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CORRECTION FOR: "BUMPY METRICS AND CLOSED PARAMETRIZED MINIMAL SURFACES IN RIEMANNIAN MANIFOLDS"

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Our purpose here is to make two corrections to the proof of the Main Theorem of [3]. The second of these corrections works only under the restriction that the dimension of the ambient manifold be at least four. Thus this dimension must be imposed on the Main Theorem:

Main Theorem. *Suppose that M is a compact connected smooth manifold of dimension at least four. Then*

(1) *For a generic choice of Riemannian metric on M, every prime minimal two-sphere $f : S^2 \to M$ is free of branch points and lies on a nondegenerate critical submanifold of dimension six which is an orbit for the G-action, where $G = PSL(2, \mathbb{C})$.*
(2) *For a generic choice of Riemannian metric on M, every prime minimal two-torus $f : T^2 \to M$ is free of branch points and lies on a nondegenerate critical submanifold of dimension two which is an orbit for the G-action, where $G = S^1 \times S^1$.*
(3) *For a generic choice of Riemannian metric on M, every prime oriented minimal surface $f : \Sigma \to M$ of genus at least two is free of branch points and is Morse nondegenerate in the usual sense.*

We first correct the discussion of tangential Jacobi fields which follows Proposition 5.2 of [3]; the following discussion should replace the derivation from (54) to (58) in that article.

It follows from Proposition 5.2 that the second variation of energy

$$E : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \longrightarrow \mathbb{R},$$

is given by the formula

$$(1) \quad d^2E(f,\omega)((X,\dot\eta),(X,\dot\eta)) = d^2E_\omega(f)(X,X) - 2\int_\Sigma \sum_{a,b} \frac{\dot\eta_{ab}}{\lambda^2}\left\langle \frac{DX}{\partial x_a}, \frac{\partial f}{\partial x_b}\right\rangle dx_1dx_2 + \int_\Sigma \frac{1}{\lambda^4}(\dot\eta_{11}^2 + \dot\eta_{12}^2)\sigma^2 dx_1dx_2,$$

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where X is tangent to $\text{Map}(\Sigma, M)$, $\dot{\eta}$ is a variation in the metric on Σ which projects to a variation in Teichmüller space $\mathcal{T}$, and, in terms of the complex parameter $z = x_1 + ix_2$ on Σ,

$$\sigma^2 = \left|\frac{\partial f}{\partial x_1}\right|^2 = \left|\frac{\partial f}{\partial x_2}\right|^2 = 2\left|\frac{\partial f}{\partial z}\right|^2. \tag{2}$$

To explain the notion of tangential Jacobi field, we need to complexify the second variation formula (1). It is convenient to describe the complexification of the tangent space to the space of metrics on Σ in terms of canonical bundles. If $\text{Met}(\Sigma)$ denotes the space of Riemannian metrics on a closed oriented surface Σ, and η is an element of $\text{Met}(\Sigma)$,

$$T_\eta(\text{Met}(\Sigma)) \otimes \mathbb{C} = \mathbf{K}^{(2,0)} \oplus \mathbf{K}^{(1,1)} \oplus \mathbf{K}^{(0,2)}, \tag{3}$$

where $\mathbf{K}$ is the canonical bundle over Σ, $\bar{\mathbf{K}}$ is its conjugate and

$$\mathbf{K}^{(p,q)} = \mathbf{K}^p \otimes \bar{\mathbf{K}}^q.$$

Of course, the holomorphic tangent bundle of Σ is isomorphic to $\bar{\mathbf{K}}$. Covariant differentiations yield operators

$$D' : \Gamma(\mathbf{K}^{(p,q)}) \to \Gamma(\mathbf{K}^{(p+1,q)}), \quad D'' : \Gamma(\mathbf{K}^{(p,q)}) \to \Gamma(\mathbf{K}^{(p,q+1)}).$$

Note that sections of the summand $\mathbf{K}^{(2,0)}$ are quadratic differentials on Σ.

