TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 12, December 2007, Pages 5959–5989
S 0002-9947(07)04252-3
Article electronically published on June 26, 2007

BIG COHEN-MACAULAY ALGEBRAS AND SEEDS

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ABSTRACT. In this article, we delve into the properties possessed by algebras, which we have termed *seeds*, that map to big Cohen-Macaulay algebras. We will show that over a complete local domain of positive characteristic any two big Cohen-Macaulay algebras map to a common big Cohen-Macaulay algebra. We will also strengthen Hochster and Huneke's "weakly functorial" existence result for big Cohen-Macaulay algebras by showing that the seed property is stable under base change between complete local domains of positive characteristic. We also show that every seed over a positive characteristic ring (R, m) maps to a balanced big Cohen-Macaulay R-algebra that is an absolutely integrally closed, m-adically separated, quasilocal domain.

1. INTRODUCTION

Methods used by M. Hochster and C. Huneke during their development of tight closure led to the remarkable result that R^+ (the integral closure of a domain R in an algebraic closure of its fraction field) is a *balanced big Cohen-Macaulay algebra* over R when R is an excellent local domain of positive characteristic [HH2]. A *big Cohen-Macaulay algebra* over a local ring (R, m) is an R-algebra B such that some system of parameters of R is a regular sequence on B. It is *balanced* if *every* system of parameters of R is a regular sequence on B. While Hochster had shown the existence of big Cohen-Macaulay modules in equal characteristic [Ho1], this new result was the first proof that big Cohen-Macaulay algebras existed. Big Cohen-Macaulay algebras also exist in equal characteristic 0 [HH5] and in mixed characteristic when $\dim R \leq 3$ [Ho3]. (The latter result follows from Heitmann's proof of the direct summand conjecture for mixed characteristic rings of dimension three [Heit].) Their existence is important as it gives new proofs for many of the local homological conjectures, such as the direct summand conjecture, monomial conjecture, and vanishing conjecture for maps of Tor.

After defining *seeds* as algebras over a local ring R that map to a (balanced) big Cohen-Macaulay R-algebra and presenting some basic properties, we characterize seeds in terms of the existence of *durable colon-killers* (see Theorem 4.8) for positive characteristic rings. One of our most useful results is given in Theorem 6.9, where we show that if R is a local Noetherian ring of positive characteristic, S is a seed over R, and T is an integral extension of S, then T is also a seed over R. We can view this theorem as a generalization of the existence of big Cohen-Macaulay algebras over complete local domains of positive characteristic; see Corollary 6.11.

Received by the editors August 22, 2005.
2000 *Mathematics Subject Classification.* Primary 13C14, 13A35; Secondary 13H10, 13B99.
Key words and phrases. Big Cohen-Macaulay algebras, tight closure.
The author was supported in part by a VIGRE grant from the National Science Foundation.

We also define a class of *minimal seeds*, which, analogously to the *minimal solid algebras* of Hochster (see [Ho2]), are seeds that have no proper homomorphic image that is also a seed. While it has only been shown that Noetherian solid algebras map onto minimal solid algebras in [Ho2], we have shown in Proposition 5.4 that every seed maps onto a minimal seed. Furthermore, like minimal solid algebras, we have shown that minimal seeds are domains in positive characteristic; see Proposition 7.2. As a result, every seed in positive characteristic over a local ring (R, m) maps to a balanced big Cohen-Macaulay algebra that is a quasilocal, absolutely integrally closed, m-adically separated domain; see Theorem 7.8.

Among the most intriguing results of this article are Theorems 8.4 and 8.10. Both results concern seeds over a complete local domain R of positive characteristic. Theorem 8.4 shows that the tensor product of two seeds is still a seed, just as the tensor product of two solid modules is still solid (see Proposition 2.14 here). As an immediate consequence, if B and B' are big Cohen-Macaulay algebras over R, then B and B' both map to a common big Cohen-Macaulay algebra C, which shows that the class of big Cohen-Macaulay algebras over R forms a directed system in this sense.

Theorem 8.10 shows that if $R \to S$ is a map of positive characteristic complete local domains, and T is a seed over R, then $T \otimes_R S$ is a seed over S, so that the seed property is stable under this manner of base change. This theorem greatly strengthens the "weakly functorial" existence result of Hochster and Huneke in [HH5, Theorem 3.9], where they show that given complete local domains of equal characteristic $R \to S$, there exists a balanced big Cohen-Macaulay R-algebra B and a balanced big Cohen-Macaulay S-algebra C such that $B \to C$ extends the map $R \to S$. Our theorem shows that if we have $R \to S$ in positive characteristic, and B is *any* big Cohen-Macaulay R-algebra, then there exists a big Cohen-Macaulay S-algebra C that fills in a commutative square:

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S. \arrow[u] \end{tikzcd}$$

2. Notation and background

All rings throughout are commutative with identity. All modules are unital.

2.1. Big Cohen-Macaulay modules and algebras.

Definition 2.1. A *big Cohen-Macaulay module* M over a local Noetherian ring (R, m) is an R-module such that some system of parameters for R is a regular sequence on M. It is *balanced* if every system of parameters is a regular sequence. (Since the definition of a regular sequence $x_1, \ldots, x_d$ requires that $(x_1, \ldots, x_d)M \neq M$, we have $mM \neq M$.) If $M = B$ is an R-algebra, then B is called a (balanced) *big Cohen-Macaulay algebra*.

The terminology "big" refers to the fact that M is not necessarily a finitely generated R-module. We say that a partial system of parameters of (R, m) is a *possibly improper regular sequence* on M if all relations on the parameters are trivial, but $mM \neq M$ does not necessarily hold. Similarly, we can have *possibly improper big Cohen-Macaulay modules*.

The question "When do big Cohen-Macaulay modules or algebras exist?" has important applications in commutative algebra. Over regular rings, the answer is simple.

Proposition 2.2 (p.77, [HH2]). *Let R be a regular Noetherian ring, and let M be an R-module. Then M is a balanced big Cohen-Macaulay module over R if and only if M is faithfully flat over R.*

In [Ho1], Hochster showed that big Cohen-Macaulay modules exist over all equicharacteristic local Noetherian rings. The first significant existence proof of big Cohen-Macaulay algebras came from a celebrated theorem of Hochster and Huneke. For a domain R, let R^+ denote the integral closure of R in an algebraic closure of its fraction field. The ring R^+ is called the *absolute integral closure* of R and is not Noetherian in general.

Theorem 2.3 (Theorem 5.15, [HH2]). *If R is an excellent local domain, then R^+ is a balanced big Cohen-Macaulay R-algebra.*

Using this result, Hochster and Huneke were also able to establish a "weakly functorial" existence of big Cohen-Macaulay algebras over all equicharacteristic local rings. The term *permissible* used in the following theorem refers to a map $R \to S$ such that every minimal prime Q of $\widehat{S}$, with $\dim \widehat{S}/Q = \dim \widehat{S}$, lies over a prime P of $\widehat{R}$ that contains a minimal prime $\mathfrak{p}$ of $\widehat{R}$ satisfying $\dim \widehat{R}/\mathfrak{p} = \dim \widehat{R}$.

Theorem 2.4 (Theorem 3.9, [HH5]). *We may assign to every equicharacteristic local ring R a balanced big Cohen-Macaulay R-algebra $\mathscr{B}(R)$ in such a way that if $R \to S$ is a permissible local homomorphism of equicharacteristic local rings, then we obtain a homomorphism $\mathscr{B}(R) \to \mathscr{B}(S)$ and a commutative diagram:*

$$\begin{tikzcd} \mathscr{B}(R) \arrow[r] & \mathscr{B}(S) \\ R \arrow[u] \arrow[r] & S. \arrow[u] \end{tikzcd}$$

A key tool in the proof of this result is the construction of big Cohen-Macaulay algebras using *algebra modifications*. Since we will make great use of algebra modifications in this article, it will be helpful to review some definitions and useful properties now.

Definition 2.5. Given a local Noetherian ring R, an R-algebra S, and a relation $sx_{k+1} = \sum_{i=1}^{k} x_i s_i$ in S, where $x_1, \dots, x_{k+1}$ is part of a system of parameters of R, the S-algebra

$$T := \frac{S[U_1, \dots, U_k]}{s - \sum_{i=1}^{k} x_i U_i} \tag{2.6}$$

is called an *algebra modification of S over R.*

We can also construct an S-algebra $\mathrm{Mod}(S/R) = \mathrm{Mod}_1(S/R)$ by adjoining infinitely many indeterminates and killing the appropriate relations (as above) so that *every* relation in S on a partial system of parameters from R is trivialized in $\mathrm{Mod}_1(S/R)$. Inductively define $\mathrm{Mod}_n(S/R) = \mathrm{Mod}(\mathrm{Mod}_{n-1}(S/R))$ and then define $\mathrm{Mod}_\infty(S/R)$ as the direct limit of the $\mathrm{Mod}_n(S/R)$. We have now formally trivialized all possible relations on systems of parameters from R and done so in a way that is universal in the following sense.

Proposition 2.7 (Proposition 3.3b, [HH5]). *Let S be an algebra over the local Noetherian ring R. Then* $\mathrm{Mod}_\infty(S/R)$ *is a possibly improper balanced big Cohen-Macaulay R-algebra. It is a proper balanced big Cohen-Macaulay algebra if and only if S maps to some balanced big Cohen-Macaulay R-algebra.*

While $\mathrm{Mod}_\infty(S/R)$ is a rather large and cumbersome object, we can study it in terms of finite sequences of algebra modifications. Given a Noetherian local ring (R, m) and an R-algebra S, we set $S^{(0)} := S$ and then inductively define $S^{(i+1)}$ to be an algebra modification of $S^{(i)}$ over R. We then obtain a finite *sequence of algebra modifications*

$$S = S^{(0)} \to S^{(1)} \to \cdots \to S^{(h)}$$

for any $h \in \mathbb{N}$. We call such a sequence *bad* if $mS^{(h)} = S^{(h)}$. We will frequently use the following proposition.

Proposition 2.8 (Proposition 3.7, [HH5]). *Let R be a local Noetherian ring, and let S be an R-algebra.* $\mathrm{Mod}_\infty(S/R)$ *is a proper balanced big Cohen-Macaulay R-algebra if and only if no finite sequence of algebra modifications is bad.*

2.2. **The Frobenius endomorphism.** Throughout this article we will often work with rings of positive prime characteristic p. We will always let e denote a nonnegative integer and let q denote p^e, a power of p. Thus, the phrase "for all q" will mean "for all powers $q = p^e$ of p."

Every characteristic p ring R comes equipped with a *Frobenius endomorphism* $F_R : R \to R$, which maps $r \mapsto r^p$. We can compose this map with itself to obtain the iterations $F_R^e : R \to R$, which map $r \mapsto r^q$. Associated to these maps are the *Peskine-Szpiro* (or *Frobenius*) *functors* $\mathbf{F}_R^e$. If we let S denote the ring R viewed as an R-module via the e^{th}-iterated Frobenius endomorphism, then $\mathbf{F}_R^e$ is the covariant functor $S \otimes_R -$ which takes R-modules to S-modules and so takes R-modules to R-modules since $S = R$ as a ring. Specifically, if $R^m \to R^n$ is a map of free R-modules given by the matrix (r_{ij}), then we may apply $\mathbf{F}_R^e$ to this map to obtain a map between the same R-modules given by the matrix (r_{ij}^q). For cyclic modules R/I, $\mathbf{F}_R^e(R/I) = R/I^{[q]}$, where

$$I^{[q]} := (a^q \mid a \in I)R$$

is the q^{th} *Frobenius power* of the ideal I. For modules $N \subseteq M$, we will denote the image of $\mathbf{F}_R^e(N)$ in $\mathbf{F}_R^e(M)$ by $N_M^{[q]}$, and we will denote the image of $u \in N$ inside of $N_M^{[q]}$ by u^q.

2.3. **Tight closure.** The operation of tight closure was developed by Hochster and Huneke in the late 1980s and early 1990s as a method for proving (often reproving with dramatically shorter proofs) and generalizing theorems for rings containing a positive characteristic field. See [HH1] for an introduction.

We will denote the complement in R of the set of minimal primes by R°.

Definition 2.9. For a Noetherian ring R of characteristic $p > 0$ and finitely generated modules $N \subseteq M$, the *tight closure* N_M^* of N in M is

$$N_M^* := \{u \in M \mid cu^q \in N_M^{[q]} \text{ for all } q \gg 1, \text{ for some } c \in R^\circ\}.$$

In the case that $M = R$ and $N = I$, $u \in I^*$ if and only if there exists $c \in R^\circ$ such that $cu^q \in I^{[q]}$, for all $q \gg 1$.

A very powerful tool in the application of tight closure is the notion of a *test element*.

Definition 2.10. For a Noetherian ring of positive characteristic p, an element $c \in R^\circ$ is called a *test element* if for all ideals I and all $u \in I^*$, we have $cu^q \in I^{[q]}$ for all $q \geq 1$. If, furthermore, c is a test element in every localization of R, then c is *locally stable*, and if it is also a test element in the completion of every localization, then c is *completely stable*.

Test elements exist in very general settings.

