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A SINGLE MINIMAL COMPLEMENT FOR THE C.E. DEGREES

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Abstract. We show that there exists a minimal (Turing) degree $b < 0'$ such that for all non-zero c.e. degrees a, $0' = a \vee b$. Since b is minimal this means that b complements all c.e. degrees other than 0 and $0'$. Since every n-c.e. degree bounds a non-zero c.e. degree, b complements every n-c.e. degree other than 0 and $0'$.

1. Introduction

In [AL1] we extended ideas developed by Slaman and Seetapun [SS1] in order to show that for every non-zero Turing degree a strictly below $0'$ there exists a minimal degree $b < 0'$ such that $0' = a \vee b$. Since b is minimal this means that b complements a. Using finite injury arguments it is also possible to show that there exists no 'finite minimal complement set', in other words that there does not exist a finite set of minimal degrees below $0'$ such that every non-zero degree strictly below $0'$ is complemented by some degree in this set. In [AL2] we significantly strengthen the latter result: we show that given any degree $c \leq 0'$ and any uniformly Δ_2 sequence of degrees $b_0, b_1, b_2, ..$ (minimal or otherwise) such that $\forall i(b_i \not\geq c)$, there exists a strictly between 0 and $0'$ such that for all $i \geq 0$, $a \vee b_i \not\geq c$. If c is c.e. and $b_0, b_1, b_2, ..$ are uniformly (strictly) below c, then there exists such an a below c. The aim of this paper is to show that there does exist a single minimal degree $b < 0'$ such that for all non-zero c.e. degrees a, $0' = a \vee b$. Since b is minimal this means that b complements all c.e. degrees other than 0 and $0'$. Since every n-c.e. degree bounds a non-zero c.e. degree this gives us the very strong result that b complements every n-c.e. degree other than 0 and $0'$. This significantly strengthens the result of Cooper, Seetapun and (independently) Li that there exists a degree $b < 0'$ which cups every non-zero c.e. degree to $0'$.

The proof is quite difficult. In order to make the intuition behind the construction clear we shall proceed to consider a number of simplified situations, building up the concepts required for the final construction in a gradual manner.

1.1. Constructing minimal degrees.

Let us begin, then, by reviewing the construction of a minimal degree by full approximation. Even for the reader who is already very familiar with the techniques involved, it is hoped that the discussion here will prove helpful in what is to follow—in this section we shall lay down the

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basic framework around which the constructions considered in all the following sections will be based. We shall also establish some useful terminology.

Let $\{\Psi_j\}_{j\geq 0}$ be an effective listing of the Turing functionals. In order to construct a set B of minimal degree we shall use the familiar method of splitting trees (for a clear account of all the basic concepts involved, see [RS1]). Thus for each $j \in \omega$ we shall attempt to construct an infinite Ψ_j splitting tree Tr_j, such that B lies on this tree. For each $j \in \omega$ there may be many candidates for Tr_j; the tree $Tr_{j,i}$ should be regarded as the i^{th} candidate for Tr_j. If for some $j \in \omega$, there exists i such that $Tr_{j,i}$ is infinite, we will be able to conclude that Ψ_j^B is total and that $B \leq_T \Psi_j^B$. If on the other hand there exists no such i for some j, then we shall be able to show that if Ψ_j^B is total it is computable. Throughout the paper it will be convenient to adopt the convention that for any $\sigma \in 2^{<\omega}$ and $j, n \in \omega$, $\Psi_j^\sigma(n)\downarrow$ only if the computation converges in less than $|\sigma|$ steps and $\Psi_j^\sigma(n')\downarrow$ for all $n' < n$.

The construction itself takes place on a tree, so that each node of the tree is assigned one strategy. This tree we consider to 'grow upwards'. At each stage s of the construction, control is initially passed to that strategy which is assigned to the node at the bottom of the tree of strategies. This strategy may then pass control to another, one level up on the tree of strategies. This latter strategy may in turn pass control to another one level up again on the tree of strategies, and so on until stage s activity is terminated for the construction with control having been passed to a finite number of strategies. There are two types of strategy involved, $\mathcal{D}$ strategies and $\mathcal{C}$ strategies. $\mathcal{D}$ strategies do almost nothing other than pass control to other $\mathcal{D}$ and $\mathcal{C}$ strategies. Their role is to clarify the organization of the tree of strategies. It is the $\mathcal{C}$ strategies which search for splittings. Since the strategies required simply in order to construct a set of minimal degree are not very complicated we shall just go ahead and describe them. Having defined the construction we shall then try to give some feel for the intuition by describing the actions of a few strategies at the base of the tree of strategies.

We use the variable H to range over the set of strategies, and we use the variable Υ to range over the set $\{\mathcal{C}, \mathcal{D}\}$. When any strategy is initially passed control, it is considered to be in state 0.

1.1.1. *The $\mathcal{D}[t,p]$ strategy.* Let $H = \mathcal{D}[t,p]$. Here $p \in 2^{<\omega}$ and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$. If $k' < k''$, then, during the course of the construction, we shall enumerate $Tr_{j_{k''},i_{k''}}$ in such a way that it may be regarded as a subtree of $Tr_{j_{k'},i_{k'}}$.

The instructions for the strategy in state 0.

Pass control to the strategy $H' = \mathcal{C}[t,p]$.

The instructions for the strategy in state 1.

The strategy will have been declared to be in state 1 by H', which will have 'delivered' two strings to H, p_1 and p_2 say, which are incompatible extensions of p. Let p_1 be the leftmost. At each stage s at which H is passed control in state 1 it 'puts B_s through p_1'—the strategy insists that $p_1 \subseteq B_s$— and always passes control to the same strategy $H'' = \mathcal{D}[t', p_1]$. We shall define t' subsequently.

In the *final* construction, it will be convenient to consider the use of a 'non-splitting tree' $Tr_{-1,0}$, which is not required to be any kind of splitting tree at all. Such considerations are irrelevant to the present discussion, but it seems reasonable nonetheless to make definitions which are in line with what will appear later—we shall do so where it is believed that no confusion will arise as a result.

Definition 1.1. Given any strategy $H = \Upsilon[t,p]$ we let $t(H)$ denote the set t. If $\Upsilon = \mathcal{D}$, then define $t^\star(H) = t$. If $\Upsilon = \mathcal{C}$, then if $t = \emptyset$ define $t^\star(H) = \emptyset$, and otherwise if $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$, $j_1 < \cdots < j_k$, then define $t^\star(H) = t - \{Tr_{j_k,i_k}\}$.

Definition 1.2. The leaves of a tree Tr are those strings $\sigma \in Tr$ which have no proper extensions in Tr. If Tr is empty, then we regard the empty string as a leaf of Tr.

Definition 1.3. We say that a string σ is compatible with a tree Tr if it is compatible with a leaf of Tr. We say that a splitting is compatible with Tr if each string in the splitting is compatible with Tr.

1.1.2. *The $\mathcal{C}[t,p]$ strategy.* Let $H = \mathcal{C}[t,p]$. Once again $p \in 2^{<\omega}$ and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$. If $t \neq \emptyset$ we regard the strategy as searching for a Ψ_{j_k} splitting compatible with $t^\star(H)$.

The instructions for the strategy in state 0.

(i) If $t = \emptyset$: choose two incompatible extensions of p, enumerate these two strings into $Tr_{-1,0}$ and 'deliver' them to that (unique) strategy H' which passes control to H, declaring H' to be in state 1. Terminate the present stage of activity for the construction.

(ii) If $t \neq \emptyset$: pass control to $H'' = \mathcal{C}[t^\star(H), p]$.

The instructions for the strategy in state 1.

The strategy will have been declared to be in state 1 by H'' (as defined in (ii) above), which will have 'delivered' two strings to H, p_1 and p_2 say, which are incompatible extensions of p. Let p_1 be the leftmost. At each stage s at which H is passed control in state 1 it performs the following steps:

(i) H performs another step in an exhaustive search procedure which looks for a Ψ_{j_k} splitting on $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$ if $k = 1$), such that both strings in the splitting extend p_1. If no such splitting q_1, q_2 has been found, then proceed to (ii). Otherwise find q_1', q_2', extensions of q_1 and q_2 respectively, which are compatible with all of the trees in t and longer than any of the strings in these trees. Enumerate q_1', q_2' into Tr_{j_k,i_k}, deliver them to that unique strategy H' which passes control to H, declaring H' to be in state 1. Terminate stage s activity for the construction.

(ii) Put B_s through p_1 and pass control to (always the same) strategy $\mathcal{D}[t', p_1]$. We shall define t' subsequently.

1.1.3. *The tree of strategies.* At the base of the tree of strategies is $\mathcal{D}[t_0, \emptyset]$, where $t_0 = \{Tr_{0,0}\}$. Above each strategy $H = \Upsilon[t,p]$ such that $t \neq \emptyset$ there are precisely two others. On the left there is that strategy which H passes control to while in state 0 and on the right there is the strategy which H passes control to while in state 1, the strategy $H' = \mathcal{D}[t', p_1]$ where p_1 is the leftmost of the two strings that are delivered to H upon being declared to be in state 1. We define t' the first time that we pass control to H' as follows. When we say that one strategy is below/above another we do not mean that it must be properly below/above this strategy. Let j be the least such that $\nexists i(Tr_{j,i} \in t)$ and such that there does not exist any $\mathcal{C}$ strategy below H searching for a Ψ_j splitting compatible with $t'' \subseteq t^\star(H)$. Let i be the least such that there is no strategy H'' that we have yet passed control to with $Tr_{j,i} \in t(H'')$ (so we choose a new candidate for Tr_j). We define $t' = \{Tr_{j,i}\} \cup t^\star(H)$. At the end of each stage s we define B_s to be the longest string that it was put through at stage s. For technical completeness, if B_s

is not put through any strings at stage s, then we define $B_s = \emptyset$—of course this will only happen a finite number of times.

1.1.4. *The true path.* In this, and in all subsequent constructions that we define, the *true path* will be the set of strategies which are passed control at all but a finite number of stages (so actually those strategies which are passed control at an infinite number of stages). In the *final* construction, however, this does not mean that the actions of each strategy are essentially convergent—any given strategy may perform an infinite number of different actions at different stages of the construction, affecting the actions of strategies above it and which strategies *they* pass control to. Of course, that is not the case in the construction that we have described in this section.

1.1.5. *The intuition.* At the base of the tree of strategies is $H_0 = \mathcal{D}[t_0, \emptyset]$, where $t_0 = \{Tr_{0,0}\}$. At stage 0 of the construction H_0 will be passed control and will pass control to $H_1 = \mathcal{C}[t_0, \emptyset]$. H_1 will then pass control to $H_2 = \mathcal{C}[\emptyset, \emptyset]$, which will choose two incompatible strings, enumerate these strings into $Tr_{-1,0}$, and deliver them to H_1, declaring this strategy to be in state 1. Let us suppose that H_0 is never declared to be in state 1. Then at every subsequent stage of the construction control will be passed to H_0, which will pass control to H_1. H_1 will pass control to $H_3 = \mathcal{D}[t_3, p_1]$, where $t_3 = \{Tr_{1,0}\}$ and p_1 is the leftmost of the two strings that were delivered to H_1 upon being declared to be in state 1. Why do we define t_3 in this way? Having been declared to be in state 1, H_1 will continue to search for a Ψ_0 splitting lying on $Tr_{-1,0}$ (which will be infinite), both strings of which extend p_1. If it was ever successful in this search, then H_0 would be declared to be in state 1 and H_1 and H_3 would never subsequently be passed control. Thus H_3 assumes that $Tr_{0,0}$ (constructed so that it may be regarded as a subtree of $Tr_{-1,0}$) will be finite and initiates the construction of a new tree $Tr_{1,0}$, which will be constructed so that it may be regarded as a subtree of $Tr_{-1,0}$ but not $Tr_{0,0}$.

Let us suppose that H_3 is eventually declared to be in state 1. Upon being declared to be in state 1 it will have two strings delivered to it, incompatible extensions of p_1 which are a Ψ_1 splitting. Let p_3 be the leftmost of these two strings. At every subsequent stage at which H_3 is passed control it will pass control to $H_4 = \mathcal{D}[t_4, p_3]$, where $t_4 = \{Tr_{1,0}, Tr_{2,0}\}$. (Actually we shall have that $t_4 = \{Tr_{1,0}, Tr_{2,i}\}$ for some $i \in \omega$. We will already have passed control to strategies which have initiated the construction of Ψ_2 splitting trees. For the sake of simplicity, however, we shall assume during the course of the present discussion that the only splitting trees under construction are those explicity mentioned.) Why do we define t_4 in this way? The fact that H_3 was eventually declared to be in state 1 means that, on this part of the tree of strategies, $Tr_{1,0}$ looks as if it will be infinite. We initiate the construction of a new splitting tree $Tr_{2,0}$, which we will construct so that it may be regarded as a subtree of $Tr_{1,0}$.

Let us suppose that H_4 is eventually declared to be in state 1. Upon being declared to be in state 1 it will have two strings delivered to it, incompatible extensions of p_3 which are a Ψ_2 splitting enumerated into $Tr_{2,0}$ and also a Ψ_1 splitting—each of the strings extending a different leaf of the tree $Tr_{1,0}$. At every subsequent stage of the construction at which H_4 is passed control it will pass control to $H_5 = \mathcal{D}[t_5, p_4]$, where $t_5 = \{Tr_{1,0}, Tr_{2,0}, Tr_{3,0}\}$ and p_4 is the leftmost of the two strings which were delivered to H_4.

The first stage at which H_5 is passed control, it will pass control to $H_6 = \mathcal{C}[t_5, p_4]$. H_6 will pass control to $H_7 = \mathcal{C}[t_4, p_4]$. H_7 will pass control to $H_8 = \mathcal{C}[t_3, p_4]$, which will pass control to $H_9 = \mathcal{C}[\emptyset, p_4]$. H_9 will choose two incompatible extensions of p_4 and deliver them to H_8, declaring this strategy to be in state 1. Suppose that H_8 is eventually successful in finding the splitting that it searches for, so that H_7 is declared to be in state 1, and has two strings delivered to it, incompatible extensions of p_4 which are a Ψ_1 splitting that has been enumerated into $Tr_{1,0}$. Let us suppose also that H_6 is never declared to be in state 1. Then at every subsequent stage of the construction, H_7 is passed control and searches for a Ψ_2 splitting on $Tr_{1,0}$, both strings of which extend the leftmost of the two strings that were delivered to it, p_7 say. H_7 passes control to $H_{10} = \mathcal{D}[t_{10}, p_7]$, where $t_{10} = \{Tr_{1,0}, Tr_{3,1}\}$. Why do we define t_{10} in this way? On this part of the tree of strategies it looks! as if $Tr_{1,0}$ will be infinite. If, however, H_7 was ever successful in finding the Ψ_2 splitting that it is searching for, then H_6 would be declared to be in state 1 and H_7 and H_{10} would never be passed control again. H_{10} therefore assumes that $Tr_{2,0}$ will be finite and initiates the construction of a new splitting tree $Tr_{3,1}$, which we shall construct so that it may be regarded as a subtree of $Tr_{1,0}$ but not $Tr_{2,0}$.

1.1.6. *$\mathcal{D}$-clusters of strategies.* Each $\mathcal{D}$ strategy H and all of those $\mathcal{C}$ strategies above H but not above any other $\mathcal{D}$ strategies strictly above H should be thought of as a '$\mathcal{D}$-cluster' of strategies. These strategies enumerate strings into the trees into $t(H)$. Suppose that $H = \mathcal{D}[t, p]$ is on the true path of the construction. If H is eventually declared to be in state 1, then the $\mathcal{C}$ strategies in this $\mathcal{D}$-cluster have enumerated splittings into each of the trees in $t(H)$ above p (such that each string in the splitting extends p). Let p_1 be the leftmost of the two strings which are delivered to H. When H is declared to be in state 1, p_1 extends a leaf of each of the trees in t. At every subsequent stage of the construction H passes control to a strategy $\mathcal{D}[t', p_1]$, where t' contains all the trees in t together with a new candidate for another splitting tree. The $\mathcal{C}$ strategies in this next $\mathcal{D}$ cluster of strategies will then look to enumerate splittings above p_1 into all of the trees in t'.

Suppose, however, that H is not declared to be in state 1. Then there is some $\mathcal{C}$ strategy in this $\mathcal{D}$-cluster of strategies, H' say, which is on the true path of the construction and which witnesses the fact that one of the trees in t is finite—H' continues to search for a splitting to enumerate into this tree without success. Above H' on the true path of the construction is a $\mathcal{D}$ strategy H'' with $t(H'')$ defined accordingly.

1.1.7. *The verification.* (Sketch) Of course a verification for the final construction will be provided in perfect detail. Since the aim of the present section, however, is simply to aid the reader in gaining some intuition for the final construction, we shall simply sketch the verification for this construction—leaving all remaining details for the reader to complete. Let us regard a strategy as being active from the point at which it is first passed control until the point at which a strategy strictly below it is declared to be in state 1. We must show first that the splitting trees are enumerated properly. In order to do so we may prove that the following statements hold, by induction on the stage s. The induction is quite simple and is left to the reader.

(i) If a $\mathcal{C}$ strategy H_0 enumerates p_1, p_2 into Tr at stage s, then these strings each extend (what was prior to this enumeration) a leaf of Tr. Suppose $Tr' \in t^\star(H_0)$. Then p_1 and p_2 extend different leaves of Tr'.

(ii) Let $Tr, Tr' \in t(H_1)$ be such that Tr is a candidate for Tr_j and Tr' is a candidate for $Tr_{j'}$, $j' > j$. Suppose that at stage s' a splitting p_1, p_2 is enumerated into Tr'. According to (i), p_1 extends a leaf of Tr, q_1 say. If at any subsequent stage s there still exists some active strategy H_2 with $Tr' \in t(H_2)$ and some string q is enumerated into Tr extending q_1, then q extends p_1.

(iii) Suppose that $H = \Upsilon[t, p]$ is passed control for the first time at stage s. Then p extends a leaf of each of the trees in t.

Statement (ii) above says essentially this—so long as there is some active strategy H_2 with $Tr' \in t(H_2)$, any string enumerated into Tr' will remain compatible with Tr. We must prove that this is the case since we must know that it is always possible to carry out the instructions given to a $\mathcal{C}$ strategy upon finding a splitting. Statement (iii) has been added to the induction hypothesis simply in order to smooth the induction.

It is clear that the instructions for each strategy at each stage of the construction are finite, that only a finite number of strategies are passed control at each stage of the construction, that the true path of the construction is an infinite set of strategies and that the approximation to B converges and B is well defined. We are thus left to show that B is a set of minimal degree.

Let $H_0, H_1, H_2, ...$ be those strategies on the true path, in the order in which they appear on the tree of strategies. We show, by induction on j, that for all j precisely one of the two following possibilities occurs. Once again, the details are not difficult and are left to the reader.

(i) $(\exists i, n)(\forall n' \geq n)[Tr_{j,i} \in t(H_{n'})]$. Then $Tr_{j,i}$ is infinite, B lies on this tree and $B \leq_T \Psi_j^B$.

(ii) There is a greatest n such that $(\exists i)[Tr_{j,i} \in t(H_n)]$. Then, letting n be such, H_n is a $\mathcal{C}$ strategy which searches for a Ψ_j splitting on an infinite tree, such that B lies on this tree. If Ψ_j^B is total, then it is computable.

1.1.8. *Terminology.* Before going on to discuss the ways in which we may adapt the framework described in this section in order to construct minimal complements, let us establish some terminology that will be useful. If a strategy is of the form $\Upsilon[t, p]$, then we shall refer to p as the 'base string' for this strategy. Generally speaking, strategies will actually be defined by a larger set of parameters; the 'base string' of the strategy is that unique parameter which is an element of $2^{<\omega}$. When a strategy is declared to be in state 1 we shall regard the strings which are delivered to the strategy as being 'in the environment' of the strategy.

1.2. **Complementing a single c.e. degree.** In this section we shall consider how to adapt the minimal degree construction described in 1.1 in order to construct a minimal complement for any individual c.e. degree (other than 0 or $0'$). We shall assume given an enumeration $\{W_s\}_{s\geq 0}$ of W which is not computable. In the *final* construction, of course, we shall have to consider *all* c.e. sets W_i, and it will be convenient to assume that $W_{i,s}$ is a string of length s. For the sake of uniformity of presentation, then, we shall assume now that for $s \in \omega$, W_s is a finite binary string of length s. We assume given an enumeration $\{K_s\}_{s\geq 0}$ of K, and we shall enumerate an approximation $\{B_s\}_{s\geq 0}$ to a set B of minimal degree b and axioms for a Turing functional Γ such that $\Gamma^{W,B} = K$.

The basic framework will be almost identical to that of the construction defined in §1.1. Once again there will be two kinds of strategy, $\mathcal{C}$ strategies and $\mathcal{D}$ strategies. Where previously we were only concerned with defining an approximation to B of minimal degree, now we must also enumerate axioms for Γ. Each strategy now has a dual role. Thus we consider strategies of the form $\mathcal{D}[t,p](n)$ and $\mathcal{C}[t,p](n)$. The n parameter here is used in order to specify the form of the axioms which are to be enumerated by the strategy. Let us initially be a little vague about the manner in which we shall enumerate axioms, saying only this. When a strategy $H = \Upsilon[t,p](n)$ is declared to be in state 1 and has two strings p_1, p_2 delivered to it, we shall call the leftmost of these two strings p_1 the 'axiom enumeration' string for H. When H puts B_s through its axiom enumeration string it will enumerate the axiom $\Gamma^{\sigma,p_1}(n) = K_s(n)$, where $\sigma = W_s \restriction |p_1|$. The 'axiom enumeration' string terminology may not really seem necessary in the context of discussing the construction of this section. Later on, however, it will be necessary to discuss the interactions of larger groups of strategies. To be able to refer to a string in the environment of one of these strategies without the use of a variable will often be convenient.

The basic obstacle that we face can be illustrated by means of a simple example. Suppose that at stage s the strategy $H = \mathcal{C}[t,p](n)$ puts B_s through p_1 and enumerates the axiom $\Gamma^{\sigma,p_1}(n) = 0$ (where $\sigma = W_s \restriction |p_1|$). H is a $\mathcal{C}$ strategy and therefore searches for a splitting to enumerate into a tree, Tr say. Suppose further, that at stage $s' > s$, H finds the splitting that it is searching for, such that both strings in the splitting extend p_1. Now we have a problem. Suppose we enumerate this splitting into Tr and then n is subsequently enumerated into K. If $\sigma \subset W$ we shall not be able to satisfy both of the conditions that a) B lies on Tr (and $p \subset B$), b) $\Gamma^{W,B} = K$.

