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EQUIVALENCE OF DOMAINS ARISING FROM DUALITY OF ORBITS ON FLAG MANIFOLDS III

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Abstract. In Gindikin and Matsuki 2003, we defined a $G_\mathbb{R}$-$K_\mathbb{C}$ invariant subset $C(S)$ of $G_\mathbb{C}$ for each $K_\mathbb{C}$-orbit S on every flag manifold $G_\mathbb{C}/P$ and conjectured that the connected component $C(S)_0$ of the identity would be equal to the Akhiezer-Gindikin domain D if S is of nonholomorphic type. This conjecture was proved for closed S in Wolf and Zierau 2000 and 2003, Fels and Huckleberry 2005, and Matsuki 2006 and for open S in Matsuki 2006. It was proved for the other orbits in Matsuki 2006, when $G_\mathbb{R}$ is of non-Hermitian type. In this paper, we prove the conjecture for an arbitrary non-closed $K_\mathbb{C}$-orbit when $G_\mathbb{R}$ is of Hermitian type. Thus the conjecture is completely solved affirmatively.

1. Introduction

Let $G_\mathbb{C}$ be a connected complex semisimple Lie group and $G_\mathbb{R}$ a connected real form of $G_\mathbb{C}$. Let $K_\mathbb{C}$ be the complexification in $G_\mathbb{C}$ of a maximal compact subgroup K of $G_\mathbb{R}$. Let $X = G_\mathbb{C}/P$ be a flag manifold of $G_\mathbb{C}$, where P is an arbitrary parabolic subgroup of $G_\mathbb{C}$. Then there exists a natural one-to-one correspondence between the set of $K_\mathbb{C}$-orbits S and the set of $G_\mathbb{R}$-orbits S' on X given by the condition:

$$S \leftrightarrow S' \Longleftrightarrow S \cap S' \text{ is non-empty and compact} \tag{1.1}$$

([M2]). For each $K_\mathbb{C}$-orbit S we defined in [GM1] a subset $C(S)$ of $G_\mathbb{C}$ by

$$C(S) = \{x \in G_\mathbb{C} \mid xS \cap S' \text{ is non-empty and compact}\}$$

where S' is the $G_\mathbb{R}$-orbit on X given by (1.1).

Akhiezer and Gindikin defined a domain $D/K_\mathbb{C}$ in $G_\mathbb{C}/K_\mathbb{C}$ as follows ([AG]). Let $\mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{m}$ denote the Cartan decomposition of $\mathfrak{g}_\mathbb{R} = \mathrm{Lie}(G_\mathbb{R})$ with respect to K. Let $\mathfrak{t}$ be a maximal abelian subspace in $i\mathfrak{m}$. Put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma\}$$

where Σ is the restricted root system of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}$. Then D is defined by

$$D = G_\mathbb{R}(\exp \mathfrak{t}^+)K_\mathbb{C}.$$

We conjectured the following in [GM1].

Conjecture 1.1 (Conjecture 1.6 in [GM1]). *Suppose that $X = G_\mathbb{C}/P$ is not $K_\mathbb{C}$-homogeneous. Then we will have $C(S)_0 = D$ for all $K_\mathbb{C}$-orbits S of non-holomorphic type on X. Here $C(S)_0$ is the connected component of $C(S)$ containing the identity.*

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Remark 1.2. When $G_{\mathbb{R}}$ is of Hermitian type, there exist two special closed $K_{\mathbb{C}}$-orbits $S_1 = K_{\mathbb{C}}B/B = Q/B$ and $S_2 = K_{\mathbb{C}}w_0B/B = \overline{Q}w_0/B$ on the full flag manifold $G_{\mathbb{C}}/B$, where $Q = K_{\mathbb{C}}B$ is the usual maximal parabolic subgroup of $G_{\mathbb{C}}$ defined by a nontrivial central element in $i\mathfrak{k}$ and w_0 is the longest element in the Weyl group. For each parabolic subgroup P containing the Borel subgroup B, two closed $K_{\mathbb{C}}$-orbits S_1P and S_2P on $G_{\mathbb{C}}/P$ are called of holomorphic type and all the other $K_{\mathbb{C}}$-orbits are called of nonholomorphic type. Especially all the non-closed $K_{\mathbb{C}}$-orbits are defined to be of nonholomorphic type.

When $G_{\mathbb{R}}$ is of non-Hermitian type, we define that all the $K_{\mathbb{C}}$-orbits are of nonholomorphic type.

Let S_{op} denote the unique open dense $K_{\mathbb{C}}$-B double coset in $G_{\mathbb{C}}$. Then S'_{op} is the unique closed $G_{\mathbb{R}}$-B double coset in $G_{\mathbb{C}}$. In this case we see that

$$C(S_{\text{op}}) = \{x \in G_{\mathbb{C}} \mid xS_{\text{op}} \supset S'_{\text{op}}\}.$$

It follows easily that $C(S_{\text{op}})$ is a Stein manifold (cf. [GM1], [H]). The connected component $C(S_{\text{op}})_0$ is often called the Iwasawa domain.

The inclusion

$$D \subset C(S_{\text{op}})_0$$

was proved in [H]. (Later [M3] gave a proof without complex analysis.) On the other hand, it was proved in [GM1], Proposition 8.1 and Proposition 8.3, that $C(S_{\text{op}})_0 \subset C(S)_0$ for all $K_{\mathbb{C}}$-P double cosets S for any P. So we have the inclusion

$$D \subset C(S)_0. \tag{1.2}$$

Hence we have only to prove the converse inclusion

$$C(S)_0 \subset D \tag{1.3}$$

for $K_{\mathbb{C}}$-orbits S of nonholomorphic type in Conjecture 1.1.

If S is closed in $G_{\mathbb{C}}$, then we can write

$$C(S) = \{x \in G_{\mathbb{C}} \mid xS \subset S'\}.$$

So the connected component $C(S)_0$ is essentially equal to the cycle space introduced in [WW]. For Hermitian cases the inclusion (1.3) for closed S was proved in [WZ2] and [WZ3]. For non-Hermitian cases it was proved in [FH] and [M4].

When S is the open $K_{\mathbb{C}}$-P double coset in $G_{\mathbb{C}}$, the inclusion (1.3) was proved in [M4] for an arbitrary P generalizing the result in [B].

Recently the inclusion (1.3) was proved in [M5] for an arbitrary orbit S when $G_{\mathbb{R}}$ is of non-Hermitian type. So the remaining problem was to prove (1.3) for non-closed and non-open orbits when $G_{\mathbb{R}}$ is of Hermitian type.

In this paper we solve this problem.

In the next section we prove the following theorem.

