TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 1, January 2008, Pages 347–367
S 0002-9947(07)04269-9
Article electronically published on August 14, 2007

# THE $(\mathbf{A_2},\mathbf{G_2})$ DUALITY IN $\mathbf{E_6}$, OCTONIONS AND THE TRIALITY PRINCIPLE

HUBERT RUBENTHALER

*A la mémoire de Maurice Drexler*

ABSTRACT. We show that the existence of a dual pair of type $(A_2, G_2)$ in $E_6$ leads to a definition of the product of octonions on a specific 8-dimensional subspace of $E_6$. This product is expressed only in terms of the Lie bracket of $E_6$. The well known triality principle becomes an easy consequence of this definition, and $G_2$ acting by the adjoint action is shown to be the algebra of derivations of the octonions. The real octonions are obtained from two specific real forms of $E_6$.

## 1. INTRODUCTION

The octonion algebra $\mathbb{O}$ over $\mathbb{C}$ is the composition algebra with (maximal) dimension 8. Since their discovery in the 19th century a large number of mathematical papers have been written on the octonions. For a survey, and also for an extensive bibliography we refer to the paper by J. Baez [Ba].

In particular, the connection between the octonions, non-associative structures and exceptional Lie algebras was very intensively studied these past decades. Many of these connections can be found in Jacobson's book on exceptional Lie algebras ([Jac]).

One of the most important results in this area is the so-called Koecher-Tits construction ([Ko], [Ti-1]), which is a procedure to construct simple Lie algebras from Jordan algebras.

Roughly speaking the Koecher-Tits construction goes as follows. If $V$ is a simple Jordan algebra over $\mathbb{C}$, then one can define a Lie algebra structure on $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = \mathfrak{g}_1 = V$ and where $\mathfrak{g}_0 = Str(V)$ is the structure algebra of $V$. For example this construction provides a model of $E_7$ if $V$ is the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$.

In order to obtain models for more simple Lie algebras, the Koecher-Tits construction was extended by several authors ([All-Fer-1], [Al-1], [Fau], [Hei], [Ka-Sko], [Lo], [Mey], [Ti-2], [Yam]) to pairs of non-associative binary algebras, or to ternary algebras.

Another construction of the exceptional Lie algebras is the celebrated Freudenthal magic square which is a recipe to associate a simple Lie algebra to a pair of composition algebras (see for example [Freu]).

Received by the editors October 4, 2004 and, in revised form, February 13, 2006.
2000 *Mathematics Subject Classification.* Primary 17A75; Secondary 17B25, 11S90.

A major step toward a common framework for all these results was carried out by B. Allison who introduced a new class of non-associative algebras with involution called *structurable algebras*, which contains the class of Jordan algebras ([Al-2]). He showed that that the Koecher-Tits construction can be generalized to these algebras. The Allison construction, which also contains the magic square, provides models for all simple Lie algebras. It must be noted that the simple algebras obtained this way have in general a 5–grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and as in the Koecher-Tits construction the space $\mathfrak{g}_1$ is a copy of the structurable algebra given at the beginning.

Using a case by case analysis of some parabolic subalgebras of the simple Lie algebras, I. L. Kantor obtained, over $\mathbb{C}$, a class of algebras without involution, which are closely related to structurable algebras ([Ka]).

Let us close this short overview of the literature with two remarks. First of all there are many other papers on this subject. For example the papers by R. B. Brown ([Bro-1], [Bro-2]), S. Garibaldi ([Gar]), H.P. Allen and J.C. Ferrar ([All-Fer-2]), J.C. Ferrar ([Fer]) are relevant. Second we should also mention here the recent work of Barton and Sudbery ([B-S]) and Lansberg and Manivel ([L-M]) who have given new constructions of the algebras occurring in the Freudenthal magic square by using what they call *triality algebras*.

We now come to the present paper. It is worth noticing that Allison's construction is reversible; this means that one can recover the structurable algebra $A$ from the Lie algebra $\mathcal{L}(A)$ obtained from $A$ (see the proof of Theorem 4 in [Al-3]). Therefore as the complex Lie algebra $E_6$ is obtained by applying Allison's contruction to the structurable algebra $\mathbb{O} \otimes (\mathbb{C} \times \mathbb{C})$, it is certainly possible to express the octonion product by using the Lie bracket in $E_6$.

The purpose of this paper is quite different. We want to use other tools, namely the theory of Prehomogeneous Vector Spaces together with purely Lie algebra techniques, to extract the octonions from $E_6$. In fact, doing so, we naturally encounter a dual pair $(A_2, G_2)$ in $E_6$, which plays an important role in our construction. The Lie algebra $G_2$ will naturally be the derivation algebra of $\mathbb{O}$, whereas the Weyl group of $A_2$ will be the triality group.

Let us now give an outline of the paper.

In section 2 we will briefly recall basic facts concerning composition algebras in general and the octonions in particular. We will also give references to the theory of Prehomogeneous Vector Spaces (abbreviated $PV$), more specifically to those attached to parabolic subalgebras of semi-simple Lie algebras, the so-called $PV$'s of parabolic type. In section 3 we will construct the $(A_2, G_2)$ dual pair in $E_6$ and give some results on the Spin groups and their related PV's. This section also contains the proof of the important bracket-multiplicative property (see Proposition 3.4.3). In section 4 we will describe an embedding $\mathbb{O} \longrightarrow E_6$ which allows us to describe the octonion multiplication in terms of Lie brackets in $E_6$ (see Theorem 4.1.1). This construction relies on the existence of the dual pair $(A_2, G_2)$ in $E_6$ and on some $PV$'s occurring there. As a corollary to our construction we will show in section 5 that two well known results on octonions can easily be obtained. The first is the so-called triality principle, the second is the determination of the automorphism group of $\mathbb{O}$. In section 6 we show that if we restrict our construction to some real forms of $E_6$ we obtain the two real forms of $\mathbb{O}$, namely the $\mathbb{O}_s$ and $\mathbb{O}_a$.

## 2. Composition algebras, octonions, and Prehomogeneous Vector Spaces

2.1. We first need to recall some basic facts about the algebra of octonions. For the proofs (and many other results) we refer the reader to the book by Springer and Veldkamp on the subject ([S-V]). Let $V$ be a finite dimensional vector space over $K = \mathbb{R}$ or $\mathbb{C}$. An *algebra structure* on $V$ is just a bilinear map

$$\begin{aligned} V \times V &\longrightarrow V, \\ (x, y) &\longmapsto x \bullet y. \end{aligned}$$

A *composition algebra* is an algebra $(V, \bullet)$, admitting an identity element, together with a non-degenerate quadratic form $Q$ on $V$ such that

$$\forall x, y \in V \quad Q(x \bullet y) = Q(x)Q(y).$$

The following well known result will be used later.

**Proposition 2.1.1.** *A finite dimensional vector space $V$ over $K = \mathbb{R}$ or $\mathbb{C}$ can be endowed with a composition algebra structure if and only if $\dim_K(V) = 1, 2, 4, 8$. If $K = \mathbb{C}$, then for a given dimension all composition algebras are isomorphic. For $K = \mathbb{R}$ and $\dim_{\mathbb{R}}(V) = 8$ there are only two non-isomorphic composition algebras: the anisotropic octonions for which $Q$ is anisotropic and the split octonions for which $Q$ is of signature $(4, 4)$. Moreover for all composition algebras, the quadratic form $Q$ which allows composition is uniquely defined by the algebra structure.*

*Proof.* See for example Proposition 1.8.1, section 1.10 and Corollary 1.2.4 in [S-V]. □

The unique composition algebra of dimension 8 over $\mathbb{C}$ will be denoted by $\mathbb{O}$ and is called the *algebra of octonions*. Similarly the unique anisotropic (resp. split) 8–dimensional composition algebra over $\mathbb{R}$ is denoted by $\mathbb{O}_a$ (resp. $\mathbb{O}_s$).

A Prehomogeneous Vector Space is a pair $(G, V)$ where $G$ is a connected algebraic group and $V$ is a finite dimensional representation space for $G$ which has an open orbit under the $G$–action. For the general theory of $PV$'s we refer to [Sa-Ki]. The author has introduced a class of $PV$'s called $PV$ of parabolic type; see [Ru-2] for a survey. These $PV$'s are associated to any parabolic subalgebra of a simple Lie algebra and play an important role in the sequel.

Throughout this paper we will use weighted Dynkin diagrams which are ordinary Dynkin diagrams where some of the vertices (roots) are circled. Several objects are uniquely defined through such a weighted Dynkin diagram: a standard parabolic subalgebra of $\mathfrak{g}$, a $\mathbb{Z}$–gradation of $\mathfrak{g}$, several Prehomogeneous Vector Spaces, and sometimes a so-called admissible sub-algebra or a dual pair in $\mathfrak{g}$. For the details concerning these objects, we refer the reader to [Ru-1] or [Ru-2].

A triple $(y, h, x)$ of elements in a Lie algebra $\mathfrak{g}$ is called an $\mathfrak{sl}_2$-triple if $[y, x] = h, [h, y] = -2y, [h, x] = 2x$ (Bourbaki's convention).

