TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 10, October 2007, Pages 4695–4709
S 0002-9947(07)04132-3
Article electronically published on April 24, 2007

COMPATIBLE VALUATIONS AND GENERALIZED MILNOR K-THEORY

IDO EFRAT

ABSTRACT. Given a field F and a subgroup S of $F^\times$ there is a minimal group $S \le H_S \le F^\times$ for which there exists an S-compatible valuation whose units are contained in H_S. Assuming that S has finite index in $F^\times$ and contains $(F^\times)^p$ for p prime, we describe H_S in computable K-theoretic terms.

1. Introduction

In many problems in field theory one needs to detect "arithmetically interesting" valuations on a field F, e.g., valuations v satisfying some variant of Hensel's lemma. An especially useful such variant is S-compatibility: given a subgroup S of the multiplicative group $F^\times$ of F, one says that v is **S-compatible** if the group $1+\mathfrak{m}_v$ of principal units of v is contained in S (here $\mathfrak{m}_v$ denotes the maximal ideal of the valuation ring O_v of v). Since S-compatibility (as well as full Henselity) is always satisfied by the trivial valuation (for which $O_v = F$), some non-triviality condition is also needed. We therefore consider a second subgroup H of $F^\times$ and require that $O_v^\times \le H$, where $O_v^\times$ is the group of invertible elements in O_v. Let $\mathrm{Val}(S,H)$ be the set of all S-compatible valuations on F which satisfy this requirement.

In practical problems, S is usually given and one looks for an S-compatible valuation with $v(F^\times)/v(S)$ as large as possible. Equivalently, one looks for a subgroup H which is as small as possible with $\mathrm{Val}(S,H) \neq \emptyset$. It follows from powerful results of Arason, Elman and Jacob [AEJ] (see also the related works [J], [Wr1], [HJ] and [E1]) that, apart from a well-understood exceptional case, there exists a *canonical minimal* group H_S with these two properties. Thus $\mathrm{Val}(S,H_S) \neq \emptyset$, and $H_S \le H$ whenever $\mathrm{Val}(S,H) \neq \emptyset$. Moreover, $\mathrm{Val}(S,H_S)$ then contains a canonical (coarsest) valuation.

A special case which has often been considered is when $S = (F^\times)^p$ for a prime number p and F contains a pth root of unity. Then a valuation v with residue characteristic $\neq p$ is S-compatible if and only if it is p-Henselian, i.e., Hensel's lemma holds for polynomials of degree p [Wd, Prop. 1.4]. Thus the problem in this case is to find a p-Henselian valuation v on F with $v(F^\times)/pv(F^\times) \cong F^\times/SO_v^\times$ as large as possible. This became especially important in the area of "birational anabelian geometry", where one recovers arithmetical properties of a field from the

Received by the editors March 24, 2005.
2000 *Mathematics Subject Classification.* Primary 19F99; Secondary 12J15, 19C99, 12J99.
This research was supported by the Israel Science Foundation grant No. 8008/02–1.

knowledge of its absolute Galois group. The above-mentioned result of [AEJ] assures that the largest possible quotient $v(F^\times)/pv(F^\times)$ which one can get is $F^\times/H_S$ (again, apart from the exceptional case). However, while H_S can be described in elementary terms (see §3), when $p \neq 2$ it is a priori not clear how to detect it *Galois-theoretically*, as is required in such problems.

In this paper we solve this problem as follows. We consider a subgroup $(F^\times)^p \leq S \leq F^\times$ and associate to it a certain group $S \leq N_S \leq F^\times$ which is defined in terms of the generalized Milnor K-ring functor $K_*^M(F)/S$, introduced in [E5]. We prove that when $(F^\times : S) < \infty$ (and apart from the exceptional case mentioned above) the groups H_S and N_S coincide. Now when $S = (F^\times)^p$ and F contains a pth root of unity, Kummer's theory and the Merkurjev–Suslin theorem imply that the ring $K_*^M(F)/S$ and the Galois cohomology ring $H^*(G_F(p), \mathbb{Z}/p)$ are isomorphic in degrees 1 and 2 (where $G_F(p)$ is the maximal pro-p Galois group of F; see [MS], [JWd, §1], [Ka]). Furthermore, N_S is actually determined by the degree ≤ 2 part of $K_*^M(F)/S$. Consequently, $G_F(p)$ indeed determines $N_S = H_S$ – in fact, in an effectively computable way.

This *quantitative* result strengthens the results of [HJ] and [Ko], which are of a *qualitative* nature – i.e., yield valuations v with $v(F^\times) \neq pv(F^\times)$. It also improves the quantitative results of [E1], which gave another bound on the possible size of $v(F^\times)/pv(F^\times)$ for p-Henselian valuations v. While the latter bound was sufficient for some applications ([EF], [E2], [E3]), it was not sharp, due to the non-canonical nature of the construction there.

As an application we go back in §8 to results of Bass-Tate [BaT], Wadsworth [Wd], and [E5], showing that the Milnor K-ring of an S-compatible valued field (F, v) can be built from the corresponding K-ring of the residue field and the group $v(F^\times)/v(S)$ as an *extension* (this construction is analogous to the construction of polynomial rings in a set of variables over a given ring; see §5). We prove a converse result, asserting that extension structures on $K_*^M(F)/S$ must originate from S-compatible valuations (Theorem 8.1).

The exceptional case mentioned above is that of **totally rigid** groups S. When $(F^\times)^p \leq S$ as before, this means in a K-theoretic language that $K_*^M(F)/S$ is an extension of either $K_*^M(\mathbb{R})/2$ (if $p = 2$) or $K_*^M(\mathbb{C})/p$. As we show in §9, totally rigid subgroups of the first kind are nothing but the **fans**, which were extensively studied in the context of ordered fields. Thus our combined K-theoretic/valuation-theoretic approach reveals a non-real analog of fans, namely, the totally rigid subgroups of the second kind.

2. Valuations

We first recall some basic facts about valuations which will be needed later on. All valuations will be in the sense of Krull, i.e., corresponding to valuation rings (see, e.g., [Bo, Ch. VI], [En], [R]), and we will not distinguish between valuations with equal valuation rings. Given valuations v, u on a field F, we say that v is **finer** than u (and u is **coarser** than v) if $O_v \subseteq O_u$. Other equivalent conditions are that $O_v^\times \leq O_u^\times$, $\mathfrak{m}_u \subseteq \mathfrak{m}_v$, or that there is an epimorphism of ordered abelian groups $\varphi\colon v(F^\times) \to u(F^\times)$ such that $u = \varphi \circ v$. In this case we write $u \leq v$. This is a partial ordering on the set of all valuations on F. We say that v and u are **comparable** if one of them is coarser than the other. The collection of all valuations on F has a tree structure with respect to $\leq$, with the trivial valuation as its minimum:

Proposition 2.1. *The above partially ordered set has the following properties:*

(a) *if v, u, u' are valuations with $u \leq v$ and $u' \leq v$ then u, u' are comparable;*
(b) *every collection v_i, $i \in I$, of valuations on F has a finest common coarsening (i.e., an infimum).*

Proof. For (a) see [Bo, Ch. VI, §4.1].

To prove (b) let O be the subring of F generated by the valuation ring O_{v_i}, $i \in I$. It is necessarily a valuation ring. The corresponding valuation is as required. $\square$

The next lemma is a special case of [E4, Cor. 2.3].

Lemma 2.2. *Let v_1, v_2 be incomparable valuations on the field F and let u be their finest common coarsening. Then $O_u^\times = (1 + \mathfrak{m}_{v_1})(1 + \mathfrak{m}_{v_2})$.*

Given subsets C_i, $i \in I$, of F let $\prod_{i \in I} C_i$ be the set of all products $\prod_{i \in I} c_i$, where $c_i \in C_i$ for all i, and $c_i = 1$ for all but finitely many i (this makes sense also when I is infinite).

