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A SHARP FORM OF THE MOSER-TRUDINGER INEQUALITY ON A COMPACT RIEMANNIAN SURFACE

YUNYAN YANG

Abstract. In this paper, a sharp form of the Moser-Trudinger inequality is established on a compact Riemannian surface via the method of blow-up analysis, and the existence of an extremal function for such an inequality is proved.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $H_0^1(\Omega)$ be the Sobolev space consisting of functions which vanish on the boundary of Ω and whose gradient is in $L^2(\Omega)$. The famous Moser-Trudinger inequality (see Moser [14]; Trudinger [17]) states the following:

$$\sup_{u\in H_0^1(\Omega),\,\|\nabla u\|_2=1}\int_\Omega e^{4\pi u^2}dx<+\infty. \tag{1.1}$$

For any $p>4\pi$, there exists a sequence $\{u_\epsilon\}_{\epsilon>0}\subset H_0^1(\Omega)$ with $\|\nabla u_\epsilon\|_2=1$ verifying that $\int_\Omega e^{pu_\epsilon^2}dx\to+\infty$. On the other hand, $\forall u\in H_0^1(\Omega)$, $\int_\Omega e^{pu^2}dx<+\infty$ for any $p>0$. Furthermore, P. L. Lions [13] obtained the following:

Theorem A (Lions). *Let $\{u_\epsilon\}_{\epsilon>0}\subset H_0^1(\Omega)$ with $\|\nabla u_\epsilon\|_2=1$ such that $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Then for any $p<1/(1-\|\nabla u_0\|_2^2)$,*

$$\limsup_{\epsilon\to0}\int_\Omega e^{4\pi p u_\epsilon^2}dx<+\infty. \tag{1.2}$$

When $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_0\neq0$, (1.2) gives more precise information than (1.1). But if $u_0=0$, (1.2) is a consequence of (1.1). However Adimurthi and Druet [1] proved the following:

Theorem B (Adimurthi-Druet). *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and let $\lambda_1(\Omega)>0$ be the first eigenvalue of the Laplacian with Dirichlet boundary condition in Ω. Then we have (i) for any $0\le\alpha<\lambda_1(\Omega)$,*

$$\sup_{u\in H_0^1(\Omega),\,\|\nabla u\|_2=1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx<+\infty,$$

and (ii) for any $\alpha\ge\lambda_1(\Omega)$,

$$\sup_{u\in H_0^1(\Omega),\,\|\nabla u\|_2=1}\int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}dx=+\infty.$$

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This result is of a different nature from Theorem A. When $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_0 \neq 0$, the inequality in Theorem B is a consequence of Theorem A. But Theorem B gives new information when $u_\epsilon \rightharpoonup 0$ weakly in $H_0^1(\Omega)$.

In this paper, we consider the same inequalities as that of Theorem B on a compact Riemannian surface. Let (Σ, g) be a compact Riemannian surface without boundary, $H^{1,2}(\Sigma)$ the completion of $C^\infty(\Sigma)$ in the norm

$$\|u\|_{H^{1,2}(\Sigma)}^2 = \int_\Sigma (|u|^2 + |\nabla u|^2) dV_g < +\infty.$$

Denote

$$\mathcal{H} = \left\{ u \in H^{1,2}(\Sigma) : \|\nabla u\|_2 = 1, \int_\Sigma u dV_g = 0 \right\};$$

here and in the sequel, $\|\cdot\|_p$ denotes the L^p-norm $(\int_\Sigma |\cdot|^p dV_g)^{1/p}$. Recall that the first eigenvalue of the Laplacian on Σ is defined by

$$\lambda_1(\Sigma) = \inf_{u \in H^{1,2}(\Sigma),\, \int_\Sigma u dV_g = 0,\, u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}. \tag{1.3}$$

Write

$$C_\alpha(\Sigma) = \sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} dV_g. \tag{1.4}$$

For simplicity, we denote

$$J_\beta^\alpha(u) = \int_\Sigma e^{\beta u^2 (1 + \alpha \|u\|_2^2)} dV_g. \tag{1.5}$$

Then we can state our main results as follows:

Theorem 1.1. *Let (Σ, g) be a compact Riemannian surface without boundary. Then we have*

(1) *For any $0 \leq \alpha < \lambda_1(\Sigma)$, $C_\alpha(\Sigma) < +\infty$.*
(2) *For any $\alpha \geq \lambda_1(\Sigma)$, $C_\alpha = +\infty$.*
(3) *For sufficiently small $\alpha > 0$, there exists a $u_\alpha \in \mathcal{H}$ such that $C_\alpha(\Sigma) = J_{4\pi}^\alpha(u_\alpha)$.*

As we have explained before, this result gives more information than the usual Moser-Trudinger inequality on a compact Riemannian surface (see for example [9], [10]). We follow the lines of the proof of Adimurthi and Druet [1]. First, we choose test functions to prove (2). Then we use blow-up analysis to prove (1), and finally use the capacity technique to prove (3).

Similarly, we have the following:

Theorem 1.2. *Let (Σ, g) be a compact Riemannian surface without boundary. Denote*

$$\mathcal{H}_1 = \{u \in H^{1,2}(\Sigma) : \int_\Sigma (|u|^2 + |\nabla u|^2) dV_g = 1\}$$

and

$$C_\alpha^1(\Sigma) = \sup_{u \in \mathcal{H}_1} \int_\Sigma e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} dV_g.$$

Then we have

(1) *For any $0 \leq \alpha < \lambda_1(\Sigma)$, $C_\alpha^1(\Sigma) < +\infty$.*
(2) *For any $\alpha \geq \lambda_1(\Sigma)$, $C_\alpha^1(\Sigma) = +\infty$.*

(3) *For sufficiently small* $\alpha > 0$, *there exists a* $u_\alpha \in \mathcal{H}_1$ *such that* $C^1_\alpha(\Sigma) = J^\alpha_{4\pi}(u_\alpha)$.

Since the proof of Theorem 1.2 is completely analogous to that of Theorem 1.1, we omit it in this paper.

Extremal functions for critical functionals can be obtained by the method of blow-up analysis. In 1984, Schoen [15] solved the Yamabe problem. In 1986, Escobar and Schoen [7] found conformal metrics with prescribed curvatures in high dimensions. In 1997, Ding, Jost, Li and Wang [5] proved the solvability of the equation $\triangle u = 8\pi - he^{8\pi u}$ on a compact Riemannian surface. For the existence of extremal functions for the classical Moser-Trudinger inequality, we would like to mention Carleson and Chang [3], Flucher [8], Lin [12], Li [10] and Li–Liu [11]. About extremals for optimal Sobolev inequalities on Riemannian manifolds, we refer the reader to Druet and Hebey [6] and the references therein.

Throughout this paper we denote the Laplacian and the gradient on Σ by $\triangle$ and ∇ , those on $\mathbb{R}^2$ by $\triangle_{\mathbb{R}^2}$ and $\nabla_{\mathbb{R}^2}$ respectively.

We organize this paper as follows: In section 2, we construct test functions to prove point (2) of Theorem 1.1. In section 3, we prove the existence of a maximizer of a subcritical functional $J^\alpha_{4\pi-\epsilon}$, and give the corresponding Euler–Lagrange equation. Section 4 contributes to the asymptotic behavior of the maximizers through blow-up analysis. An upper bound of $J^\alpha_{4\pi}$ is derived in section 5 under the assumption that blow-up occurs. In the last section, we construct a sequence of functions to show that the upper bound of $J^\alpha_{4\pi}$ is in fact greater than the one we derived in section 5.