The solution to this problem is actually perfectly simple—when H finds the splitting we shall simply 'step to the side' and wait to see whether all of the axioms enumerated on the strings in the splitting become unproblematic in some particular way, made precise by the following definition.

Definition 1.4. Suppose that at stage s we have enumerated an axiom $\Gamma^{\sigma,\tau}(n) = c$ for some $n \in \omega$, $c \in \{0,1\}$ and $\sigma, \tau \in 2^{<\omega}$. At stage $s' > s$ we say that this axiom is 'expired' if $\sigma \not\subset W_{s'}$.

If an axiom enumerated at stage s is expired at stage $s' > s$, then this means, essentially, that by stage s' it is clear that the enumeration of this axiom is no longer of relevance. Since the axiom defines the value of Γ on an initial segment incompatible with W, we can proceed just as if the axiom had never been enumerated.

So let the rightmost of the two strings delivered to $H = \mathcal{C}[t,p](n)$ be p_2 and suppose that, at stage s, H finds the splitting q_1, q_2 that it is searching for, such that both strings in the splitting extend its axiom enumeration string p_1. H does not immediately enumerate this splitting into Tr. At every subsequent stage at which H is passed control it checks to see whether all of the axioms we have enumerated of the form $\Gamma^{\sigma,\tau}(n') = c$ such that $p \subset \tau \subseteq q_i$ for $i \in \{1,2\}$ (and for any $\sigma \in 2^{<\omega}, n' \in \omega$ and $c \in \{0,1\}$) are expired. If so, then we can proceed just as we did in §1.1, enumerating this splitting into Tr and declaring that strategy which passes control to H to be in state 1. Otherwise, H passes control to the strategy $\mathcal{C}[t, p_2](n)$, which proceeds just as H did, but with base string p_2. The strategy H does not enumerate any axioms at such stages.

The apparent danger with this course of action is clear—that there might be formed an infinite sequence of $\mathcal{C}$ strategies, $H_0, H_1, H_2, ..$ such that for each $i \geq 0$ the following conditions are satisfied:

(i) H_i finds a splitting which never becomes suitable for enumeration into Tr, because the axioms enumerated on the strings in the splitting never expire;

(ii) once H_i has found a splitting it passes control to H_{i+1} at every stage at which it is passed control.

It is here that we must use the fact that W is not computable. If H_i finds a splitting which never becomes suitable for enumeration into Tr, then actually this allows us to decide the initial segment of W of the same length as the axiom enumeration string for H_i. In order to see this we can reason as follows. When H_i is declared to be in state 1 the two strings that are delivered to it are free of unexpired axioms of the relevant variety (these two strings have just been enumerated into a tree, and we will have insisted that all axioms of the relevant kind are expired prior to this enumeration). When H_i finds the splitting consider the axioms that this strategy has *already* enumerated. Subsequent to this point H_i will not enumerate any axioms. Each is of the form $\Gamma^{\sigma,\tau}(n') = c$. For all such axioms, τ is the axiom enumeration string for H_i and σ is of length $|\tau|$. If all of these axioms were to become expired at some subsequent stage, then the splitting found by H_i would become suitable for enumeration into Tr. The existence of a sequence of $\mathcal{C}$ strategies of the kind described above, then, would allow us to decide arbitrarily long initial segments of W and would thus imply the computability of W.

Before describing instructions for the modified strategies in detail, the following point is worth making. In order to deal with the possible enumeration of n into K one might think it a good idea to provide each strategy H with *three* strings, including *two* axiom enumeration strings. Then H, which is concerned with ensuring that Γ is correctly defined on argument n, could put B_s through the leftmost of the axiom enumeration strings at stages s at which n has not yet been enumerated into K (enumerating the appropriate axiom for Γ at every such stage), and put B_s through the rightmost at every subsequent stage. In fact we can deal quite easily with the enumeration of n into K without requiring that each strategy should be provided with three strings upon being declared to be in state 1. At each stage at which H is passed control we check to see whether n has been enumerated into K. If not, then, all other factors aside, we may put B_s through the axiom enumeration string and enumerate an axiom for Γ. If n has been enumerated into K, then check to see whether this strategy has previously been passed control in state 1 at a stage s' such that $K_{s'}(n) = 0$. If not, then, all other factors aside, we may put B_s through the axiom enumeration string and enumerate an axiom for Γ. Otherwise put B_s through the rightmost of the two strings delivered to H (without enumerating any axiom for Γ) and pass control to a strategy H' which is also concerned with ensuring that Γ is correctly defined on argument n.

1.2.1. *The $\mathcal{D}[t, p](n)$ strategy.* Let $H_0 = \mathcal{D}[t, p](n)$. Here $p \in 2^{<\omega}$, $n \in \omega$ and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$.

The instructions for the strategy in state 0.

Pass control to the strategy $H_1 = \mathcal{C}[t, p](n)$.

The instructions for the strategy in state 1.

H_0 will have had two strings delivered to it, p_1 and p_2. Let p_1 be the leftmost and call this the 'axiom enumeration string'. At each stage s at which H_0 is passed control in state 1 it performs the following:

a) If $K_s(n) = 0$, put B_s through the axiom enumeration string. If $s \geq |p_1|$, then enumerate the axiom $\Gamma^{\sigma,p_1}(n) = 0$, where $\sigma = W_s \upharpoonright |p_1|$ — recall that W_s is a string of length s. Pass control to $H_2 = \mathcal{D}[t_2, p_1](n+1)$. We shall define t_2 subsequently.

b) If $K_s(n) = 1$, check to see whether this strategy has ever been passed control in state 1 at a stage $s' < s$ such that $K_{s'}(n) = 0$.

If so, then put B_s through p_2. Pass control to the strategy $H_3 = \mathcal{D}[t_3, p_2](n)$. We shall define t_3 subsequently.

If not, then put B_s through the axiom enumeration string. If $s \geq |p_1|$, then enumerate the axiom $\Gamma^{\sigma,p_1}(n) = 1$, where $\sigma = W_s \upharpoonright |p_1|$. Pass control to $H_2 = \mathcal{D}[t_2, p_1](n+1)$. Again, we shall define t_2 subsequently.

1.2.2. *The $\mathcal{C}[t,p](n)$ strategy.* Let $H_0 = \mathcal{C}[t,p](n)$. Here $p \in 2^{<\omega}$, $n \in \omega$ and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$.

The instructions for the strategy in state 0.

(i) If $t = \emptyset$, choose two incompatible extensions of p, enumerate these two strings into $Tr_{-1,0}$ and 'deliver' them to that strategy H_1 which passes control to H_0, declaring H_1 to be in state 1. Terminate the present stage of activity for the construction.

(ii) If $t \neq \emptyset$, pass control to $H_2 = \mathcal{C}[t^\star(H_0), p](n)$.

The instructions for the strategy in state 1.

H_0 will have had two strings delivered to it, p_1 and p_2. Let p_1 be the leftmost and call this the 'axiom enumeration string'. At each stage s at which H_0 is passed control in state 1 it performs the following steps:

(i) Go directly to step (ii) if H_0 has already been declared 'splitting positive'. Otherwise H_0 performs another step in an exhaustive search procedure which looks for a Ψ_{j_k} splitting on $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$ if $k = 1$), such that both strings in the splitting extend p_1. If no such splitting q_1, q_2 has been found, then proceed directly to (iii). Otherwise find q_1', q_2', extensions of q_1 and q_2 respectively, which are compatible with all of the trees in $t \cup \{Tr_{-1,0}\}$ and longer than any of the strings in these trees. The strings q_1', q_2' are referred to as the splitting found by H_0, and this strategy is now regarded as being 'splitting positive'. Terminate stage s activity for the construction.

(ii) Let q_1, q_2 be the splitting found by H_0. Check to see whether all of the axioms we have enumerated of the form $\Gamma^{\sigma,\tau}(n') = c$ such that $p \subset \tau \subseteq q_i$ for $i \in \{1,2\}$ (and for any $\sigma \in 2^{<\omega}, n' \in \omega$ and $c \in \{0,1\}$) are expired. If so, then enumerate q_1, q_2 into Tr_{j_k,i_k}, deliver them to that strategy H_1 which passes control to H_0, declaring H_1 to be in state 1. Terminate stage s activity for the construction. Otherwise pass control to the strategy $H_3 = \mathcal{C}[t, p_2](n)$.

(iii) Perform the following:

a) If $K_s(n) = 0$, put B_s through the axiom enumeration string. If $s \geq |p_1|$, then enumerate the axiom $\Gamma^{\sigma,p_1}(n) = 0$, where $\sigma = W_s \upharpoonright |p_1|$. Pass control to $H_4 = \mathcal{D}[t_4, p_1](n+1)$. We shall define t_4 subsequently.

b) If $K_s(n) = 1$, check to see whether this strategy has ever been passed control in state 1 at a stage $s' < s$ such that $K_{s'}(n) = 0$.

If so, then put B_s through p_2. Pass control to $H_3 = \mathcal{C}[t, p_2](n)$.

If not, then put B_s through the axiom enumeration string. If $s \geq |p_1|$, then enumerate the axiom $\Gamma^{\sigma, p_1}(n) = 1$, where $\sigma = W_s \upharpoonright |p_1|$. Pass control to the strategy $H_4 = \mathcal{D}[t_4, p_1](n+1)$.

1.2.3. *The tree of strategies.* At the base of the tree of strategies is $\mathcal{D}[t_0, \emptyset](0)$, where $t_0 = \{Tr_{0,0}\}$.

Above each strategy $H_1 = \mathcal{D}[t_1, p](n)$ there are precisely three others. Leftmost there is that strategy which H_1 passes control to while in state 0, the strategy $H_2 = \mathcal{C}[t_1, p](n)$. Next there is that strategy which H_1 passes control to at each stage at which it puts B_s through its axiom enumeration string p_1. This is the strategy $H_3 = \mathcal{D}[t_3, p_1](n+1)$. We define t_3 the first time that we pass control to H_3 as follows. Let j be the least such that $\nexists i(Tr_{j,i} \in t_1)$ and such that there does not exist any $\mathcal{C}$ strategy below H_1 searching for a Ψ_j splitting compatible with $t'' \subseteq t^\star(H_1)$. Let i be the least such that there is no strategy H_4 that we have yet passed control to with $Tr_{j,i} \in t(H_4)$. We define $t_3 = \{Tr_{j,i}\} \cup t^\star(H_1)$. Rightmost there is that strategy which H_1 passes control to at each stage at which it puts B_s through p_2, the rightmost of the two strings delivered to H_1. This is the strategy $H_4 = \mathcal{D}[t_4, p_2](n)$. We define t_4 the first time that we pass control to H_4 in precisely the same way that we define t_3 the first time that we pass control to H_3.

Above each strategy $H_5 = \mathcal{C}[t_5, p](n)$ such that $t_5 \neq \emptyset$ there are precisely three others. Leftmost there is that strategy which H_5 passes control to while in state 0, the strategy $H_6 = \mathcal{C}[t^\star(H_5), p](n)$. Next there is that strategy which H_5 passes control to at each stage at which it puts B_s through its axiom enumeration string p_1. This is the strategy $H_7 = \mathcal{D}[t_7, p_1](n+1)$. We define t_7 the first time that H_5 passes control to H_7 in precisely the same way that we define t_3 the first time that H_1 passes control to H_3 in the above. Rightmost there is that strategy $H_8 = \mathcal{C}[t_5, p_2](n)$ which H_5 passes control to at each stage at which it puts B_s through p_2, the rightmost of the two strings delivered to H_5.

1.2.4. *Visualizing the tree of strategies.* As previously, let us consider a strategy to be active from the point at which it is first passed control until a strategy strictly below it is declared to be in state 1. The following comments are intended to aid visualization of the tree of strategies and specifically the relationship between this tree and our approximation to B. Suppose that H and H' are active strategies. If H is to the left of H' on the tree, then the base string for H is to the left of the base string for H'. If H is above H' on the tree of strategies, then the base string for H extends that of H'. When visualizing the tree of strategies one imagines nodes to which strategies are allocated. Suppose two nodes are one above the other, to the lower node is allocated the strategy H and to the upper node is allocated the strategy H'. A line between these nodes indicates that H may pass control to H'. In the context of this construction it is effective to think of this line as being the base string for H'.

1.3. **Complementing finitely many c.e. degrees.** Given enumerations of noncomputable c.e. $V_1, .., V_k$ we wish to approximate B of minimal degree and enumerate axioms for Turing functionals Γ_i, such that $\Gamma_i^{V_i, B} = K$. In fact, we need make only very minor modifications to the construction.

In order to deal with the fact that we must now enumerate axioms for more than one Turing functional, fix some computable function $f : \omega \mapsto \{1, .., k\}$ such that $f^{-1}(i)$ is infinite for each $i \in \{1,..,k\}$. We now consider strategies of the form $H = \Upsilon[..](m,n)$. When H puts B_s through its axiom enumeration string p_1 it will enumerate the axiom $\Gamma^{\sigma,p_1}_{f(m)}(n) = K_s(n)$, where $\sigma = V_{f(m),s} \restriction |p_1|$. At stages s at which H puts B_s through its axiom enumeration string it passes control to strategies of the form $\Upsilon'[..](m+1,n')$ (probably the reader is able to suggest how n' should be defined). At other stages, if H passes control to any strategy at all, it will be a strategy of the form $\Upsilon''[..](m,n)$.

The only other change that we need make to the construction is entirely obvious. Previously, when a $\mathcal{C}$ strategy H with base string p had found a splitting q_1, q_2, we would require that the following condition be satisfied before allowing the enumeration of this splitting into the relevant tree:

($\dagger_0$) All of the axioms we have enumerated of the form $\Gamma^{\sigma,\tau}(n) = c$ such that $p \subset \tau \subseteq q_{i'}$ for $i' \in \{1,2\}$ (and for any $\sigma \in 2^{<\omega}, n \in \omega$ and $c \in \{0,1\}$) are expired.

We now replace this condition with:

($\dagger_1$) All of the axioms we have enumerated of the form $\Gamma^{\sigma,\tau}_i(n) = c$ such that $p \subset \tau \subseteq q_{i'}$ for $i' \in \{1,2\}$ (and for any $\sigma \in 2^{<\omega}, n \in \omega,\ i \in \{1,\dots,k\}$ and $c \in \{0,1\}$) are expired.

Most of the new work has to be done in the verification. So suppose there is formed an infinite sequence of $\mathcal{C}$ strategies, $H_0, H_1, H_2, ..$ such that for each $j \geq 0$ the following conditions are satisfied:

(i) H_j finds a splitting which never becomes suitable for enumeration into Tr, because the axioms enumerated on the strings in the splitting never expire;

(ii) once H_j has found a splitting it passes control to H_{j+1} at every stage at which it is passed control.

For each $j \geq 0$ let q_1, q_2 be the splitting found by H_j and let $g(j)$ be the number of i such that we enumerate an axiom of the form $\Gamma^{\sigma,\tau}_i(n) = c$ such that $p \subset \tau \subseteq q_{i'}$ for $i' \in \{1,2\}$ (and for some $\sigma \in 2^{<\omega}, n \in \omega,\ i \in \{1,\dots,k\}$ and $c \in \{0,1\}$) which does not become expired. Let k' be the least such that there exist an infinite number of j, $g(j) = k'$. Then $k' \geq 1$ and for each j such that $g(j) = k'$ we are able to decide initial segments of k' of the V_i, of length at least that of the axiom enumeration string for H_j. The point is that for large j, once we reach a stage of the construction at which we know $g(j) \leq k'$ we actually know $g(j) = k'$, so that certain axioms enumerated on the splitting found by H_j cannot subsequently expire. The existence of an infinite sequence of $\mathcal{C}$ strategies as detailed above would therefore imply that at least one of the V_i is computable.

1.4. **Complementing a uniformly c.e. sequence of non-zero degrees.** Given the uniformly c.e. sequence of non-computable sets $V_1, V_2, V_3, ..$ we wish to approximate B of minimal degree and ensure that for each i, $B \oplus V_i \equiv_T \emptyset'$. Since we must now enumerate axioms for an infinite number of Turing functionals under construction, the first small modification to be made is the redefinition of f (whose role in the construction was introduced in the previous section).

Definition 1.5. Let $\langle .,. \rangle$ be the standard pairing function. For any $i, j \in \omega$ we define $f(\langle i,j \rangle) = i$.

The principal obstacle concerns the conditions that must be satisfied before the enumeration of splittings. If we continue as we did previously (insisting that axioms enumerated on the strings in a splitting for *all* Γ_i be expired before we allow enumeration), then the argument of the previous section will no longer work. The fact that there were only a finite number of V_i under consideration was essential in allowing us to define k' to be the least such that there exist an infinite number of j, $g(j) = k'$. Where an infinite number of V_i must be considered there might exist no k' for which there exist an infinite number of j, $g(j) = k'$.

In order to resolve this issue we need to induce a situation in which, during the enumeration of any *individual* splitting tree, we need only be concerned with those axioms we have enumerated for Turing functionals we contruct on behalf of a finite number of the V_i. In order to achieve this we construct a number of Turing functionals for each V_i.

Definition 1.6. For $i \in \omega$, U_i is the set of all finite sets of ordered pairs of the form $\{(y_0, z_0), .., (y_d, z_d)\}$ with $d \in \omega$, $0 \leq y_{d'} \leq i$ and $z_{d'} \in \omega$ for $0 \leq d' \leq d$ and such that if $(y, z), (y', z')$ are distinct elements of u for $u \in U_i$, then $y \neq y'$.

For any $i \in \omega$, then, any $u \in U_i$ is just a partial function defined on some arguments $\leq i$. Each $u \in U_i$ should be regarded as a 'guess' as to which trees $Tr_{j,i'}$ such that $j \leq i$ will be infinite – the 'guess' is that $Tr_{j,i'}$ will be infinite precisely if $(j, i') \in u$. We enumerate axioms for Turing functionals $\Gamma_{i,u}$ such that $u \in U_i$ and we satisfy the following: for each $i \in \omega$, if $u \in U_i$ corresponds to the correct guess as regards which trees will be infinite, then for all but finitely many $n \in \omega$, $\Gamma^{V_i,B}_{i,u}(n) = K(n)$. The guess associated with $u \in U_i$ has the following effect on where we enumerate axioms for $\Gamma_{i,u}$:

($\dagger_2$) No strategy H with base string p enumerates axioms for a functional $\Gamma_{i,u}$ if there is a $\mathcal{C}$ strategy below H which is still searching for a splitting, above a string compatible with p, to enumerate into the tree $Tr_{j,i'}$ and $(j, i') \in u$. The guess associated with u, in that case, indicates that such a strategy H will not be on the true path of the construction.

Note that, as far as the correct guess $u \in U_i$ is concerned, the condition ($\dagger_2$) is not restrictive in any problematic way—preventing the enumeration of axioms for $\Gamma_{i,u}$ only at strategies which are not on the true path.

Before enumerating a splitting q_1, q_2 into a tree, $Tr_{j,i}$ say, such that each of q_1, q_2 extends a string p which was (until this point) a leaf of $Tr_{j,i}$, we will insist that all axioms that we have enumerated of any form $\Gamma^{\sigma,\tau}_{i',u}(n) = c$ for $i' < i + j,\ u \in U_{i'},\ n \in \omega,\ c \in \{0, 1\}$ and finite binary strings σ and τ such that $p \subset \tau \subseteq q_{i''}$ for some $i'' \in \{1, 2\}$, are expired, or rather we shall insist that all such axioms are either expired or have already been labeled 'hidden' axioms, a term which we will now explain. After we have enumerated q_1, q_2 into $Tr_{j,i}$ all axioms that we have enumerated of any form $\Gamma^{\sigma,\tau}_{i',u}(n) = c$ for any $i', n \in \omega$ (not just those $i' < i + j$), $u \in U_{i'},\ c \in \{0, 1\}$ and finite binary strings σ and τ such that $p \subset \tau \subseteq q_{i''}$ for some $i'' \in \{1, 2\}$ and which are not expired axioms we shall label 'hidden' axioms if they are not already labeled as such. This is the only way in which axioms that we have enumerated may be labeled as 'hidden' and, as such, may be taken as the definition of what we mean by a 'hidden axiom'. Once an axiom has been labeled as hidden the construction will proceed just as if this axiom had never been enumerated, apart from the fact, that is, that we must still ensure that the axioms we enumerate for all of the various functionals are consistent. When a splitting is enumerated into any tree then, apart from this consistency requirement, we shall be able to proceed just as if there are no axioms enumerated on the strings in that splitting which need be taken into consideration. Any axioms enumerated on these strings will have no impact upon the way in which we subsequently define the approximation to B or the splitting trees.

For any $i \in \omega$, let $u^\star(i)$ be the set of all those pairs (j, i') such that $j \leq i$ and $Tr_{j,i'}$ is infinite. Then $u^\star(i) \in U_i$ and by ($\dagger_2$) axioms that we enumerate for $\Gamma_{i,u^\star(i)}$ will only be labeled hidden axioms when strings are enumerated into trees $Tr_{j,i'}$ such that $j + i' \leq i$ and $(j, i') \notin u^\star(i)$. But there are a finite number of such trees and each such tree is finite. Thus there will be a finite number of axioms that we enumerate for this functional which are labeled hidden axioms. For all but finitely many n we shall have that $\Gamma^{V_i,B}_{i,u^\star(i)}(n) = K(n)$.

1.5. **Complementing all c.e. degrees other than** 0 **and** $0'$**.** For each $i \in \omega$, $\{W_{i,s}\}_{s\geq 0}$ is an enumeration of W_i. For convenience we shall assume that, given any $i, s \in \omega$, $W_{i,s}$ is a finite binary string of length s. We assume given an enumeration $\{K_s\}_{s\geq 0}$ of K and we shall enumerate an approximation $\{B_s\}_{s\geq 0}$ to a set B of minimal degree b. We shall enumerate axioms for Turing functionals $\Gamma_{i,u}$ such that $u \in U_i$. If W_i is not computable, there will exist $u \in U_i$ such that, for almost all n, $\Gamma^{W_i,B}_{i,u}(n) = K(n)$.