Proposition 2.3. *Let v_i, $i \in I$, be valuations on the field F and let u be their finest common coarsening. Then:*

(a) $O_u = \prod_{i \in I} O_{v_i}$;
(b) $O_u^\times = \prod_{i \in I} O_{v_i}^\times$.

Proof. (a) We first show that $\prod_{i \in I} O_{v_i}$ is a ring. Indeed, consider elements $z = \prod_{i \in I} z_i$ and $t = \prod_{i \in I} t_i$ of $\prod_{i \in I} O_{v_i}$, with $z_i, t_i \in O_{v_i}$ as above. Clearly, $zt = \prod_{i \in I}(z_i t_i) \in \prod_{i \in I} O_{v_i}$. Also take an arbitrary $i_0 \in I$. By the ultrametric inequality, $v_{i_0}(z + t) \geq \min\{v_{i_0}(z), v_{i_0}(t)\}$ – say, $v_{i_0}(z + t) \geq v_{i_0}(z)$. Then $z + t \in zO_{i_0} \subseteq \prod_{i \in I} O_{v_i}$, as desired.

Since $\prod_{i \in I} O_{v_i}$ contains the valuation rings O_{v_i}, $i \in I$, it is itself a valuation ring. Obviously, it has no proper subrings containing all the O_{v_i}, $i \in I$.

(b) We first verify this in the case of two valuations v_1, v_2.

If v_1, v_2 are comparable, say, $v_1 \leq v_2$, then $u = v_1$, so $O_u^\times = O_{v_1}^\times \geq O_{v_2}^\times$, and the assertion is clear.

If v_1, v_2 are incomparable, then Lemma 2.2 gives

$$O_u^\times = (1 + \mathfrak{m}_{v_1})(1 + \mathfrak{m}_{v_2}) \leq O_{v_1}^\times O_{v_2}^\times.$$

On the other hand, $O_{v_i}^\times \leq O_u^\times$, $i = 1, 2$, so we are done once again.

The case of finitely many valuations v_i follows from the case of two valuations by induction.

Finally consider the general case. Since u is coarser than every v_i, we have $O_u^\times \geq \prod_{i \in I} O_{v_i}^\times$. Conversely, take $x \in O_u^\times$. (a) gives finite subsets J, J' of I with $x \in \prod_{j \in J} O_{v_j}$ and $x^{-1} \in \prod_{j \in J'} O_{v_j}$. We may replace both J and J' by $J \cup J'$ to assume that $J = J'$. Let w be the finest common coarsening of v_j, $j \in J$. By (a) again, $O_w = \prod_{j \in J} O_{v_j}$, and by what we have already seen, $O_w^\times = \prod_{j \in J} O_{v_j}^\times$. Hence $x \in O_w^\times \leq \prod_{i \in I} O_{v_i}^\times$, as desired. $\square$

Denote the residue field of a valuation v on F by $\bar{F}_v$. For a subgroup S of $F^\times$ let $\bar{S}_v$ be the image of $S \cap O_v^\times$ under the residue homomorphism $O_v \to \bar{F}_v$. It is a subgroup of $\bar{F}_v^\times$. When v is S-compatible there is a canonical short exact sequence [E5, (5.4)]

$$1 \to \bar{F}_v^\times/\bar{S}_v \to F^\times/S \to v(F^\times)/v(S) \to 0. \tag{2.1}$$

If u is another valuation on F which is coarser than v, then $O_v/\mathfrak{m}_u$ is a valuation ring on $\bar{F}_u$. We denote the corresponding valuation by v/u. One has $(\bar{F}_u)_{v/u} \cong \bar{F}_v$ canonically, and this isomorphism identifies $(\bar{S}_u)_{v/u}$ with $\bar{S}_v$ (see [Bo, Ch. VI, §4.3]).

3. Existence of valuations

In this section we summarize some of the main results of [AEJ] which will be required later on. We fix the field F and consider subgroups $S \le H$ of $F^\times$. We set:

$$\begin{aligned} O^-(S,H) &= (S-1) \setminus H, \\ O^+(S,H) &= \{x \in H \mid xO^-(S,H) \subseteq O^-(S,H)\}, \\ O(S,H) &= O^-(S,H) \cup O^+(S,H). \end{aligned}$$

Remark 3.1. For K-theoretic purposes it is actually more convenient to define $O^-(S,H)$ as $(1-S) \setminus H$. Then the main results of [AEJ] remain valid with some minor sign adaptations (see [E1]). However since we will extensively use here the results of [AEJ], we keep these definitions.

Let

$$A_S = \{x \in F^\times \mid S - xS \not\subseteq S \cup -xS\}, \qquad H_S = \langle -1, A_S \rangle.$$

For $x \in S$ we have $0 \in S - xS$ but $0 \notin S \cup -xS$. Hence $S \subseteq A_S$. In the terminology of [AEJ], A_S consists of all $x \in F$ such that $-x$ is not S-rigid. An element x of F is called S-**basic** if $x \in A_S \cup -A_S$. Thus H_S is the subgroup of $F^\times$ generated by the S-basic elements. The pair (S,H) is **pre-additive** if $1 - O^-(S,H)O^-(S,H) \subseteq S$ (see [AEJ, Lemma 2.6]).

Theorem 3.2. *The following conditions are equivalent:*

(a) $\mathrm{Val}(S,H) \neq \emptyset$*;*
(b) $O(S,H)$ *is a valuation ring corresponding to a valuation in* $\mathrm{Val}(S,H)$ *which is coarser than any other valuation in* $\mathrm{Val}(S,H)$*;*
(c) $H_S \le H$ *and* (S,H) *is pre-additive.*

Proof. (b)⇒(a): Trivial.

(a)⇒(c): Let $v \in \mathrm{Val}(S,H)$. Obviously, $-1 \in O_v^\times \le H$. By [AEJ, Prop. 1.5(1)], $A_S \subseteq SO_v^\times \le H$.

Finally, if $x \in O^-(S,H)$, then $1 + x \in S \le H$, as well as $x \notin H$. Therefore $1 + x^{-1} = x^{-1}(1+x) \notin H$, and in particular, $1 + x^{-1} \notin S$. Since v is S-compatible, $x^{-1} \notin \mathfrak{m}_v$, so $x \in O_v \setminus H$. Conclude that

$$O^-(S,H) \subseteq O_v \setminus H \subseteq O_v \setminus O_v^\times = \mathfrak{m}_v,$$

whence

$$1 - O^-(S,H)O^-(S,H) \subseteq 1 - \mathfrak{m}_v\mathfrak{m}_v \subseteq 1 + \mathfrak{m}_v \le S.$$

(c)⇒(b): By (c), H contains the S-basic elements of F. Therefore [AEJ, Th. 2.10] shows that $O(S,H)$ corresponds to a valuation in $\mathrm{Val}(S,H)$. By [AEJ, Prop. 3.2], it is coarser than any other valuation in $\mathrm{Val}(S,H)$. □

The picture given by Theorem 3.2 would be complete if we knew that $\mathrm{Val}(S, H_S)$ is non-empty. While this is not always the case, the following proposition shows that the exceptions are quite rare.

Proposition 3.3. *If* $\mathrm{Val}(S, H_S) = \emptyset$*, then*

(a) $H_S = \langle -1, S\rangle$;
(b) *there exists an intermediate group* $H_S < L \leq F^\times$ *such that* $(L : H_S) = 2$ *and* $\mathrm{Val}(S, L) \neq \emptyset$;
(c) $-1 \in S$ *or* $S + S \subseteq S$;
(d) $A_S = S$.

Proof. By assumption, $O(S, H_S)$ does not correspond to a valuation in $\mathrm{Val}(S, H_S)$. Therefore (a)–(c) follow from [AEJ, Th. 2.16].

For (d), suppose that $x \in A_S \setminus S$. Then (a) implies that $-x \in S$. Hence $-1 \notin S$, so by (c), $S - xS = S + S \subseteq S = S \cup -xS$. This contradicts $x \in A_S$. $\square$

4. κ-structures

We recall from [E5] the basic notions of the category of κ-structures, which serves as the target category for the generalized Milnor K-ring functor.

