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ANALYTICITY ON TRANSLATES OF A JORDAN CURVE

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Abstract. Let Ω be a domain in $\mathbb{C}$ which is symmetric with respect to the real axis and whose boundary is a real analytic simple closed curve. Translate $\overline{\Omega}$ vertically to get $K = \bigcup\{\overline{\Omega} + it,\ -r \le t \le r\}$ where $r > 0$ is such that $(\overline{\Omega} - ir) \cap (\overline{\Omega} + ir) = \emptyset$. We prove that if f is a continuous function on K such that for each t, $-r \le t \le r$, the function $f|(b\Omega + it)$ has a continuous extension to $\overline{\Omega} + it$ which is holomorphic on $\Omega + it$, then f is holomorphic on $\mathrm{Int}K$.

1. Introduction

Write $\Delta(a,r) = \{\zeta \in \mathbb{C}\colon\ |\zeta - a| < r\}$ and let $\Delta = \Delta(0,1)$. Translate $b\Delta$ vertically to get the strip $S = b\Delta + i\mathbb{R} = \{\zeta \in \mathbb{C} :\ -1 \le \Re\zeta \le 1\}$. Let f be a continuous function on S such that for each $t \in \mathbb{R}$ the function $f|(b\Delta + it)$ has a continuous extension to $\overline{\Delta} + it$ which is holomorphic on $\Delta + it$. Must f be holomorphic on $\mathrm{Int}S$ [5]?

A positive answer was obtained for real analytic functions by M. Agranovsky and the author [2] and independently by L. Ehrenpreis [3] and for general continuous functions by A. Tumanov [9].

To answer the question above one passes in both [2] and [9] to an associated problem in $\mathbb{C}^2$. In [2] the authors use semi-quadrics

$$\Lambda_{a,\rho} = \{(z,w)\colon\ (z-a)(w-\overline{a}) = \rho^2,\ 0 < |z-a| < \rho\}$$

which are attached to $\Sigma = \{(z,\overline{z})\colon\ z \in \mathbb{C}\}$ along the circles $\{(z,\overline{z})\colon\ z \in b\Delta(a,\rho)\}$. They use the property of $\Lambda_{a,\rho}$ that a continuous function on $b\Delta(a,\rho)$ extends holomorphically through $\Delta(a,\rho)$ if and only if the function F defined on $b\Lambda_{a,\rho} = \{(z,\overline{z})\colon\ z \in b\Delta(a,\rho)\}$ by $F(z,\overline{z}) = f(z)$ $(z \in b\Delta(a,\rho))$ has a bounded continuous extension to $\Lambda_{a,\rho} \cup b\Lambda_{a,\rho}$ which is holomorphic on $\Lambda_{a,\rho}$. Thus, when studying holomorphic extensions of a function f from circles in the plane one defines $F(z,\overline{z}) = f(z)$ in a region in Σ and then studies bounded holomorphic extensions of F from $b\Lambda_{a,\rho}$ through $\Lambda_{a,\rho}$ [2], [6].

Tumanov [9] passes to a problem in $\mathbb{C}^2$ by adding an extra variable to make the translates of the disc pairwise disjoint and then, on the union of these discs, the (smooth) manifold

$$N_1 = \{(\zeta + it, \zeta)\colon\ \zeta \in \overline{\Delta},\ t \in \mathbb{R}\},$$

he defines a continuous function F by letting, for each $t \in \mathbb{R}$, the function $\zeta \mapsto F(\zeta + it, \zeta)$ $(\zeta \in \overline{\Delta})$ be the holomorphic extension of $\zeta \mapsto f(\zeta + it)$ $(\zeta \in b\Delta)$

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through Δ. He then observes that the symmetry

$$F(\zeta+it,\zeta)=F(\zeta+it,1/\zeta)\quad(\zeta\in b\Delta, t\in\mathbb{R}), \tag{1.1}$$

which follows from the fact that $F(\zeta+it,\zeta)=f(\zeta+it)$ $(\zeta\in b\Delta,\ t\in\mathbb{R})$, makes it possible to extend F continuously to a new geometric object, the (smooth) manifold

$$N_2=\{(\zeta+it,1/\zeta)\colon\ \zeta\in\overline{\Delta}\setminus\{0\},\ t\in\mathbb{R}\},$$

by using the equality (1.1) for $\zeta\in\overline{\Delta}\setminus\{0\},\ t\in\mathbb{R}$ as a definition. Thus one gets a continuous CR function F on $N_1\cup N_2$, the union of two manifolds with common boundary

$$N_1\cap N_2=bN_1=bN_2=\{(\zeta+it,\zeta)\colon\zeta\in b\Delta,\ t\in\mathbb{R}\}.$$

Tumanov then uses methods of CR theory to show that F does not depend on the second variable, which means that f is holomorphic on $\mathrm{Int}S$. He also discovers that this is actually a finite strip problem.

Very recently Tumanov [10] studied a similar problem for a family of circles with centers sliding along a smooth curve and with smoothly changing radii. He used semi-quadrics. He obtained the result by using a classical argument of H. Lewy about holomorphic extensions of CR functions. In particular, he found a very simple proof of the theorem on the strip.

In the present paper we generalize the result of Tumanov [9] from vertical translates of circles to vertical translates of real-analytic simple closed curves which are symmetric with respect to the real axis.

2. The main results

Our first result is about analyticity on vertical translates of curves which are symmetric with respect to the real axis.

Theorem 2.1. *Let Ω be a domain in $\mathbb{C}$ which is symmetric with respect to the real axis and whose boundary is a real analytic simple closed curve. Let $r>0$ and let f be a continuous function on $K=\bigcup\{b\Omega+it,\ -r\le t\le r\}$ such that for each $t,\ -r\le t\le r$, the function $f|(b\Omega+it)$ has a continuous extension to $\overline{\Omega}+it$ which is holomorphic on $\Omega+it$. Suppose that $(\overline{\Omega}-ir)\cap(\overline{\Omega}+ir)=\emptyset$. Then f is holomorphic on* $\mathrm{Int}K$.