We now consider strategies of the form $H = \Upsilon[t, p, r](m, n)$. In this construction more versatility is required as regards the form of the axioms to be enumerated, and it is in this context that we make use of the extra parameter r. When H is passed control and puts B_s through its axiom enumeration string p' it may enumerate axioms of the form $\Gamma^{\sigma,p'}_{f(m),u}(n) = K_s(n)$ where $\sigma = W_{f(m),s} \upharpoonright r$.

Definition 1.7. Given $j \geq 0$ and $q \in 2^{<\omega}$ let n be the least natural number such that $\Psi^q_j(n) \uparrow$. We say that Ψ^q_j is of length n.

Definition 1.8. By a 4-fold Ψ_j splitting of length at least l we mean 4 strings $q_1, .., q_4$ whose images $\Psi^{q_i}_j$ $(1 \leq i \leq 4)$ are 1) of length at least l and 2) pairwise incompatible; for $1 \leq i < i' \leq 4$ if the lengths of $\Psi^{q_i}_j$ and $\Psi^{q_{i'}}_j$ are n_1 and n_2 respectively, then there exists $n_3 < n_1, n_2$ such that $\Psi^{q_i}_j(n_3) \downarrow \neq \Psi^{q_{i'}}_j(n_3) \downarrow$.

$\mathcal{C}$ strategies will now search for 4-fold splittings and when a strategy H is declared to be in state 1 it will have four strings delivered to it. Let these four strings in order from leftmost to rightmost be p_1, p_2, p_3 and p_4 (see Figure 1.1). We consider these strings to be 'in the environment' of the strategy and we label them as follows:

1) p_1 we label the 'axiomatic release';
2) p_2 we label the 'axiom enumeration string';
3) p_3 we label the 'split escape';
4) p_4 we label the 'marked escape'.

At any stage s at which H is passed control in state 1 it will put B_s through one of the strings p_2, p_3, or p_4. The axiom enumeration string plays a role identical to that which it has played in the constructions previously described. The significance of the other labeled strings will be explained in what follows.

1.5.1. *The obstacle.* Of course our principal obstacle is to be able to deal with the fact that some W_i will be computable. The constructions described in §§1.2, 1.3 and 1.4 provide much of the basic machinery required but are all based on a technique which can be summed up "when a splitting is found, step to the side and wait to see whether it becomes suitable for enumeration. We cannot be forced to step to the side infinitely often without any splitting becoming suitable for enumeration

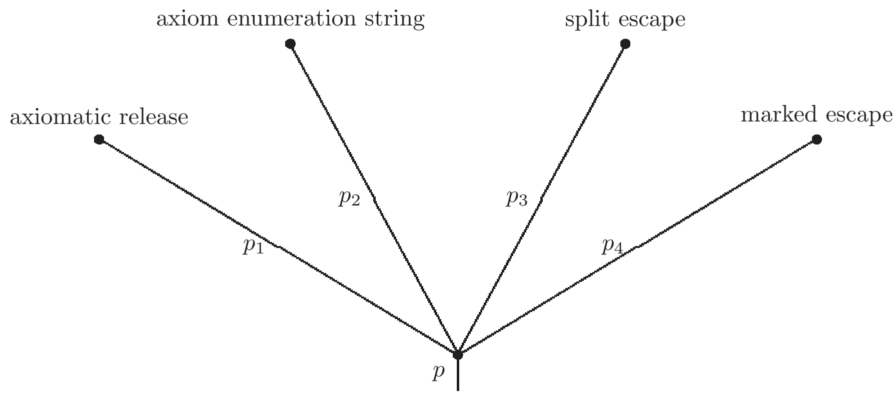


Figure 1.1

since all c.e. sets concerned are non-computable". For the present construction, however, some of the c.e. sets concerned *are* computable. We cannot simply "step to the side"; otherwise, we may be forced to do so infinitely many times without obtaining a successful outcome.

1.5.2. *The primitive intuition.* Let's first of all describe roughly what we do; then later we can come to why it works. So suppose that a $\mathcal{C}$ strategy H has base string p and searches for a splitting to enumerate into Tr_{j_k,i_k}. When H finds a splitting which is not yet suitable for enumeration we do not "step to the side". H continues to search for splittings until one of these splittings becomes appropriate for enumeration. In fact H may find an infinite number of splittings, and none of these splittings may ever become appropriate for enumeration. We must be able to show in this case that if $\Psi^B_{j_k}$ is total, it is computable. At each stage s at which it is passed control H takes each of the splittings that it has found and which are not yet suitable for enumeration into Tr_{j_k,i_k} and for each of these 4-fold splittings, $q_1,..,q_4$ say, it chooses an index $i < j_k + i_k$ to 'blame' for that splitting (we shall be using the approach described in section 1.4 whereby, during the enumeration of Tr_{j_k,i_k} we need only be concerned with those axioms we have enumerated for Turing functionals we construct on behalf of W_i such that $i < j_k + i_k$). The index i blamed for the splitting will be such that prior to the stage of the construction at which we found the splitting $q_1,..,q_4$ we have enumerated an axiom of some form $\Gamma^{\sigma,\tau}_{i,u}(n) = c$ for τ such that there exists $i' \in \{1,..,4\}$, $p \subset \tau \subseteq q_{i'}$, which (at stage s) is not expired and which has not been labeled a hidden axiom. According to this condition there may be more than one index which is appropriate for the allocation of blame. In order to decide precisely which index to blame for each splitting we shall use a procedure which we shall refer to as the 'blame procedure', and which is described in detail in §1.5.5.

Once it has decided to blame an index i for a given splitting, H then 'marks' at stage s each of those strategies above it, of any form $\Upsilon[..](m,n)$ such that $f(m) = i$, and which have enumerated the offending axioms. If one of these 'marked' strategies, $H' = \Upsilon[..](m,n)$ say, is passed control at stage s, then H' will put B_s through the 'marked escape' in the environment of that strategy, p' say, and will pass control to a strategy $H'' = \Upsilon[..](m+1,n')$ with base string p'. The important point here is that H'' has parameter $m+1$ rather than m. Thus, W_i *loses its turn to ensure* *that the appropriate functionals* $\Gamma_{i,u}$ *are defined correctly on argument* n. So we do "step to the side" after all, not immediately above the base string for H, but at the point where those axioms which we blame for the fact that the splitting is not yet suitable for enumeration have themselves been enumerated.

We must discuss two aspects of the effect that the considerations described above have on the construction. We hope that when H finds an infinite number of splittings, none of which become suitable for enumeration, the technique described above will help us in showing that $\Psi^B_{j_k}$ is computable if it is total. We shall come to this in a moment, but let us first consider (just briefly) the effect that this technique has upon the successful construction of Turing functionals $\Gamma_{i,u}$. For each splitting that is found by the strategy H and which never becomes suitable for enumeration into Tr_{j_k,i_k} there exists $i < j_k + i_k$ such that we have enumerated an axiom for some $\Gamma_{i,u}$ on one of the strings in the splitting which never becomes expired, i.e., an axiom of some form $\Gamma^{\sigma^\star,\tau}_{i,u}(n) = c$ such that $\sigma^\star \subset W_i$. Suppose that H finds an infinite number of splittings but that none of these splittings ever becomes suitable for enumeration into Tr_{j_k,i_k}. Let Y be the set of all those $i < j_k + i_k$ such that there exists an infinite number of these splittings for which there exists a stage s such that i is blamed for that splitting at stage s. In section 1.5.5, we shall discuss in some detail how it is that we are able to show that if $i \in Y$, then W_i is computable. We shall achieve this basically by ensuring that there is no upper bound on the length of those strings $\sigma^\star$ obtained from each of the splittings as above. Of course it is not the case that if W_i is computable then necessarily we have $i \in Y$.

So suppose that H finds an infinite number of splittings, none of which becomes suitable for enumeration into Tr_{j_k,i_k}. Let Y be defined as above. Clearly ($\dagger_3$) there can only be a finite number of splittings found such that all of the axioms that have been enumerated on the strings in the splitting of the relevant variety, for Turing functionals of the form $\Gamma_{i,u}$ such that $i \in Y$ and $u \in U_i$, become hidden or expired. Our technique of marking strategies, however, *will* mean that above H, those strategies which appear to be on the true path of the construction and which are successfully enumerating axioms for functionals $\Gamma_{i,u}$ such that $i \in Y$ (rather than putting B_s through the marked escape at every stage at which they are passed control), will be found higher and higher up on the tree of strategies as the construction progresses. If $\tau \subset B$, in fact, it will be the case that all axioms enumerated, on initial segments of τ and proper extensions of the base string for H, for Turing functionals of the form $\Gamma_{i,u}$ such that $i \in Y$ and $u \in U_i$, become hidden or expired. Now suppose that $\Psi^B_{j_k}$ is total. Then in any splitting one of the strings may be replaced by an initial segment of B of sufficient length. Thus there will be an infinite number of *individual* strings in the splittings found (in fact an infinite number of initial segments of B) such that all axioms of the relevant variety which have been enumerated on these strings for functionals $\Gamma_{i,u}$ such that $i \in Y$ become hidden or expired. Let this set of strings be Λ. Now consider the set of strings $\Lambda^\star = \{\Psi^p_{j_k} : p \in \Lambda\}$. Since ($\dagger_3$) holds, we shall be able to show that there exists $l \in \omega$ such that: 1) there are not l incompatible strings in $\Lambda^\star$, and 2) for every $n \in \omega$, there exists $\sigma \in \Lambda^\star$ extending the initial segment of $\Psi^B_{j_k}$ of length n. $\Psi^B_{j_k}$ is therefore computable.

1.5.3. *Splitting strings.* In the constructions of §§1.2, 1.3 and 1.4 we "stepped to the side" upon finding a splitting, and in doing so the following purpose was served.

($\dagger_4$) We were able to wait and see whether all of the axioms enumerated on the strings in the splitting, *prior* to the point at which the splitting was found, become hidden or expired—without having to consider axioms enumerated on these strings subsequent to the point at which the splitting was found. Such considerations were vital in showing that failure of the construction is sufficient to show computability of one of the c.e. sets concerned.

In order that ($\dagger_4$) should still hold we proceed (roughly) as follows. Whenever a $\mathcal{C}$ strategy, H, finds a splitting which is not yet suitable for enumeration into Tr_{j_k,i_k}, we may label those strings 'splitting strings' with respect to all strategies above H.

Definition 1.9. Given a strategy H' with base string p and a string $q \supset p$ in the environment of H', suppose that H' is passed control at stage s. We say that q is 'split at stage s' if there is some string $q' \supset p$ which is compatible with q and which is labeled a splitting string w.r.t. H' at stage s.

When any strategy $H' = \Upsilon[..,r](m,n)$, say, above H is passed control at a stage s in state 1 and finds that its axiom enumeration string is split at stage s, it will not be allowed to enumerate axioms at this stage. Let q be the axiom enumeration string for H'. If it is the case that this strategy has already enumerated axioms $\Gamma^{\sigma,q}_{f(m),u}(n) = K_s(n)$ for $\sigma \subseteq W_{f(m),s}$ (and for those functionals $\Gamma_{f(m),u}$ which are appropriate for axiom enumeration on this part of the tree of strategies), then H' may proceed to put B_s through q at stage s, since in this case there is no need to enumerate any more axioms at this stage. Otherwise, if such axioms have not been enumerated, H' may put B_s through the string which we labeled the 'split escape' for this strategy, q' say. It will then pass control to a strategy of the form $H'' = \Upsilon[.,q',r](m,n)$. In this case also, we shall therefore have that H' does not enumerate axioms at stage s.

It is clear that if we proceed in this way then ($\dagger_4$) will be satisfied, but we must consider the effect that such actions have on the successful construction of Turing functionals $\Gamma_{i,u}$. Along these lines it is important to observe that H' and H'' in the above have the same parameters r, m and n. This means that when the strategy H' is forced, at any stage s, to put B_s through its split escape and pass control to H'', $W_{f(m)}$ *has not lost its turn to ensure that the appropriate functionals* $\Gamma_{f(m),u}$ *are defined correctly on the argument* n. The action that we have taken might, nonetheless, be regarded as a form of injury. If it is the case that, at all but a finite number of stages s, H' is passed control and is forced to put B_s through its split escape before passing control to the strategy H'', clearly we must ensure that the same is not true of an infinite 'chain' of strategies $H', H'', H''...$, each defined from its predecessor in this fashion.

Definition 1.10. For any $r, m, n \in \omega$ we define a 'maximal (r, m, n) set' to be any set of strategies M which is the smallest set satisfying the following criteria for some strategy H' of the form $\Upsilon[..,r](m,n)$ (for some Υ):

(1) H' is in M;
(2) if a strategy H'' is of the form $\Upsilon'[..,r](m,n)$ and there is a stage at which it passes control to, or has control passed to it by, a strategy in M, then H'' is in M.

Definition 1.11. For $q, q' \in 2^{<\omega}$ we say $q <_l q'$ if $q \subset q'$ or if there exists a least n' such that $q(n') \downarrow \neq q'(n') \downarrow$ and for this n' it is the case that $q(n') = 0$.

More specifically, then, we shall have to show that, for any $r, m, n \in \omega$, every maximal (r, m, n) set is finite. For this reason we cannot actually proceed to label *all* of the strings in the splittings that H finds as splitting strings. But then we do not need to—often one string in a splitting can effectively be replaced by a string from another. We must make sure to label enough of these strings as splitting strings in order to ensure the following. If at any stage and for any one of the splittings that H has found it is the case that all of the axioms of the relevant variety, which had been enumerated prior to the point at which H found this splitting, have become hidden or expired axioms, then we are able to find a splitting which is suitable for enumeration into the relevant tree. In order to achieve this it suffices to proceed as follows. We label every string q_0 which is in a splitting found by H as a splitting string, unless there exists $q_1 \neq q_0$ which is also in a splitting found by H and such that both:

(a) $\Psi_{j_k}^{q_0} \subset \Psi_{j_k}^{q_1}$, or $\Psi_{j_k}^{q_0} = \Psi_{j_k}^{q_1}$ and $q_1 <_l q_0$.
(b) Let q_2 be the longest string which is an initial segment of both q_0 and q_1. Then all axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q_3}(n') = c$ for $i < j_k + i_k$, $u \in U_i$, $n' \in \omega$, $c \in \{0,1\}$, $q_2 \subset q_3 \subseteq q_1$ and $\sigma \in 2^{<\omega}$ are hidden axioms or are expired.

The string q_1 satisfies the property that it can certainly replace q_0 in any splitting of which the latter string is a member (recall, also, that H searches for a splitting to enumerate into Tr_{j_k,i_k}).

1.5.4. *Extending the splitting.* There is another problem caused by the fact that we no longer "step to the side" immediately upon finding a splitting. Suppose that H is a $\mathcal{C}$ strategy and that $t(H) = \{Tr_{j_1,i_1}, ..., Tr_{j_k,i_k}\}$ with $j_1 < \cdots < j_k$. Previously, when H found a Ψ_{j_k} splitting on $Tr_{j_{k-1},i_{k-1}}$, we could immediately find extensions of each of the strings in the splitting, compatible with all of the trees in $t(H)$ and longer than any of the strings in these trees. Then we could wait to see whether all of the axioms already enumerated on these longer strings became hidden or expired. In the event that all such axioms do become hidden or expired we could just enumerate the (longer) strings into Tr_{j_k,i_k}—these strings will still extend leaves of each of the trees in $t(H)$ and no axioms will have been enumerated on proper extensions of these strings. The problem now is that, since the point at which the splitting was found, we may have continued to enumerate strings into the trees in $t(H)$ compatible with the strings in the splitting and we may also have enumerated axioms for Turing functionals $\Gamma_{i,u}$ on proper extensions of these strings. Our task is as follows: when we find that all of the (relevant) axioms have expired or have become hidden on the strings in one of the splittings found by H we must be able to find extensions of these strings a) compatible with all of the trees in $t(H)$, b) longer than any of the strings in these trees, c) long enough that no axioms have been enumerated on proper extensions of these strings, and d) which still satisfy the condition that there are no axioms enumerated on these strings that we need be concerned with. No great creativity is required in order to solve this problem; it is really just a technical matter. It is in this context, and only in this context, that we make use of the *axiomatic release* string delivered to each strategy upon being declared to be in state 1.

It is convenient here, and elsewhere, to be able to consider a kind of 'inverse environment' function. Whenever a $\mathcal{C}$ strategy enumerates a string p into a tree

Tr it delivers p to a strategy, H_1 say. At this point, unless it is already defined, we define the 'level' of p to be H_1 and $e(p) = p$. If at any subsequent stage of the construction another $\mathcal{C}$ strategy enumerates strings $q_1, .., q_4$ such that there exists $l \in \{1, .., 4\}$ with $e(p) \subseteq q_l$ into a tree Tr' and delivers $q_1, .., q_4$ to a strategy H_2 which is strictly below that strategy which is presently defined to be the level of p on the tree of strategies, then we redefine the level of p to be H_2. In this case choose l such that $e(p) \subseteq q_l$ and redefine $e(p) = q_l$. So, generally speaking, if H_3 is the level of p, then $e(p)$ is a string extending p in the environment of H_3.

Let us return, then, to consider the strategy H which searches for a Ψ_{j_k} splitting on $Tr_{j_{k-1}, i_{k-1}}$. In fact, rather than searching for a splitting amongst the strings in $Tr_{j_{k-1}, i_{k-1}}$, H will search for a splitting amongst the strings $e(p)$ for p in this tree. Now suppose that, at some stage s, H finds that all relevant axioms on the strings in one of the splittings that it has found, $q_1, .., q_4$ say, have become hidden or have expired. We then take each of these q_i in turn and define q_i' as follows: we must find an extension of q_i which extends (not necessarily properly) a leaf of each of the trees in $t(H)$ and so is suitable for enumeration into Tr_{j_k, i_k}. When the splitting was found by H the string q_i was $e(x_i)$ for some string $x_i \in Tr_{j_{k-1}, i_{k-1}}$ (or $Tr_{-1,0}$ if $k = 1$). We have that $q_i \subseteq e(x_i)$ as it is defined at stage s. Suppose that the level of x_i is a strategy H_1. Then if q is the base string for H_1, we have $q \subset q_i$. Check to see whether H_1 has ever been passed control at a stage $s' < s$ and put $B_{s'}$ through $e(x_i)$. If not, then $q_i' = e(x_i)$. Otherwise let H_2 be that strategy which H_1 passes control to at any stage s' at which it puts $B_{s'}$ through $e(x_i)$. Let H_3 be that strategy lowest on the tree of strategies above (and including) H_2 which is in state 1. We can then define q_i' to be the axiom release for H_3. The whole point of the axiomatic release, the reason that it is suitable for the purpose described here, is that when a strategy is passed control at a stage s it will never put B_s through its axiomatic release.

1.5.5. *The blame procedure.* We shall leave it until the definition of the construction proper in order to define the 'blame procedure' precisely, but it is probably appropriate that we should give here a good impression of the manner in which this procedure will operate. The blame procedure for any strategy H stores the splittings which are found by that strategy in the order in which they are found, so that if an infinite number of splittings are found, then for any $j \in \omega$ we may refer to the j^{th} splitting. At each stage s at which H is passed control, such that H has not yet found a splitting which is suitable for enumeration into the relevant tree (Tr_{j_k, i_k} say) at stage s, the blame procedure will consider each of the splittings in turn and choose some index $i < j_k + i_k$ to blame for each. If i is blamed for the j^{th} splitting, then we shall say that 'i is blamed for j at stage s'. At each stage s at which the blame procedure is run, and for each j such that H has found at least j splittings by stage s, we shall define $g(H, j, s)$ to be the set of all those $i < j_k + i_k$ such that there are axioms (of the relevant variety) which we enumerated, prior to the stage at which this splitting was found, for functionals $\Gamma_{i,u}$ on the strings in the j^{th} splitting which (at stage s) are not yet hidden or expired. From the discussion of section 1.5.3 concerning splitting strings, it is hopefully clear why we shall only have to consider those axioms enumerated prior to the stage at which the splitting was found. The set $g(H, j, s)$, then, contains precisely those $i < j_k + i_k$ which are eligible for blame as regards the j^{th} splitting at stage s. Clearly for any $j \geq 1$ and any $s' > s$, if it is the case that both $g(H, j, s)$ and $g(H, j, s')$ are defined, then $g(H, j, s') \subseteq g(H, j, s)$.

Definition 1.12. For $j \geq 1$ we define $g^{\star}(H, j) = \{i_1, .., i_l\}$ if there exists an infinite number of stages s such that $g(H, j, s) \downarrow = \{i_1, .., i_l\}$.

Definition 1.13. We say that Y is an 'infinitely occurring index set for H' if there exists an infinite number of j such that $g^{\star}(H, j) \downarrow = Y$. We say that Y is a 'minimal infinitely occurring index set for H' if Y is an infinitely occurring index set for H and for no other infinitely occurring index set for H, Y' say, is it the case that $Y' \subset Y$.

Let us suppose that H finds an infinite number of splittings (and that none of these splittings ever becomes suitable for enumeration into Tr_{j_k, i_k}). If, for some i, there exists an infinite number of j such that there exists s, i is blamed for j at stage s, then the first thing that we shall do (with the ultimate aim of proving that W_i is computable) is to ensure that $i \in Y$ for some minimal infinitely occurring index set Y. This we shall achieve as follows.

Suppose that we wish to decide which index $i < j_k + i_k$ to blame at stage s for the j^{th} splitting found by H. First of all, we check to see whether all of the following are satisfied:

(i) H has run the blame procedure at a previous stage. If so, then let s' be the last stage at which it was run.

