathbb{R}^n)} \\ &\leq C \sum_{|\gamma|=k} \|\partial^\gamma u\|_{L^p_{\alpha-k}(\Omega)} + C\|u\|_{L^p_\alpha(\Omega)} \\ &\leq C\|u\|_{L^p_\alpha(\Omega)}.\end{aligned} \tag{5.21}$$

The third inequality uses the fact that $-1+\frac{1}{p} < \alpha - k < \frac{1}{p}$, which ensures that the extension by zero from Ω to $\mathbb{R}^n$ is a bounded operator with preservation of class. Thus, by density,

$$\mathcal{E}_k : L^p_\alpha(\Omega) \longrightarrow L^p_\alpha(\mathbb{R}^n) \tag{5.22}$$

is a bounded operator which satisfies $\mathcal{E}_k u|_\Omega = u$ for any $u \in L^p_\alpha(\Omega)$. With this in hand and relying on (5.16), it is easy to check that

$$\mathcal{E}_k : L^p_\alpha(\Omega;\Sigma) \longrightarrow L^p_\alpha(\mathbb{R}^n;\Sigma) \tag{5.23}$$

is well-defined and bounded. Thus, for each $u \in L^p_\alpha(\Omega;\Sigma)$ we have $u = \mathcal{E}_k u|_\Omega \in L^p_{\alpha,z}(\Omega;\Sigma)$. This finishes the proof of the left-to-right inclusion in (5.9).

The proof of (5.11) proceeds analogously up to (5.22). At that point, we use real interpolation to deduce that, first,

$$\mathcal{E}_k : B^{p,q}_\alpha(\Omega) \longrightarrow B^{p,q}_\alpha(\mathbb{R}^n) \tag{5.24}$$

is a bounded operator and, by arguing as before,

$$\mathcal{E}_k : B^{p,q}_\alpha(\Omega;\Sigma) \longrightarrow B^{p,q}_\alpha(\mathbb{R}^n;\Sigma) \tag{5.25}$$

is well-defined and bounded as well. With this in hand, the proof is finished as before. □

Theorem 5.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that Σ is an admissible patch for $\partial\Omega$. Then, if $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$,*

$$L^p_\alpha(\Omega;\Sigma) = \{u \in L^p_\alpha(\Omega) : \ \mathrm{Tr}\, u = 0 \ \text{on } \Sigma\}, \quad 1 < p < \infty, \tag{5.26}$$

and

$$B^{p,q}_\alpha(\Omega;\Sigma) = \{u \in B^{p,q}_\alpha(\Omega) : \ \mathrm{Tr}\, u = 0 \ \text{on } \Sigma\}, \quad 1 \leq p,q < \infty. \tag{5.27}$$

Proof. The left-to-right inclusion in (5.26) is a straightforward consequence of the definition of the space $L^p_\alpha(\Omega;\Sigma)$ and the fact that the trace operator maps $L^p_\alpha(\Omega)$ boundedly into $B^{p,p}_{\alpha-\frac{1}{p}}(\partial\Omega)$ whenever $1 < p < \infty$ and $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$.

To prove the right-to-left inclusion in (5.26), consider an arbitrary function $u \in L^p_\alpha(\Omega)$ such that $\mathrm{Tr}\, u = 0$ on Σ. Via a standard localization argument, we can assume that the following hold: $0 \in \partial\Sigma$ and there exists a cube $Q \subset \mathbb{R}^n$ centered at the origin with $\mathrm{supp}\, u \subset Q \cap \bar{\Omega}$, as well as two Lipschitz functions φ, ψ so that (2.2)-(2.3) hold.

In this context, we shall make use of the extension operator $\mathcal{E}_1$, corresponding to taking $k = 1$ in (5.13) (recall that the cone Γ and the functions ϕ, η were introduced and used on this occasion). Let $\mathcal{O}$ consist of the collection of all points

$x \in \big\{(x_1, x'', x_n) \in Q : x_n < \varphi(x') \text{ and } \psi(x'') < x_1\big\}$ subject to the following additional condition. For each $\omega \in S^{n-1} \cap \operatorname{supp} \phi$ we let $\rho(x,\omega)$ be a positive number (and matters can be arranged so that this is uniquely defined for $x \in Q$) such that $x - \rho(x,\omega)\omega \in \partial\Omega$. Then the additional condition referred to above is that $x - \rho(x,\omega)\omega \in \Sigma$. Our goal is to show that

$$\mathcal{E}_1 u \in L^p_\alpha(\mathbb{R}^n) \text{ vanishes in } \mathcal{O}. \tag{5.28}$$

To this end, let $u_j \in C^\infty(\bar\Omega)$, $\operatorname{supp} u_j \subset Q \cap \bar\Omega$, be a sequence of functions such that $u_j \to u$ in $L^p_\alpha(\Omega)$. Similarly to (5.17), for each $x \in \mathcal{O}$,

(5.29)
$$\begin{aligned}
\int_0^\infty \phi(\rho\omega)\rho^{n-1}\Big\{\Big(\frac{\partial}{\partial\rho}\Big)[u_j(x-\rho\omega)]\Big\}^{\widetilde{}}\,d\rho &= \int_{\rho(x,\omega)}^\infty \phi(\rho\omega)\rho^{n-1}\Big(\frac{\partial}{\partial\rho}\Big)[u_j(x-\rho\omega)]\,d\rho \\
&= [\phi(\rho\omega)\rho^{n-1}u_j(x-\rho\omega)]\Big|_{\rho=\rho(x,\omega)}^{\rho=\infty} - \int_{\rho(x,\omega)}^\infty u_j(x-\rho\omega)\Big(\frac{\partial}{\partial\rho}\Big)[\phi(\rho\omega)\rho^{n-1}]\,d\rho \\
&= -\phi(\rho(x,\omega)\omega)\rho(x,\omega)^{n-1}u_j(x-\rho(x,\omega)\omega) - \int_0^\infty \eta(\rho\omega)u_j\widetilde{(x-\rho\omega)}\,d\rho.
\end{aligned}$$

On previous occasions when an identity of this type was used, the hypotheses on the function u_j were such that the term $-\phi(\rho(x,\omega)\omega)\rho(x,\omega)^{n-1}u_j(x-\rho(x,\omega)\omega)$ vanished for each fixed ω. In the current context this is no longer the case, and we need to monitor its influence in subsequent calculations. We integrate it in ω over the unit sphere, then change variables from $\omega \in S^{n-1} \cap \operatorname{supp}\phi$ to $x - \rho(x,\omega)\omega \in \partial\Omega$. This yields a term of the form

$$\int_{\partial\Omega} u_j(y)J(y)\,d\sigma(y), \tag{5.30}$$

where the function J depends on ϕ, x; $\partial\Omega$ is L^∞ and supported in $\bar\Sigma$. Thus, all together,

$$\mathcal{E}_1 u_j(x) = \int_\Sigma u_j(y)J(y)\,d\sigma(y), \qquad \forall\, x \in \mathcal{O},\ j = 1, 2, \dots. \tag{5.31}$$

Since $\operatorname{Tr} u_j \to \operatorname{Tr} u$ in $B^{p,p}_{\alpha-\frac{1}{p}}(\partial\Omega)$ as $j \to \infty$, and $\operatorname{Tr} u = 0$ on Σ, it follows that $\mathcal{E}_1 u(x) = 0$ for each $x \in \mathcal{O}$ and (5.28) is therefore established.

Fix $\xi \in C^\infty_c(Q)$ with $\xi \equiv 1$ on $\operatorname{supp} u$. A careful analysis of the geometry of the set $\mathcal{O}$ reveals that we can choose a unit vector $v \in S^{n-1}$ such that for every $\varepsilon > 0$ small there exists $t_o > 0$ with the property that

$$\{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp}\xi) < t\} \setminus \Big(\mathcal{O} + \varepsilon v\Big) \text{ is disjoint from } \bar\Sigma \tag{5.32}$$

for every $0 < t < t_o$. Let $\theta \in C^\infty_c(\mathbb{R}^n)$ be a nice bump-function; as usual, set $\theta_t(x) := t^{-n}\theta(x/t)$ and introduce $u_{\varepsilon,t} := (\xi\mathcal{E}_1 u)(\cdot - \varepsilon v) * \theta_t$. From (5.32), it follows that $u_{\varepsilon,t} \in C^\infty_c(\mathbb{R}^n \setminus \bar\Sigma)$ and $u_{\varepsilon,t}|_\Omega \to (\xi\mathcal{E}_1 u)|_\Omega = u$ in $L^p_\alpha(\Omega)$ as $\varepsilon, t \to 0$. Thus, by definition, $u \in L^p_\alpha(\Omega;\Sigma)$, as desired. This concludes the proof of (5.26). The case of (5.27) is treated analogously, and this finishes the proof of the theorem. □

Remark. The above proof gives more. In order to be more specific, fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and recall that it is possible to choose a finite, open cover $\{U_j\}_j$ of $\bar\Omega$ and a family of circular, truncated cones $\{\Gamma_j\}_j$ such that $x + \Gamma_j \subset \Omega$

for every $x \in U_j \cap \partial\Omega$. Given an admissible patch $\Sigma \subseteq \partial\Omega$, call an open set $\mathcal{O}$ an *exterior, one-sided neighborhood* of Σ provided:

(i) $\mathcal{O}$ is included in $\bigcup_j U_j$, is disjoint from Ω and satisfies $\bar{\mathcal{O}} \cap \bar{\Omega} = \bar{\Sigma}$;

(ii) for j and every $x \in U_j \cap \mathcal{O}$ we have that $(x + \Gamma_j) \cap \partial\Omega$ is a non-empty subset of Σ.