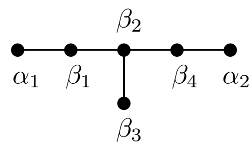
## 3. The dual pair $(A_2, G_2)$ in $E_6$

In this paragraph we develop some algebraic results on $E_6$ and $E_7$ (perhaps one should better say $E_6$ *in* $E_7$) which are needed to prove Proposition 3.4.3 which will play a central role in Theorem 4.1.1. Many results of this section are specializations of results [Ru-1] and [Ru-3], concerning either *admissible* and *C-admissible* subalgebras or *dual pairs*. More precisely all these results are connected to a dual pair

$(A_2, G_2)$ in $E_6$. Let us recall that this means just that the centralizer $Z_{E_6}(A_2)$ of $A_2$ in $E_6$ is $G_2$ and vice-versa. Here the algebra $A_2$ will be admissible (in fact even $C$–admissible) in the sense of [Ru-1] and [Ru-3]. This algebra, as all admissible sub-algebras, is closely related to some well-behaved parabolic subalgebra which we will now describe.

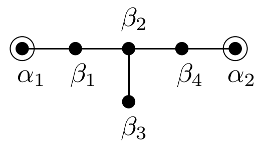
3.1. **A parabolic subalgebra of $\mathbf{E_6}$.** We first consider the simple Lie algebra $E_6$ over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $E_6$ . Let $R$ denote the roots of the pair $(E_6, \mathfrak{h})$ and let $\Psi$ be a fixed basis of simple roots. Let $R^+$ (resp. $R^-$) denote the positive roots (resp. negative roots) with respect to $\Psi$. The roots of $\Psi$ will be numbered as in the following Dynkin diagram:

(3 – 1)




We define $\theta = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and we denote by $\langle\theta\rangle$ the set of roots which are linear combinations of elements in $\theta$. We also set $\langle\theta\rangle^{\pm} = \langle\theta\rangle \cap R^{\pm}$. Let us recall the construction of the standard parabolic subalgebra of $E_6$ defined by $\theta$. It is convenient to associate to the datum $\theta$ a weighted Dynkin which is just the Dynkin diagram of $E_6$ where the roots in $\Psi \setminus \theta$ are circled. Therefore the weighted Dynkin diagram corresponding to our choice of $\theta$ is:

(3 – 2) $D =$




As we shall see this weighted Dynkin diagram contains a lot of information concerning the parabolic subalgebra.

Let $\mathfrak{h}_\theta$ be the orthogonal of $\theta$ in $\mathfrak{h}$:

$$\mathfrak{h}_\theta = \{X \in \mathfrak{h} \mid \beta_i(X) = 0, i = 1, 2, 3, 4\}.$$

Let $\mathfrak{l}_\theta = Z_{E_6}(\mathfrak{h}_\theta)$ be the centralizer of $\mathfrak{h}_\theta$. Then $\mathfrak{l}_\theta = \mathfrak{h} \oplus \sum_{\gamma \in <\theta>} E_6^\gamma$ (where as usual $E_6^\gamma$ denotes the $\gamma$ root-space in $E_6$). The algebra $\mathfrak{h}_\theta$ is then the center of $\mathfrak{l}_\theta$ and $\mathfrak{l}_\theta = \mathfrak{h}_\theta \oplus \mathfrak{l}'_\theta$ where $\mathfrak{l}'_\theta = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ is the derived algebra. One can notice that the number $(= 2)$ of circled roots is the dimension of the center of the reductive Lie algebra $\mathfrak{l}_\theta$, whereas the subdiagram corresponding to the non-circled roots (here $D_4$) is the Dynkin diagram of $\mathfrak{l}'_\theta$ . The Lie algebra $\mathfrak{h}(\theta) = \sum_{i=1}^4 \mathbb{C}H_{\beta_i} = \mathfrak{h}_\theta^\perp$ is then a Cartan subalgebra of $\mathfrak{l}'_\theta$. Also define:

$$\mathfrak{n}_\theta^+ = \sum_{\gamma \in R^+ \setminus \langle\theta\rangle^+} E_6^\gamma \quad \text{and} \quad \mathfrak{n}_\theta^- = \sum_{\gamma \in R^- \setminus \langle\theta\rangle^-} E_6^\gamma.$$

Then we get the so-called triangular decomposition: $E_6 = \mathfrak{n}_\theta^- \oplus \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta^+$, and the standard parabolic subalgebra associated to $\theta$ is $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{n}_\theta^+$. Let us define the element $H_\theta$ as the unique element in $\mathfrak{h}_\theta$ satisfying the equations:

$$\alpha_1(H_\theta) = \alpha_2(H_\theta) = 2,$$
$$\beta_i(H_\theta) = 0, \quad i = 1, 2, 3, 4.$$

As the highest root in $E_6$ is $\delta = \alpha_1 + \alpha_2 + 2\beta_1 + 2\beta_2 + 2\beta_4 + 3\beta_2$, the possible eigenvalues of $\mathrm{ad}(H_\theta)$ are $-4, -2, 0, 2, 4$, and we therefore obtain the following $\mathbb{Z}$–grading:

$$E_6 = \bigoplus_{p=-2}^{p=2} d_p(\theta) \quad \text{where} \quad d_p(\theta) = \{X \in E_6 \,|\, [H_\theta, X] = 2pX\}.$$

Then

$$\mathfrak{n}_\theta^+ = d_1(\theta) \oplus d_2(\theta),\ \mathfrak{n}_\theta^- = d_{-1}(\theta) \oplus d_{-2}(\theta),\ \mathfrak{l}_\theta = d_0(\theta).$$

We also need to refine this decomposition under the action of $\mathrm{ad}(\mathfrak{h}_\theta)$. For $\gamma \in \mathfrak{h}^*$ let us denote by $\overline{\gamma}$ the restriction of $\gamma$ to $\mathfrak{h}_\theta$. Then we define the "*block*" corresponding to $\overline{\gamma}$:

$$E_6^{\overline{\gamma}} = \{X \in E_6 \,|\, \forall H \in \mathfrak{h}_\theta,\ [H, X] = \overline{\gamma}(H)X\}.$$

The blocks should be viewed as generalisations of the classical root spaces, which are obtained by diagonalizing with respect to $\mathfrak{h}_\theta$, instead of the full Cartan subalgebra $\mathfrak{h}$. We get

$$\begin{aligned} & d_0(\theta) = E_6^{\overline{0}} = \mathfrak{l}_\theta, \\ & d_1(\theta) = E_6^{\overline{\alpha}_1} \oplus E_6^{\overline{\alpha}_2},\ d_2(\theta) = E_6^{\overline{\delta}} = E_6^{\overline{\alpha}_1 + \overline{\alpha}_2}, \text{ and symmetrically} \\ & d_{-1}(\theta) = E_6^{-\overline{\alpha}_1} \oplus E_6^{-\overline{\alpha}_2},\ d_{-2}(\theta) = E_6^{-\overline{\delta}} = E_6^{-\overline{\alpha}_1 - \overline{\alpha}_2}. \end{aligned}$$

Moreover the decomposition $d_1(\theta) = E_6^{\overline{\alpha}_1} \oplus E_6^{\overline{\alpha}_2}$ is the decomposition of $d_1(\theta)$ into irreducible $\mathfrak{l}_\theta$–modules (see for example [Ru-2], Proposition 4.2.1 p. 125 and Proposition 4.2.2 p. 127). Of course these kinds of results are true for any parabolic subalgebra in any semi-simple Lie algebra.

Hence the decomposition of $E_6$ into blocks can be visualized with the following Figure:

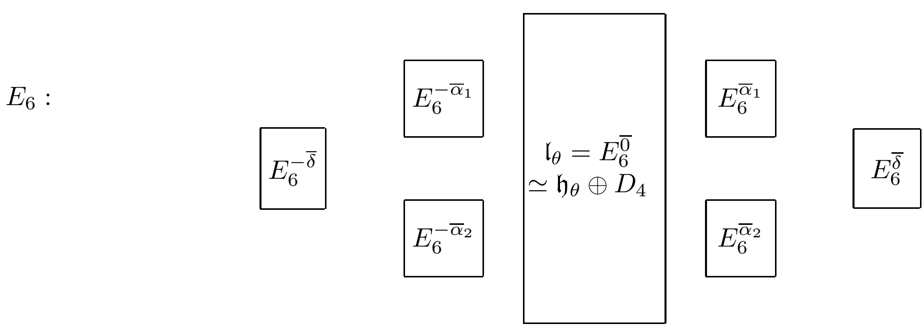
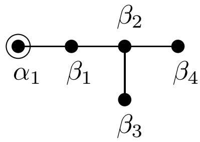


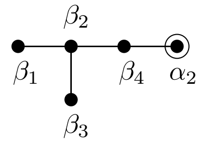

Figure I

3.2. **Construction of a $C$-admissible $\mathbf{A_2}$ subalgebra in $\mathbf{E_6}$.** Roughly speaking a semi-simple subalgebra is built up from a Cartan subalgebra and from one dimensional blocks which are just the root spaces. From the Serre relations one also knows that not all the blocks are needed, in fact only the simple roots and their negatives are needed. More precisely the basic $\mathfrak{sl}_2$–triples $(X_{-\alpha}, H_\alpha, X_\alpha)$ corresponding to the simple roots are enough to generate the whole semi-simple Lie algebra.

