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SURJECTIVITY FOR HAMILTONIAN G-SPACES IN K-THEORY

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ABSTRACT. Let G be a compact connected Lie group, and (M,ω) a Hamiltonian G-space with proper moment map μ. We give a surjectivity result which expresses the K-theory of the symplectic quotient $M/\!\!/G$ in terms of the equivariant K-theory of the original manifold M, under certain technical conditions on μ. This result is a natural K-theoretic analogue of the Kirwan surjectivity theorem in symplectic geometry. The main technical tool is the K-theoretic Atiyah-Bott lemma, which plays a fundamental role in the symplectic geometry of Hamiltonian G-spaces. We discuss this lemma in detail and highlight the differences between the K-theory and rational cohomology versions of this lemma.

We also introduce a K-theoretic version of equivariant formality and prove that when the fundamental group of G is torsion-free, every compact Hamiltonian G-space is equivariantly formal. Under these conditions, the forgetful map $K_G^*(M) \to K^*(M)$ is surjective, and thus every complex vector bundle admits a stable equivariant structure. Furthermore, by considering complex line bundles, we show that every integral cohomology class in $H^2(M;\mathbb{Z})$ admits an equivariant extension in $H_G^2(M;\mathbb{Z})$.

1. INTRODUCTION

In this article, we explore the K-theoretic analogues of several results of Kirwan from [24] involving Hamiltonian G-spaces, where G is a compact connected Lie group. Our main result is Theorem 3.1, which asserts that if M is a Hamiltonian G-space, then the G-equivariant K-theory of M surjects onto the ordinary K-theory of the symplectic quotient $M/\!\!/G$. In other words, the Kirwan map given by the composition

$$\kappa : K_G^*(M) \to K_G^*(\mu^{-1}(0)) \cong K^*(M/\!\!/G)$$

is a surjection, provided that the moment map μ is proper and 0 is a regular value. Kirwan's original theorem, which states that the analogous restriction map κ of rational cohomology rings is a surjection, is crucial for the explicit description of cohomology rings for many symplectic manifolds (see for example [24], [25], and [34, Chapter 8]).

Our primary technical tool is the K-theoretic version of the Atiyah-Bott lemma, which states that the equivariant Euler class of a G-bundle is not a zero-divisor, provided that a circle subgroup $S^1 \subset G$ fixes the base but acts non-trivially away from the zero section. If we use the norm square of the moment map to build our manifold out of equivariant strata, then the normal bundles satisfy this Atiyah-Bott

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condition, and we can then show that the norm square of the moment map is an equivariantly perfect Morse-Kirwan function. In other words, we can compute the equivariant K-theory of the whole space entirely in terms of the K-theory of the critical sets (note that the periodicity of complex K-theory eliminates the degree shifts that appear in the cohomology version).

A proof of the Atiyah-Bott lemma in the context of algebraic K-theory, for abelian actions, is given in [40]. We give a topological proof based on that of [40], extending this result to actions of compact connected Lie groups, as well as to bundles possessing only a Spin^c-structure instead of a complex structure. It turns out that the K-theoretic Atiyah-Bott lemma is stronger than its counterpart in rational cohomology; we do not need extra assumptions on the torsion of the space or of the group. We take some care to discuss these differences and indicate how we expect this K-theoretic Atiyah-Bott lemma, which is central to many results in the theory of Hamiltonian G-manifolds, to give new information in the context of symplectic geometry.

Kirwan originally proved surjectivity in the context of rational cohomology. Our K-theoretic adaptation of Kirwan's work has a slightly different flavor, with subtle differences arising from the torsion components. For example, we are no longer simply computing Betti numbers or Poincaré polynomials, and group extensions are not necessarily simply direct sums. If we were to eliminate the torsion in K-theory by tensoring with $\mathbb{Q}$, we recall from [7] that the Chern character gives an isomorphism between rational K-theory and rational cohomology. In this sense, our Theorem 3.1 can be viewed as an extension of Kirwan's results for rational cohomology. Indeed, we believe this K-theoretic extension to be more natural than a corresponding extension to integral cohomology. As we mentioned above, when working with integral cohomology, one needs to place additional constraints on the torsion to establish the Atiyah-Bott lemma and its consequences (see [39]), but such torsion constraints are not present in the K-theoretic version.

In Section 4, we discuss equivariant formality. While the usual notion of equivariant formality does not have a counterpart in equivariant K-theory, we define a slightly weaker version of equivariant formality and prove in Theorem 4.5 that it holds for all Hamiltonian G-spaces. Our primary tool here is the equivariant Künneth spectral sequence of Hodgkin, as refined by Snaith and McLeod (see [23], [38], and [32] respectively). As a consequence of equivariant formality we obtain a different sort of surjectivity, in this case for the forgetful homomorphism

$$K_G^*(M) \twoheadrightarrow K^*(M) \tag{1.1}$$

taking equivariant K-theory to non-equivariant K-theory. At the level of bundles, this implies that every complex vector bundle over a Hamiltonian G-space admits a stable equivariant structure. We go on to show in Theorem 4.7 that we can remove the word *stable* for the case of complex line bundles, establishing the surjectivity of the forgetful map

$$\mathrm{Pic}_G(M) \twoheadrightarrow \mathrm{Pic}(M) \tag{1.2}$$

for the Picard group of isomorphism classes of (equivariant) complex line bundles.

We translate our statements about line bundles and the Picard group into statements about integral cohomology in Appendix A. In particular, we show that the forgetful map

$$H_G^2(M;\mathbb{Z}) \twoheadrightarrow H^2(M;\mathbb{Z}) \tag{1.3}$$

is surjective. For this result, we use the classification of G-equivariant line bundles in terms of the equivariant cohomology $H^2_G(M;\mathbb{Z})$. This fact is non-trivial, and although it appears in various places throughout the literature, we offer a previously unpublished elementary proof due to Peter Landweber. At the core of this argument is the fact that the homomorphism $\alpha : K_G(M) \to K(M_G)$ retracts onto a map $\mathrm{Pic}_G(M) \to \mathrm{Pic}(M_G)$ of Picard groups. A technical lemma requiring only linear algebra then shows that this retraction extends to the completion of $K_G(M)$ at the augmentation ideal, and thus the map of Picard groups is in fact a retract of the Atiyah-Segal isomorphism $\widehat{\alpha} : K_G(M)^\wedge \to K(M_G)$ (see [9]). Identifying $\mathrm{Pic}(M_G)$ with $H^2_G(M;\mathbb{Z})$ gives the desired classification of G-line bundles.

We conclude with a discussion of Morse-Kirwan functions in Appendix B, showing that the results of Morse-Bott theory still hold in K-theory for such minimally degenerate functions, provided that we work in terms of strata rather than simply critical sets. Such an argument is not strictly necessary when working with the norm square of the moment map, as it is better behaved than a generic Morse-Kirwan function (see [28, 41]). However, we offer it for the sake of completeness, and in case the reader is interested in Morse-Kirwan functions other than the norm square of the moment map.

The research presented in this paper represents an ongoing project. Beyond the fundamental result of Kirwan surjectivity, another natural step is to perform computations of the kernel of the Kirwan map as in [16, 39] to explicitly compute the K-theory of symplectic quotients. This also involves expected K-theoretic analogues of results of Martin [29], in order to relate the K-theory of the symplectic quotient by a nonabelian Lie group G to that of the corresponding abelian symplectic quotient by a maximal torus $T \subseteq G$. Moreover, now that we have extended Kirwan's results to complex K-theory, it is natural to ask whether there are analogues for other generalized cohomology theories, such as real K-theory, cobordism, or elliptic cohomology. We believe that the Atiyah-Bott lemma holds for equivariant cobordism, as described in [37]. One can also ask whether there are K-theoretic extensions of similar cohomological results for other geometries possessing moment maps, such as hyper-Kähler, contact, Sasakian, or 3-Sasakian geometry.

2. The K-theoretic Atiyah-Bott lemma

The main tool used in our proof of the surjectivity theorem is Lemma 2.3 below, which is the K-theoretic analogue of the key fact behind many results in equivariant symplectic geometry. The original version of the lemma proved by Atiyah and Bott is [6, Proposition 13.4, p. 606], and is stated in terms of equivariant cohomology. An algebraic K-theory version of this lemma, for torus actions (or more precisely, for actions of diagonalizable group schemes of finite type on separated Noetherian regular algebraic spaces) can be found in [40, §4]. Our proof is modeled on this algebraic K-theory proof, but we work in the topological context, and we extend the result to general (non-abelian) compact connected Lie group actions on Spin^c-bundles. First, we require a simple technical lemma, which allows us to impose complex structures on the bundles involved.

Lemma 2.1. *Let $G = H \times S^1$ be a compact Lie group, and let $E \to X$ be a real G-vector bundle over a G-space X. Suppose that the action of the S^1-factor of G fixes precisely the zero section $X \subset E$. Then E admits a G-invariant complex structure (and is therefore orientable).*

Proof. A real vector bundle E can be viewed as a complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C}$ together with an involution corresponding to conjugation on the $\mathbb{C}$-factor. We recall that a complex $H \times S^1$-bundle over a space with trivial S^1-action can be decomposed into isotypic pieces, yielding an isomorphism of complex G-bundles

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{\lambda \in \mathbb{Z}} E_\lambda \otimes_{\mathbb{C}} \mathbb{C}_\lambda,$$

where the E_λ are complex H-bundles, and $\mathbb{C}_\lambda$ is the irreducible representation of S^1 on $\mathbb{C}$ with $z \in S^1$ acting as multiplication by z^λ. In our case, there is no trivial component with $\lambda = 0$ since S^1 fixes only the zero section. Furthermore, since $E \otimes_{\mathbb{R}} \mathbb{C}$ is self-conjugate, we have $E_{-\lambda} \cong \overline{E}_\lambda$ as complex bundles, and we will show that we can identify E with the complex bundle $\bigoplus_{\lambda>0} E_\lambda \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as real G-bundles.