Note that, in particular, $\Sigma \subset \partial\mathcal{O}$ and Σ is non-tangentially accessible from within $\mathcal{O}$.

Then $u \in L^p_\alpha(\Omega; \Sigma)$, $1 < p < \infty$, $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$, if and only if there exists $w \in L^p_\alpha(\mathbb{R}^n)$ which vanishes in some exterior one-sided neighborhood of Σ and such that $w|_\Omega = u$. A similar result is valid for the Besov versions of these spaces.

Next, we record a useful consequence of Theorem 5.3.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that Σ is an admissible patch on $\partial\Omega$. Also, fix $1 \leq p, q < \infty$ and $s \in (0, 1)$. Then the trace operator*

$$\mathrm{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega; \Sigma) \longrightarrow B^{p,q}_{s,0}(\partial\Omega \setminus \bar{\Sigma}) \tag{5.33}$$

has a continuous right inverse

$$\mathrm{Ext} : B^{p,q}_{s,0}(\partial\Omega \setminus \bar{\Sigma}) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega; \Sigma). \tag{5.34}$$

Furthermore, if $1 < p < \infty$ and $s \in (0, 1)$, (5.34) *also extends as a bounded operator*

$$\mathrm{Ext} : B^{p,p}_{s,0}(\partial\Omega \setminus \bar{\Sigma}) \longrightarrow L^p_{s+\frac{1}{p}}(\Omega; \Sigma), \tag{5.35}$$

which is a right inverse for the action of the trace in the context

$$\mathrm{Tr} : L^p_{s+\frac{1}{p}}(\Omega; \Sigma) \longrightarrow B^{p,p}_{s,0}(\partial\Omega \setminus \bar{\Sigma}). \tag{5.36}$$

Proof. That the operators (5.33) and (5.36) are well-defined and bounded follows directly from definitions and the discussion pertaining to (3.15), (3.17). As regards the existence of a right inverse in each case, we shall show that the operators (3.16) and (3.18) will do (when suitably restricted). Indeed, consider $\mathrm{Ext} : B^{p,p}_{s,0}(\partial\Omega \setminus \bar{\Sigma}) \hookrightarrow B^{p,p}_s(\partial\Omega) \longrightarrow L^p_{s+\frac{1}{p}}(\Omega)$; our goal is to show that this operator takes values in $L^p_{s+\frac{1}{p}}(\Omega; \Sigma)$. To see this, consider $f \in B^{p,p}_{s,0}(\partial\Omega \setminus \bar{\Sigma})$ and set $u := \mathrm{Ext}\, f \in L^p_{s+\frac{1}{p}}(\Omega)$. Then $\mathrm{Tr}\, u = f = 0$ on Σ so that, by Theorem 5.3, $u \in L^p_{s+\frac{1}{p}}(\Omega; \Sigma)$, as desired. The case of Besov spaces is handled analogously. $\square$

Another consequence of Theorem 5.3 is as follows.

Proposition 5.5. *Let Ω be a bounded Lipschitz domain and $\Sigma \subseteq \partial\Omega$ be an admissible patch. Then*

$$\Big[L^{p_0}_{s_0+\frac{1}{p_0}}(\Omega; \Sigma), L^{p_1}_{s_1+\frac{1}{p_1}}(\Omega; \Sigma)\Big]_\theta = L^p_{s+\frac{1}{p}}(\Omega; \Sigma), \tag{5.37}$$

where $0 < s_0, s_1 < 1$, $1 < p_0, p_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Moreover,

$$\Big(L^p_{s_0+\frac{1}{p}}(\Omega; \Sigma), L^p_{s_1+\frac{1}{p}}(\Omega; \Sigma)\Big)_{\theta,q} = B^{p,q}_{s+\frac{1}{p}}(\Omega; \Sigma) \tag{5.38}$$

if $0 < s_0 \neq s_1 < 1$, $1 < p, q < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$.

Finally,

$$\Big[B^{p_0,q_0}_{s_0+\frac{1}{p_0}}(\Omega;\Sigma), B^{p_1,q_1}_{s_1+\frac{1}{p_1}}(\Omega;\Sigma)\Big]_\theta = B^{p,q}_{s+\frac{1}{p}}(\Omega;\Sigma) \tag{5.39}$$

if $0<s_0,s_1<1$, $1<p_0,p_1<\infty$, $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$, *and*

$$\Big(B^{p,q_0}_{s_0+\frac{1}{p}}(\Omega;\Sigma), B^{p,q_1}_{s_1+\frac{1}{p}}(\Omega;\Sigma)\Big)_{\theta,q} = B^{p,q}_{s+\frac{1}{p}}(\Omega;\Sigma) \tag{5.40}$$

if $0<s_0\neq s_1<1$, $1\leq p,q_0,q_1,q<\infty$, $0<\theta<1$, $s=(1-\theta)s_0+\theta s_1$.

Before presenting the proof of the above proposition, we recall an abstract interpolation result. The part dealing with the complex method of interpolation has been first proved in [34], whereas the extension to the real method of interpolation is from [35].

Lemma 5.6. *Let X_i, Y_i, Z_i, $i=0,1$, be two triplets of Banach spaces such that $X_0\cap X_1$ is dense in both X_0 and X_1, and similarly for Z_0, Z_1. Suppose that $Y_i\hookrightarrow Z_i$, $i=0,1$ and there exists a linear operator D such that $D: X_i\to Z_i$ boundedly for $i=0,1$. Define the spaces*

$$X_i(D):=\{u\in X_i:\, Du\in Y_i\},\quad i=0,1, \tag{5.41}$$

equipped with the graph norm. Finally, suppose that there exists a continuous linear mapping $G: Z_i\to X_i$ with the property $D\circ G=I$, the identity on Z_i for $i=0,1$. Then, for each $0<\theta<1$,

$$[X_0(D),X_1(D)]_\theta=\{u\in[X_0,X_1]_\theta:\, Du\in[Y_0,Y_1]_\theta\},\qquad \theta\in(0,1). \tag{5.42}$$

Furthermore, for each $0<\theta<1$ and $0<q\leq\infty$,

$$(X_0(D),X_1(D))_{\theta,q}=\{u\in(X_0,X_1)_{\theta,q}:\, Du\in(Y_0,Y_1)_{\theta,q}\}. \tag{5.43}$$

We are now ready to present the

Proof of Proposition 5.5. We shall only prove (5.37) as all the other interpolation identities are handled similarly. To this end, for $i=0,1$ set $X_i:=L^{p_i}_{s_i+\frac{1}{p_i}}(\Omega)$, $Y_i:=B^{p_i,p_i}_{s_i,0}(\partial\Omega\setminus\bar{\Sigma})$, $Z_i:=B^{p_i,p_i}_{s_i}(\partial\Omega)$ and consider $D:=\mathrm{Tr}$. From the discussion in connection with (3.15)-(3.18), we know that $G:=\mathrm{Ext}$ is a right inverse for D. Since $X_i(D)=L^{p_i}_{s_i+\frac{1}{p_i}}(\Omega;\Sigma)$ by Theorem 5.3, Lemma 5.6 applies and, in concert with Proposition 4.2, yields (5.37). $\square$

Remark. By proceeding as in [20] it is possible to prove similar interpolation identities for smaller values of the smoothness index. However, these spaces are less useful in the treatment of mixed boundary value problems since the trace operator may not be well-defined in this latter context.

6. Functional analysis tools

In this section we present a series of results useful in the future obtained using functional analysis tools regarding dual spaces and Banach envelopes of the function spaces introduced in the previous section. This relies on material from [38] to which we refer the reader for a more detailed exposition.

Definition 6.1. Let X be a quasi-Banach space which is dual-rich (i.e., whose dual separates its points). Then the Banach envelope of X denoted by $\widehat{X}$ is the minimal enlargement of X to a Banach space. In particular the inclusion

$$\iota : X \hookrightarrow \widehat{X} \text{ is continuous with dense range.} \tag{6.1}$$

It turns out that applying "hat" has good functorial properties such as preserving the linearity, boundedness and the quality of being an isomorphism for operators between quasi-Banach spaces. More specifically, we have the following.

Proposition 6.2. *Let $T : X \to Y$ be a bounded, linear operator between two dual-rich quasi-Banach spaces. Then T extends to a bounded linear operator $\hat{T} : \widehat{X} \to \widehat{Y}$. Moreover, if T is an isomorphism, then so is $\hat{T}$.*

See [38] for more details; here we also want to point out that, in general,

$$\big(\widehat{X}\big)^* = X^* \tag{6.2}$$

and that the conditions (6.1)-(6.2) identify the Banach space $\hat{X}$ uniquely (up to an isomorphism). Below we record yet another useful criterion for computing Banach envelopes.