Note that our assumptions imply that $K=\bigcup\{\overline{\Omega}+it,\ -r\le t\le r\}$.

We will deduce Theorem 2.1 from a more general result below which involves vertical translates of general domains and their images under conjugation.

Let D be a domain in $\mathbb{C}$ bounded by a real-analytic simple closed curve. Let S be the vertical strip defined by $S=\bigcup\{bD+it,\ t\in\mathbb{R}\}=\{\zeta\in\mathbb{C}\colon\ \alpha\le\Re\zeta\le\beta\}$. Write $D^*=\{\overline{\zeta}\colon\ \zeta\in D\}$. Obviously, $\bigcup\{bD^*+is,\ s\in\mathbb{R}\}=S$.

Theorem 2.2. *Let $\lambda\colon[\alpha,\beta]\to\mathbb{R}$ be a continuous function and let a,b,c,d be real numbers such that $\overline{D}+ia$, $\overline{D^*}+ic$ are both contained in $\{t+is\colon\ s<\lambda(t),\ \alpha\le t\le\beta\}$ and such that $\overline{D}+ib$, $\overline{D^*}+id$ are both contained in $\{t+is\colon\ s>\lambda(t),\ \alpha\le t\le\beta\}$. Let*

$$Q_1=\bigcup\{bD+it\colon\ a\le t\le b\},\ Q_2=\bigcup\{bD^*+is\colon\ c\le s\le d\}$$

and let f be a continuous function on $Q_1 \cup Q_2$ such that

$$\left.\begin{array}{c}\textit{for each } t, a \le t \le b, \textit{ the function } f|(bD+it) \textit{has a continuous} \\ \textit{extension to } \overline{D}+it \textit{ which is holomorphic on} D+it,\end{array}\right\} \tag{2.1}$$

$$\left.\begin{array}{c}\textit{for each } s, \quad c \le s \le d, \textit{ the function } f|(bD^*+is) \textit{ has a continuous} \\ \textit{extension to } \overline{D^*}+is \textit{ which is holomorphic on } D^*+is.\end{array}\right\} \tag{2.2}$$

Then the function f is holomorphic on $\mathrm{Int}Q_1 \cup \mathrm{Int}Q_2$.

Our assumptions about a, b, c, d mean that $\overline{D}+ia$, $\overline{D^*}+ic$ both lie below the curve $\ell = \{t+i\lambda(t)\colon\ \alpha \le t \le \beta\}$ and $\overline{D}+ib$, $\overline{D^*}+id$ both lie above the curve ℓ. Note that this implies that $Q_1 = \bigcup\{\overline{D}+it\colon\ a \le t \le b\}$ and $Q_2 = \bigcup\{\overline{D^*}+is\colon\ c \le s \le d\}$. Theorem 2.1 follows from Theorem 2.2 by putting $\Omega = D = D^*$, $a = c = -r$, $b = d = r$ and $\lambda \equiv 0$, that is, $\ell = [\alpha, \beta]$.

3. From circles to general curves

In this section we describe the idea for passing from circles to general curves. Let Ω be a domain bounded by a real-analytic simple closed curve which is symmetric with respect to the real axis. With no loss of generality assume that Ω contains the origin. Let f be a continuous function on $\bigcup\{b\Omega + it\colon\ t \in \mathbb{R}\}$ such that for each $t \in \mathbb{R}$, the function $\zeta \mapsto f(\zeta + it)$ $(\zeta \in b\Omega)$ has a continuous extension to $\overline{\Omega}$ which is holomorphic on Ω.

Semi-quadrics are related to circles, so they cannot be used to study the analyticity of functions on a family of translates of a given curve that is not a circle. We look again at the way in which Tumanov [9] adds the extra variable in the case of the circles. An important point in his setting is that on $b\Delta$ the conjugation $z \mapsto \overline{z}$ extends to the map $z \mapsto 1/z$ which carries $\Delta \setminus \{0\}$ biholomorphically onto $\mathbb{C} \setminus \overline{\Delta}$. This is not the case for general curves, so for domains Ω more general than a disc it seems difficult to work with the manifold $\{(\zeta + it, \zeta)\colon\ \zeta \in \overline{\Omega}, t \in \mathbb{R}\}$ used in [9] when Ω is a disc. However, since the reflection (1.1) takes place only in the second variable the idea is to replace the manifold $\{(\zeta + it, \zeta)\colon\ \zeta \in \overline{\Omega}, t \in \mathbb{R}\}$ with a manifold that is attached to the cylinder $\{(z, w)\colon\ |w| = 1\}$. To do this we take the conformal map $\Phi\colon \Omega \mapsto \Delta$ that satisfies $\Phi(\overline{\zeta}) = \overline{\Phi(\zeta)}$ $(\zeta \in \Omega)$, $\Phi(0) = 0$, notice that Φ extends to a diffeomorphism Φ from $\overline{\Omega}$ to $\overline{\Delta}$ and define the smooth manifold N_1 by

$$N_1 = \{(\zeta + it, \Phi(\zeta))\colon\ \zeta \in \overline{\Omega},\ t \in \mathbb{R}\}.$$

We define a continuous function F on N_1 by letting, for each $t \in \mathbb{R}$, the function $\zeta \mapsto F(\zeta + it, \Phi(\zeta))$ $(\zeta \in \overline{\Omega})$ be the holomorphic extension of $\zeta \mapsto f(\zeta + it)$ $(\zeta \in b\Omega)$ through Ω.