Denote the tensor algebra of an abelian group Γ (considered as a $\mathbb{Z}$-algebra) by $\mathrm{Tens}(\Gamma)$. Let $\kappa = \bigoplus_{r=0}^\infty \kappa_r = \mathrm{Tens}(\{\pm 1\})$, and let ϵ be the non-trivial element of $\kappa_1 \cong \{\pm 1\}$. A **κ-structure** consists of a graded ring $A = \bigoplus_{r=0}^\infty A_r$ and a graded ring homomorphism $\kappa \to A$ such that:

(i) $A_0 = \mathbb{Z}$ and the homomorphism $\kappa \to A$ is the identity in degree 0;
(ii) A_1 generates A as a ring;
(iii) the image ϵ_A of ϵ in A_1 satisfies $a^2 = \epsilon_A a = a\epsilon_A$ for all $a \in A_1$.

For $a, b \in A_1$, (iii) gives $ab + ba = (a + b)^2 - a^2 - b^2 = 0$, so A is anti-commutative. The **trivial** κ-structure $\mathbf{0}$ is defined by $\mathbf{0}_0 = \mathbb{Z}$ and $\mathbf{0}_r = 0$ for $r \geq 1$. A **morphism** $A \to B$ of κ-structures is a graded ring homomorphism which commutes with the structural homomorphisms $\kappa \to A$, $\kappa \to B$.

The tensor product in the category of graded rings is defined as usual by

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{r=0}^{\infty} \Big(\bigoplus_{i+j=r} A_i \otimes_{\mathbb{Z}} B_j \Big),$$

with the multiplication law

$$(a \otimes b)(a' \otimes b') = (-1)^{i'j} aa' \otimes bb'$$

for $a \in A_i$, $a' \in A_{i'}$, $b \in B_j$, $b' \in B_{j'}$. Given κ-structures A, B, we define their tensor product in the category of *κ-structures* to be $A \otimes_\kappa B = (A \otimes_{\mathbb{Z}} B)/I$, where I is the homogeneous ideal generated by $\epsilon_A \otimes 1_B - 1_A \otimes \epsilon_B$. The structural homomorphism $\kappa \to A \otimes_\kappa B$ is given by $\epsilon \mapsto \epsilon_A \otimes 1_B + I = 1_A \otimes \epsilon_B + I$.

Next, for an abelian group Γ let $\kappa[\Gamma]$ be the quotient of $\mathrm{Tens}(\kappa_1 \oplus \Gamma)$ by the homogeneous ideal generated by all elements $\epsilon \otimes \gamma - \gamma \otimes \gamma$, where $\gamma \in \Gamma$. The obvious embedding $\kappa_1 \hookrightarrow \kappa_1 \oplus \Gamma$ induces a graded ring homomorphism $\kappa \to \kappa[\Gamma]$, making $\kappa[\Gamma]$ into a κ-structure. More generally, for a κ-structure $\bar{A}$ we define the **extension** $\bar{A}[\Gamma] = \bar{A} \otimes_\kappa \kappa[\Gamma]$ of $\bar{A}$ by Γ. One may identify $(\bar{A}[\Gamma])_1 = \bar{A}_1 \oplus \Gamma$. If Γ' is another abelian group, then $(\bar{A}[\Gamma])[\Gamma'] \cong \bar{A}[\Gamma \oplus \Gamma']$ canonically [E5, Cor. 1.3].

4.1 Examples. (a) $\mathbf{0}[\Gamma]$ is just the alternating $\mathbb{Z}$-algebra over Γ [Lg, Ch. XVI, §6].

(b) Let $\bar{A}$ be a κ-structure, let $\Gamma=\langle\gamma\rangle$ be a cyclic group, and let $A=\bar{A}[\Gamma]$. For $r\geq 1$, the group A_r is generated by the image of $\bigoplus_{i+j=r}(\bar{A}_i\otimes_{\mathbb{Z}}\Gamma^{\otimes j})$. By (iii) above, $\gamma^j=\epsilon_{\bar{A}}^{j-1}\gamma$ in A. It follows that $A_r=\bar{A}_r\oplus(\bar{A}_{r-1}\otimes_{\mathbb{Z}}\Gamma)$. Under this identification, the product $A_1\times A_1\to A_2$ is given by

$$(\bar{a},i\gamma)\cdot(\bar{b},j\gamma)=(\bar{a}\cdot\bar{b},(j\bar{a}-i\bar{b}+ij\epsilon_{\bar{A}})\otimes\gamma) \tag{4.1}$$

for $\bar{a},\bar{b}\in\bar{A}_1$ and $i,j\in\mathbb{Z}$.

The construction of extensions is intimately related to the following quite subtle notion, which goes back to Szymiczek [S] and Ware ([Wr1], [Wr2]).

Definition. Let n be a positive integer, and let A be a κ-structure. We say that $a\in A_1$ is n**-rigid** if:

(1) $\langle a\rangle\cong\mathbb{Z}/n$; and
(2) for every $b\in A_1$ with $a\cdot b=0$ in A_2 there exist integers i,j, at least one of which is prime to n, such that $j(\epsilon_A+a)=ib$.

Note that when $n=p$ is prime and $pA_1=0$, (2) means that for every $b\in A_1$ with $a\cdot b=0$ the elements ϵ_A+a and b of A_1 are $\mathbb{F}_p$-linearly dependent. Also note that when $p\neq 2$ we actually have $\epsilon_A=0$. The first fundamental connection between rigidity and extensions is the following fact.

Proposition 4.2. *Let n be a positive integer, and let $\bar{A}$ be a κ-structure such that $n\bar{A}_1=0$. Let Γ be a free $\mathbb{Z}/n$-module and let $A=\bar{A}[\Gamma]$. Suppose that the projection of $a\in A_1$ to Γ has order n. Then a is n-rigid.*

Proof. We may decompose $\Gamma=\Gamma_1\oplus\Gamma_2$, where $\Gamma_2\cong\mathbb{Z}/n$ and where the projection of a in Γ_2 has order n. As $A\cong(\bar{A}[\Gamma_1])[\Gamma_2]$, it is enough to consider the case where $\Gamma=\Gamma_2=\langle\gamma\rangle$ is cyclic of order n. Then the computations of Example 4.1(b) apply.

We write $a=(\bar{a},i\gamma)\in A_1=\bar{A}_1\oplus\Gamma$, with $\bar{a}\in\bar{A}_1$ and $i\in\mathbb{Z}$. By assumption, $\langle a\rangle\cong\mathbb{Z}/n$, and furthermore, n and i are relatively prime. Note that if $\epsilon_{\bar{A}}\neq 0$, then n is even, so i is odd and $i\epsilon_{\bar{A}}=\epsilon_{\bar{A}}$. Obviously, this also holds when $\epsilon_{\bar{A}}=0$.

Now let $b=(\bar{b},j\gamma)\in A_1=\bar{A}_1\oplus\Gamma$ and suppose that $a\cdot b=0$ in A_2. By (4.1), $i\bar{b}=j\bar{a}+ij\epsilon_{\bar{A}}=j(\bar{a}+\epsilon_{\bar{A}})$. Therefore

$$ib=(i\bar{b},ij\gamma)=j(\bar{a}+\epsilon_{\bar{A}},i\gamma)=j(a+\epsilon_A)\quad,$$

as required. □

4.3 Examples. (1) Let p be a prime number and let Γ be an abelian group of exponent p. Let $A=\mathbf{0}[\Gamma]$. By Proposition 4.2, every non-zero element of A_1 is p-rigid.

(2) Let Γ be an abelian group of exponent 2 and let $A=\kappa[\Gamma]$. Then every non-zero element a of A_1 is 2-rigid. Indeed, for $a=\epsilon_A$ this is trivial from the definition of rigidity, while for $a\neq 0,\epsilon_A$ it follows again from Proposition 4.2.