The non-trivial irreducible real representations of S^1 are of the form $\mathbb{R}^2_\lambda$ for integers $\lambda > 0$, where $\mathbb{R}^2_\lambda \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}_\lambda \oplus \mathbb{C}_{-\lambda}$. Such a representation is called *complex*, as the commuting-field of complex S^1-equivariant endomorphisms of $\mathbb{R}^2_\lambda \otimes_{\mathbb{R}} \mathbb{C}$ is 2-dimensional and can be identified with $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \overline{\mathbb{C}}$. Continuing to work with the complexifications, we have the decomposition into isotypic components described in [9, §8] and [36, §2],

$$E \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{\lambda>0} \mathrm{Hom}_{\mathbb{C}}^{S^1}\left(\mathbb{R}^2_\lambda \otimes_{\mathbb{R}} \mathbb{C},\, E \otimes_{\mathbb{R}} \mathbb{C}\right) \otimes_{\mathrm{End}_{\mathbb{C}}^{S^1}(\mathbb{R}^2_\lambda \otimes_{\mathbb{R}} \mathbb{C})} \left(\mathbb{R}^2_\lambda \otimes_{\mathbb{R}} \mathbb{C}\right),$$

which respects the conjugation involution. Restricting to the real parts fixed under conjugation, we have the corresponding decomposition

$$E \cong \bigoplus_{\lambda>0} \mathrm{Hom}_{\mathbb{R}}^{S^1}\left(\mathbb{R}^2_\lambda, E\right) \otimes_{\mathrm{End}_{\mathbb{R}}^{S^1}(\mathbb{R}^2_\lambda)} \mathbb{R}^2_\lambda.$$

Finally, identifying $\mathbb{R}^2_\lambda \cong \mathbb{C}_\lambda$ for all integers $\lambda > 0$ gives us the identification $\mathrm{End}_{\mathbb{R}}^{S^1}(\mathbb{C}_\lambda) \cong \mathbb{C}$ for the commuting-field. We can therefore make the identification

$$E \cong \bigoplus_{\lambda>0} \mathrm{Hom}_{\mathbb{R}}^{S^1}(\mathbb{C}_\lambda, E) \otimes_{\mathbb{C}} \mathbb{C}_\lambda, \tag{2.1}$$

where the S^1-invariant complex structure on each component comes from the $\mathbb{C}_\lambda$-factors. The action of the group H on each component comes from the first factor, with H acting trivially on the $\mathbb{C}_\lambda$-factors. The complex structure is therefore H-invariant as well.

If E and X are smooth, we can construct this complex structure explicitly from the vector field ξ on E generating the S^1-action. Since the S^1-action maps each fiber onto itself, the infinitesimal action ξ is a vertical vector field. Furthermore, since S^1 acts linearly on the fibers, the vector field ξ is given by fiberwise multiplication by an endomorphism $I_x : E_x \to E_x$ for each point $x \in X$ in the base. On the isotypic component of E transforming like $\mathbb{R}^2_\lambda$, we can then choose our complex structure to be the rescaling $J = \frac{1}{\lambda} I$, which yields the identification $\mathbb{R}^2_\lambda \cong \mathbb{C}_\lambda$ used above. □

Remark 2.2. If in addition $E \to X$ is a smooth vector bundle, then the conditions of Lemma 2.1 allow us to identify E with the normal bundle for the H-equivariant inclusion of the S^1 fixed point set $X = E^{S^1} \hookrightarrow E$ via the zero section. It is well known that the normal bundle to a connected component of the fixed point set for an S^1-action admits a complex structure, essentially by the proof given above, and Lemma 2.1 simply generalizes this fact.

We are now ready to prove our K-theory analogue of the Atiyah-Bott lemma.

Lemma 2.3. *Let a compact connected Lie group G act fiberwise linearly on a* Spin^c*-vector bundle $\pi : E \to X$ over a compact G-manifold X. Assume that a circle subgroup $S^1 \subset G$ acts by restriction on E so that its fixed point set is precisely the zero section X. Choose an invariant metric on E and let D and S denote the disc and sphere bundles, respectively. Then the long exact sequence for the pair (D,S) in equivariant K-theory splits into short exact sequences*

$$0 \longrightarrow K_G^*(D,S) \longrightarrow K_G^*(D) \longrightarrow K_G^*(S) \longrightarrow 0.$$

Proof. We will show that the first homomorphism $K_G^*(D,S) \to K_G^*(D)$ is injective. By the equivariant Thom isomorphism (see [3]), we have for the domain $K_G^*(D,S) \cong K_G^*(X)$, and since the disc bundle D retracts equivariantly onto the zero section X, we likewise have for the codomain $K_G^*(D) \cong K_G^*(X)$. In terms of these isomorphisms, as shown in the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_G^*(D,S) \arrow[r] & K_G^*(D) \arrow[r] \arrow[d, "\cong"] & K_G^*(S) \arrow[r] & \cdots \\
& K_G^*(X) \arrow[r, "\otimes e_G(E)"] \arrow[u, "\cong"', "\text{Thom}"] & K_G^*(X) & &
\end{tikzcd}$$

this first map is multiplication by the G-equivariant Euler class,

$$e_G(E) = [\mathbb{S}_E^+] - [\mathbb{S}_E^-] \in K_G^*(X),$$

where $\mathbb{S}_E^+$ and $\mathbb{S}_E^-$ are the two complex half-spin representations determined by the Spin^c-structure on E (note that it follows from the conditions of the lemma that $\operatorname{rank} E$ is even). If E is in fact a complex bundle, then it admits a canonical Spin^c-structure with half-spin representations

$$\mathbb{S}_E^+ = \Lambda_{\mathbb{C}}^{\mathrm{even}}(E^*), \qquad \mathbb{S}_E^- = \Lambda_{\mathbb{C}}^{\mathrm{odd}}(E^*),$$

and corresponding Euler class

$$e_G(E) = \lambda_{-1}(E^*) = \sum\nolimits_p (-1)^p \left[\Lambda_{\mathbb{C}}^p(E^*)\right]. \tag{2.2}$$

See [11] for a discussion of the K-theory Euler class in the non-equivariant case. To establish the injectivity of this map, we will demonstrate that this K-theoretic Euler class is not a zero-divisor. (We note that a related result concerning the invertibility of the equivariant K-theory Euler class appears as [8, Lemma 2.7], dating back to the early days of K-theory.)

Let T be a maximal torus of G containing the S^1-subgroup fixing X. Given any G-bundle over X, by restricting the action from G to T we obtain a T-bundle, giving us the forgetful map $K_G^*(X) \to K_T^*(X)$. This homomorphism is injective by [3, Proposition 4.9], in which Atiyah argues that the K-theory pushforward map defined in terms of the Dolbeault complex for G/T gives a left inverse. The image of the forgetful map is precisely the Weyl invariants in $K_T^*(X)$. By the naturality of the Euler class, the image of $e_G(E) \in K_G^*(X)$ is $e_T(E) \in K_T^*(X)$, and so we can therefore restrict our attention to T and show that $e_T(E)$ is not a zero-divisor.

Factoring out our designated circle, the torus decomposes as $T \cong T' \times S^1$, where $T' = T/S^1$. Since the S^1-action fixes X, we obtain a $\mathbb{Z}_2$-graded ring isomorphism

$$K_T^*(X) \cong K_{T'}^*(X) \otimes K_{S^1}^*(\mathrm{pt}) \cong K_{T'}^*(X)[z, z^{-1}] \tag{2.3}$$

(see [36]), where $K_{S^1}^0(\mathrm{pt}) \cong R(S^1) \cong \mathbb{Z}[z, z^{-1}]$ is the representation ring of S^1, and $K_{S^1}^1(\mathrm{pt}) = 0$ vanishes. Indeed, any complex S^1-bundle V over X decomposes in terms of the S^1-action as a finite sum $V \cong \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda \otimes \mathbb{C}_\lambda$, where the V_λ are complex T'-vector bundles over X, and $\mathbb{C}_\lambda$ is the representation of S^1 on $\mathbb{C}$ with $z \in S^1$ acting as multiplication by z^λ.

By Lemma 2.1, the bundle E admits a T-invariant complex structure, which by (2.1) decomposes into isotypic components for the S^1-action as $E \cong \bigoplus_{\lambda>0} E_\lambda \otimes \mathbb{C}_\lambda$. This complex structure induces a canonical Spin^c-structure on E, whose corresponding half-spin representations, and consequently its Euler class (2.2), differ from that of the given Spin^c structure (or the given complex structure) by a line bundle. The K-theory class of this line bundle is of the form $[L]z^k \in K^*_{T'}(X)[z, z^{-1}]$, for some complex T'-equivariant line bundle L and integer k. Recalling that the Euler class (2.2) for complex bundles is multiplicative, we compute

$$\begin{aligned} e_T(E) &= [L]z^k\, e_T\Big(\bigoplus\nolimits_{\lambda>0} E_\lambda \otimes \mathbb{C}_\lambda\Big) = [L]z^k \prod\nolimits_{\lambda>0} e_T(E_\lambda \otimes \mathbb{C}_\lambda) \\ &= [L]z^k \prod\nolimits_{\lambda>0} \big(1 - E_i^* z^{-\lambda} + \cdots \pm \Lambda^{\operatorname{rank} E_\lambda}(E_\lambda^*)\, z^{-\lambda \operatorname{rank} E_\lambda}\big). \end{aligned} \tag{2.4}$$

To show that $e_T(E)$ is not a zero-divisor in $K^*_T(X)$, we show that each of its factors is not a zero-divisor. First, we recall that the class $[L]z^k \in K^*_T(X)$ of a complex line bundle is a unit in K-theory and thus not a zero-divisor. Secondly, a polynomial $p(z) = 1 + a_1 z^{-\lambda} + \cdots + a_n z^{-n\lambda}$ for $\lambda > 0$ with constant term 1 and coefficients $a_i \in K^*_{T'}(X)$ cannot be a zero-divisor in $K^*_T(X)$. Explicitly, suppose that $p(z)q(z) = 0$ for a finite Laurent series $q(z) = \sum_i b_i z^i \in K^*_{T'}(X)[z, z^{-1}]$. Then,

$$\begin{aligned} 0 &= \sum\nolimits_i \big(b_i z^i + a_1 b_i z^{i-\lambda} + \cdots + a_n b_i z^{i-n\lambda}\big) \\ &= \sum\nolimits_i (b_i + a_1 b_{i+\lambda} + \cdots + a_n b_{i+n\lambda})\, z^i, \end{aligned}$$

and thus we can write each coefficient of $q(z)$ as

$$b_i = -a_1 b_{i+\lambda} - \cdots - a_n b_{i+n\lambda}$$

in terms of higher degree coefficients. However, if $q(z) = \sum_i b_i z^i$ is a finite Laurent series, this immediately implies that $b_i = 0$ for all i, and thus $q(z) = 0$. □

Remark 2.4. The proof of the equivariant cohomology version of this lemma given by Atiyah and Bott in [6] uses the filtration $F_p = H^{\geq p}_{T'}(X) \otimes H^*_{S^1}(\mathrm{pt})$ of $H^*_T(X)$ by the cohomological degree of the $H^*_{T'}(X)$-factor. With respect to this filtration, the projection onto the degree-zero component of the associated graded algebra is the restriction map $H^*_T(X) \to H^*_{S^1}(\mathrm{pt})$. Acting on the associated graded algebra, multiplication by the equivariant Euler class descends to multiplication by its image in $H^*_{S^1}(\mathrm{pt})$. By this argument, all that is necessary is to show that the equivariant Euler class is not a zero-divisor over a point.