The real inner product on $T\Sigma$ extends to a complex bilinear inner product $\langle \cdot, \cdot \rangle$ on the sum of all $\mathbf{K}^{(p,q)}$'s, which pairs $\mathbf{K}^{(p,q)}$ with $\mathbf{K}^{(q,p)}$ or to a Hermitian inner product with respect to which all the $\mathbf{K}^{(p,q)}$'s are orthogonal. The real structure determines a conjugation $C : \mathbf{K}^{(p,q)} \to \mathbf{K}^{(q,p)}$.

In the context of Fréchet manifolds, the Lie algebra of the group $\text{Diff}(\Sigma)$ of diffeomorphisms of Σ is just the space of vector fields on Σ. Complex-valued vector fields are sections of $\mathbf{K} \oplus \bar{\mathbf{K}}$, and as described in [1], the action of $\Gamma(\mathbf{K} \oplus \bar{\mathbf{K}})$ on $T_\eta(\text{Met}(\Sigma))$ is given by

$$D' + D'' : \Gamma(\mathbf{K}^{(1,0)}) \to \Gamma(\mathbf{K}^{(2,0)}) \oplus \Gamma(\mathbf{K}^{(1,1)}),$$
$$D' + D'' : \Gamma(\mathbf{K}^{(0,1)}) \to \Gamma(\mathbf{K}^{(1,1)}) \oplus \Gamma(\mathbf{K}^{(0,2)}).$$

Thus for example, if one restricts attention to the holomorphic tangent bundle $\bar{\mathbf{K}}$, the key piece of this action is

$$D'' : Z \mapsto \frac{DZ}{\partial \bar{z}} d\bar{z},$$

The other component $D'(Z)$ simply affects the scale factor in the metric represented by $\mathbf{K}^{(1,1)}$. The cokernel of $D' : \Gamma(\mathbf{K}^{(1,0)}) \to \Gamma(\mathbf{K}^{(2,0)})$ can be thought of as the cotangent space to Teichmüller space consisting of holomorphic quadratic differentials, which is isomorphic to the tangent space consisting of extremal Beltrami differentials.

To analyze the second term in the variation formula (1), we write

$$\sum \frac{DX}{\partial x_a} dx_a = \frac{DX}{\partial z} dz + \frac{DX}{\partial \bar{z}} d\bar{z} \tag{4}$$

and if $h = (1/2)(\dot{\eta}_{11} - \dot{\eta}_{22}) - i\dot{\eta}_{12}$,

$$\sum \dot{\eta}_{ab} \frac{\partial f}{\partial x_b} dx_a = h \frac{\partial f}{\partial \bar{z}} dz + \bar{h} \frac{\partial f}{\partial z} d\bar{z}. \tag{5}$$

Taking the inner product of (4) with (5) allows us to write the second term of (1) as

$$-4\int_\Sigma \left\langle \frac{DX}{\partial z}, \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 - 4\int_\Sigma \left\langle \frac{DX}{\partial \bar{z}}, \frac{h}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right\rangle dx_1dx_2, \tag{6}$$

where h transforms under change of complex parameter in such a way that

$$hdz^2 \quad \leftrightarrow \quad \frac{2\bar{h}}{\lambda^2}\frac{d\bar{z}}{dz} \tag{7}$$

are quadratic or Beltrami differentials, sections of $\mathbf{K}^{(2,0)}$ or $\mathbf{K}^{(-1,1)}$ respectively. (In taking the inner product, we use the fact that $\langle dx_a, dx_b\rangle = (1/\lambda^2)\delta_{ab}$.) Similarly, $\bar{h}d\bar{z}^2$ is a section of $\mathbf{K}^{(0,2)}$. The third term in (1) is

$$\int_\Sigma \frac{1}{\lambda^4}(|\dot{\eta}_{11}|^2 + |\dot{\eta}_{12}|^2)\sigma^2 dx_1dx_2 = \int_\Sigma \left(\left| \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z}\right|^2 + \left| \frac{h}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right|^2 \right) dx_1dx_2.$$