Given a κ-structure A we define a homomorphism $\mathrm{Bock}_A\colon A_1\to A_2$ by

$$\mathrm{Bock}_A(a)=a^2=\epsilon_A\cdot a\quad.$$

Generalizing standard terminology (see, e.g., [NSW, p. 191]), we call Bock_A the **Bockstein map** of A.

Lemma 4.4. *Let Γ be an abelian group of exponent 2, and let $A = \kappa[\Gamma]$. Then Bock_A is injective.*

Proof. Suppose that $0 \neq a \in A_1$ satisfies $\mathrm{Bock}_A(a) = 0$. As we have just seen, a is 2-rigid. Since $a^2 = 0$, the elements a and $a + \epsilon$ of A_1 must be $\mathbb{F}_2$-linearly dependent. As $a, \epsilon \neq 0$, this can happen only if $a = \epsilon$. But $\mathrm{Bock}_A(\epsilon) = \epsilon^2 \neq 0$, a contradiction. □

5. Generalized Milnor K-rings

Let F be a field and S a subgroup of $F^\times$. For $r \geq 0$ let $\mathrm{St}_{F,r}(S)$ be the subgroup of $(F^\times/S)^{\otimes r} = F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times$ (r times) generated by all elements $a_1 S \otimes \cdots \otimes a_r S$ such that $1 \in a_i S + a_j S$ for some $i \neq j$. Generalizing standard terminology, we call such generators **Steinberg elements**. As in [E5] we set

$$K_r^M(F)/S = (F^\times/S)^{\otimes r}/\mathrm{St}_{F,r}(S).$$

Then $K_*^M(F)/S = \bigoplus_{r=0}^{\infty} K_r^M(F)/S$ is a κ-structure with respect to the multiplication induced from the tensor algebra over $F^\times$, and with the structural morphism $\kappa \to K_*^M(F)/S$ given by $\epsilon \mapsto -S \in F^\times/S$ (see [E5, §2]). We call it the **Milnor K-ring of F modulo S**. In particular, $K_0^M(F)/S = \mathbb{Z}$ and $K_1^M(F)/S = F^\times/S$. When $S = \{1\}$ this is the classical Milnor K-ring $K_*^M(F)$ defined in [Mi]. When $S = (F^\times)^m$ for a positive integer m, the rings $K_*^M(F)/S$ and $K_*^M(F)/m$ coincide except in degree 0.

Given $a_1, \ldots, a_r \in F^\times$ we denote the image of $a_1 S \otimes \cdots \otimes a_r S$ in $K_r^M(F)/S$ by $\{a_1, \ldots, a_r\}_S$. The map $\{a_1, \ldots, a_r\}_S$ is multi-linear and anti-commutative. Also, since the identity $\{a, -a\} = 0$ for $a \in F^\times$ is well-known to hold in $K_2^M(F)$ [Mi, §1], one has $\{a, -a\}_S = 0$ in $K_*^M(F)/S$. Therefore $\{a, a\}_S = \{-1, a\}_S = \{a, -1\}_S$.

For an intermediate group $S \leq S' \leq F^\times$ there is a canonical **restriction** morphism

$$\mathrm{Res}\colon K_*^M(F)/S \to K_*^M(F)/S',$$

which maps a generator $\{a_1, \ldots, a_r\}_S$ of $K_r^M(F)/S$ to $\{a_1, \ldots, a_r\}_{S'}$.

The S-compatible valuations reflect in the K-ring as extensions, in the sense of §4. This is the following special case of [E5, Th. 5.1], which in turn extends earlier results of Bass and Tate [BaT, I, Prop. 4.3] and Wadsworth [Wd, §2].

Theorem 5.1. *Let F be a field, p a prime number, and $(F^\times)^p \leq S \leq F^\times$ an intermediate group. Let v be an S-compatible valuation. Then*

$$K_*^M(F)/S \cong (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)].$$

The arithmetic structure of F as an ordered field is also reflected in its K-theory in a rather simple way as follows. A subgroup S of $F^\times$ is called a (proper) **preordering** if it is additively closed, contains $(F^\times)^2$, but does not contain -1. It is called an **ordering** if in addition $F^\times = S \cup -S$.

Proposition 5.2. (a) *S is an ordering if and only if $K_*^M(F)/S \cong \kappa$ as κ-structures.*

(b) *For distinct orderings S_1, S_2 on F one has $K_*^M(F)/(S_1 \cap S_2) \cong \kappa[\mathbb{Z}/2]$.*

Proof. (a) See [E5, Prop. 3.2].

(b) Set $S = S_1 \cap S_2$ and choose $a_1 \in S_1 \setminus S_2$. Then $a_2 := -a_1 \in S_2 \setminus S_1$, and $F^\times/S$ consists of the cosets of $1, -1, a_1, a_2$. Since $K_*^M(F)/S$ is a κ-structure,

$\{a_i,a_i\}_S=\{-1,a_i\}_S$, $i=1,2$, and $\{a_1,a_2\}_S=0$. It follows that for $r\geq 2$ the group $K^M_r(F)/S$ consists only of the elements

$$0\quad,\quad\{-1,-1,\dots,-1\}_S\quad,\quad\{-1,\dots,-1,a_1\}_S\quad,\quad\{-1,\dots,-1,a_2\}_S.$$

Furthermore, consider the restriction morphism $\mathrm{Res}\colon K^M_*(F)/S\to K^M_*(F)/S_i$ ($\cong\kappa$, by (a)). When $i=1$ it maps only the first and the third elements in this list trivially, and when $i=2$ it maps only the first and the fourth elements trivially. It follows that these four elements are distinct.

On the other hand, Examples 4.1(b) shows that $(\kappa[\mathbb{Z}/2])_r$ is a group of order 4 generated by ϵ^r and $\epsilon^{r-1}\gamma$, where γ is the generator of $\mathbb{Z}/2$. The multiplicative structure of $\kappa[\mathbb{Z}/2]$ is then determined by the equality $\gamma^2=\epsilon\gamma$. Consequently the κ-structures $K^M_*(F)/S$ and $\kappa[\mathbb{Z}/2]$ are isomorphic (with a_1 corresponding to γ). $\square$

6. Totally rigid subgroups

Let F be a field and S a subgroup of $F^\times$. The identity $\{a,-a\}_S=0$ implies that every tensor product $a_1S\otimes\cdots\otimes a_rS$ such that $a_1,\dots,a_r\in F^\times$ and $a_iS=-a_jS$ for some $1\leq i<j\leq r$ is contained in $\mathrm{St}_{F,r}(S)$. We say that S is **totally rigid** if for every r the group $\mathrm{St}_{F,r}(S)$ is in fact generated by such tensor products.

Proposition 6.1. (a) *Assume that $-1\in S$. Then S is totally rigid if and only if $K^M_*(F)/S\cong\mathbf{0}[\Gamma]$ for some abelian group Γ.*

(b) *Assume that $-1\notin S$ and $(F^\times)^2\leq S$. Then S is totally rigid if and only if $K^M_*(F)/S\cong\kappa[\Gamma]$ for some group Γ.*

Proof. (a) This follows from Examples 4.1(a), taking $\Gamma=F^\times/S$.

(b) This follows from the definition of $\kappa[\Gamma]$, where Γ is taken to be a complement of the subgroup generated by $-S$ in $F^\times/S$. $\square$

Proposition 6.2. *If $A_S=S$, then S is totally rigid.*

Proof. Let $a_1S\otimes\cdots\otimes a_rS$ be a standard generator of $\mathrm{St}_{F,r}(S)$, i.e., $1=a_is+a_js'$ for some $i\neq j$ and some $s,s'\in S$. If $a_i\in S$ or $a_j\in S$, then this generator is trivial. If $a_i,a_j\notin S=A_S$, then $a_js'=1-a_is\in S-a_iS\subseteq S\cup -a_iS$ and $a_js'\notin S$. Hence $a_jS=-a_iS$. $\square$

We can now give a K-theoretic answer to the problem considered in §3 whether $\mathrm{Val}(S,H_S)$ is non-empty.