Following the general construction of admissible subalgebras ([Ru-1], [Ru-3]), we will now construct, in a way analogous to Serre relations, a subalgebra of type $A_2$ in $E_6$ by putting a very specific one dimensional space in each of the blocks different from $E_6^{\overline{0}}$ in Figure I. The center $\mathfrak{h}_\theta$ of the block $E_6^{\overline{0}} = \mathfrak{l}_\theta$ will be the Cartan subalgebra of this $A_2$.

For this we separately consider the pairs of blocks $(E_6^{\overline{\alpha}_1}, E_6^{-\overline{\alpha}_1})$ and $(E_6^{\overline{\alpha}_2}, E_6^{-\overline{\alpha}_2})$. The pair $(E_6^{\overline{\alpha}_1}, E_6^{-\overline{\alpha}_1})$ (resp. $(E_6^{\overline{\alpha}_1}, E_6^{-\overline{\alpha}_1})$) already occurs as a pair of blocks in the subalgebra of type $D_5$ of $E_6$ corresponding to the subdiagram

$D_1 =$ 

(resp. $D_2 =$ 

)

This means that if we denote by $D_{5,1}$ and $D_{5,2}$ the subalgebras of type $D_5$ of $E_6$ corresponding to $D_1$ and $D_2$ and perform in these algebras the construction of the standard parabolic subalgebras associated to $\theta$, we obtain two triangular decompositions:

$$D_{5,1} = E_6^{-\overline{\alpha}_1} \oplus \mathfrak{l}_\theta^1 \oplus E_6^{\overline{\alpha}_1} \quad (\text{resp. } D_{5,2} = E_6^{-\overline{\alpha}_2} \oplus \mathfrak{l}_\theta^2 \oplus E_6^{\overline{\alpha}_2}),$$

where $\mathfrak{l}_\theta^{1\prime} = \mathfrak{l}_\theta^{2\prime} = D_4$. We also need the elements in $D_{5,1}$ and $D_{5,2}$ which are analogous to the element $H_\theta$ in $E_6$. This goes as follows.

Let $\Psi_1 = \{\alpha_1\} \cup \theta$ and $\Psi_2 = \{\alpha_2\} \cup \theta$. Let $H_{\overline{\alpha}_1}$ (resp. $H_{\overline{\alpha}_2}$) be the unique element in $\sum_{\alpha \in \Psi_1} \mathbb{C} H_\alpha$ (resp. $\sum_{\alpha \in \Psi_2} \mathbb{C} H_\alpha$) satisfying the equations:

$$\alpha_1(H_{\overline{\alpha}_1}) = 2 \text{ and } \beta_j(H_{\overline{\alpha}_1}) = 0 \quad (j = 1, 2, 3, 4),$$
$$(\text{resp. } \alpha_2(H_{\overline{\alpha}_2}) = 2 \text{ and } \beta_j(H_{\overline{\alpha}_2}) = 0 \quad (j = 1, 2, 3, 4)).$$

As the prehomogeneous vector spaces $(\mathfrak{l}_\theta^1, E_6^{\overline{\alpha}_1})$ and $(\mathfrak{l}_\theta^1, E_6^{\overline{\alpha}_1})$ are irreducible and regular, it is known ([Ru-2], Th. 4.3.2 p.132) that there exist two $\mathfrak{sl}_2$–triples of the form $(X_{-\overline{\alpha}_1}, H_{\overline{\alpha}_1}, X_{\overline{\alpha}_1})$ and $(X_{-\overline{\alpha}_2}, H_{\overline{\alpha}_2}, X_{\overline{\alpha}_2})$ where $X_{\pm\overline{\alpha}_1} \in E_6^{\pm\overline{\alpha}_1}$ and $X_{\pm\overline{\alpha}_2} \in E_6^{\pm\overline{\alpha}_2}$. All possible choices for such $\mathfrak{sl}_2$–triples are conjugate under $\exp \operatorname{ad}(\mathfrak{l}_\theta)$. These two $\mathfrak{sl}_2$–triples, which play the same role as the $\mathfrak{sl}_2$–triples used in the Serre relations, generate inside $E_6$ a so called $C$–admissible subalgebra isomorphic to $\mathfrak{sl}_3$ which we will simply denote $A_2$ ([Ru-1] Th. 3.1, [Ru-3] section 3). More precisely if we define $X_{\overline{\delta}} = [X_{\overline{\alpha}_1}, X_{\overline{\alpha}_2}]$ and $X_{-\overline{\delta}} = [X_{-\overline{\alpha}_1}, X_{-\overline{\alpha}_2}]$, then $(X_{\pm\overline{\delta}}, X_{\pm\overline{\alpha}_1}, X_{\pm\overline{\alpha}_2}, H_{\overline{\alpha}_1}, H_{\overline{\alpha}_2})$ is a Chevalley basis for $A_2$. Notice also that $\mathfrak{h}_\theta = \mathbb{C} H_{\overline{\alpha}_1} \oplus \mathbb{C} H_{\overline{\alpha}_2}$ is a Cartan subalgebra for $A_2$. The subalgebra $A_2$ is called the $C$–admissible subalgebra associated to $\theta$, and it is uniquely defined up to conjugation under $\mathfrak{l}_\theta'$. It turns out that the centralizer of $A_2$ in $E_6$ is a simple Lie subalgebra of type $G_2$ (we will simply denote it $G_2$) (see [Ru-3], p. 32-33). From the general construction of dual pairs (see [Ru-3], Th. 4.3. p.27) we conclude the striking fact that the pair $(A_2, G_2)$ is a dual pair in $E_6$. In fact $G_2 \subset \mathfrak{l}_\theta' = [\mathfrak{l}_\theta, \mathfrak{l}_\theta]$ and $G_2$ is the generic isotropy subalgebra in the $PV$ $(\mathfrak{l}_\theta, d_1(\theta))$ of the generic element $X_{\overline{\alpha}_1} + X_{\overline{\alpha}_2}$. The $A_2$–graded structure of $E_6$ is visualized by Figure II at the end of section 4.

It is known ([Ru-4] section 8, p. 388) that the six spaces $E_6^{\pm\overline{\alpha}_1}$, $E_6^{\pm\overline{\alpha}_2}$,$E_6^{\pm\overline{\delta}}$ which are $\mathfrak{l}_\theta$–invariant (under the adjoint action) are also irreducible under $\mathfrak{l}_\theta' = D_4$. In fact one can easily see that $E_6^{\overline{\alpha}_1}$, $E_6^{\overline{\alpha}_2}$,$E_6^{-\overline{\delta}}$ are the three non-equivalent 8–dimensional representations of $\mathfrak{l}_\theta' = D_4$. This can, for example, be read directly from the combinatorics of the extended Dynkin diagram of $D_4$ ([Ru-2] Prop. 4.2.2. or [Ru-4]).

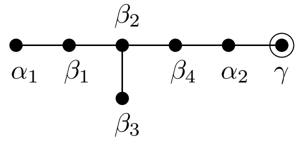
Using the Killing form of $E_6$, the $\mathfrak{l}_\theta$–modules $E_6^{-\overline{\alpha}_1}$, $E_6^{-\overline{\alpha}_2}$,$E_6^{-\overline{\delta}}$ appear naturally as the dual $\mathfrak{l}_\theta$–modules of $E_6^{\overline{\alpha}_1}$, $E_6^{\overline{\alpha}_2}$,$E_6^{\overline{\delta}}$ respectively. But as all representations of $\mathfrak{l}'_\theta = D_4$ are self-dual we have the following $\mathfrak{l}'_\theta$-isomorphisms:

$$(3-3) \qquad E_6^{-\overline{\alpha}_1} \simeq E_6^{\overline{\alpha}_1}, E_6^{-\overline{\alpha}_2} \simeq E_6^{\overline{\alpha}_2}, E_6^{-\overline{\delta}} \simeq E_6^{\overline{\delta}}.$$

3.3. **The automorphism of $\mathbf{E_6}$ coming from $\mathbf{E_7}$.** We will now show that there exists an inner automorphism $A$ of $E_7$ which implements all the isomorphisms in $(3-3)$.

For this purpose we need to use the usual embedding of $E_6$ into $E_7$ coming from the inclusion of the diagrams. Consider first the following weighted Dynkin diagram of $E_7$:

(3 – 4)




and let us perform the construction of the standard parabolic subalgebra associated to this diagram in the same way as we did in section 3.1. First let $\widetilde{\mathfrak{h}}$ be a Cartan subalgebra of $E_7$ such that $\widetilde{\Psi} = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4, \gamma\}$ is a set of simple roots in the root system $R(E_7, \widetilde{\mathfrak{h}})$. The set of non-circled roots $\widetilde{\theta} = \{\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4\}$ (which is equal to $\Psi$ in the notation of section 3.1) will therefore be the set defining this parabolic subalgebra.

Let $\widetilde{\mathfrak{h}}_{\widetilde{\theta}}$ be the orthogonal of $\widetilde{\theta}$ in $\widetilde{\mathfrak{h}}$. For any $\lambda \in \widetilde{\mathfrak{h}}^*$ we will denote by $\overline{\overline{\lambda}}$ its restriction to $\widetilde{\mathfrak{h}}_{\widetilde{\theta}}$ (the simple "bar" has already been used for restriction to $\mathfrak{h}_\theta$ in $E_6$). If $H_{\overline{\overline{\gamma}}}$ denotes the unique element in $\widetilde{\mathfrak{h}}_{\widetilde{\theta}}$ such that $\gamma(H_{\overline{\overline{\gamma}}}) = 2$, then we have $\widetilde{\mathfrak{h}}_{\widetilde{\theta}} = \mathbb{C}H_{\overline{\overline{\gamma}}}$.