Theorem 2.11 (Theorem 6.1, [HH4])**.** *If R is a reduced, excellent local ring of positive characteristic, then R has a completely stable test element.*

Below is a collection of important tight closure results that we will refer to during this article. These properties help make tight closure a "good" closure operation for solving problems related to the local homological conjectures in positive characteristic.

Theorem 2.12. *Let R, S be Noetherian rings of positive prime characteristic p, and let $N \subseteq M$ be finitely generated R-modules.*

(a) [HH1, Theorem 4.4, Proposition 8.7] *If R is regular, then $N^*_M = N$.*
(b) ***(colon-capturing)*** [HH1, Theorem 4.7] *Let R be module-finite and torsion-free over a regular domain A. Let $x_1, \ldots, x_n \in A$ be parameters in R. Then $(x_1, \ldots, x_{n-1})R :_R x_n \subseteq ((x_1, \ldots, x_{n-1})R)^*$.*
(c) ***(persistence)*** [HH4, Theorem 6.24] *Let R, S be excellent rings with $R \to S$ a homomorphism. If $w \in N^*_M$, then $1 \otimes w \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^*_{S \otimes_R M}$.*
(d) [HH3, Corollary 5.23] *Let S be a module-finite extension of R. If $1 \otimes u \in \mathrm{Im}(S \otimes_R N \to S \otimes_R M)^*_{S \otimes_R M}$, calculated over S, then $u \in N^*_M$. In particular, $(IS)^*_S \cap R \subseteq I^*_R$ for all ideals I of R.*

2.4. **Solid algebras and modules.** Hochster introduced the notion of *solid* modules and algebras in [Ho2] in an attempt to define a characteristic free notion of tight closure.

Definition 2.13. If R is a Noetherian domain, then an R-module M is *solid* if $\mathrm{Hom}_R(M, R) \neq 0$. If $M = S$ is an R-algebra, then S is *solid* over R if it is solid as an R-module.

The following properties have analogues in the world of seeds and big Cohen-Macaulay modules. See Corollary 4.6 and Theorems 8.4 and 8.10.

Proposition 2.14 (Section 2, [Ho2])**.** *Let R be a Noetherian domain.*

(a) *Let S be a module-finite domain extension of R, and let M be an S-module. Then M is solid over S if and only if M is solid over R.*
(b) *If M and N are solid R-modules, then $M \otimes_R N$ is solid.*
(c) ***(persistence of solidity)*** *Let $R \to S$ be any map of Noetherian domains. If M is a solid R-module, then $S \otimes_R M$ is a solid S-module.*

Whether seeds and solid algebras are the same in positive characteristic is an interesting open question. An answer to one direction was given by Hochster.

Theorem 2.15 (Corollary 10.6, [Ho2])**.** *Let R be a complete local domain. An R-algebra that has an R-algebra map to a big Cohen-Macaulay algebra over R is solid.*

We also need a connection between tight closure and big Cohen-Macaulay algebras.

Theorem 2.16 (Theorem 11.1, [Ho2])**.** *Let R be a complete local domain of positive characteristic, and let $N \subseteq M$ be finitely generated R-modules. If $u \in M$, then $u \in N^*_M$ if and only if there exists a (balanced) big Cohen-Macaulay R-algebra B such that $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$.*

3. Definition and properties of seeds

Definition 3.1. For any local Noetherian ring (R, m), an R-algebra S is called a *seed* over R if S maps to a big Cohen-Macaulay R-algebra.

Bartijn and Strooker [BS, Theorem 1.7] show that the m-adic separated completion of a big Cohen-Macaulay algebra (module) is a balanced big Cohen-Macaulay algebra (module). The definition of seed is then unchanged by requiring a balanced big Cohen-Macaulay algebra. Using this terminology, Proposition 2.8 implies that S is a seed if and only if S does not have a bad sequence of algebra modifications. Based on this characterization of seeds, a direct limit of seeds is still a seed.

Lemma 3.2. *Let (R, m) be a local Noetherian ring, and let $S = \varinjlim_\lambda S_\lambda$ be a direct limit of a directed set of R-algebras. Then S is a seed if and only if each S_λ is a seed.*

Proof. Since S is an S_λ-algebra for all λ, if S is a seed, then so is each S_λ. Conversely, suppose that S is not a seed. We will find an S_λ that also has a bad sequence of modifications.

As S is not a seed, it has a bad sequence of algebra modifications. It is then straightforward to show that this bad sequence over S induces a bad sequence of modifications over some S_λ because a finite sequence involves only finitely many relevant elements. These elements generate an S_λ over R that is not a seed. $\square$

In positive characteristic, we can use the Frobenius endomorphism and its iterates to map any seed to a reduced and perfect balanced big Cohen-Macaulay algebra. We will let R^∞ denote the direct limit of the directed system

$$R \to \mathbf{F}(R) \to \mathbf{F}^2(R) \to \cdots \to \mathbf{F}^e(R) \to \cdots,$$

where $\mathbf{F}^e$ is the iterated Peskine-Szpiro functor. If R is reduced, then R^∞ can be obtained by adjoining all q^{th} roots to R. Notice that $R^\infty = (R_{\mathrm{red}})^\infty$, where R_{red} is the quotient of R obtained by killing all nilpotents, and that R^∞ is always reduced and contains all of its q^{th} roots.

Lemma 3.3. *Let R be a local Noetherian ring of positive characteristic p. If B is a balanced big Cohen-Macaulay R-algebra, then B^∞ is a reduced balanced big Cohen-Macaulay algebra containing all of its q^{th} roots. Moreover, if B is quasilocal, then B^∞ is also quasilocal.*

Proof. A direct limit of balanced big Cohen-Macaulay algebras is a balanced big Cohen-Macaulay algebra. If B is quasilocal, then all maps in the direct limit are local so that B^∞ is quasilocal. $\square$

We will show later (Proposition 7.6) that any reduced seed (in any characteristic) can be modified into a reduced balanced big Cohen-Macaulay algebra.

Lemma 3.4. *Let (R, m) be a local Noetherian ring. If B is a (balanced) big Cohen-Macaulay R-algebra and $\mathfrak{p}$ is any prime ideal of B containing mB, then $B_{\mathfrak{p}}$ is also a (balanced) big Cohen-Macaulay R-algebra. Moreover, if B is reduced (resp., R has positive characteristic and B is reduced and perfect), then $B_{\mathfrak{p}}$ is still reduced (resp., reduced and perfect).*

Proof. Given $x_{k+1}(r/u) \in (x_1, \ldots, x_k)B_{\mathfrak{p}}$, where $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, we may assume $u = 1$ in showing that the relation is trivial. Therefore, there exists a $v \in B \setminus \mathfrak{p}$ such that $x_{k+1}(rv) \in (x_1, \ldots, x_k)B$. Since B is big Cohen-Macaulay, $rv \in (x_1, \ldots, x_k)B$, and thus $r/1 \in (x_1, \ldots, x_k)B_{\mathfrak{p}}$ as needed. Furthermore, since $mB \neq B$ and $\mathfrak{p} \supseteq mB$, we see that $mB_{\mathfrak{p}} \neq B_{\mathfrak{p}}$, and so $B_{\mathfrak{p}}$ is a (balanced) big Cohen-Macaulay algebra.

The other claims follow from the following easy lemma. □

Lemma 3.5. *If S is any reduced ring and U is a multiplicatively closed set in S, then $U^{-1}S$ is also reduced. If, in addition, S has positive characteristic and is perfect, then $U^{-1}S$ is also perfect.*

Proof. The first claim is well known.

Now suppose that S has positive characteristic and is perfect. Given s/u in $U^{-1}S$, with $s \in S$ and $u \in U$, we have $s = a^q$ and $u = b^q$, where a and b are in S. Then ab^{q-1}/u is an element of $U^{-1}S$ and is a q^{th} root of s/u. □

We can also use the separated completion of a big Cohen-Macaulay R-algebra with respect to the maximal ideal of R to give us an m-adically separated balanced big Cohen-Macaulay algebra while preserving the other properties we have worked with earlier.

Lemma 3.6. *If A is a reduced and perfect ring of positive characteristic p, and I is an ideal of A, then the I-adic completion $\widehat{A}$ of A is reduced and perfect.*

Proof. By definition,

$$\widehat{A} = \{a = (a_1, a_2, a_3, \ldots) \in \prod_j A/I^j \mid a_k \equiv a_j \pmod{I^j}, \quad \forall k > j\}.$$

If $a^n = 0$ in $\widehat{A}$, then there exists q, a power of p, such that $a^q = 0$, so that $a_k^q \in I^k$ for all k. Given any index j, there exists an integer $k(j)$ such that $I^{k(j)} \subseteq (I^j)^{[q]}$. Therefore, for any j, we can find $k(j) \geq j$ such that $a_{k(j)}^q \in (I^j)^{[q]}$. Since A is perfect, $a_{k(j)} \in I^j$, and since $k(j) \geq j$, we have $a_j \in I^j$. Hence, $a = 0$, and $\widehat{A}$ is reduced.

Given $a = (a_1, a_2, \ldots) \in \widehat{A}$, we will now find an element $b \in \widehat{A}$ such that $b^q = a$. Indeed, let $b = (a_{k(1)}^{1/q}, a_{k(2)}^{1/q}, \ldots)$, where $k(j)$ is chosen so that $k(j) \geq j$, $k(j) \geq k(j-1)$, and $I^{k(j)} \subseteq (I^j)^{[q]}$. If $i \geq k(j)$, then $a_i \equiv a_{k(j)} \pmod{I^{k(j)}}$, so that $a_i \equiv a_{k(j)} \pmod{(I^j)^{[q]}}$. Since A is perfect, we can take q^{th} roots to see that $a_i^{1/q} \equiv a_{k(j)}^{1/q} \pmod{I^j}$, which shows that b is a well-defined element of $\widehat{A}$. Finally, $b^q = (a_{j(1)}, a_{j(2)}, \ldots)$, which is easily seen to be equal to a. □

This lemma is the last piece we need to show that seeds map to big Cohen-Macaulay algebras with certain rather useful properties. We will show later that seeds map to big Cohen-Macaulay algebras with even stronger properties. See Theorem 7.8.

Proposition 3.7. *Let (R, m) be a Noetherian local ring of positive characteristic. Every seed over R maps to a balanced big Cohen-Macaulay R-algebra B that is reduced, perfect, quasilocal, and m-adically separated.*

Proof. Use Lemmas 3.3, 3.4, 3.6, and [BS, Theorem 1.7]. □

4. Colon-killers and seeds

Hochster and Huneke used colon-killers (also called *Cohen-Macaulay multipliers*) in [HH2] as tools for proving the existence of big Cohen-Macaulay algebras in positive characteristic. Not surprisingly, the existence of such elements in algebras over a local ring will be useful in determining whether an algebra is a seed or not. We shall work with a generalized version of their definition and will define a special class of colon-killers that will help us determine when an R-algebra is a seed.

Definition 4.1. Let R be a local Noetherian ring, S an R-algebra, and M an arbitrary S-module. An element $c \in S$ is a *colon-killer* for M over R if

$$c((x_1, \ldots, x_k)M :_M x_{k+1}) \subseteq (x_1, \ldots, x_k)M,$$

for each partial system of parameters $x_1, \ldots, x_{k+1}$ in R.

We will soon prove that a colon-killer for an S-module M over R has a power that is a colon-killer for M over S when S is an integral extension of R. First, we need the next lemma connecting colon-killers and Koszul homology.

Lemma 4.2. *Let R be a local Noetherian ring, let S be an R-algebra, and let M be an arbitrary S-module. If $c \in S$ is nonzero, then the following are equivalent:*

(i) *Some power of c is a colon-killer for M over R.*
(ii) *Some power of c kills the Koszul homology modules $H_i(x_1, \ldots, x_k; M)$ for all $i \geq 1$ and all partial systems of parameters $x_1, \ldots, x_k$.*
(iii) *Some power of c kills $H_1(x_1, \ldots, x_k; M)$ for all partial systems of parameters $x_1, \ldots, x_k$.*

Proof. (ii) $\Rightarrow$ (iii) is obvious. For (iii) $\Rightarrow$ (i), let $\mathbf{x} = x_1, \ldots, x_k$ be part of a system of parameters for R, and let $\mathbf{x}' = x_1, \ldots, x_{k-1}$. We obtain a short exact sequence

$$0 \to \frac{H_i(\mathbf{x}'; M)}{x_k H_i(\mathbf{x}'; M)} \to H_i(\mathbf{x}; M) \to \mathrm{Ann}_{H_{i-1}(\mathbf{x}'; M)} x_k \to 0 \tag{4.3}$$

for all i from [BH, Corollary 1.6.13(a)]. In the case $i = 1$, we see that there is a surjection of $H_1(\mathbf{x}; M)$ onto the module $((\mathbf{x}')M :_M x_k)/(\mathbf{x}')M$, which implies that the latter module is killed by the same power of c that kills the former.