Theorem 1.3. *Suppose that $G_{\mathbb{R}}$ is of Hermitian type and let S be a non-closed $K_{\mathbb{C}}$-P double coset in $G_{\mathbb{C}}$. Then there exist $K_{\mathbb{C}}$-B double cosets $\widetilde{S}_1$ and $\widetilde{S}_2$ contained in the boundary $\partial S = S^{cl} - S$ of S such that*

$$x(\widetilde{S}_1 \cup \widetilde{S}_2)^{cl} \cap {S'_0}^{cl} \neq \phi$$

for all the elements x in the boundary of D. Here S_0 denotes the dense $K_{\mathbb{C}}$-B double coset in S.

Remark 1.4. It seems that $\widetilde{S_1}$ and $\widetilde{S_2}$ are always distinct $K_{\mathbb{C}}$-orbits. But we do not need this distinctness.

Corollary 1.5. *Suppose that $G_{\mathbb{R}}$ is of Hermitian type and let S be a non-closed $K_{\mathbb{C}}$-P double coset in $G_{\mathbb{C}}$. Then $C(S)_0 = D$.*

Proof. Let S_0 be as in Theorem 1.3. Let Ψ denote the set of the simple roots in the positive root system for B. For each $\alpha \in \Psi$ we define a parabolic subgroup

$$P_\alpha = B \sqcup B w_\alpha B$$

of $G_{\mathbb{C}}$. By [GM2], Lemma 2, we can take a sequence $\{\alpha_1, \ldots, \alpha_m\}$ of simple roots such that

$$\dim_{\mathbb{C}} S_0 P_{\alpha_1} \cdots P_{\alpha_k} = \dim_{\mathbb{C}} S_0 + k$$

for $k = 0, \ldots, m = \operatorname{codim}_{\mathbb{C}} S_0$. Then it is shown in [M5], Theorem 1.4, that

$$x \in C(S) \cap D^{cl} \Longrightarrow xS^{cl} \cap S'_{\mathrm{op}} P_{\alpha_m} \cdots P_{\alpha_1} = xS \cap S'_0. \tag{1.4}$$

Let x be an element in the boundary of D. Then it follows from Theorem 1.3 that

$$x(\partial S) \cap {S'_0}^{cl} \neq \phi.$$

If x is also contained in $C(S)$, then it follows from (1.4) that

$$x(\partial S) \cap S'_{\mathrm{op}} P_{\alpha_m} \cdots P_{\alpha_1} = \phi.$$

Since ${S'_0}^{cl}$ is contained in the closed set $S'_{\mathrm{op}} P_{\alpha_m} \cdots P_{\alpha_1}$, we have

$$x(\partial S) \cap {S'_0}^{cl} = \phi,$$

a contradiction. Hence $x \notin C(S)$. Thus we have proved $C(S)_0 \subset D$. □

Section 3 is devoted to the explicit computation of the case where $G_{\mathbb{R}} = Sp(2, \mathbb{R})$. We use Proposition 3.2 in the proof of Lemma 2.4 in Section 2. Another simple example of the $SU(2,1)$-case is explicitly computed in [M4] Example 1.5.

2. Proof of Theorem 1.3

Let $\mathfrak{j}$ be a maximal abelian subspace of $i\mathfrak{k}$. Let Δ denote the root system of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j})$. Since $G_{\mathbb{R}}$ is a group of Hermitian type, there exists a nontrivial central element Z of $i\mathfrak{k}$ and we can write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$$

where $\Delta_n^+ = \{\alpha \in \Delta \mid \alpha(Z) > 0\}$, $\mathfrak{n} = \bigoplus_{\alpha \in \Delta_n^+} \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha)$ and $* \mapsto \overline{*}$ denotes the conjugation in $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_{\mathbb{R}}$. Let Q be the maximal parabolic subgroup of $G_{\mathbb{C}}$ defined by $Q = K_{\mathbb{C}} \exp \mathfrak{n}$. Let Δ^+ be a positive system of Δ containing Δ_n^+. Then it defines a Borel subgroup $B = B(\mathfrak{j}, \Delta^+)$ of $G_{\mathbb{C}}$ contained in Q.

Let P be a parabolic subgroup of $G_{\mathbb{C}}$ containing B. Let S be a non-closed $K_{\mathbb{C}}$-P double coset in $G_{\mathbb{C}}$ and let S_0 denote the dense $K_{\mathbb{C}}$-B double coset in S. By [M1], Theorem 2, we can write

$$S_0 = K_{\mathbb{C}} c_{\gamma_1} \cdots c_{\gamma_k} w B$$

with some $w \in W$ and a strongly orthogonal system $\{\gamma_1, \ldots, \gamma_k\}$ of roots in Δ_n^+. Here W is the Weyl group of Δ and

$$c_{\gamma_j} = \exp(X - \overline{X})$$

with some $X \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \gamma_j)$ such that $c_{\gamma_j}^2$ is the reflection with respect to γ_j.

Let Θ denote the subset of Ψ such that $P = BW_\Theta B$ where W_Θ is the subgroup of W generated by $\{w_\alpha \mid \alpha \in \Theta\}$. Let Δ_Θ denote the subset of Δ defined by

$$\Delta_\Theta = \{\beta \in \Delta \mid \beta = \sum_{\alpha\in\Theta} n_\alpha \alpha \text{ for some } n_\alpha \in \mathbb{Z}\}.$$

If $\gamma_j \in w\Delta_\Theta$ for all $j = 1, \ldots, k$, then it follows that $c_{\gamma_j} \in wPw^{-1}$ for all $j = 1, \ldots, k$ and therefore

$$Sw^{-1} = S_0Pw^{-1} = K_\mathbb{C} c_{\gamma_1} \cdots c_{\gamma_k} wPw^{-1} = K_\mathbb{C} wPw^{-1}$$

becomes closed in $G_\mathbb{C}$, contradicting the assumption. Hence there exists a j such that $\gamma_j \notin w\Delta_\Theta$. Replacing the order of $\gamma_1, \ldots, \gamma_k$, we may assume that

$$\gamma_1 \notin w\Delta_\Theta.$$

Let $\mathfrak{l}$ denote the complex Lie subalgebra of $\mathfrak{g}_\mathbb{C}$ generated by $\mathfrak{g}_\mathbb{C}(\mathfrak{j}, \gamma_1) \oplus \mathfrak{g}_\mathbb{C}(\mathfrak{j}, -\gamma_1)$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and let L be the analytic subgroup of $G_\mathbb{C}$ for $\mathfrak{l}$. Then we have $(L \cap K_\mathbb{C})c_{\gamma_1}(L \cap wBw^{-1}) = (L \cap K_\mathbb{C})c_{\gamma_1}^{-1}(L \cap wBw^{-1})$ since both of the double cosets are open dense in L. Hence we have

$$S_0 = K_\mathbb{C} c_{\gamma_1} \cdots c_{\gamma_k} wB = K_\mathbb{C} c_{\gamma_1}^{-1} c_{\gamma_2} \cdots c_{\gamma_k} wB = K_\mathbb{C} c_{\gamma_1} \cdots c_{\gamma_k} w_{\gamma_1} wB.$$

If $\gamma_1 \notin w\Delta^+$, then $\gamma_1 \in w_{\gamma_1} w\Delta^+$. So we may assume

$$\gamma_1 \in w\Delta^+,$$

replacing w with $w_{\gamma_1} w$ if necessary. Let ℓ denote the real rank of $G_\mathbb{R}$.