We can reprise this proof in K-theory using the filtration of $K^*_T(X)$ by powers of the kernel of the restriction map $K^*_T(X) \to K^*_{S^1}(\mathrm{pt})$. For $T = S^1$, this is the filtration $F_p = [\tilde{K}^*(X)]^p \otimes K^*_{S^1}(\mathrm{pt})$ using powers of the reduced K-theory in place of the cohomological degree. Once again, the problem is reduced to showing that the equivariant Euler class is not a zero-divisor over a point. A single fiber E_{pt} is a finite dimensional representation of S^1, which splits as a direct sum $E_{\mathrm{pt}} \cong \bigoplus_i \mathbb{C}_{\lambda_i}$

of one-dimensional irreducible representations, and (2.4) becomes

$$e_{S^1}(E_{\mathrm{pt}}) = z^k\, e_{S^1}\Big(\bigoplus\nolimits_i \mathbb{C}_{\lambda_i}\Big) = z^k \prod\nolimits_i e_{S^1}(\mathbb{C}_{\lambda_i}) = z^k \prod\nolimits_i \big(1 - z^{-\lambda_i}\big).$$

Since the Euler class over a point is a finite Laurent polynomial whose lowest (and highest) degree term has coefficient ± 1, it is not a zero-divisor in $\mathbb{Z}[z, z^{-1}]$. Furthermore, this Euler class is not divisible by any prime $p \in \mathbb{Z}$ and is thus a primitive element of $K^*_{S^1}(\mathrm{pt})$.

The original Atiyah-Bott lemma in [6] for cohomology is actually weaker than our result for K-theory. In Lemma 2.3, we need not make any assumptions on the torsion of the K-theory $K^*(X)$, since the K-theoretic Euler class restricted to a point is always primitive. In contrast, the cohomology Euler class may not be primitive, because $e_{S^1}(E_{\mathrm{pt}}) = \prod_i \lambda_i u \in H^*_{S^1}$, where u is the generator of $H^2_{S^1}$, may be divisible by primes $p \in \mathbb{Z}$. Therefore, to state the analogous theorem for equivariant cohomology, we either require that $H^*(X;\mathbb{Z})$ be torsion-free, or we explicitly kill the torsion by working with $H^*(X;\mathbb{Q})$.

Atiyah and Bott also require a second condition on the torsion. In order to establish the injection $H^*_G(X;\mathbb{Z}) \to H^*_T(X;\mathbb{Z})$, they require that the cohomology $H^*(G;\mathbb{Z})$ of the Lie group G be torsion-free in order that $H^*(BG;\mathbb{Z})$ be torsion-free and the fibration $G/T \to BT \to BG$ behave like a product for integral cohomology, i.e., $H^*(BT;\mathbb{Z}) \cong H^*(BG;\mathbb{Z}) \otimes H^*(G/T;\mathbb{Z})$. However, as the representation ring K^*_G is always torsion-free, there is no need for this condition when working with K-theory. Furthermore, by Hodgkin's Theorem (see [22, 32]), if a compact connected Lie group has no torsion in its fundamental group, then its K-theory is automatically torsion free.

In light of these observations, our K-theoretic Atiyah-Bott lemma is actually stronger than its cohomological counterpart. It works without localization or the need to tensor with the rationals, and it also works for all compact manifolds X and compact connected Lie groups G, not just those whose cohomology is torsion-free. This is because Thom and Euler classes are better behaved in K-theory, allowing us to formulate our arguments with no mention of torsion. From another point of view, recall that the Atiyah-Hirzebruch spectral sequence (see [7]) implies that the order of the torsion subgroup in K-theory is at most equal to the order of the torsion subgroup in integral cohomology. We then see that passing from integral cohomology to K-theory eliminates just enough torsion for the Atiyah-Bott lemma to work.

3. Kirwan surjectivity in K-theory

The goal of this section is to prove a K-theoretic extension of the surjectivity theorem of Kirwan. We now briefly recall the setting of these results. Let G be a compact connected Lie group. A *Hamiltonian G-space* is a symplectic manifold (M, ω) on which G acts by symplectomorphisms, together with a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$ satisfying Hamilton's equation

$$\langle d\mu, X \rangle = \imath_{X^\sharp}\omega, \quad \forall X \in \mathfrak{g},$$

where $X^\sharp$ denotes the vector field on M generated by $X \in \mathfrak{g}$. We will assume throughout that the moment map $\mu : M \to \mathfrak{g}^*$ is proper, i.e., the preimage of a compact set is compact. We further assume that 0 is a regular value so that G acts locally freely on the level set $\mu^{-1}(0)$, i.e., the G-action has finite stabilizers. In this situation the *symplectic quotient* or Marsden-Weinstein reduction of M at 0 is

$$M/\!\!/G := \mu^{-1}(0) \,/\, G,$$

viewed as the standard quotient if the action is free, or an orbifold otherwise.

The surjectivity theorem of Kirwan gives a method of computing the rational cohomology ring of this symplectic quotient using that of the original symplectic manifold M (see [24]). We do the same here, except in K-theory. Our main theorem is as follows.

Theorem 3.1. *Let (M,ω) be a Hamiltonian G-space with proper moment map $\mu : M \to \mathfrak{g}^*$. Assume that 0 is a regular value of μ so that the group G acts locally freely on $\mu^{-1}(0)$. Then the Kirwan map κ induced by the inclusion $\iota : \mu^{-1}(0) \hookrightarrow M$,*

$$\begin{tikzcd} K_G^*(M) \arrow[r, "\iota^*"] \arrow[dr, "\kappa"'] & K_G^*\big(\mu^{-1}(0)\big) \arrow[d, "\cong"] \\ & K^*(M/\!\!/G) \end{tikzcd}$$

is a surjection.

If the G-action on the level set $\mu^{-1}(0)$ is not free, then we take $K^*(M/\!\!/G)$ to be the orbifold K-theory, which for a global quotient is given, and in some cases defined, by

$$K^*_{\mathrm{orb}}\big(\mu^{-1}(0) \,/\, G\big) \cong K_G^*\big(\mu^{-1}(0)\big). \tag{3.1}$$

See, for example, [1] for further discussion of orbifold K-theory.

Our strategy for the proof of Theorem 3.1 is to follow Kirwan's original proof of surjectivity for rational cohomology [24] and verify that the cohomological aspects of her argument extend to K-theory. With this in mind, we now summarize Kirwan's proof. In [24, §4], Kirwan considers the norm square of the moment map, whose minimum $\mu^{-1}(0)$ is automatically the level set of interest, and shows that it is a minimally degenerate Morse function (i.e., "is Morse in the sense of Kirwan"). In [24, §10], she establishes that the results of Morse-Bott theory still hold in this generalized setting. In [24, §5], Kirwan proves that this Morse function is equivariantly perfect. She then uses this result to establish surjectivity via an inductive argument using a stratification built out of the Morse function.

We begin with a few definitions. Let $M = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$ be a smooth stratification of M indexed by a partially ordered set $\mathcal{B}$. For a smooth stratification, the strata S_β are locally closed submanifolds of M satisfying the closure property

$$\overline{S}_\beta \subseteq M_{\geq \beta}, \tag{3.2}$$

i.e., the closure of S_β is contained in the strata above S_β for each $\beta \in \mathcal{B}$. Extending the partial ordering on $\mathcal{B}$ to a total ordering, consider the unions

$$M_{<\beta} := \bigsqcup\nolimits_{\gamma<\beta} S_\gamma, \qquad M_{\leq\beta} := \bigsqcup\nolimits_{\gamma\leq\beta} S_\gamma, \qquad M_{\geq\beta} := \bigsqcup\nolimits_{\gamma\geq\beta} S_\gamma.$$

The ordered collection of subsets $M_{\leq\beta}$ then gives a filtration of M. By analogy to the Atiyah-Hirzebruch filtration (see [7]), we obtain a filtration of $K^*(M)$ and a spectral sequence satisfying

$$E_1 = \bigoplus\nolimits_{\beta\in\mathcal{B}} K^*\big(M_{\leq\beta}, M_{<\beta}\big), \qquad E_\infty = \mathrm{Gr}\ K^*(M), \tag{3.3}$$

which converges to the associated graded algebra of the K-theory of M. For a smooth stratification, we have a normal bundle N_β to S_β in M, and due to the closure property (3.2) we can choose a tubular neighborhood U_β of S_β in M diffeomorphic to N_β such that

$$U_\beta \subset M_{\leq\beta}, \qquad U_\beta \setminus S_\beta \subset M_{<\beta}. \tag{3.4}$$

Indeed, due to the closure property (3.2), any point sufficiently close to S_β in a higher stratum must be in the closure of S_β, and hence must lie along a direction tangent to S_β. The normal directions to S_β must therefore initially extend into the lower strata. By excision, we therefore have

$$K^*(M_{\leq\beta}, M_{<\beta}) \cong K^*(N_\beta, N_\beta \setminus S_\beta). \tag{3.5}$$

(In terms of K-theory with compact supports, this is $K_c^*(N_\beta) = \tilde{K}^*(N_\beta^+)$.) If N_β is complex (or Spin^c), then we have a Thom isomorphism

$$K^*(N_\beta, N_\beta \setminus S_\beta) \cong K^{*-d(\beta)}(S_\beta), \tag{3.6}$$

where the degree $d(\beta)$ of the stratum is the rank of its normal bundle N_β. Combining the isomorphisms (3.5) and (3.6), our spectral sequence (3.3) becomes

$$E_1 \cong \bigoplus\nolimits_{\beta\in\mathcal{B}} K^{*-d(\beta)}(S_\beta), \qquad E_\infty = \mathrm{Gr}\ K^*(M). \tag{3.7}$$

The same discussion holds for equivariant K-theory for a G-space with a G-invariant stratification.