If $\zeta \in b\Omega$, then $\overline{\zeta} \in b\Omega$, and if $t \in \mathbb{R}$, then $\zeta + it = \overline{\zeta} + is$, where $s = t + (\zeta - \overline{\zeta})/i \in \mathbb{R}$, so $F(\zeta + it, \Phi(\zeta)) = f(\zeta + it) = f(\overline{\zeta} + is) = F(\overline{\zeta} + is, \Phi(\overline{\zeta})) = F(\overline{\zeta} + is, \overline{\Phi(\zeta)}) = F(\overline{\zeta} + is, 1/\Phi(\zeta)) = F(\zeta + it, 1/\Phi(\zeta))$, so

$$F(\zeta + it, \Phi(\zeta)) = F(\zeta + it, 1/\Phi(\zeta)) \quad (\zeta \in b\Omega,\ t \in \mathbb{R}), \tag{3.1}$$

which makes it possible to extend F continuously to a new geometric object, the smooth manifold

$$N_2 = \{(\zeta + it, 1/\Phi(\zeta))\colon\ \zeta \in \overline{\Omega} \setminus \{0\},\ t \in \mathbb{R}\},$$

by using the equality (3.1) for $\zeta \in \overline{\Omega} \setminus \{0\}$ as a definition. Thus we get a continuous CR function F on $N_1 \cup N_2$, the union of two manifolds with the common boundary

$$N_1 \cap N_2 = bN_1 = bN_2 = \{(\zeta + it, \Phi(\zeta)) \colon \ \zeta \in b\Omega, \ t \in \mathbb{R}\}.$$

We then show that the classical argument of H. Lewy, which Tumanov used with semiquadrics, works also in the present situation. This helps us to prove that F depends only on the first variable, which implies that f is holomorphic.

In fact, our main result, Theorem 2.2, is somewhat more general than the one just described. Its proof, although technically a bit complicated, uses essentially the idea above.

4. The manifold N

We now begin with the proof of Theorem 2.2. With no loss of generality we assume that $0 \in D$ and that the imaginary axis intersects bD transversely.

Let $\Phi \colon \ D \to \Delta$ be a conformal map such that $\Phi(0) = 0$. Since bD is real-analytic the map Φ extends to a biholomorphic map from a neighbourhood of $\overline{D}$ to a neighbourhood of $\overline{\Delta}$. Define $\Psi \colon \ D^* \to \Delta$ by

$$\Psi(\zeta) = \overline{\Phi(\overline{\zeta})} \ \ (\zeta \in D^*).$$

The map Ψ maps D^* conformally onto Δ and extends to a biholomorphic map from a neighbourhood of $\overline{D^*}$ to a neighbourhood of $\overline{\Delta}$.

Define

$$N_1 = \{(\xi + it, \Phi(\xi)) \colon \ \xi \in \overline{D}, \ t \in \mathbb{R}\},$$
$$N_2 = \{(\zeta + is, 1/\Psi(\zeta)) \colon \ \zeta \in \overline{D^*} \setminus \{0\}, \ s \in \mathbb{R}\}$$

and set $N = N_1 \cup N_2$. Write

$$\Phi^{-1}(w) = p(w) + iq(w) \ \ (w \in \overline{\Delta})$$

where p and q are real functions. Then $N_1 = \{(p(w)+iq(w)+it, w) \colon \ w \in \overline{\Delta}, \ t \in \mathbb{R}\} = \{(p(w) + it, w) \colon \ w \in \overline{\Delta}, \ t \in \mathbb{R}\} = \{(p(w), w) \colon \ w \in \overline{\Delta}\} + \mathbb{R}(i, 0)$. If $w = 1/\Psi(\zeta)$, then $\overline{\Phi(\overline{\zeta})} = \Psi(\zeta) = 1/w$, so $\zeta = \overline{\Phi^{-1}(1/\overline{w})} = p(1/\overline{w}) - iq(1/\overline{w})$, which implies that $N_2 = \{(p(1/\overline{w}) - iq(1/\overline{w}) + is, w) \colon \ w \in \mathbb{C} \setminus \Delta, \ s \in \mathbb{R}\} = \{(p(1/\overline{w}) + is, w) \colon \ w \in \mathbb{C} \setminus \Delta, \ s \in \mathbb{R}\} = \{(p(1/\overline{w}), w) \colon \ w \in \mathbb{C} \setminus \Delta\} + \mathbb{R}(i, 0)$. Define

$$\theta(w) = \begin{cases} p(w) & (w \in \overline{\Delta}), \\ p(1/\overline{w}) & (w \in \mathbb{C} \setminus \Delta). \end{cases}$$

The function θ is well defined and continuous on $\mathbb{C}$ since $w = 1/\overline{w}$ $(w \in b\Delta)$. The function θ is invariant with respect to $w \to 1/\overline{w}$, the reflection across $b\Delta$. Note that $\theta|\overline{\Delta}$ is smooth as it extends to a harmonic function in a neighbourhood of $\overline{\Delta}$. Similarly, $\theta|(\mathbb{C} \setminus \Delta)$ is smooth. We have

$$N = \{(\theta(w) + it, w) \colon \ w \in \mathbb{C}, t \in \mathbb{R}\}, \tag{4.1}$$

which shows that we obtain N by taking the graph $\{(\theta(w), w) \colon \ w \in \mathbb{C}\}$ of θ in $\mathbb{R} \times \mathbb{C} = \mathbb{R} \times \{0\} \times \mathbb{C}$ and then forming the union of all translates of this graph in the extra perpendicular direction $(i, 0)$, that is,

$$N = \{(\theta(w), w) \colon \ w \in \mathbb{C}\} + \mathbb{R}(i, 0).$$

Since

$$N_1 = \{(\theta(w), w)\colon\ w \in \overline{\Delta}\} + \mathbb{R}(i, 0),$$
$$N_2 = \{(\theta(w), w)\colon\ w \in \mathbb{C} \setminus \Delta\} + \mathbb{R}(i, 0), \tag{4.2}$$

we see that N is the union of manifolds N_1 and N_2 with boundary which meet along the common boundary

$$N_1 \cap N_2 = bN_1 = bN_2 = \{(\theta(w), w)\colon\ w \in b\Delta\} + \mathbb{R}(i, 0).$$