For $\lambda \in \widetilde{\mathfrak{h}}^*$ we also define the blocks in $E_7$ relatively to $\overline{\overline{\lambda}}$ by

$$E_7^{\overline{\overline{\lambda}}} = \{X \in E_7 \,|\, \forall H \in \mathfrak{h}^\perp, [H, X] = \overline{\overline{\lambda}}(H)X\}.$$

Then $E_7^{\overline{\overline{0}}} = E_6 \oplus \widetilde{\mathfrak{h}}_{\widetilde{\theta}}$ and the parabolic subalgebra of $E_7$ associated to $\widetilde{\theta}$ is $\mathfrak{p}_{\widetilde{\theta}} = E_7^{\overline{\overline{0}}} \oplus E_7^{\overline{\overline{\gamma}}}$. Rather than this parabolic subalgebra we will consider the $\mathbb{Z}$–grading of length 3 associated to $\widetilde{\theta}$:

$$(3-5) \qquad E_7 = E_7^{-\overline{\overline{\gamma}}} \oplus E_7^{\overline{\overline{0}}} \oplus E_7^{\overline{\overline{\gamma}}}.$$

It is well known that the adjoint representation of $E_6$ on $E_7^{\overline{\overline{\gamma}}}$ (or on $E_7^{-\overline{\overline{\gamma}}}$) is the irreducible 27-dimensional representation of $E_6$ (this can also be obtained by general arguments; see section 4.2 p. 125 in [Ru-2]). Using again Th. 4.3.2 p. 132 in [Ru-2], we see that there exist $X_{\overline{\overline{\gamma}}} \in E_7^{\overline{\overline{\gamma}}}$ and $X_{-\overline{\overline{\gamma}}} \in E_7^{-\overline{\overline{\gamma}}}$ such that $(X_{-\overline{\overline{\gamma}}}, H_{\overline{\overline{\gamma}}}, X_{\overline{\overline{\gamma}}})$ is an $\mathfrak{sl}_2$–triple. The algebra of type $A_1$ defined by this triple is another example of a $C$–admissible subalgebra.

Define

$$(3-6) \qquad A = \exp \operatorname{ad} X_{\overline{\overline{\gamma}}} \exp \operatorname{ad} X_{-\overline{\overline{\gamma}}} \exp \operatorname{ad} X_{\overline{\overline{\gamma}}}.$$

Then we have:

**Proposition 3.3.1.** *The automorphism $A$ of the Lie algebra $E_7$ restricts to an involution of $E_6$. Moreover one can choose $X_{\overline{\overline{\gamma}}}$ and $X_{-\overline{\overline{\gamma}}}$ such that the restrictions of*

*A realize the unique (up to constants) $\mathfrak{l}'_\theta$–isomorphisms between $E_6^{\overline{\alpha}_1}$ and $E_6^{-\overline{\alpha}_1}$, $E_6^{\overline{\alpha}_2}$ and $E_6^{-\overline{\alpha}_2}$, and $E_6^{\overline{\delta}}$ and $E_6^{-\overline{\delta}}$. One can also choose the $\mathfrak{sl}_2$–triples $(X_{-\overline{\alpha}_1}, H_{\overline{\alpha}_1}, X_{\overline{\alpha}_1})$, $(X_{-\overline{\alpha}_2}, H_{\overline{\alpha}_2}, X_{\overline{\alpha}_2})$, $(X_{-\overline{\delta}}, H_{\overline{\delta}}, X_{\overline{\delta}})$ defining $A_2$ such that $AX_{\overline{\alpha}_1} = X_{-\overline{\alpha}_1}$, $AX_{\overline{\alpha}_2} = X_{-\overline{\alpha}_2}$ and $AX_{\overline{\delta}} = X_{-\overline{\delta}}$.*

*Proof.* The $PV$ $\left(E_7^{\overline{\overline{0}}} = E_6 \oplus \mathbb{C}H_{\overline{\overline{\gamma}}}, E_7^{\overline{\overline{\gamma}}}\right)$ associated to the diagram $(3-4)$ is regular and of commutative parabolic type . From Prop. 4.4.5 p. 146 in [Ru-2] it is known that $A_{|_{E_6}}$ is an involution whose set of fixed points $S_{\overline{\overline{\gamma}}}$ is:

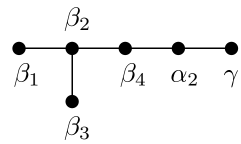
$$S_{\overline{\overline{\gamma}}} = Z_{E_6}(X_{\overline{\overline{\gamma}}}) = Z_{E_6}(X_{-\overline{\overline{\gamma}}}) \simeq F_4.$$

This proves the first assertion.

We will now specify $X_{\overline{\overline{\gamma}}}$ and $X_{-\overline{\overline{\gamma}}}$. From Théorème 2.8 in [M-R-S] it is known that $X_{\overline{\overline{\gamma}}}$ can be chosen such that

$$X_{\overline{\overline{\gamma}}} = \sum_i X_{\lambda_i}$$

where the root vectors $X_{\lambda_i}$ correspond to a maximal family $\{\lambda_i\}$ of strongly orthogonal roots inside the set of positive roots $R(E_7, \widetilde{\mathfrak{h}})^+$ which are not in $E_6$ (or equivalently, which are in $E_7^{\overline{\overline{\gamma}}}$). From Table 2 in [M-R-S] we know that in the case under consideration such a set has only 3 elements. Let $\nu = 2\alpha_1 + 2\alpha_2 + 3\beta_1 + 4\beta_2 + 2\beta_3 + 3\beta_4 + \gamma$ be the highest root of $E_7$. Also let $\mu = \gamma + 2\alpha_2 + 2\beta_4 + 2\beta_2 + \beta_3 + \beta_1$ be the highest root of the subalgebra $D_6$ of $E_7$ corresponding to the subdiagram




From the extended Dynkin diagram of $D_6$ and $E_7$ it is easy to see that $\{\gamma, \mu, \nu\}$ is such a set of strongly orthogonal roots. Therefore we can choose:

$$X_{\overline{\overline{\gamma}}} = X_\gamma + X_\mu + X_\nu, \qquad X_{-\overline{\overline{\gamma}}} = X_{-\gamma} + X_{-\mu} + X_{-\nu}.$$

As we already did before, we denote by $D_4$ the subalgebra of $D_6 \subset E_6 \subset E_7$ defined by the roots $\{\beta_1, \beta_2, \beta_3, \beta_4\}$. From the extended Dynkin diagrams of $E_7$ and of $D_6$ it is also easy to see that $D_4$ commutes with $X_\gamma, X_\mu$ and $X_\nu$, in other words we obtain the inclusion $D_4 \subset S_{\overline{\overline{\gamma}}}$. Therefore $A_{|_{E_6}}$ commutes with the adjoint action of $D_4$ on $E_6$ and therefore preserves the $D_4$–isotypic components in $E_6$. Hence

$$A(E_6^{\overline{\alpha}_1}) \subset E_6^{-\overline{\alpha}_1} \oplus E_6^{\overline{\alpha}_1}, A(E_6^{\overline{\alpha}_2}) \subset E_6^{-\overline{\alpha}_2} \oplus E_6^{\overline{\alpha}_2}, A(E_6^{\overline{\delta}}) \subset E_6^{-\overline{\delta}} \oplus E_6^{\overline{\delta}}.$$

But $\mathbb{C}H_{\overline{\alpha}_1} \oplus \mathbb{C}H_{\overline{\alpha}_2} = \mathfrak{h}_\theta$ is orthogonal (for the Killing form on $E_7$) to $\mathfrak{h}(\theta) = \mathbb{C}H_{\beta_1} \oplus \mathbb{C}H_{\beta_2} \oplus \mathbb{C}H_{\beta_3} \oplus \mathbb{C}H_{\beta_4}$ which is a Cartan subalgebra of $D_4$. As $D_4 \subset S_{\overline{\overline{\gamma}}} \simeq F_4$ we obtain that $\mathfrak{h}(\theta)$ is also a Cartan subalgebra of $S_{\overline{\overline{\gamma}}}$. Hence $[\mathfrak{h}_\theta, \mathfrak{h}(\theta)] = 0$ implies $B(\mathfrak{h}_\theta, S_{\overline{\overline{\gamma}}}) = 0$. Therefore $AH = -H$ for all $H \in \mathfrak{h}_\theta$. This implies that:

$$A(E_6^{\overline{\alpha}_1}) = E_6^{-\overline{\alpha}_1}, \quad A(E_6^{\overline{\alpha}_2}) = E_6^{-\overline{\alpha}_2} \text{ and } A(E_6^{\overline{\delta}}) = E_6^{-\overline{\delta}}.$$

As $A$ commutes with $G_2 \subset D_4$, $AX_{\overline{\alpha}_1}$ belongs to the centralizer $Z_{E_6}(G_2) = A_2$ (here we use the fact that $(A_2, G_2)$ is a dual pair in $E_6$), hence $AX_{\overline{\alpha}_1} = cX_{-\overline{\alpha}_1}$, with a non-zero complex constant $c$. Then a simple classical calculation shows that if we set $X_{\overline{\alpha}_1} := \frac{1}{\sqrt{c}} X_{\overline{\alpha}_1}$ and $X_{-\overline{\alpha}_1} := \sqrt{c} X_{-\overline{\alpha}_1}$, we obtain an $\mathfrak{sl}_2$–triple with the required property. The same is true for $\overline{\alpha}_2$ and $\overline{\delta}$. $\square$

The action of the automorphism $A$ on $E_6$ is summarized in Figure II and the end of section 4.