Definition 3.2. A smooth stratification $M = \bigsqcup_{\beta\in\mathcal{B}} S_\beta$ is called *(equivariantly) perfect* for K-theory if the spectral sequence (3.7) for (equivariant) K-theory collapses at the E_1 page, or equivalently if we have short exact sequences of the form

$$0 \longrightarrow K^{*-d(\beta)}\big(S_\beta\big) \longrightarrow K^*\big(M_{\leq\beta}\big) \longrightarrow K^*\big(M_{<\beta}\big) \longrightarrow 0 \tag{3.8}$$

for each $\beta \in \mathcal{B}$ (similarly for equivariant K-theory).

In other words, whenever we add a stratum, the equivariant K-theory of the new space including that stratum is an extension of, and therefore surjects onto, the equivariant K-theory of the union of all the lower strata. Although our primary goal in this section is to prove a surjectivity result, we will instead attack this long exact sequence from the opposite direction, proving that the leftmost maps of these exact sequences are injective. We will do this using our results from Section 2.

We can refine this definition slightly if the stratification is obtained from a proper Morse function (or more generally from a Morse-Bott or Morse-Kirwan function). Let $f : M \to \mathbb{R}$ be a proper Morse function on a Riemannian manifold M and consider its negative gradient flow. Let $\{C_\beta\}$ denote the connected components of the critical set of f, and for each C_β define the stratum S_β to be the set of points of M which flow down to C_β via their paths of steepest descent. (Note that this is a decomposition of M by stable manifolds, not the unstable manifolds as is usually the case in Morse theory.) The critical sets are partially ordered by $\beta \leq \gamma$ if $f(C_\beta) \leq f(C_\gamma)$. Extending this to a total order, this gives us a smooth stratification of M satisfying the closure property (3.2). Furthermore, the inclusion $C_\beta \hookrightarrow S_\beta$ of the critical set into the stratum induces an isomorphism $K^*(S_\beta) \cong K^*(C_\beta)$. (For Morse-Bott functions, there is a deformation retraction from the stratum onto the critical set. For a general Morse-Kirwan function, we obtain a retraction at the level of (equivariant) K-theory, as we describe in Appendix B.) The spectral sequence (3.7) then becomes

$$E_1 \cong \bigoplus\nolimits_{\beta\in\mathcal{B}} K^{*-d(\beta)}(C_\beta), \qquad E_\infty = \mathrm{Gr}\ K^*(M). \tag{3.9}$$

In the equivariant case, we require that both the Morse function and the Riemannian metric be G-invariant, giving us a G-invariant stratification.

Definition 3.3. A Morse (or Morse-Bott or Morse-Kirwan) function $f : M \to \mathbb{R}$ on a Riemannian manifold is called *(equivariantly) perfect* for K-theory if the spectral sequence (3.9) for (equivariant) K-theory collapses at the E_1 page, or equivalently if we have short exact sequences of the form

$$0 \longrightarrow K^{*-d(\beta)}\big(C_\beta\big) \longrightarrow K^*\big(M_{\leq\beta}\big) \longrightarrow K^*\big(M_{<\beta}\big) \longrightarrow 0 \tag{3.10}$$

for each $\beta \in \mathcal{B}$ (and similarly for equivariant K-theory).

Remark 3.4. We note that our definition of (equivariant) perfection given above differs slightly from the usual one. The usual definition of (equivariant) perfection is in terms of (equivariant) Poincaré polynomials—in particular, that the (equivariant) Morse inequalities are in fact equalities. However, that definition is concerned with only the ranks of the free components, as given by the Betti numbers in rational or real cohomology, and not the torsion that may be present in K-theory. Since we are here concerned with statements that include the torsion components, we opt for the definition in terms of the short exact sequences.

Before getting to our main results for this section, we first recall an elementary lemma, proved in [36, §2] and [31, p. 144]:

Lemma 3.5. *Let $H \subset G$ be a Lie subgroup and let X be a compact H-space. Then $G \times_H X = (G \times X)/H$ is a bundle over the homogeneous space G/H with fiber X, and $K_G^*(G \times_H X) \cong K_H^*(X)$.*

In the special case where X is a point, then this is simply the isomorphism $K_G(G/H) \cong K_H(\mathrm{pt}) = R(H)$. We are now ready to show that the norm square of the moment map is equivariantly perfect.

Theorem 3.6. *Let (M, ω) be a Hamiltonian G-space with proper moment map $\mu : M \to \mathfrak{g}^*$. Given a G-invariant inner product on the coadjoint representation $\mathfrak{g}^*$, the norm square of the moment map $f = \|\mu\|^2$ is an equivariantly perfect Morse-Kirwan function for K-theory. In addition, the degrees $d(\beta)$ of the critical sets are all even and may thus be dropped when working with complex K-theory.*

Proof. Since the moment map $\mu : M \to \mathfrak{g}^*$ is G-equivariant, the moment map image of the critical set of $f := \|\mu\|^2$ must be composed of complete coadjoint orbits. Since G is compact, each coadjoint orbit intersects the positive Weyl chamber (given a choice of Cartan subalgebra and positive root system) exactly once, except for those orbits which intersect the boundary of the Weyl chamber. We can therefore index the critical sets $\{C_\beta\}$ by a finite subset $\mathcal{B} \subset \mathfrak{t}_+$ of a fixed positive Weyl chamber, defining C_β for $\beta \in \mathcal{B}$ to be the set of all critical points of f with moment map image in the coadjoint orbit of β. Then, for any $\beta \in \mathcal{B}$, the function f takes the constant value $f(C_\beta) = \|\beta\|^2$ on the critical set C_β. We observe that the critical sets C_β may not be connected, as there may be several connected components of the critical set of f with moment map image in the same coadjoint orbit. (Actually, the different connected components of C_β may have distinct degrees, so Kirwan further decomposes the critical set as the disjoint union of $C_\beta = \bigsqcup_d C_{\beta,d}$ according to degree.) Since μ is proper, its norm square $f := \|\mu\|^2$ is proper, and the critical sets C_β are compact.

Choose a G-invariant metric on M. Since the norm square of the moment map is a G-invariant function, the stratification $M = \bigsqcup_{\beta\in\mathcal{B}} S_\beta$ obtained from the negative gradient flow of f is G-invariant. We can therefore consider the spectral sequence (3.9) for equivariant K-theory.

Let N_β be the normal bundle to S_β in M, and let $D(S_\beta)$ and $S(S_\beta)$ denote the disc and sphere bundles, respectively, of N_β. We now claim that there is a commutative diagram

(3.11)
$$\begin{tikzcd}
\cdots \arrow[r] & K_G^*\big(M_{\le\beta}, M_{<\beta}\big) \arrow[r, "a"] \arrow[d, "\cong"] & K_G^*\big(M_{\le\beta}\big) \arrow[r] \arrow[d, "i^*"] & K_G^*\big(M_{<\beta}\big) \arrow[r] & \cdots \\
\cdots \arrow[r] & K_G^*\big(D(S_\beta), S(S_\beta)\big) \arrow[r, "b"] & K_G^*\big(D(S_\beta)\big) \arrow[r] & K_G^*\big(S(S_\beta)\big) \arrow[r] & \cdots
\end{tikzcd}$$

whose rows are long exact sequences. The left vertical arrow is an isomorphism by excision (3.5), and the middle vertical arrow is the pullback homomorphism induced by the inclusion $i : D(S_\beta) \to U_\beta \hookrightarrow M_{\le\beta}$ of the tubular neighborhood U_β from (3.4). In order to prove that the top long exact sequence splits, it is sufficient to show that the bottom long exact sequence splits. Indeed, if the arrow b injective, then the composition of the arrow a with the pullback i^* is injective, and thus the arrow a must be injective.

Recalling Lemma 2.3, in order to show that the long exact sequence for the Thom complex $\big(D(S_\beta), S(S_\beta)\big)$ splits, we must find a circle subgroup of G which acts on the normal bundle N_β fixing only the zero section. This does not necessarily hold over the whole stratum S_β. However, the negative gradient flow of the norm square of the moment map always converges when M is a Hamiltonian G-space with a proper moment map. (This result is due to an unpublished manuscript of Duistermaat, whose argument is reproduced in [28]. Another proof can be found in [41, Appendix A].) The negative gradient flow therefore gives an equivariant deformation retraction of the stratum S_β onto the critical set C_β. (Kirwan does not make use of this fact in [24, §10]. Instead she establishes Morse-theoretic results for minimally degenerate functions which do not necessarily satisfy this special property; we describe her argument, translated to the K-theory setting, in Appendix B.) Furthermore, Kirwan observes in [24, §3] that each critical subset C_β is of the form $G \times_{\mathrm{Stab}(\beta)} \big(Z_\beta \cap \mu^{-1}(\beta)\big)$, where Z_β is a critical set of the nondegenerate Morse-Bott function $\mu^\beta = \langle \mu, \beta\rangle$, the component of the moment map μ in the β direction. Here $\mathrm{Stab}(\beta) \subset G$ is the isotopy subgroup stabilizing the element $\beta \in \mathfrak{t}_+ \subset \mathfrak{g}$. By Lemma 3.5, we have an isomorphism

$$K_G^*(C_\beta) \cong K^*_{\mathrm{Stab}(\beta)}\big(Z_\beta \cap \mu^{-1}(\beta)\big). \tag{3.12}$$

Restricting the normal bundle N_β first to the critical set C_β and then to the fiber $Z := Z_\beta \cap \mu^{-1}(\beta)$ over the identity coset of the homogeneous space $G/\operatorname{Stab}(\beta)$, let $D(Z)$ and $S(Z)$ denote the disc and sphere bundles of $N_\beta|_Z$. The bottom long exact sequence of (3.11) is therefore isomorphic to the long exact sequence for the