Proposition 6.3. *Let X_i, $i = 1, 2$, be two quasi-Banach spaces and Y_i, $i = 1, 2$, be two Banach spaces so that the inclusions $X_i \hookrightarrow Y_i$ are well-defined and continuous. Next, let $T : Y_1 \to Y_2$ be a linear, bounded operator which has a linear, continuous inverse to the right, i.e., $R : Y_2 \to Y_1$ with $T \circ R = I$. Assume that $T|_{X_1} : X_1 \to X_2$ and $R|_{X_2} : X_2 \to X_1$ are well-defined and bounded; cf. the diagram below:*

$$\begin{tikzcd}
X_1 \arrow[r, "\iota"] \arrow[d, shift left, "T|_{X_1}"] & Y_1 \arrow[d, shift left, "T"] \\
X_2 \arrow[r, "\iota"] \arrow[u, shift left, "R|_{X_2}"] & Y_2 \arrow[u, shift left, "R"]
\end{tikzcd} \tag{6.3}$$

Then, if $\widehat{X_1} = Y_1$, we also have $\widehat{X_2} = Y_2$.

Turning to specific spaces, we first record a result proved in [38].

Proposition 6.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. For $\frac{n-1}{n} < p < 1$, we have*

$$\widehat{h^{1,p}(\partial\Omega)} = B^{1,1}_{1-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega) \quad \textit{and} \quad \widehat{h^{p}(\partial\Omega)} = B^{1,1}_{-(n-1)\left(\frac{1}{p}-1\right)}(\partial\Omega). \tag{6.4}$$

We seek to establish analogous results for the Hardy and Besov spaces naturally intervening in the mixed boundary problems we have in mind.

Proposition 6.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, for any $\Sigma \subseteq \partial\Omega$ admissible patch and $\frac{n-1}{n} < p < 1$, we have*

$$\widehat{h^p_0(\Sigma)} = B^{1,1}_{-s,0}(\Sigma), \quad \textit{where} \quad s = (n-1)\left(\frac{1}{p} - 1\right), \tag{6.5}$$

and

$$\widehat{h^{1,p}(\Sigma)} = B^{1,1}_{1-s}(\Sigma), \quad \textit{where} \quad s = (n-1)\left(\frac{1}{p} - 1\right). \tag{6.6}$$

Proof. Let s and p be as in (6.5). From Proposition 6.4 we know that (6.5)-(6.6) hold when $\Sigma = \partial\Omega$. Also, observe that the inclusion operators from $h^p_0(D)$ and $B^{1,1}_{-s,0}(D)$ into $h^p(\partial\Omega)$ and $B^{1,1}_{-s}(\partial\Omega)$, respectively, are continuous right inverses for the operator P_D acting on $h^p(\partial\Omega)$ and $B^{1,1}_{-s}(\partial\Omega) = \widehat{h^p}(\partial\Omega)$, respectively. Then (6.5) follows by appealing to Proposition 6.3. The proof of (6.6) is similar, and this concludes the proof of the proposition. $\square$

For the applications we have in mind, the following version of Proposition 6.5 is better suited. We slightly change notation and write $\{X\}\widehat{}$ instead of $\widehat{X}$, whenever convenient.

Proposition 6.6. *Under the same hypotheses as in Proposition* 6.5*, including the indices s and p,*

$$\Big\{h^p_0(\Sigma) \cap \{1\}^\perp\Big\}\widehat{} = B^{1,1}_{-s,0}(\Sigma) \cap \{1\}^\perp, \qquad \Big\{h^{1,p}(\Sigma)/\mathbb{R}\Big\}\widehat{} = B^{1,1}_{1-s}(\Sigma)/\mathbb{R}. \tag{6.7}$$

Proof. This is proved starting from Proposition 6.5 with the help of Proposition 6.3 in a manner similar to the way a number of results have already been established in §3 (cf., e.g., the last part in the proof of Proposition 4.6). $\square$

In the last part of this section we record a useful result regarding the stability of isomorphisms on complex interpolation scales.

Proposition 6.7. *Let X_0, X_1 be a compatible couple of quasi-Banach spaces and assume that $X_0 + X_1$ is analytically convex. Also, consider a bounded, linear operator $T : X_j \to X_j$, $j = 0, 1$. If we now set $X_\theta := [X_0, X_1]_\theta$ for each $\theta \in (0,1)$, then T induces a bounded linear operator*

$$T_\theta : X_\theta \longrightarrow X_\theta, \qquad \theta \in (0,1), \tag{6.8}$$

in a natural fashion.

Assume next that there exists $\theta_o \in (0,1)$ such that T_{θ_o} is an isomorphism (respectively, Fredholm). Then there exists $\varepsilon > 0$ such that T_θ continues to be an isomorphism (respectively, Fredholm) whenever $|\theta - \theta_o| < \varepsilon$.

Furthermore, T_θ^{-1} agrees with $T_{\theta'}^{-1}$ on $X_\theta \cap X_{\theta'}$ for any $\theta, \theta' \in (\theta_o - \varepsilon, \theta_o + \varepsilon)$.

This is a version of the results proved in [29] (which, in turn, refines the work in [51]). We refer to that paper for a more extensive discussion and references to related matters.

7. The Neumann-to-Dirichlet operator

The main goal of this section is to introduce the Neumann-to-Dirichlet operator Υ and to show the connection between the invertibility of Υ and the well-posedness of the mixed boundary problem on various function spaces. Furthermore, when the domain in question is also creased (cf. Definition 2.3), we establish sharp invertibility results on the Besov scale for the operator Υ.

Recall the function spaces introduced in Section 3 and the definition of the non-tangential maximal function introduced in (2.4). We shall be concerned with the Neumann problem

$$\left\{\begin{array}{l}\Delta u=0 \quad \text{in} \quad \Omega,\\[4pt] \dfrac{\partial u}{\partial\nu}=f \text{ on } \partial\Omega,\end{array}\right. \tag{7.1}$$

when f belongs to various spaces of distributions on $\partial\Omega$ and appropriate smoothness conditions (or growth restrictions) are imposed on u.

Let $\mathcal{H}_\varepsilon$ stand for the interior of the hexagonal region in the plane with vertices

$$(0,0),\quad (\varepsilon,0),\quad \Big(1,\frac{1}{2}-\varepsilon\Big),\quad (1,1),\quad (1-\varepsilon,1),\quad \Big(0,\frac{1}{2}+\varepsilon\Big). \tag{7.2}$$

We shall also work with a slight variation of this hexagon, which we denote by $\widetilde{\mathcal{H}_\varepsilon}$, consisting of the old region $\mathcal{H}_\varepsilon$ to which we append the two horizontal sides, corresponding to $p=1$ and $1-\varepsilon<s<1$ and, respectively, $p=\infty$ and $0<s<\varepsilon$.

The well-posedness of (7.1) on Sobolev-Besov-Hardy spaces is well-understood at the present time. In the following theorem we summarize such well-posedness results proved in [24], [56], [11], [6], [16]. To state this, we need one more piece of notation. For a generic space of distributions on the boundary of a Lipschitz domain $\Omega\subset\mathbb{R}^n$, say $\mathcal{X}(\partial\Omega)\subset\Big(\mathrm{Lip}\,(\partial\Omega)\Big)^*$, we set

$$\mathcal{X}(\partial\Omega)\cap\{1\}^\perp:=\{f\in\mathcal{X}(\partial\Omega):\,\langle f,1\rangle=0\}. \tag{7.3}$$

Theorem 7.1. *Let Ω be a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon>0$ depending on the domain Ω such that the boundary value problem* (7.1) *has a unique, modulo constants, solution (which also satisfies natural estimates) in each of the following cases:*

(i) *when $f\in L^p(\partial\Omega)\cap\{1\}^\perp$ with $p\in(1,2+\varepsilon)$, in the class of functions satisfying $(\nabla u)^*\in L^p(\partial\Omega)$;*
(ii) *when $f\in B^{p,p}_{s-1}(\partial\Omega)\cap\{1\}^\perp$ with $(s,1/p)\in\widetilde{\mathcal{H}_\varepsilon}$, in the class of functions belonging to $B^{p,p}_{s+1/p}(\Omega)$;*
(iii) *when $f\in B^{p,p}_{s-1}(\partial\Omega)\cap\{1\}^\perp$ with $(s,1/p)\in\mathcal{H}_\varepsilon$, in the class of functions belonging to $L^p_{s+1/p}(\Omega)$;*
(iv) *when $f\in h^p(\partial\Omega)\cap\{1\}^\perp$ with $p\in(1-\varepsilon,1]$, in the class of functions satisfying $(\nabla u)^*\in L^p(\partial\Omega)$.*

Assume that $\Omega\subset\mathbb{R}^n$, $n\geq 3$, is a bounded, creased domain and recall the partition of $\partial\Omega$ into two disjoint admissible patches D and N from Definition 2.3. In this scenario, whenever meaningful, we introduce the *Neumann-to-Dirichlet operator* Υ as

$$\Upsilon f:=u\big|_D,\ \text{modulo constants, where } u \text{ solves (7.1) with datum } f. \tag{7.4}$$

An immediate consequence of this definition and the above theorem is the following.