For (i) $\Rightarrow$ (ii), assume without loss of generality that c itself is a colon-killer for M over R. We will use induction on k to show that $c^{2^{k-1}}$ kills $H_i(x_1, \ldots, x_k; M)$ for $i \geq 1$. If $k = 1$, then $H_1(x_1; M)$ is the only nonzero Koszul homology module, and it is isomorphic to $\mathrm{Ann}_M x_1 = (0 :_M x_1)$. Since c is a colon-killer for M, c kills $H_1(x_1; M)$. Now let $k \geq 2$, $\mathbf{x} = x_1, \ldots, x_k$, $\mathbf{x}' = x_1, \ldots, x_{k-1}$, and suppose that $c^{2^{k-2}}$ kills $H_i(\mathbf{x}'; M)$ for $i \geq 1$. Using the sequence (4.3), we see that $c^{2^{k-1}}$ kills $H_i(\mathbf{x}; M)$ for all $i \geq 2$ by the inductive hypothesis, and $c^{2^{k-2}+1}$ kills $H_1(\mathbf{x}; M)$ by the inductive hypothesis together with c being a colon-killer. Therefore, if $N = 2^{\dim R - 1}$, then c^N kills all of the relevant Koszul homology modules. □

From this lemma, we can obtain our result on colon-killers.

Proposition 4.4. *Let S be a Noetherian local ring that is an integral extension of a local Noetherian ring R. Let M be an arbitrary S-module. If $c \in R$ kills the Koszul homology modules $H_i(x_1, \dots, x_k; M)$ for all $i \geq 1$ and all partial systems of parameters $x_1, \dots, x_k$ in R, then c^N kills $H_i(y_1, \dots, y_k; M)$ for all $i \geq 1$ and all partial systems of parameters $y_1, \dots, y_k$ in S, for some N. Consequently, if c is a colon-killer for M over R, then a power of c is a colon-killer for M over S.*

Proof. Based on the previous lemma, it is enough to prove the first claim. Let $\mathbf{y} = y_1, \dots, y_k$ be part of a system of parameters for S. Since $R \hookrightarrow S$ is integral, $R/((\mathbf{y}) \cap R) \hookrightarrow S/(\mathbf{y})$ is also integral, so that

$$\dim R/((\mathbf{y}) \cap R) = \dim S/(\mathbf{y}) = \dim S - k = \dim R - k.$$

We claim that $(\mathbf{y}) \cap R$ contains a partial system of parameters $\mathbf{x} = x_1, \dots, x_k$ for R. Indeed, we proceed by induction on k, where the case $k = 0$ is trivial. We can then assume without loss of generality that $k = 1$ and obtain our result from the general fact that if I is an ideal of R such that $\dim R/I < \dim R$, then I contains a parameter of R. If not, then for every $x \in I$, there exists a prime ideal $\mathfrak{p}$ of R such that $x \in \mathfrak{p}$ and $\dim R/\mathfrak{p} = \dim R$. Therefore, I is contained in the union of such prime ideals, and so by prime avoidance, I is contained in a prime $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$. We have a contradiction, which implies that I contains a parameter.

Thus, there exists part of a system of parameters $\mathbf{x} = x_1, \dots, x_k$ of R such that $(\mathbf{x})S \subseteq (\mathbf{y})$. Then a power of c kills $H_i(\mathbf{y}; M)$ for all $i \geq 1$ by the following lemma, where the power depends only on k, which in turn is bounded by $\dim R$. $\square$

Lemma 4.5. *Let S be any ring and M any S-module. Suppose $(x_1, \dots, x_k)S \subseteq (y_1, \dots, y_k)S = (\mathbf{y})S$ and that $c \in S$ kills $H_i(x_1, \dots, x_m; M)$ for all $i \geq 1$ and all $1 \leq m \leq k$. Then $c^{D(m)}$ kills $H_{k+1-m}(\mathbf{y}; M)$ for $1 \leq m \leq k$, where $D(1) = 1$, and $D(m) = 2^{k-2}D(m-1) + 2$ for $m \geq 2$.*

Proof. We will use induction on m. For $m = 1$, we need c to kill $H_k(\mathbf{y}; M)$, but

$$H_k(\mathbf{y}; M) \cong \operatorname{Ann}_M(\mathbf{y}) \subseteq \operatorname{Ann}_M(x_1) \cong H_1(x_1; M),$$

which implies what we want.

Now, let $m \geq 2$. The hypothesis on c together with Lemma 4.2 implies that c is a colon-killer for M with respect to subsequences of $x_1, \dots, x_k$, but this implies that c is a colon-killer for M/x_1M with respect to subsequences of $x_2, \dots, x_k$. The proof of Lemma 4.2 then implies that $c^{2^{k-2}}$ kills $H_i(x_2, \dots, x_m; M/x_1M)$ for all $i \geq 1$ and all $2 \leq m \leq k$.

For the induction, suppose that $c^{D(m-1)}$ kills $H_{k+2-m}(\mathbf{y}; M)$. Consider the exact sequence

$$0 \to \operatorname{Ann}_M x_1 \to M \xrightarrow{x_1} M \to M/x_1M \to 0,$$

from which we obtain two short exact sequences

$$\begin{gathered} 0 \to \operatorname{Ann}_M x_1 \to M \to x_1M \to 0, \\ 0 \to x_1M \to M \to M/x_1M \to 0. \end{gathered}$$

These sequences then induce long exact sequences in Koszul homology:

$$\begin{gathered} H_{i+1}(\mathbf{y}; M) \xrightarrow{f_{i+1}} H_{i+1}(\mathbf{y}; x_1M) \to H_i(\mathbf{y}; \operatorname{Ann}_M x_1) \to H_i(\mathbf{y}; M) \xrightarrow{f_i} H_i(\mathbf{y}; x_1M), \\ H_{i+1}(\mathbf{y}; x_1M) \xrightarrow{g_{i+1}} H_{i+1}(\mathbf{y}; M) \to H_{i+1}(\mathbf{y}; \tfrac{M}{x_1M}) \to H_i(\mathbf{y}; x_1M) \xrightarrow{g_i} H_i(\mathbf{y}; M), \end{gathered}$$

which in turn yield short exact sequences for all $i \geq 0$:

$$(*_i) \qquad 0 \to \frac{H_{i+1}(\mathbf{y}; x_1 M)}{f_{i+1}(H_{i+1}(\mathbf{y}; M))} \to H_i(\mathbf{y}; \mathrm{Ann}_M x_1) \to \ker(f_i) \to 0,$$

$$(\#_i) \qquad 0 \to \frac{H_{i+1}(\mathbf{y}; M)}{g_{i+1}(H_{i+1}(\mathbf{y}; x_1 M))} \to H_{i+1}(\mathbf{y}; M/x_1 M) \to \ker(g_i) \to 0.$$

Since the map of homology

$$H_i(\mathbf{y}; M) \xrightarrow{f_i} H_i(\mathbf{y}; x_1 M) \xrightarrow{g_i} H_i(\mathbf{y}; M)$$

is induced by the composition $M \to x_1 M \to M$, which is multiplication by x_1, and since $x_1 \in (\mathbf{y})S$, $g_i \circ f_i = 0$ for all $i \geq 0$.

Since $c\mathrm{Ann}_M x_1 = 0$ by hypothesis, $(*_i)$ implies that

$$cH_{i+1}(\mathbf{y}; x_1 M) \subseteq f_{i+1}(H_{i+1}(\mathbf{y}; M)) \subseteq \ker(g_{i+1}),$$

for all $i \geq 0$. Applying the inductive hypothesis to $M/x_1 M$ implies that $c^{2^{k-2}D(m-1)}$ kills $H_{k+2-m}(\mathbf{y}; M/x_1 M)$. Then $(\#_{k+1-m})$ shows that $c^{2^{k-2}D(m-1)} \ker(g_{k+1-m}) = 0$. Therefore, $c^{2^{k-2}D(m-1)+1}$ kills $H_{k+1-m}(\mathbf{y}; x_1 M)$. To finish, notice that

$$c^{2^{k-2}D(m-1)+1} = c^{D(m)-1},$$

which kills the image of $H_{k+1-m}(\mathbf{y}; M)$ inside $H_{k+1-m}(\mathbf{y}; x_1 M)$ under the map f_{k+1-m}. Thus, $c^{D(m)-1} H_{k+1-m}(\mathbf{y}; M)$ is contained in $\ker(f_{k+1-m})$, which is killed by c, using $(*_{k+1-m})$, and so $c^{D(m)}$ kills $H_{k+1-m}(\mathbf{y}; M)$, as needed. □

Since 1 is a colon-killer for a balanced big Cohen-Macaulay module, the previous result gives us a result about big Cohen-Macaulay modules and base change.

Corollary 4.6. *Let S be a Noetherian local ring that is also an integral extension of a local Noetherian ring R. If M is an S-module and a balanced big Cohen-Macaulay R-module, then M is a balanced big Cohen-Macaulay S-module.*

We will now introduce another notion of a colon-killer that will be very useful for us in the following sections when we need to determine whether a ring is a seed.

Definition 4.7. For a local Noetherian ring (R, m) and an R-algebra S, an element $c \in S$ is called a *weak durable colon-killer* over R if for some system of parameters $x_1, \ldots, x_n$ of R,

$$c((x_1^t, \ldots, x_k^t)S :_S x_{k+1}^t) \subseteq (x_1^t, \ldots, x_k^t)S,$$

for all $1 \leq k \leq n-1$ and all $t \in \mathbb{N}$, and if for any $N \geq 1$, there exists $k \geq 1$ such that $c^N \notin m^k S$. An element $c \in S$ will be simply called a *durable colon-killer* over R if it is a weak durable colon-killer for every system of parameters of R.

Notice that if $S = R$, then all colon-killers in R that are not nilpotent are durable colon-killers. So, if R is reduced, all nonzero colon-killers are durable colon-killers. We can now use the existence of durable colon-killers to characterize when an algebra is a seed by adapting the proof of [Ho2, Theorem 11.1].

Theorem 4.8. *Let (R, m) be a local Noetherian ring of positive characteristic p, and let S be an R-algebra. Then S is a seed if and only if there is a map $S \to T$ such that T has a (weak) durable colon-killer c.*

Proof. If S is a seed, then $S \to B$, for some balanced big Cohen-Macaulay R-algebra B. As pointed out above, 1 is a durable colon-killer in B, so $T = B$ will suffice. For the converse, we will modify the proof of [Ho2, Theorem 11.1] to obtain our result. We will show that the existence of a (weak) durable colon-killer in an S-algebra T implies that S is a seed. (All parenthetical remarks will apply to the case that T only possesses a weak durable colon-killer.)

Suppose that $S \to T$ is a map such that T has a (weak) durable colon-killer c (with respect to a fixed system of parameters in R). Let $S^{(0)} := S$, and given $S^{(i)}$ for $0 \le i \le t-1$, let

$$S^{(i+1)} := \frac{S^{(i)}[U_1^{(i)}, \dots, U_{k_i}^{(i)}]}{s^{(i)} - \sum_{j=1}^{k_i} x_j^{(i)} U_j^{(i)}},$$

where $x_1^{(i)}, \cdots, x_{k_i+1}^{(i)}$ is a system of parameters for R (the fixed system of parameters in the latter case), and $x_{k_i+1}^{(i)} s^{(i)} = \sum_{j=1}^{k_i} x_j^{(i)} s_j^{(i)}$ is a relation in $S^{(i)}$. Then

$$S = S^{(0)} \to S^{(1)} \to S^{(2)} \to \cdots \to S^{(t)}$$

is a finite sequence of algebra modifications. Suppose to the contrary that S is not a seed and the sequence is bad, so that $1 \in mS^{(t)}$. (In the weak case, we are supposing that S does not map to an S-algebra where the fixed system of parameters is a regular sequence.) We can then write

$$1 = \sum_{j=1}^{n} r_j w_j, \tag{4.9}$$

where $r_j \in m$ and $w_j \in S^{(t)}$ for all j.

We will construct inductively homomorphisms $\psi_e^{(i)}$ from each $S^{(i)}$ to $\mathbf{F}^e(T)$ forming a commutative diagram:

$$\begin{tikzcd}
\mathbf{F}^e(T)_c \arrow[r, equal] & \mathbf{F}^e(T)_c \arrow[r, equal] & \mathbf{F}^e(T)_c \arrow[r, equal] & \cdots \arrow[r, equal] & \mathbf{F}^e(T)_c \\
S^{(0)} \arrow[u, "\psi_e^{(0)}"] \arrow[r] & S^{(1)} \arrow[u, "\psi_e^{(1)}"] \arrow[r] & S^{(2)} \arrow[u, "\psi_e^{(2)}"] \arrow[r] & \cdots \arrow[r] & S^{(t)}. \arrow[u, "\psi_e^{(t)}"]
\end{tikzcd}$$

In order to construct the maps we need to keep track of bounds, independent of $q = p^e$, associated with the images of certain elements of each $S^{(i)}$. For all $1 \le i \le t$, we will use reverse induction to define a finite subset Γ_i of $S^{(i)}$ and positive integers $b(i)$. We will then inductively define positive integers $\beta(i)$ and $B(i)$, which will be the necessary bounds.