Thom complex $\big(D(Z), S(Z)\big)$ in Stab(β)-equivariant K-theory:

$$(3.13)\quad \cdots \to K^*_{\mathrm{Stab}(\beta)}\big(D(Z), S(Z)\big) \to K^*_{\mathrm{Stab}(\beta)}\big(D(Z)\big) \to K^*_{\mathrm{Stab}(\beta)}\big(S(Z)\big) \to \cdots .$$

Let T_β be the subtorus of Stab(β) generated by the element $\beta \in \mathfrak{g}$. (If β is an irrational element that does not generate a circle, then we let T_β be the closure of the one parameter subgroup generated by β, which still gives us a torus.) We recall that the critical set Z_β of the moment map component μ^β is a (union of certain components of the) fixed point set of T_β. It follows that this subtorus $T_\beta \subset \mathrm{Stab}(\beta)$ fixes $Z_\beta \cap \mu^{-1}(\beta)$ and acts on the normal bundle N_β with no non-zero fixed vectors. Kirwan argues in [24, §4] that the normal bundle N_β is a complex vector bundle, with complex structure inherited from the G-invariant almost complex structure on M induced by the symplectic form and metric. Therefore, by the K-theoretic Atiyah-Bott Lemma 2.3, the long exact sequence (3.13) splits into short exact sequences. It follows that the long exact sequences in (3.11) also split, which establishes the short exact sequences (3.10). The norm square of the moment map is therefore equivariantly perfect in K-theory. Furthermore, as the normal bundle N_β is complex, its real rank is even, and so the degrees $d(\beta)$ are even and do not affect complex K-theory. □

Now that we have established that the norm square of the moment map is equivariantly perfect, we can use the corresponding stratification to prove Kirwan surjectivity.

Proof of Theorem 3.1. Finally, we prove Kirwan surjectivity via an inductive argument. Building the space M one stratum at a time, the short exact sequences (3.10) give us a chain of surjections

$$\cdots \twoheadrightarrow K^*_G\big(M_{\leq\beta}\big) \twoheadrightarrow K^*_G\big(M_{<\beta}\big) \twoheadrightarrow \cdots \twoheadrightarrow K^*_G(S_0),$$

which ends with the bottom stratum S_0. The stratum S_0 is open and dense in M and retracts onto the minimal critical set $C_0 = \mu^{-1}(0)$. Therefore $K^*_G\big(M_{\leq\beta}\big) \twoheadrightarrow K^*_G(S_0) \cong K^*_G(\mu^{-1}(0))$ for every $\beta \in \mathcal{B}$. Since the maps $K^*_G\big(M_{\leq\beta}\big) \twoheadrightarrow K^*_G\big(M_{<\beta}\big)$ are surjections for all β, there is no $\lim^1$ term (see [19, §3.F]), and we may conclude that

$$K^*_G(M) = K^*_G\big(\varinjlim M_{\leq\beta}\big) = \varprojlim K^*_G\big(M_{\leq\beta}\big).$$

Hence the restriction map gives a surjection

$$\kappa : K^*_G(M) \twoheadrightarrow K^*_G\big(\mu^{-1}(0)\big).$$

Finally, if 0 is a regular value, then $\mu^{-1}(0)$ is a manifold, and we can compose this surjection with the isomorphism $K^*_G(\mu^{-1}(0)) \cong K^*(\mu^{-1}(0)/G)$, using orbifold K-theory as in (3.1) if necessary. □

4. Equivariant formality

In this section we discuss the K-theoretic analogue of the Kirwan-Ginzburg equivariant formality theorem for compact Hamiltonian G-spaces. Recall that the equivariant cohomology of a G-space M is defined in terms of the Borel construction by $H^*_G(M) := H^*(M_G)$, where $M_G = M \times_G EG$. In [17], Goresky, Kottwitz, and MacPherson call a G-space M *equivariantly formal* if the Leray-Serre spectral sequence for the cohomology of the fibration $M \to M_G \to BG$ collapses at the E_2

page. When working with coefficients in a field, equivariant formality is equivalent to the existence of an isomorphism

$$H^*_G(M) \cong H^*(M) \otimes H^*_G(\mathrm{pt}) \tag{4.1}$$

as modules over $H^*_G(\mathrm{pt})$ or the corresponding statement about the Betti numbers or Poincaré polynomials.

Unfortunately, neither of these versions of equivariant formality applies to equivariant K-theory. Requiring an isomorphism analogous to (4.1) is too strong a condition due to the possible presence of torsion. On the other hand, a statement in terms of the Leray-Serre spectral sequence applies only to the Borel construction of equivariant K-theory, given by $\mathcal{K}^*_G(M) := K^*(M_G)$. For the Atiyah-Segal equivariant bundle construction of $K^*_G(M)$, we do not have such a spectral sequence at our disposal. Instead, we consider the following weakened version of equivariant formality:

Definition 4.1. We call a G-space M *weakly equivariantly formal* for K-theory if the map

$$K^*_G(M) \otimes_{R(G)} \mathbb{Z} \to K^*(M), \tag{4.2}$$

induced by the forgetful homomorphism $K^*_G(M) \to K^*(M)$, is an isomorphism.

Here, the representation ring $R(G)$ acts on $\mathbb{Z}$ via the augmentation homomorphism $\epsilon : R(G) \to \mathbb{Z}$ taking a virtual representation to its dimension. So, rather than attempting to factor $R(G)$ out of the equivariant K-theory, we instead divide by the $R(G)$-action, or more precisely by the action of the augmentation ideal $I(G) = \operatorname{Ker} \epsilon$. The relationship between the forgetful homomorphism and the map (4.2) is given by the commutative diagram

$$\begin{tikzcd} K^*_G(M) \arrow[d, two heads] \arrow[dr] & \\ K^*_G(M) \otimes_{R(G)} \mathbb{Z} \arrow[r] & K^*(M) \end{tikzcd}$$

from which we obtain the following immediate consequences:

Proposition 4.2. *If a G-space M is weakly equivariantly formal for K-theory, then the forgetful map $K^*_G(M) \to K^*(M)$ is surjective, and thus every complex vector bundle E over M admits a stable equivariant structure, i.e., the G-action on M lifts to a fiberwise linear G-action on $E \oplus \underline{k}$ for some trivial bundle $\underline{k} = M \times \mathbb{C}^k$. In addition, the kernel of the forgetful map is $I(G) \cdot K^*_G(M)$.*

We observe that our statement (4.2) of weak equivariant formality is closely related to the Künneth formula in equivariant K-theory. In non-equivariant K-theory, we have a Künneth formula (see [2]), which expresses the K-theory of a product $K^*(X \times Y)$ in terms of a short exact sequence involving the product of the K-theories $K^*(X) \otimes K^*(Y)$ together with torsion information. On the other hand, equivariant K-theory is a module over the representation ring $R(G)$, and for two G-spaces X and Y, the equivariant K-theory of the product $K^*_G(X \times Y)$ can be expressed in terms of the tensor product $K^*_G(X) \otimes_{R(G)} K^*_G(Y)$ over $R(G)$, together with $R(G)$-torsion information. In [23], Hodgkin constructs a Künneth spectral sequence, related to an Eilenberg-Moore spectral sequence, starting with

$$E_2(X, Y) = \operatorname{Tor}^*_{R(G)}\big(K^*_G(X), K^*_G(Y)\big),$$

where the Tor groups are the derived functors of the tensor product (see [13] or any homological algebra textbook). Here, we use Hodgkin's convention of labeling the higher Tor groups with a negative superscript in place of the more standard positive subscript, writing

$$\mathrm{Tor}^{-p}_{R(G)}(A,B) = \mathrm{Tor}^{R(G)}_{p}(A,B) \text{ with } p \geq 0,$$

which places Hodgkin's Künneth spectral sequence in the left half-plane. Under appropriate circumstances (elaborated on in [38] and [32]), this spectral sequence converges to $K^*_G(X \times Y)$. In particular, the E_2 page contains the tensor product

$$\mathrm{Tor}^{0}_{R(G)}\big(K^*_G(X), K^*_G(Y)\big) = K^*_G(X) \otimes_{R(G)} K^*_G(Y)$$

over the representation ring.

In [23, §9], Hodgkin considers the special case $Y = G$, where

$$\begin{aligned} K^*_G(G) &\cong K^*(\mathrm{pt}) \cong \mathbb{Z}, \text{ and} \\ K^*_G(X \times G) &\cong K^*(X). \end{aligned}$$

The Künneth spectral sequence then starts with the $E_2(X,G)$ page containing

$$\mathrm{Tor}^{0}_{R(G)}\big(K^*_G(X), \mathbb{Z}\big) = K^*_G(X) \otimes_{R(G)} \mathbb{Z},$$

and since $Y = G$ is a free G-space, the spectral sequence converges to $K^*(X)$, provided that G is compact, connected, and $\pi_1(G)$ is torsion-free (see [23, Theorem 8.1]). The map (4.2) is then the edge homomorphism for this spectral sequence, so weak equivariant formality is precisely the statement that the edge homomorphism is an isomorphism. This leads us to the following alternative spectral sequence condition sufficient to guarantee weak equivariant formality:

Proposition 4.3. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. A compact G-space M is weakly equivariant formal for K-theory if*

$$\mathrm{Tor}^{p}_{R(G)}\big(K^*_G(M), \mathbb{Z}\big) = 0$$

for all $p \neq 0$, i.e., the higher $R(G)$-torsion vanishes.