Lemma 2.1. *There exists a maximal strongly orthgonal system $\{\beta_1, \ldots, \beta_\ell\}$ of roots in Δ_n^+ satisfying the following conditions:*

(i) *If γ_1 is a long root of Δ, then $\beta_1 = \gamma_1$ and $\gamma_2, \ldots, \gamma_k \in \mathbb{R}\beta_2 \oplus \cdots \oplus \mathbb{R}\beta_\ell$.* (*If the roots in Δ have the same length, then we define that all the roots are long roots.*)

(ii) *If γ_1 is a short root of Δ, then $\gamma_1 \in \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$ and $\gamma_2, \ldots, \gamma_k \in \mathbb{R}\beta_3 \oplus \cdots \oplus \mathbb{R}\beta_\ell$.*

Proof. First suppose that $\mathfrak{g}_\mathbb{R}$ is of type AIII, DIII, EIII, EVII or DI (of real rank 2). Then the roots in Δ have the same length. So we have only to take $\beta_j = \gamma_j$ for $j = 1, \ldots, k$ and choose an orthogonal system $\{\beta_1, \ldots, \beta_\ell\}$ of roots in Δ_n^+ containing $\{\beta_1, \ldots, \beta_k\}$.

Next suppose that $\mathfrak{g}_\mathbb{R} \cong \mathfrak{sp}(\ell, \mathbb{R})$. Write

$$\Delta = \{\pm e_r \pm e_s \mid 1 \le r < s \le \ell\} \sqcup \{\pm 2e_r \mid 1 \le r \le \ell\}$$

and

$$\Delta_n^+ = \{e_r + e_s \mid 1 \le r < s \le \ell\} \sqcup \{2e_r \mid 1 \le r \le \ell\}$$

as usual using an orthonormal basis $\{e_1, \ldots, e_\ell\}$ of $\mathfrak{j}^*$. If $\gamma_1 = 2e_r$, then $\{\beta_2, \ldots, \beta_\ell\}$ $= \{2e_s \mid s \neq r\}$ satisfies condition (i). If $\gamma_1 = e_r + e_s$ with $r \neq s$, then we put $\beta_1 =$ $2e_r$ and $\beta_2 = 2e_s$. Assertion (ii) is clear if we put $\{\beta_3, \ldots, \beta_\ell\} = \{2e_p \mid p \neq r, s\}$.

Finally suppose that $\mathfrak{g}_\mathbb{R} = \mathfrak{so}(2, 2p-1)$ with $p \ge 2$. Then the real rank of $\mathfrak{g}_\mathbb{R}$ is two, and we can write

$$\Delta = \{\pm e_r \pm e_s \mid 1 \le r < s \le p\} \sqcup \{\pm e_r \mid 1 \le r \le p\}$$

and

$$\Delta_n^+ = \{e_1 \pm e_s \mid 2 \le s \le p\} \sqcup \{e_1\}$$

with an orthonormal basis $\{e_1, \dots, e_p\}$ of $\mathfrak{j}^*$. If $k = 2$, then we have $\gamma_1 = \beta_1 = e_1 \pm e_s$ and $\gamma_2 = \beta_2 = e_1 \mp e_s$ with some s. If $k = 1$ and $\gamma_1 = e_1 \pm e_s$, then $\beta_1 = \gamma_1$ and $\beta_2 = e_1 \mp e_s$. If $k = 1$ and $\gamma_1 = e_1$, then we may put $\beta_1 = e_1 + e_2$ and $\beta_2 = e_1 - e_2$. $\square$

Definition 2.2. (i) Define a subroot system Δ_1 of Δ as follows.

If γ_1 is a long root of Δ, then we put

$$\Delta_1 = \{\pm\beta_1\} = \{\pm\gamma_1\}.$$

On the other hand if γ_1 is a short root of Δ, then we put

$$\Delta_1 = \Delta \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)$$

(which is of type C_2).

(ii) Put $\Delta_2 = \{\alpha \in \Delta \mid \alpha \text{ is orthogonal to } \Delta_1\}$.

(iii) Let $\mathfrak{l}_j$ denote the complex Lie subalgebras of $\mathfrak{g}_\mathbb{C}$ generated by $\bigoplus_{\alpha\in\Delta_j} \mathfrak{g}_\mathbb{C}(\mathfrak{j}, \alpha)$ for $j = 1, 2$.

(iv) Let L_1 and L_2 denote the analytic subgroups of $G_\mathbb{C}$ for $\mathfrak{l}_1$ and $\mathfrak{l}_2$, respectively.

It follows from Lemma 2.1 that

$$c_{\gamma_1} \in L_1 \quad \text{and that} \quad c_{\gamma_2} \cdots c_{\gamma_k} \in L_2.$$

Let X_j be nonzero root vectors in $\mathfrak{g}_\mathbb{C}(\mathfrak{j}, \beta_j)$ for $j = 1, \dots, \ell$. Then we can define a maximal abelian subspace

$$\mathfrak{t} = \mathbb{R}(X_1 - \overline{X_1}) \oplus \cdots \oplus \mathbb{R}(X_\ell - \overline{X_\ell})$$

in $i\mathfrak{m}$ and a maximal abelian subspace

$$\mathfrak{a} = \mathbb{R}(X_1 + \overline{X_1}) \oplus \cdots \oplus \mathbb{R}(X_\ell + \overline{X_\ell})$$

in $\mathfrak{m}$ as in [GM1], Section 2. Since the restricted root system $\Sigma(\mathfrak{t})$ is of type BC_ℓ or C_ℓ, the set $\mathfrak{t}^+$ is defined by the long roots in $\Sigma(\mathfrak{t})$. Hence it is of the form

$$\mathfrak{t}^+ = \{Y_1 + \cdots + Y_\ell \mid Y_j \in \mathfrak{t}_j^+\}$$

where $\mathfrak{t}_j^+ = \{s(X_j - \overline{X_j}) \mid -(\pi/4) < s < \pi/4\}$ by a suitable normalization of X_j for $j = 1, \dots, \ell$.