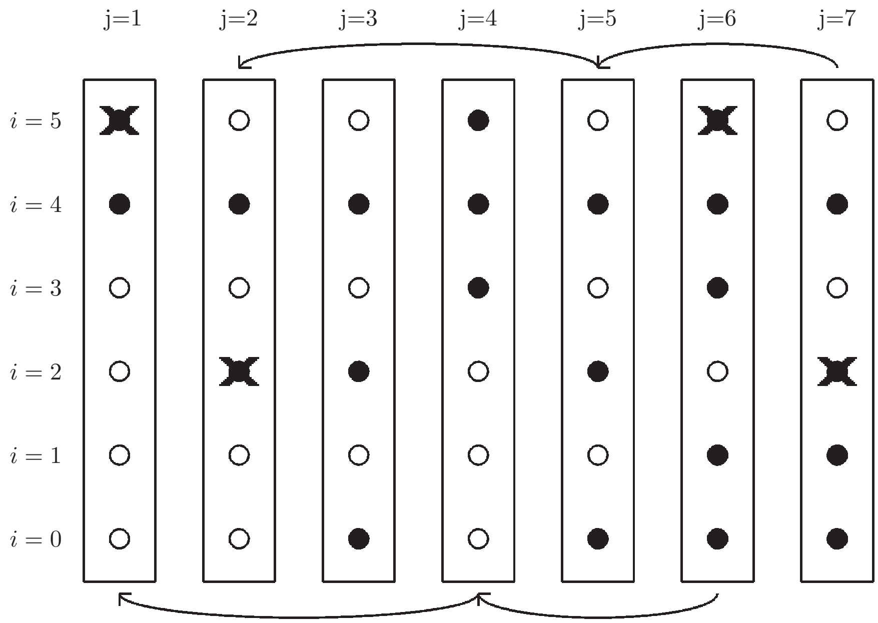
(ii) At stage s', H had already found at least j splittings. If so, then let i be such that we blamed i for j at stage s'.

(iii) $i \in g(H, j, s)$.

If all of these conditions are satisfied, then let i be as above and blame i for j at stage s. Otherwise proceed as follows. Test to see whether there exists $j_1 < j$ such that $g(H, j_1, s) \subset g(H, j, s)$. If not, then just choose some $i \in g(H, j, s)$ and blame i for j at stage s. Otherwise let j_1 be the greatest such number. Then test to see whether there exists $j_2 < j_1$ such that $g(H, j_2, s) \subset g(H, j_1, s)$. If not, then find i such that we blamed i for j_1 at stage s and blame i for j at stage s. Otherwise let j_2 be the greatest such number. Test to see whether there exists $j_3 < j_2$ such that $g(H, j_3, s) \subset g(H, j_2, s)$. If not, then find i such that we blamed i for j_2 at stage s and blame i for j at stage s, and so on.

Why do we first check whether the statements (i), (ii) and (iii) apply? We wish to ensure that for any strategy H' it is either the case that H' is marked by H at all but a finite number of stages, or that H' is only marked by H at a finite number of stages. Since any strategy can only be marked by the finite number of strategies below it, this will ensure that for any strategy H' it is either the case that H' is marked (by some strategy) at all but a finite number of stages or that, at all but a finite number of stages H' is not marked by any strategies at all. Amongst other things this is clearly necessary in showing that we can define a true path for the construction (once again, the true path of the construction will be those strategies passed control at all but a finite number of stages).

Explanation of Figure 1.2: This diagram illustrates how we might decide which index to blame for $j = 6$ and for $j = 7$. For each $1 \leq j \leq 7$, the coloured circles indicate those i in $g(H, j, s)$. For $j = 1, 2, 6, 7$ *only*, we have illustrated which i is blamed for the splitting at stage s with a cross.



In order to show that this procedure achieves what it is supposed to, choose $j_1^\star$ large enough such that for no $j \geq j_1^\star$ is it the case that $g^\star(H, j)$ is not an infinitely occurring index set. Choose $j_2^\star$ large enough such that for every Y which is a minimal infinitely occurring index set for H, there exists j with $j_1^\star \leq j \leq j_2^\star$ such that $g^\star(H, j) = Y$. Choose s large enough such that for all $1 \leq j \leq j_2^\star$, $g(H, j, s) = g^\star(H, j)$. Now suppose that for $j > j_2^\star$, $s' > s$ and some $i < j_k + i_k$, we blame i for j for the first time at stage s'. Consider the iteration that took place. If $g(H, j, s')$ is a minimal infinitely occurring index set, then clearly we may conclude that i is in a minimal infinitely occurring index set. Otherwise there exists $j_1' < j$ such that $g(H, j_1', s') \subset g(H, j, s')$ and $g(H, j_1', s')$ is a minimal infinitely occurring index set. Let j_1 be the greatest number (of any kind) $< j$ such that $g(H, j_1, s') \subset g(H, j, s')$. If $g(H, j_1, s')$ is a minimal infinitely occurring index set, then clearly we may conclude, since $i \in g(H, j_1, s')$, that i is in a minimal infinitely occurring index set. Otherwise there exists $j_2' < j_1$ such that $g(H, j_2', s') \subset g(H, j_1, s')$ and $g(H, j_2', s')$ is a minimal infinitely occurring index set. Let j_2 be the greatest number $< j_1$ (of any kind) such that $g(H, j_2, s') \subset g(H, j_1, s')$, and so on. At some point in this iteration we must find j_u such that $g(H, j_u, s')$ is a minimal infinitely occurring index set, and it must then be the case that $i \in g(H, j_u, s')$.

Let i be such that there exists an infinite number of j such that there exists an s such that i is blamed for j at stage s. Then, by the argument above, we may choose Y which is a minimal infinitely occurring index set for H and such that $i \in Y$. Let $j_1^\star$ be as above. Suppose that for $j \geq j_1^\star$ we find at some stage s that $g(H, j, s) = Y$. Clearly ($\dagger_5$), it cannot be the case at any subsequent stage s' that $g(H, j, s') \subset g(H, j, s)$. In order to show how we can use this fact in order to decide an initial segment of W_i we shall consider a simple example. Suppose that, at the stage at which we found the j^{th} splitting, the only axioms of the relevant variety which we had enumerated on the strings in this splitting, for functionals $\Gamma_{i,u}$ such that $u \in U_i$, and which were not then hidden or expired axioms were the two axioms $\Gamma^{\sigma,\tau}_{i,u'}(n) = c$ and $\Gamma^{\sigma',\tau'}_{i,u''}(n') = c'$. Clearly σ and σ' must be compatible. Let us suppose that $|\sigma| < |\sigma'|$. Then by ($\dagger_5$) the first of these two axioms cannot become an expired axiom, so that at stage s we may conclude that $\sigma \subset W_i$. Suppose that at some subsequent stage the first of these two axioms becomes hidden. At this stage we may conclude that $\sigma' \subset W_i$. Let $j_1 < j_2 < ..$ be all those $j \geq j_1^\star$ such that $g^\star(H, j) = Y$. For each j_n let σ_n be the longest string which we are able to deduce is an initial segment of W_i, by considering those axioms that have been enumerated on the strings in the j_n^{th} splitting in the manner described above. For each $n \geq 1$ there is a strategy of some form $H_n = \Upsilon_n[.., r_n](m_n, .)$, say, such that $f(m_n) = i$, $r_n = |\sigma_n|$ and which enumerates an axiom (of the relevant variety) on one of the strings in the j_n^{th} splitting, which never becomes hidden or expired.

Suppose that there exists an upper bound on the length of the strings σ_n. We have already stated in §1.5.3 that for all $r, m, n \in \omega$ every maximal (r, m, n) set will be finite. If the first strategy passed control in any maximal (r, m, n) set is passed control at stage s, then we shall have that $r = s$ simply because we shall define things so. Since a finite number of strategies is passed control at any stage of the construction (and there is an upper bound on the length of the strings σ_n) the set of strategies $\{H_n\}_{n\geq 1}$ must therefore be finite. Now in the course of searching for splittings for the strategy H we shall have to address the issue: what exactly is to be regarded as a new splitting? If we have previously found the splitting $q_1, .., q_4$, do we subsequently consider $q_1', .., q_4'$ such that each q_i' is simply an extension of q_i, to be a 'new' splitting which must be taken into consideration? In searching for splittings, then, we shall use a specific procedure (answering such questions), which we shall call the 'splitting search procedure', and which will 'output' the splittings that it finds (and which are not necessarily appropriate yet for enumeration) one at a time. The splitting search procedure will ensure ($\dagger_6$) that for any strategy H' which enumerates an axiom which never becomes hidden or expired, and with axiom enumeration string q say, there are a finite number of splittings outputted such that both:

(a) at least one string in the splitting extends q;
(b) not every string in the splitting extends q.

Since the set of strategies $\{H_n\}_{n\geq 1}$ is finite it must be the case, for at least one of the strategies H_n, that there are an infinite number of splittings outputted by the splitting search procedure for H such that at least one string in the splitting extends the axiom enumeration string for H_n. Let H_n be such a strategy and let the axiom enumeration string for this strategy be q. But then, by ($\dagger_6$), it must be the case for the strategy H_n that every string in an infinite number of the splittings outputted by H extends q. This means that $q \subset B$, so that, in fact, for all but a finite number of the splittings found by H it must be the case that at least one string in the splitting extends q. But there exists an infinite number of j such that there exists s for which i is blamed for j at stage s. Thus H_n would be marked at an infinite number of stages. We have stated previously that this will mean H_n is marked at all but a finite number of stages, which gives us the required contradiction since in that case $q \not\subset B$.

1.5.6. *The z function.* From that which we have already said it follows that if W_i is non-computable, then any individual $\mathcal{C}$ strategy will only mark a finite number of strategies of any form $\Upsilon[..](m, n)$ such that $f(m) = i$. The danger remains that for fixed $n' \in \omega$ an infinite number of strategies on the true path will mark strategies on the true path of some form $\Upsilon[..](m', n')$ such that $f(m') = i$. We shall use the function z, as defined below, precisely in order to remove this possibility.

Definition 1.14. For any $j \in \omega$, $z(j) = \{(i, n) : \langle i, n\rangle \leq j\}$.

Thus any strategy which is searching for Ψ_j splittings, say, will not be allowed to mark a strategy of any form $\Upsilon[..](m', n')$ such that $f(m') = i'$ and $(i', n') \in z(j)$. So we shall restrict the strings amongst which it is searching for splittings to those on which we have not enumerated axioms of any form $\Gamma^{\sigma,\tau}_{i',u}(n') = c$ such that $(i', n') \in z(j)$ and which are not hidden or expired. In defining $t(H)$, the first time we pass control to any strategy H, we must therefore deviate just slightly from what occurred in the construction of §1.1. Above a strategy of the form $\Upsilon[..](m', n')$ such that $f(m') = i'$ and such that $(i', n') \in z(j)$, which puts B_s through its axiom enumeration string at stage s, we may have to introduce at stage s a new candidate for Tr_j. We shall be able to show (and hopefully it is clear anyway) that, for any $j \in \omega$, there exists a finite number of strategies on the true path of the construction which put B_s through their axiom enumeration strings at all but a finite number of stages, and which search for Ψ_j splittings. Our use of the function z therefore achieves what we require of it.

1.5.7. *Some technicalities.* Suppose H with base string p searches for a splitting to enumerate into Tr_{j_k,i_k}. In §1.5.4 we discussed the provisions that must be made in order to ensure that, once all of the relevant axioms that have been enumerated on the strings in a splitting have become hidden or expired, we are able to find extensions of each of these strings which are suitable for enumeration. One aspect of the approach outlined was that, rather than searching for a splitting lying on $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$), we search for a splitting amongst the strings $e(p')$ for p' lying on this tree. Now the introduction of the z function in §1.5.6 suggests that we should further refine the set of strings amongst which we search for splittings. In fact the splitting search procedure for H operates as follows, if run at stage s of the construction. It first calculates $e(p')$ for every string p' that has been enumerated into $Tr_{j_{k-1},i_{k-1}}$ or the 'non-splitting tree' if $k-1 = 0$ and which extends the axiom enumeration string for H. Let the set of these images q which satisfy also the condition that there have been no axioms of the form $\Gamma^{\sigma,q'}_{i,u}(n') = c$ enumerated for $(i, n') \in z(j_k)$ and $p \subset q' \subseteq q$ (and for any $u \in U_i$, $c \in \{0, 1\}$ and any finite binary string σ) which are not hidden or expired, be called $\Pi(H, s)$. The splitting search procedure then proceeds to search for splittings amongst the strings in $\Pi(H, s)$.

In doing so, however, we open ourselves up to a new danger which must be avoided. If H is able to find only a finite number of splittings and if there are no strategies above H on the true path of any form $\Upsilon[..](m', n')$ such that $(f(m'), n') \in z(j_k)$ and which put $B_{s'}$ through their axiom enumeration string at an infinite number of stages s', then if $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$ if $k = 1$) is infinite we need to be able to show that either $\Psi^B_{j_k}$ is not total or it is computable. In order to do so, we wish to produce an infinite c.e. set of strings including an infinite number of initial segments of B and in which there does not exist a Ψ_{j_k} splitting. With the nature of the sets $\Pi(H, s)$, however, this becomes more problematic than it might initially seem since any string in the set $\Pi(H, s)$ will not necessarily have extensions (proper or otherwise) in sets $\Pi(H, s')$ for $s' > s$.

The problem is resolved by making another use of the 'splitting string' machinery introduced in §1.5.3. We proceed roughly as follows. We label those strings q in $\Pi(H,s)$ as splitting strings w.r.t. H' for every strategy H' above and including H on the tree of strategies unless there exists $q' \neq q$ in $\Pi(H,s)$ with $\Psi^q_{j_k} \subset \Psi^{q'}_{j_k}$ or $\Psi^q_{j_k} = \Psi^{q'}_{j_k}$ and $q' <_l q$. This does not have any significant effect upon the proof required in order to show that each maximal (r,m,n) set is finite, the only real change being that we must now acknowledge that there are two distinct ways in which a string may come to be labeled a splitting string. Moreover, the problem is solved: if at some stage s the splitting search procedure is run and there exists $q \in \Pi(H,s)$ with $\Psi^q_{j_k} = p'$, say, then at the next stage s' at which the procedure is run (if there exists such) there will be a string $q' \in \Pi(H,s')$ with $p' \subseteq \Psi^{q'}_{j_k}$. If the splitting search procedure only finds a finite number of splittings, then there can only be a finite number of incompatible strings in the set $\{\Psi^q_{j_k} : \exists s'[q \in \Pi(H,s')]\}$. If $\Psi^B_{j_k}$ is total, then it is computable.

The fact that we now search for splittings amongst the strings in $\Pi(H,s)$ causes another problem, which is also easily solved. When we finally do enumerate a splitting into Tr_{j_k,i_k} we wish that each string in this splitting should extend a different leaf of each of the trees in $t^\star(H)$. According to the definitions we have given, however, it is quite possible for $e(p')$ and $e(p'')$ to be incompatible where p and p' are compatible. Even were we to rectify this problem, there are further complications introduced by the fact that, when we find that all relevant axioms have become hidden or expired on the strings in a splitting, we then have to replace each of these strings with another which is suitable as a replacement (in that the newly formed set of strings will still form a splitting!) and which has been labeled a splitting string at all necessary stages—here we refer the reader back to the discussion of §1.5.3. An effective (if perhaps rather heavy handed) way to deal with the problem is to demand that four $\mathcal{C}$ strategies should each find a splitting, and that we should then form the required splitting by choosing one string from each of the four splittings found. Thus we make use of the following lemma.

Lemma 1.1. *Suppose that $\Lambda_1,..,\Lambda_4$ are sets each consisting of four pairwise incompatible strings and such that if $1 \leq i < i' \leq 4$, then every string in $\Lambda_{i'}$ is longer than every string in Λ_i. Then we may choose from each Λ_i a string q_i such that q_i is incompatible with $q_{i'}$ if $i \neq i'$.*

When a strategy $H = \mathcal{C}[t,p,r](m,n)$ finds a splitting on which all relevant axioms have become hidden or are expired, we shall declare this strategy to be 'successful'. At subsequent stages at which the strategy is passed control it may then (other factors aside) put B_s through its split escape p' and pass control to the strategy $\mathcal{C}[t,p',r](m,n)$. At each stage at which H is passed control we consider the smallest set of strategies S satisfying the following criteria:

(1) $H \in S$;
(2) if a strategy H' is of the form $\mathcal{C}[t,p'',r](m,n)$ and has control passed to it by, or has passed control to, a strategy in S, then $H' \in S$.

Once four of the strategies in S have been declared successful (each finding a splitting of greater length than the last) we shall form a splitting for enumeration into Tr_{j_k,i_k} by choosing one string from each of the splittings found by these strategies.

Finally, then, let us briefly consider what criterion the splitting search procedure (for H with base string p which searches for a splitting to enumerate into Tr_{j_k,i_k}) should use in deciding precisely which sets of strings are to be outputted as 'new' splittings. We need a condition which is sufficiently permissive so that (i) should there only be a finite number of outputted splittings, then we are able to show that $\Psi_{j_k}^B$ is computable if total, and (ii) the argument outlined in §1.5.2 will follow in the case that H finds an infinite number of splittings, none of which ever become suitable for enumeration. On the other hand we need to ensure that $(\dagger_6)$ of §1.5.5 is satisfied. The solution is that, for any strategy H' which enumerates an axiom which never becomes hidden or expired, and with axiom enumeration string q say, we allow to be outputted a predetermined number of splittings such that (a) at least one string in the splitting extends q and (b) not every string in the splitting extends q, for each possibility as regards those i such that all axioms we have enumerated for functionals $\Gamma_{i,u}$ on the strings in the splitting are already hidden or expired. More specifically, we consider a number of 'counters', $\beta,\ \beta',\ \kappa$, and we define the function υ as follows. If H' is a strategy above (or equal to) H and is the j^{th} such to first be passed control, then $\upsilon(H,H')=j$. For all strategies H,H' and all $Y\subseteq\omega$, $\beta(H,H',Y)$ and $\beta'(H,H',Y)$ are initially (at the beginning of stage 0 of the construction) equal to 0 and $\kappa(H,H',Y)$ is initially the empty set.

At each stage at which it is run the procedure searches to see whether there exist four strings $q_1,..,q_4\in\Pi(H,s)$ which satisfy all of the following.

(1) $\Psi_{j_k}^{q_i}$ is incompatible with $\Psi_{j_k}^{q_{i'}}$ for $1\leq i<i'\leq 4$.
(2) There is no splitting which has been outputted by this splitting search procedure which is such that each of $q_1,..,q_4$ extends a different string of this splitting.
(3) For each $1\leq i\leq 4$ let Y_i be the set of all those $i'<j_k+i_k$ such that every axiom we have enumerated of the form $\Gamma_{i',u}^{\sigma,p'}(n')=c$ for $u\in U_{i'}$, $n'\in\omega,\ c\in\{0,1\}$ and finite binary strings σ and p' such that $p\subset p'\subseteq q_i$, is either hidden or is expired. Let $Y=\bigcap_{i=1}^4 Y_i$. Then for every strategy H' which is such that it has enumerated an axiom which is not hidden or expired and there exists i' such that $q_{i'}$ extends the axiom enumeration string for this strategy, but such that it is not the case that all of $q_1,..,q_4$ extend this string, either (a) there exists i such that $\beta(H,H',Y_i)<\upsilon(H,H')$ and $q_i\notin\kappa(H,H',Y_i)$, or (b) $\beta'(H,H',Y)<\upsilon(H,H')$.

If there are four such strings we 'output' this splitting and then proceed as follows. Let $Y_i,\ 1\leq i\leq 4$ be defined as in (3) above. For every strategy H' which has enumerated an axiom which is not hidden or expired, such that there exists i' for which $q_{i'}$ extends the axiom enumeration string for this strategy, but such that it is not the case that all of $q_1,..,q_4$ extend this string, proceed as follows:

(i) If (a) applies in (3) above, then take each i such that $\beta(H,H',Y_i)<\upsilon(H,H')$ and q_i is not in the set $\kappa(H,H',Y_i)$, increase $\beta(H,H',Y')$ by 1 for all $Y'\subseteq Y_i$ and enumerate q_i into $\kappa(H,H',Y')$ for all $Y'\subseteq Y_i$.

(ii) If (b) applies in (3) above, then increase $\beta'(H,H',Y')$ by 1 for all $Y'\subseteq Y$.

1.5.8. *Outcomes and injury on the true path.* The true path of the construction is the infinite set of strategies which are passed control at all but a finite number of stages of the construction. Thus, for any given strategy, there is either a stage of the construction after which it is never passed control, or there is a stage of the construction after which it is always passed control and always passes control to the same strategy, having put B_s through the same string in its environment. Let us consider, then, the injury that takes place along the true path of the construction. For any j there will exist a finite number of trees $Tr_{j,i}$ which belong to $t(H)$ for strategies H on the true path. It may be the case that there exists i such that $Tr_{j,i} \in t(H)$ for all but finitely many strategies H on the true path. In this case $Tr_{j,i}$ is infinite and $B \leq_T \Psi_j^B$. Otherwise there will exist a highest strategy H on the true path such that there exists i with $Tr_{j,i} \in t(H)$. H will search for a Ψ_j splitting (on) an infinite tree which is suitable for enumeration into $Tr_{j,i}$. The failure of H to find such a splitting suffices to show that Ψ_j^B is computable if it is total. In addition to there being finite injury *amongst* the splitting trees along the true path, there is also finite injury to these trees caused by our use of the z function.

Now suppose that W_i is not computable. For each n there will be a finite number of strategies on the true path of the form $\Upsilon[..](m,n)$ such that $f(m) = i$ and for all but a finite number of n the highest of these, H say, will successfully ensure that $\Gamma_{i,u}$ is correctly defined on argument n, where u is that element of U_i which constitutes the correct guess as regards which of the trees $Tr_{j,i'}$ such that $j \leq i$ will be infinite. For those strategies of the form $\Upsilon[..](m,n)$ such that $f(m) = i$, which are on the true path and which are strictly below H, it may be the case that they are never declared to be in state 1, or that they are declared to be in state 1 but eventually always put B_s through the split escape, or that they are declared to be in state 1 but are marked at all but a finite number of stages of the construction and so eventually always put B_s through the marked escape in the environment of that strategy.

At this point it is probably best that we begin to articulate the specifics of the construction.

2. The $\mathcal{D}[t,p,r](m,n)$ strategy

Let $H = \mathcal{D}[t,p,r](m,n)$. Here $n, m, r \in \omega$, p is a finite binary string and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$.

Definition 2.1. Given a strategy H we say that $u \in U_i$ is 'appropriate for axiom enumeration at H' if there are no strategies $\mathcal{C}[t'..]$ below or equal to H with $t' = \{Tr_{j'_1,i'_1}, .., Tr_{j'_{k'},i'_{k'}}\}$, $j'_1 < \cdots < j'_{k'}$ and $(j'_{k'}, i'_{k'}) \in u$.