2. The test functions

In this section, following Adimurthi and Druet [1], we choose test functions to prove point (2) of Theorem 1.1. Let u_0 be a weak solution of

$$\begin{cases} -\triangle u_0 = \lambda_1(\Sigma) u_0 \quad \text{in} \quad \Sigma, \\ \int_\Sigma u_0 dV_g = 0, \quad \|u_0\|_2^2 = 1. \end{cases}$$

By elliptic estimates, $u_0 \in C^\infty(\Sigma)$. The fact that $\int_\Sigma u_0 dV_g = 0$ implies that there exists some $p \in \Sigma$ such that $u_0(p) > 0$, and a domain $U \subset \Sigma$ such that $p \in U$ and $u_0 \geq u_0(p)/2$ in U. Choose an isothermal coordinate system (V, ψ) around p such that $V \subset U$, $\psi : V \to \mathbb{B}_\delta = \{x \in \mathbb{R}^2 : |x| \leq \delta\}$, $\psi(p) = 0$. In this coordinate system, the metric g can be represented by $g = e^{2f}(dx_1^2 + dx_2^2)$, where f is a smooth function with $f(0) = 0$.

For any $x \in \mathbb{B}_\delta$, let

$$m_\epsilon(x) = \begin{cases} \sqrt{\frac{1}{4\pi}\log\frac{1}{\epsilon}}, & |x| \leq \delta\sqrt{\epsilon}, \\ \frac{1}{\sqrt{\pi \log \frac{1}{\epsilon}}} \log \frac{\delta}{|x|}, & \delta\sqrt{\epsilon} < |x| \leq \delta. \end{cases}$$

We set

$$u_\epsilon = \begin{cases} m_\epsilon \circ \psi & \text{in} \quad \psi^{-1}(\mathbb{B}_\delta), \\ l_\epsilon \varphi & \text{in} \quad \Sigma \setminus \psi^{-1}(\mathbb{B}_\delta), \end{cases}$$

where $\varphi \in C_0^\infty(\Sigma \setminus \psi^{-1}(\mathbb{B}_\delta))$ and l_ϵ is a real number such that $\int_\Sigma u_\epsilon dV_g = 0$.

It is not difficult to check that

$$l_\epsilon = O(1/(\log\tfrac{1}{\epsilon})^{1/2}), \qquad \|\nabla u_\epsilon\|_2^2 = 1 + O(1/\log\frac{1}{\epsilon}),$$
$$\|u_\epsilon\|_1 = O(1/(\log\tfrac{1}{\epsilon})^{1/2}), \qquad \|u_\epsilon\|_2^2 = O(1/\log\frac{1}{\epsilon}).$$

Setting $v_\epsilon = u_\epsilon + t_\epsilon u_0$ with $t_\epsilon \to 0$, we get $t_\epsilon^2 \log\frac{1}{\epsilon} \to +\infty$ and $t_\epsilon^2(\log\frac{1}{\epsilon})^{1/2} \to 0$. Then we have

$$\begin{aligned}
\|v_\epsilon\|_2^2 &= \|u_\epsilon\|_2^2 + t_\epsilon^2\|u_0\|_2^2 + 2t_\epsilon\int_\Sigma u_\epsilon u_0 dV_g \\
&= t_\epsilon^2 + 2t_\epsilon\int_\Sigma u_\epsilon u_0 dV_g + O(1/\log\frac{1}{\epsilon}),\\
\|\nabla v_\epsilon\|_2^2 &= \|\nabla u_\epsilon\|_2^2 + t_\epsilon^2\|\nabla u_0\|_2^2 + 2t_\epsilon\int_\Sigma \nabla u_\epsilon \nabla u_0 dV_g \\
&= 1 + 2\lambda_1(\Sigma)t_\epsilon\int_\Sigma u_\epsilon u_0 dV_g + \lambda_1(\Sigma)t_\epsilon^2 + O(1/\log\frac{1}{\epsilon}),\\
\frac{1}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) &= 1 + (\alpha - \lambda_1(\Sigma))\left(t_\epsilon^2 + 2t_\epsilon\int_\Sigma u_\epsilon u_0 dV_g\right)\\
&\quad + \ o(t_\epsilon/(\log\frac{1}{\epsilon})^{1/2}).
\end{aligned}$$

We have for $\alpha \geq \lambda_1(\Sigma)$,

$$\frac{1}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) \geq 1 + o(t_\epsilon/(\log\frac{1}{\epsilon})^{1/2}).$$

Note that on $\psi^{-1}(\mathbb{B}_{\delta\sqrt{\epsilon}})$,

$$\begin{aligned}
4\pi\frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right) &\geq 4\pi\left(t_\epsilon^2 u_0^2 + \frac{1}{4\pi}\log\frac{1}{\epsilon} + 2t_\epsilon(\frac{1}{4\pi}\log\frac{1}{\epsilon})^{1/2}u_0\right)\\
&\quad \times(1 + o(t_\epsilon/(\log\frac{1}{\epsilon})^{1/2}))\\
&\geq \log\frac{1}{\epsilon} + t_\epsilon(\log\frac{1}{\epsilon})^{1/2}(4\sqrt{\pi}u_0 + o(1)).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_\Sigma e^{4\pi\frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right)} dV_g &\geq \int_{\psi^{-1}(\mathbb{B}_{\delta\sqrt{\epsilon}})} \frac{1}{\epsilon} e^{t_\epsilon\sqrt{\log\frac{1}{\epsilon}}(4\sqrt{\pi}u_0 + o(1))} dV_g\\
&\geq C(\delta)e^{t_\epsilon\sqrt{\log\frac{1}{\epsilon}}(2\sqrt{\pi}u_0(p)+o(1))}.
\end{aligned}$$

Since $u_0(p) > 0$, then $\int_\Sigma e^{4\pi\frac{v_\epsilon^2}{\|\nabla v_\epsilon\|_2^2}\left(1+\alpha\frac{\|v_\epsilon\|_2^2}{\|\nabla v_\epsilon\|_2^2}\right)} dV_g \to +\infty$ as $\epsilon \to 0$. This completes the proof of (2) of Theorem 1.1. □

3. Existence of maximizers for subcritical functionals

In this section, we will prove the existence of maximizers of subcritical functionals. Similar to P. L. Lions [13], we have the following:

Lemma 3.1. *Let $\{u_\epsilon\}_{\epsilon>0}$ be a sequence of functions in $\mathcal{H}$ with $u_\epsilon \rightharpoonup u_0$ weakly in $H^{1,2}(\Sigma)$. Then for any $p < 1/(1 - \|\nabla u_0\|_2^2)$, $\limsup_{\epsilon\to 0}\int_\Sigma e^{4\pi p u_\epsilon^2} dV_g < +\infty$.*

Proof. Clearly we have $\int_\Sigma u_0 dV_g = 0$. If $u_0 \neq 0$, then one can see that

$$\|\nabla(u_\epsilon - u_0)\|_2^2 \to 1 - \|\nabla u_0\|_2^2 < 1.$$

Hence we have for $p < 1/(1 - \|\nabla u_0\|_2^2)$,

$$\begin{aligned}\int_\Sigma e^{4\pi p u_\epsilon^2} dV_g &\leq \int_\Sigma e^{4\pi p(1+\delta)(u_\epsilon - u_0)^2 + 4\pi p(1+1/\delta)u_0^2} dV_g \\ &\leq \left(\int_\Sigma e^{4\pi \frac{(u_\epsilon - u_0)^2}{\|\nabla(u_\epsilon - u_0)\|_2^2}} dV_g\right)^{1/r} \left(\int_\Sigma e^{4\pi p' u_0^2} dV_g\right)^{1/s}\end{aligned}$$

for some $\delta > 0$ and $p' > p$ provided that ϵ is sufficiently small , where $1/r + 1/s = 1$. By the Orlicz embedding, $e^{u_0^2}$ is bounded in $L^q(\Sigma)$ for any $q > 1$. A result of Fontana [9] gives $\sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi u^2} dV_g < +\infty$. Hence

$$\limsup_{\epsilon \to 0} \int_\Sigma e^{4\pi p u_\epsilon^2} dV_g < +\infty. \tag{3.1}$$