The complement of the graph of θ in $\mathbb{R} \times \mathbb{C}$ has two components: $\{(t, w)\colon\ t > \theta(w),\ w \in \mathbb{C}\}$ and $\{(t, w)\colon\ t < \theta(w),\ w \in \mathbb{C}\}$, which, by (4.1), implies that $\mathbb{C}^2 \setminus N$ has two components:

$$P_1 = \{(t + is, w)\colon\ t > \theta(w),\ s \in \mathbb{R}, w \in \mathbb{C}\},$$
$$P_2 = \{(t + is, w)\colon\ t < \theta(w),\ s \in \mathbb{R}, w \in \mathbb{C}\}.$$

5. Intersecting N with complex lines

We will apply the reasoning of H. Lewy concerning holomorphic extensions of CR functions. To this end, we look first at the intersections of N with complex lines $L(z) = \{(z, w)\colon\ w \in \mathbb{C}\}$. We shall use the fact that since bD is real-analytic and compact there are at most finitely many points $\zeta \in bD$ such that the tangent line to bD at ζ is parallel to the imaginary axis.

For $z \in S$ write

$$\tilde{E}(z) = N \cap L(z), \quad \tilde{E}_j(z) = N_j \cap L(z), \quad j = 1, 2$$

and

$$\Lambda(z) = \{w \in \mathbb{C}\colon\ (z, w) \in N\}, \quad \Lambda_j(z) = \{w \in \mathbb{C}\colon\ (z, w) \in N_j\},\ j = 1, 2,$$

so that

$$\tilde{E}(z) = \{z\} \times \Lambda(z), \quad \tilde{E}_j(z) = \{z\} \times \Lambda_j(z),\ j = 1, 2.$$

For each $t \in \mathbb{R}$ we have $N + t(i, 0) = N,\ N_j + t(i, 0) = N_j,\ j = 1, 2$, so it follows that

$$\Lambda(z) = \Lambda(\Re z), \quad \Lambda_j(z) = \Lambda_j(\Re z) \quad (j = 1, 2,\ z \in S).$$

Thus, it is enough to study $\Lambda(\tau),\ \Lambda_j(\tau),\ j = 1, 2$, where $\alpha \le \tau \le \beta$.

The set $\tilde{E}(\tau) = \{\tau\} \times \Lambda(\tau)$, contained in $\mathbb{R}^3 = \mathbb{R} \times \{0\} \times \mathbb{C}$, is the intersection of $\{(\theta(w), w)\colon\ w \in \mathbb{C}\}$, the graph of θ, with the two-plane (in fact, the complex line), $\{\tau\} \times \mathbb{C}$. Since θ is invariant with respect to the reflection across $b\Delta$ it follows that we get $\Lambda_2(\tau)$ by reflecting $\Lambda_1(\tau) \subset \overline{\Delta}$ across $b\Delta$. So it is enough to study $\Lambda_1(\tau)$. Clearly

$$\Lambda_1(\tau) = \Phi(K(\tau)) \ \text{ where } \ K(\tau) = (\tau + i\mathbb{R}) \cap \overline{D}.$$

If $\tau \in [\alpha, \beta]$ is such that $\tau + i\mathbb{R}$ meets bD transversely, as happens for all but finitely many τ, then $\Lambda_1(\tau)$ consists of finitely many pairwise disjoint closed arcs with endpoints on $b\Delta$ but otherwise contained in Δ which meet $b\Delta$ transversely. By transversality and by the fact that Φ extends across bD as a conformal map, these arcs change smoothly with τ as long as $\tau + i\mathbb{R}$ meets bD transversely. If $\tau \in (\alpha, \beta)$ is such that $\tau + i\mathbb{R}$ does not meet bD transversely, then $\Lambda(\tau)$ consists of a finite number of arcs with endpoints on $b\Delta$ and pairwise disjoint interiors plus a possibly finite set on $b\Delta$. There may be points on $b\Delta$ which are common endpoints of two (but not more than two) of these arcs. Since we get $\Lambda_2(\tau)$ by reflecting $\Lambda_1(\tau)$ across $b\Delta$ it follows that if $\tau \neq 0$ and if $\tau + i\mathbb{R}$ is transverse to bD, then $\Lambda(\tau)$ consists of

finitely many pairwise disjoint simple closed curves, symmetric with respect to $b\Delta$. If $\tau \neq 0$ and $\tau + i\mathbb{R}$ does not meet bD transversely, then $\Lambda(\tau)$ consists of finitely many pairwise disjoint simple closed curves, symmetric with respect to $b\Delta$ plus a possibly finite subset of $b\Delta$. There may be points on $b\Delta$ that are common points of two, but not more than two of these curves. Except for these points, the curves are pairwise disjoint. Clearly $\Lambda(\alpha)$ and $\Lambda(\beta)$ are finite sets.

Since $i\mathbb{R}$ meets bD transversely, $\Lambda_1(0)$ consists of finitely many pairwise disjoint closed arcs with endpoints on $b\Delta$ but otherwise contained in Δ which meet $b\Delta$ transversely. One of these arcs passes through the origin so its image under the reflection across $b\Delta$ passes through infinity. Thus, $\Lambda(0) \cup \{\infty\}$ consists of finitely many pairwise disjoint simple closed curves on the Riemann sphere, one of which contains infinity.