Corollary 7.2. *Let $\Omega\subset\mathbb{R}^n$, $n\geq 3$, be a bounded creased domain with connected boundary. Assume that $\partial\Omega$ is partitioned into two disjoint admissible patches D*

and N as in Definition 2.3. Then there exists $\varepsilon > 0$ depending only on the domain Ω such that the operators

$$\begin{aligned}
&\Upsilon : L^p(\partial\Omega) \cap \{1\}^\perp \longrightarrow L^p_1(D)\Big/\mathbb{R}, \quad \forall\, p \in (1, 2+\varepsilon),\\
&\Upsilon : h^p(\partial\Omega) \cap \{1\}^\perp \longrightarrow h^{1,p}(D)\Big/\mathbb{R}, \quad \forall\, p \in (1-\varepsilon, 1],\\
&\Upsilon : B^{p,p}_{s-1}(\partial\Omega) \cap \{1\}^\perp \longrightarrow B^{p,p}_s(D)\Big/\mathbb{R}, \quad \forall\, (s, 1/p) \in \widetilde{\mathcal{H}_\varepsilon}
\end{aligned} \tag{7.5}$$

are well-defined, linear and bounded. Furthermore, the actions of these operators are compatible with one another.

Next introduce the mixed Neumann/Dirichlet boundary value problems with Neumann data in $L^p(N)$, $B^{p,p}_s(N)$, and respectively $h^p(N)$ for the corresponding range of s and p from (7.5).

Proposition 7.3. *Let Ω be a bounded Lipschitz domain with connected boundary. Assume that $D, N \subset \partial\Omega$ are two non-empty, disjoint admissible patches satisfying $\bar{D} \cap \bar{N} = \partial D = \partial N$ and $\bar{D} \cup \bar{N} = \partial\Omega$.*

Then there exists $\varepsilon > 0$ depending only on the domain Ω such that the following hold.

(i) For each $p \in (1, 2+\varepsilon)$, the mixed problem

$$\begin{cases}
\Delta u = 0 \quad \textit{in} \quad \Omega,\\
\dfrac{\partial u}{\partial \nu}\Big|_N = f \in L^p(N),\\
u\,\big|_D = g \in L^p_1(D),\\
(\nabla u)^* \in L^p(\partial\Omega)
\end{cases} \tag{7.6}$$

is well-posed if and only if the operator

$$\Upsilon : L^p_0(D) \cap \{1\}^\perp \longrightarrow L^p_1(D)\Big/\mathbb{R} \tag{7.7}$$

is an isomorphism.

(ii) For each $p \in (1-\varepsilon, 1]$, the mixed problem

$$\begin{cases}
\Delta u = 0 \quad \textit{in} \quad \Omega,\\
\dfrac{\partial u}{\partial \nu}\Big|_N = f \in h^p(N),\\
u\,\big|_D = g \in h^{1,p}(D),\\
(\nabla u)^* \in L^p(\partial\Omega)
\end{cases} \tag{7.8}$$

is well-posed if and only if the operator

$$\Upsilon : h^p_0(D) \cap \{1\}^\perp \longrightarrow h^{1,p}(D)\Big/\mathbb{R} \tag{7.9}$$

is an isomorphism.

(iii) For each $(s, 1/p) \in \mathcal{H}_\varepsilon$, the mixed problem

$$\begin{cases}
\Delta u = 0 \quad \textit{in} \quad \Omega,\\
\dfrac{\partial u}{\partial \nu}\Big|_N = f \in B^{p,p}s-1(N),\\
u\,\big|_D = g \in B^{p,p}_s(D),\\
u \in B^{p,p}_{s+\frac{1}{p}}(\Omega) \cap L^p_{s+\frac{1}{p}}(\Omega)
\end{cases} \tag{7.10}$$

is well-posed if and only if the operator

$$\Upsilon : B^{p,p}_{s-1,0}(D) \cap \{1\}^{\perp} \longrightarrow B^{p,p}_{s}(D)\Big/\mathbb{R} \tag{7.11}$$

is an isomorphism.

Proof. Consider the direct implication in (i). Let $\psi \in L^p_0(D) \cap \{1\}^{\perp}$ be such that $\Upsilon\psi$ is constant on D, and let u solve (7.1) with boundary datum ψ. Then $u\big|_D$ is constant and $\partial_\nu u\big|_N = \psi\big|_N = 0$. From the well-posedness of the mixed boundary problem (7.6) it follows that u is constant on Ω. In particular, $\psi = \partial_\nu u = 0$, which shows that the operator (7.7) is one-to-one.

To see that this operator is also onto, fix an arbitrary $g \in L^p_1(D)$. Since the problem (7.6) with boundary data $f = 0$ and this particular g is well-posed, it has a unique solution, which we denote by u. In particular, $u\big|_D = g$. Going further, set $\psi := \partial_\nu u \in L^p(\partial\Omega)$. Since, by design, $\partial_\nu u\big|_N = 0$, it follows that in fact $\psi \in L^p_0(D) \cap \{1\}^{\perp}$. Now, u solves the problem (7.1) with boundary datum ψ so that, by (7.4), $\Upsilon\psi$ differs from $u\big|_D = g$ by a constant. This shows that Υ in (7.7) is surjective and, ultimately, an isomorphism.

We now tackle the converse implication. More specifically, we assume that the operator (7.7) is an isomorphism and seek to show that (7.6) is well-posed, starting with uniqueness. By linearity, it suffices to show that any solution u of the homogeneous version of (7.6) vanishes. To see this, set $\psi := \partial_\nu u \in L^p_0(D) \cap \{1\}^{\perp}$ (here we use the fact that $\partial_\nu u\big|_N = 0$). Thus, u solves the Neumann problem with boundary datum ψ so that $\Upsilon\psi$ and $u\big|_D = 0$ differ by a constant. Since Υ in (7.7) is assumed to be one-to-one this further entails $\partial_\nu u = \psi = 0$ on $\partial\Omega$, which forces u to be a constant in Ω. However, $u\big|_D = 0$ so that, ultimately, $u = 0$ in Ω.

As far as the claim (i) in the statement of the theorem is concerned, it remains to show, under the same hypotheses as above, the existence of a solution for (7.6). To this end, given f, g as in (7.6), consider $F \in L^p(\partial\Omega)$ such that $F\big|_N = f$. In fact, matters can be arranged so that this extension of f lies in $L^p(\partial\Omega) \cap \{1\}^{\perp}$. Assuming that this is the case, we let v be a solution of the Neumann problem (7.1) with boundary datum F. Also, let $\psi \in L^p_0(D) \cap \{1\}^{\perp}$ be such that $\Upsilon\psi$ and $v\big|_D - g$ differ by a constant. Finally, let w be a solution of the Neumann problem (7.1) with boundary datum ψ and introduce $u := v - w$. It is straightforward to check that u minus a suitable constant solves (7.6). This finishes the proof of the claim made in (i).

The remaining points are dealt with in a similar fashion. □

The issue of the invertibility of the operator Υ in (7.7) and (7.9) has been considered by R. Brown and J. Sykes. Building on the earlier work in [5], in [53] they have proved the following key result.

Proposition 7.4. *Let Ω be a bounded creased domain in $\mathbb{R}^n$, $n \geq 3$, whose boundary is partitioned into two disjoint admissible patches D and N as in Definition* 2.3. *Then the mixed boundary value problem* (7.6) *is well-posed for each $p \in (1, 2]$, whereas the mixed boundary value problem* (7.8) *is well-posed for $p = 1$.*

Recall the hexagonal region $\mathcal{H}_\varepsilon$ from (7.2). The main result of this section is

Theorem 7.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded creased domain whose boundary decomposes into two pieces D and N in the sense of Definition* 2.3. *Then there* *exists $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ such that for each $(s, 1/p) \in \mathcal{H}_\varepsilon$ the mixed boundary value problem* (7.10) *is well-posed.*

To set the stage for the proof of the above theorem, we dispose of a couple of technical preliminaries.

Lemma 7.6. *Let Ω be a bounded creased domain in $\mathbb{R}^n$, $n \geq 3$, whose boundary is partitioned into two disjoint admissible patches D and N as in Definition* 2.3*. Then there exists $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ such that the operator* (7.7) *is an isomorphism for each $p \in (1, 2+\varepsilon)$, and the operator* (7.9) *is an isomorphism for each $p \in (1-\varepsilon, 1]$.*

Proof. That the operators (7.7), (7.9) are isomorphisms for $1 < p \leq 2$ and $p = 1$, respectively, follows from Proposition 7.3 and Proposition 7.4. In order to carry out the extension of these results to $2 < p < 2+\varepsilon$ for (7.7) and to $1-\varepsilon < p < 1$ for (7.9) we invoke Proposition 7.2, Proposition 6.7 plus the fact that the scales of spaces which are naturally associated with the operators under discussion are stable under complex interpolation (a fact already established in Proposition 4.6). □

A useful result regarding the dual operators Υ^* of Υ from the first two lines of (7.5) is the following.