If Z is a section of $f^*TM \otimes \mathbb{C}$, then

$$\begin{aligned} d^2E(f,\omega)((Z,\dot{\eta}),(\bar{Z},\dot{\eta})) &= d^2E_\omega(f)(Z,\bar{Z}) \\ &- 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial \bar{z}}, \frac{h}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right\rangle dx_1dx_2 - 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 \\ &+ \int_\Sigma \left(\left| \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z}\right|^2 + \left| \frac{h}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right|^2 \right) dx_1dx_2. \end{aligned} \tag{8}$$

When we complexify the tangent space to $\mathrm{Met}(\Sigma)$, we allow the sections of $\mathbf{K}^{(2,0)}$ and $\mathbf{K}^{(0,2)}$ to be different, and thus we specify two quadratic differentials hdz^2 and $\bar{k}d\bar{z}^2$. Then for

$$\left(Z, hdz^2, \bar{k}d\bar{z}^2\right) \in T_f\mathrm{Map}(\Sigma, M) \oplus \Gamma(\mathbf{K}^{(2,0)}) \oplus \Gamma(\mathbf{K}^{(0,2)}),$$

which has conjugate $\left(\bar{Z}, kdz^2, \bar{h}d\bar{z}^2\right)$, since conjugation interchanges $\mathbf{K}^{(2,0)}$ and $\mathbf{K}^{(0,2)}$, the second variation formula becomes

$$\begin{aligned} d^2E(f,\omega)\left(\left(Z, hdz^2, \bar{k}d\bar{z}^2\right), \left(\bar{Z}, kdz^2, \bar{h}d\bar{z}^2\right)\right) &= d^2E_\omega(f)(Z,\bar{Z}) \\ &- 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial \bar{z}}, \frac{h}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right\rangle dx_1dx_2 - 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{k}}{\lambda^2}\frac{\partial f}{\partial z}\right\rangle dx_1dx_2 \\ &+ \int_\Sigma \left(\left| \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z}\right|^2 + \left| \frac{k}{\lambda^2}\frac{\partial f}{\partial \bar{z}}\right|^2 \right) dx_1dx_2. \end{aligned} \tag{9}$$

According to (16) from [3],

$$d^2E_\omega(f)(Z,\bar{Z}) = 4\int_\Sigma \left[\left| \frac{DZ}{\partial \bar{z}}\right|^2 - \left\langle R\left(Z, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z}\right\rangle \right] dx_1dx_2.$$

Substituting into the previous formula yields

$$
\begin{aligned}
(10)\quad & d^2E(f,\omega)\left(\left(Z, hdz^2, \bar{k}d\bar{z}^2\right), \left(\bar{Z}, kdz^2, \bar{h}d\bar{z}^2\right)\right) \\
&= \int_\Sigma \left[\left| 2\frac{DZ}{\partial \bar{z}} - \frac{\bar{h}}{\lambda^2}\frac{\partial f}{\partial z} \right|^2 - \left\langle R\left(Z, \frac{\partial f}{\partial z}\right)\frac{\partial f}{\partial \bar{z}}, \bar{Z} \right\rangle \right] dx_1 dx_2 \\
&\qquad - 4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{k}}{\lambda^2}\frac{\partial f}{\partial z} \right\rangle dx_1 dx_2 + \int_\Sigma \left| \frac{k}{\lambda^2}\frac{\partial f}{\partial \bar{z}} \right|^2 dx_1 dx_2 .
\end{aligned}
$$

Although the formula is somewhat complicated, note how naturally the action of the diffeomorphism group on $\mathrm{Met}(\Sigma)$ is incorporated into the first term.