We will first use this criterion to establish weak equivariant formality for Hamiltonian T-spaces, where T is a torus. Then, to make the transition from abelian to non-abelian actions, we will use the following lemma (see also [30]):

Lemma 4.4. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let M be a compact G-space. Then M is weakly equivariantly formal with respect to G if and only if it is weakly equivariantly formal with respect to a maximal torus $T \subset G$.*

Proof. We begin by observing that

$$K^*_T(M) \cong K^*_G(G/T \times M). \tag{4.3}$$

At the level of vector bundles, a T-equivariant bundle $\pi : E \to M$ induces a G-equivariant bundle $G \times_T E$ over $G/T \times M$. Here G acts on the left on the G-factor of $G \times_T E$, while it acts via the diagonal action on $G/T \times M$. The projection map is given by $(g,v) \mapsto (gT,\, g \cdot \pi(v))$ for $g \in G$ and $v \in E$, which is well defined since for any $t \in T$,

$$\big(gt^{-1},\, tv\big) \sim (g,v) \longmapsto \big(gt^{-1}T,\, gt^{-1}t \cdot \pi(v)\big) = (gT,\, g \cdot \pi(v))\,,$$

and is G-equivariant since for any $h \in G$,

$$h \cdot (g, v) = (hg, v) \longmapsto (hgT,\, hg \cdot \pi(v)) = h \cdot (g,\, g \cdot \pi(v)).$$

(Note that this bundle has the same total space as in Lemma 3.5, but different base and projection.) Conversely, given a G-equivariant bundle over $G/T \times M$, its restriction to the identity coset $eT \times M$ is a T-equivariant bundle over M. These maps of vector bundles are inverses of each other and extend to K-theory.

In [38, §3], Snaith argues that under a technical hypothesis, later verified by McLeod in [32], Hodgkin's Künneth spectral sequence yields the identity

$$K^*_G(G/T \times M) \cong R(T) \otimes_{R(G)} K^*_G(M),$$

i.e., in this case the higher Tor groups vanish and the Künneth spectral sequence collapses. Combining this with the identity (4.3), we find that G-equivariant and T-equivariant K-theories of M are related by

$$K^*_T(M) \cong K^*_G(M) \otimes_{R(G)} R(T).$$

Comparing this to our weak equivariant formality condition (4.2), we obtain

$$K^*_T(M) \otimes_{R(T)} \mathbb{Z} \cong \big(K^*_G(M) \otimes_{R(G)} R(T)\big) \otimes_{R(T)} \mathbb{Z} \cong K^*_G(M) \otimes_{R(G)} \mathbb{Z},$$

and thus weak equivariant formality with respect to G and T are equivalent. $\square$

As an immediate corollary to this lemma, when considering equivariant formality for K-theory, we can replace the Lie group G with any compact connected subgroup of maximal rank in G. We are now ready to prove our equivariant formality theorem. The basic idea of the proof, using the fact that a generic component of the moment map is equivariantly perfect with respect to a maximal torus, is based on [24, Proposition 5.8]. However, we replace Kirwan's argument involving Poincaré polynomials with an argument examining the higher Tor groups. There is a similar proof in the context of algebraic K-theory in [40, §5], using the Bialynicki-Birula stratification and replacing Hodgkin's topological Künneth spectral sequence with an algebraic K-theory version due to Merkur′ev [33].

Theorem 4.5. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. If M is a compact Hamiltonian G-space, then M is weakly equivariantly formal for K-theory, the forgetful map $K^*_G(M) \to K^*(M)$ is surjective, and every complex vector bundle over M admits a stable equivariant structure.*

Proof. Let M be a compact Hamiltonian G-space. By Lemma 4.4 we need only establish weak equivariant formality with respect to a maximal torus $T \subset G$. Choosing a generic element $X \in \mathfrak{t} \subset \mathfrak{g}$ in the corresponding Cartan subalgebra, the component $\mu^X = \langle \mu, X \rangle$ of the moment map is a Morse-Bott function whose critical sets are the connected components of the fixed point set M^T (see [4, Lemma 2.2]). Since the normal bundle to the fixed point set automatically satisfies the conditions of the Atiyah-Bott lemma (see Remark 2.2), we find that the function μ^X is equivariantly perfect for K-theory. We can therefore compute $K^*_T(M)$ as was done using formula (3.10) by taking a series of extensions of the form

$$0 \longrightarrow K^*_T\big(M_i^T\big) \longrightarrow K^*_T\big(M_{\leq i}\big) \longrightarrow K^*_T\big(M_{<i}\big) \longrightarrow 0\,,$$

where i indexes the connected components of the fixed point set M^T. Since T acts trivially on its fixed point set, for each of the critical sets we have

$$K^*_T\big(M_i^T\big) \cong K^*\big(M_i^T\big) \otimes R(T)$$

and thus their higher Tor groups vanish, i.e.,

$$\operatorname{Tor}^p_{R(T)}\big(K^*_T(M_i^T), \mathbb{Z}\big) = 0 \text{ for } p \neq 0.$$

By the long exact sequence for the Tor functor (see [13, §VIII.1] for a discussion of the algebra), this property continues to hold upon taking a series of extensions, and so for all of M we have

$$\operatorname{Tor}^p_{R(T)}\big(K^*_T(M), \mathbb{Z}\big) = 0 \text{ for } p \neq 0.$$

It follows from Proposition 4.3 that M is weakly equivariantly formal for K-theory. The remaining assertions follow from Proposition 4.2. □

Remark 4.6. Kirwan proves the standard version of equivariant formality for rational cohomology in [24, §5]. The Leray-Serre spectral sequence gives an upper bound on the size (in terms of the Betti numbers) of the equivariant cohomology $H^*_G(M;\mathbb{Q})$. On the other hand, knowing that the components of the moment map are equivariantly perfect Morse-Bott functions, the Morse inequalities for the non-equivariant cohomology provide a lower bound on the size of $H^*_G(M;\mathbb{Q})$, establishing the result. As a corollary, Kirwan notes that the components of the moment map are also perfect in the non-equivariant case. Ginzburg in [15] works with de Rham cohomology and reverses this argument, showing that the moment map components are perfect and then using that to establish equivariant formality. As a consequence, on a compact Hamiltonian G-space, any rational cohomology class admits an equivariant extension, and any closed form admits an equivariantly closed extension.

A deeper question is whether we can remove the word *stable* from Theorem 4.5. In other words, does every complex vector bundle over a Hamiltonian G-space admit a lift of the G-action, without requiring any additional degrees of freedom? In [26], Kostant shows that the moment map contains precisely the data required to lift the G-action on M to a G-action on the prequantum line bundle $L \to M$ satisfying $c_1(L) = [\omega]$. In the following we show that every complex line bundle admits an equivariant structure, at least with respect to a finite cover of G. (See [21, 35] for related discussions of lifts of group actions to line bundles.)

Theorem 4.7. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let M be a compact Hamiltonian G-space.*

1. *Every complex line bundle over M admits a fiberwise linear lift of the G-action on M.*
2. *If M is connected, then the isomorphism classes of the equivariant structures on a fixed complex line bundle over M are in one-to-one correspondence with the character group of G.*
3. *If G is semi-simple, then each complex line bundle over M has a unique equivariant structure.*

Proof. We recall that the Picard group $\operatorname{Pic}(M)$ of isomorphism classes of complex line bundles over M is a retract of $K(M)$. In particular, the determinant homomorphism $\det : K(M) \twoheadrightarrow \operatorname{Pic}_(M)$ induced by the map taking a complex vector bundle E to its top exterior power $\det E = \Lambda^{\operatorname{rank} E}(E)$ is surjective. The same holds in the equivariant case, and we have a retraction $\det_G : K_G(M) \twoheadrightarrow \operatorname{Pic}_G(M)$. Using

Theorem 4.5, we can construct the commutative diagram

$$\begin{tikzcd} K_G(M) \arrow[r, two heads] \arrow[d, two heads, "\det_G"'] & K(M) \arrow[d, two heads, "\det"] \\ \mathrm{Pic}_G(M) \arrow[r] & \mathrm{Pic}(M) \end{tikzcd}$$

from which we see that the forgetful map $\mathrm{Pic}_G(M) \to \mathrm{Pic}(M)$ is also surjective.

The kernel of the forgetful map consists of all equivariant structures on the trivial complex line bundle $M\times\mathbb{C}$. Restricted to a single G-orbit $G\cdot x$ in M, such G-actions are necessarily of the form $(G\cdot x)\times\mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is a 1-dimensional representation of G. Over all of M, an equivariant structure on the trivial bundle corresponds to a continuous map $M/G \to \mathrm{Pic}_G(\mathrm{pt})$ from the orbit space to the character group of all such 1-dimensional representations (i.e., continuous homomorphisms from G to the circle group). Since the character group $\mathrm{Pic}_G(\mathrm{pt})$ is discrete, the representation $\mathbb{C}_\lambda$ must be constant on each connected component of M. The equivariant structures therefore correspond to $\mathrm{Pic}_G(\pi_0 M)$, and if M is connected they correspond to the character group $\mathrm{Pic}_G(\mathrm{pt})$.

In summary, we obtain the short exact sequence of Picard groups

$$1 \longrightarrow \mathrm{Pic}_G(\pi_0 M) \longrightarrow \mathrm{Pic}_G(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow 1$$

whenever M is a compact Hamiltonian G-space with respect to a compact connected Lie group G with $\pi_1(G)$ torsion-free. For the final assertion, we recall that a compact semi-simple Lie group has no one-dimensional representations other than the trivial representation, and thus its character group is trivial. □

Example 4.8. If the G-action on M is not Hamiltonian, then equivariant formality may fail. Let $M = T^2$ with $G = S^1$ acting by rotating one of the two circles freely. This is not a Hamiltonian action. The equivariant K-theory is

$$K^*_{S^1}(T^2) \cong K^*(T^2/S^1) \cong K^*(S^1),$$

which does not surject onto the standard K-theory $K^*(T^2)$. Also, since S^1 acts freely on T^2, the representation ring $R(S^1)$ acts on $K_{S^1}(T^2)$ via its image under the augmentation homomorphism, and we obtain

$$K^*_{S^1}(T^2) \otimes_{R(S^1)} \mathbb{Z} \cong K^*_{S^1}(T^2).$$

For line bundles we have

$$\mathrm{Pic}_{S^1}(T^2) \cong \mathrm{Pic}(T^2/S^1) \cong \mathrm{Pic}(S^1) = 0,$$

while $\mathrm{Pic}(T^2) \cong \mathbb{Z}$. Thus there exist non-trivial line bundles over T^2 which do not admit equivariant structures, as all equivariant line bundles must be trivial.

Remark 4.9. The reader may be curious why all the results in this section require the fundamental group of G to be free abelian. This condition appears in Hodgkin's derivation of the Künneth spectral sequence in [23], and in Remark A.4 below we show explicitly how this condition is used in one particular case. Most significantly, the condition on the torsion of $\pi_1(G)$ is the K-theoretic analogue of the condition that $H^*(G;\mathbb{Z})$ be torsion free, which we encountered at the end of Section 2 when discussing the torsion constraints of the cohomological Atiyah-Bott lemma. In fact, by Hodgkin's Theorem (see [22, 32]), if a compact connected Lie group has no torsion in its fundamental group, then its K-theory is automatically torsion-free.