Definition 2.2. At any point in the construction and for any $i, n \in \omega$, $u \in U_i$ and $\tau \in 2^{<\omega}$, we define $\gamma(\tau, i, u, n)$ to be all of those $\sigma \in 2^{<\omega}$ such that we have enumerated an axiom of the form $\Gamma_{i,u}^{\sigma',\tau'}(n) = c$ for some $c \in \{0,1\}$ and $\sigma', \tau' \in 2^{<\omega}$ such that $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$. Note that this set may change as we carry out the construction.

The strategy is initially considered to be in state 0.

2.1. **The instructions for the strategy in state 0.**

If the strategy is passed control in state 0 at stage s, it simply passes control to the strategy $\mathcal{C}[t,p,r](m,n)$.

2.2. **The instructions for the strategy in state 1.**

If the strategy is in state 1, it will have been provided with four incompatible strings all extending p (these strings will have been 'delivered' to it). In order from leftmost to rightmost let these strings be $p_1, .., p_4$. We say that these strings are 'in the environment' of this strategy. We label these strings as follows:

1) p_1 we label the 'axiomatic release';
2) p_2 we label the 'axiom enumeration string';
3) p_3 we label the 'split escape';
4) p_4 we label the 'marked escape'.

If the strategy is passed control in state 1 at stage s, we carry out the following instructions.

a) If H is not marked at stage s, $K_s(n) = 0$ and p_2 is not split at stage s, then put B_s through p_2 and for any $u \in U_{f(m)}$ which is appropriate for axiom enumeration at H and which is such that $W_{f(m),s} \notin \gamma(p_2, f(m), u, n)$ enumerate the axiom $\Gamma^{\sigma,p_2}_{f(m),u}(n) = 0$, where σ is the prefix of $W_{f(m),s}$ of length r.

b) If H is not marked at stage s, $K_s(n) = 0$ and p_2 is split at stage s, then check to see if this strategy has ever been passed control in state 1 at a stage $s' < s$ and put $B_{s'}$ through p_2. If not, then put B_s through p_3. Otherwise let s' be the greatest stage $< s$ at which this strategy was passed control and put $B_{s'}$ through p_2. If the prefix of $W_{f(m),s'}$ of length r is equal to the prefix of $W_{f(m),s}$ of length r, then put B_s through p_2. Otherwise put B_s through p_3.

c) If H is not marked at stage s, $K_s(n) = 1$ and p_2 is not split at stage s, then let s' be the least stage such that $K_{s'}(n) = 1$. Check to see whether this strategy was ever passed control before stage s'. If not, then put B_s through p_2 and for any $u \in U_{f(m)}$ which is appropriate for axiom enumeration at H and which is such that $W_{f(m),s} \notin \gamma(p_2, f(m), u, n)$ enumerate the axiom $\Gamma^{\sigma,p_2}_{f(m),u}(n) = 1$, where σ is the prefix of $W_{f(m),s}$ of length r. Otherwise put B_s through p_3.

d) If H is not marked at stage s, $K_s(n) = 1$ and p_2 is split at stage s, then let s' be the least stage such that $K_{s'}(n) = 1$. Check to see whether this strategy was ever passed control before stage s'. If so, then put B_s through p_3. Otherwise check to see whether this strategy has ever been passed control in state 1 at a previous stage s'' and put $B_{s''}$ through p_2. If not, then put B_s through p_3. Otherwise let s'' be the greatest stage less than s at which this happened. If the prefix of $W_{f(m),s''}$ of length r is equal to the prefix of $W_{f(m),s}$ of length r, then put B_s through p_2 and otherwise put B_s through p_3.

e) If H is marked at stage s, then put B_s through p_4.

Suppose that H has put B_s through p_i. If there is a previous stage s' at which this strategy has put $B_{s'}$ through p_i, then transfer control to the same strategy as the last stage at which it did so. Otherwise run the procedure 'TRANSFER' with inputs $\mathcal{D}[t, p, r](m, n)$ and p_i in order to decide which strategy to transfer control to.

3. The 'TRANSFER' procedure

When a strategy H puts B_s through a string in its environment for the first time, the TRANSFER procedure decides which strategy H should pass control to. In effect, then, this procedure defines the tree of strategies once we have specified which strategy is to be allocated to the node at its base, and which strategies are to be passed control while in state 0. Recall that any strategy H passed control to at a stage s will not put B_s through the string which we have labeled its axiomatic release. Suppose that we are given inputs $\Upsilon[t, p, r](m, n)$ and (the finite binary string) p' at stage s. Then we transfer control to the strategy $\Upsilon'[t', p', r'](m', n')$, where Υ', t', r', m' and n' are defined as follows.

1) If $\Upsilon = \mathcal{D}$, then $\Upsilon' = \mathcal{D}$.

2) If $\Upsilon = \mathcal{C}$, then, if p' is the split or the marked escape for the strategy $\Upsilon[t, p, r](m, n)$, $\Upsilon' = \mathcal{C}$ and if p' is the axiom enumeration string for this strategy, then $\Upsilon' = \mathcal{D}$.

3) If $\Upsilon = \mathcal{C}$ and p' is the split or the marked escape for the strategy $\Upsilon[t, p, r](m, n)$, then $t' = t$.

4) If p' is the split escape for $\Upsilon[t, p, r](m, n)$, then $r' = r, m' = m$ and $n' = n$.

5) If p' is either the marked escape or the axiom enumeration string for the strategy $\Upsilon[t, p, r](m, n)$, then $m' = m+1$ and $r' = s$. Find the number of strategies below and including $\Upsilon[t, p, r](m, n)$ on the tree of strategies of any form $\Upsilon''[..](m'', n'')$ such that $f(m'') = f(m + 1)$ and for any n'', Υ'' and which put B_s through the axiom enumeration string for that strategy at stage s. We define n' to be that number.

6) If $\Upsilon = \mathcal{D}$, or $\Upsilon = \mathcal{C}$ and p' is the axiom enumeration string for the strategy $\mathcal{C}[t, p, r]$ (m, n), then let t' be defined as follows. Let us call the strategy $\Upsilon[t, p, r](m, n)$, H. When we say that one strategy is below/above another we do not mean that it must be properly below/above this strategy. Then find the least number j such that $\nexists i'(Tr_{j,i'} \in t^\star(H))$ and either:

(a) there is no (as yet unsuccessful) strategy $\mathcal{C}[t''..]$ below the strategy H with $t'' = \{Tr_{j'_1,i'_1}, .., Tr_{j'_{k'},i'_{k'}}\}$, $j'_1 < \cdots < j'_{k'}$ (say), and such that $j'_{k'} = j$;

(b) there is an (as yet unsuccessful) strategy $\mathcal{C}[t''..]$ below the strategy H with $t'' = \{Tr_{j'_1,i'_1}, .., Tr_{j'_{k'},i'_{k'}}\}$, $j'_1 < \cdots < j'_{k'}$ (say) and such that $j'_{k'} = j$ but if $\mathcal{C}[t'', ..]$ is the highest of these on the tree of strategies below H, then $k' - 1 \neq 0$ and $Tr_{j'_{k'-1},i'_{k'-1}} \notin t^\star(H)$;

(c) there is an (as yet unsuccessful) strategy $\mathcal{C}[t''..]$ below the strategy H with $t'' = \{Tr_{j'_1,i'_1}, .., Tr_{j'_{k'},i'_{k'}}\}$, $j'_1 < \cdots < j'_{k'}$ (say) and such that $j'_{k'} = j$. Furthermore there exists a strategy $\Upsilon''[..](m^\star, n^\star)$ above this $\mathcal{C}$ strategy and below H such that $(f(m^\star), n^\star) \in z(j)$, which put B_s through its axiom enumeration string at stage s and such that there does not exist a strategy $\Upsilon'''[t''', ..]$ strictly above $\Upsilon''[..](m^\star, n^\star)$ and below H such that $\exists j'$, $Tr_{j,j'} \in t'''$.

Let i be the least number such that there is no strategy $\Upsilon''[t'', ..]$ that we have yet passed control to with $Tr_{j,i} \in t''$ (so we choose a new candidate for Tr_j). We define:

$$t' = \{Tr_{j,i}\} \cup \{Tr_{j',i'} : Tr_{j',i'} \in t^\star(H), j' < j\}.$$

[Explanation: in understanding the motivation behind (c) above we refer the reader to §1.5.6, while (a) and (b) are simply a repeat of what took place in the construction of §1.1. The instructions of 1) to 5) simply follow the intuition outlined in the introduction.]

4. The $\mathcal{C}[t, p, r](m, n)$ strategy

Let $H = \mathcal{C}[t, p, r](m, n)$. Here $n, m, r \in \omega$, p is a finite binary string and t is a finite set of candidates for the Tr_j, $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, with $j_1 < \cdots < j_k$.

The instructions for the strategy at stage s in the case that $t = \emptyset$ are very simple: choose four incompatible extensions of p, $p_1, .., p_4$ say, and enumerate these into the 'non-splitting tree', $Tr_{-1,0}$. Declare that strategy H' which passed control to H to be in state 1, 'deliver' $p_1,..,p_4$ to H' and terminate stage s activity for the entire construction. So suppose that $t \neq \emptyset$.

The strategy (if $t \neq \emptyset$) is initially considered to be in state 0.

4.1. **The instructions for the strategy in state 0.** If the strategy is passed control at stage s in state 0, then it immediately passes control to the strategy $\mathcal{C}[t',p,r](m,n)$, where $t' = t^{\star}(H)$.

4.2. **The instructions for the strategy in state 1.** If the strategy is in state 1, it will have been provided with four incompatible strings all extending p (these strings will have been delivered to it). In order from leftmost to rightmost, let these strings be $p_1,..,p_4$. We say that these strings are 'in the environment' of H. We label these strings as follows:

1) p_1 we label the 'axiomatic release';
2) p_2 we label the 'axiom enumeration string';
3) p_3 we label the 'split escape';
4) p_4 we label the 'marked escape'.

If the strategy is passed control in state 1 at stage s, then we carry out the following instructions.

Step 1) If either the strategy has already been declared successful or the strategy is already marked at stage s, then proceed to step 5). Otherwise we must decide what 'splitting state' the strategy is in. In order to do this consider the smallest set of strategies S satisfying the following criteria:

(1) $H \in S$;
(2) if a strategy H' is of the form $\mathcal{C}[t,..]$ and has control passed to it by, or has passed control to, another strategy in S, then $H' \in S$.

The splitting state of the strategy, d say, is 4 minus the number of strategies in S which have been declared successful. If this is the first time that this strategy has been in splitting state d, then (re)define l_H to be larger than any number yet mentioned in the construction. Proceed to step 2).

4.3. **The 'level' of a string.** These definitions were made in the introduction. We make them again here, in order that this section should be as self-contained as possible. Whenever a $\mathcal{C}$ strategy enumerates a string p' into a tree $Tr_{j,i}$, it delivers p' (and three other strings) to a strategy, H' say. At this point, unless it is already defined, we define the 'level' of p' to be H' and $e(p') = p'$. If at any subsequent stage of the construction another $\mathcal{C}$ strategy enumerates strings $q_1,..,q_4$ such that there exists $l \in \{1,..,4\}$ with $e(p') \subseteq q_l$ into a tree $Tr_{j',i'}$ and delivers $q_1,..,q_4$ to a strategy H'' which is strictly below that strategy which is presently defined to be the level of p' on the tree of strategies, then we redefine the level of p' to be H''. Choose l such that $e(p') \subseteq q_l$ and redefine $e(p') = q_l$. Intuitively speaking, then, in defining the level of a string p' we are just considering a kind of 'inverse environment function'.

4.4. **The splitting search procedure for $\mathcal{C}[t,p,r](m,n)$.** If the strategy H has not been passed control at any previous stage s' and put $B_{s'}$ through the axiom enumeration string, then proceed to step 5). Otherwise proceed as follows.

A) If the splitting search procedure for H is run at stage s of the construction, then it first calculates $e(p')$ for every string p' that has been enumerated into

$Tr_{j_{k-1},i_{k-1}}$ or the 'non-splitting tree' if $k-1=0$ and which extends the axiom enumeration string for H. Let the set of these images q which satisfy also the condition that there have been no axioms of the form $\Gamma^{\sigma,q'}_{i,u}(n')=c$ enumerated for $(i,n')\in z(j_k)$ and $p\subset q'\subseteq q$ (and for any $u\in U_i,\ c\in\{0,1\}$ and any finite binary string σ) which are not hidden or expired, be called $\Pi(H,s)$. We then label every string $q\in\Pi(H,s)$ a 'splitting string at stage s w.r.t. H'' for every strategy H' above and including H on the tree of strategies unless (1), (2) or (3) below apply.

(1) There exists $q'\neq q$ in $\Pi(H,s)$ with $\Psi^{q}_{j_k}\subset\Psi^{q'}_{j_k}$.
(2) There exists $q'\neq q$ in $\Pi(H,s)$ with $\Psi^{q}_{j_k}=\Psi^{q'}_{j_k}$ and $q'<_l q$.
(3) $\left|\Psi^{q}_{j_k}\right|<l_H$ $(=l_{\mathcal{C}[t,p,r](m,n)}$ as defined previously).

[The explanation for this use of splitting strings was given in §1.5.7.]

B) We define the function υ as follows. If H' is a strategy above (or equal to) H and is the j^{th} such to first be passed control, then $\upsilon(H,H')=j$. For all strategies H,H' and all $Y\subseteq\omega$, $\beta(H,H',Y)$ and $\beta'(H,H',Y)$ are initially (at the beginning of stage 0 of the construction) equal to 0 and $\kappa(H,H',Y)$ is initially the empty set. The procedure then searches to see whether there exist four strings $q_1,..,q_4\in\Pi(H,s)$ which satisfy all of the following.

(1) $l_H\leq\left|\Psi^{q_i}_{j_k}\right|$, for $1\leq i\leq 4$.
(2) $\Psi^{q_i}_{j_k}$ is incompatible with $\Psi^{q_{i'}}_{j_k}$ for $1\leq i<i'\leq 4$.
(3) There is no splitting which has been outputted by this splitting search procedure since H entered its present splitting state d which is such that each of $q_1,..,q_4$ extends a different string of this splitting.
(4) For each $1\leq i\leq 4$ let Y_i be the set of all those $i'<j_k+i_k$ such that every axiom we have enumerated of the form $\Gamma^{\sigma,p'}_{i',u}(n')=c$ for $u\in U_{i'},\ n'\in\omega,\ c\in\{0,1\}$ and finite binary strings σ and p' such that $p\subset p'\subseteq q_i$, is either hidden or is expired. Let $Y=\bigcap_{i=1}^{4}Y_i$. Then for every strategy H' which is such that it has enumerated an axiom which is not hidden or expired and such that there exists i' for which $q_{i'}$ extends the axiom enumeration string for this strategy, but such that it is not the case that all of $q_1,..,q_4$ extend this string, either a) there exists i such that $\beta(H,H',Y_i)<\upsilon(H,H')$ and $q_i\notin\kappa(H,H',Y_i)$, or b) $\beta'(H,H',Y)<\upsilon(H,H')$.

If there are four such strings, then we first 'output' this splitting and then proceed as follows. Let $Y_i,\ 1\leq i\leq 4$ be defined as in (4) above. For every strategy H' which has enumerated an axiom which is not hidden or expired, such that there exists i' for which $q_{i'}$ extends the axiom enumeration string for this strategy, but such that it is not the case that all of $q_1,..,q_4$ extend this string, proceed as follows:

(i) If a) applies in (4) above, then take each i such that $\beta(H,H',Y_i)<\upsilon(H,H')$ and q_i is not in the set $\kappa(H,H',Y_i)$, increase $\beta(H,H',Y')$ by 1 for all $Y'\subseteq Y_i$ and enumerate q_i into $\kappa(H,H',Y')$ for all $Y'\subseteq Y_i$.

(ii) If b) applies in (4) above, then increase $\beta'(H,H',Y')$ by 1 for all $Y'\subseteq Y$.

If there are then more sets of strings which satisfy these criteria, noting that the conditions for the satisfaction of (3) and (4) will now have changed, the procedure also outputs these splittings one at a time and makes the necessary changes to the functions β,β' and κ as just described.

[The counters $\beta,\ \beta'$ and κ were introduced in §1.5.7, and some motivation for their definition was provided in that section.]

C) Every time that the procedure outputs a splitting we make the following definitions. Let $q_1, .., q_4$ be the j^{th} splitting that this procedure has outputted since H entered its present splitting state d and a splitting that this search procedure has outputted at stage s. The first two of these definitions we make for use in defining the construction. The last is a simple variant that we make now for use in the verification.

Define the set $Ax(H, j, d)$ to be all those axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q'}(n') = c$ for some $i < j_k + i_k,\ u \in U_i,\ n' \in \omega,\ p \subset q' \subseteq q_l$ for some $1 \leq l \leq 4$, $c \in \{0, 1\}$ and some finite binary string σ, which are not hidden axioms and which are not expired.

For each $i < j_k + i_k$ we define the set $Ax_i(H, j, d)$ to be all those axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q'}(n') = c$ for $u \in U_i,\ n' \in \omega,\ p \subset q' \subseteq q_l$ for some $1 \leq l \leq 4$, for $c \in \{0, 1\}$ and some finite binary string σ, which are not hidden axioms and which are not expired.

For each $1 \leq l \leq 4$ and $i < j_k + i_k$ we define $Ax_i^{\star}(H, j, q_l, d)$ to be all those axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q'}(n') = c$ for $u \in U_i,\ n' \in \omega,\ p \subset q' \subseteq q_l$, $c \in \{0, 1\}$ and $\sigma \in 2^{<\omega}$, which are not hidden or expired.

Step 2) Run the splitting search procedure for this strategy. Then check to see whether there exists a splitting which has been outputted by the splitting search procedure since H entered its present splitting state d and which satisfies the condition that if this was the j^{th} such splitting outputted by this procedure, then all of the axioms in $Ax(H, j, d)$ are either hidden or expired. If so, then proceed to step 4). Otherwise proceed to step 3).

4.5. **The blame procedure for $\mathcal{C}[t, p, r](m, n)$.** The blame procedure stores every splitting that has been outputted by the splitting search procedure for this strategy since it entered its present splitting state d in the order in which they were outputted, so that we may refer to the j^{th} outputted splitting. At each stage s at which it is run it takes these splittings in order and carries out the following instructions.

For the j^{th} splitting, call the strings in the splitting $q_1, .., q_4$. Find all of the $i < j_k + i_k$ such that there is an axiom in $Ax_i(H, j, d)$ which is not hidden and which is not expired. Let these i be $\{i_1, .., i_g\}$ and define $g(H, j, s) = \{i_1, .., i_g\}$. Check to see whether all of the following three conditions are satisfied:

(1) This blame procedure has been run before. If so, let the last stage at which it was run be the stage s'.
(2) At stage s' the splitting search procedure had outputted at least j splittings since H entered its present splitting state d.
(3) $\exists i \in g(H, j, s)$ which was blamed for j at stage s'.

If all three conditions are satisfied, then let i be as in (3) above and 'blame i for j at stage s'. Otherwise run the following iteration.

Test to see whether there exists $r_1 < j$ such that $g(H, r_1, s) \subset g(H, j, s)$. If not, then choose $i \in g(H, j, s)$ and blame i for j at stage s. Otherwise let r_1 be the greatest such number. If there does not exist $r_2 < r_1$ such that $g(H, r_2, s) \subset g(H, r_1, s)$, then find i such that we blamed i for r_1 at stage s. Blame i for j at stage s. Otherwise let r_2 be the greatest such number. Test to see whether there exists $r_3 < r_2$ such that $g(H, r_3, s) \subset g(H, r_2, s)$. If not, then find i such that we blamed i for r_2 at stage s. Blame i for j at stage s. Otherwise let r_3 be the greatest such number, and so on.

Once we have decided which index to blame for j, i say, we then 'mark at stage s' all strategies above (and including) H of any form $\Upsilon[.., r'](m', n')$ which satisfy all of (1)-(3) below:

(1) $f(m') = i$ and $(m', n') \notin z(j_k)$;
(2) there exists $q_{i'}$, $1 \leq i' \leq 4$ which extends the axiom enumeration string for this strategy;
(3) this strategy has enumerated an axiom that is not expired and is not hidden.

Having carried out these instructions for all of the required splittings we then proceed as follows. Let A be the set of all of the strings in the splittings outputted by the splitting search procedure since H entered splitting state d. Label every string $q_0 \in A$ a 'splitting string at stage s w.r.t. H'' for all strategies H' above (and including) H, unless there exists $q_1 \neq q_0$ in A such that both:

a) $\Psi_{j_k}^{q_0} \subset \Psi_{j_k}^{q_1}$, or $\Psi_{j_k}^{q_0} = \Psi_{j_k}^{q_1}$ and $q_1 <_l q_0$.

b) Let q_2 be the longest string which is an initial segment of both q_0 and q_1. Then all axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q_3}(n') = c$ for $i < j_k + i_k$, $u \in U_i$, $n' \in \omega$, $c \in \{0,1\}$, $q_2 \subset q_3 \subseteq q_1$ and $\sigma \in 2^{<\omega}$ are hidden axioms or are expired.

For any $q_0 \in A$, if there exists $q_1 \neq q_0$ in A which satisfies a) and b) above, then we say that q_1 satisfies $(\star, q_0)$ for H at stage s.

Step 3) Perform the blame procedure for this strategy. Then proceed to step 5).

Step 4) There exists j such that all of the axioms in $Ax(H, j, d)$ are hidden or are expired. Declare the strategy to be successful. Let the four strings in the j^{th} splitting outputted by the splitting search procedure for this strategy since it entered its present splitting state d be $p_1, .., p_4$. For each $1 \leq i \leq 4$ we define $r(H, p_i, s)$ as follows, bearing in mind the manner in which the blame procedure will label splitting strings. For each i, $r(H, p_i, s)$ will be a string with which we can replace p_i in order to form the required splitting. Let $s' \leq s$ be the stage at which the splitting search procedure outputted the splitting $p_1, .., p_4$. If $s' = s$, then $r(H, p_i, s) = p_i$. Otherwise if p_i has been labeled a splitting string by the blame procedure for H at stage s' and at every subsequent stage $s'' < s$ at which the blame procedure has been run (if there exist any such), then $r(H, p_i, s) = p_i$. If this is not the case, then let s_1 be the first such stage at which p_i was not labeled a splitting string. Then we may choose p' such that at stage s_1, p' satisfied $(\star, p_i)$ for H and was labeled a splitting string. If p' has been labeled a splitting string by the blame procedure for H at every subsequent stage $s_1 < s'' < s$ at which the blame procedure has been run, then $r(H, p_i, s) = p'$. Otherwise let s_2 be the first such stage at which p' was not labeled a splitting string. Then we may choose p'' such that at stage s_2, p'' satisfied $(\star, p')$ for H and was labeled a splitting string, etc. Note that the string $r(H, p_i, s)$ has been labeled a splitting string at every stage at which H has been passed control and has run the blame procedure for this strategy since it was last outputted as part of a splitting by the splitting search procedure for this strategy.