Lemma 7.7. *Retain the same hypotheses as in Lemma* 7.6*. Then whenever p is such that $1-\varepsilon < p < 1$ and $s = (n-1)\left(\frac{1}{p}-1\right)$ the following diagram is commutative:*

$$\begin{array}{ccc} \left(h^{1,p}(D)/\mathbb{R}\right)^* & \xrightarrow{\Upsilon^*} & \left(h^p_0(D) \cap \{1\}^\perp\right)^* \\ \uparrow{\scriptstyle\tilde{\Psi}} & & \downarrow{\scriptstyle\Phi} \\ B^{\infty,\infty}_{s-1,0}(D) \cap \{1\}^\perp & \xrightarrow{\Upsilon} & B^{\infty,\infty}_s(D)/\mathbb{R} \end{array} \tag{7.12}$$

where the functions $\tilde{\Psi}$ and Φ are those defined in (4.34) *and* (4.24)*.*

Furthermore, for $1 < p < 2+\varepsilon$ and $1/p + 1/p' = 1$, the following diagram is also commutative:

$$\begin{array}{ccc} \left(L^{1,p}(D)/\mathbb{R}\right)^* & \xrightarrow{\Upsilon^*} & \left(L^p_0(D) \cap \{1\}^\perp\right)^* \\ \uparrow{\scriptstyle\tilde{\Psi}} & & \downarrow{\scriptstyle\Phi} \\ L^{p'}_{-1,0}(D) \cap \{1\}^\perp & \xrightarrow{\Upsilon} & L^{p'}(D)/\mathbb{R} \end{array} \tag{7.13}$$

where, this time, the maps $\tilde{\Psi}$, Φ are those introduced in (4.35)*,* (4.36)*.*

Proof. Consider first (7.12) and let $\Lambda \in B^{\infty,\infty}_{s-1,0}(D) \cap \{1\}^\perp$. Denote by u a solution of the Neumann problem (7.1) with datum Λ. We start by using (4.24) to conclude that $\Phi\left(\Upsilon^*(\tilde{\Psi}(\Lambda))\right) = \left[\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))}\,|_D\right]$, where $[\cdot]$ stands for "modding out" constants. In this notation, the commutativity of (7.12) reads as

$$\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))}\,|_D \ \text{ and } u|_D \text{ differ by a constant.} \tag{7.14}$$

To this end, we start by using (4.24) to conclude that the identity $\Phi\left(\Upsilon^*(\tilde{\Psi}(\Lambda))\right) = \left[\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))}|_D\right]$ holds. Recall the operator P_D defined as in (4.12) (with Σ replaced by D). From the definition of $\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))}$ in (4.23) and duality we have that

$$\langle\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))},\xi\rangle = \langle\Upsilon^*(\tilde{\Psi}(\Lambda)),(\pi(P_D(\xi)))\rangle = \langle\tilde{\Psi}(\Lambda),\Upsilon(\pi(\xi))\rangle, \tag{7.15}$$

for any $\xi \in h^p(\partial\Omega)$, $\operatorname{supp}\xi \subseteq D$. Employing the support condition on ξ in (7.15) we have used the fact that $P_D(\xi) = \xi$. Next, denote by w a solution of the Neumann problem with datum $\pi(\xi) \in h^p(D) \cap \{1\}^\perp$. Then $w|_{\partial\Omega} \in h^{1,p}(\partial\Omega)$, and by (7.4), $\Upsilon(\pi(\xi))$ and $w|_D$ differ by a constant. This and (4.30) further imply that

$$\langle\tilde{\Psi}(\Lambda),\Upsilon(\pi(\xi))\rangle = \langle\Lambda, w|_{\partial\Omega}\rangle. \tag{7.16}$$

Since u and w are harmonic functions in Ω, a simple application of Green's theorem gives that $\langle\tilde{\Psi}(\Lambda),\Upsilon(\pi(\xi))\rangle = \langle\Lambda, w|_{\partial\Omega}\rangle$ and using (7.15), (7.16) we obtain

$$\langle\phi_{\Upsilon^*(\tilde{\Psi}(\Lambda))},\xi\rangle = \langle u,\pi(\xi)\rangle. \tag{7.17}$$

Finally, it is straightforward to check that (7.17) gives (7.14). This completes the proof of the commutativity of (7.12). The commutativity of the second diagram (7.13) follows in a similar manner, and the proof of Lemma 7.7 is finished. □

We are now ready to present the

Proof of Theorem 7.5. From (*iii*) in Proposition 7.3 we know that it suffices to show that

$$\exists\,\varepsilon > 0 \text{ such that the operator (7.11) is an isomorphism } \forall\,(s,1/p) \in \mathcal{H}_\varepsilon. \tag{7.18}$$

To begin with, Lemma 7.6 ensures that there exists $\varepsilon > 0$ such that the operator

$$\Upsilon : L^p_0(D)\cap\{1\}^\perp \longrightarrow L^p_1(D)\Big/\mathbb{R} \text{ is an isomorphism } \forall\,p \in (1,2+\varepsilon). \tag{7.19}$$

Based on this, duality, Lemma 7.7 and the fact that the vertical arrows in (7.13) are isomorphisms, we may further conclude that

$$\Upsilon : L^{p'}_{-1,0}(D)\cap\{1\}^\perp \longrightarrow L^{p'}(D)\Big/\mathbb{R} \text{ is an isomorphism } \forall\,p \in (1,2+\varepsilon). \tag{7.20}$$

In a similar fashion,

$$\Upsilon : B^{\infty,\infty}_{s-1,0}(D)\cap\{1\}^\perp \longrightarrow B^{\infty,\infty}_{s}(D)\Big/\mathbb{R} \text{ is an isomorphism } \forall\,s \in (0,\varepsilon), \tag{7.21}$$

for a possibly different $\varepsilon > 0$.

Next, based on Proposition 6.6 and Proposition 6.2 we may also conclude that

$$\Upsilon : B^{1,1}_{-s,0}(D)\cap\{1\}^\perp \longrightarrow B^{1,1}_{1-s}(D)\Big/\mathbb{R} \text{ is an isomorphism } \forall\,s \in (0,\varepsilon). \tag{7.22}$$

With (7.19), (7.21), (7.20) and (7.22) in hand, (7.18) follows by repeated applications of the real and complex methods of interpolation. In this segment of our analysis, (a version of) Proposition 4.2 is used. □

Remark. Let $\mathcal{S}$ stand for the harmonic single layer potential operator associated with the Lipschitz domain Ω, and set $Sf := \operatorname{Tr}\mathcal{S}f$, $(-\frac{1}{2}I + K^*)f := \partial_\nu\mathcal{S}f$ for its trace and normal derivative on $\partial\Omega$. For a more detailed discussion see, e.g., [10], [15]. It has been proved in [16] that for every bounded Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. If $(s,1/p) \in \mathcal{H}_\varepsilon$, then any harmonic function u belonging to either $B^{p,p}_{s+\frac{1}{p}}(\Omega)$ or $L^p_{s+\frac{1}{p}}(\Omega)$ can be represented in the form $u = \mathcal{S}f$ for some unique $f \in B^{p,p}_{s-1}(\partial\Omega)$. It follows from this and Theorem 7.5 that for any bounded creased domain $\Omega \subset \mathbb{R}^n$ whose boundary decomposes into two pieces D and N (in the sense of Definition 2.3), there exists $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ with the following significance. For any $(s, 1/p) \in \mathcal{H}_\varepsilon$, the operator

$$T : B^{p,p}_{s-1}(\partial\Omega) \ni f \mapsto \Big(Sf\Big|_D\,,\ \Big(-\tfrac{1}{2}I + K^*\Big)f\Big|_N\Big) \in B^{p,p}_s(D) \oplus B^{p,p}_{s-1}(N) \tag{7.23}$$

is an isomorphism. Consequently, the solution of the mixed boundary value problem described in the statement of Theorem 7.5 has the following integral representation formula:

$$u = \mathcal{S}\Big(T^{-1}(f,g)\Big). \tag{7.24}$$

8. The Poisson problem with mixed conditions

The first order of business is to define a concept of normal derivative which is suitable in the context of the Poisson problem with mixed boundary conditions. Recall the spaces introduced according to the general recipe in (5.3) and that, in general, $\mathrm{Lip}_c(\mathcal{X})$ stands for the collection of all Lipschitz functions defined on $\mathcal{X}$ with compact support.

Proposition 8.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with connected boundary and suppose that $D, N \subset \partial\Omega$ are two non-empty, disjoint admissible patches satisfying $\bar{D} \cap \bar{N} = \partial D = \partial N$ and $\bar{D} \cup \bar{N} = \partial\Omega$.*

Fix $p, p' \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $s \in (0,1)$ and consider $u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$ and $F \in \Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*$ such that $\Delta u = F\Big|_\Omega$, where $F\Big|_\Omega$ is the distribution on Ω defined by the requirement that $\langle F\Big|_\Omega, \xi\rangle = F(\xi)$ for each $\xi \in C^\infty_c(\Omega) \hookrightarrow B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$. Finally, recall the extension operator Ext *from Proposition* 5.4 *(considered here for $\Sigma = D$).*

Then $\partial^F_\nu u\,|_N$ given by

$$\langle \partial^F_\nu u\,|_N\,, \phi\rangle := \langle \nabla u, \nabla \mathrm{Ext}\,\tilde{\phi}\rangle + \langle F, \mathrm{Ext}\,\tilde{\phi}\rangle, \qquad \phi \in \mathrm{Lip}_c(N), \tag{8.1}$$

is an element in $B^{p,p}_{s-1}(N)$. Above, the tilde denotes extension by zero to $\partial\Omega$. Also, the first pairing in (8.1) *is the distributional pairing on N, while the other two are the duality ones between $B^{p,p}_{s+\frac{1}{p}-1}(\Omega)$ and $B^{p',p'}_{1-s-\frac{1}{p}}(\Omega)$ on the one hand, and $\Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*$ and $B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$ on the other hand.*

Furthermore, this definition of $\partial^F_\nu u\,|_N$ is independent of the particular extension operator used in (8.1) *and the estimate*

$$\big\|\partial^F_\nu u\,|_N\big\|_{B^{p,p}_{s-1}(N)} \le C(\Omega, D, N, p, s)\Big(\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} + \|F\|_{\big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega;D)\big)^*}\Big) \tag{8.2}$$

holds. Finally, a similar definition can be given for $\partial^F_\nu u\,|_N$ in the case when $u \in L^p_{s+\frac{1}{p}}(\Omega)$ and $F \in \Big(L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*$ is such that $\Delta u = F\Big|_\Omega$. Once again, $\partial^F_\nu u\,|_N \in B^{p,p}_{s-1}(N)$ and it enjoys similar properties as before.