Put $T^+ = \exp \mathfrak{t}^+$ and $A = \exp \mathfrak{a}$. Then it is shown in [GM1], Lemma 2.1, that $AQ = T^+Q$ and hence that

$$G_\mathbb{R}Q = KAQ = KT^+Q$$

by the Cartan decomposition $G_\mathbb{R} = KAK$. The closure of $G_\mathbb{R}Q$ in $G_\mathbb{C}$ is written as

$$(G_\mathbb{R}Q)^{cl} = G_\mathbb{R}Q \sqcup G_\mathbb{R}c_{\beta_1}Q \sqcup G_\mathbb{R}c_{\beta_1}c_{\beta_2}Q \sqcup \cdots \sqcup G_\mathbb{R}c_{\beta_1} \cdots c_{\beta_\ell}Q$$

where $c_{\beta_j} = \exp(\pi/4)(X_j - \overline{X_j})$ for $j = 1, \dots, \ell$ ([WZ1], Theorem 3.8). We also see that

$$G_\mathbb{R}c_{\beta_1} \cdots c_{\beta_k}Q = Kc_{\beta_1} \cdots c_{\beta_k}T_{k+1}^+ \cdots T_\ell^+Q \tag{2.1}$$

where $T_j^+ = \exp \mathfrak{t}_j^+$ since we can consider the action of the Weyl group $W_K(T)$ on T which is of type BC_ℓ.

By the map

$$\iota : xK_\mathbb{C} \mapsto (xQ, x\overline{Q})$$

the complex symmetric space $G_{\mathbb{C}}/K_{\mathbb{C}}$ is embedded in $G_{\mathbb{C}}/Q\times G_{\mathbb{C}}/\overline{Q}$ ([WZ2]). It is shown in [BHH], Section 3, and [GM1], Proposition 2.2, that

$$\iota(D/K_{\mathbb{C}})=G_{\mathbb{R}}Q/Q\times G_{\mathbb{R}}\overline{Q}/\overline{Q}.$$

Lemma 2.3. *Suppose that*

$$\iota(xK_{\mathbb{C}})\in G_{\mathbb{R}}c_{\beta_1}Q/Q\times G_{\mathbb{R}}\overline{Q}/\overline{Q}$$

and that γ_1 is a long root of Δ_n^+. (If the roots in Δ have the same length, then we define that all the roots are long roots.) Define a $K_{\mathbb{C}}$-B double coset $\widetilde{S}_1$ by

$$\widetilde{S}_1=K_{\mathbb{C}}c_{\gamma_2}\cdots c_{\gamma_k}wB.$$

Then $\widetilde{S}_1$ is contained in $\partial S=S^{cl}-S$ and

$$x\widetilde{S}_1\cap S_0'\neq\phi.$$

Proof. It is clear that we may replace x by any elements in the double coset $G_{\mathbb{R}}xK_{\mathbb{C}}$. By the left $G_{\mathbb{R}}$-action we may assume that $x\in\overline{Q}$. By the right $K_{\mathbb{C}}$-action we may moreover assume that $x\in\overline{N}$ since $\overline{Q}=\overline{N}K_{\mathbb{C}}$. Since $K=K_{\mathbb{C}}\cap G_{\mathbb{R}}$ normalizes $\overline{N}$, we may assume by (2.1) that

$$xQ=c_{\beta_1}t_2\cdots t_\ell Q$$

with some $t_j\in T_j^+$ for $j=2,\dots,\ell$. As in [WZ2], we write

$$c_{\beta_1}=c_{\gamma_1}=c=c^-c^+\quad\text{and}\quad t_j=t_j^-t_j^+\text{ for }j=2,\dots,\ell$$

with $c^-,\ t_j^-\in\overline{N}$ and $c^+,\ t_j^+\in Q$. Then we have

$$x=c^-t_2^-\cdots t_\ell^-.$$

It follows from Lemma 2.1 and Definition 2.2 that $c_{\gamma_2}\cdots c_{\gamma_k}\in L_2$. Since $\mathrm{Ad}(c_{\gamma_2}\cdots c_{\gamma_k})\mathfrak{j}$ is θ-stable, the double cosets

$$S_{L_2}=(L_2\cap K_{\mathbb{C}})c_{\gamma_2}\cdots c_{\gamma_k}(L_2\cap wBw^{-1})$$

and

$$S'_{L_2}=(L_2\cap G_{\mathbb{R}})c_{\gamma_2}\cdots c_{\gamma_k}(L_2\cap wBw^{-1})$$

correspond by the duality ([M1], Theorem 2).

It follows from Lemma 2.1 (i) and Definition 2.2 that

$$c^\pm\in L_1\quad\text{and}\quad t_2^\pm,\dots,t_\ell^\pm\in L_2.$$

It follows moreover from Definition 2.2 (i) that $\mathfrak{l}_1\cong\mathfrak{sl}(2,\mathbb{C})$.

Write $y=t_2^-\cdots t_\ell^-$. Then we have

$$yQ=t_2\cdots t_\ell Q\subset T^+Q\subset G_{\mathbb{R}}Q$$

and

$$y\overline{Q}=\overline{Q}\subset G_{\mathbb{R}}\overline{Q}.$$

Hence we have

$$y\in L_2\cap(C(S_1)\cap C(S_2))=L_2\cap D$$

by [GM1], (1.3). By the inclusion (1.2) this implies that the set $yS_{L_2}\cap S'_{L_2}$ is nonempty and closed in L_2. Take an element z of $yS_{L_2}\cap S'_{L_2}$.

Since $\gamma_1 \in w\Delta^+$, we have $c^+ \in wBw^{-1}$. Since $c^+ \in L_1$ commutes with elements in L_2, we have

$$\begin{aligned} cz \in cyS_{L_2} &= c^-c^+y(L_2 \cap K_{\mathbb{C}})c_{\gamma_2}\cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\ &= c^-y(L_2 \cap K_{\mathbb{C}})c_{\gamma_2}\cdots c_{\gamma_k}c^+(L_2 \cap wBw^{-1}) \\ &\subset c^-yK_{\mathbb{C}}c_{\gamma_2}\cdots c_{\gamma_k}wBw^{-1} = x\widetilde{S}_1w^{-1}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} cz \in cS'_{L_2} &= c(L_2 \cap G_{\mathbb{R}})c_{\gamma_2}\cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\ &= (L_2 \cap G_{\mathbb{R}})c_{\gamma_1}c_{\gamma_2}\cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \subset S'_0w^{-1}. \end{aligned}$$

Hence $x\widetilde{S}_1 \cap S'_0 \neq \phi$. It is clear that $\widetilde{S}_1 \subset S_0^{cl} = S^{cl}$ because

$$(L_1 \cap K_{\mathbb{C}})(L_1 \cap wBw^{-1}) \subset ((L_1 \cap K_{\mathbb{C}})c(L_1 \cap wBw^{-1}))^{cl} = L_1.$$