First, let

$$\Gamma_t := \{w_1, \dots, w_n\},$$

where the w_j are from relation (4.9). Now, given Γ_{i+1} (with $0 \le i \le t-1$), each element can be written as a polynomial in the $U_j^{(i)}$ with coefficients in $S^{(i)}$. Let $b(i+1)$ be the largest degree of any such polynomial. For $i \ge 1$, let Γ_i be the set of all coefficients of these polynomials together with $s^{(i)}, s_1^{(i)}, \cdots, s_{k_i}^{(i)}$. Now define $\beta(1) := 1$, $B(1) := b(1)$, and given $B(i)$ for $1 \le i \le t-1$, let

$$\beta(i+1) := B(i) + 1 \qquad \text{and} \qquad B(i+1) := \beta(i+1)b(i+1) + B(i).$$

Notice that, as claimed, all $\beta(i)$ and $B(i)$ are independent of q.

Fix $q = p^e$. By hypothesis, we have a map $S^{(0)} = S \to T$ that can be naturally extended to a map $\psi_e^{(0)} : S^{(0)} \to \mathbf{F}^e(T)_c$ by composing with $T \to \mathbf{F}^e(T)_c$. We next define a map $\psi_e^{(1)} : S^{(1)} \to \mathbf{F}^e(T)_c$ that extends $\psi_e^{(0)}$, maps the $U_j^{(0)}$ to the cyclic $\mathbf{F}^e(T)$-submodule $c^{-1}\mathbf{F}^e(T) = c^{-\beta(1)}\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$, and maps Γ_1 to $c^{-b(1)}\mathbf{F}^e(T) = c^{-B(1)}\mathbf{F}^e(T)$ inside $\mathbf{F}^e(T)_c$. To do this we need only find appropriate images of the $U_j^{(0)}$ such that the image of $s^{(0)} - \sum_{j=1}^{k_0} x_j^{(0)} U_j^{(0)}$ maps to 0.

Since

$$x_{k_0+1}^{(0)} s^{(0)} = \sum_{j=1}^{k_0} x_j^{(0)} s_j^{(0)},$$

we have

$$(x_{k_0+1}^{(0)})^q \psi_e^{(0)}(s^{(0)}) = \sum_{j=1}^{k_0} (x_j^{(0)})^q \psi_e^{(0)}(s_j^{(0)})$$

in $\mathbf{F}^e(T)_c$, where the image of $\psi_e^{(0)}$ is contained in the image of $\mathbf{F}^e(T)$ inside $\mathbf{F}^e(T)_c$. As c is a (weak) durable colon-killer in T, and so also a (weak) colon-killer in $\mathbf{F}^e(T)$,

$$c\psi_e^{(0)}(s^{(0)}) = \sum_{j=1}^{k_0} (x_j^{(0)})^q \sigma_j^{(0)},$$

where the $\sigma_j^{(0)}$ are in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. If we define $\psi_e^{(1)}$ such that $U_j^{(0)} \mapsto c^{-1}\sigma_j^{(0)}$, then we have accomplished our goal for $\psi_e^{(1)}$, because the elements of Γ_1 can be written as polynomials in the $U_j^{(0)}$ of degree at most $b(1) = B(1)$ with coefficients in S.

Now suppose that for some $1 \le i \le t-1$ we have a map $\psi_e^{(i)} : S^{(i)} \to \mathbf{F}^e(T)_c$, where the $U_j^{(i-1)}$ all map to $c^{-\beta(i)}\mathbf{F}^e(T)$, and Γ_i maps to $c^{-B(i)}\mathbf{F}^e(T)$. We will extend $\psi_e^{(i)}$ to a map from $S^{(i+1)}$ such that each $U_j^{(i)}$ maps to $c^{-\beta(i+1)}\mathbf{F}^e(T)$, and Γ_{i+1} maps to $c^{-B(i+1)}\mathbf{F}^e(T)$.

In order to simplify notation, we drop many of the (i) labels on parameters. Then

$$S^{(i+1)} = \frac{S^{(i)}[U_1, \ldots, U_k]}{s - \sum_{j=1}^k x_j U_j}.$$

Since s and the s_j (in the relation $x_{k+1}s = \sum_{j=1}^k x_j s_j$ in $S^{(i)}$) are in Γ_i, we can write

$$\psi_e^{(i)}(s) = c^{-B(i)}\sigma \qquad \text{and} \qquad \psi_e^{(i)}(s_j) = c^{-B(i)}\sigma_j, \tag{4.10}$$

where σ and the σ_j are elements in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Hence,

$$x_{k+1}^q \psi_e^{(i)}(s) = \sum_{j=1}^k x_j^q \psi_e^{(i)}(s_j)$$

in $\mathbf{F}^e(T)_c$. Multiplying through by $c^{B(i)}$ yields

$$x_{k+1}^q \sigma = \sum_{j=1}^k x_j^q \sigma_j$$

in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Using our (weak) colon-killer c, we have

$$c\sigma = \sum_{j=1}^{k} x_j^q \tau_j,$$

where τ_j is an element in the image of $\mathbf{F}^e(T)$ in $\mathbf{F}^e(T)_c$. Therefore,

$$\psi_e^{(i)}(s) = \sum_{j=1}^{k} x_j^q (c^{-B(i)-1}\tau_j)$$

in $\mathbf{F}^e(T)_c$.

We now have a well-defined map $\psi_e^{(i+1)} : S^{(i+1)} \to \mathbf{F}^e(T)_c$ extending $\psi_e^{(i)}$ given by

$$\psi_e^{(i+1)}(U_j) = c^{-B(i)-1}\tau_t = c^{-\beta(i+1)}\tau_t$$

such that the U_j map to $c^{-\beta(i+1)}\mathbf{F}^e(T)$, and Γ_{i+1} maps to $c^{-B(i+1)}\mathbf{F}^e(T)$ since

$$B(i+1) = \beta(i+1)b(i+1) + B(i),$$

and these elements can be written as polynomials in the U_j of degree at most $b(i+1)$ with coefficients in Γ_i.

We can finally conclude that, for all $q = p^e$, there exists a map

$$\psi_e^{(t)} : S^{(t)} \to \mathbf{F}^e(T)_c$$

such that the equation (4.9) that puts $1 \in mS^{(t)}$ maps to

$$1 = \sum_{j=1}^{n} r_j^q \psi_e^{(t)}(w_j).$$

If we let $B := B(t)$, then each $\psi_e^{(t)}(w_j)$ is in $c^{-B}\mathbf{F}^e(T)$ as Γ_t contains these elements. Multiplying through by c^B, we see that $c^B \in m^{[q]}T$ for all $q \geq 1$, where B is independent of q, which implies that $c^B \in m^k T$ for all $k \geq 1$. Since c is a (weak) durable colon-killer, we have a contradiction. Therefore, no such finite sequence of modifications of S is bad.

In the case of the durable colon-killer, we see that S maps to a balanced big Cohen-Macaulay algebra over R. In the weak case, S maps to a big Cohen-Macaulay algebra that can be made balanced by [BS, Theorem 1.7]. In either event, S is a seed. □

Later we will use this result as a piece of the proof that integral extensions of seeds are seeds. We will also use durable colon-killers to obtain our results concerning when the seed property is preserved by base change.

5. Minimal seeds

Definition 5.1. For a Noetherian local ring (R, m), an R-algebra S is a *minimal seed* if S is a seed over R and no proper homomorphic image of S is a seed over R.

Example 5.2. (1) If R is a Cohen-Macaulay ring, then R is a minimal seed.
(2) If R is an excellent local domain of positive prime characteristic, then R^+ is a balanced big Cohen-Macaulay algebra over R and a minimal seed.

We also point out the following easy, but useful, fact about minimal seeds.

Lemma 5.3. *A seed S over a local Noetherian ring R is a minimal seed if and only if every map from S to a (balanced) big Cohen-Macaulay R-algebra B is injective if and only if every map to a seed over R is injective.*

A very important question about minimal seeds is whether or not every seed maps to a minimal seed.

Proposition 5.4. *Let R be a local Noetherian ring, and let S be a seed over R. Then S/I is a minimal seed for some ideal $I \subseteq S$.*

Proof. Let Σ be the set of all ideals J of S such that S/J is a seed. If Σ contains a maximal element I, then S/I will be a minimal seed. Let $J_1 \subset J_2 \subset \cdots$ be a chain of ideals in Σ, and let $J = \bigcup_k J_k$. Then $S/J = \varinjlim_k S/J_k$, and since each S/J_k is a seed, Lemma 3.2 implies that S/J is a seed as well. By Zorn's Lemma, Σ has a maximal element I. $\square$

Now that we know minimal seeds exist, we would like to know whether they are domains or not. After dealing with integral extensions of seeds, we will prove in Section 7 that in positive characteristic, minimal seeds are domains. In the meantime, we will point out that minimal seeds are reduced in positive characteristic.

Proposition 5.5. *Let S be a minimal seed over a local ring R of positive characteristic. Then S is a reduced ring.*

Proof. By Lemma 3.3, there exists a reduced big Cohen-Macaulay algebra B such that $S \to B$. Since S is minimal, Lemma 5.3 implies that $S \hookrightarrow B$. As B has no nilpotents, neither does S. $\square$

6. Integral extensions of seeds

The primary goal of this section is to prove that integral extensions of seeds are seeds in positive characteristic. Since all integral extensions are direct limits of module-finite extensions, with Lemma 3.2 we can concentrate on module-finite extensions of seeds. We begin the argument by proving that the problem can be reduced to a much more specific problem, which we attack by constructing a durable colon-killer in a certain module-finite extension of a big Cohen-Macaulay algebra. Our first reduction of the problem will be that we can assume we are considering a module-finite extension of a balanced big Cohen-Macaulay algebra that is reduced, quasilocal, and m-adically separated.

Lemma 6.1. *Let (R, m) be a local Noetherian ring of positive characteristic, and let S be a seed with T a module-finite extension of S. Suppose that any module-finite extension of a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay algebra is a seed. Then T is a seed.*

Proof. By Propositions 5.4 and 5.5, S/I is a minimal reduced seed for some ideal I. Since S/I is reduced, I is a radical ideal and so is an integrally closed ideal. Therefore, $IT \cap S = I$ so that S/I injects into T/IT, which is thus a module-finite extension of S/I. Since T is a seed if T/IT is a seed, we can now assume that S is a reduced seed.

By Lemma 5.3 and Proposition 3.7, there exists a commutative square

$$\begin{tikzcd} B \arrow[r] & C \\ S \arrow[u, hook] \arrow[r, hook] & T, \arrow[u] \end{tikzcd}$$

where B is a reduced, quasilocal, and m-adically separated balanced big Cohen-Macaulay algebra, and $C := T \otimes_S B$. If we can show that the upper map $B \to C$ is injective, then C will be a module-finite extension of B, and our hypotheses will imply that C and T are seeds. Since B is reduced, the next lemma allows us to reach our goal. □

Lemma 6.2. *If S is a ring, T is an integral extension of S, and B is a reduced extension of S, then B injects into $C := T \otimes_S B$.*

Proof. We will first prove the claim in the case that S, T, and B are all domains. Let K be the algebraic closure of the fraction field of S, and let L be the algebraic closure of the fraction field of B. Since T is an integral extension domain of S, we have the following diagram:

$$\begin{tikzcd} B \arrow[rr, hook] & & L \\ S \arrow[u, hook] \arrow[r, hook] & T \arrow[r, hook] & K. \arrow[u, hook] \end{tikzcd}$$

Under the injection $K \hookrightarrow L$, the ring T maps isomorphically to some subring T' of L. Now let C' be the S-subalgebra of L generated by B and T'. Since $C = T \otimes_S B$, we have a (surjective) map $C \to C'$ and a diagram

$$\begin{tikzcd} & & C' \\ B \arrow[r] \arrow[urr, hook, bend left] & C \arrow[ur] & \\ S \arrow[u, hook] \arrow[r, hook] & T \arrow[u] \arrow[uur, hook, bend right] & \end{tikzcd}$$

Since B injects into C', B also injects into C.

For the general case, since B is reduced, it will suffice to show that the kernel of $B \to C$ is contained in every prime ideal of B. Let $\mathfrak{p}$ be a prime ideal of B, and let $\mathfrak{p}_0 := \mathfrak{p} \cap S$. Since T is integral over S, there exists a prime ideal $\mathfrak{q}_0$ of T lying over $\mathfrak{p}_0$. If we put $D := T/\mathfrak{q}_0 \otimes_{S/\mathfrak{p}_0} B/\mathfrak{p}$, then we obtain the following commutative diagram:

$$\begin{tikzcd}
& T/\mathfrak{q}_0 \arrow[rr] & & D \\
T \arrow[ur] \arrow[rr] & & C \arrow[ur] & \\
& S/\mathfrak{p}_0 \arrow[uu, hook] \arrow[rr, hook] & & B/\mathfrak{p} \arrow[uu] \\
S \arrow[uu, hook] \arrow[ur] \arrow[rr, hook] & & B \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Since $T/\mathfrak{q}_0$ is still an integral extension of $S/\mathfrak{p}_0$ and $B/\mathfrak{p}$ is a domain extension of $S/\mathfrak{p}_0$, the domain case of the proof shows that $B/\mathfrak{p}$ injects into D. Therefore, if b is in the kernel of $B \to C$, then b is in the kernel of $B \to D$, which implies that $b \in \mathfrak{p}$, as desired. □

To finish our argument that module-finite extensions of seeds are seeds, we will show that we can extend the map $B \hookrightarrow C$ to another module-finite extension $B^{\#} \hookrightarrow C^{\#}$ such that $B^{\#}$ is a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay algebra. Our new rings will also have a nonzero element $b \in B^{\#}$ such that b multiplies $C^{\#}$ into a finitely generated free $B^{\#}$-submodule of $C^{\#}$. We will then finally show that b is a durable colon-killer in $C^{\#}$ so that Theorem 4.8 implies that C is a seed.