Suppose that (Z, hdz^2) is a pair, where Z is a section of the line bundle $\mathbf{L}$ described in [3] and hdz^2 is a holomorphic quadratic differential, which satisfy the equation

$$
(11)\qquad \frac{DZ}{\partial \bar{z}} = \frac{\bar{h}}{2\lambda^2}\frac{\partial f}{\partial z}.
$$

We can then set $k = 0$ and obtain a Jacobi field $(Z, hdz^2, 0)$ for (10). To see this, we need to check that

$$
d^2E(f,\omega)\left(\left(Z, hdz^2, 0\right), \left(\bar{Z}_1, k_1dz^2, \bar{h}_1d\bar{z}^2\right)\right) = 0
$$

for all $(\bar{Z}_1, k_1dz^2, \bar{h}_1d\bar{z}^2)$. Note first that the curvature term vanishes when Z is a section of $\mathbf{L}$, no matter what Z_1 is. Thus the equation (11) implies that the only terms that could be nonzero are those which appear in

$$
d^2E(f,\omega)\left(\left(Z, hdz^2, 0\right), \left(0, k_1dz^2, 0\right)\right),
$$

and it is apparent from (11) that the only term in this expression that does not immediately vanish is

$$
-4\mathrm{Re}\int_\Sigma \left\langle \frac{DZ}{\partial z}, \frac{\bar{k}_1}{\lambda^2}\frac{\partial f}{\partial z} \right\rangle dx_1 dx_2 .
$$

This term vanishes because $\mathbf{L} \oplus \bar{\mathbf{L}}$ is a parallel decomposition of $(f^*TM)^\top \otimes \mathbb{C}$, the tangential part of $(f^*TM) \otimes \mathbb{C}$, and

$$
\left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right\rangle = 0 \quad \text{or equivalently,} \quad \langle \mathbf{L}, \mathbf{L} \rangle = 0.
$$

We call the pair (Z, hdz^2) a *tangential Jacobi field*.

Note that (10) is invariant under the conjugate linear conjugation operation C. Thus when $\left(Z, hdz^2, 0\right)$ is a Jacobi field, so is $\left(\bar{Z}, 0, \bar{h}d\bar{z}^2\right)$, as well as the real part

$$
\left(\frac{1}{2}(Z + \bar{Z}), \frac{1}{2}hdz^2, \frac{1}{2}\bar{h}d\bar{z}^2\right),
$$

and the imaginary part.

Equation (11) is the same as equation (58) from [3], except that h has been replaced by $h/2$. Indeed, one can check that there is a missing factor of $1/2$ on the right-hand side of equation (67), and hence on the right-hand side of (68) and the equation between them. There are missing factors of 2 on the right-hand side of (43) and (59). The rest of the article seems to be unaffected, except $\bar{h}$ should be replaced by $\bar{h}/2$ in equations corresponding to (11).

The second correction involves the proof of Lemma 7.1. Part III of this proof is incorrect, and hence the Lemma is proven only when f has only finitely many points of self-intersection. But this suffices when the dimension of M is at least four, because we can then show directly that for generic choice of metric on M, the self-intersection set of the restriction of any conformal harmonic map $f : \Sigma \to M$ to $\Sigma - \{\text{branch points}\}$ consists of only transverse double points. There are most likely many ways of proving this. We present one which is in the spirit of §10 of [3] below.

If $p \in \Sigma$, we let $D_\epsilon(p)$ denote the open ball of radius ϵ about p in Σ. Suppose that we are given a branch type Λ, as described in §9 of [3], a collection $\{p_1, \dots, p_k\}$ of points, and an $\epsilon > 0$. For any such choice, we can then consider the space

$$\mathcal{U} = \mathrm{Map}_{\Lambda,\epsilon.\{p_1,\dots,p_k\}}(\Sigma, M)$$

of maps $f : \Sigma \to M$ which have branch type Λ such that the branch locus of f lies within

$$D_\epsilon(p_1) \cup \cdots \cup D_\epsilon(p_k).$$

It suffices to show that for generic choice of Riemannian metric on M, any prime harmonic conformal map $(f, \omega) \in \mathcal{U} \times \mathcal{T}$ has the property that the restriction of f to

$$\Sigma_0 = \Sigma - (D_\epsilon(p_1) \cup \cdots \cup D_\epsilon(p_1)) \tag{12}$$

has transverse double points as its only points of self-intersection.