Proof. By Proposition 5.4,

$$\mathrm{Ext}: B^{p',p'}_{1-s,0}(N) \longrightarrow B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D) \tag{8.3}$$

is well-defined, linear and bounded whenever $p, p' \in (1,\infty)$, $\frac{1}{p}+\frac{1}{p'}=1$, $s\in(0,1)$. Also, $\{\tilde{\phi} : \phi \in \mathrm{Lip}_c(N)\}$ is a dense subspace of $B^{p',p'}_{1-s,0}(N)$, the dual of which can be naturally identified with $B^{p,p}_{s-1}(N)$ (cf. (4.37)). Hence, (8.1) defines $\partial^F_\nu u\,|_N$ as an element in $B^{p,p}_{s-1}(N)$. Furthermore, a quick inspection of this argument shows that (8.2) holds.

It remains to prove that the definition of (8.1) is independent of the particular choice of the extension operator Ext. To this end, fix $\phi \in \mathrm{Lip}_c(N)$ and let $\Phi \in B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$ be an arbitrary function such that $\mathrm{Tr}\,\Phi = \tilde{\phi}$. We desire to show that

$$\langle \partial^F_\nu u\,|_N\,, \phi\rangle = \langle \nabla u, \nabla\Phi\rangle + \langle F, \Phi\rangle. \tag{8.4}$$

By linearity and the definition of $\partial^F_\nu u\,|_N$, it suffices to show that

$$0 = \langle \nabla u, \nabla\Phi\rangle + \langle F, \Phi\rangle \tag{8.5}$$

whenever $\Phi \in B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$ satisfies $\mathrm{Tr}\,\Phi = 0$. Consider such a function and, using Proposition 3.12 in [25], take a sequence $(\Phi_j)_j$, $\Phi_j \in C^\infty_c(\Omega)$ such that

$$\Phi_j \longrightarrow \Phi \quad \text{in} \quad B^{p',p'}_{2-s-\frac{1}{p}}(\Omega). \tag{8.6}$$

It follows that $\Phi_j \longrightarrow \Phi$ in $B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$ since the latter space is, by definition, a closed subspace of $B^{p',p'}_{2-s-\frac{1}{p}}(\Omega)$. Furthermore, $\nabla\Phi_j \longrightarrow \nabla\Phi$ in $B^{p',p'}_{1-s-\frac{1}{p}}(\Omega) = \left(B^{p,p}_{-1+s+\frac{1}{p}}(\Omega)\right)^*$ by [55]. Since $\nabla u \in B^{p,p}_{-1+s+\frac{1}{p}}(\Omega)$, using the distributional derivative of the Laplacian and the fact that $\Delta u = F\,|_\Omega$ as distributions in Ω, we may write

$$\begin{aligned}\langle \nabla u, \nabla\Phi\rangle + \langle F, \Phi\rangle &= \lim_{j\to\infty} \left(\langle \nabla u, \nabla\Phi_j\rangle + \langle F, \Phi_j\rangle\right)\\ &= \lim_{j\to\infty}\left(-\langle \Delta u, \Phi_j\rangle + \langle F\,|_\Omega\,, \Phi_j\rangle\right) = 0,\end{aligned} \tag{8.7}$$

as wanted. The case of Sobolev spaces is similar, and this finishes the proof of the Proposition 8.1. □

Remark. The above proof gives a little bit more. Specifically, retain the same geometric context as in Proposition 8.1 and fix $p, p' \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $s\in(0,1)$.

Then, for any $u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$, $F \in \left(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\right)^*$ such that $\Delta u = F\Big|_\Omega$, and for any $v \in B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)$, the following Green-type formula holds:

$$\langle \partial^F_\nu u\,|_N\,, \mathrm{Tr}\, v\rangle = \langle \nabla u, \nabla v\rangle + \langle F, v\rangle. \tag{8.8}$$

Note that $\partial^F_\nu u\,|_N \in B^{p,p}_{s-1}(N)$ and, by Theorem 5.3, $\mathrm{Tr}\, v \in B^{p',p'}_{1-s,0}(N)$. Accordingly, the first pairing above is well-defined.

The same type of result holds when $u \in L^p_{s+\frac{1}{p}}(\Omega)$, $F \in \left(L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)\right)^*$ satisfy $\Delta u = F\Big|_\Omega$, and when $v \in L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)$.

The main result of this section is the following.

Theorem 8.2. *Let Ω be a bounded creased domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega$ decomposes into two pieces D and N (as in Definition 2.3). Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the mixed boundary value problems*

$$(8.9)\qquad \begin{cases} u \in L^p_{s+\frac{1}{p}}(\Omega), \\ \Delta u = F\Big|_{\Omega},\ F \in \Big(L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*, \\ \operatorname{Tr} u\,\big|_D = g \in B^{p,p}_s(D), \\ \partial^F_\nu u\,\big|_N = f \in B^{p,p}_{s-1}(N) \end{cases}$$

and

$$(8.10)\qquad \begin{cases} u \in B^{p,p}_{s+\frac{1}{p}}(\Omega), \\ \Delta u = F\Big|_{\Omega}, F \in \Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*, \\ \operatorname{Tr} u\,\big|_D = g \in B^{p,p}_s(D), \\ \partial^F_\nu u\,\big|_N = f \in B^{p,p}_{s-1}(N) \end{cases}$$

are well-posed whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$ (here, as before, $1/p + 1/p' = 1$).

Proof. Fix $\varepsilon > 0$ such that the results in Section 7 hold. Next, consider an arbitrary $F \in \Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^*$ and let

$$(8.11)\qquad \widehat{F} : L^{p'}_{2-s-\frac{1}{p}}(\Omega) \longrightarrow \mathbb{R}$$

be an extension of F to an element in $\Big(L^{p'}_{2-s-\frac{1}{p}}(\Omega)\Big)^*$ (whose existence is guaranteed by the Hahn-Banach Theorem). If we denote by R_Ω the restriction operator from $\mathbb{R}^n$ to Ω, it follows that

$$(8.12)\qquad \widehat{F} \circ R_\Omega \in \Big(L^{p'}_{2-s-\frac{1}{p}}(\mathbb{R}^n)\Big)^* = L^p_{-2+s+\frac{1}{p}}(\mathbb{R}^n).$$

Furthermore, $\operatorname{supp} \widehat{F} \circ R_\Omega \subseteq \bar{\Omega}$. Apply the Newtonian potential operator in $\mathbb{R}^n$ to the distribution $\widehat{F} \circ R_\Omega$ and then truncate with a compactly supported function which is identically one in a neighborhood of $\bar{\Omega}$. Call the resulting function v, so that

$$(8.13)\qquad v \in L^p_{s+\frac{1}{p}}(\mathbb{R}^n) \quad \text{and} \quad \Delta v\Big|_{\Omega} = F\Big|_{\Omega} \quad \text{as distributions in} \quad \Omega.$$

Now let w be the solution of the mixed boundary problem

$$(8.14)\qquad \begin{aligned} &\Delta w = 0 \text{ in } \Omega, \quad \operatorname{Tr} w\,\big|_D = g - \operatorname{Tr} v\,\big|_D \in B^{p,p}_s(D), \\ &\partial^F_\nu w\,\big|_N = f - \partial^F_\nu v\,\big|_N \in B^{p,p}_{s-1}(N). \end{aligned}$$

By virtue of Theorem 7.5, the problem (8.14) has a unique solution $w \in L^p_{s+\frac{1}{p}}(\Omega)$ whenever $(s, 1/p) \in \mathcal{H}_\varepsilon$. It is then straightforward to check that $u := v + w$ is a solution of the original problem (8.9). Natural estimates for this solution are implicit in the above construction, and uniqueness follows directly from Theorem 7.5.

The reasoning for (8.10) is similar and this completes the proof. $\square$

An immediate consequence of Theorem 8.2 is the following.

Corollary 8.3. *Under the same geometrical assumptions as in Theorem* 8.2*, there exists* $\varepsilon = \varepsilon(\Omega, D, N) > 0$ *such that whenever* $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ *the mixed boundary value problem*

$$\begin{cases} \Delta u = F\Big|_{\Omega}, \ F \in \Big(L_1^{p'}(\Omega; D)\Big)^*, \\ \operatorname{Tr} u\,\big|_D = 0, \\ \partial_\nu^F u\,\big|_N = 0 \end{cases} \tag{8.15}$$

has a unique solution in the Sobolev space $L_1^p(\Omega)$.