We start the process by showing that we can adjoin indeterminates and then localize our ring B without losing any of its key properties.

Definition 6.3. Let $(B, \mathfrak{p})$ be a quasilocal ring. If $n, s \in \mathbb{N}$ and $\{t_{ij} \mid i \leq n, j \leq s\}$ is a set of indeterminates over B, then

$$B^{\#(n,s)} := B[t_{ij} \mid i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]}.$$

Lemma 6.4. *Let $(B, \mathfrak{p})$ be a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay R-algebra, where (R, m) is a local Noetherian ring. If $n, s \in \mathbb{N}$, then $B^{\#(n,s)}$ is also a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay R-algebra.*

Proof. For the duration of the proof, n and s will be fixed, so we will simply write $B^{\#}$ instead of $B^{\#(n,s)}$. We will let $\mathbf{t}$ denote the set of all t_{ij}.

As B is reduced, $B[\mathbf{t}]$ is reduced after adjoining indeterminates. By Lemma 3.5, $B^{\#}$ will also be reduced. As $\mathfrak{p}$ is prime in B, the extension of $\mathfrak{p}$ to $B[\mathbf{t}]$ is also prime so that it makes sense to localize at this ideal and end up with $B^{\#}$ quasilocal.

The construction of $B^{\#}$ implies that $B^{\#}$ is a faithfully flat extension of B. Therefore, for any ideals I and J of B, we have $IB^{\#} :_{B^{\#}} JB^{\#} = (I :_B J)B^{\#}$ (see [N2, Theorem 18.1]). This fact implies that every system of parameters of R will be a possibly improper regular sequence on $B^{\#}$, because B is a balanced big Cohen-Macaulay algebra. Moreover, the faithful flatness also implies that $mB^{\#} \neq B^{\#}$ as $mB \neq B$. Hence, $B^{\#}$ is a balanced big Cohen-Macaulay R-algebra.

To show that $B^{\#}$ is also m-adically separated will take a little bit more effort. Suppose that an element $F \in B^{\#}$ is in $m^N B^{\#}$ for every N. Multiplying through by its denominator, we obtain such an element from $B[\mathbf{t}]$, so that we may assume without loss of generality that F is a polynomial in $B[\mathbf{t}]$. Thus, for every N there exists $G_N \notin \mathfrak{p}B[\mathbf{t}]$ such that $G_N F \in m^N B[\mathbf{t}]$. It suffices to show that any polynomial $G \notin \mathfrak{p}B[\mathbf{t}]$ is not a zerodivisor modulo $m^N B[\mathbf{t}]$ for any N. If we put $D := B/m^N B$, then the image $\overline{G}$ of G in $D[\mathbf{t}]$ is a polynomial not in $\mathfrak{p}D[\mathbf{t}]$; i.e., $\overline{G}$ is a polynomial whose coefficients generate the unit ideal of D. To finish, it suffices to apply the following general lemma. $\square$

Lemma 6.5. *If D is any ring and G is a polynomial in $D[t_\lambda \mid \lambda \in \Lambda]$ such that the coefficients of G generate the unit ideal in D, then G is not a zerodivisor.*

Proof. The lemma reduces to the Noetherian case, where it follows from Corollary 2 on p. 152 of [Mat]. $\square$

Lemma 6.6. *If $(B, \mathfrak{p})$ is a quasilocal ring, and $B^{\#(n,s)} = B[t_{ij} \mid i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]}$ for some n and some s, then for any $k \leq n$,*

$$B^{\#(n,s)} \cong (B^{\#(k,s)})^{\#(n-k,s)}.$$

Proof. Let $\mathbf{x}$ denote the indeterminates t_{ij} such that $1 \leq i \leq k$ and $1 \leq j \leq s$, and let $\mathbf{y}$ denote the remaining indeterminates t_{ij}. Let $C := B^{\#(k,s)} = B[\mathbf{x}]_{\mathfrak{p}B[\mathbf{x}]}$, Q be the maximal ideal of C, and $D := (B^{\#(k,s)})^{\#(n-k,s)} = C[\mathbf{y}]_{QC[\mathbf{y}]}$.

Since D is a B-algebra, there exists a unique ring homomorphism $B[\mathbf{x}, \mathbf{y}] \to D$ that maps the indeterminates t_{ij} to their natural images in D. We claim that the units in $B[\mathbf{x}, \mathbf{y}]$ map to units in D under this map. Indeed, if $f(\mathbf{x}, \mathbf{y})$ is a unit in $B[\mathbf{x}, \mathbf{y}]$, then f has some coefficient that is in $B \setminus \mathfrak{p}$. If we rewrite

$$f(\mathbf{x}, \mathbf{y}) = g_k(\mathbf{x})\mathbf{y}^k + \cdots + g_1(\mathbf{x})\mathbf{y} + g_0(\mathbf{x}),$$

then some $g_i(\mathbf{x})$ is in $C \setminus Q$ so that the image of f is not in the maximal ideal of D. We therefore have a ring homomorphism $\phi : B^{\#(n,s)} \to D$. Using the previous lemma, it is easy to verify that ϕ is injective. It is also routine to check that ϕ is surjective. $\square$

Using the construction $B^{\#(n,s)} = B[t_{ij} \mid i \leq n, j \leq s]_{\mathfrak{p}B[t_{ij}]}$, where $(B, \mathfrak{p})$ is a quasilocal ring, we also define

$${}^{\#(n,s)}M := B^{\#(n,s)} \otimes_B M$$

for any B-module M. Since $B^{\#(n,s)}$ is faithfully flat over B, when $M = C$ is a module-finite extension of B, we also have that ${}^{\#(n,s)}C$ is a module-finite extension of $B^{\#(n,s)}$.

Lemma 6.7. *Let B be a reduced quasilocal ring, and let M be a B-module generated by $m_1, \ldots, m_s$ in M. Then there exists $k \leq s$ such that $b({}^{\#(k,s)}M) \subseteq G$, where b is a nonzero element of $B^{\#(k,s)}$ and G is a finitely generated free $B^{\#(k,s)}$-submodule of ${}^{\#(k,s)}M$.*

Proof. Throughout the proof, define

$$g_i := t_{i1}m_1 + \cdots + t_{is}m_s$$

in any $B^{\#(n,s)}$, where $i \leq n$. Note that there exists a maximum $0 \leq n \leq s$ such that the set $\{g_1, \ldots, g_n\}$ generates a $B^{\#(n,s)}$-free submodule of ${}^{\#(n,s)}M$, where $B^{\#(0,s)} = B$, since ${}^{\#(n,s)}M$ has s generators. If $\alpha = (t_{ij})_{1 \leq i,j \leq s}$, then $\det(\alpha)$ is not in the unique maximal ideal of $B^{\#(s,s)}$, so that α is an invertible matrix.

As $m_1, \ldots, m_s$ generate ${}^{\#(s,s)}M$ over $B^{\#(s,s)}$ and α is invertible, $g_1, \ldots, g_s$ also generate ${}^{\#(s,s)}M$. If the g_i are linearly independent over $B^{\#(s,s)}$, then ${}^{\#(s,s)}M$ is a free module, and we can use $k = s$, $b = 1$, and $G = {}^{\#(s,s)}M$ to fulfill our claim.

Otherwise, the maximum value n is strictly less than s, and we put $k := n + 1$. In this case, there exists a nonzero $b' \in B^{\#(k,s)}$ such that

$$b' g_k \in (g_1, \ldots, g_{k-1})({}^{\#(k,s)}M).$$

Indeed, since n was chosen to be a maximum and $k = n + 1$, there must be a nontrivial relation $b' g_k = b_1 g_1 + \cdots + b_{k-1} g_{k-1}$ in ${}^{\#(k,s)}M$. If $b' = 0$, then we have a nontrivial relation on $g_1, \ldots, g_{k-1}$. As $B^{\#(k,s)} \cong (B^{\#(k-1,s)})^{\#(1,s)}$ (by Lemma 6.6), we see that $B^{\#(k,s)}$ is faithfully flat over $B^{\#(k-1,s)}$, and so the nontrivial relation on $g_1, \ldots, g_{k-1}$ in ${}^{\#(k,s)}M$ implies that there is a nontrivial relation on $g_1, \ldots, g_{k-1}$ in ${}^{\#(k-1,s)}M$, a contradiction. Hence, b' is nonzero as claimed. Notice that the same argument implies that $g_1, \ldots, g_{k-1}$ still generate a finitely generated free submodule G of ${}^{\#(k,s)}M$.

We now claim that there exists a nonzero $b \in B^{\#(k,s)}$ such that $b({}^{\#(k,s)}M) \subseteq G$. If we put $M_0 := ({}^{\#(k,s)}M)/G$, and replace $m_1, \ldots, m_s$ and g_k by their images in M_0, then b' kills g_k. We intend to show that the annihilator of M_0 cannot be zero. Suppose to the contrary that no nonzero element of $B^{\#(k,s)}$ kills M_0. Then we have an injective map $B^{\#(k,s)} \hookrightarrow M_0^{\oplus s}$ defined by $b \mapsto (bm_1, \ldots, bm_s)$.

After clearing the denominator on b', we may assume that b' is a polynomial in $B[\mathbf{t}]$, where $\mathbf{t}$ denotes the set of all t_{ij} with $i \leq k$ and $j \leq s$. Write $b' = \sum_\nu c_\nu \mathbf{t}^\nu$, where ν is an $n \times s$ matrix of integers, and each c_ν is in B. Let A_0 be the prime ring of B, and let A be the A_0-subalgebra of B (finitely) generated by the nonzero c_ν. Then A is Noetherian and reduced. If we let $\mathfrak{q}$ be the contraction of $\mathfrak{p}$ to A and replace A by the local ring $A_\mathfrak{q}$, then $(A, \mathfrak{q})$ is a reduced local Noetherian subring of B. We then obtain an injective map $A^\# \hookrightarrow B^{\#(k,s)}$, where $A^\#$ denotes $A^{\#(k,s)}$.

Since b' is in $A^\#$ (and still nonzero), we can define an $A^\#$-module by

$$N := \frac{A^\# m_1 \oplus \cdots \oplus A^\# m_s}{b'(t_{k1}m_1 + \cdots + t_{ks}m_s)}.$$

There is then a natural map $N \to M_0$ that induces a commutative square:

$$\begin{tikzcd} B^{\#(k,s)} \arrow[r, hook] & M_0^{\oplus s} \\ A^\# \arrow[u, hook] \arrow[r] & N^{\oplus s}. \arrow[u] \end{tikzcd}$$

This implies that the map $A^\# \to N^{\oplus s}$ defined by $a \mapsto (am_1, \ldots, am_s)$ is injective (which also shows that $N \neq 0$). Therefore, we may now assume without loss of generality that B is a reduced, local Noetherian ring.

As B is reduced and Noetherian, B has finitely many minimal primes $Q_1, \ldots, Q_h$ such that $\bigcap_i Q_i = 0$. Since b' is a nonzero polynomial in $B^{\#(k,s)}$, some coefficient of b' is not in some minimal prime Q. Thus, the image of b' is still nonzero in $(B_Q)^{\#(k,s)}$. Moreover, if $M_0' := (B_Q)^{\#(k,s)} \otimes_{B^{\#(k,s)}} M_0$, then M_0' is a finitely generated $(B_Q)^{\#(k,s)}$-module with $b'(t_{k1}m_1 + \cdots + t_{ks}m_s) = 0$ and with an injection $(B_Q)^{\#(k,s)} \hookrightarrow (M_0')^{\oplus s}$ since $(B_Q)^{\#(k,s)} \cong (B^{\#(k,s)})_{QB^{\#(k,s)}}$. (Again, this fact implies that M_0' is nonzero.) Since B is reduced and Q is minimal, B_Q is a field, and so we can now assume that $B = K$ is a field.

In this final case, $K^{\#(k,s)} \cong K(\mathbf{t})$, and M_0 is a nonzero module over a field so that M_0 is a nonzero free $K^{\#(k,s)}$-module. Therefore, if $b'(t_{k1}m_1 + \cdots + t_{ks}m_s) = 0$ in M_0, then $t_{k1}m_1 + \cdots + t_{ks}m_s = 0$ in M_0. This is impossible, however, since the t_{ij} are algebraically independent.

The resulting contradiction implies that M_0 is killed by some nonzero element $b \in B^{\#(k,s)}$ in our original set-up, and since M_0 was originally $({}^{\#(k,s)}M)/G$, where G is free over $B^{\#(k,s)}$, the proof is complete. □

We are now ready to show that module-finite extensions of sufficiently nice big Cohen-Macaulay algebras are indeed seeds. As mentioned above, the key fact will be that the element b constructed above is a durable colon-killer.