Corollary 6.3. *Suppose that S is not totally rigid. Then* $\mathrm{Val}(S,H_S)\neq\emptyset$.

Proof. Apply Proposition 6.2 and Proposition 3.3(d). $\square$

From now on we fix a prime number p and assume that $(F^\times)^p\leq S\leq F^\times$. Let

$$T_S=\Big\{x\in F^\times\ \Big|\ 1-x\notin\bigcup_{i=0}^{p-1}x^iS\Big\}\quad.$$

6.4 Remarks. (1) When $-1\in S$ one has $T_S\subseteq A_S$.

(2) Suppose that $T_S\subseteq S$. Then $\mathrm{St}_{F,r}(S)$ is generated by the tensor products $a_1S\otimes\cdots\otimes a_rS$ such that $a_iS=a_jS$ for some $1\leq i<j\leq r$. If in addition $-1\in S$, then we deduce that S is totally rigid.

We now define the group N_S as follows: it is the subgroup of $F^\times$ generated by -1, S, and by all $x \in F^\times$ such that $\{x\}_S$ is *not* p-rigid in $K_*^M(F)/S$. More specifically, N_S is generated by -1, S, and by all $x \in F^\times$ for which there exists $y \in F^\times$ such that $\{x, y\}_S = 0$ and $y \notin \bigcup_{i=0}^{p-1}(-x)^i S$.

Lemma 6.5. (a) *If $-1 \in S$, then $T_S \subseteq N_S$.*
(b) *If $p = 2$, then $A_S \subseteq N_S$.*

Proof. (a) Take $1 \neq x \in T_S$. Then $\{x, 1-x\} = 0$ and

$$1 - x \notin \bigcup_{i=0}^{p-1} x^i S = \bigcup_{i=0}^{p-1} (-x)^i S,$$

so $x \in N_S$.

(b) Let $x \in A_S$. Thus $S - xS \not\subseteq S \cup -xS$. Hence there exists $x' \in xS$ such that $y = 1 - x' \notin S \cup -x'S$. Conclude that $x' \in N_S$, whence also $x \in N_S$. $\square$

We will also need the following estimate:

Proposition 6.6. *If $-1 \in S$ and $T_S \subseteq S$, then $(H_S : S)|p$.*

Proof. When the pair (S, H_S) is pre-additive this is proved in [E1, Cor. 3.3] (following [HJ]).

When (S, H_S) is not pre-additive, $\mathrm{Val}(S, H_S) = \emptyset$ by Theorem 3.2. Conclude from Proposition 3.3 that $A_S = S$, so $H_S = S$ and we are done once again. $\square$

Proposition 6.7. *For S totally rigid:*
(a) *$N_S = \langle -1, S\rangle$;*
(b) *$(H_S : S)|p$.*

Proof. (a) By Examples 4.3(1), (2) and Proposition 6.1 all non-zero elements of $F^\times/S$ are p-rigid in $K_*^M(F)/S$. The assertion follows.

(b) Assume first that $-1 \in S$. Then $N_S = S$ by (a), so by Lemma 6.5(a), $T_S \subseteq S$. Now use Proposition 6.6.

Next assume that $-1 \notin S$. As $(F^\times)^p \leq S$ necessarily $p = 2$. By Lemma 6.5(b), $A_S \subseteq N_S$. Use (a) once again to conclude that $H_S = \langle -1, S\rangle$, whence $(H_S : S) = 2$. $\square$

7. Rigidity and valuations

Let again p be a prime number and $(F^\times)^p \leq S \leq F^\times$ an intermediate group. The subgroup N_S gives a bound on the possible sizes of the quotients $v(F^\times)/v(S)$, with v an S-compatible valuation:

Proposition 7.1. *For every S-compatible valuation v on F one has $N_S \leq SO_v^\times$ and $(v(F^\times) : v(S)) \leq (F^\times : N_S)$.*

Proof. By Theorem 5.1, $K_*^M(F)/S \cong (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)]$ as κ-structures. By Proposition 4.2, elements of $F^\times/S$ whose projection via v in $v(F^\times)/v(S)$ is non-zero are p-rigid in $K_*^M(F)/S$. Conclude from the definition of N_S that $v(N_S) \leq v(S)$, i.e., $N_S \leq SO_v^\times$. As $S \leq N_S$ we actually have $v(S) = v(N_S)$. Consequently,

$$(v(F^\times) : v(S)) = (v(F^\times) : v(N_S)) = (F^\times : N_S O_v^\times) \leq (F^\times : N_S). \qquad \square$$

In Theorem 3.2 we have seen that if $S \leq H \leq F^\times$ and $\mathrm{Val}(S,H) \neq \emptyset$, then $H_S \leq H$. Proposition 7.1 implies that the subgroup N_S has the same property:

Corollary 7.2. *If $S \leq H \leq F^\times$ and $\mathrm{Val}(S,H) \neq \emptyset$, then $N_S \leq H$.*

Again, this raises the question whether $\mathrm{Val}(S,N_S) \neq \emptyset$. This will be shown (under some assumptions) as a part of Theorem 7.5 below. We will first need the following key connection between the structure of $K_*^M(F)/S$ and valuations.

Lemma 7.3. *Suppose that $\langle -1,S\rangle \leq M < N_S$ and $(N_S : M) = p$. Then:*

(a) *$T_M \subseteq M$;*
(b) *there exists $M \leq L \leq F^\times$ such that $(L : M)|p$ and $\mathrm{Val}(M,L) \neq \emptyset$.*

Proof. (a) Take $x \notin M$. We need to show that the cosets $\{x\}_M$ and $\{1-x\}_M$ are $\mathbb{F}_p$-linearly dependent in $F^\times/M$.

If $\{x\}_S$ and $\{1-x\}_S$ are both not p-rigid in $K_*^M(F)/S$, then $x, 1-x \in N_S$. Since $\dim_{\mathbb{F}_p}(N_S/M) \leq 1$, the cosets $\{x\}_M$ and $\{1-x\}_M$ are then $\mathbb{F}_p$-linearly dependent in N_S/M $(\leq F^\times/M)$, and we are done.

Next suppose that at least one of $\{x\}_S, \{1-x\}_S$ is p-rigid in $K_*^M(F)/S$. From $\{x, 1-x\}_S = 0$ we then deduce that either $\{-x\}_S, \{1-x\}_S$ or $\{x\}_S, \{-(1-x)\}_S$ are $\mathbb{F}_p$-linearly dependent in $F^\times/S$. But $-1 \in M$, so in both cases, $\{x\}_M$ and $\{1-x\}_M$ are $\mathbb{F}_p$-linearly dependent in $F^\times/M$.

(b) By (a) and Proposition 6.6, $(H_M : M)|p$. When $\mathrm{Val}(M,H_M) \neq \emptyset$ we can therefore take $L = H_M$.

When $\mathrm{Val}(M,H_M) = \emptyset$ Proposition 3.3 shows that $H_M = M$, $p = 2$, and there is a subgroup L as required. □

The next main step towards the proof of Theorem 7.5 is the following proposition.

Proposition 7.4. *Suppose that $-1 \in S < N_S$ and $(F^\times : S) < \infty$. Then there exists an S-compatible valuation v on F such that $N_S = SO_v^\times$.*

Proof. Let $\mathcal{M}$ be the collection of all intermediate groups $S \leq M < N_S$ with $(N_S : M) = p$. The assumptions imply that $\mathcal{M}$ is finite and non-empty. For every $M \in \mathcal{M}$ Lemma 7.3(b) yields a group $M \leq L_M \leq F^\times$ with $(L_M : M)|p$ and a valuation $v_M \in \mathrm{Val}(M,L_M)$. Let v be the finest common coarsening of the valuations v_M, $M \in \mathcal{M}$ (Proposition 2.1(b)). Then

$$1 + \mathfrak{m}_v \leq \bigcap_{M \in \mathcal{M}} (1 + \mathfrak{m}_{v_M}) \leq \bigcap_{M \in \mathcal{M}} M = S,$$

so v is S-compatible. By Proposition 7.1, $N_S \leq SO_v^\times$. To show that this is an equality we distinguish between two cases.