If $u_0 = 0$, (3.1) is an immediate corollary of Fontana's result. □

Lemma 3.2. *Let $0 \leq \alpha < \lambda_1(\Sigma)$. For any $\epsilon > 0$, there exists a $u_\epsilon \in C^\infty(\Sigma) \cap \mathcal{H}$ such that*

$$J_{4\pi-\epsilon}^\alpha(u_\epsilon) = \sup_{u \in \mathcal{H}} J_{4\pi-\epsilon}^\alpha(u).$$

Proof. For any fixed $\epsilon > 0$, we choose a maximizing sequence $\{u_i\} \subset \mathcal{H}$ such that

$$J_{4\pi-\epsilon}^\alpha(u_i) \to \sup_{u \in \mathcal{H}} J_{4\pi-\epsilon}^\alpha(u) \quad \text{as} \quad i \to +\infty.$$

Since $\{u_i\}$ is bounded in $H^{1,2}(\Sigma)$, we have

$$\begin{aligned} u_i &\rightharpoonup u_\epsilon \quad \text{weakly} \quad \text{in} \quad H^{1,2}(\Sigma), \\ u_i &\to u_\epsilon \quad \text{strongly} \quad \text{in} \quad L^2(\Sigma), \\ u_i &\to u_\epsilon \quad \text{a.e.} \quad \text{in} \quad \Sigma. \end{aligned}$$

Hence

$$f_i = e^{(4\pi-\epsilon)u_i^2(1+\alpha\|u_i\|_2^2)} \to f_\epsilon = e^{(4\pi-\epsilon)u_\epsilon^2(1+\alpha\|u_\epsilon\|_2^2)} \quad \text{a.e.} \quad \text{in} \quad \Sigma.$$

If we suppose $u_\epsilon = 0$, then we have $1 + \alpha\|u_i\|_2^2 \to 1$. Since $\int_\Sigma e^{4\pi u_i^2} dV_g < +\infty$, we have f_i is bounded in $L^p(\Sigma)$ for some $p > 1$ and $f_i \to 1$ in $L^1(\Sigma)$. Hence $\mathrm{Vol}(\Sigma) = \sup_{u \in \mathcal{H}} J_{4\pi-\epsilon}^\alpha(u)$, which is impossible. Therefore $u_\epsilon \neq 0$. By Lemma 3.1, we have for $p < 1/(1 - \|\nabla u_\epsilon\|_2^2)$,

$$\limsup_{i \to +\infty} \int_\Sigma e^{4\pi p u_i^2} dV_g < +\infty.$$

Since $0 \leq \alpha < \lambda_1(\Sigma)$, we have

$$1 + \alpha\|u_i\|_2^2 \to 1 + \alpha\|u_\epsilon\|_2^2 < \frac{1}{1 - \|\nabla u_\epsilon\|_2^2}.$$

Hence f_i is bounded in $L^p(\Sigma)$ for some $p > 1$. Since $f_i \to f_\epsilon$ a.e. in Σ, then $f_i \to f_\epsilon$ strongly in $L^1(\Sigma)$. Therefore $\int_\Sigma f_\epsilon dV_g = \sup_{u \in \mathcal{H}} J_{4\pi-\epsilon}^\alpha(u)$, and $u_\epsilon \in \mathcal{H}$. □

It is not difficult to check that u_ϵ satisfies

$$\begin{cases} -\triangle u_\epsilon = \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon u_\epsilon - \frac{\mu_\epsilon}{\lambda_\epsilon} \\ \int_\Sigma u_\epsilon dV_g = 0, \quad \|\nabla u_\epsilon\|_2 = 1 \\ \alpha_\epsilon = (4\pi - \epsilon)(1 + \alpha \|u_\epsilon\|_2^2) \\ \beta_\epsilon = (1 + \alpha \|u_\epsilon\|_2^2)/(1 + 2\alpha \|u_\epsilon\|_2^2) \\ \gamma_\epsilon = \alpha/(1 + 2\alpha \|u_\epsilon\|_2^2) \\ \lambda_\epsilon = \int_\Sigma u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dV_g \\ \mu_\epsilon = \beta_\epsilon \int_\Sigma u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g. \end{cases} \tag{3.2}$$

4. Blow-up analysis

In this section, we will use blow-up analysis to understand the asymptotic behavior of the maximizers u_ϵ. We proceed as Adimurthi and Druet did in [1]. Firstly we have

Lemma 4.1. $\liminf_{\epsilon \to 0} \lambda_\epsilon > 0$.

Proof. Using the elementary inequality $e^t \leq 1 + te^t$ for $t \geq 0$, one has

$$\int_\Sigma e^{\alpha_\epsilon u_\epsilon^2} dV_g \leq Vol(\Sigma) + \alpha_\epsilon \lambda_\epsilon.$$

On the other hand,

$$\lim_{\epsilon \to 0} \int_\Sigma e^{\alpha_\epsilon u_\epsilon^2} dV_g = \sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} dV_g > Vol(\Sigma).$$

The above two inequalities, together with the fact that α_ϵ is bounded, imply the result. □

Lemma 4.2. *$\mu_\epsilon/\lambda_\epsilon$ is bounded.*

Proof. By (3.2), we have

$$\begin{aligned} \frac{|\mu_\epsilon|}{\lambda_\epsilon} &\leq \beta_\epsilon \int_{|u_\epsilon| \geq 1} \frac{|u_\epsilon|}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon^2} dV_g + \beta_\epsilon \int_{|u_\epsilon| < 1} \frac{|u_\epsilon|}{\lambda_\epsilon} e^{\alpha_\epsilon u_\epsilon^2} dV_g \\ &\leq 1 + \frac{e^{\alpha_\epsilon}}{\lambda_\epsilon} Vol(\Sigma) \leq C; \end{aligned}$$

here we have used Lemma 4.1 and $\beta_\epsilon \leq 1$. □

Let $c_\epsilon = |u_\epsilon(x_\epsilon)| = \max_\Sigma |u_\epsilon|$. If c_ϵ is bounded, by the standard elliptic estimates, Theorem 1.1 holds. Without loss of generality, we may assume in the following that

$$x_\epsilon \to p, \quad u_\epsilon(x_\epsilon) \to +\infty \tag{4.2}$$

as $\epsilon \to 0$. Here and in the sequel, we do not distinguish sequence and subsequence; the reader can understand it easily from the context.

Lemma 4.3. *$u_\epsilon \rightharpoonup 0$ weakly in $H^{1,2}(\Sigma)$, $u_\epsilon \to 0$ strongly in $L^2(\Sigma)$, and $|\nabla u_\epsilon|^2 dx \rightharpoonup \delta_p$ in the sense of measure, where δ_p is the Dirac measure at p.*

Proof. We may assume $u_\epsilon \rightharpoonup u_0$ weakly in $H^{1,2}(\Sigma)$. Obviously $\int_\Sigma u_0 dV_g = 0$. If we suppose $u_0 \neq 0$, then we have

$$1 + \alpha\|u_\epsilon\|_2^2 \to 1 + \alpha\|u_0\|_2^2 \leq 1 + \|\nabla u_0\|^2 < \frac{1}{1 - \|\nabla u_0\|_2^2}.$$

By Lemma 3.1, one has $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^p(\Sigma)$ for some $p > 1$ provided that ϵ is sufficiently small. Applying the elliptic estimates to equation (3.2), one gets c_ϵ is bounded, and a contradiction. Hence $u_0 = 0$, whence $\alpha_\epsilon \to 4\pi$, $\beta_\epsilon \to 1$ and $\gamma_\epsilon \to \alpha$. Assume $|\nabla u_\epsilon|^2 dV_g \rightharpoonup \mu$ in the sense of measure. If $\mu \neq \delta_p$ for all $p \in \Sigma$, then the usual truncation and covering arguments imply that $e^{\alpha_\epsilon u_\epsilon^2}$ is bounded in $L^q(\Sigma)$ for some $q > 1$. Applying elliptic estimates again to equation (3.2), we have u_ϵ is bounded in $L^\infty(\Sigma)$, which contradicts (4.2). □

Take an isothermal coordinate system (Ω, ϕ) near p such that $\phi(p) = 0$. In such coordinates, the metric g has the representation $g = e^{2f}(dx_1^2 + dx_2^2)$ with $f(0) = 0$. Let $r_\epsilon^2 = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^2} e^{-\alpha_\epsilon c_\epsilon^2}$ and $\Omega_\epsilon = \{x \in \mathbb{R}^2 : x_\epsilon + r_\epsilon x \in \Omega\}$. Let $\psi_\epsilon(x) = u(x_\epsilon + r_\epsilon x)/c_\epsilon$ and

$$\varphi_\epsilon(x) = c_\epsilon(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon), \quad x \in \Omega_\epsilon.$$

Then we have the following:

Lemma 4.4. *For any* $0 < \alpha < 4\pi$*, we have* $r_\epsilon^2 e^{\alpha c_\epsilon^2} \to 0$.