Lemma 6.8. *Let (R, m) be a local Noetherian ring of positive characteristic, and let B be a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay R-algebra. If C is a module-finite extension of B, then C is a seed.*

Proof. By Lemma 6.4 and the remarks made before the previous lemma, for any k, we have a commutative square:

$$\begin{tikzcd} B^{\#(k,s)} \arrow[r, hook] & {}^{\#(k,s)}C \\ B \arrow[u] \arrow[r, hook] & C, \arrow[u] \end{tikzcd}$$

where the top map is also a module-finite extension of a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay R-algebra, and C is generated by s elements as a B-module. After applying the previous lemma, we may assume that there exists a nonzero element $b \in B$ such that b multiplies C into a finitely generated free B-submodule G. In order to see that C is a seed, we show that b is a durable colon-killer in C.

Indeed, let $x_1, \ldots, x_{t+1}$ be part of a system of parameters in R and suppose that $u \in (x_1, \ldots, x_t)C :_C x_{t+1}$. Then by construction $bux_{t+1} \in (x_1, \ldots, x_t)G$, so as an element of G we have $(bu) \in (x_1, \ldots, x_t)G :_G x_{t+1}$. Since B is a balanced big Cohen-Macaulay R-algebra and since G is a free B-module, G is a balanced big Cohen-Macaulay R-module. Hence, $bu \in (x_1, \ldots, x_t)G \subseteq (x_1, \ldots, x_t)C$ as G is a submodule of C.

Now, if $b^N \in \bigcap_t m^t C$ for some N, then $b^{N+1} \in \bigcap_t m^t G$. Since B is m-adically separated, $\bigcap_t m^t B = 0$, and since G is free over B, we also have $\bigcap_t m^t G = 0$. As G is a submodule of C and the map $B \to C$ is an injection, $b^{N+1} = 0$ in B, but B reduced implies that $b = 0$, a contradiction. Therefore, b is a durable colon-killer, and C is a seed by Theorem 4.8. □

We have now gathered together all of the tools that we will need to prove the primary result of this section.

Theorem 6.9. *Let (R, m) be a local Noetherian ring of positive characteristic. If S is a seed and T is an integral extension of S, then T is a seed.*

Proof. By Lemma 3.2, we may assume that T is a module-finite extension of S because integral extensions are direct limits of module-finite extensions. By Lemma 6.1, we may assume that $S = B$ is a reduced, quasilocal, m-adically separated balanced big Cohen-Macaulay algebra. Finally, Lemma 6.8 implies that T is a seed. □

Remark 6.10. We feel it is worthwhile to point out that the hypothesis that our base ring has positive characteristic is only required in two places: (1) the existence of a durable colon-killer implies that an algebra is a seed, and (2) all seeds map to a reduced big Cohen-Macaulay algebra. These facts have proofs that rely heavily on the use of the Frobenius endomorphism, but are the only two that we can prove only in positive characteristic.

We can also view the above theorem as a generalization of the existence of big Cohen-Macaulay algebras over complete local domains of positive characteristic.

Corollary 6.11 (Hochster-Huneke)**.** *If R is a complete local domain of positive characteristic, then there exists a balanced big Cohen-Macaulay algebra B over R.*

Proof. By the Cohen structure theorem, R is a module-finite extension of a regular local ring A. Since A is clearly a seed over itself, Theorem 6.9 implies that R is a seed over A as well. Let B be a balanced big Cohen-Macaulay algebra over A such that B is also an R-algebra. By Corollary 4.6, B is also a balanced big Cohen-Macaulay algebra over R. □

7. More properties of seeds

In this section, we will show that all seeds in positive characteristic can be mapped to quasilocal balanced big Cohen-Macaulay algebra domains that are absolutely integrally closed and m-adically separated. We start off the section by delivering the promised proof that minimal seeds are domains. First, we show that we can reduce to the case of a finitely generated minimal seed.

Lemma 7.1. *Let R be a local Noetherian ring. If all finite type minimal seeds are domains, then all minimal seeds are domains.*

Proof. Let S be an arbitrary minimal seed over R. Then $S = \varinjlim_{\lambda \in \Lambda} S_\lambda$, where Λ indexes the set of all finitely generated subalgebras of S. Suppose that S is not a domain and that $ab = 0$ in S with $a, b \neq 0$. Since S is a minimal seed, S/aS and S/bS are not seeds. Let $\Lambda(a)$ and $\Lambda(b)$ be the subsets of Λ indexing all finitely generated subalgebras of S that contain a and b, respectively. Then $S/aS = \varinjlim_{\lambda \in \Lambda(a)} S_\lambda/aS_\lambda$, with a similar result for S/bS. Since S/aS and S/bS are not seeds, Lemma 3.2 implies that there exists an S_α containing a and an S_β containing b such that S_α/aS_α and S_β/bS_β are not seeds. Therefore, there exists a common S_γ containing a and b such that S_γ is not a seed modulo aS_γ nor modulo bS_γ. We also have $ab = 0$ in S_γ since $ab = 0$ in S. As S is a seed, S_γ is a seed. Since S_γ is also finitely generated as an R-algebra, S_γ maps onto a finitely generated minimal seed T. Therefore, $ab = 0$ in T, and as T is a domain by hypothesis, $a = 0$ or $b = 0$ in T. Suppose without loss of generality that $a = 0$. This implies that the map $S_\gamma \to T$ factors through S_γ/aS_γ, which is not a seed and so cannot map to any seed. We have a contradiction, and so S is a domain after all. □

Proposition 7.2. *Let R be a local Noetherian ring of positive characteristic p. If S is a minimal seed over R, then S is a domain.*

Proof. By the previous lemma, we can assume that S is finitely generated over R. By Proposition 5.5, S is Noetherian and reduced. Let $\overline{S}$ be the normalization of S in its total quotient ring. Then $\overline{S}$ is a finite direct product of normal domains by Serre's Criterion (see [E, Theorem 11.5]) and is an integral extension of S. By our main result of the last section, Theorem 6.9, $\overline{S}$ is also a seed. Since $\overline{S}$ is a seed and a finite product $D_1 \times \cdots \times D_t$ of domains, we will be done once we have proven the following lemma. ☐

Lemma 7.3. *Let (R, m) be a Noetherian local ring. If $S = S_1 \times \cdots \times S_t$, then S is a seed over R if and only if S_i is a seed over R, for some i.*

Proof. Clearly, if some S_i is a seed, then S is also a seed. Suppose then that S is a seed, but no S_i is a seed. Since S is a direct product, if $S \to B$, a balanced big Cohen-Macaulay algebra, then $B \cong B_1 \times \cdots \times B_t$, where each B_i is an S_i-algebra. We first claim that each B_i is a possibly improper balanced big Cohen-Macaulay algebra. Indeed, let $x_{k+1}b = \sum_{j=1}^{k} x_j b_j$ be a relation in B_i on a partial system of parameters $x_1, \dots, x_{k+1}$ from R, and let e_i be the idempotent associated to B_i in B. Therefore, $x_{k+1}(be_i) = \sum_{j=1}^{k} x_j(b_j e_i)$ is a relation in B, and so $be_i = \sum_{j=1}^{k} x_j c_j$, for elements c_j in B. Multiplying this equation by e_i yields $be_i = \sum_{j=1}^{k} x_j(c_j e_i)$ since $e_i^2 = e_i$. If we let c'_j be the image of $c_j e_i$ in B_i, for all j, then $b = \sum_{j=1}^{k} x_j c'_j$ in B_i, as claimed.

Now, if, as assumed, each S_i is not a seed, then $1 \in mB_i$, for all i. Thus $e_i \in mB$, for all i, and so $1 = \sum_i e_i \in mB$, a contradiction. Therefore, some S_i must be a seed if S is a seed. ☐

As a corollary, we will prove that each seed maps to a big Cohen-Macaulay algebra that is also a domain. We must first show that a domain seed can be modified into a balanced big Cohen-Macaulay algebra domain. We will use the "$\mathfrak{A}$-transform"

$$\Theta = \Theta(x, y; S) := \{u \in S_{xy} \mid (xy)^N u \subseteq \operatorname{Im}(S \to S_{xy}),\ \text{for some } N\}.$$

See [N1, Chapter V] and [Ho2, Section 12] for an introduction to the properties of Θ. The most useful property for us is that if x, y form part of a system of parameters in a local Noetherian ring R and S is a seed over R, then x, y become a regular sequence on Θ (see [Ho2, Lemma 12.4]). As a result any map from S to a balanced big Cohen-Macaulay R-algebra B factors through Θ.

Suppose now that S is a seed over a local Noetherian ring R, and let

$$T = \frac{S[U_1, \dots, U_k]}{(s - s_1U_1 - \cdots - s_kU_k)}$$

be an algebra modification of S, where $x_{k+1}s = x_1s_1 + \cdots + x_ks_k$ is a relation in S on a partial system of parameters $x_1, \dots, x_{k+1}$ in R. When $k \geq 2$, we also have an induced relation on $x_1, \cdots, x_{k+1}$ in $\Theta = \Theta(x_1, x_2; S)$ so that

$$T' = \frac{\Theta[U_1, \dots, U_k]}{(s - s_1U_1 - \cdots - s_kU_k)}$$

is an algebra modification of Θ over R. We will call T' an *enhanced algebra modification of S over R induced by the relation $x_{k+1}s = x_1s_1 + \cdots + x_ks_k$.*

Since any map from S to a balanced big Cohen-Macaulay algebra B factors through Θ, we obtain a commutative diagram

$$\begin{tikzcd} \Theta \arrow[r] & T' \arrow[dr] & \\ S \arrow[u] \arrow[r] & T \arrow[u] \arrow[r] & B, \end{tikzcd}$$

which shows that maps from algebra modifications of seeds to balanced big Cohen-Macaulay algebras factor through the enhanced modification of S when T is a modification with respect to a relation on 3 or more parameters from R. With this factorization, we can adapt the process described in [HH5, Section 3] to construct a balanced big Cohen-Macaulay algebra from a given seed as a very large direct limit of enhanced and ordinary modifications.

Lemma 7.4. *Let (R, m) be a local Noetherian ring, and let S be a domain (resp., reduced). If T is an enhanced algebra modification of S over R, then T is also a domain (resp., reduced).*

Proof. Let T be induced by the relation $x_{k+1}s = x_1s_1+\cdots+x_ks_k$ in S, where $k \geq 2$. Then x_1, x_2 forms a possibly improper regular sequence on $\Theta = \Theta(x_1, x_2; S)$ (see [Ho2, Lemma 12.4]), and so $x_1, s - x_1U_1 - \cdots - x_kU_k$ is a possibly improper regular sequence on $\Theta[U_1, \ldots, U_k]$. (Any polynomial $f(U)$ that kills $s - x_1U_1 - \cdots - x_kU_k$ modulo x_1 has a highest degree term as a polynomial in U_2, but this term is killed by x_2 modulo x_1. Since x_2 is not a zerodivisor modulo x_1, the term must be divisible by x_1. Hence, $f(U)$ is divisible by x_1, and $s - x_1U_1 - \cdots - x_kU_k$ is not a zerodivisor modulo x_1.) Therefore, x_1 is not a zerodivisor on

$$\Theta[U_1, \ldots, U_k]/(s - x_1U_1 - \cdots - x_kU_k).$$

The result now follows from the following short lemma. □

Lemma 7.5. *Let A be a domain (resp., reduced). If a and x are elements of A such that x is not a zerodivisor on $A' := A[U]/(a - xU)$, then A' is also a domain (resp., reduced).*

Proof. Since x is not a zerodivisor on A', we have an inclusion $A' \hookrightarrow (A')_x$ so that it suffices to show that $(A')_x$ is a domain (resp., reduced). It is, however, easy to verify that $(A')_x \cong A_x$ via the map that sends U to a/x (even without any hypotheses on A or x). Since A is a domain (resp., reduced), so is A_x. □

As promised, we now prove that one can modify a domain or reduced seed into a balanced big Cohen-Macaulay algebra with the same property. Notice that the result is characteristic free.

Proposition 7.6. *Let R be a local Noetherian ring. If S is a seed and a domain (resp., reduced), then S maps to a balanced big Cohen-Macaulay algebra that is a domain (resp., reduced).*

Proof. Any element of S killed by a parameter of R will be in the kernel of any map to a balanced big Cohen-Macaulay R-algebra B, so that the map $S \to B$ factors through the quotient of S modulo the ideal of elements killed by a parameter of R. When S is a domain, this ideal is the zero ideal, and when S is reduced, this ideal is radical. Hence the quotient is still a domain (resp., reduced). Without loss of generality, we may then assume that no element of S is killed by a parameter of R. Thus, any nontrivial relation $x_{k+1}s = x_1s_1 + \cdots + x_ks_k$ in S on part of a system of parameters $x_1, \cdots, x_{k+1}$ from R will have $k \geq 1$.

If $k = 1$, then an algebra modification with respect to that relation factors through the $\mathfrak{A}$-transform $\Theta(x_1, x_2; S)$. If $k \geq 2$, then we can factor any algebra modification through an enhanced algebra modification. Therefore, we can map S to a balanced big Cohen-Macaulay R-algebra that is constructed as a very large direct limit of sequences of enhanced algebra modifications and $\mathfrak{A}$-transforms.