For $1 \leq i \leq 4$ define $q_i = r(H, p_i, s)$. We then take each of these q_i in turn and define q_i' as follows: we must find an extension of q_i which extends (not necessarily properly) a leaf of each of the trees in t and so is suitable for enumeration into Tr_{j_k, i_k}. At the last stage at which it was outputted as part of a splitting by the splitting search procedure the string q_i was $e(q)$ for some string $q \in Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$ if $k = 1$). Choose one such string and call it x_i (then $q_i \subseteq e(x_i)$ as it is defined at stage s). Suppose that the level of x_i is a strategy H' (then if q is the base string for H', we have $q \subset q_i$). Check to see whether H' has ever been passed control at a stage $s' < s$ and put $B_{s'}$ through $e(x_i)$. If not, then $q_i' = e(x_i)$. Otherwise let H'' be that strategy which H' passes control to at any stage s' at which it puts $B_{s'}$ through $e(x_i)$. Let H''' be that strategy lowest on the tree of strategies above (and including) H'' which is in state 1. Define q_i' to be the axiom release for H'''. The four strings $q_1', .., q_4'$ are then called the resultant splitting for the strategy H. Let $H' = \Upsilon[., p', .]$ be that strategy which passes control to that strategy in S (as defined in step 1 of the construction) which is lowest on the tree of strategies. For $1 \leq i \leq 4$ we define $y(H', q_i') = q_i$ and $x(H', q_i') = x_i$ - we make these definitions only for use in the verification.

If the strategy H is in splitting state 1 and has been declared successful, then we declare the strategy H' to be in state 1. There must be four strategies in S which have been declared successful. Label these $H_1, .., H_4$ in the order that they were declared successful so that H_1 was the first to be declared successful and H_4 the last. Take one string, q_4'' say, from the resultant splitting for H_4 and then one, q_3'' say, from the resultant splitting for H_3 such that $\Psi_{j_k}^{q_4''}$ and $\Psi_{j_k}^{q_3''}$ are incompatible. Then take one string, q_2'' say, from the resultant splitting for H_2 such that $\Psi_{j_k}^{q_2''}$ is incompatible with $\Psi_{j_k}^{q_3''}$ and $\Psi_{j_k}^{q_4''}$, etc. Thus we provide a 4-fold j_k-splitting for H', $q_1'', .., q_4''$, and we then enumerate this splitting into Tr_{j_k,i_k} and 'deliver' $q_1'', .., q_4''$ to H'. Any axioms that we have enumerated of the form $\Gamma_{i,u}^{\sigma,q}(n') = c$ for $p' \subset q \subseteq q_l''$, $1 \leq l \leq 4$, (with p' as defined above and for any i and $u \in U_i$) and which are not hidden or expired axioms we label 'hidden axioms'.

Whichever splitting state the strategy H is in, if we have declared it to be successful, then we terminate stage s activity for the entire construction.

Step 5) Carry out the following instructions.

a) If H is not marked at stage s, has not been declared successful, $K_s(n) = 0$ and p_2 is not split at stage s, then put B_s through p_2 and for any $u \in U_{f(m)}$ which is appropriate for axiom enumeration at H and which is such that $W_{f(m),s} \notin \gamma(p_2, f(m), u, n)$ enumerate the axiom $\Gamma_{f(m),u}^{\sigma,p_2}(n) = 0$, where σ is the prefix of $W_{f(m),s}$ of length r.

b) If H is not marked at stage s, has not been declared successful, $K_s(n) = 0$ and p_2 is split at stage s, then check to see if this strategy has ever been passed control in state 1 at a stage $s' < s$ and put $B_{s'}$ through p_2. If not, then put B_s through p_3. Otherwise let s' be the greatest stage $< s$ at which this occurred. If the prefix of $W_{f(m),s'}$ of length r is equal to the prefix of $W_{f(m),s}$ of length r, then put B_s through p_2. Otherwise put B_s through p_3.

c) If H is not marked at stage s, has not been declared successful, $K_s(n) = 1$ and p_2 is not split at stage s, then let s' be the least such that $K_{s'}(n) = 1$. Check to see whether this strategy was ever passed control before stage s'. If not, then put B_s through p_2 and for any $u \in U_{f(m)}$ which is appropriate for axiom enumeration at H and which is such that $W_{f(m),s} \notin \gamma(p_2, f(m), u, n)$ enumerate the axiom $\Gamma_{f(m),u}^{\sigma,p_2}(n) = 1$, where σ is the prefix of $W_{f(m),s}$ of length r. Otherwise put B_s through p_3.

d) If H is not marked at stage s, has not been declared successful, $K_s(n) = 1$ and p_2 is split at stage s, then let s' be the least stage such that $K_{s'}(n) = 1$. Check to see whether this strategy was ever passed control before stage s'. If so, then put B_s through p_3. Otherwise check to see whether this strategy has ever been passed control in state 1 at a previous stage s'' and put $B_{s''}$ through p_2. If not, then put B_s through p_3. Otherwise let s'' be the greatest stage less than s at which this happened. If the prefix of $W_{f(m),s''}$ of length r is equal to the prefix of $W_{f(m),s}$ of length r, then put B_s through p_2 and otherwise put B_s through p_3.

e) If H is marked at stage s, then put B_s through p_4.

f) If H has been declared successful and is not marked at stage s, then put B_s through p_3.

Suppose that H has put B_s through p_i. If there is a previous stage s' at which this strategy has put $B_{s'}$ through p_i, then transfer control to the same strategy as the last stage at which it did so. Otherwise run the procedure 'TRANSFER' with inputs $\mathcal{C}[t, p, r](m, n)$ and p_i in order to decide which strategy to transfer control to.

5. The construction and the tree of strategies

In order to complete our description of the construction we need only add that at the beginning of every stage control is passed to that strategy at the bottom of the tree of strategies, the strategy $\mathcal{D}[t, \emptyset, 0](0, 0)$ where $t = \{Tr_{0,0}\}$. Above every strategy H_0 such that $t(H_0) \neq \emptyset$ there are precisely four others—in order from leftmost to rightmost let these be called $H_1, .., H_4$. H_1 is that strategy which H_0 passes control to while in state 0 (H_1 is defined in the instructions for the strategy). H_2, H_3 and H_4 are those strategies which H_0 passes control to when it puts B_s through its axiom enumeration string, its split escape and its marked escape respectively. These strategies are defined by the TRANSFER procedure.

6. The verification

We let one 'step' of the construction be the activity carried out by one strategy at one stage of the construction. It is clear that the instructions to be carried out at any step of the construction are finite.

Definition 6.1. Given any strategy H let $H_1, .., H_k$ be the $\mathcal{D}$ strategies below (and, if H is a $\mathcal{D}$ strategy, then including) H on the tree of strategies. We define $T(H) = \bigcup_{j=1}^{k} t(H_j)$.

Definition 6.2. Given any finite binary string p we say at any point in the construction that 'no axioms have been enumerated properly extending p' if no axioms have been enumerated of the form $\Gamma_{i,u}^{\sigma,p'}(n) = c$ for $i, n \in \omega,\ u \in U_i,\ c \in \{0, 1\}$ and finite binary strings σ and p' such that $p \subset p'$. For any $i \in \omega$ and $u \in U_i$ we say that 'no $\Gamma_{i,u}$ axioms have been enumerated properly extending p' if no axioms have been enumerated of the form $\Gamma_{i,u}^{\sigma,p'}(n) = c$ for $n \in \omega,\ c \in \{0, 1\}$ and finite binary strings σ and p' such that $p \subset p'$.

Lemma 6.1. *1) When a strategy H with base string p is passed control for the first time, p extends (not necessarily properly) a leaf of each of the trees in $T(H) \cup \{Tr_{-1,0}\}$ and no axioms have been enumerated properly extending p (before any* *strings have been enumerated into a tree we regard the empty string as a leaf of the tree).*

2) If a strategy $H = \mathcal{C}[t,p,.](m,n)$ with $t = \{Tr_{j_1,i_1},..,Tr_{j_k,i_k}\}$, $j_1 < \cdots < j_k$, (or with $t = \emptyset$) enumerates a splitting $p_1,..,p_4$ into Tr_{j_k,i_k} (or $Tr_{-1,0}$):

a) If $t \neq \emptyset$, then for each $Tr \in t^\star(H) \cup \{Tr_{-1,0}\}$, $p_1,..,p_4$ each extends different (pairwise incompatible) leaves of Tr.

b) For all $Tr \in T(H) \cup \{Tr_{-1,0}\}$, each of $p_1,..,p_4$ extends a leaf of Tr.

c) No axioms have been enumerated properly extending p_i, $1 \leq i \leq 4$ and if $t = \emptyset$, then no axioms have been enumerated properly extending p.

3) Suppose that at step s' a strategy H' entered state 1 and was provided with four strings, $q_1,..,q_4$ say, and suppose further that the tree $Tr \in t^\star(H') \cup \{Tr_{-1,0}\}$. According to 2), at the end of step s' each q_i extended a different leaf of Tr, $q_i^\star$ say. Suppose further that at step $s > s'$ a strategy H below H' with $Tr \in t^\star(H) \cup \{Tr_{-1,0}\}$ enters state 1. Let the four strings that are delivered to this strategy be $p_1,..,p_4$. For all $1 \leq i \leq 4$ it is either the case that there exists j such that p_i extends q_j or that p_i extends a leaf of Tr incompatible with $q_1^\star,..,q_4^\star$.

Proof. The proof is by induction on the step s of the construction. For $s = 0$ the result holds trivially. So assume that we have reached step $s > 0$ of the construction and that the result holds for all $s' < s$. □

Sublemma 6.1. *Suppose that at step $s_1 < s$ control was passed to a strategy H with base string q. Let $s_2 \leq s$ be the least step $\geq s_1$ at which a strategy strictly below H entered state 1 if such exists and let $s_2 = s$ otherwise. Suppose further that at step s_3 with $s_1 < s_3 < s_2$ a strategy $H' \neq H$ which is not above H on the tree of strategies entered state 1 and had delivered to it $p_1,..,p_4$ which were enumerated into a tree $Tr \in T(H) \cup \{Tr_{-1,0}\}$. We have by the induction hypothesis that $p_1,..,p_4$ each extended what were, before step s_3, leaves of this tree. Let these leaves be $q_1,..,q_m$. Then before step s_3 a string q' had been enumerated into Tr which is incompatible with $q_1,..,q_m$ and such that $q' \subseteq q$.*

Proof. Since $s_3 < s_2$, H' cannot be below H on the tree of strategies. So consider, at step s_3, that strategy highest on the tree of strategies which is below H and H'. It must be of one of the two following kinds:

(1) $\mathcal{D}[t',..]$ with $Tr \in t' \cup \{Tr_{-1,0}\}$ and four strings in its environment $p'_1,..,p'_4$;
(2) a $\mathcal{C}$ strategy H'' with $Tr \in t^\star(H'') \cup \{Tr_{-1,0}\}$ and four strings in its environment $p'_1,..,p'_4$.

□

In either case we have by the induction hypothesis that $p'_1,..,p'_4$ each extends different, (pairwise) incompatible strings in Tr. Now if q extends p'_i and $p_1,..,p_4$ all extend p'_j, then $i \neq j$ so that the result follows.

Sublemma 6.2. *Suppose that at step $s_1 < s$ control was passed to a strategy H with base string q. Let $s_2 \leq s$ be the least step $\geq s_1$ at which a strategy strictly below H entered state 1 if such exists and let $s_2 = s$ otherwise. Suppose furthermore that at step s_3 with $s_1 < s_3 < s_2$ an axiom $\Gamma_{i,u}^{\sigma,q'}(n) = c$ for some $i,n \in \omega$, $u \in U_i$, $c \in \{0,1\}$ and finite binary strings σ and q' was enumerated by some strategy $H' \neq H$ which is not above or below H on the tree of strategies. Then q is incompatible with q'.*

Proof. Consider that strategy H'' which is highest on the tree of strategies and below H and H'. This strategy must have been in state 1 at step s_1 and q and q' extend incompatible strings in the environment of this strategy.

So suppose that a strategy H with base string p is passed control for the first time at step s. If there is no strategy strictly below H which is in state 1 at step s, then statement 1) of the lemma follows immediately for the inductive step since in this case no strategy has yet entered state 1. Otherwise let H' be that strategy highest on the tree of strategies strictly below H which is in state 1 at step s. Then p is a string in the environment of H'. We have by the induction hypothesis that when H' first entered state 1, p extended a leaf of each of the trees in $T(H)$ $\cup\{Tr_{-1,0}\}$ and that no axioms had been enumerated properly extending p (since s is the first step at which H has been passed control, any tree $Tr \in T(H) - T(H')$ was empty before step s). Since then no strategy below H' can have entered state 1; otherwise, H would not be passed control at step s. Let the strings in the environment of H' be p, q_1, q_2, q_3. Since H' entered state 1 the strategies above H' can, of all the trees in $T(H) \cup \{Tr_{-1,0}\}$, only have enumerated strings into trees in $t^\star(H') \cup \{Tr_{-1,0}\}$. We have by the induction hypothesis that p, q_1, q_2, q_3 extend (pairwise) incompatible strings in these trees and the strategies above H' can only have enumerated strings into trees in $t^\star(H') \cup \{Tr_{-1,0}\}$ extending one of the q_i. Thus by Sublemma 6.1 (applied to the strategy H'), when H is passed control at step s, p extends a leaf of each of the trees in $T(H) \cup \{Tr_{-1,0}\}$. Since H' entered state 1 it is clear by Sublemma 6.2 that no axioms have been enumerated properly extending p. □

Sublemma 6.3. *Suppose that a strategy H with base string q entered state* 1 *at step $s_1 < s$ and that at step s_1 there existed $Tr \in T(H)$ and a string q' which had been enumerated into Tr such that $q \subset q'$. Then $Tr \in t^\star(H)$.*

Proof. We have by the induction hypothesis that the first time H was passed control, q extended a leaf in each of the trees in $T(H)$. At step s_1 the strategies above H can, of all the trees in $T(H)$, only have enumerated strings into those trees in $t^\star(H)$. We have by Sublemma 6.1 that if when H enters state 1 there exists $Tr \in T(H)$ and q' which has been enumerated into Tr s.t $q \subset q'$, then q' was enumerated into Tr by a strategy above H. Thus $Tr \in t^\star(H)$. □

Sublemma 6.4. *1) Suppose that a strategy H such that $t^\star(H) \neq \emptyset$ entered state* 1 *for the first time at step $s' \leq s$ and had four strings delivered to it, $p_1, .., p_4$ say. Then for each $1 \leq i \leq 4$ the level H' of the string $x(H, p_i)$ before any activity is carried out at step s', is a strategy above H on the tree of strategies.*

2) At any step $s' \leq s$ and for any string p, if the level of p is defined at step s', then it is a strategy with base string $q \subset p$.

Proof. It may be seen by a simple induction that at any step of the construction and for any string p, if it is the case that $e(p)\downarrow$, then $p \subseteq e(p)$, and furthermore that if at some step $e(p) \downarrow = \sigma$ and at a later stage $e(p) \downarrow = \sigma'$ then $\sigma \subseteq \sigma'$. So in order to prove statement 1) of the sublemma consider each of the p_i in turn. The string $y(H, p_i)$ (as defined in step 4 of the description of the $\mathcal{C}$ strategy) extends the axiom enumeration string of a $\mathcal{C}$ strategy above H. Thus, by the previous observation, $e(x(H, p_i))$ extends this string so that H' (the level of the string $x(H, p_i)$ before any activity is carried out at step s') cannot be a strategy below H. Suppose that

H' is a strategy that is neither above nor below H. Then consider that strategy highest on the tree of strategies which is below H and H', H'' say. This must be a strategy in state 1. There must be precisely one string in the environment of H'' which is compatible with $e(x(H,p_i))$, namely that compatible with the base string for H, but then by the induction hypothesis on statement 3) of the lemma and by Sublemma 6.3 we may conclude that H' must be passed control after H'' entered state 1 and that therefore $e(x(H,p_i))$ and the base string for H extend different and incompatible strings in the environment of H''. This gives us the required contradiction.

In order to prove statement 2) of the sublemma, we observe that the level of any string p first becomes defined when it is delivered to some strategy H. At this point it is clear that if q is the base string for H, then $q \subset p$. If the level of p is ever subsequently redefined to be a strategy H', then H' is a strategy below H, so that if q' is the base string for H', then $q' \subseteq q$.

So suppose that at step $s' < s$ a strategy H' with base string q entered state 1 and was provided with four strings $q_1, .., q_4$ and suppose further that the tree $Tr \in t^\star(H') \cup \{Tr_{-1,0}\}$. We have by the induction hypothesis that when activity was completed at step s' each q_i extended a different (incompatible) leaf of Tr, $q_i^\star$ say. Suppose further that at step s a strategy H below H' with $Tr \in t^\star(H) \cup \{Tr_{-1,0}\}$ enters state 1. Let the four strings that are delivered to this strategy be $p_1, .., p_4$. Now consider each of the $x(H,p_i)$ and the level of this string, H_i say, before any activity is carried out at step s. Then by Sublemma 6.4, H_i is a strategy above H. Suppose $Tr \neq Tr_{-1,0}$. Let $t^\star(H) = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$. Since $Tr \in t^\star(H')$, H' is strictly above H and had been passed control before H entered state 1, we have that $Tr \in t^\star(H) - \{Tr_{j_k,i_k}\}$. Now $x(H,p_i)$ is a string in $Tr_{j_{k-1},i_{k-1}}$ and thus by Sublemmas 6.3 and 6.4, $Tr_{j_{k-1},i_{k-1}} \in t^\star(H_i)$. Therefore $Tr \in t^\star(H_i) \cup \{Tr_{-1,0}\}$.

If $H_i = H'$, then statement 3) of the lemma follows immediately (for the inductive step and as regards this string p_i). If H_i is a strategy strictly above H' that had control passed to it after H' entered state 1, then $e(x(H,p_i))$ (before any activity is carried out at step s) extends one of $q_1, .., q_4$. If H_i is a strategy strictly above H' that had control passed to it before H' entered state 1, then since none of the q_i extend $e(x(H,p_i))$ we have by the induction hypothesis that $e(x(H,p_i))$ extends a string in Tr incompatible with each of the $q_i^\star$. Since p_i extends $e(x(H,p_i))$, the result follows.

If H_i is a strategy strictly below H' that was in state 1 before H' entered state 1, then $e(x(H,p_i))$ (before any activity is carried out at step s) is either incompatible with q or is an initial segment of q. In the former case, since p_i extends $e(x(H,p_i))$, p_i is incompatible with q. If $e(x(H,p_i)) \subseteq q$, then let that strategy which H_i passes control to at any stage s'' at which it puts $B_{s''}$ through $e(x(H,p_i))$ be called H''. In forming p_i we then find that strategy, H''', lowest on the tree of strategies above and including H'' which has entered state 1 by step s. If H''' is a strategy strictly below H' which had already entered state 1 at step s' (the step at which H' entered state 1), then the axiom release string for this strategy is incompatible with q. If H''' is a strategy strictly below H' which entered state 1 after step s', then we have by the induction hypothesis that the axiom release string for this strategy either extends one of the q_i or extends a string of Tr incompatible with each of the $q_i^\star$. If $H''' = H'$, then p_i is the axiom release string for this strategy.

If H_i is a strategy below H' that entered state 1 after step s', then we have by the induction hypothesis that $e(x(H,p_i))$ either extends some q_i or extends a string in Tr incompatible with each of the $q_i^\star$.

If H_i is not a strategy below or above H', then consider that strategy, H'' say, highest on the tree of strategies which is below H' and H_i. There are three possibilities:

1) H'' is a strategy below H' which had entered state 1 at step s'. Furthermore H_i had control passed to it after this strategy entered state 1. Then let the strings in the environment of H'' be $q_1',..,q_4'$. We have by the induction hypothesis that each q_l' extends a different (incompatible) string in Tr. If q extends q_l' and $e(x(H,p_i))$ extends q_j', then $l \neq j$.

2) H'' is a strategy below H' which had entered state 1 at step s'. Furthermore H_i had control passed to it before this strategy entered state 1. Then let the strings in the environment of H'' be $q_1',..,q_4'$. Since none of these strings extend $e(x(H,p_i))$, we have by the induction hypothesis that they each extend a string in Tr incompatible with a string in this tree which $e(x(H,p_i))$ extends. There exists j such that q extends q_j'.

3) H'' is a strategy below H' which had not entered state 1 at step s'. Furthermore H_i had control passed to it after this strategy entered state 1. Then let the strings in the environment of H'' be $q_1',..,q_4'$. We have by the induction hypothesis that each of these strings either extends some q_j or extends a string in Tr incompatible with each of the $q_j^\star$. Thus $e(x(H,p_i))$ and p_i either extend some q_j or extend a string in Tr incompatible with each of the $q_j^\star$.

So suppose the strategy $H^\star = \mathcal{C}[t,p,.]$ with $t = \emptyset$ or $t = \{Tr_{j_1,i_1},..,Tr_{j_k,i_k}\}$, $j_1 < \cdots < j_k$, enumerates a splitting $p_1,..,p_4$ into Tr_{j_k,i_k} (or $Tr_{-1,0}$) at step s. If $t = \emptyset$, then statement 2) of the lemma follows immediately for step s since we have already proved statement 1) for the induction step. So suppose $t \neq \emptyset$. Let $H_1,..,H_4$ be the $\mathcal{C}$ strategies which found the resultant splittings from which $p_1,..,p_4$ were taken. For each $1 \leq i \leq 4$ the set $T' = T(H_i) - t^\star(H_i)$ is the same. Consider each of the p_i in turn. We have by the induction hypothesis that the first time H_i was passed control the base string for this strategy extended a leaf in each of these trees. Since the strategies strictly above H_i (prior to step s) cannot enumerate strings into the trees in T' we have by Sublemma 6.1 that this is still the case at step s. Now if $k-1 \neq 0$, then $p_1,..,p_4$ (pairwise) extend incompatible strings that have been enumerated into $Tr_{j_{k-1},i_{k-1}}$ and thus by the induction hypothesis, for each tree $Tr \in t^\star(H^\star) \cup \{Tr_{-1,0}\}$, $p_1,..,p_4$ (pairwise) extend incompatible strings that have been enumerated into Tr. We are thus left to prove that $p_1,..,p_4$ extend leaves in these trees and that no axioms have been enumerated properly extending these strings.