3.4. **The bracket-multiplicative property.** This section is essentially devoted to the proof of Proposition 3.4.3 which we call the **bracket-multiplicative property** of the Killing form of $E_6$ and which will play a crucial role in the sequel. It will turn out that this property is exactly equivalent to the composition law $Q(x \bullet y) = Q(x)Q(y)$ (see Section 2.1) for the octonion product we are going to define on $E_6^{\overline{\alpha}_1}$ in the next section.

We begin with the following well known lemma.

**Lemma 3.4.1.** *Let $V_1, V_2, V_3$ be the three non-equivalent irreducible representations of dimension 8 of the Lie algebra $\mathfrak{so}(8)$. Each of them extends to an irreducible representation of the complex group $Spin(8)$. The representations $(Spin(8) \times \mathbb{C}^*, V_i)$ ($i = 1, 2, 3$) are irreducible regular prehomogeneous vector spaces whose fundamental relative invariants are quadratic forms $Q_i$. Let $i_0 \in \{1, 2, 3\}$ and let $x \in V_{i_0}$ be such that $Q_{i_0}(x) \neq 0$ (this means that $x$ is generic in $V_{i_0}$). Let $Spin(8)_x$ be the isotropy subgroup of $x$, and $\mathfrak{so}(8)_x$ the corresponding isotropy subalgebra. Then $\mathfrak{so}(8)_x$ is of type $B_3$ (i.e. isomorphic to $\mathfrak{so}(7)$) and for $i \neq i_0$, the restriction of $V_i$ to $\mathfrak{so}(8)_x$ is the irreducible Spin representation of $\mathfrak{so}(8)_x \simeq \mathfrak{so}(7)$. The group $Spin(8)_x$ is isomorphic to $Spin(7)$. Moreover the representations $(Spin(8)_x \times \mathbb{C}^*, V_i)$ ($i \neq i_0$) are still irreducible $PV'$s with the $Q_i$'s as fundamental relative invariants (they therefore have the same open orbit as $(Spin(8) \times \mathbb{C}^*, V_i)$ ($i \neq i_0$)).*

*Proof.* The $PV$ part of the statement is classical. It relies essentially on the fact that the PV's $(Spin(8) \times \mathbb{C}^*, V_i)$ are equivalent (in the sense of [Sa-Ki], Definition 4 p. 36) to the PV $(SO(8) \times \mathbb{C}^*, \mathbb{C}^8)$ ([Sa-Ki] p. 20, or [Ki]). The other statement is a consequence of the construction of the *Spin* representation in the odd dimensional case ([Sa-Ki] p. 19-20 and p. 114, [Che]). □

Now we need to precisely define some subalgebras, groups and subgroups which will be used later. First recall that we have defined $D_4$ to be the subalgebra of $E_6$ corresponding to the roots $\{\beta_1, \beta_2, \beta_3, \beta_4\}$. The generators $(X_{-\overline{\alpha}_1}, H_{\overline{\alpha}_1}, X_{\overline{\alpha}_1})$, $(X_{-\overline{\alpha}_2}, H_{\overline{\alpha}_2}, X_{\overline{\alpha}_2})$, $(X_{-\overline{\delta}}, H_{\overline{\delta}}, X_{\overline{\delta}})$ of $A_2$ are chosen to satisfy the conditions in Proposition 3.3.1. Recall also that $G_2$ is the centralizer of $A_2$ in $E_6$. The algebra $G_2$ can also be defined as the centralizer of $X_{\overline{\alpha}_1}$ and $X_{\overline{\alpha}_2}$ in $D_4$ or $\mathfrak{l}_\theta$. We will denote by $B_3$ the centralizer of $X_{\overline{\alpha}_1}$ in $D_4$ (this is of course coherent with the type of this algebra and is a consequence of Lemma 3.4.1). Therefore $G_2$ can be viewed as the centralizer of $X_{\overline{\alpha}_2}$ in $B_3$.

Let $G$ be the adjoint group of the Lie algebra $E_6$ and let $L_\theta$ be the analytical subgroup of $G$ with Lie algebra $\mathfrak{l}_\theta$. The three irreducible representations of dimension 8 of $D_4$ (or of $\mathfrak{l}_\theta$), namely $E_6^{\overline{\alpha}_1}, E_6^{\overline{\alpha}_2}, E_6^{\overline{\delta}}$, will clearly integrate as representations of the corresponding analytical subgroup via the Adjoint action. Therefore, as it is well known that these three representation cannot be integrated on a group isomorphic to $SO(8)$ but only on its double cover $Spin(8)$, it is justified to denote by $\mathbf{Spin(8)}$ the analytical subgroup corresponding to $D_4$ (we use boldfaced letters to mention that this group is a precisely defined subgroup of $G$). Then we have $L_\theta = \mathbf{Spin(8)}. \exp \operatorname{ad} \mathfrak{h}_\theta$, as an almost direct product. From Lemma 3.4.1 it is also justified to denote by $\mathbf{Spin(7)}$ the centralizer of $X_{\overline{\alpha}_1}$ in $\mathbf{Spin(8)}$. Of course the Lie algebra of $\mathbf{Spin(7)}$ is now exactly $B_3$. Finally we denote by $\mathbf{exp(G_2)}$ the analytic subgroup of $L_\theta$ corresponding to $G_2$.

Concerning these groups we have the following lemma.

**Lemma 3.4.2.** *The group* $\mathbf{exp(G_2)}$ *is the centralizer of* $X_{\overline{\alpha}_2}$ *in* $\mathbf{Spin(7)}$ *(or the centralizer of* $X_{\overline{\alpha}_1}$ *and* $X_{\overline{\alpha}_2}$ *in* $\mathbf{Spin(8)}$*). The Adjoint action of* $\mathbf{Spin(7)}$ *on* $E_6^{\overline{\alpha}_2}$ *is the spin representation, and it is a faithful representation which contains* $(-Id)_{|_{E_6^{\overline{\alpha}_2}}}$. *If* $-1$ *denotes the element of* $\mathbf{Spin(7)}$ *acting by* $(-Id)_{|_{E_6^{\overline{\alpha}_2}}}$, *then the only proper closed subgroup of* $\mathbf{Spin(7)}$ *containing* $\mathbf{exp(G_2)}$ *is* $\mathbf{exp(G_2)} \times \{\pm 1\}$.

*Proof.* This lemma is certainly well known, therefore we only sketch the proof. In order to prove that $\mathbf{exp(G_2)} = \mathbf{Spin(7)}_{X_{\overline{\alpha}_2}}$ it is enough to prove that $\mathbf{Spin(7)}_{X_{\overline{\alpha}_2}}$ is connected. This is well known; see [Ig], Prop. 4 p. 1015. We deduce from Lemma 3.4.1 that the Adjoint action of $\mathbf{Spin(7)}$ on $E_6^{\overline{\alpha}_2}$ is the *Spin* representation, and this representation is also known to be faithful as all odd dimension *Spin* representations (see [Ig] p. 999, or [Che]).

Then consider the covering map $\rho : \mathbf{Spin(7)} \longrightarrow SO(7)$. As any connected complex Lie group having $G_2$ as a Lie algebra is simply connected, the map $\rho : \mathbf{exp(G_2)} \longrightarrow \rho(\mathbf{exp(G_2)}) = \exp(G_2)$ is an isomorphism, and as the covering map is a two-fold cover, we get $\rho^{-1}(\exp(G_2) = \mathbf{exp(G_2)} \times \{\pm 1\}$. As $\exp(G_2)$ is a maximal subgroup of $SO(7)$ it is now easy to see that the only possible closed subgroup of $\mathbf{Spin(7)}$ containing strictly $\mathbf{exp(G_2)}$ is $\mathbf{exp(G_2)} \times \{\pm 1\}$. □

Here comes the key result.

**Proposition 3.4.3** (Bracket-multiplicative property)**.** *Let* $B$ *be the Killing form on* $E_6$ *and define*

$$\forall\, x, y \in E_6, \quad b(x,y) = -\frac{1}{48}B(x,y).$$

*Then* $\forall\, x \in E_6^{\overline{\alpha}_1}, \forall\, y \in E_6^{\overline{\alpha}_2}$, *we have:*

$$b([x,y], A[x,y]) = b(x, Ax)b(y, Ay).$$

*Proof.* An easy calculation made in $A_2 \simeq \mathfrak{sl}_3$ shows that $\operatorname{ad} H_{\overline{\alpha}_1}$ acts by $-\mathrm{Id}$ on $E_6^{\overline{\alpha}_2}$ and by Id on $E_6^{\overline{\delta}}$. Hence $B(H_{\overline{\alpha}_1}, H_{\overline{\alpha}_1}) = \mathrm{tr}_{|_{E_6}}(\operatorname{ad} H_{\overline{\alpha}_1})^2 = 2(4 \times 8 + 8 + 8) = 96$. Therefore, from the well known identity $B(X_{\overline{\alpha}_1}, X_{-\overline{\alpha}_1}) = -\frac{1}{2}B(H_{\overline{\alpha}_1}, H_{\overline{\alpha}_1})$ we deduce that $b(X_{\overline{\alpha}_1}, X_{-\overline{\alpha}_1}) = 1$. Similarly we obtain $b(X_{\overline{\alpha}_2}, X_{-\overline{\alpha}_2}) = 1$ and $b(X_{\overline{\delta}}, X_{-\overline{\delta}}) = 1$.