For a given branch type Λ, let

$$\mathcal{P}_\Lambda \subset \mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M)$$

denote the space of (f, ω, g) where (f, ω) is a prime harmonic conformal map with respect to g. Using the notation of §9, we note that

$$\mathcal{P}_\Lambda \subset \mathcal{B}_\Lambda(\Sigma, M) = \mathrm{Map}'_\Lambda(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M),$$

and over $\mathcal{B}_\Lambda(\Sigma, M)$ we have a bundle of smoothly varying family of line bundles $\mathbf{L}$ with a corresponding smoothly family of spaces of potential tangential Jacobi fields $\mathcal{J}(\mathbf{L})$. We can extend $\mathcal{J}(\mathbf{L})$ over a neighborhood $\mathcal{V}$ of $\mathcal{B}_\Lambda(\Sigma, M)$ in

$$\mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M),$$

as described in §10. If Σ has genus zero or one, we break the symmetry given by complex automorphisms of Σ by replacing $\mathcal{P}_\Lambda$ and $\mathcal{B}_\Lambda(\Sigma, M)$ by submanifolds of codimension six and two respectively, just as in §8 of [3].

We claim that $\mathcal{P}_\Lambda$ is contained in a submanifold

$$\mathcal{R}_\Lambda \subset \mathcal{V} \subset \mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M)$$

which has Fredholm projection to $\mathrm{Met}(\Sigma)$ of Fredholm index zero. Indeed, this can be shown by modifying the proof of Lemma 10.1, replacing the space $\mathcal{N}(\mathbf{L})$ by $\mathcal{J}(\mathbf{L})$. (In the modified argument we would not need Lemma 7.1.)

In more detail, we let $\mathcal{E} \to \mathcal{V}$ denote the subbundle of the tangent bundle to $\mathrm{Map}(\Sigma, M) \times \mathcal{T}$ whose fiber at $(f, \dot\omega, g)$ is

$$\{(X, \dot\omega) \in T_{(f,\dot\omega)}(\mathrm{Map}(\Sigma, M) \times \mathcal{T}) : \; X \text{ is } \perp \text{ to } \mathrm{Re}\mathcal{J}(\mathbf{L}) \; \},$$

and let p denote the bundle projection from $\mathrm{Map}(\Sigma, M) \times \mathcal{T}$ to $\mathcal{E}$. If

$$F : \mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M) \longrightarrow T(\mathrm{Map}(\Sigma, M) \times \mathcal{T})$$

is the Euler-Lagrange map, we set $\mathcal{R}_\Lambda = (p \circ F)^{-1}(\mathcal{Z})$, where $\mathcal{Z}$ is the zero section of $\mathcal{E}$. As in [3], we can say that up to a finite-dimensional error elements of $\mathcal{R}_\Lambda$

are conformal harmonic maps with branch type Λ, and in particular conformal harmonic maps with branch type Λ are contained in $\mathcal{R}_\Lambda$. By the arguments presented in [3], one sees that $p \circ F$ is transversal to the zero section and hence $\mathcal{R}_\Lambda$ is indeed a submanifold as claimed.