Since S is a domain (resp., a reduced ring) and since the $\mathfrak{A}$-transform of a domain (resp., reduced ring) is a domain (resp., reduced), Lemma 7.4 implies that all enhanced algebra modifications and $\mathfrak{A}$-transforms in any sequence will continue to be domains (resp., reduced). Hence, S maps to a balanced big Cohen-Macaulay algebra B that is a direct limit of domains (resp., reduced rings), and so B itself is a domain (resp., reduced). □

Corollary 7.7. *Let R be a local Noetherian ring of positive characteristic. If S is a seed, then S maps to a balanced big Cohen-Macaulay algebra domain.*

Proof. By Proposition 5.4, S maps to a minimal seed, and Proposition 7.2 implies that the minimal seed is a domain. The previous lemma then implies that a minimal seed can be mapped to a balanced big Cohen-Macaulay algebra that is a domain. □

We can also show that all seeds map to big Cohen-Macaulay algebras with a host of nice properties. We will use an uncountable limit ordinal number in the proof and refer the reader to [HJ, Chapters 7 and 8] for the properties of such ordinal numbers.

Theorem 7.8. *Let (R, m) be a local Noetherian ring of positive characteristic. If S is a seed, then S maps to an absolutely integrally closed, m-adically separated, quasilocal balanced big Cohen-Macaulay algebra domain B.*

Proof. We will construct B as a direct limit of seeds indexed by an uncountable ordinal number. Let β be an uncountable initial ordinal of cardinality $\aleph_1$. Using transfinite induction, we will define an S-algebra S_α, for each ordinal number $\alpha < \beta$, and then we will define B to be the direct limit of all such S_α.

Let $S_0 = S$. Given a seed S_α, we can form a sequence

$$S_\alpha \to S_\alpha^{(1)} \to S_\alpha^{(2)} \to S_\alpha^{(3)} \to S_\alpha^{(4)} =: S_{\alpha+1},$$

where $S_\alpha^{(1)}$ is a minimal seed (and so a domain by Proposition 7.2), $S_\alpha^{(2)} = (S_\alpha^{(1)})^+$ (an integral extension of a seed and so a seed by Theorem 6.9), $S_\alpha^{(3)}$ is a quasilocal balanced big Cohen-Macaulay R-algebra (which $S_\alpha^{(2)}$ maps to by Lemma 3.4), and $S_\alpha^{(4)}$ is the m-adic completion of $S_\alpha^{(3)}$ (which is m-adically separated and a balanced big Cohen-Macaulay algebra by [BS, Theorem 1.7]). If α is a limit ordinal, then we will define

$$S_\alpha := \varinjlim_{\gamma<\alpha} S_\gamma.$$

Given our definition for S_α, for each ordinal $\alpha < \beta$, we define

$$B := S_\beta = \varinjlim_{\alpha<\beta} S_\alpha.$$

Since each S_α maps to a domain $S_\alpha^{(1)}$, and conversely, each $S_\alpha^{(1)}$ maps to $S_{\alpha+1}$, the ring B can be written as a direct limit of domains and is, therefore, also a domain. Similarly, B is also a direct limit of absolutely integrally closed domains (using $S_\alpha^{(2)}$ for each $\alpha < \beta$). If we let L be the algebraic closure of the fraction field of B and let K_α be the algebraic closure of the fraction field of $S_\alpha^{(2)}$ for each $\alpha < \beta$, then an element $u \in L$ satisfying a monic polynomial equation over B is the image of an element v in some K_α that satisfies a monic polynomial equation over $S_\alpha^{(2)}$

for some α. Since $S_\alpha^{(2)}$ is absolutely integrally closed, v is in $S_\alpha^{(2)}$, and so its image u in L is also in B. Therefore, B is absolutely integrally closed.

Using the rings $S_\alpha^{(3)}$, we can see that B is the direct limit of quasilocal balanced big Cohen-Macaulay R-algebras, and so B is itself a quasilocal balanced big Cohen-Macaulay algebra. Indeed, it is easy to see that

$$(x_1, \ldots, x_k)B :_B x_{k+1} \subseteq (x_1, \ldots x_k)B$$

for each partial system of parameters $x_1, \ldots, x_{k+1}$ of R as this fact is true in each $S_\alpha^{(3)}$. Furthermore, $mB \neq B$, as the opposite would imply that $mS_\alpha^{(3)} = S_\alpha^{(3)}$ for some α, which is impossible in a balanced big Cohen-Macaulay algebra. It is also straightforward to verify that a direct limit of quasilocal rings is quasilocal.

Finally, to see that B is m-adically separated, we note that B is a direct limit of the m-adically separated rings $S_\alpha^{(4)}$. Suppose that $u \in \bigcap_k m^k B$. Then for each k, there exists an ordinal $\alpha(k)$ such that $u \in m^k S_{\alpha(k)}^{(4)}$. Let α be the union of all the $\alpha(k)$. Since $\alpha(k) < \beta$ for all k, and β is uncountable, we see that $\alpha < \beta$. Therefore, $u \in \bigcap_k m^k S_\alpha$, and so $u \in \bigcap_k m^k S_\alpha^{(4)} = 0$. $\square$

8. Tensor products and base change

In this section, we provide positive answers to two previously open questions about big Cohen-Macaulay algebras. Given two big Cohen-Macaulay R-algebras B and B' over a complete local domain, does there exist a big Cohen-Macaulay algebra C such that

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & B' \arrow[u] \end{tikzcd}$$

commutes?

Another open question involves base change $R \to S$ between complete local domains. Given a big Cohen-Macaulay R-algebra B, can B be mapped to a big Cohen-Macaulay S-algebra C such that the diagram

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

commutes?

We will show that both of these questions have positive answers in positive characteristic. First, we address the question of why the tensor product of seeds is a seed. We will derive our result from the case of a regular local base ring and make use of tight closure and test elements for the general case.

By applying $T \otimes_S -$ to the sequence $0 \to I :_S x/I \to S/I \xrightarrow{x} S/I$, one has:

Lemma 8.1. *If S and T are any commutative rings such that T is flat over S, I is an ideal of S, and $x \in S$, then $IT :_T x = (I :_S x)T$.*

Lemma 8.2. *If C is a (balanced) big Cohen-Macaulay algebra over a local ring (S, n) and D is faithfully flat over C, then D is a (balanced) big Cohen-Macaulay S-algebra.*

Proof. Let $x_1, \ldots, x_{k+1}$ be part of a system of parameters for S, and let d be an element of $(x_1, \ldots, x_k)D :_D x_{k+1}$. Using the lemma above,

$$d \in ((x_1, \ldots, x_k)C :_C x_{k+1})D \subseteq ((x_1, \ldots, x_k)C)D = (x_1, \ldots, x_k)D,$$

because C is a (balanced) big Cohen-Macaulay S-algebra. Finally, as D is faithfully flat over C, and $nC \neq C$, $nD \neq D$ either. □

If A is a regular local ring, and B and B' are balanced big Cohen-Macaulay A-algebras, then B is faithfully flat over A by Proposition 2.2. Therefore, $B \otimes_A B'$ is faithfully flat over B', and since B' is a balanced big Cohen-Macaulay A-algebra, we can use the previous lemma to conclude:

Lemma 8.3. *If A is a regular local Noetherian ring, and B and B' are balanced big Cohen-Macaulay A-algebras, then $B \otimes_A B'$ is a balanced big Cohen-Macaulay A-algebra as well. Consequently, if S and S' are seeds over A, then $S \otimes_A S'$ is a seed over A.*

We can now establish our first result, concerning tensor products of seeds.

Theorem 8.4. *Let (R, m) be a complete local domain of positive characteristic. If $(S_i)_{i \in I}$ is an arbitrary family of seeds over R, then $\bigotimes_{i \in I} S_i$ is also a seed over R. Consequently, if B and B' are (balanced) big Cohen-Macaulay R-algebras, then there exists a balanced big Cohen-Macaulay R-algebra C filling the commutative diagram:*

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & B'. \arrow[u] \end{tikzcd}$$

Proof. Since a direct limit of seeds is a seed by Lemma 3.2, we may assume that I is a finite set. By induction, we may assume that I consists of two elements. We may then also assume that $S_1 = B$ and $S_2 = B'$ are balanced big Cohen-Macaulay R-algebras.

By the Cohen Structure Theorem, R is a module-finite extension of a complete regular local ring A. We reduce to the case that R is a separable extension of A. For any $q = p^e$, $R[A^{1/q}]$ is still a module-finite extension of $A^{1/q}$, a complete regular local ring. Furthermore, $B[A^{1/q}]$ and $B'[A^{1/q}]$ are clearly still balanced big Cohen-Macaulay $R[A^{1/q}]$-algebras, and if $B[A^{1/q}] \otimes_{R[A^{1/q}]} B'[A^{1/q}]$ is a seed over $R[A^{1/q}]$, then it is also a seed over R, which will show that $B \otimes_R B'$ is a seed over R. We may therefore replace A and R by $A^{1/q}$ and $R[A^{1/q}]$ for any $q \geq 1$. We claim that for any $q \gg 1$, we obtain a separable extension $A^{1/q} \to R[A^{1/q}]$. Indeed, suppose that R is not separable over A. If K is the fraction field of A, then $K \otimes_A R$ is a finite product of finite field extensions of K, one of which is not separable over K. It suffices to let L be a finite inseparable field extension of K and show that $L[K^{1/q}]$ is separable over $K^{1/q}$, for some $q \gg 1$. For any element y of L whose minimal polynomial is inseparable, we can find q sufficiently large so that the minimal polynomial of y in $L[K^{1/q}]$ over $K^{1/q}$ becomes separable. Since L is a finite extension, for any q sufficiently large, $L[K^{1/q}]$ becomes separable over $K^{1/q}$, which implies that $R[A^{1/q}]$ will be separable over $A^{1/q}$.

We can now assume without loss of generality that R is separable over A. Let J be the ideal of $R \otimes_A R$ generated by all elements killed by an element of A. Then

$R_0 := (R \otimes_A R)/J$ is a module-finite extension of A and a reduced ring as R_0 is also a separable extension of A.

Since B and B' are balanced big Cohen-Macaulay R-algebras and R is a module-finite extension of A, we also have that B and B' are balanced big Cohen-Macaulay A-algebras. By Lemma 8.3, $B \otimes_A B'$ is also a balanced big Cohen-Macaulay A-algebra. The R-algebra structures of B and B' induce a natural map from $R \otimes_A R$ to $B \otimes_A B'$. Since the latter ring is balanced big Cohen-Macaulay over A, the ideal J is contained in the kernel of the map to $B \otimes_A B'$. Therefore, this map factors through R_0.

Since R is a domain and a homomorphic image of $R \otimes_A R$ (and so of R_0), the kernel of $R_0 \to R$ is a prime ideal P, and the kernel of $B \otimes_A B' \to B \otimes_R B'$ is the extended ideal $P(B \otimes_A B')$. Since R_0 is a module-finite extension of A, $\dim R = \dim R_0$, so that P is a minimal prime of R_0. We therefore obtain a commutative diagram

$$\begin{tikzcd} B \otimes_A B' \arrow[r] & B \otimes_R B' \\ R_0 \arrow[u] \arrow[r] & R \arrow[u] \end{tikzcd}$$

where the horizontal maps are the result of killing the minimal prime ideal P of R_0 (resp., $P(B \otimes_A B')$).

We next construct a durable colon-killer in $B \otimes_R B'$. Since R_0 is reduced and P is a minimal prime, there exists $c' \notin P$ such that c' kills P and $P(B \otimes_A B')$. As $c' \notin P$, its image in R is nonzero. Since R is a complete local domain, R has a test element $c'' \neq 0$ by Theorem 2.11. Let $c = c'c''$, a nonzero test element of R, and let d be a lifting of c to R_0 so that $dP(B \otimes_A B') = 0$.

Now, suppose that $x_1, \ldots, x_n$ is a system of parameters of A, and suppose that $x_{k+1}u \in (x_1, \ldots, x_k)(B \otimes_R B')$ for some $k \leq n-1$. Then

$$x_{k+1}u \in ((x_1, \ldots, x_k) + P)(B \otimes_A B'),$$

and $x_{k+1}du \in (x_1, \ldots, x_k)(B \otimes_A B')$. Since $B \otimes_A B'$ is a balanced big Cohen-Macaulay A-algebra, $du \in (x_1, \ldots, x_k)(B \otimes_A B')$, and so $cu \in (x_1, \ldots, x_k)(B \otimes_R B')$, as c and d have the same image in $B \otimes_R B'$. Therefore, c is a colon-killer for systems of parameters of A in $B \otimes_R B'$. By Proposition 4.4, we may replace c by a power that is a colon-killer for systems of parameters of R.

Finally, suppose that $c^N \in \bigcap_k m^k(B \otimes_R B')$. Then [Ho2, Theorem 8.6] and Theorem 2.15 imply that $c^N \in \bigcap_k ((m^k)^*)$ since $B \otimes_R B'$ is solid over R by Proposition 2.14(b). As c is a test element in R, $c^{N+1} \in \bigcap_k m^k = 0$, a contradiction. Hence, c is a durable colon-killer in $B \otimes_R B'$, and so $B \otimes_R B'$ is a seed over R by Theorem 4.8. $\square$

We now proceed to the question of whether being a seed over a complete local domain of positive characteristic is a property that is preserved by base change to another complete local domain. If $R \to S$ is a map of complete local rings, then [AFH, Theorem 1.1] says that the map factors through a complete local ring R' such that $R \to R'$ is faithfully flat with regular closed fiber, and $R' \to S$ is surjective. It therefore suffices to treat the cases of a flat local map with regular closed fiber and the case of a surjective map with kernel a prime ideal. We start with an elementary lemma and then prove the result for the flat local case.