Consider each of the p_i in turn and suppose that $Tr \in t^\star(H^\star) \cup \{Tr_{-1,0}\}$. Let S be as in the description of the $\mathcal{C}$ strategy $H^\star$ (at step s) and let H be that strategy which passes control to that strategy in S which is lowest on the tree of strategies. In forming p_i from $y(H,p_i)$ we first found the level, let us call it H_i, of $x(H,p_i)$ and $e(x(H,p_i))$. Again we have, by Sublemmas 6.3 and 6.4, that $Tr \in t^\star(H_i) \cup \{Tr_{-1,0}\}$. When H_i entered state 1 the four strings it was provided with, $q_1,..,q_4$, (pairwise) extended incompatible leaves of Tr, $q_1^\star,..,q_4^\star$ say. We then find that strategy lowest on the tree of strategies above that strategy which H_i passes control to at any stage s'' at which it puts $B_{s''}$ through $e(x(H,p_i))$ which has entered state 1. If there is no such strategy, then $p_i = e(x(H, p_i))$ and otherwise we choose p_i to be the axiom release string for that strategy. In either case by Sublemma 6.1, if s' is the greatest step $< s$ at which H_i was passed control, then (since it is clear that no strategy above H_i could violate this condition) at step s', p_i extended a leaf of Tr, and by Sublemma 6.2 it is clear that no axioms have been enumerated properly extending p_i at step s'.

Now let $s_1, .., s_l$ be all of those steps after s' as above and before s at which a strategy below H_i has entered state 1 for the first time and order them so that $s' < s_1 < ... < s_l < s$. Let $H^1, .., H^l$ be the corresponding strategies (so that H^1 is below H_i, H^2 is below H^1, and so on). Since $H^\star$ is passed control at step s, all of the H^j $(1 \leq j \leq l)$ are above H. Also $Tr \in t^\star(H^j) \cup \{Tr_{-1,0}\}$ for $1 \leq j \leq l$. We have by statement 3) of the induction hypothesis that when each of the H^j enters state 1 the strings that are delivered to it extend strings of Tr incompatible with $q_1^\star, .., q_4^\star$. Now we have by Sublemmas 6.1 and 6.2 that at steps s'' with $s' \leq s'' < s_1$ no strategy can violate the requirement $(\star)$ that p_i extends a leaf of Tr and no axioms have been enumerated properly extending p_i. Then by Sublemmas 6.1 and 6.2 again (since it is clear that no strategy above or below H^1 can violate requirement $(\star)$ at such steps), we have that at steps s'' with $s_1 \leq s'' < s_2$ no strategy could violate the requirement $(\star)$, and so on. We may conclude by an obvious inductive argument that, at step s, p_i extends a leaf in Tr and no axioms have been enumerated properly extending p_i. □

Lemma 6.2. *At any stage of the construction control is only passed to a finite number of strategies.*

Proof. At any stage s of the construction control can only be passed to a finite number of strategies before a strategy $\mathcal{C}[t, ..]$ with $t = \emptyset$ is passed control, whereupon stage s activity is terminated. Of course other strategies may also terminate stage s activity. □

Lemma 6.3. *Suppose that the strategy H is passed control at an infinite number of stages and that there is a stage at which H enters state* 1. *Let the strings in the environment of H be $p_1, .., p_4$. Then there exists $s \in \omega$, $s(H)$ say, and $1 \leq i \leq 4$ such that if H is passed control at stage $s' \geq s$, then it puts $B_{s'}$ through p_i. Thus $B(n) = \lim_{s\to\infty} B_s(n)$ is defined for all n and there exists an infinite number of strategies on the true path.*

Proof. The proof is by induction on the level of H on the tree of strategies. Let H be as in the statement of the lemma. Then we may take $s_1 > s(H')$ for all strategies H' strictly below H. It is helpful first to prove the following sublemma. □

Sublemma 6.5. *Assume the induction hypothesis of the lemma. Suppose that the $\mathcal{C}$ strategy H' is below (or equal to) the strategy H in the statement of the lemma. Then there exists s such that either 1) or 2) below applies:*

1) at every stage $s' \geq s$, H is marked by H' at stage s';

2) at every stage $s' \geq s$, H is not marked by H' at stage s'.

Proof. If the base string for H is not compatible with the axiom enumeration string of the strategy H', then the result follows immediately. So suppose that this is not the case. If $H' = H$ and is declared successful, then the result follows immediately. So suppose also that this is not the case. Suppose that H is a strategy $\Upsilon[.., r](m, n)$. Let s be the least stage such that the prefix of $W_{f(m),s}$ of length r is equal to the prefix of $W_{f(m)}$ of length r. If H does not enumerate any axioms at any stage $\geq s$, then it cannot be marked by H' at any stage $s' \geq s$. Now if at some stage an axiom that H has enumerated becomes hidden, then every axiom that it has enumerated and which is not expired becomes hidden and H then enumerates no more axioms. In this case it is clear that there cannot be an infinite number of stages at which H' marks H. So assume that no axiom that H enumerates ever becomes a hidden axiom and that H enumerates an axiom after stage s as above. Take s_2 large enough such that no strategy below (and including) H enters a new splitting state after stage s_2. Consider first the case that H' is strictly below H. Take $s' > s_1, s_2, s$ (with s_1 defined as above). If H' marks H at any stage $s'' > s'$, then it marks it at every stage $s''' \geq s''$. So consider the case $H = H'$. Then we may take s_3 large enough such that either $\forall s' \geq s_3$, H is marked by a strategy strictly below it at stage s', or $\forall s' \geq s_3$, H is not marked by a strategy strictly below it at stage s'. If the former case applies, then it is clear that the result follows. Otherwise take $s' > s_1, s_2, s_3, s$. If H marks H at any stage $s'' > s'$, then it marks it at every stage $s''' \geq s''$.

So suppose that the strategy H as in the statement of the lemma is of the form $\Upsilon[..,r](m,n)$. We may conclude by Sublemma 6.5, since if a strategy is marked at any stage, then it is marked by one of the finite number of strategies below it (not necessarily properly) on the tree of strategies, that for the strategy H there exists s such that either:

(1) $\forall(s' \geq s)$ if H is passed control at stage s', then it is marked at stage s', or
(2) $\forall(s' \geq s)$ if H is passed control at stage s', then it is not marked at stage s'.

If case (1) above applies, then at all stages $s' \geq s$ at which the strategy is passed control it will put $B_{s'}$ through the marked escape for this strategy. Suppose case (2) applies. If H is a $\mathcal{C}$ strategy and there is a stage at which it is declared successful, then the result follows immediately. So suppose otherwise. If there exists $s' \geq s$ such that whenever the strategy is passed control at a stage s'' after stage s' and is choosing which string to put $B_{s''}$ through it finds that the axiom enumeration string for this strategy is split, then it is clear that there exists s' such that either:

(1) at any stage $s'' \geq s'$ at which the strategy is passed control it puts $B_{s''}$ through the axiom enumeration string for this strategy, or
(2) at any stage $s'' \geq s'$ at which the strategy is passed control it puts $B_{s''}$ through the split escape for this strategy.

If there does not exist $s' \geq s$ such that whenever the strategy is passed control at a stage s'' after stage s' and is choosing which string to put $B_{s''}$ through it finds that the axiom enumeration string for this strategy is split, then, since there does exist s' such that for all $s'' \geq s'$ the prefix of $W_{f(m),s''}$ of length r is a prefix of $W_{f(m)}$, there exists s' such that $\forall(s'' \geq s')$ if H is passed control at stage s'', then it puts $B_{s''}$ through the axiom enumeration string for this strategy, unless $K(n) = 1$ and if s'' is the least stage such that $K_{s''}(n) = 1$, then H was passed control before stage s''. In the latter case there exists s' such that at any stage $s'' \geq s'$ at which H is passed control it puts $B_{s''}$ through the split escape for this strategy. □

Lemma 6.4. *For any $r, m, n \in \omega$, every maximal (r, m, n) set is finite.*

Proof. When a strategy enters state 1 the strategy that it passed control to while in state 0 will never be passed control again. It thus suffices to assume that we have an infinite 'chain' of strategies of the following form and to derive a contradiction.

So suppose that the infinite 'chain' of strategies, $H_0, H_1, H_2, ..$ satisfies the following criteria:

(1) $\forall i \in \omega$ there exists a stage s at which H_i passes control to H_{i+1};
(2) $\forall i \in \omega$ either a) H_i never enters state 1, or b) at any stage s at which H_i passes control to H_{i+1} it puts B_s through its split escape;
(3) H_0 is a strategy of the form $\Upsilon[.., r](m, n)$.

By Lemma 6.3 we may choose s large enough so that:

(1) if $K(n) = 1$, then $K_s(n) = 1$;
(2) $\forall(s' \geq s)$ the strategy H_0 is passed control at stage s';
(3) the prefix of $W_{f(m),s}$ of length r is a prefix of $W_{f(m)}$.

Let i' be the largest such that $H_{i'}$ has been passed control before stage s. Clearly there can be a finite number of the strategies H_i which are $\mathcal{C}$ strategies and which are declared successful. If it is the case that for an infinite number of the H_i there is a stage s' at which they are passed control and put $B_{s'}$ through the axiom enumeration string for this strategy, then the result follows, since then for some H_i with $i > i'$ which is never declared successful there must exist a stage $s' > s$ such that H_i is passed control at stage s' and puts $B_{s'}$ through its axiom enumeration string. Then at any stage $s'' \geq s'$ at which this H_i is passed control it either puts $B_{s''}$ through the marked escape or the axiom enumeration string for this strategy. So suppose otherwise. Choose $i^\star > i'$ such that for no strategy H_i with $i > i^\star$ does there exist a stage s' at which this strategy is passed control and puts $B_{s'}$ through its axiom enumeration string or a stage s' at which this strategy is declared successful.

Let $\mathcal{C}_1, .., \mathcal{C}_l$ be those $\mathcal{C}$ strategies strictly below H_0 on the tree of strategies which put $B_{s'}$ through their axiom enumeration strings at any stage s' at which H_0 is passed control. At any stage $s' > s$ at which H_0 is passed control consider the number of strings q which are labeled splitting strings at stage s' by one of the strategies $\mathcal{C}_j$, $1 \leq j \leq l$, and such that there exists some H_i with $i > i^\star$, H_i is a strategy with base string p (say), $p \subset q$ and q is compatible with the axiom enumeration string for H_i. Each such string is labeled a splitting string at stage s' by either the splitting search procedure for some $\mathcal{C}_j$ or the blame procedure for some $\mathcal{C}_j$ $(1 \leq j \leq l)$. But if four or more of these strings are labeled splitting strings at stage s' by the splitting search procedure for $\mathcal{C}_j$, then any four of these strings, $p_1, .., p_4$ say, constitute a splitting of the sort that this splitting search procedure is looking for and will have been outputted as a splitting by this splitting search procedure at stage s' unless (possibly) either:

a) at least one, but not all, of the p_i extends the axiom enumeration string of a strategy H, which has enumerated an axiom which is not hidden or expired;

b) there is some splitting $q_1, .., q_4$ which has already been outputted by this splitting search procedure since $\mathcal{C}_j$ entered its present splitting state such that each of the p_i extends a different $q_{i'}$.

If a) applies, then H must be a strategy H_i for some $i > i^\star$. But H_i has not enumerated an axiom which is not hidden or expired. Whether b) applies or not we can conclude that there is a splitting which has been outputted by the splitting search procedure for $\mathcal{C}_j$ which is such that the only axioms which have been enumerated on the strings in this splitting and which are not expired or hidden axioms are axioms enumerated by strategies below H_0. Since $\mathcal{C}_j$ has not been declared successful the blame procedure for this strategy must have marked at stage s' one of those strategies below H_0 which puts $B_{s''}$ through its axiom enumeration string at any stage s'' at which H_0 is passed control. But then H_0 could not be passed control at stage s'.

So suppose that four or more of these strings have been labeled splitting strings at stage $s' > s$ by the blame procedure for $\mathcal{C}_j$. Then choose any four of these strings, $p_1, .., p_4$ say. Suppose that $\mathcal{C}_j$ is a strategy with base string p', is searching for $\Psi_{j'}$ splittings and $t^\star(\mathcal{C}_j) = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$. Now there exists a string, q'_i say, which has been enumerated into Tr_{j_k,i_k} (or $Tr_{-1,0}$ if $t^\star(\mathcal{C}_j) = \emptyset$) before stage s' and such that when $\mathcal{C}_j$ was passed control at stage s', $e(q'_i)$ extended p_i. Let q'_i be the shortest such string for each $1 \leq i \leq 4$. Define $q_i = e(q'_i)$ as it was defined when $\mathcal{C}_j$ was passed control at stage s'. For each $1 \leq i \leq 4$ it is the case that p_i has been labeled a splitting string at every stage s'' at which $\mathcal{C}_j$ has been passed control and has put $B_{s''}$ through its axiom enumeration string, since it was last outputted as part of a splitting. If p_i is compatible with the axiom enumeration string for $H_{i'}$ for $i' > i^\star$, then no axioms have been enumerated properly extending the axiom enumeration string for $H_{i'}$. Thus each q_i satisfies the property that there have been no axioms enumerated of the form $\Gamma^{\sigma,p''}_{i'',u}(n') = c$ for $(i'', n') \in z(j')$, $c \in \{0,1\}$ and finite binary strings σ and p'' such that $p' \subset p'' \subseteq q_i$ and which are not hidden or expired. We conclude that $q_1, .., q_4$ would be outputted as a splitting by the splitting search procedure for $\mathcal{C}_j$ at stage s' unless (possibly) either a) or b) apply as above. But then we reach the required contradiction in precisely the same way as before.

Any chain of strategies, such that for each strategy in the chain there is a stage at which it passes control to the next and such that no strategy in the chain enters state 1, must be finite (there can be at most one $\mathcal{D}$ strategy in such a chain and when a strategy $\mathcal{C}[t, ..](., .)$ in state 0 passes control to a strategy $\mathcal{C}[t', ..](., .)$ we have that $t' \subset t$). Clearly if the chain we have been considering is to be infinite, then there must be an infinite number of i such that H_i enters state 1 at some stage. If, for $i > i^\star$ such that H_i enters state 1, neither a) nor b) below apply, then if H_i is a strategy with base string p, then for all but a finite number of stages s', H_i is passed control at stage s' and finds that there is some string q such that $p \subset q$, q is compatible with the axiom enumeration string for this strategy and has been labeled a splitting string by one of the strategies $\mathcal{C}_j$. We have shown that at any stage $s' > s$ there must exist less than $8l$ strings q which are labeled splitting strings at stage s' by one of the strategies $\mathcal{C}_j$, $1 \leq j \leq l$, and such that there exists some H_i with $i > i^\star$, H_i is a strategy with base string p (say), $p \subset q$ and q is compatible with the axiom enumeration string for H_i. Since the axiom enumeration strings for the strategies H_i are (pairwise) incompatible we may conclude that there exists $i > i^\star$ such that H_i satisfies either a) or b) below.

a) H_i is a $\mathcal{D}$ strategy and there is a stage s' at which it is passed control and puts $B_{s'}$ through its axiom enumeration string.

b) H_i is a $\mathcal{C}$ strategy and there is a (first) stage s' at which it is passed control in state 1 and finds while performing step 5) of the instructions for this strategy that its axiom enumeration string is not split at stage s' and that the strategy is not marked at stage s'. (Note that no strategy performs the splitting search procedure until after a stage s' at which it is passed control and puts $B_{s'}$ through its axiom enumeration string.) $\square$

This gives us the required contradiction.

Lemma 6.5. *Given any $\mathcal{C}$ strategy H and any $i \geq 0$, if there exists an infinite number of j such that there exists s such that i is blamed for j by the blame procedure for H at stage s, then W_i is computable.*

Proof. This now follows entirely from the proof given in the introduction. We need only make more explicit how the strings σ_n should be defined. Let $j_1, j_2, j_3, ..$ be as defined as in the introduction (just before we supposed that there was no upper bound on the lengths of the strings σ_n). For each $n \geq 1$ consider those axioms in $Ax(H, j_n, d)$, of the form $\Gamma^{\sigma,p'}_{i,u}(n') = c$ say, which do not become hidden and for which σ is shortest (for any $u \in U_i$, $n' \geq 0$, $c \in \{0,1\}$ and finite binary strings σ and p'). Define $\sigma_n = \sigma$. Strictly speaking we assumed in Sublemma 6.5 that the strategy H was passed control at an infinite number of stages, but now that we have proved there exists a true path for the contruction the same proof works for the general case. $\square$

Lemma 6.6. *Suppose that a 'chain' of $\mathcal{C}$ strategies $H_0, H_1, H_2, ..$ is such that for each $i \geq 0$, if there is a strategy H_{i+1} in the chain, then there exists a stage s at which H_i passes control to H_{i+1}. Then there are a finite number of strategies in the chain.*

Proof. Suppose that such a chain of strategies is infinite. Then there exists $i^\star$ such that there does not exist a strategy H which is either in this chain or below H_0 and a tree $Tr_{j,i} \in T(H)$ such that $j + i > i^\star$. No strategy of any form $\Upsilon[..](m,n)$ such that $f(m) = i^\star$ can be marked by any strategy H which is either in this chain or below H_0. Suppose that there exists an infinite number of i such that H_i passes control to H_{i+1} at stages s such that H_i is marked at stage s. Then there exists i' such that $H_{i'}$ is a strategy of some form $\Upsilon[.., r](m,n)$ such that $f(m) = i^\star$. But then all of the strategies $H_{i'}, H_{i'+1}, H_{i'+2}, ..$ are in the same maximal (r,m,n) set and any such set is finite.

Thus there exists a finite number of i such that H_i passes control to H_{i+1} at stages s such that H_i is marked at stage s. Let i'' be the greatest number such that $H_{i''}$ is such a strategy. Then for some (r,m,n), $H_{i''+1}, H_{i''+2}, ..$ etc. are all in the same maximal (r,m,n) set. But any such set is finite. $\square$

Lemma 6.7. *Suppose that a strategy H with base string p is declared to be in state 1 at stage s and that four strings, $p_1, .., p_4$ are delivered to it and enumerated into $Tr_{j,i}$. At this point all axioms that have been enumerated of any form $\Gamma^{\sigma,p'}_{i',u}(n) = c$ such that $p \subset p' \subseteq p_{i''}$ for some $1 \leq i'' \leq 4$ and which are not already hidden or expired axioms, we label hidden axioms. For all such axioms we have $i' \geq j + i$.*

Proof. If $Tr_{j,i}$ is the tree $Tr_{-1,0}$, then the result follows immediately by Lemma 6.1 (in this case there is in fact no need to label any axioms as hidden). Otherwise consider each of the $p_{i''}$ in turn. If $H' = \mathcal{C}[., p'', .]$ is the strategy which found the resultant splitting from which $p_{i''}$ was taken, then it is clear that all axioms that have been enumerated of any form $\Gamma^{\sigma,p'}_{i',u}(n) = c$ such that $p \subset p' \subseteq p''$ were either hidden or expired before stage s. All axioms that have been enumerated of the form $\Gamma^{\sigma,p'}_{i',u}(n) = c$ for $p'' \subset p' \subseteq y(H, p_{i''})$ and $i' < j + i$ were hidden or were expired before any axioms are labeled hidden at stage s (this can easily be seen by a simple induction). At all stages s' since $y(H, p_{i''})$ was last outputted as part of a splitting by the splitting search procedure for H', at which H' has been passed control and has put $B_{s'}$ through its axiom enumeration string, $y(H, p_{i''})$ has been labeled a splitting string. Thus all axioms that have been enumerated of the form $\Gamma^{\sigma,p'}_{i',u}(n) = c$ for $p'' \subset p' \subseteq e(x(H, p_{i''}))$ and $i' < j + i$ were hidden or were expired before any axioms are labeled hidden at stage s (by Sublemma 6.4, before any activity is carried out at stage s, $x(H, p_{i''})$ and therefore $y(H, p_{i''})$ properly extend the base string of that strategy which is the level of $x(H, p_{i''})$). If $e(x(H, p_{i''})) = p_{i''}$, then the result follows. Otherwise let H' be the level of $x(H, p_{i''})$ (before $p_{i''}$ is enumerated into $Tr_{j,i}$). Let H'' be that strategy which H' passes control to at any stage s'' at which it puts $B_{s''}$ through $e(x(H, p_{i''}))$. We find that strategy H''' lowest on the tree of strategies above and including H'' which has entered state 1 by stage s and choose $p_{i''}$ to be the axiom release for this strategy. If s'' is the stage at which this strategy entered state 1, then it is clear that after stage s'' activity was over any axioms that we had enumerated of any form $\Gamma^{\sigma,p'}_{i',u}(n) = c$ for $e(x(H, p_{i''})) \subset p' \subseteq p_{i''}$ and $i' < j + i$ were hidden or were expired. Since then no axioms of this form have been enumerated. □

Lemma 6.8. *For any $i \in \omega$ and $u \in U_i$ the axioms that are enumerated for $\Gamma_{i,u}$ are consistent; i.e., if we enumerate the two axioms $\Gamma^{\sigma,p}_{i,u}(n) = c$ and $\Gamma^{\sigma',p'}_{i,u}(n') = c'$, then if $n = n'$ and $c \neq c'$, either σ and σ' are incompatible or p and p' are incompatible.*

Proof. Suppose that a strategy H enumerates an axiom $\Gamma^{\sigma,p}_{i,u}(n) = c$ at stage s. If s' is the first stage at which H was passed control in state 1 we have, by 2) of Lemma 6.1, that at stage s' no axioms had been enumerated properly extending the axiom enumeration string for H. Since stage s' no strategy below H can have entered state 1 for the first time; otherwise, H would not be passed control at stage s. It is clear that no strategy strictly above or strictly below H on the tree of strategies could have enumerated an axiom of the form $\Gamma^{\sigma',p'}_{i,u}(n) = c'$ for p' compatible with p (and for some $c' \in \{0, 1\}$ and finite binary string σ'), since H entered state 1. Thus by Sublemma 6.2 when H is passed control at stage s no axioms have been enumerated of the form $\Gamma^{\sigma',p'}_{i,u}(n) = c'$ for p' properly extending p. Since H enumerates the given axiom, we have that prior to the enumeration of this axiom, $W_{i,s} \notin \gamma(p, i, u, n)$ so that no axioms have been enumerated of the form $\Gamma^{\sigma',p'}_{i,u}(n) = c'$ with σ' compatible with σ and p' compatible with p. □

Lemma 6.9. *Suppose that a strategy $H = \Upsilon[t, p, r](m, n)$ is passed control at an infinite number of stages and that at an infinite number of stages s this strategy puts B_s through its axiom enumeration string. Suppose furthermore that, for this n and for some $u \in U_{f(m)}$ which is appropriate for axiom enumeration at H, at no stage in the construction is an axiom of any form $\Gamma^{\sigma,p'}_{f(m),u}(n) = c$ (i.e., for any $c \in \{0, 1\}$ and any finite binary strings σ and p') labeled a hidden axiom. Then $\Gamma^{W_{f(m)},B}_{i,u}(n) = K(n)$.*

Proof. Let s be the first stage at which the prefix of $W_{f(m),s}$ of length r is a prefix of $W_{f(m)}$. Let s' be the least stage $\geq s$ such that H is passed control at stage s' and puts $B_{s'}$ through its axiom enumeration string, p' say. At this stage H will enumerate the axiom $\Gamma^{\sigma,p'}_{f(m),u}(n) = K_{s'}(n)$, where σ is the prefix of $W_{f(m)}$ of length r. Now $p' \subset B$ and $K_{s'}(n) = K(n)$ since there are an infinite number of stages s'' at which H is passed control and puts $B_{s''}$ through its axiom enumeration string. □

Lemma 6.10. *For any $j \in \omega$ either there exists $i \in \omega$ such that $Tr_{j,i}$ is infinite or there is a highest strategy H on the true path of the construction such that there exists i with $Tr_{j,i} \in t(H)$. In the latter case this strategy H is a $\mathcal{C}$ strategy which puts B_s through its axiom enumeration string at an infinite number of stages s.*

Proof. Let $\mathcal{D}_1, \mathcal{D}_2, ..$ be those $\mathcal{D}$ strategies on the true path of the construction (those $\mathcal{D}$ strategies passed control at an infinite number of stages) ordered so that for $m < n$, $\mathcal{D}_m$ is below $\mathcal{D}_n$ on the tree of strategies; by Lemma 6.6 there are an infinite number of strategies in this sequence. The proof is by induction on j. So suppose that for all $j' < j$ there exists $h(j')$ such that either:

(1) there exists an i such that for all $n \geq h(j')$ we have $Tr_{j',i} \in t(\mathcal{D}_n)$, or
(2) for all $n \geq h(j')$, there does not exist an i with $Tr_{j',i} \in t(\mathcal{D}_n)$.