For each $z,\ \alpha < \Re z < 0$ the set $Y(z) = P_1 \cap L(z)$ is a bounded open subset of $L(z)$ whose boundary $E(z) = bY(z)$ is the part of $\tilde{E}(z) = \{z\} \times \Lambda(z)$ consisting of curves (recall that in addition to these curves, $\tilde{E}(z)$ may contain an additional finite set contained in $\{z\} \times b\Delta$). The complex line $L(z)$ has a natural orientation. We orient $E(z)$ as the boundary of $Y(z)$ in $L(z)$. Similarly, for $0 < \Re z < \beta$ we orient $E(z)$ as the boundary of $Y(z) = P_2 \cap L(z)$ in $L(z)$. This determines the orientation of $\Lambda(z),\ \alpha < \Re z < \beta,\ \Re z \neq 0$, or more precisely, the part of $\Lambda(z)$ consisting of curves, and the orientation of $K(\tau),\ \alpha < \tau < \beta, \tau \neq 0$, upwards if $\alpha < \tau < 0$ and downwards if $0 < \tau < \beta$.

If $0 < \tau < \beta$ and a point (τ, w) is above the graph of θ, that is, contained in P_1, then $[\tau, \beta] \times \{w\}$ is contained in P_1. A consequence of this is

Proposition 5.1. *Suppose that $0 < \tau < \beta$ and let $q_0 \in L(\tau + i\lambda(\tau)) \cap P_1$. Then there is a path $t \mapsto q(t)$ $(\tau \leq t \leq \beta)$ such that $q(\tau) = q_0$ and $q(t) \in L(t + i\lambda(t)) \cap P_1$ $(\tau \leq t \leq \beta)$.*

Proof. We have $q_0 = (\tau + i\lambda(\tau), w)$ where $\tau > \theta(w)$. Define $q(t) = (t + i\lambda(t), w)$ $(\tau \leq t \leq \beta)$. It is easy to see that q has all the required properties. □

A similar proposition holds for $\alpha < \tau < 0$ and for P_2 in the place of P_1.

6. Continuity of an integral

We shall need

Proposition 6.1. *Let $z_0 \in S$ and suppose that G is a continuous function on a neighbourhood of $\tilde{E}(z_0)$ in N. Then the function*

$$\Theta(z) = \int_{\Lambda(z)} G(z, w) dw,$$

defined in a neighbourhood of z_0 in S, is continuous at z_0.

Proof. We prove the continuity of

$$\Theta_1(z) = \int_{\Lambda_1(z)} G(z, w) dw = \int_{K(\Re z)} G(z, \Phi(\zeta)) \Phi'(\zeta) d\zeta.$$

The proof for $\Theta_2(z) = \int_{\Lambda_2(z)} G(z, w) dw$ will be analogous; note that $\Theta = \Theta_1 + \Theta_2$ since $\Lambda_1(z)$ and $\Lambda_2(z)$ meet in a finite set. Recall that Φ and Φ' extend holomorphically into a neighbourhood of $\overline{D}$, so the continuity of Θ_1 depends on how $K(t) = \overline{D} \cap (t + i\mathbb{R})$ changes with t near $t_0 = \Re z_0$. Assume for a moment that

$\alpha < t_0 < \beta$. There are at most finitely many $t,\ \alpha \le t \le \beta$ such that $t + i\mathbb{R}$ does not intersect bD transversely. Thus there is an $\eta > 0$ such that $t + i\mathbb{R}$ intersects bD transversely for every $t,\ 0 < |t - t_0| < \eta$. In particular, for each $t,\ t_0 - \eta < t < \eta,\ K(t)$ is a finite collection of pairwise disjoint closed segments with endpoints varying continuously with t. Since $bD \cap (t_0 + i\mathbb{R})$ is a finite set and since bD is compact it follows that each of these endpoints has a limit as $t \nearrow t_0$. As $t \nearrow t_0$ some segments may degenerate into points in the limit $K(t_0 - 0)$ and some pairs of segments may get a common endpoint. Clearly $K(t_0 - 0) \subset K(t_0)$ and $K(t_0) \setminus K(t_0 - 0) \subset bD \cap (t_0 + i\mathbb{R})$. Since the set $bD \cap (t_0 + i\mathbb{R})$ is finite it follows that $K(t_0 - 0) \subset K(t_0)$ is a finite set. Thus, $\lim_{z \to z_0, \Re z \le \Re z_0} \Theta(z) = \Theta(z_0)$. Similarly we show that $\lim_{z \to z_0, \Re z \ge \Re z_0} \Theta(z) = \Theta(z_0)$ which proves that Θ is continuous at z_0. The same (one-sided) reasoning applies if $z_0 \in bS$. The proof is complete. $\square$

7. The manifold M and the function F

We now define a submanifold of N that is more closely related to our problem. Write

$$M_1 = \{(\xi + it, \Phi(\xi)) \colon\ \xi \in \overline{D},\ a \le t \le b\},$$
$$M_2 = \{(\zeta + is,\ 1/\Psi(\zeta)) \colon\ \zeta \in \overline{D^*} \setminus \{0\},\ c \le s \le d\}$$

and let $M = M_1 \cup M_2$. Note that M_1 is a smooth manifold with boundary $bM_1 = \{(\xi + it, \Phi(\xi)) \colon\ \xi \in bD,\ a \le t \le b\} \cup \{(\xi + ia, \Phi(\xi)) \colon\ \xi \in \overline{D}\} \cup \{(\xi + ib, \Phi(\xi)) \colon\ \xi \in \overline{D}\}$. It is a submanifold of N_1. Similarly, M_2 is a smooth manifold with boundary $bM_2 = \{(\zeta + is, 1/\Psi(\zeta)) \colon\ \zeta \in bD^*,\ c \le s \le d\} \cup \{(\zeta + ic, 1/\Psi(\zeta)) \colon\ \zeta \in \overline{D^*} \setminus \{0\}\} \cup \{(\zeta + id, 1/\Psi(\zeta)) \colon\ \zeta \in \overline{D^*} \setminus \{0\}\}$.