Lemma 8.5. *Let (A, m) be a local Noetherian ring, and let B be a flat, local Noetherian R-algebra. If $y_1, \ldots, y_t$ is a regular sequence on B/mB, then $y_1, \ldots, y_t$ is a regular sequence on B, and $B/(y_1, \ldots, y_t)B$ is flat over A.*

Proof. The proof is immediate by induction on t, where the base case of $t = 1$ is given by [Mat, (20.F)]. ☐

With this lemma, we are ready to prove our base change result for flat local maps. It is perhaps interesting to note that our argument only requires that the closed fiber is Cohen-Macaulay, not regular. Unlike the surjective case, we will not need to assume that our rings have positive characteristic, are complete, or are domains.

Proposition 8.6. *Let $R \to S$ be a flat local map of Noetherian local rings with a Cohen-Macaulay closed fiber S/mS, where m is the maximal ideal of R. If T is a seed over R, then $T \otimes_R S$ is a seed over S.*

Proof. It suffices to assume that $T = B$ is a balanced big Cohen-Macaulay R-algebra and show that a single system of parameters of S is a regular sequence on $B \otimes_R S$.

Fix a system of parameters $x_1, \ldots, x_d$ for R. Since S is faithfully flat over R, we have the dimension equality $\dim S = \dim R + \dim S/mS$. Hence, the images of $x_1, \ldots, x_d$ in S can be extended to a full system of parameters $x_1, \ldots, x_d, \ldots, x_n$ of S, where $x_{d+1}, \ldots, x_n$ is a system of parameters for S/mS. As $x_1, \ldots, x_d$ form a regular sequence on B for any $1 \leq k \leq d - 1$, we have an exact sequence:

$$0 \to B/(x_1, \ldots, x_k)B \xrightarrow{x_{k+1}} B/(x_1, \ldots, x_k)B,$$

and since S is flat over R, the sequence remains exact after tensoring with S. Thus, x_{k+1} is not a zerodivisor on

$$(B/(x_1, \ldots, x_k)B) \otimes_R S \cong (B \otimes_R S)/(x_1, \ldots, x_k)(B \otimes_R S)$$

for any $1 \leq k \leq d - 1$.

It now suffices to show that $x_{d+1}, \ldots, x_n$ is a regular sequence on the quotient

$$\overline{B} := (B \otimes_R S)/(x_1, \ldots, x_k)(B \otimes_R S).$$

Let $I := (x_1, \ldots, x_d)R$, and let $\overline{R} := R/I$, and $\overline{S} := S/IS$. Then $\overline{S}$ is faithfully flat over $\overline{R}$, and since $x_{d+1}, \ldots, x_n$ is a system of parameters for the Cohen-Macaulay ring S/mS, it is a regular sequence on $S/mS \cong \overline{S}/\overline{m}\overline{S}$, where $\overline{m} = m/I$, the maximal ideal of $\overline{R}$. We can now apply Lemma 8.5 to $\overline{R}$ and $\overline{S}$ to conclude that $x_{d+1}, \ldots, x_n$ is a regular sequence on $\overline{S}$ and that $\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}$ is flat over $\overline{R}$ for all $d + 1 \leq k \leq n$.

For any $d \leq k \leq n - 1$, we have a short exact sequence

$$0 \to \overline{S}/(x_{d+1}, \ldots, x_k)\overline{S} \xrightarrow{x_{k+1}} \overline{S}/(x_{d+1}, \ldots, x_k)\overline{S} \to \overline{S}/(x_{d+1}, \ldots, x_k, x_{k+1})\overline{S} \to 0,$$

where $x_{d+1}, \ldots, x_k$ is the empty sequence when $k = d$. Since $\overline{S}/(x_{d+1}, \ldots, x_{k+1})\overline{S}$ is flat over $\overline{R}$, we have $\operatorname{Tor}_1^{\overline{R}}(\overline{B}, \overline{S}/(x_{d+1}, \ldots, x_{k+1})\overline{S}) = 0$. Therefore,

$$0 \to \overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}) \xrightarrow{x_{k+1}} \overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S})$$

is exact, and since

$$\overline{B} \otimes_{\overline{R}} (\overline{S}/(x_{d+1}, \ldots, x_k)\overline{S}) \cong (\overline{B} \otimes_{\overline{R}} \overline{S})/(x_{d+1}, \ldots, x_k)(\overline{B} \otimes_{\overline{R}} \overline{S}),$$

$x_{d+1}, \ldots, x_n$ is a possibly improper regular sequence on $\overline{B} \otimes_{\overline{R}} \overline{S}$. We can now finally see that $x_1, \ldots, x_d, x_{d+1}, \ldots, x_n$ is a possibly improper regular sequence on

$B \otimes_R S$ as $\overline{B} \otimes_{\overline{R}} \overline{S} \cong (B \otimes_R S)/I(B \otimes_R S)$. If, however, $(B \otimes_R S)/n(B \otimes_R S) = 0$, where n is the maximal ideal of S, then $B \otimes_R (S/nS) = 0$, which implies that the product $(B/mB) \otimes_{R/m} (S/nS) = 0$ over the field R/m. Therefore, $B/mB = 0$ or $S/nS = 0$, but neither of these occurs, so we have a contradiction. Hence, $x_1, \ldots, x_n$ is a regular sequence on $B \otimes_R S$, and so $B \otimes_R S$ is a seed. $\square$

We will now handle the case of a surjective map $R \to S$ of complete local domains of positive characteristic. First we demonstrate the result when R is normal and then show how the problem can be reduced to the normal case using Theorem 6.9.

In the normal case, we will make use of test elements again. When R is normal, the defining ideal I of the singular locus has height at least 2. Since R_c is regular for any $c \in I$, [HH4, Theorem 6.1] implies that some power c^N is a test element when R is a reduced excellent local ring. Hence, if P is a height 1 prime of R, there exists a test element c of R not in P. We record this fact in the following lemma.

Lemma 8.7. *Let R be a normal excellent local ring of positive characteristic. If P is a height 1 prime of R, then there exists a test element $c \in R \setminus P$.*

Lemma 8.8. *Let (R, m) be a complete local, normal domain of positive characteristic, and let $S = R/P$, where P is a height 1 prime of R. If T is a seed over R, then T/PT is a seed over S.*

Proof. We can assume that $B = T$ is a balanced big Cohen-Macaulay R-algebra. Since R is normal and ht $P = 1$, we see that R_P is a DVR. Therefore, PR_P is a principal ideal generated by the image of $x \in R$. Each element of P is then multiplied into xR by some element of $R \setminus P$, and since P is finitely generated, there exists $c' \in R \setminus P$ such that $c'P \subseteq xR$. By Lemma 8.7, there exists a test element $c'' \in R \setminus P$, and so if we put $c := c'c''$, then c is a test element, $cP \subseteq xR$, and c is not in P.

We claim that c is a weak durable colon-killer for B/PB. Extend x to a full system of parameters $x, x_2, \ldots, x_d$ for R. Then $x_2^t, \ldots, x_d^t$ is a system of parameters for S for any $t \in \mathbb{N}$. Suppose that $bx_{k+1}^t \in (x_2^t, \ldots, x_k^t)B/PB$ for some $k \leq d-1$ and some t. This relation lifts to a relation $bx_{k+1}^t \in (x_2^t, \ldots, x_k^t)B + PB$ in B, and so $cbx_{k+1}^t \in (x, x_2^t, \ldots, x_k^t)B$. Since B is a balanced big Cohen-Macaulay R-algebra, we have $cb \in (x, x_2^t, \ldots, x_k^t)B$, and so $cb \in (x_2^t, \ldots, x_k^t)B/PB$.

To finish, suppose that $c^N \in \bigcap_k (m/P)^k B/PB$. We can then lift to B to obtain $c^N \in \bigcap_k (m^k + P)B$. Since R is a complete local domain, and B is a balanced big Cohen-Macaulay R-algebra, Theorem 2.16 implies that $c^N \in \bigcap_k (m^k + P)^*$. Since c was chosen to be a test element, we have $c^{N+1} \in \bigcap_k (m^k + P) = P$, a contradiction as $c \notin P$. Therefore, the image of c in B/PB is a weak durable colon-killer, and so B/PB is a seed over S by Theorem 4.8. $\square$

We finally treat the case of an arbitrary surjection of complete local domains by reducing to the case of the previous lemma.

Proposition 8.9. *Let $R \to S$ be a surjective map of positive characteristic complete local domains. If T is a seed over R, then $T \otimes_R S$ is a seed over S.*

Proof. We can immediately assume that the kernel of $R \to S$ is a height 1 prime P of R. Let R' be the normalization of R in its fraction field. Then R' is also a complete local domain. Since R' is an integral extension of R, there exists a height 1 prime Q lying over P.

By Corollary 7.7, T maps to a big Cohen-Macaulay R-algebra domain B, and so we may replace T by B and assume that T is a domain. We then have an integral extension $T[R']$ of T inside the fraction field of T. Since T is a seed over R, Theorem 6.9 implies that $T[R']$ is also a seed over R. Therefore, $T[R']$ maps to a balanced big Cohen-Macaulay R-algebra C (which is also an R'-algebra), and so Corollary 4.6 implies that C is a balanced big Cohen-Macaulay R'-algebra. We now have the commutative diagram:

$$\begin{tikzcd}
 & C \arrow[rr] & & C/QC \\
T \arrow[ur] \arrow[rr] & & T/PT \arrow[ur] & \\
 & R' \arrow[uu] \arrow[rr] & & R'/Q \arrow[uu] \\
R \arrow[uu] \arrow[ur] \arrow[rr] & & S \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

since $S = R/P$ and Q lies over P. By Lemma 8.8, C/QC is a seed over R'/Q. Since R'/Q is an integral extension of $S = R/P$, every system of parameters of S is a system of parameters of R'/Q, and so C/QC is also a seed over S, but this implies that T/PT is also a seed over S, as needed. $\square$

Now that we have shown that the property of being a seed over a complete local domain is preserved by flat base change with a regular closed fiber (Proposition 8.6) and by surjections (Proposition 8.9), we may apply [AFH, Theorem 1.1] to factor any map of complete local domains into these two maps. We therefore arrive at the following theorem, which answers the base change question asked at the beginning of the section.

Theorem 8.10. *Let $R \to S$ be a local map of positive characteristic complete local domains. If T is a seed over R, then $T \otimes_R S$ is a seed over S. Consequently, if B is a big Cohen-Macaulay R-algebra, then there exists a balanced big Cohen-Macaulay S-algebra C filling the commutative diagram:*

$$\begin{tikzcd}
B \arrow[r] & C \\
R \arrow[u] \arrow[r] & S. \arrow[u]
\end{tikzcd}$$

9. Seeds and tight closure in positive characteristic

Using Theorems 8.4 and 8.10, we can now use the class of all balanced big Cohen-Macaulay R-algebras $\mathscr{B}(R)$, where R is a complete local domain of characteristic p, to define a closure operation for all Noetherian rings of positive characteristic. A key point is that $\mathscr{B}(R)$ is a directed family and has the right base change properties. (It is easy to see that the two definitions of $\natural$-closure below coincide for complete local domains.)

Definition 9.1. Let R be a complete local domain of positive characteristic, and let $N \subseteq M$ be finitely generated R-modules. Let $N^{\natural}_M$ be the set of all elements

$u \in M$ such that $1 \otimes u \in \operatorname{Im}(B \otimes_R N \to B \otimes_R M)$ for some big Cohen-Macaulay R-algebra B.

Let S be a Noetherian ring of positive characteristic, and let $N \subseteq M$ be finitely generated S-modules. Let $N_M^\natural$ be the set of all $u \in M$ such that for all S-algebras T, where T is a complete local domain, $1 \otimes u \in \operatorname{Im}(T \otimes_S N \to T \otimes_S M)^\natural_{T \otimes_S M}$. We will call $N_M^\natural$ the $\natural$-*closure* of N in M.

By Theorem 2.16, our new closure operation is equivalent to tight closure for complete local domains of positive characteristic, but the results above imply many of the properties one would want in a good closure operation directly from the properties of big Cohen-Macaulay algebras, independent of tight closure. These properties include persistence, $(IS)^\natural_S \cap R \subseteq I^\natural_R$ for module-finite extensions, $I^\natural = I$ for ideals in a regular ring, phantom acyclicity, and colon-capturing. (See Theorem 2.12 and [HH1, Section 9] for the statement of these properties for tight closure. See [D] for the statements and proofs for $\natural$-closure.)

These results add evidence to the idea that such a closure operation can be defined for more general classes of rings.

Acknowledgments

I thank my advisor, Mel Hochster, for all of the time, help, and insight given to me during the production of this work. I also thank the entire Mathematics Department at the University of Michigan for the five years of education and support provided to me while I was a graduate student.

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