Proof. This corresponds to taking $s + \frac{1}{p} = 1$ in the portion of Theorem 8.2 referring to (8.9). Elementary algebra then shows that this problem is well-posed in this context provided that $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ for some $\varepsilon > 0$. □

Let us take a closer look at boundary value problems such as (8.15) and denote by G the solution operator of the problem

$$\Delta u = F\,\big|_{\Omega}, \quad \operatorname{Tr} u\,\big|_D = 0, \quad \partial_\nu^F u\,\big|_N = 0, \tag{8.16}$$

i.e., $G(F) = u$. In terms of the Green operator G, the results proved in this section (in concert with Theorem 5.3) yield the following:

Corollary 8.4. *Let* Ω *be a bounded creased domain in* $\mathbb{R}^n$*,* $n \geq 3$*, whose boundary decomposes into two pieces* D *and* N *as in Definition* 2.3*. Then there exists some* $\varepsilon = \varepsilon(\Omega) > 0$ *such that the Green operator extends as a bounded operator*

$$\begin{aligned} G &: \Big(L^{p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^* \longrightarrow L^p_{s+\frac{1}{p}}(\Omega; D), \\ G &: \Big(B^{p',p'}_{2-s-\frac{1}{p}}(\Omega; D)\Big)^* \longrightarrow B^{p,p}_{s+\frac{1}{p}}(\Omega; D), \end{aligned} \tag{8.17}$$

for each $(s, 1/p) \in \mathcal{H}_\varepsilon$.

We conclude this section with a discussion regarding the sharpness of the range of indices in Corollary 8.4. First, the spaces intervening in (8.17) form complex interpolation scales, thanks to Proposition 5.5. Therefore we may conclude that

$$\mathcal{O} := \{(s, 1/p) \in (0,1) \times (0,1) : \ \text{the operators (8.17) are bounded}\} \tag{8.18}$$

is a (geometrically) convex set. Based on Proposition 6.7 it is possible to show that the region $\mathcal{O}$ is open. Also, by Lax-Milgram's lemma, the point $(1/2, 1/2)$ belongs to $\mathcal{O}$ and, since the operator G is formally selfadjoint, the region $\mathcal{O}$ is symmetric with respect to the point $(1/2, 1/2)$. Consider now $\{p \in (1, \infty) : (1 - 1/p, 1/p) \in \mathcal{O}\}$ which, by the above analysis, is an open interval of the form (p_o, q_o) with $1/p_o + 1/q_o = 1$. Finally, Corollary 8.3 guarantees that $q_o > 3$ whereas from [12] we know that there exist convex polyhedral domains in $\mathbb{R}^3$ for which q_o can be as close to 3 as we please. In summary, Corollary 8.4 is in the nature of best possible.

9. Applications

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. If $\mathcal{X}(\Omega)$ denotes a space of distributions in Ω we write $\mathcal{X}(\Omega, \mathbb{R}^n)$ for the space of $\mathcal{X}(\Omega)$-valued vector fields in Ω. Going further, let div denote the usual divergence operator, i.e. $\operatorname{div} u = \sum_{j=1}^n \partial_j u_j$

if $u = (u_1, \ldots, u_n)$. Also, for each vector field $u \in L^p_\alpha(\Omega, \mathbb{R}^n)$ with $\operatorname{div} u = \Lambda\Big|_\Omega$ (as distributions in Ω) where $\Lambda \in \Big(L^{p'}_{1-\alpha}(\Omega)\Big)^*$ we define $\nu \cdot u \in B^{p,p}_{\alpha-\frac{1}{p}}(\partial\Omega) = \Big(B^{p',p'}_{-\alpha+\frac{1}{p}}(\partial\Omega)\Big)^*$ by setting

$$\langle \nu \cdot u, \phi\rangle := \langle \Lambda, \operatorname{Ext}\phi\rangle + \langle u, \nabla \operatorname{Ext}\phi\rangle, \qquad \forall\, \phi \in B^{p',p'}_{-\alpha+\frac{1}{p}}(\partial\Omega) \tag{9.1}$$

with $\operatorname{Ext} : B^{p',p'}_{-\alpha+\frac{1}{p}}(\partial\Omega) \to L^{p'}_{1-\alpha}(\Omega)$ a linear right inverse for the trace operator. Since $\phi \in B^{p',p'}_{-\alpha+\frac{1}{p}}(\partial\Omega)$ entails $\nabla\operatorname{Ext}\phi \in L^{p'}_{-\alpha}(\Omega) = \Big(L^p_\alpha(\Omega)\Big)^*$, it follows that all pairings in (9.1) are well-defined and, moreover,

$$\|\nu \cdot u\|_{B^{p,p}_{\alpha-\frac{1}{p}}(\partial\Omega)} \leq C\Big(\|u\|_{L^p_\alpha(\Omega,\mathbb{R}^n)} + \|\Lambda\|_{(L^{p'}_{1-\alpha}(\Omega))^*}\Big). \tag{9.2}$$

The definition of $\nu\cdot u$ depends not only on $u \in L^p_\alpha(\Omega, \mathbb{R}^n)$ but also on the extension of the distribution $\operatorname{div} u \in L^p_{\alpha-1}(\Omega)$ to a functional $\Lambda \in \Big(L^{p'}_{1-\alpha}(\Omega)\Big)^*$. On the other hand, by once again relying on Proposition 3.12 in [25] and proceeding as in the proof of Proposition 8.1, it can be shown that the definition (9.1) is independent of the particular choice of the extension operator Ext. In fact, if $1 < p < \infty$ and $-1 + 1/p < \alpha < 1/p$, then the following general integration by parts formula holds:

$$\langle \nu \cdot u, \operatorname{Tr} v\rangle = \langle \Lambda, v\rangle + \langle u, \nabla v\rangle \tag{9.3}$$

for any $u \in L^p_\alpha(\Omega, \mathbb{R}^n)$ with $\operatorname{div} u = \Lambda\Big|_\Omega$ for some $\Lambda \in \Big(L^{p'}_{1-\alpha}(\Omega)\Big)^*$, and any scalar function $v \in L^{p'}_{1-\alpha}(\Omega)$.

To state our first main result in this section, for each admissible patch $\Sigma \subseteq \partial\Omega$ and $1 < p < \infty$, $-1 + 1/p < \alpha < 1/p$, consider the space

$$L^p_\alpha(\Omega; \Sigma; \operatorname{div} 0) = \{u \in L^p_\alpha(\Omega, \mathbb{R}^n) : \ \operatorname{div} u = 0 \text{ in } \Omega,\ \ \nu \cdot u = 0 \text{ on } \Sigma\}. \tag{9.4}$$

Above, $\nu \cdot u$ is defined as in (9.1) with $\Lambda = 0$. In fact, we shall always adopt this convention whenever dealing with divergence-free vector fields.

Theorem 9.1. *Let Ω be a bounded creased domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega$ decomposes into two pieces D and N as in Definition 2.3. Then there exists some $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ such that*

$$L^p_\alpha(\Omega, \mathbb{R}^n) = L^p_\alpha(\Omega; N; \operatorname{div} 0) \oplus \nabla L^p_{\alpha+1}(\Omega; D) \tag{9.5}$$

(topologically and algebraically), whenever $(1 + \alpha - \frac{1}{p}, \frac{1}{p}) \in \mathcal{H}_\varepsilon$.

Furthermore, a similar decomposition holds for $B^{p,p}_\alpha(\Omega, \mathbb{R}^n)$.

Proof. Set $s := 1 + \alpha - \frac{1}{p} \in (0, 1)$ so that $(s, \frac{1}{p}) \in \mathcal{H}_\varepsilon$. Also, let Π stand for the Newtonian potential operator in $\mathbb{R}^n$. Then for each $u \in L^p_\alpha(\Omega, \mathbb{R}^n)$ introduce

$$Pu := u - \nabla \operatorname{div} \Pi\, u - \nabla v, \tag{9.6}$$

where $v \in L^p_{s+\frac{1}{p}}(\Omega)$ is a solution to the mixed boundary value problem

$$
\begin{cases} \Delta v = 0 \text{ in } \Omega, \\ \operatorname{Tr} v\Big|_D = -\operatorname{Tr}(\operatorname{div}\Pi\, u)\Big|_D \in B^{p,p}_s(D), \\ \partial^0_\nu v\Big|_N = [\nu\cdot(u - \nabla\operatorname{div}\Pi\, u)]\Big|_N \in B^{p,p}_{s-1}(N). \end{cases} \tag{9.7}
$$

Above, the dot product is taken in the sense of (9.1) (note that the vector field $u - \nabla\operatorname{div}\Pi\, u \in L^p_\alpha(\Omega,\mathbb{R}^n)$ is divergence-free).

It follows that $Pu \in L^p_\alpha(\Omega,\mathbb{R}^n)$ satisfies $\operatorname{div} Pu = 0$ in Ω and $\nu\cdot Pu = 0$ on N. Consequently, $Pu \in L^p_\alpha(\Omega;N;\operatorname{div}0)$. Then $u = Pu + \nabla[\operatorname{div}\Pi\, u + v]$ yields a decomposition of u as in (9.5) since $\operatorname{div}\Pi\, u + v \in L^p_{\alpha+1}(\Omega)$ has, by design, a trace which vanishes on D. Thus, $\operatorname{div}\Pi\, u + v \in L^p_{\alpha+1}(\Omega;D)$, by Theorem 5.3. Note also that naturally accompanying estimates for the pieces involved are implicit from what we have done so far.