Suppose that f is a continuous function on $Q_1 \cup Q_2$ which satisfies (2.1) and (2.2). For each $t,\ a \le t \le b$, let g_t be a continuous extension of $\xi \mapsto f(\xi + it)$ $(\xi \in bD)$ to $\overline{D}$ which is holomorphic on D and for each $s,\ c \le s \le d,$ let h_s be the continuous extension of $\zeta \mapsto f(\zeta + is)$ $(\zeta \in bD^*)$ to $\overline{D^*}$ which is holomorphic on D^*. Define the function G on M_1 by

$$G(\xi + it, \Phi(\xi)) = g_t(\xi) \quad (\xi \in \overline{D},\ a \le t \le b)$$

and the function H on M_2 by

$$H(\zeta + is, 1/\Psi(\zeta)) = h_s(\zeta) \quad (\zeta \in \overline{D^*} \setminus \{0\},\ c \le s \le d).$$

In particular, on the part of bM_1 contained in bN_1 we have

$$G(\xi + it, \Phi(\xi)) = f(\xi + it) \quad (\xi \in bD,\ a \le t \le b),$$

and on the part of bM_2 contained in bN_2 we have

$$H(\zeta + is, 1/\Psi(\zeta)) = f(\zeta + is) \quad (\zeta \in bD^*,\ c \le s \le d).$$

Suppose that $(z, w) \in M_1 \cap M_2$. Then there are $\xi \in bD,\ \zeta \in bD^*$ and $t, s,\ a \le t \le b,\ c \le s \le d$, such that $(\xi + it, \Phi(\xi)) = (z, w) = (\zeta + is, 1/\Psi(\zeta)) = (\zeta + is, \overline{\Psi(\zeta)}) = (\zeta + is, \Phi(\overline{\zeta}))$, which implies that $\zeta = \overline{\xi}$ and $\xi + it = \overline{\xi} + is$. Thus, $G(z, w) = G(\xi + it, \Phi(\xi)) = f(\xi + it) = f(\overline{\xi} + is) = H(\overline{\xi} + is, 1/\Psi(\overline{\xi})) = H(\zeta + is, 1/\Psi(\zeta)) = H(z, w)$. It follows that

$$F(z, w) = \begin{cases} G(z, w) & ((z, w) \in M_1) \\ H(z, w) & ((z, w) \in M_2) \end{cases}$$

is a well defined continuous function on $M_1 \cup M_2$ which is holomorphic on each holomorphic leaf of M_1 and on each holomorphic leaf of M_2.

Our aim is to show that F depends only on z which will imply that f is holomorphic on $\mathrm{Int}Q_1 \cup \mathrm{Int}Q_2$.

8. Integrals of CR functions on M

Denote by π_1 the projection $\pi_1(z,w) = z$. With no loss of generality assume that $\lambda(0) = 0$. Our assumptions imply that there is an $\eta > 0$ such that if

$$\Sigma = \{t + is\colon\ \lambda(t) - \eta < s < \lambda(t) + \eta,\ \alpha < t < \beta\},$$

then $L(z) \cap M = L(z) \cap N$ for all $z \in \Sigma$, that is, $\pi_1^{-1}(\Sigma) \cap M = \pi_1^{-1}(\Sigma) \cap N$. Put $\Sigma_1 = \{z \in \Sigma,\ \Re z < 0\}$, $\Sigma_2 = \{z \in \Sigma,\ \Re z > 0\}$. Recall that for $z \in \Sigma_1$ the set $E(z)$ is the boundary of $Y(z) = P_1 \cap L(z)$ in $L(z)$ and for $z \in \Sigma_2$ the set $E(z)$ is the boundary of $Y(z) = P_2 \cap L(z)$ in $L(z)$. Let

$$A_j = \bigcup\{Y(z)\colon\ z \in \Sigma_j\} = \pi_1(\Sigma_j) \cap P_j \quad (j = 1, 2).$$

For each $j = 1, 2$, A_j is an open subset of $\Sigma_j \times \mathbb{C}$ whose relative boundary is $N \cap (\Sigma_j \times \mathbb{C})$. Using Proposition 5.1 we see that the complement of $\overline{A}_j$ in $\Sigma_j \times \mathbb{C}$ is connected, $j = 1, 2$.

We shall prove that the function F extends holomorphically into A_1 and into A_2. We begin to follow the reasoning of H. Lewy. In [7] this was done for smooth functions on smooth manifolds and for more general, including continuous, functions on smooth manifolds this was done in [8]. We cannot refer to these results directly since in our case the manifold is not smooth but consists of two smooth pieces. However, these two pieces are both foliated by analytic discs which simplifies the situation. We provide the details to make the proof self-contained.