We observe that if $\pi_1(G)$ contains torsion, then by the Hurewicz theorem the homology group $H_1(G;\mathbb{Z})$ also contains the same torsion. By the universal coefficient theorem, we then obtain torsion in $H^2(G;\mathbb{Z})$, which in turn transgresses to give torsion in $H^3(BG;\mathbb{Z}) \cong H^3_G(\mathrm{pt})$. However, twistings of K-theory, as described in [10, 14] are classified by elements of $H^3_G(M)$. Our condition on the torsion of $\pi_1(G)$ ensures that $H^3_G(\mathrm{pt})$ vanishes, and thus we do not encounter any twistings of $K^*_G(M)$ arising solely from the Lie group G. We expect that many of our results of this section still hold without the torsion constraint, albeit in terms of twisted K-theory.

Appendix A. Equivariant cohomology and line bundles

Our results from Section 4 involving equivariant line bundles and the equivariant Picard group can also be stated in terms of equivariant cohomology. We recall that G-equivariant line bundles over a compact G-space M are classified up to isomorphism by $H^2_G(M;\mathbb{Z})$. This is a non-trivial result, which appears in the literature in several sources, including [18, Theorem C.47] and [27]. Here we present an elementary proof due to Peter Landweber. We recall that for compact M, the first Chern class $c_1 : \mathrm{Pic}(M) \to H^2(M;\mathbb{Z})$ is an isomorphism taking tensor products of line bundles to sums of cohomology classes, as we have

$$\mathrm{Pic}(M) \cong [M, \mathrm{BU}_1] \cong [M, K(\mathbb{Z}, 2)] \cong H^2(M;\mathbb{Z}).$$

We now establish the equivariant version of this statement.

Theorem A.1. *Let G be a compact Lie group acting on a compact manifold M. The equivariant first Chern class*

$$c_1^G : \mathrm{Pic}_G(M) \to H^2_G(M;\mathbb{Z}) \tag{A.1}$$

is an isomorphism between the equivariant Picard group of isomorphism classes of complex G-line bundles on M and the equivariant cohomology $H^2_G(M;\mathbb{Z}) = H^2(M_G;\mathbb{Z})$, defined in terms of the Borel construction $M_G = EG \times_G M$.

Proof. To prove this result, we compare the G-equivariant first Chern class (A.1) on M to the standard first Chern class for the Borel construction M_G. We recall that the K-theory of a general (possibly non-compact) space X is defined homotopically by

$$K(X) := [X, \mathbb{Z} \times \mathrm{BU}].$$

For any X, the first Chern class gives us a retraction

$$c_1 : K(X) \to H^2(X;\mathbb{Z}), \tag{A.2}$$

taking the homotopy class $[f]$ of a map $f : X \to \mathbb{Z} \times \mathrm{BU}$ to the pullback f^*a of the preferred generator $a \in H^2(\mathrm{BU};\mathbb{Z})$. (Alternatively, we can take the composition of f with a representative of the class $a \in [\mathrm{BU}, K(\mathbb{Z}, 2)]$, whose homotopy class gives $c_1[f] \in [X, K(\mathbb{Z}, 2)]$.) The right inverse of the homomorphism (A.2) is the injection

$$H^2(X;\mathbb{Z}) \cong [X, \mathrm{BU}_1] \longrightarrow [X, \mathbb{Z} \times \mathrm{BU}] \cong K(X), \tag{A.3}$$

induced by the inclusion $U_1 \hookrightarrow U$, with BU_1 mapping into the $1 \times \mathrm{BU}$ connected component. We observe that if X is compact, then this homomorphism (A.3) is the composition

$$H^2(X;\mathbb{Z}) \xrightarrow{c_1^{-1}} \mathrm{Pic}(X) \longrightarrow K(X)$$

of the inverse of the first Chern class with the homomorphism taking a complex line bundle L to its K-theory class $[L]$. However, we shall consider these maps in the non-compact case $X = M_G$. On the other hand, the equivariant Picard group $\mathrm{Pic}_G(M)$ is a retract of the equivariant K-theory $K_G(M)$, as we discussed in the proof of Theorem 4.7.[1]

We recall that the equivariant Chern classes of a G-bundle $E \to M$ are defined by

$$c_i^G(E) := c_i(EG \times_G E) \in H^{2i}(M_G) = H_G^{2i}(M).$$

In terms of K-theory, let $\alpha : K_G(M) \to K(M_G)$ denote the Atiyah-Segal homomorphism, induced by the map taking a G-bundle E to the associated bundle $EG \times_G E$ over $M_G = EG \times_G M$. We then have $c_i^G[E] = c_i(\alpha[E])$, which gives us the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}_G(M) \arrow[r, "c_1^G"] \arrow[d, hook] & H_G^2(M;\mathbb{Z}) \arrow[d, hook] \\
K_G(M) \arrow[r, "\alpha"] \arrow[d, two heads, "\det"'] & K(M_G) \arrow[d, two heads, "c_1"] \\
\mathrm{Pic}_G(M) \arrow[r, "c_1^G"] & H_G^2(M;\mathbb{Z})
\end{tikzcd}$$

where both vertical compositions are identity maps. From this we observe that the equivariant first Chern class c_1^G is a retract of the Atiyah-Segal homomorphism α.

In Lemma A.2 below, we prove that the determinant homomorphism extends to the completion $K_G(M)^\wedge$ of the equivariant K-theory $K_G(M)$ with respect to the augmentation ideal $I(G) \cdot K_G(M)$. It follows that $\mathrm{Pic}_G(M)$ is a retract of $K_G(M)^\wedge$ as well. Our commutative diagram then becomes:

$$\begin{tikzcd}
\mathrm{Pic}_G(M) \arrow[rr, "c_1^G"] \arrow[d, hook] & & H_G^2(M;\mathbb{Z}) \arrow[d, hook, dashed] \\
K_G(M) \arrow[r, hook] \arrow[d, two heads, "\det"'] & K_G(M)^\wedge \arrow[r, dashed, "\widehat{\alpha}", "\cong"'] \arrow[dl, dashed, two heads, "\widehat{\det}"'] & K(M_G) \arrow[d, two heads, "c_1"] \\
\mathrm{Pic}_G(M) \arrow[rr, "c_1^G"] & & H_G^2(M;\mathbb{Z})
\end{tikzcd}$$

The equivariant first Chern class c_1^G is therefore a retract of the map $\widehat{\alpha} : K_G(M)^\wedge \to K(M_G)$, which by the Atiyah-Segal completion theorem is an isomorphism (see [9]). As a consequence, the map c_1^G is an isomorphism, with inverse given by the composition of the dashed arrows. $\square$

To complete the proof, we must establish the following technical lemma, whose proof requires only linear algebra.

Lemma A.2. *The determinant homomorphism* $\det : K_G(M) \to \mathrm{Pic}_G(M)$ *extends to a homomorphism* $\widehat{\det} : K_G(M)^\wedge \to \mathrm{Pic}_G(M)$ *on the completion at the augmentation ideal.*

[1]The map $c_1 : K(M_G) \to H_G^2(X;\mathbb{Z})$ can also be interpreted geometrically as the map $\det : K(M_G) \twoheadrightarrow \mathrm{Pic}(M_G)$.

Proof. Consider the augmentation homomorphism $K_G \to \mathbb{Z}$ taking a virtual G-module to its integral dimension. The augmentation ideal $I(G)$ is the kernel of this map. The completion of the equivariant K-theory with respect to the $I(G)$-adic topology is given by

$$K_G(M)^\wedge = \varprojlim K_G(M) \,/\, I(G)^n \cdot K_G(M).$$

We claim that the determinant is trivial on $I(G)^2 \cdot K_G(M)$, and it thus extends to the $I(G)$-adic completion $K_G(M)^\wedge$. Let $E \cong P_1 \oplus \cdots \oplus P_p$ and $F \cong Q_1 \oplus \cdots \oplus Q_q$ be two complex G-bundles, each splitting as a direct sum of complex G-line bundles. Their tensor product is then $E \otimes F \cong \bigoplus_{i,j} P_i \otimes Q_j$, which has determinant

$$\det(E \otimes F) = \Big(\prod\nolimits_i P_i^{\otimes q}\Big) \otimes \Big(\prod\nolimits_j Q_j^{\otimes p}\Big) = (\det E)^{\operatorname{rank} F} \otimes (\det F)^{\operatorname{rank} E}. \tag{A.4}$$

By the equivariant splitting principle (see [36]), the formula (A.4) holds for all G-bundles. Furthermore, since the determinant of a sum is the product of the determinants in the Picard group, we can extend the formula (A.4) to virtual G-bundles in place of E and F. In particular if V, W are both virtual G-modules with rank 0, and E is an arbitrary virtual G-bundle, then $W \otimes E$ has rank 0, and we obtain

$$\det(V \otimes W \otimes E) = (\det V)^0 \otimes (\det W \otimes E)^0 = 1,$$

the identity element in $\mathrm{Pic}_G(M)$. □

Theorem A.3. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let M be a compact Hamiltonian G-space. Then we have a short exact sequence*

$$0 \longrightarrow H^2_G(\mathrm{pt}) \otimes H^0(M;\mathbb{Z}) \longrightarrow H^2_G(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow 0, \tag{A.5}$$

i.e., the forgetful map is surjective and its kernel is isomorphic to a direct sum of one copy of $H^2_G(\mathrm{pt};\mathbb{Z})$ for each connected component of M. From this, we obtain the following:

1. *Every cohomology class in $H^2(M;\mathbb{Z})$ admits an equivariant extension.*
2. *If M is connected, then the isomorphism classes of the equivariant structures on a given cohomology class in $H^2(M;\mathbb{Z})$ are in one-to-one correspondence with $H^2_G(\mathrm{pt};\mathbb{Z})$.*
3. *If G is semi-simple, then the forgetful map is an isomorphism, and thus every class in $H^2(M;\mathbb{Z})$ admits a unique equivariant extension.*