As $A$ commutes with $\mathbf{Spin(8)}$, we note that $x \longmapsto b(x, Ax)$ is a non-trivial relative invariant of the $PV$ $(L_\theta, E_6^{\overline{\alpha}_1})$. More precisely, from our normalization we know that $x \longmapsto b(x, Ax)$ is the unique relative invariant taking the value 1 on $X_{\overline{\alpha}_1}$. Similarly $y \longmapsto b(y, Ay)$ is the unique relative invariant of the $PV$ $(L_\theta, E_6^{\overline{\alpha}_2})$ taking the value 1 on $X_{\overline{\alpha}_2}$, and $z \longmapsto b(z, Az)$ is the unique relative invariant of the $PV$ $(L_\theta, E_6^{\overline{\delta}})$ taking the value 1 on $X_{\overline{\delta}}$.

Now let $y$ be generic in $E_6^{\overline{\alpha}_2}$ and for $x \in E_6^{\overline{\alpha}_1}$ define $\varphi_y(x) = b([x,y], A[x,y])$. Let $Z_{\mathbf{Spin(8)}}(y)$ be the centralizer of $y$ in $\mathbf{Spin(8)}$. We know from Lemma 3.4.1 that $Z_{\mathbf{Spin(8)}}(y) \simeq Spin(7)$. If $g \in Z_{\mathbf{Spin(8)}}(y)$ we have $\varphi_y(gx) = b([gx,y], A[gx,y]) = b(g[x,y], Ag[x,y]) = b(g[x,y], gA[x,y]) = \varphi_y(x)$. Hence $\varphi_y$ is a relative invariant of the $PV$ $(Z_{L_\theta}(y), E_6^{\overline{\alpha}_1})$. From Lemma 3.4.1 we obtain that

$$b([x,y], A[x,y]) = c(y)b(x, Ax)$$

where $c(y)$ is a constant depending on $y$. Now taking $x$ as generic in $E_6^{\overline{\alpha}_1}$, a similar proof shows that $c(y)$ is a relative invariant of the $PV$ $(Z_{L_\theta}(x), E_6^{\overline{\alpha}_2})$, and therefore, again using Lemma 3.4.1, we get $c(y) = cb(y, Ay)$, where $c \in \mathbb{C}$. Then for all $x \in E_6^{\overline{\alpha}_1}$ and all $y \in E_6^{\overline{\alpha}_y}$ we have:

$$b([x,y], A[x,y]) = cb(x, Ax)b(y, Ay).$$

Then taking, $x = X_{\overline{\alpha}_1}$, $y = X_{\overline{\alpha}_2}$ (and therefore $[x,y] = X_{\overline{\delta}}$), we obtain $c = 1$. $\square$

## 4. The octonion product on $E_6^{\overline{\alpha}_1}$

4.1. In this section we will define a composition law denoted by $\bullet$ on $E_6^{\overline{\alpha}_1}$ and show that $(E_6^{\overline{\alpha}_1}, \bullet)$ is the octonion algebra over $\mathbb{C}$. This composition law will be defined only in terms of the Lie bracket in $E_6$, but it depends on the $C$–admissible subalgebra $A_2$ constructed in section 3.2, which moreover satisfies the conditions of Proposition 3.3.1. The precise statement is Theorem 4.1.1 below.

We first need to define some Weyl group elements. As seen in section 3.2 the restricted roots $\pm\overline{\alpha}_1, \pm\overline{\alpha}_2, \pm\overline{\delta}$ form a root system of type $A_2$. The elements of the Weyl group $W$ of $A_2$ (which is isomorphic to $S_3$ the permutation group of three variables) are classically realized as restrictions to $\mathfrak{h}_\theta$ of inner automorphisms of $E_6$. More precisely, let

$$\begin{aligned} w_1 &= \exp \operatorname{ad} X_{\overline{\alpha}_1} \exp \operatorname{ad} X_{-\overline{\alpha}_1} \exp \operatorname{ad} X_{\overline{\alpha}_1}, \\ w_2 &= \exp \operatorname{ad} X_{\overline{\alpha}_2} \exp \operatorname{ad} X_{-\overline{\alpha}_2} \exp \operatorname{ad} X_{\overline{\alpha}_2}, \\ w_3 &= \exp \operatorname{ad} X_{\overline{\delta}} \exp \operatorname{ad} X_{-\overline{\delta}} \exp \operatorname{ad} X_{\overline{\delta}}. \end{aligned}$$

Then $w_{i\,|_{\mathfrak{h}_\theta}}$ $(i = 1, 2, 3)$ are the reflections associated to $\overline{\alpha}_1, \overline{\alpha}_2, \overline{\delta}$ respectively.

Let us also recall that the automorphism $A$ which has been defined in $(3-6)$ is an inner automorphism of $E_7$ which preserves $E_6$. Moreover $A$ maps bijectively $E_6^{\overline{\alpha}_1}$ onto $E_6^{-\overline{\alpha}_1}$ and is the unique $\mathfrak{l}'_\theta$–isomorphism between these spaces which sends $X_{\overline{\alpha}_1}$ on $X_{-\overline{\alpha}_1}$ (see Proposition 3.3.1).

**Theorem 4.1.1.** *Let us define the product of two elements of $E_6^{\overline{\alpha}_1}$ as follows:*

$$(4-1)\ \ \forall x_1, x_2 \in E_6^{\overline{\alpha}_1} \quad x_1 \bullet x_2 = w_2[x_1, w_1 w_2 x_2] = [X_{-\overline{\alpha}_2}, [x_1, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]]].$$

*The map $(x_1, x_2) \longmapsto x_1 \bullet x_2$ is bilinear from $E_6^{\overline{\alpha}_1} \times E_6^{\overline{\alpha}_1}$ into $E_6^{\overline{\alpha}_1}$. Moreover $(E_6^{\overline{\alpha}_1}, \bullet)$ is (isomorphic to) the octonion algebra over $\mathbb{C}$, with identity element $X_{\overline{\alpha}_1}$ and with norm $Q_1$ defined by*

$$\forall x \in E_6^{\overline{\alpha}_1} \quad Q_1(x) = b(x, Ax).$$

*Proof.* First of all notice that the equality

$$w_2[x_1, w_1 w_2 x_2] = [X_{-\overline{\alpha}_2}, [x_1, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]]]$$

follows from an easy calculation, essentially made in $\mathfrak{sl}_3$. It is also obvious that this product is bilinear.

Let us now check that $x_1 \bullet x_2 \in E_6^{\overline{\alpha}_1}$. From basic facts for $\mathfrak{sl}_3$ we know that $w_1 w_2(\overline{\alpha}_1) = \overline{\alpha}_2$, and an easy calculation shows that $w_1 w_2 : E_6^{\overline{\alpha}_1} \longrightarrow E_6^{\overline{\alpha}_2}$ is an isomorphism sending $X_{\overline{\alpha}_1}$ on $X_{\overline{\alpha}_2}$. Hence

$$[x_1, w_1 w_2 x_2] \in [E_6^{\overline{\alpha}_1}, E_6^{\overline{\alpha}_2}] \subset E_6^{\overline{\alpha}_1 + \overline{\alpha}_2} = E_6^{\overline{\delta}}.$$

As $w_2$ maps $E_6^{\overline{\delta}}$ bijectively on $E_6^{\overline{\alpha}_1}$, we see that the product $x_1 \bullet x_2$ of two elements of $E_6^{\overline{\alpha}_1}$ is again in $E_6^{\overline{\alpha}_1}$. Another way to prove this would have been, using the second expression for the product $\bullet$ and the $A_2$–grading, to remark that successively we have $[X_{\overline{\alpha}_2}, x_2] \in E_6^{\overline{\delta}}$, $[X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]] \in E_6^{\overline{\alpha}_2}$, $[x_1, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]] \in E_6^{\overline{\delta}}$, and finally

$$x_1 \bullet x_2 = [X_{-\overline{\alpha}_2}, [x_1, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]]] \in E_6^{\overline{\alpha}_1}.$$

To visiualize the definition of $\bullet$, see Figure II below.