Proof. A straightforward calculation shows that, for $0 < \alpha < 4\pi$,

$$\begin{aligned} r_\epsilon^2 e^{\alpha c_\epsilon^2} &= \frac{e^{(\alpha - \alpha_\epsilon)c_\epsilon^2}}{\beta_\epsilon c_\epsilon^2} \int_\Sigma u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dV_g \\ &\leq \frac{1}{\beta_\epsilon c_\epsilon^2} \int_\Sigma u_\epsilon^2 e^{\alpha u_\epsilon^2} dV_g \end{aligned}$$

for sufficiently small ϵ. Obviously $u_\epsilon^2 e^{\alpha u_\epsilon^2}$ is bounded in $L^1(\Sigma)$, which gives the result immediately. □

By (3.2), we have

$$(4.3) \qquad \begin{cases} -\triangle\psi_\epsilon = \frac{1}{c_\epsilon^2}\psi_\epsilon e^{\alpha_\epsilon(u_\epsilon^2 - c_\epsilon^2)} + r_\epsilon^2\gamma_\epsilon\psi_\epsilon - r_\epsilon^2\frac{\mu_\epsilon}{c_\epsilon\lambda_\epsilon}, \\ -\triangle\varphi_\epsilon = \psi_\epsilon e^{\alpha_\epsilon\varphi_\epsilon(1+\psi_\epsilon)} + c_\epsilon r_\epsilon^2\gamma_\epsilon u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon r_\epsilon^2\mu_\epsilon/\lambda_\epsilon. \end{cases}$$

It is easy to see that $\triangle\psi_\epsilon \to 0$ in $L^2(B_R(0))$, $|\psi_\epsilon| \leq 1$ and $\psi_\epsilon(0) = 1$. Elliptic estimates and Liouville's theorem give $\psi_\epsilon \to 1$ in $C^2(B_{R/2}(0))$. Applying elliptic estimates to (4.3), we obtain $\varphi_\epsilon \to \varphi$ in $C^2_{loc}(\mathbb{R}^2)$, where φ satisfies

$$(4.4) \qquad \begin{cases} -\Delta_{\mathbb{R}^2}\varphi = e^{8\pi\varphi} \text{ in } \mathbb{R}^2, \\ \varphi(0) = \sup_{\mathbb{R}^2}\varphi = 0, \\ \int_{\mathbb{R}^2} e^{8\pi\varphi}dx \leq 1. \end{cases}$$

By the uniqueness result of Chen and Li [4], we have $\varphi(x) = -\frac{1}{4\pi}\log(1 + \pi|x|^2)$ and $\int_{\mathbb{R}^2} e^{8\pi\varphi}dx = 1$.

Define $u_{\epsilon,\beta} = \min\{\beta c_\epsilon, u_\epsilon\}$. Then we have

Lemma 4.5. $\forall 0 < \beta < 1, \quad \limsup_{\epsilon\to 0}\|\nabla u_{\epsilon,\beta}\|_2^2 = \beta.$

Proof. By the fact that $\psi_\epsilon \to 1$ in $C^2_{loc}(\mathbb{R}^2)$, we have

$$B_{Rr_\epsilon}(x_\epsilon) \subset \{x \in \Sigma : u_\epsilon(x) \geq \beta c_\epsilon\}$$

for any fixed $R > 0$ and sufficiently small ϵ. Using the divergence theorem and equation (3.2), we have

$$\begin{aligned}\int_\Sigma |\nabla(u_\epsilon - \beta c_\epsilon)^+|^2 dV_g &= -\int_\Sigma (u_\epsilon - \beta c_\epsilon)^+ \triangle u_\epsilon dV_g \\ &= \int_{u_\epsilon \geq \beta c_\epsilon} (u_\epsilon - \beta c_\epsilon)\frac{1}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g + o_\epsilon(1) \\ &\geq \int_{B_{Rr_\epsilon}(x_\epsilon)} (u_\epsilon - \beta c_\epsilon)\frac{1}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g + o_\epsilon(1) \\ &= (1-\beta)\int_{B_R(0)} e^{8\pi\varphi} dx + o_\epsilon(1) + o_\epsilon(R),\end{aligned}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$, and $o_\epsilon(R) \to 0$ for any fixed R as $\epsilon \to 0$. Letting $\epsilon \to 0$ first, then $R \to +\infty$, we obtain

$$\liminf_{\epsilon \to 0} \int_\Sigma |\nabla(u_\epsilon - \beta c_\epsilon)^+|^2 dV_g \geq 1 - \beta.$$

Using the divergence theorem, equation (3.2), Lemma 4.2, Lemma 4.3 and the fact that $\varphi_\epsilon \to \varphi$ in $C^2_{loc}(\mathbb{R}^2)$, we have

$$\begin{aligned}\int_\Sigma |\nabla u_{\epsilon,\beta}|^2 dV_g &= -\int_\Sigma u_{\epsilon,\beta} \triangle u_\epsilon dV_g \\ &= \int_\Sigma u_{\epsilon,\beta} \frac{1}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g + o_\epsilon(1) \\ &\geq \int_{B_{R\epsilon}(x_\epsilon)} \frac{1}{\lambda_\epsilon} \beta c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g + o_\epsilon(1) \\ &= \beta \int_{B_R(0)} e^{8\pi\varphi} dx + o_\epsilon(1) + o_\epsilon(R).\end{aligned}$$

Letting $\epsilon \to 0$ first, then $R \to +\infty$, we have $\liminf_{\epsilon\to 0} \int_\Sigma |\nabla u_{\epsilon,\beta}|^2 dV_g \geq \beta$. Noting that

$$\int_\Sigma |\nabla(u_\epsilon - \beta c_\epsilon)^+|^2 dV_g + \int_\Sigma |\nabla u_{\epsilon,\beta}|^2 dV_g = 1,$$

we get the result. □

Lemma 4.6. $\limsup_{\epsilon\to 0} J^\alpha_{4\pi-\epsilon}(u_\epsilon) \leq Vol(\Sigma) + \limsup_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}$.