Let $\Delta(u,r)$ be contained in either Σ_1 or Σ_2 and assume that Θ is a continuous function on $\pi_1^{-1}(\Delta(u,r)) \cap N$ which is holomorphic on each holomorphic leaf. The function

$$z \mapsto Q(z) = \int_{\Lambda(z)} \Theta(z,w)dw$$

is, as we know, well defined and by Proposition 6.1 it is continuous on $\Delta(u,r)$. Recall that there are at most finitely many real values τ such that $\tau + i\mathbb{R}$ is not transverse to bD. So, if we want to prove that Q is holomorphic on $\Delta(u,r)$ it is enough to prove that Q is holomorphic in a neighbourhood of each $z_0 \in \Delta(u,r)$ such that $z_0 + i\mathbb{R}$ intersects bD transversely. Let z_0 be such a point. Let $\rho > 0$ be such that for each $z \in \overline{\Delta(z_0,\rho)}$, $z + i\mathbb{R}$ meets bD transversely. Passing to a smaller ρ if necessary we may assume that there is a $\gamma > 0$ such that whenever U is a closed disc contained in $\Delta(z_0,\rho)$ of radius not exceeding γ, the circle $bU + it$, $t \in \mathbb{R}$, either misses bD or meets bD in one point or in two points. Let U be such a disc. By transversality, there is a positive integer ν such that for each $z \in U$ the set $L(z) \cap N = L(z) \cap M$ consists of ν pairwise disjoint simple closed curves, each being the union of two smooth arcs with endpoints on $\{z\} \times b\Delta$, one having its interior contained in $\{z\} \times \Delta$, and the other having its interior contained in $\{z\} \times (\mathbb{C} \setminus \overline{\Delta})$, which change smoothly with z. So $\pi_1^{-1}(U) \cap N = \pi_1^{-1}(U) \cap M = \bigcup\{L(z) \cap N\colon\ z \in U\}$ is an open subset of N which has ν components whose closures are pairwise disjoint; the boundary of this set is $\pi_1^{-1}(bU) \cap N = \bigcup\{L(z) \cap N\colon z \in bU\}$. Let Ω be one of these components. Write $T = bN_1 = bN_2$. The set Ω consists of three

pairwise disjoint parts: the domains $\Omega_1 = \Omega \cap \mathrm{Int} N_1$, $\Omega_2 = \Omega \cap \mathrm{Int} N_2$ and the two dimensional surface $\Omega \cap T$. For each $z \in bU$, $L(z) \cap b\Omega$ is a simple closed curve, so $b\Omega$ is a torus and Ω is a solid torus in N.
We now want to show that

$$\int_{b\Omega} \Theta(z,w) dz \wedge dw = 0. \tag{8.1}$$

Note first that $\Omega \cap T$ is the common part of $b\Omega_1$ and $b\Omega_2$, so to prove (8.1) it is enough to prove that

$$\int_{b\Omega_1} \Theta(z,w) dz \wedge dw = 0 \tag{8.2}$$

and

$$\int_{b\Omega_2} \Theta(z,w) dz \wedge dw = 0. \tag{8.3}$$

Consider (8.2). The properties of U imply that Ω_1 can be written as the union of a continuous family of pairwise disjoint analytic discs

$$A_t = \{(\zeta + it, \Phi(\zeta)) \colon \ \zeta \in D, \zeta + it \in U\} = \{(\zeta, \Phi(\zeta)) + it \colon \ \zeta \in D \cap (-it + U)\},$$

and $b\Omega_1$ is the union of their pairwise disjoint boundaries

$$bA_t = \{(\zeta, \Phi(\zeta)) + it \colon \zeta \in b(D \cap (-it + U))\}.$$

These analytic discs, if nonempty, are of two sorts: either their boundaries are smooth simple closed curves which meet $\Omega \cap T$ in at most one point (which happens if $U \subset D + it$), or their boundaries are simple closed curves consisting of two arcs, one contained in $\mathrm{Int} N_1$ and the other contained in T (which happens if U meets $D + it$ but is not contained in $D + it$). Recall that $N_1 = \{(\Upsilon(w) + it, w) \colon \ w \in \overline{\Delta}, \ t \in \mathbb{R}\}$, where the conformal map $\Upsilon = \Phi^{-1}$ extends to a biholomorphic map from $R\Delta$ for some $R > 1$ to a neighbourhood of $\overline{D}$. Define the function ρ by $\rho(z,w) = (1/i)(z - \Upsilon(w))$ and notice that ρ is real on N_1. Then $\Theta(z,w) dz \wedge dw = d\rho \wedge \mu$, where $\mu = -(1/i)(\Theta(z,w)/\Upsilon'(w)) dz$ which, by using the Fubini theorem on each of the three smooth pieces of $b\Omega_1$ and adding the results, implies that

$$\int_{b\Omega_1} \Theta(z,w) dz \wedge dw = \int_I \Big[\int_{bA_t} \mu \Big] dt, \tag{8.4}$$

where $I \subset \mathbb{R}$ is the segment of all t such that $\Omega_1 \cap \{(\zeta + it, \Phi(\zeta)) \colon \ \zeta \in D\}$ is not empty. For each $t \in I$ we have

$$\int_{bA_t} \mu = -\frac{1}{i} \int_{bA_t} \Theta(z,w) \frac{1}{\Upsilon'(w)} dz,$$

where the integral on the right vanishes by the Cauchy theorem since the function $(z,w) \mapsto \Theta(z,w)/\Upsilon'(w)$ is continuous on $\overline{A}_t$ and holomorphic on A_t. This proves that the integral on the left in (8.4) vanishes. We repeat the reasoning for N_2 to get (8.3). This proves (8.1). Thus,

$$\int_{bU} \Big[\int_{b\Lambda(z)} \Theta(z,w) dw \Big] dz = 0$$

for every disc $U \subset \Delta(z_0, \rho)$, which, by the Morera theorem, implies that the function Q is holomorphic on $D(z_0, \rho)$. This proves that Q is holomorphic on $\Sigma_1 \cup \Sigma_2$.