Now we will prove $\widetilde{S}_1 \not\subset S$. Consider the map

$$\varphi : K_{\mathbb{C}}\backslash G_{\mathbb{C}}/B \ni K_{\mathbb{C}}gB \mapsto B\theta(g)^{-1}gB \in B\backslash G_{\mathbb{C}}/B$$

introduced in [Sp] where θ is the holomorphic involution in $G_{\mathbb{C}}$ defining $K_{\mathbb{C}}$. We have

$$\varphi(\widetilde{S}_1) = Bw^{-1}w_{\gamma_2}\cdots w_{\gamma_k}wB$$

and

$$\varphi(S) = \varphi(S_0P) \subset Pw^{-1}w_{\gamma_1}\cdots w_{\gamma_k}wP = BW_\Theta w^{-1}w_{\gamma_1}\cdots w_{\gamma_k}wW_\Theta B.$$

So we have only to show

$$w^{-1}w_{\gamma_2}\cdots w_{\gamma_k}w \notin W_\Theta w^{-1}w_{\gamma_1}\cdots w_{\gamma_k}wW_\Theta. \tag{2.2}$$

Let Z be an element in $\mathfrak{j}$ defining P. This implies that Z is dominant for Δ^+ and that $\{\alpha \in \Psi \mid \alpha(Z) = 0\} = \Theta$. Let w_1 and w_2 be elements in W_Θ. Let $B(\ ,\ )$ denote the Killing form on $\mathfrak{g}$ and let Y_{γ_1} denote the element in $\mathfrak{j}$ such that

$$\gamma_1(Y) = B(Y, Y_{\gamma_1}) \quad \text{for all } Y \in \mathfrak{j}.$$

Then we have

$$\begin{aligned} &B(Z, w^{-1}w_{\gamma_2}\cdots w_{\gamma_k}wZ) - B(Z, w_1w^{-1}w_{\gamma_1}w_{\gamma_2}\cdots w_{\gamma_k}ww_2Z) \\ &= B(wZ - w_{\gamma_1}wZ, w_{\gamma_2}\cdots w_{\gamma_k}wZ) \\ &= \frac{2B(Y_{\gamma_1}, wZ)}{B(Y_{\gamma_1}, Y_{\gamma_1})}B(Y_{\gamma_1}, w_{\gamma_2}\cdots w_{\gamma_k}wZ) \\ &= \frac{2B(Y_{\gamma_1}, wZ)^2}{B(Y_{\gamma_1}, Y_{\gamma_1})} > 0 \end{aligned}$$

since $\gamma_1 \notin w\Delta_\Theta$. Thus we have proved (2.2). $\square$

Lemma 2.4. *Suppose that*

$$\iota(xK_{\mathbb{C}}) \in G_{\mathbb{R}}c_{\beta_1}Q/Q \times G_{\mathbb{R}}\overline{Q}/\overline{Q}$$

and that γ_1 is a short root of Δ_n^+. (We assume that $\mathfrak{g}_{\mathbb{R}} \cong \mathfrak{sp}(\ell, \mathbb{R})$ or $\mathfrak{so}(2, 2p-1)$ with $p \geq 2$.) Define a $K_{\mathbb{C}}$-B double coset $\widetilde{S}_1$ by $\widetilde{S}_1 = K_{\mathbb{C}}gc_{\gamma_2}\cdots c_{\gamma_k}wB$ where

$$g = \begin{cases} e & \text{if } \gamma_1 \text{ is the simple short root of } \Delta_1^+, \\ c_\beta & \text{if } \gamma_1 \text{ is the non-simple short root of } \Delta_1^+. \end{cases}$$

Here $\Delta_1^+ = \Delta_1 \cap w\Delta^+$ *and* β *is the long simple root of* Δ_1^+*. Then* $\widetilde{S}_1$ *is contained in* $\partial S = S^{cl} - S$ *and*

$$x\widetilde{S}_1 \cap {S_0'}^{cl} \neq \phi.$$

Proof. It follows from Lemma 2.1 (ii) and Definition 2.2 that

$$c_{\beta_1}^{\pm}, t_2^{\pm} \in L_1 \quad \text{and} \quad t_3^{\pm}, \ldots, t_\ell^{\pm} \in L_2.$$

It follows moreover from Definition 2.2 (i) that $\mathfrak{l}_1 \cong \mathfrak{sp}(2, \mathbb{C})$.

Write $y = t_3^- \cdots t_\ell^-$. Then by the same argument as in the proof of Lemma 2.3 we see that the set $yS_{L_2} \cap S'_{L_2}$ is nonempty and closed in L_2. Take an element z of $yS_{L_2} \cap S'_{L_2}$.

The positive system Δ_1^+ of Δ_1 consists of two long roots and two short roots. Since $\gamma_1 \in \Delta_1^+$, γ_1 is either of these two short roots. Write $x_1 = c_{\beta_1}^- t_2^-$.

First assume that γ_1 is the simple short root of Δ_1^+. Then it follows from Proposition 3.2 (i) in the next section that

$$x_1(L_1 \cap K_{\mathbb{C}})(L_1 \cap wBw^{-1}) \cap ((L_1 \cap G_{\mathbb{R}})c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl} \tag{2.3}$$

is nonempty. Note that $L_1 \cap wBw^{-1}$ and γ_1 correspond to $w_{\beta_2}Bw_{\beta_2}^{-1}$ and δ in the next section, respectively. Let z_1 be an element of (2.3). Then we have

$$\begin{aligned} z_1 z &\in x_1(L_1 \cap K_{\mathbb{C}})(L_1 \cap wBw^{-1})yS_{L_2} \\ &= x_1(L_1 \cap K_{\mathbb{C}})(L_1 \cap wBw^{-1})y(L_2 \cap K_{\mathbb{C}})c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\ &= x_1 y(L_1 \cap K_{\mathbb{C}})(L_2 \cap K_{\mathbb{C}})c_{\gamma_2} \cdots c_{\gamma_k}(L_1 \cap wBw^{-1})(L_2 \cap wBw^{-1}) \\ &\subset xK_{\mathbb{C}}c_{\gamma_2} \cdots c_{\gamma_k} wBw^{-1} = x\widetilde{S}_1 w^{-1} \end{aligned}$$

and

$$\begin{aligned} z_1 z &\in ((L_1 \cap G_{\mathbb{R}})c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl} S'_{L_2} \\ &= ((L_1 \cap G_{\mathbb{R}})c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}(L_2 \cap G_{\mathbb{R}})c_{\gamma_2} \cdots c_{\gamma_k}(L_2 \cap wBw^{-1}) \\ &\subset (G_{\mathbb{R}}c_{\gamma_1}c_{\gamma_2} \cdots c_{\gamma_k} wBw^{-1})^{cl} = {S_0'}^{cl} w^{-1}. \end{aligned}$$

So we have $x\widetilde{S}_1 \cap {S_0'}^{cl} \neq \phi$. We can prove $\widetilde{S}_1 \subset S^{cl} - S$ by the same arguments as in the proof of Lemma 2.3.