Proof. For any $0 < \beta < 1$, an elementary computation gives

$$\begin{aligned}J^\alpha_{4\pi-\epsilon}(u_\epsilon) &= \int_{u_\epsilon < \beta c_\epsilon} e^{\alpha_\epsilon u_\epsilon^2} dV_g + \int_{u_\epsilon \geq \beta c_\epsilon} e^{(\alpha_\epsilon) u_\epsilon^2} dV_g \\ &\leq \int_\Sigma e^{\alpha_\epsilon u_{\epsilon,\beta}^2} dV_g + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2}.\end{aligned}$$

By Lemma 4.5, we can see that $e^{\alpha_\epsilon u_{\epsilon,\beta}^2}$ is bounded in $L^p(\Sigma)$ for some $p > 1$, whence $\int_\Sigma e^{\alpha_\epsilon u_{\epsilon,\beta}^2} dV_g \to Vol(\Sigma)$ as $\epsilon \to 0$. Therefore

$$J_{4\pi-\epsilon}^{\alpha}(u_\epsilon) \le Vol(\Sigma) + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} + o_\epsilon(1),$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$. Letting $\epsilon \to 0$ first, then $\beta \to 1$, we obtain the result. □

Similar to [1] and [10], we have the following:

Lemma 4.7. *For any $\phi \in C^\infty(\Sigma)$ we have*

$$\lim_{\epsilon\to 0} \int_\Sigma \phi \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dx = \phi(p). \tag{4.5}$$

Proof. We divide Σ into three parts:

$$\Sigma = (\{u_\epsilon > \beta c_\epsilon\} \setminus B_{Rr_\epsilon}(x_\epsilon)) \cup (\{u_\epsilon \le \beta c_\epsilon\} \setminus B_{Rr_\epsilon}(x_\epsilon)) \cup B_{Rr_\epsilon}(x_\epsilon),$$

for some $0 < \beta < 1$. Denote the integrals on the left side of (4.5) on the above three domains by I_1, I_2 and I_3 respectively. Then

$$\begin{aligned} |I_1| &\le \frac{1}{\beta} \sup_\Sigma |\phi| \int_{\{u_\epsilon > \beta c_\epsilon\}\setminus B_{Rr_\epsilon}(x_\epsilon)} \frac{1}{\lambda_\epsilon} u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dV_g \\ &\le \frac{1}{\beta} \sup_\Sigma |\phi| \left(1 - \int_{B_{Rr_\epsilon}(x_\epsilon)} \frac{1}{\lambda_\epsilon} u_\epsilon^2 e^{\alpha_\epsilon u_\epsilon^2} dV_g\right) \\ &\le \frac{1}{\beta} \sup_\Sigma |\phi| \left(1 - \int_{B_R(0)} e^{8\pi\varphi} dx + o_\epsilon(R)\right), \end{aligned}$$

where $o_\epsilon(R) \to 0$ as $\epsilon \to 0$ for any fixed R. Letting $\epsilon \to 0$ first, and then $R \to +\infty$, one has $I_1 \to 0$. Then

$$\begin{aligned} |I_2| &\le \sup_\Sigma |\phi| \frac{c_\epsilon}{\lambda_\epsilon} \int_\Sigma |u_\epsilon| e^{\alpha_\epsilon u_{\epsilon,\beta}^2} dV_g \\ &\le \sup_\Omega |\phi| \frac{c_\epsilon}{\lambda_\epsilon} \parallel u_\epsilon \parallel_{L^{\frac{1+\beta}{1-\beta}}(\Sigma)} \parallel e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2} \parallel_{L^{\frac{1+\beta}{2\beta}}(\Sigma)} \\ &\le C\frac{c_\epsilon}{\lambda_\epsilon} \end{aligned}$$

for some constant C depending only on β and Σ; here we have used the Hölder inequality and the Sobolev imbedding theorem. Lemma 4.6 implies that $\lambda_\epsilon / c_\epsilon \to +\infty$, whence $c_\epsilon/\lambda_\epsilon \to 0$. Hence we have $I_2 \to 0$ as $\epsilon \to 0$. Then

$$\begin{aligned} I_3 &= \int_{B_{Rr_\epsilon}(x_\epsilon)} \phi \frac{c_\epsilon}{\lambda_\epsilon} u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g \\ &= \phi(x_\epsilon + r_\epsilon \xi) \left(\int_{B_R(0)} e^{8\pi\varphi} dx + o_\epsilon(R)\right), \end{aligned}$$

where $\xi \in B_R(0)$. As before, letting $\epsilon \to 0$ first, then $R \to +\infty$, we get $I_3 \to \varphi(p)$. Combining all the above estimates gives (4.5). □

We need a result of Brezis and Merle [2]:

Theorem C (Brezis-Merle). *Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let $u \in H_0^1(\Omega)$ be a weak solution of $-\triangle u = f(x)$ in Ω, with $f \in L^1(\Omega)$. Then for any $\delta \in (0, 4\pi)$,*

$$\int_\Omega \exp\left[\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}\right] dx \le \frac{4\pi^2}{\delta}(\mathrm{diam}\Omega).$$

Modifying the argument of Struwe [16], we have the following:

Lemma 4.8. *Assume $u \in C^\infty(\Sigma)$ is a solution of*

$$\left\{\begin{array}{l} -\triangle u = f(x) \quad \text{in} \quad \Sigma, \\ \|u\|_1 \le c_0\|f\|_1. \end{array}\right. \tag{4.6}$$

Then for any $1 < q < 2$, $\|\nabla u\|_q \le C(q, c_0, \Sigma)\|f\|_1$.

Proof. Without loss of generality, we assume $\|f\|_1 = 1$. $\forall p \in \Sigma$, take an isothermal coordinate system (Ω, ϕ) near p such that $\phi(p) = 0$, and $g = e^{2h}(dx_1^2 + dx_2^2)$ with $h(0) = 0$. In this coordinate system, $\triangle = e^{-2h}\triangle_{\mathbb{R}^2}$, and then

$$-\triangle_{\mathbb{R}^2} u = e^{2h} f \quad \text{in} \quad \Omega. \tag{4.7}$$

Let $v \in H_0^1(\Omega)$ be a solution of (4.7). Then we have by Theorem C that $e^{|v|}$ is bounded in $L^s(\Omega)$ for some $s > 0$, whence v is bounded in $L^1(\Omega)$. Clearly $u - v$ is harmonic in Ω. By the mean value theorem, $\forall \tilde{\Omega} \Subset \Omega$, $\forall x \in \tilde{\Omega}$, we have for $r < dist(\partial\Omega, \partial\tilde{\Omega})$,

$$\begin{aligned} |(u-v)(x)| &= \frac{1}{Vol(\mathbb{B}_r(x))}\left|\int_{\mathbb{B}_r(x)} (u-v)(x)dx\right| \\ &\le \frac{1}{\pi r^2}\left(\int_\Sigma |u|dV_g + \int_\Omega |v|dx\right), \end{aligned}$$

which implies that $u - v$ is bounded in $\tilde{\Omega}$. Therefore

$$\int_{\tilde{\Omega}} e^{s|u(x)|}dx \le \int_{\tilde{\Omega}} e^{s|v(x)|+s|(u-v)(x)|}dx \le C.$$

A covering argument implies that $e^{|u|}$ is bounded in $L^{s_0}(\Sigma)$ for some $s_0 > 0$.

Define $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = -\min\{u(x), 0\}$ for $x \in \Sigma$. Testing equation (4.6) by $\log\frac{1+2u^+}{1+u^+}$, we have

$$\int_\Sigma \frac{|\nabla u^+|^2}{(1+2u^+)(1+u^+)}dV_g \le \log 2.$$

The same inequality holds for u^-. Hence

$$\int_\Sigma \frac{|\nabla u|^2}{(1+2|u|)(1+|u|)}dV_g \le \log 2.$$

Thus we have

$$\begin{aligned} \int_\Sigma |\nabla u|^q dV_g &\le \int_\Sigma \frac{|\nabla u|^2}{(1+2|u|)^2}dV_g + \int_\Sigma (1+2|u|)^{\frac{2q}{2-q}}dV_g \\ &\le \int_\Sigma \frac{|\nabla u|^2}{(1+2|u|)(1+|u|)}dV_g + \int_\Sigma e^{s_0|u|}dV_g + C \\ &\le C \end{aligned}$$

for some constant C depending only on q, c_0 and Σ. Here we use the same C to denote various constants. This completes the proof of the lemma. □

Lemma 4.9. *$c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $H^{1,q}(\Sigma)$ and $c_\epsilon u_\epsilon \to G$ strongly in $L^2(\Sigma)$ for any $1 < q < 2$, where G is a Green function satisfying the following:*

$$\begin{cases} -\Delta G = \delta_p + \alpha G - \frac{1}{Vol(\Sigma)}, \\ \int_\Sigma G dV_g = 0. \end{cases} \tag{4.8}$$

Furthermore, $c_\epsilon u_\epsilon \to G$ in $C^2_{loc}(\Sigma \setminus \{p\})$.