In order to finish the proof of the theorem, we remember from Proposition 2.1.1 that it will be enough (as $\dim E_6^{\overline{\alpha}_1} = 8$) to prove that the $\bullet$–product on $E_6^{\overline{\alpha}_1}$ admits an identity and satisfies the composition rule

$$\forall x_1, x_2 \in E_6^{\overline{\alpha}_1} \quad Q_1(x_1 \bullet x_2) = Q_1(x_1)Q_1(x_2).$$

More precisely we have to prove that

$$(4-2) \qquad \forall x_1, x_2 \in E_6^{\overline{\alpha}_1} \qquad b(x_1 \bullet x_2, A(x_1 \bullet x_2)) = b(x_1, Ax_1)b(x_2, Ax_2).$$

– Let us first prove that $X_{\overline{\alpha}_1}$ is an identity for the product $\bullet$. We have :

$$\begin{aligned} X_{\overline{\alpha}_1} \bullet x_2 &= [X_{-\overline{\alpha}_2}, [X_{\overline{\alpha}_1}, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]]] = [X_{-\overline{\alpha}_2}, [[X_{\overline{\alpha}_1}, X_{-\overline{\alpha}_1}], [X_{\overline{\alpha}_2}, x_2]]] \\ &= [X_{-\overline{\alpha}_2}, [-H_{\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]] = [X_{-\overline{\alpha}_2}, -\overline{\delta}(H_{\overline{\alpha}_1})[X_{\overline{\alpha}_2}, x_2]] \\ &= [X_{-\overline{\alpha}_2}, -[X_{\overline{\alpha}_2}, x_2]] = -[H_{\overline{\alpha}_2}, x_2] = -\overline{\alpha}_1(H_{\overline{\alpha}_2})x_2 = x_2. \end{aligned}$$

Similarly one can prove that $x_1 \bullet X_{\overline{\alpha}_1} = x_1$. Hence $X_{\overline{\alpha}_1}$ is an identity element for the product $\bullet$.

– Let us now prove that relation $(4-2)$ holds. We have :

$$\begin{aligned} b(x_1 \bullet x_2, A(x_1 \bullet x_2)) &= b\big(w_2[x_1, w_1w_2x_2], Aw_2[x_1, w_1w_2x_2]\big) \\ &= b\big(w_2[x_1, w_1w_2x_2], w_2w_2^{-1}Aw_2[x_1, w_1w_2x_2]\big) \\ (4-3) \qquad &= b\big([x_1, w_1w_2x_2], w_2^{-1}Aw_2[x_1, w_1w_2x_2]\big). \end{aligned}$$

Consider the automorphism $w_2^{-1}Aw_2$ of $E_6$. As $w_2 : E_6^{\overline{\delta}} \longrightarrow E_6^{\overline{\alpha}_1}$, and $w_2^{-1} : E_6^{-\overline{\alpha}_1} \longrightarrow E_6^{-\overline{\delta}}$, we obtain that $w_2^{-1}Aw_2$ maps $E_6^{\overline{\delta}}$ on $E_6^{-\overline{\delta}}$.

Recall that the subgroup $\overline{W}$ of $Aut(\mathfrak{g}, \mathfrak{h})$ generated by $w_1, w_2, w_3$ stabilizes $D_4$ and $\mathbf{Spin(8)}$ (because $\overline{W}$ stabilizes $\mathfrak{h}_\theta$ and because $D_4$ is the centralizer of $\mathfrak{h}_\theta$ in $E_6$). Recall also that $A$ commutes with the elements of $\mathbf{Spin(8)}$.

Therefore for $g \in \mathbf{Spin(8)}$ and for $z \in E_6^{\overline{\delta}}$ we obtain:

$$\begin{aligned} w_2^{-1}Aw_2gz &= w_2^{-1}Aw_2gw_2^{-1}w_2z = w_2^{-1}w_2gw_2^{-1}Aw_2z \\ &= gw_2^{-1}Aw_2z. \end{aligned}$$

This means that $w_2^{-1}Aw_2$ intertwines the D$_4$–actions between $E_6^{\overline{\delta}}$ and $E_6^{-\overline{\delta}}$. Therefore there exists a constant $\lambda$ such that $w_2^{-1}Aw_{2|E_6^{\overline{\delta}}} = \lambda A_{|E_6^{\overline{\delta}}}$ (Proposition 3.3.1). A simple calculation shows that $w_2^{-1}Aw_2X_{\overline{\delta}} = X_{-\overline{\delta}} = AX_{\overline{\delta}}$. Hence $\lambda = 1$ and $w_2^{-1}Aw_{2|E_6^{\overline{\delta}}} = A_{|E_6^{\overline{\delta}}}$.

Therefore we get from $(4-3)$:

$$b\big(x_1 \bullet x_2, A(x_1 \bullet x_2)\big) = b\big([x_1, w_1w_2x_2], A[x_1, w_1w_2x_2]\big).$$

Then, from the bracket-multiplicative property seen in Proposition 3.4.3 we obtain:

$$b\big(x_1 \bullet x_2, A(x_1 \bullet x_2)\big) = b(x_1, Ax_1)b\big(w_1w_2x_2, Aw_1w_2x_2\big).$$

But the map

$$x_2 \longmapsto b\big(w_1w_2x_2, Aw_1w_2x_2\big)$$

$E_6$ :

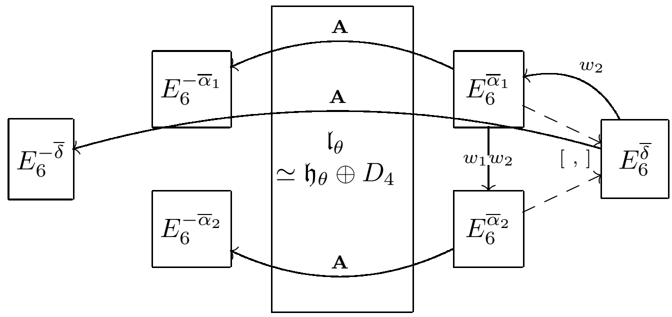



FIGURE II

is invariant under $\mathbf{Spin(8)}$. Moreover, as $w_1 w_2 X_{\overline{\alpha}_1} = X_{\overline{\alpha}_2}$, we see that this map takes the value 1 on $X_{\overline{\alpha}_1}$. Hence

$$\forall x \in E_6^{\overline{\alpha}_1} \quad b\big(w_1 w_2 x, A w_1 w_2 x\big) = b(x, Ax).$$

This proves $(4-2)$ and the theorem. □

## 5. THE TRIALITY PRINCIPLE, THE AUTOMORPHISM GROUP AND THE DERIVATION ALGEBRA OF $(E_6^{\overline{\alpha}_1}, \bullet)$

5.1. The aim of this section is to provide proofs for three results concerning octonions which are easier than the usual ones. It also seems to us that the following formulation of these classical results sheds new light on the structures involved.

**Proposition 5.1.1** (Triality Principle). *Let $g_1 \in L_\theta$. Then there exists $g_2, g_3 \in L_\theta$ such that*

$$\forall x, y \in E_6^{\overline{\alpha}_1} \quad g_1(x \bullet y) = g_2 x \bullet g_3 y.$$

*Proof.* From Theorem 4.1.1 we have:

$$\begin{aligned} g_1(x \bullet y) &= g_1 w_2[x, w_1 w_2 y] = w_2[w_2^{-1} g_1 w_2 x, w_1 w_2 w_2^{-1} w_1^{-1} w_2^{-1} g_1 w_2 w_1 w_2 y] \\ &= g_2 x \bullet g_3 y, \end{aligned}$$

where $g_2 = w_2^{-1} g_1 w_2$ and $g_3 = w_2^{-1} w_1^{-1} w_2^{-1} g_1 w_2 w_1 w_2$. (Compare with the proof of Theorem 3.2.1. p. 42 of [S-V]). □

**Theorem 5.1.2.** *Let* $\operatorname{Aut}(E_6^{\overline{\alpha}_1}, \bullet)$ *be the group of automorphisms of* $(E_6^{\overline{\alpha}_1}, \bullet)$. *Then* $\operatorname{Aut}(E_6^{\overline{\alpha}_1}, \bullet) = \mathbf{exp(G_2)}$, *acting on* $E_6^{\overline{\alpha}_1}$ *by the Adjoint action.*

*Proof.* First let $g \in \mathbf{exp(G_2)}$. As for $x, y \in E_6^{\overline{\alpha}_1}$ we have $x \bullet y = w_2[x, w_1 w_2 y]$; it is obvious to see that $g(x \bullet y) = (gx) \bullet (gy)$, and therefore the Adjoint action of $g$ on $E_6^{\overline{\alpha}_1}$ defines an automorphism of the octonions. Moreover as the action of $\mathbf{Spin(8)}_{X_{\overline{\alpha}_2}}$ on $E_6^{\overline{\alpha}_1}$ (which is another realization of the $Spin(7)$ representation) is faithful, we obtain an embedding $\mathbf{exp(G_2)} \longrightarrow \operatorname{Aut}(E_6^{\overline{\alpha}_1}, \bullet)$.