Proof. These results follow immediately from Theorem 4.7, together with the first Chern class isomorphisms $\mathrm{Pic}(M) \cong H^2(M;\mathbb{Z})$ and $\mathrm{Pic}_G(M) \cong H^2_G(M;\mathbb{Z})$ from Theorem A.1. □

Remark A.4. Alternatively, we can prove Theorem A.3 using the Leray-Serre spectral sequence for the fibration $M \to M_G \to BG$. As we noted in Remark 4.6, a Hamiltonian G-space M is equivariantly formal for rational cohomology, so with rational coefficients this spectral sequence collapses at the E_2 term, and the rational differentials $d_k^{\mathbb{Q}}$ all vanish. In particular, the image of the forgetful map $H^2_G(M) \to H^2(M)$ is $E_\infty^{0,2}$, and thus this map is surjective if and only if $H^2(M)$ persists to E_∞, or equivalently if all differentials d_k vanish on $H^2(M)$. Using integral cohomology, since $\pi_1(G)$ is free, we see that $H_1(G;\mathbb{Z})$ is free, and that $H^2(G;\mathbb{Z})$ is likewise free by the universal coefficient theorem. Since G is connected, its classifying space BG is simply connected, and it follows that $H^1(BG;\mathbb{Z}) = H^3(BG;\mathbb{Z}) = 0$ and $H^2(BG;\mathbb{Z})$ is free. Applying the universal coefficient theorem once again, we see that the cohomology $H^1(M;\mathbb{Z})$ of the fiber is also free. The E_2 page is then

$$\begin{tikzcd}
H^2(M;\mathbb{Z}) \arrow[drr, "d_2"] \arrow[ddrrr, "d_3"] & & & \\
H^1(M;\mathbb{Z}) & 0 & H^2\big(BG;\, H^1(M;\mathbb{Z})\big) & 0 \\
H^0(M;\mathbb{Z}) & 0 & H^2\big(BG;\, H^0(M;\mathbb{Z})\big) & 0
\end{tikzcd}$$

where the transgression d_3 actually acts on the kernel of d_2. Since the image of d_2 is free, and since tensoring with the rationals gives $(\operatorname{Im} d_2) \otimes \mathbb{Q} = \operatorname{Im} d_2^{\mathbb{Q}} = 0$, we see that d_2 vanishes on $H^2(M;\mathbb{Z})$. Furthermore, the transgressions d_3 and d_k for $k > 3$ all clearly vanish on $H^2(M;\mathbb{Z})$. As a result, all of $H^2(M;\mathbb{Z})$ persists to E_∞, and thus $H^2_G(M;\mathbb{Z})$ surjects onto $H^2(M;\mathbb{Z})$.

By a similar argument, the transgression $d_2 : H^1(M;\mathbb{Z}) \to H^2(BG; H^0(M;\mathbb{Z}))$ vanishes as its range is free, and so $H^2(BG; H^0(M;\mathbb{Z}))$ persists to E_∞. We thus obtain the short exact sequence

$$0 \longrightarrow H^2\big(BG; H^0(M;\mathbb{Z})\big) \longrightarrow H^2(M_G;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}) \longrightarrow 0,$$

which is equivalent to (A.5). We note that this argument shows explicitly why we require that G have a free fundamental group, as well as giving a different formulation of the kernel of the forgetful map.

Appendix B. Morse-Kirwan functions

In [24, §10], Kirwan extends the results of Morse theory (or more properly, Morse-Bott theory) to a larger class of Morse functions, which she calls "minimally degenerate" and which are now commonly called Morse-Kirwan functions (or sometimes "Morse in the sense of Kirwan").

Definition B.1. A *Morse-Kirwan function* on a manifold M is a smooth function $f : M \to \mathbb{R}$ whose critical set decomposes as the union of disjoint closed critical subsets on which f takes constant values, such that each of these critical subsets C is contained in a locally closed submanifold Σ_C with the following properties:

(1) The critical subset C is the subset of Σ_C minimizing the function f.
(2) The neighborhood Σ_C has an orientable normal bundle.
(3) The tangent space $T_x\Sigma_C$ to Σ_C at $x \in C$ is maximal among all subspaces of the tangent space T_xM on which the Hessian $H_x(f)$ is positive semidefinite.

For such a Morse-Kirwan function, the critical sets may not be manifolds, as they are for Morse-Bott functions, but rather they may have singularities. However, we are allowed to resolve these singularities by flowing upwards slightly to obtain a manifold neighborhood of the critical set, whose normal bundle consists of the downward directions needed for Morse theory.

Kirwan shows in [24, §10] that such minimally degenerate Morse functions yield smooth stratifications, and thus a form of Morse theory still holds, at least at the level of the stratification. The only complication is that for a general Morse-Kirwan function, the negative gradient flow of a point need not converge to a unique limit; rather, its set of limit points is a closed, connected subset of the critical set of f. (On the other hand, the gradient flow of the norm square of the moment map does indeed converge, as described in an unpublished work of Duistermaat, reproduced in [28], and also proved in [41, Appendix A].) Instead, Kirwan argues that the inclusion $C \hookrightarrow S_C$ of a critical set into its stratum is a retraction in (equivariant) Čech cohomology. We now show, reprising carefully Kirwan's argument from [24, §10], that the inclusion $C \hookrightarrow S_C$ is also a retraction in (equivariant) K-theory. Let M be a G-manifold, and consider a G-invariant Morse-Kirwan function $f : M \to \mathbb{R}$. For small $\delta \geq 0$, we consider the compact neighborhood

$$N_\delta = \{x \in S_C \,|\, f(x) \leq f(C) + \delta\}$$

of C in S_C. Note that $N_0 = C$, and in the following we approximate the critical set C as $\lim_{\delta \to 0} N_\delta$.

Lemma B.2. *The gradient flow of f induces a G-equivariant deformation retraction of the stratum S_C onto each of the neighborhoods N_δ for $\delta > 0$.*

Proof. Let $\gamma : M \times \mathbb{R}^{\geq 0} \to M$ denote the gradient flow of f, satisfying $\partial_t \gamma_t(x) = -(\operatorname{grad} f)|_{\gamma_t(x)}$. Fixing $\delta > 0$, for each point $x \in S_C$, we define

$$\tau(x) := \inf\{t \geq 0 \,|\, \gamma(x,t) \in N_\delta\}$$

to be the time at which the gradient flow of x first enters the neighborhood N_δ. In particular, we have $\tau(x) = 0$ if and only if $x \in N_\delta$, and we see that $\tau(x)$ is well defined as S_C consists of all points which flow down to the critical set C. Letting $\epsilon : [0,1) \to [0,\infty)$ be the homeomorphism $\epsilon(s) = s/(1-s)$, we define a homotopy $F : S_C \times [0,1] \to S_C$ by

$$F_s(x) = \begin{cases} \gamma_{\tau(x)}(x) & \text{if } s = 1 \text{ or } \epsilon(s) \geq \tau(x), \\ \gamma_{\epsilon(s)}(x) & \text{otherwise.} \end{cases}$$

Then $F_0(x)$ is the identity map on S_C, as $\epsilon(0) = 0$ and $\gamma_0(x) = x$. On the other hand, the map $F_1(x)$ is the retraction of the stratum S_C onto the neighborhood N_δ obtained by taking each $x \in S_C$ to the point at which its gradient flow first enters N_δ.

As for the G-equivariance, since the Morse-Kirwan function f is G-invariant, the G-action maps the neighborhood N_δ to itself, and the gradient flow $\gamma_t(x)$ is G-equivariant. For $x \in S_C$ and $g \in G$, it follows that $\tau(gx) = \tau(x)$, and thus $\gamma_{\tau(gx)}(gx) = g\gamma_{\tau(x)}(x)$. The homotopy $F_s(x)$ is therefore a G-equivariant deformation retraction of S_C onto N_δ. □

Lemma B.3. *The inclusion $C \hookrightarrow S_C$ of a critical set into its stratum induces an isomorphism $K^*_G(S_C) \cong K^*_G(C)$ in equivariant K-theory.*

Proof. By Lemma B.2, the inclusions $N_\delta \hookrightarrow S_C$ induce isomorphisms $K^*_G(S_C) \cong K^*_G(N_\delta)$ in equivariant K-theory for all $\delta > 0$. To compute the equivariant K-theory of the critical set, $K^*_G(C) = K^*_G(N_0)$, observe that the compact neighborhoods N_δ form a directed inverse system with inverse limit C as $\delta \to 0$:

$$C = \varprojlim N_\delta.$$

In our case, we have inclusions $N_{\delta'} \hookrightarrow N_\delta$ for $\delta' < \delta$ and the simplified condition that $C = \bigcap_{\delta>0} N_\delta$. Furthermore, it follows from Lemma B.2 that the corresponding restriction maps $K_G^*(N_\delta) \to K_G^*(N_{\delta'})$ are isomorphisms. So, we invoke the continuity axiom for equivariant K-theory, which asserts that for compact Hausdorff spaces, the equivariant K-theory of an inverse limit is the direct limit of the equivariant K-theories (see [36, Proposition 2.11]), to obtain the isomorphism

$$K_G^*(C) = K_G^*\big(\varprojlim N_\delta\big) = \varinjlim K_G^*(N_\delta) \cong \varinjlim K_G^*(S_C) = K_G^*(S_C)$$

induced by the inclusion $C \hookrightarrow S_C$. □

Remark B.4. To establish the continuity axiom in equivariant K-theory, we recall from [12] that the continuity axiom is equivalent to the extension and reduction theorems. Let A be a closed subset of a compact Hausdorff space X. The extension theorem says that any K-theory class on A can be extended to a class on a compact neighborhood of A. The reduction theorem says that if a K-theory class on X vanishes when restricted to A, then it likewise vanishes on some compact neighborhood of A. However, for K-theory, these results are both true at the level of vector bundles. See for example the texts [5, §1.4] or [20, proof of Proposition 2.9] for discussions of reduction for standard K-theory. These results then remain true when passing to the Grothendieck group. Segal argues along these lines in [36], in which he proves the continuity property of equivariant K-theory in the case we require above where the inverse limit is actually the intersection.

We expect a similar form of Morse theory to hold for general Morse-Kirwan functions in any cohomology theory which has Thom classes and satisfies the continuity axiom. In addition, when working with the norm square of the moment map, we can drop the continuity condition.

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