Proof. By (3.2), we have

$$-\triangle(c_\epsilon u_\epsilon) = \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} + \gamma_\epsilon c_\epsilon u_\epsilon - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon}. \tag{4.9}$$

Integration on both sides of the above equation gives $c_\epsilon \mu_\epsilon / \lambda_\epsilon \to 1/Vol(\Sigma)$. Let v_ϵ be a solution of

$$\begin{cases} -\triangle v_\epsilon = \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon} - A_\epsilon, \\ \int_\Sigma v_\epsilon dV_g = 0, \end{cases} \tag{4.10}$$

where

$$A_\epsilon = \frac{1}{Vol(\Sigma)} \int_\Sigma \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon}.$$

By Lemma 4.2 and Lemma 4.7, A_ϵ is bounded. Applying the Green representation formula to (4.10), we get the $L^1(\Sigma)$-bound of v_ϵ. By Lemma 4.8, v_ϵ is bounded in $H^{1,q}(\Sigma)$ for any $1 < q < 2$, whence v_ϵ is bounded in $L^s(\Sigma)$ for any $s > 1$. Subtracting (4.10) from (4.9), we have

$$-\triangle(c_\epsilon u_\epsilon - v_\epsilon) = \gamma_\epsilon (c_\epsilon u_\epsilon - v_\epsilon) + \gamma_\epsilon v_\epsilon + A_\epsilon. \tag{4.11}$$

Testing equation (4.11) by $c_\epsilon u_\epsilon - v_\epsilon$, we have

$$\int_\Sigma |\nabla(c_\epsilon u_\epsilon - v_\epsilon)|^2 dV_g = \gamma_\epsilon \int_\Sigma (c_\epsilon u_\epsilon - v_\epsilon)^2 dV_g + \gamma_\epsilon \int_\Sigma v_\epsilon (c_\epsilon u_\epsilon - v_\epsilon) dV_g. \tag{4.12}$$

Note that $\gamma_\epsilon \to \alpha < \lambda_1(\Sigma)$ and $|v_\epsilon(c_\epsilon u_\epsilon - v_\epsilon)| \le \delta(c_\epsilon u_\epsilon - v_\epsilon)^2 + \frac{1}{4\delta} v_\epsilon^2$ for any $\delta > 0$, one can choose $\delta < (\lambda_1(\Sigma) - \alpha)/2$ and get by (4.12),

$$\int_\Sigma |\nabla(c_\epsilon u_\epsilon - v_\epsilon)|^2 dV_g \le C \int_\Sigma v_\epsilon^2 dV_g \tag{4.13}$$

for some constant C depending only on $\lambda_1(\Sigma) - \alpha$, provided that ϵ is sufficiently small. (4.13), together with the Poincaré inequality, gives that $c_\epsilon u_\epsilon - v_\epsilon$ is bounded in $H^{1,2}(\Sigma)$. Since v_ϵ is bounded in $H^{1,q}(\Sigma)$ for any $1 < q < 2$, we have $c_\epsilon u_\epsilon$ is also bounded in $H^{1,q}(\Sigma)$. Passing to a subsequence, we can assume

$$c_\epsilon u_\epsilon \rightharpoonup G \quad \text{weakly in} \quad H^{1,q}(\Sigma),$$
$$c_\epsilon u_\epsilon \to G \quad \text{strongly in} \quad L^2(\Sigma)$$

for some $G \in H^{1,q}(\Sigma)$. Testing (4.9) by $\phi \in C^\infty(\Sigma)$, we have

$$\begin{aligned} \int_\Sigma \nabla\phi \nabla(c_\epsilon u_\epsilon) dV_g &= \int_\Sigma \phi \frac{\beta_\epsilon}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{\alpha_\epsilon u_\epsilon^2} dV_g + \int_\Sigma \phi \gamma_\epsilon c_\epsilon u_\epsilon dV_g - c_\epsilon \frac{\mu_\epsilon}{\lambda_\epsilon} \int_\Sigma \phi dV_g \\ &\to \phi(p) + \alpha \int_\Sigma \phi G dV_g - \frac{1}{Vol(\Sigma)} \int_\Sigma \phi dV_g. \end{aligned}$$

Hence (4.8) holds.

For any fixed $\delta > 0$, choose a cut-off function $\eta \in C_0^\infty(\Sigma \setminus B_\delta(p))$ such that $\eta \equiv 1$ on $\Sigma \setminus B_{2\delta}(p)$. By Lemma 4.3, we have $\|\nabla(\eta u_\epsilon)\|_2 \to 0$ as $\epsilon \to 0$. Hence $e^{\eta^2 u_\epsilon^2}$ is bounded in $L^q(\Sigma \setminus B_\delta(p))$, whence $e^{u_\epsilon^2}$ is bounded in $L^q(\Sigma \setminus B_{2\delta}(p))$ for any $q > 1$. Note that $c_\epsilon \mu_\epsilon / \lambda_\epsilon \to 1/Vol(\Sigma)$ as $\epsilon \to 0$, and $c_\epsilon u_\epsilon$ is bounded in $L^r(\Sigma)$ for any $r > 2$ by the Sobolev embedding theorem. We can see from (4.9) that $-\triangle(c_\epsilon u_\epsilon)$ is bounded in $L^{q_0}(\Sigma \setminus B_{2\delta}(p))$ for some $q_0 > 2$. Applying the elliptic estimates to (4.9), we have $c_\epsilon u_\epsilon \to G$ in $C^1(\Sigma \setminus B_{3\delta}(p))$. Again by the elliptic estimates, $c_\epsilon u_\epsilon \to G$ in $C^2(\Sigma \setminus B_{4\delta}(p))$. Hence the second assertion of the lemma holds. □

The proof of (1) of Theorem 1.1 follows immediately from Lemma 4.9.

5. Upper bound estimates

In this section, we use the capacity technique to derive an upper bound of $J_{4\pi}^\alpha$ under the assumption that $c_\epsilon \to +\infty$. The fact that capacity technique can be used here was first discovered by Li [10].

Let (Ω, ϕ) be an isothermal coordinate system near p such that $\phi(p) = 0$ as in section 4; we still denote $\phi(x_\epsilon)$ by x_ϵ for simplicity. In such coordinates, $g = e^{2f}(dx_1^2 + dx_2^2)$ with $f(0) = 0$. Then we have

$$|\nabla u_\epsilon|^2 dV_g = |\nabla_{\mathbb{R}^2}(u_\epsilon \circ \phi^{-1})|^2 dx_1 dx_2.$$

Let $\mathbb{B}_r = \mathbb{B}_r(x_\epsilon) \subset \mathbb{R}^2$ be the standard ball centered at x_ϵ with radius r.

In section 4, we have proved that $c_\epsilon(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon) \to \varphi$ in $C^2_{loc}(\mathbb{R}^2)$, and $c_\epsilon u_\epsilon \to G$ in $C^2_{loc}(\Sigma \setminus \{p\})$. Denote $s_\epsilon = \sup_{\partial \mathbb{B}_\delta} u_\epsilon \circ \phi^{-1}$ and $i_\epsilon = \inf_{\partial \mathbb{B}_{Rr_\epsilon}} u_\epsilon \circ \phi^{-1}$. Then we obtain

$$\begin{aligned} s_\epsilon &= \frac{1}{c_\epsilon}\left(-\frac{1}{2\pi}\log\delta + A_p + o_\delta(1) + o_\epsilon(1)\right), \\ i_\epsilon &= c_\epsilon + \frac{1}{c_\epsilon}\left(-\frac{1}{4\pi}\log(1+\pi R^2) + o_\epsilon(R) + o_\epsilon(1)\right), \end{aligned}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$, and $o_\epsilon(R) \to 0$ for any fixed $R > 0$ as $\epsilon \to 0$. Define a function space

$$\mathcal{T}_\epsilon = \{u \in H^{1,2}(\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon}) : u|_{\partial\mathbb{B}_\delta} = s_\epsilon,\, u|_{\partial\mathbb{B}_{Rr_\epsilon}} = i_\epsilon\}.$$