Next assume that γ_1 is the non-simple short root of Δ_1^+. Then it follows from Proposition 3.2 (ii) in the next section that

$$x_1(L_1 \cap K_{\mathbb{C}})c_\beta(L_1 \cap wBw^{-1}) \cap ((L_1 \cap G_{\mathbb{R}})c_{\gamma_1}(L_1 \cap wBw^{-1}))^{cl}$$

is nonempty. Note that $L_1 \cap wBw^{-1}$, γ_1 and β correspond to B, δ and β_2 in the next section, respectively. By the same argument as above we can prove

$$x\widetilde{S}_1 \cap {S_0'}^{cl} \neq \phi.$$

It follows from Remark 3.3 that $\widetilde{S}_1 \subset S^{cl}$. Finally we will prove that $\widetilde{S}_1 \not\subset S$. Using the same argument as in the proof of Lemma 2.3, we have only to show

$$w^{-1}w_\beta w_{\gamma_2} \cdots w_{\gamma_k} w \notin W_\Theta w^{-1} w_{\gamma_1} \cdots w_{\gamma_k} w W_\Theta. \tag{2.4}$$

Let Z and Y_{γ_1} be as in the proof of Lemma 2.3. Define $Y_\beta \in \mathfrak{j}$ so that

$$\beta(Y) = B(Y, Y_\beta) \quad \text{for all } Y \in \mathfrak{j}.$$

Then we have

$$
\begin{aligned}
&B(Z, w^{-1}w_\beta w_{\gamma_2}\cdots w_{\gamma_k}wZ) - B(Z, w_1w^{-1}w_{\gamma_1}w_{\gamma_2}\cdots w_{\gamma_k}ww_2Z)\\
&= B(w_\beta wZ - w_{\gamma_1}wZ, w_{\gamma_2}\cdots w_{\gamma_k}wZ)\\
&= B(wZ - w_{\gamma_1}wZ, w_{\gamma_2}\cdots w_{\gamma_k}wZ) - B(wZ - w_\beta wZ, w_{\gamma_2}\cdots w_{\gamma_k}wZ)\\
&= \frac{2B(Y_{\gamma_1}, wZ)}{B(Y_{\gamma_1}, Y_{\gamma_1})}B(Y_{\gamma_1}, w_{\gamma_2}\cdots w_{\gamma_k}wZ) - \frac{2B(Y_\beta, wZ)}{B(Y_\beta, Y_\beta)}B(Y_\beta, w_{\gamma_2}\cdots w_{\gamma_k}wZ)\\
&= \frac{2B(Y_{\gamma_1}, wZ)^2}{B(Y_{\gamma_1}, Y_{\gamma_1})} - \frac{2B(Y_\beta, wZ)^2}{B(Y_\beta, Y_\beta)} > 0
\end{aligned}
$$

for $w_1, w_2 \in W_\Theta$ since

$$B(Y_{\gamma_1}, wZ) > 0, \quad 0 \le B(Y_\beta, wZ) \le B(Y_{\gamma_1}, wZ) \quad \text{and} \quad B(Y_\beta, Y_\beta) = 2B(Y_{\gamma_1}, Y_{\gamma_1}).$$

Thus we have proved (2.4). $\square$

Using the conjugation on $G_{\mathbb{C}}$ with respect to the real form $G_{\mathbb{R}}$, the following follows from Lemma 2.3 and Lemma 2.4.

Corollary 2.5. *Suppose that*

$$\iota(xK_{\mathbb{C}}) \in G_{\mathbb{R}}Q/Q \times G_{\mathbb{R}}\overline{c_{\beta_1}}\,\overline{Q}/\overline{Q}.$$

Then there exists a $K_{\mathbb{C}}$-B double coset $\widetilde{S}_2$ contained in ∂S such that

$$x\widetilde{S}_2 \cap {S_0'}^{cl} \neq \phi.$$

Proof of Theorem 1.3. Let S be a non-closed $K_{\mathbb{C}}$-P double coset in $G_{\mathbb{C}}$. Then it follows from Lemma 2.3, Lemma 2.4 and Corollary 2.5 that there exist $K_{\mathbb{C}}$-B double cosets $\widetilde{S}_1$ and $\widetilde{S}_2$ contained in ∂S such that

$$x(\widetilde{S}_1 \cup \widetilde{S}_2) \cap {S_0'}^{cl} \neq \phi \tag{2.5}$$

for all $x \in \partial D$ satisfying

$$xK_{\mathbb{C}} \in \iota^{-1}((G_{\mathbb{R}}c_{\beta_1}Q/Q \times G_{\mathbb{R}}\overline{Q}/\overline{Q}) \sqcup (G_{\mathbb{R}}Q/Q \times G_{\mathbb{R}}\overline{c_{\beta_1}}\,\overline{Q}/\overline{Q})). \tag{2.6}$$

Suppose that

$$y(\widetilde{S}_1 \cup \widetilde{S}_2)^{cl} \cap {S_0'}^{cl} = \phi$$

for some $y \in \partial D$. Then there exists a neighborhood U of y in $G_{\mathbb{C}}$ such that

$$x(\widetilde{S}_1 \cup \widetilde{S}_2)^{cl} \cap {S_0'}^{cl} = \phi$$

for all $x \in U$. But this contradicts (2.5) because the right hand side of (2.6) is dense in $\partial(D/K_{\mathbb{C}})$. $\square$

3. $Sp(2,\mathbb{R})$-case

Let $G_{\mathbb{C}} = Sp(2,\mathbb{C}) = \{g \in GL(4,\mathbb{C}) \mid {}^tgJg = J\}$ where

$$J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

Let

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} g & 0 \\ 0 & {}^tg^{-1} \end{pmatrix} \,\middle|\, g \in GL(2,\mathbb{C}) \right\} \quad \text{and} \quad G_{\mathbb{R}} = G_{\mathbb{C}} \cap U(2,2) \cong Sp(2,\mathbb{R}).$$

Put $U_+ = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $U_- = \mathbb{C}e_3 \oplus \mathbb{C}e_4$ by using the canonical basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{C}^4$. Then we have

$$K_{\mathbb{C}} = Q \cap \overline{Q}$$

where $Q = \{g \in G_{\mathbb{C}} \mid gU_+ = U_+\}$ and $\overline{Q} = \{g \in G_{\mathbb{C}} \mid gU_- = U_-\}$.