Case I: For each $M \in \mathcal{M}$ there exists $M' \in \mathcal{M}$ such that v_M and $v_{M'}$ are incomparable. For each M we choose such M' and denote the finest common coarsening of v_M and $v_{M'}$ by u_M. Then v is also the finest common coarsening of the u_M for $M \in \mathcal{M}$. As $M \neq M'$ Lemma 2.2 gives:

$$O_{u_M}^\times = (1 + \mathfrak{m}_{v_M})(1 + \mathfrak{m}_{v_{M'}}) \leq MM' = N_S \quad .$$

Conclude from Proposition 2.3(b) that $O_v^\times = \prod_{M \in \mathcal{M}} O_{u_M}^\times \leq N_S$, whence $N_S = SO_v^\times$ in this case.

Case II: There exists $M \in \mathcal{M}$ such that v_M is comparable to $v_{M'}$ for all $M' \in \mathcal{M}$. Then v is the finest common coarsening of all the $v_{M'}$ which are coarser than v_M. These $v_{M'}$ form a chain with respect to coarsening (Proposition 2.1(a)). Since $\mathcal{M}$ is finite, $v = v_{M'}$ for some M'. Hence $O_v^\times = O_{v_{M'}}^\times \leq L_{M'}$. Therefore $M' < N_S \leq SO_v^\times \leq L_{M'}$. Since $(L_{M'} : M')|p$ this implies that $N_S = SO_v^\times$ in this case as well. □

We now come to our main result:

Theorem 7.5. *Suppose that $(F^\times)^p \leq S \leq F^\times$ and that S is not totally rigid. Moreover, assume that $p = 2$ or $(F^\times : S) < \infty$. Then:*

(a) $\mathrm{Val}(S, N_S) \neq \emptyset$*;*
(b) $N_S = H_S$*.*

Proof. (a) We break the proof into three cases.

Case I: $p = 2$. Then $A_S \subseteq N_S$, by Lemma 6.5(b). Hence $H_S \leq N_S$. Since S is not totally rigid, Corollary 6.3 gives $\emptyset \neq \mathrm{Val}(S, H_S) \subseteq \mathrm{Val}(S, N_S)$.

Case II: $p \neq 2$ and $S < N_S$. Then $-1 \in (F^\times)^p \leq S$ and $(F^\times : S) < \infty$. We may therefore apply Proposition 7.4.

Case III: $p \neq 2$ and $S = N_S$. Then again $-1 \in S$, so by Lemma 6.5(a), $T_S \subseteq N_S = S$. But according to Remarks 6.4(2) this can happen only when S totally rigid, a contradiction.

(b) Corollary 6.3 gives again $\mathrm{Val}(S, H_S) \neq \emptyset$, so by Corollary 7.2, $N_S \leq H_S$. On the other hand, (a) and Theorem 3.2 show that $H_S \leq N_S$. □

Corollary 7.6. *Under the assumptions of Theorem 7.5, there is an S-compatible valuation v on F such that $N_S = SO_v^\times$ and $(v(F^\times) : v(S)) = (F^\times : N_S)$.*

Proof. Theorem 7.5 yields a valuation $v \in \mathrm{Val}(S, N_S)$. Thus v is S-compatible and $O_v^\times \leq N_S$. By Proposition 7.1, $N_S = SO_v^\times$, whence the last equality. □

In the exceptional case where S is totally rigid, we have $N_S = \langle -1, S\rangle$ (Proposition 6.7(a)). Then the following slightly weaker version of Theorem 7.5 and Corollary 7.6 holds:

Proposition 7.7. *Suppose that $(F^\times)^p \leq S \leq F^\times$ is totally rigid. Then there exists an S-compatible valuation v on F such that $N_S = \langle -1, S\rangle$ is a subgroup of $SO_v^\times$ of index dividing p. In particular, $(v(F^\times) : v(S)) \geq (F^\times : \langle -1, S\rangle)/p$.*

Proof. If there exists a valuation $v \in \mathrm{Val}(S, H_S)$, then $S \leq \langle -1, S\rangle \leq SO_v^\times \leq H_S$, so by Proposition 6.7(b), $(SO_v^\times : \langle -1, S\rangle)|(H_S : S)|p$.

If $\mathrm{Val}(S, H_S) = \emptyset$, then by Proposition 3.3, $H_S = \langle -1, S\rangle$, $p = 2$, $A_S = S$, and there exists a group $H_S < L \leq F^\times$ such that $(L : H_S) = 2$ and $\mathrm{Val}(S, L) \neq \emptyset$. Choose $v \in \mathrm{Val}(S, L)$. As $H_S = \langle -1, S\rangle \leq SO_v^\times \leq L$, we obtain $(SO_v^\times : \langle -1, S\rangle)|2$ in this case as well.

For the last assertion of the theorem we compute:

$$\begin{aligned}(v(F^\times) : v(S)) = (F^\times : SO_v^\times) &= (F^\times : \langle -1, S\rangle)/(SO_v^\times : \langle -1, S\rangle) \\ &\geq (F^\times : \langle -1, S\rangle)/p.\end{aligned}$$ □

8. Extensions and valuations

We can now prove the following converse of Theorem 5.1.

Theorem 8.1. *Let p be a prime number, let $(F^\times)^p \leq S \leq F^\times$, and suppose that $p = 2$ or $(F^\times : S) < \infty$. Furthermore, assume that $K_*^M(F)/S \cong \bar{A}[\Gamma]$ for some κ-structure $\bar{A} \not\cong \mathbf{0}, \kappa$ and an abelian group Γ of exponent p. Then there exists an S-compatible valuation v on F such that $(v(F^\times) : v(S)) \geq |\Gamma|$.*

Proof. We have a sequence of group epimorphisms

$$F^\times \to F^\times/S \xrightarrow{\sim} \bar{A}_1 \oplus \Gamma \ \to \Gamma,$$

the first and third maps being the obvious projections. It follows from Proposition 4.2 that N_S is mapped trivially to Γ under the composed epimorphism. Hence $(F^\times : N_S) \geq |\Gamma|$.

If S is not totally rigid, then we are done by Corollary 7.6.

So suppose that S is totally rigid. Then $K_*^M(F)/S \cong \bar{A}'[\Gamma']$, where $\bar{A}'$ is either $\mathbf{0}$ or κ and Γ' is an abelian group of exponent p (Proposition 6.1). The previous argument shows that $(F^\times : N_S) \geq |\Gamma'|$. For an S-compatible valuation v as in Proposition 7.7 we therefore have:

$$(v(F^\times) : v(S)) \geq (F^\times : \langle -1, S\rangle)/p = (F^\times : N_S)/p \geq |\Gamma'|/p.$$

Thus it suffices to show that $|\Gamma'|/p \geq |\Gamma|$.

Indeed, we have $\bar{A}'[\Gamma'] \cong \bar{A}[\Gamma]$. If $\bar{A}' = \mathbf{0}$, then $\Gamma' \cong \bar{A}_1 \oplus \Gamma$ and $\bar{A}_1$ is a non-trivial group of exponent p, so the claim is clear. On the other hand, if $\bar{A}' = \kappa$, then $p = 2$ and $\epsilon_{\bar{A}}^r \neq 0$ for all r. As $\bar{A} \neq \mathbf{0}, \kappa$ necessarily $|\bar{A}_1| \geq 4$. Hence $\kappa_1 \oplus \Gamma' \cong \bar{A}_1 \oplus \Gamma$ gives $|\Gamma'|/2 \geq |\Gamma|$, as desired. □

9. Fans

A **fan** on the field F is a subset T of F such that $T \setminus \{0\}$ is a preordering on F, and for every $b \in F^\times \setminus (-T)$ one has $T + bT = T \cup bT$ ([BK], [L, Th. 5.5]). This notion, which has been extensively studied and used in the context of ordered fields, turns out to be essentially equivalent to the notion of a totally rigid preordering. Many other equivalent definitions can be given using the notions discussed so far.