It is not difficult to see that

$$\inf_{u \in \mathcal{T}_\epsilon} \int_{\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon}} |\nabla_{\mathbb{R}^2} u|^2 dx_1 dx_2$$

is attained by $w(x)$ satisfying

$$\begin{cases} \triangle_{\mathbb{R}^2} w = 0 \quad \text{in} \quad \mathbb{B}_\delta \setminus \overline{\mathbb{B}}_{Rr_\epsilon}, \\ w|_{\partial\mathbb{B}_\delta} = s_\epsilon, \\ w|_{\partial\mathbb{B}_{Rr_\epsilon}} = i_\epsilon. \end{cases}$$

One can check that

$$w(x) = \frac{s_\epsilon(\log|x - x_\epsilon| - \log(Rr_\epsilon)) + i_\epsilon(\log\delta - \log|x - x_\epsilon|)}{\log\delta - \log(Rr_\epsilon)},$$

whence

$$\int_{\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon}} |\nabla_{\mathbb{R}^2} w|^2 dx_1 dx_2 = \frac{2\pi(s_\epsilon - i_\epsilon)^2}{\log\delta - \log(Rr_\epsilon)}. \tag{5.1}$$

Let $\tilde{u}_\epsilon = \max\{s_\epsilon, \min\{u_\epsilon, i_\epsilon\}\}$. Then $\tilde{u}_\epsilon \circ \phi^{-1} \in \mathcal{T}_\epsilon$, whence

$$\begin{aligned}&\int_{\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon}} |\nabla_{\mathbb{R}^2} w|^2 dx_1 dx_2 \\ &\le \int_{\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon}} |\nabla_{\mathbb{R}^2} (\tilde{u}_\epsilon \circ \phi^{-1})|^2 dx_1 dx_2 \\ &= \int_{\phi^{-1}(\mathbb{B}_\delta)\setminus \phi^{-1}(\mathbb{B}_{Rr_\epsilon})} |\nabla \tilde{u}_\epsilon|^2 dV_g \\ &\le 1 - \int_{\Sigma \setminus \phi^{-1}(\mathbb{B}_\delta)} |\nabla u_\epsilon|^2 dV_g - \int_{\phi^{-1}(\mathbb{B}_{Rr_\epsilon})} |\nabla u_\epsilon|^2 dV_g.\end{aligned} \tag{5.2}$$

Now we compute $\int_{\phi^{-1}(\mathbb{B}_{Rr_\epsilon})} |\nabla u_\epsilon|^2 dV_g$ and $\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta)} |\nabla u_\epsilon|^2 dV_g$. Recall that $G = -\frac{\log r}{2\pi} + \beta$, where $\beta = A_p + O(r)$ and $\beta \in C^1(\Sigma)$. Using equation (4.8) and the divergence theorem, we have

$$\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta)} |\nabla G|^2 dV_g = -\frac{1}{2\pi}\log\delta + A_p + \alpha\|G\|_2^2 + o_\epsilon(\delta) + o_\epsilon(1) + o_\delta(1),$$

where $o_\epsilon(\delta) \to 0$ for any fixed $\delta > 0$ as $\epsilon \to 0$. By Lemma 4.9, $c_\epsilon u_\epsilon \to G$ in $C^2_{loc}(\Omega \setminus \{p\})$, and we obtain

$$\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta)} |\nabla u_\epsilon|^2 dV_g = \frac{1}{c_\epsilon^2}\left(-\frac{\log\delta}{2\pi} + A_p + \alpha\|G\|_2^2 + o_\epsilon(\delta) + o_\epsilon(1) + o_\delta(1)\right). \tag{5.3}$$

On the other hand,

$$\int_{B_R(0)} |\nabla\varphi|^2 dx = \frac{1}{4\pi}\log(1+\pi R^2) - \frac{1}{4\pi} + o_R(1),$$

where $o_R(1) \to 0$ as $R \to +\infty$. Hence by the fact that

$$c_\epsilon(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon) \to \varphi(x) \quad \text{in} \quad C^2_{loc}(\mathbb{R}^2),$$

we have

$$\int_{\phi^{-1}(\mathbb{B}_{Rr_\epsilon})} |\nabla u_\epsilon|^2 dV_g = \frac{1}{c_\epsilon^2}\left(\frac{1}{4\pi}\log(1+\pi R^2) - \frac{1}{4\pi} + o_R(1) + o_\epsilon(R) + o_\epsilon(1)\right). \tag{5.4}$$

Recalling $r_\epsilon^2 = \lambda_\epsilon/(\beta_\epsilon c_\epsilon^2)e^{-\alpha_\epsilon c_\epsilon^2}$, one gets

$$\frac{1}{2\pi}(\log(\delta) - \log(Rr_\epsilon)) = \frac{\log\delta - \log R}{2\pi} - \frac{1}{4\pi}\log\frac{1}{\beta_\epsilon} - \frac{1}{4\pi}\log\frac{\lambda_\epsilon}{c_\epsilon^2} + \frac{\alpha_\epsilon c_\epsilon^2}{4\pi}. \tag{5.5}$$

From (5.1) to (5.5), we obtain

$$\begin{aligned}&c_\epsilon^2 + 2(-\frac{1}{4\pi}\log(1+\pi R^2) + \frac{1}{2\pi}\log\delta - A_p) + o_\epsilon(\delta) + o_\epsilon(R) + o_\epsilon(1) \\ &\le \left(\frac{\log\delta - \log R}{2\pi} - \frac{1}{4\pi}\log\frac{\lambda_\epsilon}{c_\epsilon^2} + \frac{\alpha_\epsilon}{4\pi}c_\epsilon^2 + o_\epsilon(1)\right) \times \Bigg(1- \\ &\frac{\frac{1}{4\pi}\log(1+\pi R^2) - \frac{1}{4\pi} - \frac{1}{2\pi}\log\delta + A_p + \alpha\|G\|_2^2 + o_\epsilon(\delta) + o_\epsilon(R) + o_\delta(1) + o_R(1)}{c_\epsilon^2}\Bigg),\end{aligned}$$

which implies that

$$\log \frac{\lambda_\epsilon}{c_\epsilon^2} \leq 4\pi\alpha c_\epsilon^2 \|u_\epsilon\|_2^2 + 4\pi A_p + 1 + \log\pi - 4\pi\alpha\|G\|_2^2 + o_\epsilon(\delta) + o_\epsilon(R) + o_\epsilon(1) + o_\delta(1) + o_R(1). \tag{5.6}$$

By Lemma 4.9,

$$\lim_{\epsilon\to 0} c_\epsilon^2 \|u_\epsilon\|_2^2 = \|G\|_2^2.$$

Letting $\epsilon \to 0$ first, then $R \to +\infty$ and $\delta \to 0$, we obtain by (5.6),

$$\limsup_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \leq \pi e^{1+4\pi A_p},$$

whence by Lemma 4.6,

$$\sup_{u\in\mathcal{H}} J_{4\pi}^\alpha(u) = \limsup_{\epsilon\to 0} \int_\Sigma e^{\alpha_\epsilon u_\epsilon^2} dV_g \leq Vol(\Sigma) + \pi e^{1+4\pi A_p}.$$

In fact we have proved the following:

Proposition 5.1. *Under the assumption that $c_\epsilon \to +\infty$, it follows that*

$$\sup_{u\in\mathcal{H}} J_{4\pi}^\alpha(u) \leq Vol(\Sigma) + \pi e^{1+4\pi A_p}.$$

6. The existence result

In this section, we will construct a blow-up sequence ϕ_ϵ such that $\|\nabla\phi_\epsilon\|_2 = 1$ and

$$\int_\Sigma e^{4\pi(\phi_\epsilon - \bar\phi_\epsilon)^2(1+\alpha\|\phi_\epsilon-\bar\phi_\epsilon\|_2^2)} dV_g > Vol(\Sigma) + \pi e^{1+4\pi A_p} \tag{6.1}$$

for sufficiently small α, $\epsilon > 0$, where $\bar\phi_\epsilon = \frac{1}{Vol(\Sigma)}\int_\Sigma \phi_\epsilon dV_g$. The contradiction between (6.1) and Proposition 5.1 implies that c_ϵ is bounded. Applying elliptic estimates to equation (3.2), we have $u_\epsilon \to u_\alpha$ in $C^\infty(\Sigma)$ for some $u_\alpha \in C^\infty(\Sigma)\cap\mathcal{H}$. Hence the point (3) of Theorem 1.1 holds.