The full flag manifold X of $G_{\mathbb{C}}$ consists of the flags (V_1, V_2) in $\mathbb{C}^4$ where $\dim V_j = j$, $V_1 \subset V_2$ and ${}^t uJv = 0$ for all $u, v \in V_2$. Let B denote the Borel subgroup of $G_{\mathbb{C}}$ defined by

$$B = \{g \in G_{\mathbb{C}} \mid g\mathbb{C}e_1 = \mathbb{C}e_1 \text{ and } gU_+ = U_+\}.$$

Then the full flag manifold X is identified with $G_{\mathbb{C}}/B$ by the map

$$gB \mapsto (V_1, V_2) = (g\mathbb{C}e_1, gU_+).$$

There are eleven $K_{\mathbb{C}}$-orbits

$$\begin{aligned}
S_1 &= \{(V_1, V_2) \mid V_2 = U_+\}, \\
S_2 &= \{(V_1, V_2) \mid V_2 = U_-\}, \\
S_3 &= \{(V_1, V_2) \mid V_1 \subset U_+,\ \dim(V_2 \cap U_-) = 1\}, \\
S_4 &= \{(V_1, V_2) \mid V_1 \subset U_-,\ \dim(V_2 \cap U_+) = 1\}, \\
S_5 &= \{(V_1, V_2) \mid V_1 \subset U_+\} - (S_1 \sqcup S_3), \\
S_6 &= \{(V_1, V_2) \mid V_1 \subset U_-\} - (S_2 \sqcup S_4), \\
S_7 &= \{(V_1, V_2) \mid \dim(V_2 \cap U_+) = \dim(V_2 \cap U_-) = 1\} - (S_3 \sqcup S_4), \\
S_8 &= \{(V_1, V_2) \mid V_1 \cap U_+ = \{0\},\ \dim(V_2 \cap U_+) = 1,\ V_2 \cap U_- = \{0\}\}, \\
S_9 &= \{(V_1, V_2) \mid V_1 \cap U_- = \{0\},\ \dim(V_2 \cap U_-) = 1,\ V_2 \cap U_+ = \{0\}\}, \\
S_{10} &= \{(V_1, V_2) \mid V_2 \cap U_\pm = \{0\},\ {}^t vJ\tau(v) = 0 \text{ for } v \in V_1\}, \\
S_{\mathrm{op}} &= \{(V_1, V_2) \mid V_2 \cap U_\pm = \{0\},\ {}^t vJ\tau(v) \neq 0 \text{ for } v \in V_1 - \{0\}\}
\end{aligned}$$

on X where

$$\tau(v) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} v$$

for $v \in \mathbb{C}^4$. These orbits are related as follows ([MO], Fig. 12):

$$\begin{tikzcd}
S_1 \arrow[dr, "2"'] & & S_3 \arrow[dl, "2"] \arrow[dr, "1"'] & & S_4 \arrow[dl, "1"] \arrow[dr, "2"'] & & S_2 \arrow[dl, "2"] \\
& S_5 \arrow[d, "1"'] & & S_7 \arrow[d, "2"'] & & S_6 \arrow[d, "1"'] & \\
& S_8 \arrow[drr, "2"'] & & S_{10} \arrow[d, "1"'] & & S_9 \arrow[dll, "2"] & \\
& & & S_{\mathrm{op}} & & &
\end{tikzcd}$$

Let P_1 and P_2 be the parabolic subgroups of $G_{\mathbb{C}}$ defined by

$$P_1 = Q \quad \text{and} \quad P_2 = \{g \in G_{\mathbb{C}} \mid g\mathbb{C}e_1 = \mathbb{C}e_1\},$$

respectively. Then the above diagram implies, for example, that

$$S_1 P_2 = S_5 P_2 \quad \text{and that} \quad \dim S_1 = \dim S_5 - 1$$

by the arrow attached with the number 2 joining S_1 and S_5.

On the other hand define subsets

$$C_+ = \{z \in \mathbb{C}^4 \mid (z,z) > 0\}, \quad C_- = \{z \in \mathbb{C}^4 \mid (z,z) < 0\}$$
$$\text{and} \quad C_0 = \{z \in \mathbb{C}^4 \mid (z,z) = 0\}$$

of $\mathbb{C}^4$ using the Hermitian form $(w,z) = \overline{w_1}z_1 + \overline{w_2}z_2 - \overline{w_3}z_3 - \overline{w_4}z_4$ defining $U(2,2)$. For $v \in \mathbb{C}^4$ define subspaces

$$v^J = \{u \in \mathbb{C}^4 \mid {}^t v J u = 0\} \quad \text{and} \quad v^\perp = \{u \in \mathbb{C}^4 \mid (v,u) = 0\}$$

of $\mathbb{C}^4$. Then C_0 is devided as $C_0 = C_0^s \sqcup C_0^r$ where

$$C_0^s = \{v \in C_0 \mid v^J = v^\perp\} \quad \text{and} \quad C_0^r = \{v \in C_0 \mid v^J \neq v^\perp\}.$$

The $G_{\mathbb{R}}$-orbits on X are

$$\begin{aligned}
S_1' &= \{(V_1,V_2) \mid V_2 - \{0\} \subset C_+\}, \\
S_2' &= \{(V_1,V_2) \mid V_2 - \{0\} \subset C_-\}, \\
S_3' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_+,\ V_2 \cap C_- \neq \phi\}, \\
S_4' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_-,\ V_2 \cap C_+ \neq \phi\}, \\
S_5' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_+,\ V_2 \cap C_0^s \neq \{0\}\}, \\
S_6' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_-,\ V_2 \cap C_0^s \neq \{0\}\}, \\
S_7' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_0^r,\ V_2 \not\subset C_0\}, \\
S_8' &= \{(V_1,V_2) \mid V_1 \subset C_0^s,\ V_2 \cap C_+ \neq \phi\}, \\
S_9' &= \{(V_1,V_2) \mid V_1 \subset C_0^s,\ V_2 \cap C_- \neq \phi\}, \\
S_{10}' &= \{(V_1,V_2) \mid V_1 - \{0\} \subset C_0^r,\ V_2 \subset C_0\}, \\
S_{\mathrm{op}}' &= \{(V_1,V_2) \mid V_1 \subset C_0^s,\ V_2 \subset C_0\}.
\end{aligned}$$

Here the $K_{\mathbb{C}}$-orbit S_j and the $G_{\mathbb{R}}$-orbit S_j' correspond by the duality for each $j = 1, \ldots, 10, \mathrm{op}$.

Take a maximal abelian subspace

$$\mathfrak{j} = \left\{ Y(a_1,a_2) = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & -a_2 \end{pmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\}$$

of $i\mathfrak{m}$. Using the linear forms $e_j : Y(a_1,a_2) \mapsto a_j$ for $j = 1,2$, we can write

$$\Delta = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\} \quad \text{and} \quad \Delta_n^+ = \{2e_1, 2e_2, e_1 + e_2\}.$$

Write $\beta_1 = 2e_1,\ \beta_2 = 2e_2$ and $\delta = e_1 + e_2$. Take root vectors $X_1 = -E_{13}$ of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j},\beta_1)$ and $X_2 = -E_{24}$ of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j},\beta_2)$ where E_{ij} $(i,j = 1,\ldots,4)$ denotes the matrix units.