9. Holomorphic extensions of F and the completion of the proof

We continue to follow the reasoning of H. Lewy. For each $z \in \Sigma_1 \cup \Sigma_2$ and each $w \in \mathbb{C} \setminus \Lambda(z)$ (that is, for each w such that $(z,w) \notin N$) define

$$\Xi(z,w) = \frac{1}{2\pi i} \int_{\Lambda(z)} \frac{F(z,\zeta)}{\zeta - w} d\zeta.$$

For a fixed z the function $w \mapsto \Xi(z,w)$ is holomorphic on $\mathbb{C} \setminus \Lambda(z)$. Now, fix $z_0 \in \Sigma_1$ and $w \in \mathbb{C} \setminus \Lambda(z_0)$. We show that $z \mapsto \Xi(z,w)$ is holomorphic in a neighbourhood of z_0. There is a $\rho > 0$ such that $\Delta(z_0,\rho) \subset \Sigma_1$ and $(z,w) \notin N$ $(z \in \Delta(z_0,\rho))$, so the function $(z,\zeta) \mapsto F(z,\zeta)/(\zeta - w)$ is continuous on $\bigcup\{L(z) \cap N\colon\ z \in \Delta(z_0,\rho)\}$ and holomorphic on each holomorphic leaf. By the reasoning in Section 8 it follows that $z \mapsto \Xi(z,w)$ is holomorphic on $\Delta(z_0,\rho)$. This shows that Ξ is holomorphic on $\pi_1^{-1}(\Sigma_1) \setminus N$. Fix a large w. We know that $z \mapsto \Xi(z,w)$ is continuous on $\Sigma_1 \cup [\alpha + i(-\delta,\delta)]$ and holomorphic on Σ. Since $\Lambda(\alpha)$ is a finite set it follows that $\Xi(z,w)$ approaches 0 as z approaches a point of $\alpha + i(-\delta,\delta)$. It follows that $\Xi(z,w) \equiv 0$ $(z \in \Sigma_1)$. Since this holds for all sufficiently large w, the connectedness of $(\Sigma_1 \times \mathbb{C}) \setminus \overline{A_1}$ implies that $\Xi \equiv 0$ on $(\Sigma_1 \times \mathbb{C}) \setminus \overline{A_1}$, so

$$\frac{1}{2\pi i} \int_{\Lambda(z)} \frac{F(z,\zeta)}{\zeta - w} d\zeta \equiv 0 \quad (z \in \Sigma_1,\ w \in \mathbb{C} \setminus \overline{Y(z)}).$$

It follows, in particular, that for all $z \in \Sigma_1$ the function $w \mapsto F(z,w)$, defined on $E(z) = bY(z)$, has a continuous extension into $Y(z) \cup bY(z)$ which is holomorphic on $Y(z)$. The same reasoning shows this for all $z \in \Sigma_2$.

Recall that $\Lambda(0) \cup \{\infty\}$ consists of finitely many pairwise disjoint simple closed curves on the Riemann sphere, one of which passes through infinity. By transversality, $\Lambda(t)$ changes continuously with t near 0 and contains ∞ only if $t = 0$. As $t \nearrow 0$ the open sets $Y(t)$ and their oriented boundaries converge to a domain Y^- and its oriented boundary bY^- which, as a set, coincides with $\Lambda(0)$. As $t \searrow 0$ the open sets $Y(t)$ and their oriented boundaries $\Lambda(t)$ converge to a domain Y^+ and its oriented boundary bY^+ which, as a set, coincides with $\Lambda(0)$. We have $Y^- \cup \Lambda(0) \cup Y^+ = \mathbb{C}$. Since the function F is bounded and continuous on M it follows that as $t \nearrow 0$, the holomorphic extensions of $F|M \cap L(t + i\lambda(t))$ converge to the holomorphic extension of $F|M \cap L(0)$, a bounded continuous function on $\{0\} \times \overline{Y^-}$, holomorphic on $\{0\} \times Y^-$. In particular, $F|M \cap L(0)$ extends to a bounded continuous function on $\{0\} \times \overline{Y^-}$ which is holomorphic on $\{0\} \times Y^-$. Similarly, $F|M \cap L(0)$ extends to a bounded continuous function on $\{0\} \times \overline{Y^+}$ which is holomorphic on $\{0\} \times \overline{Y^+}$. Consequently $F|M \cap L(0)$ extends to a bounded continuous function on $\{0\} \times \mathbb{C}$ which is holomorphic on $\{0\} \times [Y^+ \cup Y^-]$. This function extends holomorphically across $\{0\} \times \Lambda(0)$ to a bounded entire function on $L(0)$, which, by the Liouville theorem, must be constant. This implies that $F|M \cap L(0)$ is a constant. In the same way we show that $F|M \cap L(is)$ is constant for each s, $-\eta < s < \eta$. It follows that there is an $r > 0$ such that on $r\Delta$ the holomorphic extensions of all functions $f|(it + bD)$ to $it + D$, for all t such that $r\Delta \subset it + D$, coincide, and that on $r\Delta$ the holomorphic extensions of all functions $f|(it + bD^*)$ to $it + D^*$, for all t such that $r\Delta \subset it + D^*$, coincide. This implies first that f is holomorphic on $\bigcup\{it + bD\colon r\Delta \subset it + D,\ a < t < b\} \cap \mathrm{Int}S$, and then, by translating bD further along $i\mathbb{R}$, that f is holomorphic on $\bigcup\{it + bD\colon\ a < t < b\} \cap \mathrm{Int}S = \mathrm{Int}Q_1$. Similarly, we show that f is holomorphic on $\mathrm{Int}Q_2$. The proof is complete.

10. Remarks

It remains an open problem to prove Theorem 2.1 without the assumption that Ω is symmetric with respect to the real axis. It is known that one cannot drop the assumption that $(\overline{\Omega} - ir) \cap (\overline{\Omega} + ir) = \emptyset$. In fact the function

$$f(z) = \begin{cases} z^2/\overline{z} & (z \neq 0) \\ 0 & (z = 0) \end{cases}$$

is continuous on $\mathbb{C}$ and extends holomorphically from each circle which either surrounds the origin or contains the origin, yet f is not holomorphic.

After the present paper was submitted for publication the author received the preprint [1] from Mark Agranovsky in which he proves a similar result for real analytic functions for considerably more general families of curves.

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