To prove (6.1), we set $\tilde\beta = G + \frac{1}{2\pi}\log r - A_p$; hence $\tilde\beta = O(r)$. Here $r(x) = dist(x,p)$. Set

$$\phi_\epsilon = \begin{cases} \frac{c+\frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi\frac{r^2}{\epsilon^2})+B\right)}{\sqrt{1+\frac{\alpha}{c^2}\|G\|_2^2}} & \text{for} \quad r \leq R\epsilon, \\ \frac{1}{\sqrt{c^2+\alpha\|G\|_2^2}}(G - \eta\tilde\beta) & \text{for} \quad R\epsilon < r < 2R\epsilon, \\ \frac{1}{\sqrt{c^2+\alpha\|G\|_2^2}} G & \text{for} \quad r \geq 2R\epsilon, \end{cases}$$

where $\eta \in C_0^\infty(B_{2R\epsilon}(p))$ is a cutoff function, $\eta = 1$ on $B_{R\epsilon}(p)$, $\|\nabla\eta\|_{L^\infty} = O(\frac{1}{R\epsilon})$, B is a constant to be determined later, and R, c depending on ϵ will also be chosen later such that $R\epsilon \to 0$ and $R \to +\infty$. In order to ensure that $\phi_\epsilon \in H^{1,2}(\Sigma)$, we set

$$c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi R^2) + B\right) = \frac{1}{c}(-\frac{1}{2\pi}\log(R\epsilon) + A_p),$$

which gives

$$2\pi c^2 = -2\log\epsilon - 2\pi B + 2\pi A_p + \frac{1}{2}\log\pi + O(\frac{1}{R^2}). \tag{6.2}$$

By a direct calculation,

$$\int_\Sigma |\nabla\phi_\epsilon|^2 dV_g = \frac{1}{c^2+\alpha\|G\|_2^2}\Big(-\frac{\log\epsilon}{2\pi}+\frac{\log\pi}{4\pi}-\frac{1}{4\pi}+A_p+\alpha\|G\|_2^2 + O(\frac{1}{R^2})+O(R\epsilon\log(R\epsilon))\Big).$$

To ensure $\|\nabla\phi_\epsilon\|_2 = 1$, we set

$$c^2 = -\frac{\log\epsilon}{2\pi}+\frac{\log\pi}{4\pi}-\frac{1}{4\pi}+A_p+O(\frac{1}{R^2})+O(R\epsilon\log(R\epsilon)). \tag{6.3}$$

By (6.2) and (6.3), we have

$$B = \frac{1}{4\pi}+O(\frac{1}{R^2})+O(R\epsilon\log(R\epsilon)).$$

Setting $R = -\log\epsilon$, one gets $\bar\phi_\epsilon = O((R\epsilon)^2\log\epsilon)$. A straightforward calculation shows

$$\int_{B_{R\epsilon}(p)} e^{4\pi(\phi_\epsilon-\bar\phi_\epsilon)^2(1+\alpha\|\phi_\epsilon-\bar\phi_\epsilon\|_2^2)}dV_g \geq \pi e^{1+4\pi A_p}+\alpha^2\|G\|_2^4 O(\frac{1}{\log\epsilon})+O(\frac{\log\log\epsilon}{(\log\epsilon)^2}).$$

On the other hand,

$$\begin{aligned}\int_{\Sigma\setminus B_{R\epsilon}(p)} e^{4\pi(\phi_\epsilon-\bar\phi_\epsilon)^2(1+\alpha\|\phi_\epsilon-\bar\phi_\epsilon\|_2^2)}dV_g &\geq \int_{\Sigma\setminus B_{2R\epsilon}(p)}(1+4\pi(\phi_\epsilon-\bar\phi_\epsilon)^2)dV_g\\ &\geq Vol(\Sigma)+\frac{8\pi^2}{\log\frac{1}{\epsilon}}\int_\Sigma G^2dx+o(\frac{1}{\log\epsilon}).\end{aligned}$$

Hence, we have

$$J_{4\pi}^\alpha(\phi_\epsilon-\bar\phi_\epsilon) > Vol(\Sigma)+\pi e^{1+4\pi A_p}$$

for sufficiently small α, $\epsilon > 0$, and (6.1) holds. □

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References

[1] Adimurthi and O. Druet: Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. Comm. in PDE., **29**(1): 295-322, 2004. MR2038154 (2005a:46064)
[2] H. Brezis and F. Merle: Uniform Estimates and Blow-up Behavior for Solutions of $-\Delta u = V(x)e^u$ in Two Dimensions, Commun In PDE 16, 1223-1253 (1991). MR1132783 (92m:35084)
[3] L. Carleson and A. Chang: On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math., **110**: 113-127, 1986. MR0878016 (88f:46070)
[4] W. Chen and C. Li: Classification of solutions of some nonlinear elliptic equations. Duke Math. J. **63**: 615-622, 1991. MR1121147 (93e:35009)
[5] W. Y. Ding, J. Jost, J. Li and G. Wang: The differential equation $-\Delta u = 8\pi - 8\pi he^u$ on a compact Riemann Surface. Asian J. Math., **1**(2): 230-248, 1997. MR1491984 (98k:58062)
[6] O. Druet and E. Hebey: The AB program in geometric analysis. Sharp Sobolev inequalities and related problems. Memoirs of the AMS, **106**: 761, 2002. MR1938183 (2003m:58028)
[7] J. Escobar and R. Schoen: Conformal metrics with prescribed scalar curvature. Inventiones Math., **86**: 243-253, 1986. MR0856845 (88b:58030)
[8] M. Flucher: Extremal functions for the Trudinger-Moser inequality in 2 dimensions. Comment. Math. Helv., **67**: 471-497, 1992. MR1171306 (93k:58073)

[9] L. Fontana: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. Comm. Math. Helv., **68**: 415- 454, 1993. MR1236762 (94h:46048)

[10] Y. Li: Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. J. Partial Differential Equations **14**, no. 2, 163–192, 2001. MR1838044 (2002h:58033)

[11] Y. Li and P. Liu: A Moser-Trudinger inequality on the boundary of a compact Riemann surface. Math. Z., **250**: 363-386, 2005. MR2178789

[12] K.C. Lin: Extremal functions for Moser's inequality. Trans. Amer. Math. Sco., **348**:2663-2671, 1996. MR1333394 (96i:58043)

[13] P. L. Lions: The concentration-compactness principle in the calculus of variations, the limit case, part I. Rev. Mat. Iberoamericana, **1**: 145-201, 1985. MR0834360 (87c:49007)

[14] J. Moser: A sharp form of an Inequality by N. Trudinger. Ind. Univ. Math. J., **20**:1077-1091, 1971. MR0301504 (46:662)

[15] R. Schoen: Conformal deformation of a Riemannian metric to a constant scalar curvature. J. Differential Geometry, **20**: 479-495, 1984. MR0788292 (86i:58137)

[16] M. Struwe: Positive solutions of critical semilinear elliptic equations on non-contractible planar domains. J. Eur. Math. Soc., **2**(4):329–388, 2000. MR1796963 (2001h:35070)

[17] N. S. Trudinger: On embeddings into Orlicz spaces and some applications. J. Math. Mech., **17**: 473-484, 1967. MR0216286 (35:7121)

Department of Mathematics, Information School, Renmin University of China, Beijing 100872, People's Republic of China

E-mail address: yunyan_yang2002@yahoo.com.cn