Theorem 9.1. *Let S be a preordering on F. The following conditions are equivalent:*

- (a) *S is totally rigid;*
- (b) *$S \cup \{0\}$ is a fan;*
- (c) *for every $a \in F^\times \setminus (S \cup -S)$ one has $S - aS \subseteq S \cup -aS$;*
- (d) *$A_S = S$;*
- (e) *$H_S = \langle -1, S\rangle$;*
- (f) *$N_S = \langle -1, S\rangle$;*
- (g) *$K_*^M(F)/S \cong \kappa[\Gamma]$ for some abelian group Γ of exponent 2.*

Proof. (a)⇒(f): This is Proposition 6.7(a).

(f)⇒(d): Here $p = 2$, so Lemma 6.5(b) gives $A_S \subseteq N_S = \langle -1, S\rangle$. Furthermore, for $a \in -S$ we have $S - aS = S + S \subseteq S = S \cup -aS$, i.e., $a \notin A_S$. Thus $A_S \cap (-S) = \emptyset$.

(d)⇒(a): This is Proposition 6.2.

(d)⇔(e), (d)⇒(c): Straightforward.

(c)$\Rightarrow$(b): Set $T = S \cup \{0\}$ and take $b \in F^\times \setminus (-T)$ and $t_1, t_2 \in T$. Let $c = t_1 + bt_2$. We need to show that $c \in T \cup bT$. If $b \in S$, then $c \in T + T = T$, and we are done. Also, if $t_1 = 0$ or $t_2 = 0$, then the claim is clear. So suppose that $b \notin S$ and $t_1, t_2 \in S$. Then $\pm b \in F^\times \setminus (S \cup -S)$. By (c), $c \in S + bS \subseteq S \cup bS \subseteq T \cup bT$.

(b)$\Rightarrow$(d): We always have $S \subseteq A_S$. Conversely, take $a \in F^\times \setminus S$ and set again $T = S \cup \{0\}$. (b) implies that $S - aS \subseteq T - aT = T \cup -aT$. But $0 \notin S - aS$, so $S - aS \in S \cup -aS$, i.e., $a \notin A_S$.

(a)$\Leftrightarrow$(g): This is Proposition 6.1(b). $\square$

10. Examples of totally rigid groups

The next list of examples shows that for S totally rigid (so $N_S = \langle -1, S\rangle$ by Proposition 6.7(a)), the set $\mathrm{Val}(S, \langle -1, S\rangle)$ can be both empty or non-empty. In fact, each of these cases is possible when $-1 \in S$ as well as when $S \cup \{0\}$ is a fan. This shows that the assumption in Theorem 7.5(a) that S is not totally rigid cannot be removed.

All our examples will be of the following general type: we take an appropriate field $\bar F$ of characteristic $\neq p$ and the ordered group $\Gamma = \mathbb{Z}^m$ with its lexicographic order, where m is a cardinal number. We take $F = \bar F((\Gamma))$ to be the field of all formal power series $\alpha = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with $a_\gamma \in \bar F$ and well-ordered support $\{\gamma \in \Gamma \mid a_\gamma \neq 0\}$. As was shown by Hahn [H], F indeed forms a field with respect to the natural operations. We identify $\bar F$ with the subfield of all constant series in F. When $\alpha \neq 0$ one defines $v(\alpha)$ to be the minimal element of the support of α. The map v is a valuation on F with value group Γ and residue field $\bar F$. By a result of Krull [Kr, Satz 26], (F, v) has no proper immediate extensions, whence it is Henselian. We will also take an appropriate subgroup $(\bar F^\times)^p \leq \bar S \leq \bar F^\times$ and let $S = (F^\times)^p \bar S$. By Hensel's lemma, $1 + \mathfrak{m}_v \leq (F^\times)^p \leq S$, i.e., v is S-compatible. Also $v(S) = p\Gamma$ and $\bar S_v = \bar S$, so by Theorem 5.1:

$$K_*^M(F)/S \cong (K_*^M(\bar F)/\bar S)[(\mathbb{Z}/p)^m] \quad .$$

Example 10.1. S is totally rigid, $-1 \in S$, and $\mathrm{Val}(S, S) \neq \emptyset$. Take any $\bar F$ as above and $\bar S = \bar F^\times$. Then $K_*^M(\bar F)/\bar S = \mathbf{0}$, so $K_*^M(F)/S \cong \mathbf{0}[(\mathbb{Z}/p)^m]$. By Proposition 6.1(a), S is totally rigid. The groups $O_v^\times$ and $S \cap O_v^\times$ have the same image $\bar F_v^\times$ under the residue homomorphism. Hence $O_v^\times \leq (1 + \mathfrak{m}_v)(S \cap O_v^\times) \leq S$, so $v \in \mathrm{Val}(S, S)$.

Example 10.2. $S \cup \{0\}$ is a fan and $\mathrm{Val}(S, \langle -1, S\rangle) \neq \emptyset$. Take $p = 2$, $\bar F = \mathbb{R}$, and $\bar S = (\mathbb{R}^\times)^2$. Since $\bar S$ is an ordering on $\mathbb{R}$, we have $K_*^M(\bar F)/\bar S = \kappa$ (Proposition 5.2(a)). Thus $K_*^M(F)/S \cong \kappa[(\mathbb{Z}/2)^m]$, so by Proposition 6.1(b), S is totally rigid. In light of Lemma 4.4, the Bockstein map here is injective. By [E5, Prop. 3.3] this implies that S is a preordering, so by Theorem 9.1, $S \cup \{0\}$ is a fan.

Finally, the residue map induces an isomorphism $SO_v^\times/S \cong \bar F_v^\times/\bar S_v = \mathbb{R}^\times/(\mathbb{R}^\times)^2$. Hence $(SO_v^\times : S) = 2 = (\langle -1, S\rangle : S)$. As $\langle -1, S\rangle \leq SO_v^\times$, necessarily $\langle -1, S\rangle = SO_v^\times$, i.e., $v \in \mathrm{Val}(S, \langle -1, S\rangle)$.

Example 10.3. S totally rigid, $-1 \in S$, and $\mathrm{Val}(S, S) = \emptyset$. Dirichlet's theorem on primes in arithmetical progressions gives a prime number l such that $l \equiv 1 \pmod{2p}$. Take $\bar F = \mathbb{F}_l$ and $\bar S = (\mathbb{F}_l^\times)^p$. Then $S = (F^\times)^p$ and $-1 \in (\mathbb{F}_l^\times)^p \leq S$. The group $\mathbb{F}_l^\times/(\mathbb{F}_l^\times)^p$ is cyclic and non-trivial, whence of order p. Further, $K_r^M(\mathbb{F}_l) = 0$ for $r \geq 2$ [Mi, Example 1.5]. Therefore $K_*^M(\mathbb{F}_l)/\bar S \cong \mathbf{0}[\mathbb{Z}/p]$ (see Example 4.1(b)).

It follows that

$$K_*^M(F)/S \cong (\mathbf{0}[\mathbb{Z}/p])[(\mathbb{Z}/p)^m] \cong \mathbf{0}[(\mathbb{Z}/p)^{m+1}].$$

Conclude from Proposition 6.1(a) that S is totally rigid.

Now suppose that $u \in \mathrm{Val}(S,S)$. To derive a contradiction we let w be the finest common coarsening of v and u. We claim that $O_w^\times \leq S$.

Indeed, if u were strictly finer than v, then $O_{u/v} = O_u/\mathfrak{m}_v$ would be a proper valuation ring on $\bar{F}_v = \mathbb{F}_l$ (see §2), which is impossible. If u is coarser than v, then $w = u$, so $O_w^\times = O_u^\times \leq S$. Finally, if v and u are incomparable, then Lemma 2.2 gives $O_w^\times = (1+\mathfrak{m}_v)(1+\mathfrak{m}_u) \leq S$. This proves the claim.