It remains to prove that such a splitting of u is unique. By linearity, it suffices to show that if $w \in L^p_\alpha(\Omega;N;\operatorname{div}0)$ and $\phi \in L^p_{\alpha+1}(\Omega;D)$ are such that $w + \nabla\phi = 0$, then, necessarily, $w = \nabla\phi = 0$ in Ω. Indeed, it follows that $\nabla\phi = -w$ and, hence, $\Delta\phi = 0$ since w is divergence-free. On the other hand, $\partial^0_\nu\phi\,|_N = 0$ since $\nu\cdot w = 0$ on N and $\operatorname{Tr}\phi = 0$ on D by Theorem 5.3. Hence, the well-posedness of the mixed boundary value problem (8.9) forces $\phi = 0$ in Ω, which also entails $w = 0$ in Ω, as desired.

The case of vector fields in Besov spaces is treated similarly. □

For Sobolev spaces with integer amount of smoothness we have the following.

Corollary 9.2. *Under the same geometrical assumptions as in Theorem* 9.1*, there exists* $\varepsilon = \varepsilon(\Omega,D,N) > 0$ *such that the (algebraic and topologic) decomposition*

$$
L^p(\Omega,\mathbb{R}^n) = L^p(\Omega;N;\operatorname{div}0) \oplus \nabla L^p_1(\Omega;D) \tag{9.8}
$$

is valid whenever $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$.

Proof. This corresponds to taking $\alpha = 0$ in Theorem 9.1. □

Recall the Green operator G, defined as the solution operator for the problem (8.16). Mapping properties of G on Sobolev and Besov scales have been established in Corollary 8.4. Here we desire to study the boundedness of ∇G on the scale of Lebesgue spaces. The theorem below extends the main result in [9], where the case $N = \emptyset$ has been first treated.

Theorem 9.3. *Let* Ω *be a bounded creased domain in* $\mathbb{R}^n$*,* $n \geq 3$*, whose boundary decomposes into two pieces* D *and* N *as in Definition* 2.3*. Then there exists* $\varepsilon = \varepsilon(\partial\Omega,D,N) > 0$ *such that*

$$
\nabla G : L^p(\Omega) \longrightarrow L^{p^*}(\Omega) \tag{9.9}
$$

is a bounded mapping whenever

$$
1 < p < \frac{3n}{n+3} + \varepsilon \quad \text{and} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{9.10}
$$

Proof. Fix $1 < p < n$, $f \in L^p(\Omega)$ and pick $F \in L^p_2(\Omega)$ such that $\Delta F = f$ in Ω. Using the Sobolev embedding theorem we have that

$$F \in L^p_2(\Omega) \hookrightarrow L^q_{s+\frac{1}{q}}(\Omega) \quad \text{and} \quad \nabla F \in L^p_1(\Omega) \hookrightarrow L^q_{s+\frac{1}{q}-1}(\Omega) \tag{9.11}$$

whenever

$$\frac{1}{q} - \frac{s+\frac{1}{q}}{n} = \frac{1}{p} - \frac{2}{n}, \quad s \in (0,1), \quad q \in (1,\infty). \tag{9.12}$$

Note that $\mathrm{Tr}\, F\,|_D \in B^{q,q}_s(D)$. Also, recall that, according to (9.1), $\nu \cdot \nabla F$ can be defined as an element in $B^{q,q}_{s-1}(\partial\Omega)$ since $\mathrm{div}\,(\nabla F) = f\Big|_\Omega$ as distributions in Ω, where we view f as an element in $\Big(L^q_{2-s-\frac{1}{q}}(\Omega)\Big)^*$ defined as $L^q_{2-s-\frac{1}{q}}(\Omega) \ni g \mapsto \int_\Omega fg$. Note that the latter integral pairing is well-defined since, by the Sobolev embedding theorem,

$$f \in L^p(\Omega) \hookrightarrow \Big(L^q_{2-s-\frac{1}{q}}(\Omega)\Big)^* \quad \text{whenever} \quad s, q \quad \text{satisfy} \quad (9.12). \tag{9.13}$$

In particular $\nu \cdot \nabla F\,|_N \in B^{q,q}_{s-1}(N)$. Having dispensed with these preliminaries, we next consider the mixed boundary value problem

$$\begin{cases} w \in L^q_{s+\frac{1}{q}}(\Omega), \\ \Delta w = 0 \text{ in } \Omega, \\ \mathrm{Tr}\, w\Big|_D = \mathrm{Tr}\, F\Big|_D \in B^{q,q}_s(D), \\ \partial^0_\nu w\Big|_N = \nu \cdot \nabla F\Big|_N \in B^{q,q}_{s-1}(N). \end{cases} \tag{9.14}$$

By Theorem 7.5, there exists $\varepsilon = \varepsilon(\partial\Omega, D, N) > 0$ such that (9.14) is well-posed whenever $(s, 1/q) \in \mathcal{H}_\varepsilon$, i.e.,

$$-\frac{\varepsilon}{2} < \frac{1}{q} - \frac{s}{2} < \frac{1+\varepsilon}{2}, \qquad 1 < q < \infty, \quad 0 < s < 1. \tag{9.15}$$

Assuming that both (9.12) and (9.15) hold, we now observe that $F - w \in L^q_{s+\frac{1}{q}}(\Omega)$ satisfies $\Delta(F-w) = f\Big|_\Omega$ as distributions in Ω where, once again, we regard f as functional in $\Big(L^q_{2-s-\frac{1}{q}}(\Omega; D)\Big)^*$ given by $L^q_{2-s-\frac{1}{q}}(\Omega; D) \ni g \mapsto \int_\Omega fg$ (the integral pairing is well-defined thanks to (9.13)). Moreover, matters have been arranged so that $\mathrm{Tr}\,(F-w)\,|_D = 0$.

We now claim that $\partial^f_\nu (F-w)\,|_N = 0$ also. To see this, for an arbitrary function $\Phi \in \mathrm{Lip}\,(\bar\Omega)$ with $\mathrm{supp}\,\Phi \cap \partial\Omega \subset N$ we may write, based on Green's formula (8.8),

$$\langle \partial^f_\nu (F-w)\,|_N\,, \mathrm{Tr}\,\Phi\rangle = \langle \nabla(F-w), \nabla(\Phi|_\Omega)\rangle + \langle f, \Phi|_\Omega\rangle. \tag{9.16}$$

Note that $\Phi|_\Omega \in L^q_{2-s-\frac{1}{q}}(\Omega; D)$, so the last pairing above is well-defined. Let us consider the first term in the right-hand side of (9.16) separately:

$$\begin{aligned} \langle \nabla(F-w), \nabla(\Phi|_\Omega)\rangle &= \langle \nabla F, \nabla(\Phi|_\Omega)\rangle - \langle \nabla w, \nabla(\Phi|_\Omega)\rangle \\ &= \langle \nu \cdot \nabla F, \mathrm{Tr}\,\Phi\rangle - \langle f, \Phi|_\Omega\rangle - \langle \nabla w, \nabla(\Phi|_\Omega)\rangle. \end{aligned} \tag{9.17}$$

On the other hand, since $\partial^0_\nu w|_N = \nu \cdot \nabla F|_N$ and $\mathrm{supp}\,\Phi \cap \partial\Omega \subset N$ we obtain

$$\langle \nu \cdot \nabla F, \mathrm{Tr}\,\Phi\rangle = \langle \partial^0_\nu w, \mathrm{Tr}\,\Phi\rangle = \langle \nabla w, \nabla(\Phi|_\Omega)\rangle \tag{9.18}$$

so that, altogether, (9.16)-(9.18) yield $\langle \partial_\nu^f(F-w)\,|_{N}\,, \mathrm{Tr}\,\Phi\rangle = 0$ for any $\Phi \in \mathrm{Lip}\,(\bar{\Omega})$ with $\mathrm{supp}\,\Phi \cap \partial\Omega \subset N$. This proves that $\partial_\nu^f(F-w)\,|_{N} = 0$, as claimed above.

Thus, by definition, $Gf = F - w$, and the fact that (9.9) is well-defined and bounded is going to be a consequence of

$$\nabla F - \nabla w \in L^q_{s+\frac{1}{q}-1}(\Omega) \hookrightarrow L^{p^*}(\Omega), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{9.19}$$

The embedding in (9.19) holds provided $\frac{1}{p^*} = \frac{1}{q} - \frac{s+1/q}{n} + \frac{1}{n}$, which is automatically satisfied whenever (9.12) is assumed and, in addition,

$$s + \frac{1}{q} > 1. \tag{9.20}$$

In summary, if $1 < p < n$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, then the operator (9.9) is bounded so long as (9.12), (9.15) and (9.20) are true. (These conditions are best understood on a two-dimensional diagram.) Elementary algebra now shows that, for a given $p \in (1, n)$, it is always possible to choose q and s satisfying (9.12), (9.15) and (9.20) if and only if $1 < p < \frac{3n}{n+3} + \varepsilon$ for some small $\varepsilon > 0$, which is precisely the condition in the statement of the theorem. This concludes the proof of Theorem 9.3. $\square$

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