To see that the projection $\pi : \mathcal{R}_\Lambda \to \mathrm{Met}(M)$ has Fredholm index zero, we need to calculate the tangent space and find that if $(f, \omega, h) \in \mathcal{B}_\Lambda(\Sigma, M)$ the tangent space to $\mathcal{R}_\Lambda$ at this point is

$$\{(X, \dot{\omega}, h) \in T_f \mathrm{Map}(\Sigma, M) \oplus T_\omega \mathcal{T} \oplus T_g \mathrm{Met}(M) : \\ L'(X, \dot{\omega}) + p \circ \pi_V \circ (D_2 F)_{(f,\omega,g)}(h) = 0\},$$

where $L' = \pi_V \circ D_1(p \circ F)|\mathcal{E}$ is the restricted Jacobi operator, a Fredholm operator, which just like L, has Fredholm index zero. It is now straightforward to check that the Fredholm projection to $\mathrm{Met}(M)$ is Fredholm with Fredholm index zero by a modification of the argument for Proposition 3.2 of [3].

As in the proof of Lemma 6.1 of [3], we can show that at each point of $\mathcal{P}_\Lambda$, the projection onto the first factor

$$\pi_0 : \mathcal{R}_\Lambda \to \mathrm{Map}(\Sigma, M)$$

has a differential which projects onto a complement to the family of spaces $\mathrm{Re}\mathcal{J}(\mathbf{L})$. By continuity, the same holds for points in a neighborhood $\mathcal{V}'$ of $\mathcal{P}_\Lambda$ in $\mathcal{V}$. Henceforth we let $\mathcal{R}_\Lambda$ denote the smaller submanifold $\mathcal{R}_\Lambda \cap \mathcal{V}'$.

We construct a countable open cover of $\mathrm{Map}(\Sigma, M) \times \mathcal{T} \times \mathrm{Met}(M)$ by product open balls $U_i \times V_i$,

$$U_i \subset \mathrm{Map}(\Sigma, M) \times \mathcal{T}, \qquad V_i \subset \mathrm{Met}(M),$$

chosen so that if $U_i \times V_i$ intersects $\mathcal{R}_\Lambda$,

(1) it is the domain for a submanifold chart for $\mathcal{R}_\Lambda$ and
(2) the projection $\mathcal{R}_\Lambda \cap (U_i \times V_i) \to V_i$ is proper.

The last fact can be arranged by Theorem 1.6 of [4].

It follows from standard transversality theory for finite-dimensional manifolds (see §2 of Chapter 3 of [2]) that if Σ_0 is defined by (12),

$$\mathcal{T}C = \{f \in \mathrm{Map}(\Sigma_0, M) : \ f \text{ has transversal crossings }\}$$

is an open dense subset of $\mathrm{Map}(\Sigma_0, M)$. Since the projection π_0 to $\mathrm{Map}(\Sigma_0, M)$ is a submersion, $\pi_0^{-1}(\mathcal{T}C) \cap (U_i \times V_i)$ intersects $\mathcal{R}_\Lambda \cap (U_i \times V_i)$ in an open dense subset. The projection $\mathcal{R}_\Lambda \cap (U_i \times V_i) \to V_i$ is a proper Fredholm map of Fredholm index zero, and its noncritical values form a residual subset V_i' of V_i by the Sard-Smale theorem. If g lying in a residual subset V_i'' of V_i, the maps in $\pi_2^{-1}(g) \cap \mathcal{R}_\Lambda$ (where π_2 is the projection to V_i) have transversal crossings. Note that $W_i = V_i'' \cup (\mathrm{Met}(M) - \overline{V}_i)$ is residual. Metrics g which lie in the intersections of the W_i's, a residual subset of $\mathrm{Met}(M)$, have the property that if $\pi_2^{-1}(g) \cap \mathcal{P}_\Lambda$ then the restriction of f to

$$\Sigma_0 = \Sigma - (D_\epsilon(p_1) \cup \cdots \cup D_\epsilon(p_1))$$

has transverse double points as its only points of self-intersection.

Since the choice of Λ, of $\epsilon > 0$ and of points in Σ was arbitrary, we conclude that for generic choice of metric on M, the self-intersection set of the restriction of any conformal harmonic map $f : \Sigma \to M$ to $\Sigma - \{\text{branch points}\}$ consists of only transverse double points, which is what we needed.

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