Consequently, $\bar{F}_w^\times = \bar{S}_w$. Using the comments at the end of §2 we now get the contradiction

$$\mathbb{F}_l^\times = \bar{F}_v^\times = (\bar{F}_w^\times)_{v/w} = (\bar{S}_w)_{v/w} = \bar{S}_v = (\mathbb{F}_l^\times)^p.$$

Example 10.4. $S \cup \{0\}$ is a fan and $\mathrm{Val}(S, \langle -1, S\rangle) = \emptyset$. Let $p = 2$, let $\bar{F} = \mathbb{Q}(\sqrt{2})$, and let $\bar{S} = \bar{S}_1 \cap \bar{S}_2$, where $\bar{S}_1, \bar{S}_2$ are the distinct orderings on $\bar{F}$. By Proposition 5.2(b), $K_*^M(\bar{F})/\bar{S} \cong \kappa[\mathbb{Z}/2]$. Therefore

$$K_*^M(F)/S \cong (\kappa[\mathbb{Z}/2])[(\mathbb{Z}/2)^m] \cong \kappa[(\mathbb{Z}/2)^{m+1}].$$

By Proposition 6.1(b), S is totally rigid. Lemma 4.4 and [E5, Prop. 3.3] show again that S is a preordering, i.e., $S \cup \{0\}$ is a fan.

Now suppose that $u \in \mathrm{Val}(S, \langle -1, S\rangle)$. To get a contradiction, we break the discussion as before into three cases.

If v and u are incomparable, then their finest common coarsening w satisfies $-1 \in O_w^\times = (1+\mathfrak{m}_v)(1+\mathfrak{m}_u) \leq S$ (by Lemma 2.2), which is not the case.

If v is finer than u, then $O_v^\times \leq O_u^\times \leq \langle -1, S\rangle$. The residue map induces an isomorphism $SO_v^\times/S \cong \bar{F}_v^\times/\bar{S}_v = \bar{F}^\times/\bar{S}$. We get the obvious contradiction

$$4 = (\bar{F}^\times : \bar{S}) = (SO_v^\times : S) \bigm| (\langle -1, S\rangle : S) = 2.$$

Finally suppose that u is strictly finer than v. Then u/v is a non-trivial valuation on $\bar{F} = \bar{F}_v$. As $1+\mathfrak{m}_u \leq S$ we have $1+\mathfrak{m}_{u/v} \leq \bar{S}_v = \bar{S}_1 \cap \bar{S}_2$. Now any non-trivial valuation on the number field $\bar{F}$ is π-adic for some prime element π (see [Bo, Ch. VI, §1.4, Prop. 3]). Hence $(u/v)(l) > 0$ for the prime number l lying under π, so $1 - l \in 1+\mathfrak{m}_{u/v} \leq \bar{S}_1$, and we again obtained a contradiction.

References

[AEJ] J.K. Arason, R. Elman and B. Jacob, *Rigid elements, valuations, and realization of Witt rings*, J. Algebra **110** (1987), 449–467. MR0910395 (89a:11041)

[BaT] H. Bass and J. Tate, *The Milnor ring of a global field*, In: Algebraic K-Theory, Part II (H. Bass, ed.), (Battelle Institute Conference 1972; Springer Lect. Notes Math. 342 , 1973, pp. 349–446. MR0442061 (56:449)

[BK] E. Becker and E. Köpping, *Reduzierte quadratische Formen und Semiordnungen reeller Körper*, Abh. Math. Sem. Univ. Hamburg **46** (1977), 143–177. MR0506028 (58:21934)

[Bo] N. Bourbaki, *Commutative Algebra, Chapters 1–7*, Springer, Berlin, 1989. MR0979760 (90a:13001)

[E1] I. Efrat, *Construction of valuations from K-theory*, Math. Res. Letters **6** (1999), 335–344. MR1713134 (2001i:12011)

[E2] I. Efrat, *The local correspondence over absolute fields – an algebraic approach*, Inter. Math. Res. Notices **2000:23** (2000), 1213–1223. MR1809368 (2002g:12003)

[E3] I. Efrat, *Demuškin fields with valuations*, Math. Z. **243** (2003), 333–353. MR1961869 (2004d:11116)

[E4] I. Efrat, *A generalization of Marshall's equivalence relation*, Trans. Amer. Math. Soc. **358** (2006), 2561–2577. MR2204044

[E5] I. Efrat, *Quotients of Milnor K-rings, orderings, and valuations*, Pac. J. Math. **226** (2006), 259–276. MR2247864

[EF] I. Efrat and I. Fesenko, *Fields Galois-equivalent to a local field of positive characteristic*, Math. Res. Letters **6** (1999), 345–356. MR1713135 (2001i:12004)

[En] O. Endler, *Valuation Theory*, Springer, Berlin, 1972. MR0357379 (50:9847)

[H] H. Hahn, *Über die nichtarchimedischen Grössensysteme*, Sitz.-Ber. d. Wiener Akad., Math.-Nat. Klasse, Abt. IIa **116** (1907), 601–653.

[HJ] Y.S. Hwang and B. Jacob, *Brauer group analogues of results relating the Witt ring to valuations and Galois theory*, Canad. J. Math. **47** (1995), 527–543. MR1346152 (97a:12004)

[J] B. Jacob, *On the structure of pythagorean fields*, J. Algebra **68** (1981), 247–267. MR0608534 (82g:12020)

[JWd] B. Jacob and A. Wadsworth, *A new construction of noncrossed product algebras*, Trans. Amer. Math. Soc. **293** (1986), 693–722. MR0816320 (87g:16027)

[Ka] B. Kahn, *La conjecture de Milnor (d'aprés V. Voevodsky)*, Sém. Bourbaki 1996/97, Astérisque **245** (1997), 379–418. MR1627119 (2000a:19002)

[Ko] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterisation of p-adic fields)*, J. reine angew. Math. **465** **1995**, 165–182. MR1344135 (96m:12003)

[Kr] W. Krull, *Allgemeine Bewertungstheorie*, J. reine angew. Math. **167** (1931), 160–196.

[L] T.Y. Lam, *Orderings, valuations and quadratic forms*; Conf. Board of the Mathematical Sciences **52** , AMS, 1983. MR0714331 (85e:11024)

[Lg] S. Lang, *Algebra*, Addison–Wesley Publishing Company, Reading, Massachusetts, 1984. MR0783636 (86j:00003)

[MS] A.S. Merkurjev and A.A. Suslin, *K-cohomology of Brauer-Severi varieties and the norm residue homomorphism*, Math. USSR Izv. **21** (1983), 307–340; English translation (Russian).

[Mi] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. math. **9** (1970), 318–344. MR0260844 (41:5465)

[NSW] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of Number Fields*, Springer, Berlin–Heidelberg, 2000. MR1737196 (2000j:11168)

[R] P. Ribenboim, *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal, 1968. MR0249425 (40:2670)

[S] K. Szymiczek, *Quadratic forms over fields*, Dissertationes Math. (Rosprawy Mat.) **152** (1977). MR0450199 (56:8495)

[Wd] A.R. Wadsworth, *p-henselian fields: K-theory, Galois cohomology, and graded Witt rings*, Pac. J. Math. **105** (1983), 473–496. MR0691616 (84m:12026)

[Wr1] R. Ware, *Valuation rings and rigid elements in fields*, Canad. J. Math. **33** (1981), 1338–1355. MR0645230 (83i:10028)

[Wr2] R. Ware, *Galois groups of maximal p-extensions*, Trans. Amer. Math. Soc. **333** (1992), 721–728. MR1061780 (92m:12008)

Department of Mathematics, Ben-Gurion University of the Negev, P.O. Box 653, Be'er-Sheva 84105, Israel

E-mail address: `efrat@math.bgu.ac.il`