Then take $k > h(j')$ for all $j' < j$ and large enough so that there is no strategy on the true path of the construction above or including $\mathcal{D}_k$ of any form $\Upsilon[..](m,n)$ such that $(f(m), n) \in z(j)$ and which puts B_s through its axiom enumeration string at an infinite number of stages s. If there does not exist an $i \in \omega$ such that $Tr_{j,i} \in t(\mathcal{D}_k)$, then this is also the case for all $k' > k$ and the result follows. Suppose that there does exist an $i \in \omega$ such that $Tr_{j,i} \in t(\mathcal{D}_k)$. If $Tr_{j,i} \in t(\mathcal{D}_{k'})$ for all $k' \geq k$, then the result follows. So suppose that there exists a least number $k' > k$ such that $Tr_{j,i} \notin t(\mathcal{D}_{k'})$. Then for all $k'' > k'$ there does not exist an i' such that $Tr_{j,i'} \in t(\mathcal{D}_{k''})$ and there is a $\mathcal{C}$ strategy searching for a Ψ_j splitting in the '$\mathcal{D}_{k'-1}$ cluster' (terminology used in the introduction) which puts B_s through its axiom enumeration string at an infinite number of stages s. □

Lemma 6.11. *Suppose that W_i is non-computable. Then there exists $u \in U_i$ such that for all but finitely many n, $\Gamma^{W_i,B}_{i,u}(n) = K(n)$.*

Proof. Let u be the set of all those pairs (y, z) such that $y \leq i$ and $Tr_{y,z}$ is infinite. Then $u \in U_i$ and u is appropriate for axiom enumeration at every strategy on the true path. By Lemma 6.7, axioms for $\Gamma_{i,u}$ can only be labeled hidden axioms when strings are enumerated into trees $Tr_{y,z}$ with $y + z \leq i$ and such that $(y, z) \notin u$. There are a finite number of such trees and each such tree is finite. Thus there are only a finite number of axioms that are enumerated for $\Gamma_{i,u}$ and which are later labeled hidden axioms. By Lemma 6.9, it suffices to show that for each $n \in \omega$ there exists a strategy on the true path, of some form $\Upsilon[..](m,n)$ say, such that $f(m) = i$ and which puts B_s through its axiom enumeration string at an infinite number of stages s. Suppose that the result holds for all $n' < n$. Since, for any $r', m', n' \in \omega$, every maximal (r', m', n') set is finite, there are an infinite number of strategies on the true path, of the form $\Upsilon[..](m', n')$ say, such that $f(m') = i$. Suppose that none of these strategies is of the kind that we are looking for. Then there are an infinite number of strategies on the true path of the form $\Upsilon[..](m', n)$ such that $f(m') = i$. Such strategies can only be marked by strategies $\mathcal{C}[t..]$ on the true path with $t = \{Tr_{j_1,i_1}, .., Tr_{j_k,i_k}\}$ say, $j_1 < \cdots < j_k$, and $(i, n) \notin z(j_k)$. But by Lemma 6.10 there are only a finite number of such strategies and, since W_i is non-computable each such can only mark a finite number of strategies of the form $\Upsilon[..](m', n)$ such that $f(m') = i$. So let H be a strategy on the true path of the form $\Upsilon[..](m',n)$ with $f(m')=i$ which is such that no strategy in the maximal (r',m',n) set to which H belongs is ever marked. The maximal (r',m',n) set to which H belongs is finite, so that there is some strategy in this set which is on the true path and which puts B_s through its axiom enumeration string at an infinite number of stages s, which gives us the required contradiction. $\square$

Lemma 6.12. *B is a set of minimal degree.*

Proof. Suppose we are given $j \in \omega$. If there exists an i such that $Tr_{j,i}$ is infinite, then there exists an infinite number of strategies $\mathcal{D}[t,p,.]$ on the true path with $Tr_{j,i} \in t$. For each such strategy p is an initial segment of B and the strategies in this '$\mathcal{D}[t,p,.]$ cluster' (terminology used in the introduction) must enumerate strings into $Tr_{j,i}$ extending p. Thus B lies on $Tr_{j,i}$, Ψ_j^B is total and $B \leq_T \Psi_j^B$.

So suppose that there does not exist an i such that $Tr_{j,i}$ is infinite. By Lemma 6.10 we may let $H=\mathcal{C}[t,p,.](.)$ be that strategy on the true path highest on the tree of strategies such that there exists an i with $Tr_{j,i} \in t$. Suppose $t = \{Tr_{j_1,i_1},..,Tr_{j_k,i_k}\}$ with $j_1 < \cdots < j_k$. Then $j_k = j,\ i_k = i$ and:

(1) If $k \neq 1$, then $Tr_{j_{k-1},i_{k-1}}$ is infinite and B lies on this tree. It is clear that $Tr_{-1,0}$ is infinite and that B lies on this tree.
(2) There do not exist any strategies on the true path above or including H of the form $\Upsilon[..](m,n)$ say, which put B_s through their axiom enumeration string at an infinite number of stages s and such that $(f(m),n) \in z(j)$.

Clearly H is never declared successful and there are an infinite number of stages s at which this strategy is passed control and puts B_s through its axiom enumeration string. Let d be the lowest splitting state that H is ever in. Let s_1 be the first stage at which H is passed control in splitting state d. Define $\Lambda = \bigcup_{s \geq s_1} \Pi(H,s)$ (where $\Pi(H,s)$ is defined as in the description of the splitting search procedure). For any $l \geq 0$ there exists $q \in \Lambda$ of length greater than l which is an initial segment of B. Define $\Lambda^\star = \{\Psi_j^q : q \in \Lambda\}$. If there exists some k' such that there are not at least k' incompatible strings in $\Lambda^\star$, then if Ψ_j^B is total it is computable. Otherwise, if there does not exist such a k', we wish to show that the splitting search procedure for H outputs an infinite number of splittings. So suppose that it outputs a finite number of splittings. Then let $r \in \omega$ be such that for every strategy H' above H and every $Y \subseteq \{0,..,j_k+i_k\}$, it is always the case that $\beta(H,H',Y) < r$. We may choose $s_2 > s_1$ such that after stage s_2 the splitting search procedure outputs no more splittings. Let those strategies H' above H such that $\upsilon(H,H') < r$, and such that if q is the axiom enumeration string for H', then $q \subset B$, be called $H_1,..,H_l$. Let those strategies H' above H such that $\upsilon(H,H') < r$, and such that if q is the axiom enumeration string for H', then $q \not\subset B$, be called $H_1',..,H_{l'}'$. Then only a finite number of strings will ever be enumerated into $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$) extending the axiom enumeration strings of the strategies $H_1',..,H_{l'}'$ and it is clear that for any string p' in $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$), $e(p')$ will only be redefined a finite number of times. Now the splitting search procedure for H labels splitting strings in such a way that if at some stage s at which it is run there exists $p' \in \Pi(H,s)$ with $\Psi_j^{p'} = q$, say, and $|q| \geq l_H$, then if s' is the next stage at which it is run, at stage s' there exists $p'' \in \Pi(H,s')$ with $q \subseteq \Psi_j^{p''}$. Since for any k' there exist k'

incompatible strings in $\Lambda^\star$ there exists a stage $s_3 > s_2$ and four incompatible strings in $\Pi(H, s_3)$, $q_1, .., q_4$ say, such that:

(1) $l_H \leq \left| \Psi_j^{q_{i'}} \right|$, for $1 \leq i' \leq 4$.
(2) $\Psi_j^{q_{i'}}$ is incompatible with $\Psi_j^{q_{i''}}$ for $1 \leq i' < i'' \leq 4$.
(3) There is no splitting which has been outputted by the splitting search procedure for H since this strategy entered splitting state d which is such that each of $q_1, .., q_4$ extends a different string of this splitting.
(4) For $1 \leq i' \leq l$, all of the $q_{i''}$ extend the axiom enumeration string of the strategy $H_{i'}$. For $1 \leq i' \leq l'$, none of the $q_{i''}$ extends the axiom enumeration string of the strategy $H_{i'}$. None of the $q_{i'}$ are outputted as part of a splitting by the splitting search procedure for H.

This gives us the required contradiction. □

So suppose that the splitting search procedure for H outputs an infinite number of splittings. Let Y be the set of all those i' such that there exist an infinite number of j' such that there exists s, i' is blamed for j' at stage s by the blame procedure for H. Define $Y' = \{i' : i' < i + j,\ i' \notin Y\}$. Now if for some string $p' \supset p$ (recall that H has base string p) it is the case that there is an axiom enumerated of some form $\Gamma_{i',u}^{\sigma,p''}(n) = c$ such that $i' \in Y$ and $p \subset p'' \subseteq p'$ which never becomes hidden or expired, then p' is not an initial segment of B because otherwise there would be some strategy H' above H on the true path, of the form $\Upsilon[..](m, n)$ say, such that $f(m) = i'$ which puts B_s through its axiom enumeration string at an infinite number of stages and which enumerates an axiom which does not become hidden or expired. But then there exists a j' such that $\forall(j'' \geq j')$, there exists a string in the j''^{th} splitting outputted by the splitting search procedure for H since this strategy entered splitting state d which extends the axiom enumeration string for H' and, since $i' \in Y$, H' would be marked by H at an infinite number of stages which gives us a contradiction.

Choose j_1 large enough such that for no $j' \geq j_1$ does there exist an s such that i' is blamed for j' at stage s for any $i' \in Y'$. Now consider the set $\Lambda^\star$ of all those strings $\Psi_j^{p'}$ such that $(\star, p', j')$ holds for some j': p' was outputted by the splitting search procedure for the strategy H as part of the j'^{th} splitting since this strategy entered splitting state d with $j' \geq j_1$ and for all $i' \in Y$ it is the case that all axioms in $Ax_{i'}^\star(H, j', p', d)$ become hidden or expired. Define Λ to be the set of all those strings p' such that there exists j' such that $(\star, p', j')$ holds.

If there are only a finite number of strings in Λ, then if Ψ_j^B is total it is computable. For suppose otherwise. We shall produce another string in Λ, an initial segment of B. Then there exists $r \in \omega$ such that for all strategies H' above and including H, $\beta(H, H', Y)$ is always less than r (note that κ is vital in ensuring that this is the case). Once again let those strategies H' above H such that $\upsilon(H, H') < r$, and such that if q is the axiom enumeration string for H' then $q \subset B$, be called $H_1, .., H_l$. Let those strategies H' above H such that $\upsilon(H, H') < r$, which enumerate an axiom which does not become hidden or expired and such that if q is the axiom enumeration string for H' then $q \not\subset B$, be called $H_1', .., H_{l'}'$. We can choose $s_2 > s_1$ (with s_1 as defined previously) large enough such that:

(1) At all stages $s \geq s_2$, H is passed control and puts B_s through its axiom enumeration string.

(2) At no stage $s \geq s_2$ is there is splitting outputted by the splitting search procedure such that some string in the splitting extends the axiom enumeration string of one of the strategies $H'_{i'}$, $1 \leq i' \leq l'$.
(3) If H' is a strategy such that $\upsilon(H, H') < r$ with axiom enumeration string $q \not\subset B$ which does not enumerate an axiom which does not become hidden or expired, then by stage s_2 every axiom that H' has enumerated is either hidden or expired and at no stage $s > s_2$ does H' enumerate any axioms.

Choose $l^\star$ to be larger than any number yet mentioned in the course of the construction at the end of stage s_2. Now if any string q which has been outputted as part of a splitting is such that $|\Psi_j^q| > l^\star$, then we can be sure that q, and in fact $r(H, q, s)$ at any subsequent stage s, was only outputted as part of a splitting after stage s_2. Choose $p^\star \subset B$ long enough such that it properly extends every string in Λ which is a prefix of B, and long enough such that it properly extends the axiom enumeration strings of the strategies $H_1, .., H_l$. Let Z be the (finite) set of all those strings $\Psi_j^{p'}$ such that p' is a string which is outputted as part of a splitting by the splitting search procedure for H since this strategy entered splitting state d and p' is incompatible with $p^\star$. Since the splitting search procedure for H outputs an infinite number of splittings and Ψ_j^B is total and non-computable, there exist four strings $q_1, .., q_4$ such that each is outputted as part of a splitting by the splitting search procedure since H entered splitting state d, and which satisfy:

(1) For each $1 \leq i' \leq 4$, $\Psi_j^{q_{i'}}$ is of length greater than $l^\star$.
(2) For each $1 \leq i' \leq 4$, for each string q in Z it is either the case that $\Psi_j^{q_{i'}}$ properly extends q or that it is incompatible with q.
(3) For $1 \leq i' < i'' \leq 4$, $\Psi_j^{q_{i'}}$ is incompatible with $\Psi_j^{q_{i''}}$.

Since Ψ_j^B is total there exists an initial segment of B, p' say, and a stage $s_3 > s_2$ such that before stage s_3 each of the strings $q_1, .., q_4$ has been outputted as part of a splitting by the splitting search procedure for H and:

(1) $\Psi_j^{p'}$ is of length greater than l_H.
(2) $\Psi_j^{p'}$ is incompatible with (at least) three of the $\Psi_j^{q_{i'}}$.
(3) By stage s_3 all axioms that have been previously enumerated of the form $\Gamma_{i',u}^{\sigma,p''}(n) = c$ for $i' \in Y,\ u \in U_{i'},\ \ c \in \{0,1\},\ n \in \omega$ (or such that $(i', n) \in z(j)$) and finite binary strings σ and p'' such that $p \subset p'' \subseteq p'$ are hidden or are expired (recall that H has base string p).
(4) When H is passed control at stage s_3, $p' = e(q)$ for some string q in $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$).

Reorder the $q_{i'}$ so that for $1 \leq i' \leq 3$, $\Psi_j^{p'}$ is incompatible with $\Psi_j^{q_{i'}}$. Then for each $1 \leq i' \leq 3$ define $q'_{i'} = r(H, q_{i'}, s_3)$. For each $1 \leq i' \leq 3$ there exists a shortest string, $x_{i'}$ in $Tr_{j_{k-1},i_{k-1}}$ (or $Tr_{-1,0}$ as appropriate) such that when H is passed control at stage s_3, $e(x_{i'})$ extends $q'_{i'}$. For each i', $1 \leq i' \leq 3$ define $q^\star_{i'} = e(x_{i'})$ as it was defined when H was passed control at stage s_3. Since each $q'_{i'}$ was outputted as part of a splitting by the splitting search procedure for H after stage s_2, none of the strings $p', q^\star_1, q^\star_2, q^\star_3$ extends the axiom enumeration string of a strategy $H'_{i'}$ $1 \leq i' \leq l'$. All axioms that have been enumerated of the form $\Gamma_{i'',u}^{\sigma,p''}(n) = c$ for $i'' \in \omega,\ u \in U_{i''},\ c \in \{0,1\},\ n \geq 0$, such that $(i'', n) \in z(j)$ and for finite binary strings σ and p'' such that $p \subset p'' \subseteq q^\star_{i'}$ are hidden or expired. All four of these strings properly extend $p^\star$ and thus properly extend the axiom enumeration strings of each of the strategies $H_1, .., H_l$. Thus these four strings would be outputted as a splitting by the splitting search procedure at stage s_3, unless there is already some splitting which has been outputted at this stage such that each of the $q^\star_{i'}$ extends a different string of the splitting. In either case we have shown that there exists another string in Λ, which gives us the required contradiction.

So suppose that there is an infinite number of strings in Λ. We wish to show that if there exists a k' such that there are not k' incompatible strings in $\Lambda^\star$, then if Ψ^B_j is total it is computable. But this follows easily since for every initial segment of B, p' say, there exists a string in Λ which extends p'.

It thus suffices to show that there exists k' such that there are not k' incompatible strings in $\Lambda^\star$. So suppose otherwise. Then let r be such that for all strategies H' above H, it is always the case that $\beta'(H, H', Y) < r$. Let s_2 and $H_1, .., H_l$, $H'_1, .., H'_{l'}$ be as we defined them in the proof that there are an infinite number of strings in Λ, but in terms of the presently defined value of r.

Let j_1 be as defined previously, and choose $p^\star \subset B$ longer than any of the strings in those splittings $1 \leq j' \leq j_1$ and longer than the axiom enumeration strings of any of the strategies $H_1, .., H_l$. Let Z be as defined previously, in terms of the present definition of $p^\star$.

Choose $l^\star$ to be larger than any number mentioned in the course of the construction by the end of stage s_2. Since for any k' there exist k' incompatible strings in $\Lambda^\star$, we can choose four strings $p_1, .., p_4$ from Λ such that:

1. for each $1 \leq i' \leq 4$, $\left|\Psi^{p_{i'}}_j\right| > l^\star$;
2. for $1 \leq i' < i'' \leq 4$, $\Psi^{p_{i'}}_j$ is incompatible with $\Psi^{p_{i''}}_j$;
3. for each $1 \leq i' \leq 4$, for each string q in Z it is either the case that $\Psi^{p_{i'}}_j$ properly extends q or that it is incompatible with q.

For each $1 \leq i' \leq 4$ let $j'_{i'}$ be such that $(\star, p_{i'}, j'_{i'})$ holds. Choose $s_3 > s_2$ large enough such that before stage s_3, for each $1 \leq i' \leq 4$ and for all $i'' \in Y$, all axioms in $Ax^\star_{i''}(H, j'_{i'}, p_{i'}, d)$ were hidden or were expired and such that the $j'^{th}_{i'}$ splitting outputted by the splitting search procedure for H since it entered splitting state d has been outputted before stage s_3. For each $1 \leq i' \leq 4$ define $q_{i'} = r(H, p_{i'}, s_3)$. For each i' there exists a shortest string $x_{i'}$ in $Tr_{j_{k-1}, i_{k-1}}$ (or $Tr_{-1,0}$ as appropriate) such that at stage s_3, $e(x_{i'})$ extends (not necessarily properly) $q_{i'}$. For each i' define $q^\star_{i'} = e(x_{i'})$ (as it was defined when H was passed control at stage s_3). If s is the last stage at which $q_{i'}$ was outputted as part of a splitting by the splitting search procedure for H, then at all subsequent stages s' at which H has been passed control and has put $B_{s'}$ through its axiom enumeration string, $q_{i'}$ has been labeled a splitting string. All axioms that have been enumerated of the form $\Gamma^{\sigma, p'}_{i'', u}(n) = c$ for $i'' \in Y,\ u \in U_{i''},\ c \in \{0, 1\},\ n \geq 0$ (or such that $(i'', n) \in z(j)$) and finite binary strings σ and p' such that $p \subset p' \subseteq q^\star_{i'}$ are hidden or expired. Now $q^\star_1, .., q^\star_4$ do not extend the axiom enumeration strings of any of the strategies $H'_1, .., H'_{l'}$. Each of these strings extends $p^\star$, so it cannot be the case that each string extends a different string of the j'^{th} splitting outputted by the splitting search procedure for H since it entered splitting state d for some $j' < j_1$. For the same reason, all four of $q^\star_1, .., q^\star_4$ extend the axiom enumeration strings of the strategies $H_1, .., H_l$.

Thus $q^\star_1, .., q^\star_4$ will be outputted as a splitting by the splitting search procedure for H at stage s_3 unless there is already a splitting that it has outputted at this stage such that each of the $q^\star_{i'}$ extends a different string of this splitting. In either case there exists $j' \geq j_1$ such that there exists $i' \in Y'$ and a stage s such that i' is blamed for j' at stage s. This gives us the required contradiction.

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