Define

$$t_1(s) = \exp s(X_1 - \overline{X_1}) = \exp s(E_{31} - E_{13}) = \begin{pmatrix} \cos s & 0 & -\sin s & 0 \\ 0 & 1 & 0 & 0 \\ \sin s & 0 & \cos s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$t_2(s) = \exp s(X_2 - \overline{X_2}) = \exp s(E_{42} - E_{24}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos s & 0 & -\sin s \\ 0 & 0 & 1 & 0 \\ 0 & \sin s & 0 & \cos s \end{pmatrix}$$

for $s \in \mathbb{R}$. Then we can write the Akhiezer-Gindikin domain D as

$$D = G_{\mathbb{R}} T^+ K_{\mathbb{C}}$$

where $T^+ = \{t_1(s_1)t_2(s_2) \mid |s_1| < \pi/4,\ |s_2| < \pi/4\}$. Write $c_{\beta_j} = t_j(\pi/4)$ and $w_{\beta_j} = t_j(\pi/2)$ for $j = 1, 2$. Then we can write

$$S_j = K_{\mathbb{C}} gB \quad \text{and} \quad S'_j = G_{\mathbb{R}} gB$$

for $j = 1, \dots, 10, \mathrm{op}$ with the following representatives g ([M1], Theorem 2):

j	1	2	3	4	5	6	7	8	9	10	op
g	e	$w_{\beta_1} w_{\beta_2}$	w_{β_2}	w_{β_1}	c_{β_2}	$c_{\beta_2} w_{\beta_1}$	$c_\delta w_{\beta_2}$	c_{β_1}	$c_{\beta_1} w_{\beta_2}$	c_δ	$c_{\beta_1} c_{\beta_2}$

Here

$$c_\delta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \exp \frac{\pi}{4}(X_\delta - \overline{X_\delta})$$

with $X_\delta = -(E_{14} + E_{23}) \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \delta)$.

The standard maximal flag manifold $G_{\mathbb{C}}/Q$ is identified with the space Y of two dimensional subspaces V_+ of $\mathbb{C}^4$ such that ${}^t uJv = 0$ for all $u, v \in V_+$ by the map

$$G_{\mathbb{C}}/Q \ni gQ \mapsto V_+ = gU_+ \in Y.$$

Similarly we also identify $G_{\mathbb{C}}/\overline{Q}$ with Y by the map

$$G_{\mathbb{C}}/\overline{Q} \ni g\overline{Q} \mapsto V_- = gU_- \in Y.$$

As in Section 2 the complex symmetric space $G_{\mathbb{C}}/K_{\mathbb{C}}$ is naturally identified with the open subset

$$\{(V_+, V_-) \in G_{\mathbb{C}}/Q \times G_{\mathbb{C}}/\overline{Q} \mid V_+ \cap V_- = \{0\}\}$$

of $G_{\mathbb{C}}/Q \times G_{\mathbb{C}}/\overline{Q} \cong Y \times Y$ by the map

$$\iota : gK_{\mathbb{C}} \mapsto (V_+, V_-) = (gU_+, gU_-).$$

Then the Akhiezer-Gindikin domain $D/K_{\mathbb{C}}$ is identified with

$$G_{\mathbb{R}}Q/Q \times G_{\mathbb{R}}\overline{Q}/\overline{Q} = \{(V_+, V_-) \in Y \times Y \mid V_+ - \{0\} \subset C_+ \text{ and } V_- - \{0\} \subset C_-\}.$$

Let $xK_{\mathbb{C}}$ be an element of $\partial(D/K_{\mathbb{C}})$ such that $\iota(xK_{\mathbb{C}}) \in G_{\mathbb{R}} c_{\beta_1} Q/Q \times G_{\mathbb{R}}\overline{Q}/\overline{Q}$. Then it follows from Lemma 2.3 that

$$xK_{\mathbb{C}} gB \cap G_{\mathbb{R}} c_{\beta_1} gB \neq \phi$$

for $g = e, w_{\beta_2}$ and c_{β_2}. This implies that

$$xS_1 \cap S'_8 \neq \phi, \tag{3.1}$$

$$xS_3 \cap S'_9 \neq \phi \tag{3.2}$$

and that

$$xS_5 \cap S'_{\mathrm{op}} \neq \phi. \tag{3.3}$$

Since ${S'_7}^{cl} = \{(V_1, V_2) \mid V_1 \subset C_0\} \supset S'_9$, it follows from (3.2) that

$$xS_3 \cap {S'_7}^{cl} \neq \phi. \tag{3.4}$$

On the other hand since ${S'_{10}}^{cl} \supset S'_{\mathrm{op}}$, it follows from (3.3) that

$$xS_5 \cap {S'_{10}}^{cl} \neq \phi. \tag{3.5}$$

Remark 3.1. (i) If $\iota(xK_{\mathbb{C}}) \in G_{\mathbb{R}}Q/Q \times G_{\mathbb{R}}\overline{c_{\beta_1}}\overline{Q}/\overline{Q}$, then we can prove

$$xS_2 \cap S'_9 \neq \phi, \quad xS_4 \cap S'_8 \neq \phi, \quad xS_6 \cap S'_{\mathrm{op}} \neq \phi,$$
$$xS_4 \cap {S'_7}^{cl} \neq \phi \quad \text{and} \quad xS_6 \cap {S'_{10}}^{cl} \neq \phi$$

in the same way.

(ii) If we apply [M4], Theorem 1.3, to this case, then we have

$$x \in \partial D \Longrightarrow x(S_5 \sqcup S_6)^{cl} \cap S'_{\mathrm{op}} \neq \phi.$$

So we see that the results in this paper are refinements of this theorem for Hermitian cases.

By (3.4) and (3.5) we proved the following.

Proposition 3.2. *If $\iota(xK_{\mathbb{C}}) \in G_{\mathbb{R}}c_{\beta_1}Q/Q \times G_{\mathbb{R}}\overline{Q}/\overline{Q}$, then we have*:
(i) $xK_{\mathbb{C}}w_{\beta_2}B \cap (G_{\mathbb{R}}c_\delta w_{\beta_2}B)^{cl} \neq \phi$.
(ii) $xK_{\mathbb{C}}c_{\beta_2}B \cap (G_{\mathbb{R}}c_\delta B)^{cl} \neq \phi$.

Remark 3.3. It is clear that $K_{\mathbb{C}}w_{\beta_2}B = S_3 \subset S_7^{cl} = (K_{\mathbb{C}}c_\delta w_{\beta_2}B)^{cl}$ and that $K_{\mathbb{C}}c_{\beta_2}B = S_5 \subset S_{10}^{cl} = (K_{\mathbb{C}}c_\delta B)^{cl}$.

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