Conversely let $\gamma \in \operatorname{Aut}(E_6^{\overline{\alpha}_1}, \bullet)$. Then $\gamma$ centralizes the identity element: $\gamma X_{\overline{\alpha}_1} = X_{\overline{\alpha}_1}$. Moreover an easy proof shows that $\gamma$ is necessarily an isometry for the quadratic form $b(X, AX)$ (see [S-V], Corollary 1.2.4. p.6). Therefore there exists $g \in \mathbf{Spin(7)} = \mathbf{Spin(8)}_{X_{\overline{\alpha}_1}}$ such that

$$\forall x \in E_6^{\overline{\alpha}_1} \qquad \gamma x = gx.$$

Let us now write the condition for $g$ to be an automorphism:
(5 − 1)
$$\forall x, y \in E_6^{\overline{\alpha}_1}, \quad g[X_{-\overline{\alpha}_2}, [x, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, y]]]] = [X_{-\overline{\alpha}_2}, [gx, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, gy]]]].$$
Let us denote by
$$\Phi : E_6^{\overline{\alpha}_1} \otimes E_6^{\overline{\alpha}_1} \longrightarrow E_6^{\overline{\alpha}_1}$$
the linear map induced by the octonion product
$$(x, y) \mapsto x \bullet y = [X_{-\overline{\alpha}_2}, [x, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, y]]]].$$

Suppose now that any $g \in \mathbf{Spin(7)}$ defines an automorphism. Then from $(5-1)$ we see that $\Phi$ becomes $\mathbf{Spin(7)}$-equivariant. But the decomposition of $E_6^{\overline{\alpha}_1}$ under $\mathbf{Spin(7)}$ is well known:
$$E_6^{\overline{\alpha}_1} = \mathbb{C}X_{\overline{\alpha}_1} \oplus U_{\overline{\alpha}_1}$$
where $\mathbb{C}X_{\overline{\alpha}_1}$ is the trivial representation and where $U_{\overline{\alpha}_1}$ is the irreducible (vectorial) representation of dimension 7. Hence
$$E_6^{\overline{\alpha}_1} \otimes E_6^{\overline{\alpha}_1} \simeq \mathbb{C}X_{\overline{\alpha}_1} \oplus U_{\overline{\alpha}_1} \oplus U_{\overline{\alpha}_1} \oplus (U_{\overline{\alpha}_1} \otimes U_{\overline{\alpha}_1}).$$
As $U_{\overline{\alpha}_1} \otimes U_{\overline{\alpha}_1} \simeq W_1 \oplus W_2$, where $W_1 \simeq B_3$ is the Adjoint representation and where $W_2$ is the irreducible 28–dimensional representation on symmetric $7 \times 7$ matrices[1], we must have $\Phi(U_{\overline{\alpha}_1} \otimes U_{\overline{\alpha}_1}) = 0$. But the hyperplane $U_{\overline{\alpha}_1}$ is certainly different from the “singular” quadric in $E_6^{\overline{\alpha}_1}$ defined by $b(X, AX) = 0$, and hence there exist generic elements of the $PV$ $(L_\theta, E_6^{\overline{\alpha}_1})$ which are contained in $U_{\overline{\alpha}_1}$. This shows that $U_{\overline{\alpha}_1} \bullet U_{\overline{\alpha}_1} \neq 0$ and implies that $\Phi(U_{\overline{\alpha}_1} \otimes U_{\overline{\alpha}_1}) \neq 0$. Therefore the automorphism group of the octonions is a strict closed subgroup of $\mathbf{Spin(7)}$ containing $\mathbf{exp(G_2)}$. Then from Lemma 3.4.2 we know that it must be either $\mathbf{exp(G_2)}$ or $\mathbf{exp(G_2)} \times \{\pm 1\}$. But $-1$ acts on $E_6^{\overline{\alpha}_1}$ by $-\mathrm{Id}$ and therefore cannot define an automorphism. Hence the automorphism group is $\mathbf{exp(G_2)}$. □

As a consequence, we obtain:

**Theorem 5.1.3.** *Let* $\mathrm{Der}(E_6^{\overline{\alpha}_1}, \bullet)$ *be the Lie algebra of derivations of* $(E_6^{\overline{\alpha}_1}, \bullet)$. *Then* $\mathrm{Der}(E_6^{\overline{\alpha}_1}, \bullet) = G_2$ *acting on* $E_6^{\overline{\alpha}_1}$ *by the adjoint action.*

## 6. Real forms of $E_6$ and real octonions

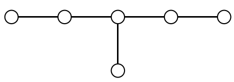
6.1. Let $E_{6,\mathbb{R}}$ be a real form of $E_6$ and let $E_{6,\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, where $\mathfrak{k}$ is a maximal compact subalgebra. We denote by $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and choose a maximal abelian subalgebra $\mathfrak{h}_\mathbb{R}$ of $E_{6,\mathbb{R}}$ containing $\mathfrak{a}$. The complexification $\mathfrak{h}$ of $\mathfrak{h}_\mathbb{R}$ is then a Cartan subalgebra of $E_6$. Concerning the roots $R$ of the pair $(E_6, \mathfrak{h})$ we will use the same notation as in section 3.

Let $\rho : \mathfrak{h}^* \longrightarrow \mathfrak{a}^*_\mathbb{C}$ be the restriction map. Let $R_0$ be the set of roots $\alpha \in R$ such that $\rho(\alpha) = 0$. It is well known that $R_0$ is a root system (the so-called *compact* roots ). We will denote by $\Sigma = \Sigma(E_{6,\mathbb{R}}, \mathfrak{a})$ the set of restricted roots (*i.e.* the set of non-zero restrictions of $R$) and will also choose a basis $\Psi$ of $R$ such that

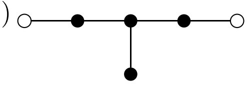
1) $\Psi_0 = \Psi \cap R_0$ is a basis of $R_0$,
2) $\rho(\Psi \setminus \Psi_0) = \pi$ is a basis of $\Sigma$.

[1]In fact, as $U_{\overline{\alpha}_1} \simeq U^*_{\overline{\alpha}_1}$, we have $U_{\overline{\alpha}_1} \otimes U_{\overline{\alpha}_1} \simeq \mathrm{Hom}(U_{\overline{\alpha}_1}, U_{\overline{\alpha}_1})$, and then the preceding decomposition is just the Cartan decomposition of $\mathrm{Hom}(U_{\overline{\alpha}_1}, U_{\overline{\alpha}_1})$ .

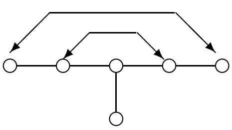
As is well known ([Wa]) the real forms of $E_6$ are then classified by their Satake diagram, which is the Dynkin diagram of $R$ (or $E_6$) where the vertices in $\Psi_0$ are coloured black, the remainder white and two vertices representing elements $\alpha, \beta \in \Psi \setminus \Psi_0$ such that $\rho(\alpha) = \rho(\beta)$ are joined by an arrow. The non-compact Satake diagrams for $E_6$ are listed in the following (see [Wa]):

1) 

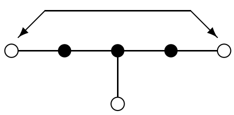
$E_{6,C_2}$ split

2) 

$E_{6,F_4}$

3) 

$E_{6,A_1\times A_5}$, quasi-split

4) 

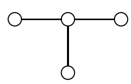
$E_{6,D_5\times\mathbb{R}}$

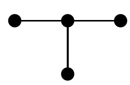
FIGURE III

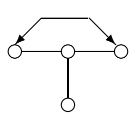
(Here the second index indicates the type of the maximal compact subalgebra $\mathfrak{k}$.)

For $E \subset \pi$ we denote by $\mathfrak{p}_E$ the standard parabolic subalgebra of $E_{6,\mathbb{R}}$ attached to $E$ ([Wa]). We are interested in algebras $\mathfrak{p}_E$ which are real forms of $\mathfrak{p}_\theta$, where $\mathfrak{p}_\theta$ is the standard parabolic subalgebra of $E_6$ defined in section 3.1.

Obviously the compact real form has no proper parabolic subalgebra. On the other hand each of the other forms has a unique parabolic subalgebra which is a real form of $\mathfrak{p}_\theta$. From now on we will fix $E = \rho(\theta)\setminus\{0\}$; then we have $\mathfrak{p}_{E^{\mathbb{C}}} = \mathfrak{p}_\theta$ (for a proof see [Ru-5], Lemme 3.2.1). Let $\mathfrak{a}_E = \{H \in \mathfrak{a}, \lambda(H) = 0, \forall\lambda \in E\}$ and let $\mathfrak{l}_E$ be the centralizer of $\mathfrak{a}_E$ in $E_{6,\mathbb{R}}$. Then $\mathfrak{l}_E$ is a real form of of $\mathfrak{l}_\theta$ ([Ru-5], Théorème 4.3) and $\mathfrak{l}'_E = [\mathfrak{l}_E, \mathfrak{l}_E]$ is a real form of $D_4$. From the Satake diagrams in Figure III it is easy to obtain the following:

1)$\mathfrak{l}'_E =$  $= \mathfrak{so}(4,4)$, split real form of $\mathrm{D}_4$ for $E_{6,C_2}$,

2)$\mathfrak{l}'_E =$  $= \mathfrak{so}(8)$, compact real form of $\mathrm{D}_4$ for $E_{6,F_4}$,

3)$\mathfrak{l}'_E =$  $= \mathfrak{so}(5,3)$, quasi-split form of $D_4$ for $E_{6,A_1\times A_5}$,

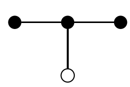
4)$\mathfrak{l}'_E =$  $= \mathfrak{so}(7,1)$ for $E_{6,D_5\times\mathbb{R}}$

FIGURE IV

(the numbers in this list refer to Figure III).

For $\lambda \in \Sigma$, we denote by $E_{6,\mathbb{R}}^{\lambda}$ the corresponding root space in $E_{6,\mathbb{R}}$. For $\mu \in \Sigma \setminus \langle E \rangle$, define

$$w_{\mu}(E) = \{\lambda \in \Sigma, \lambda = \mu \bmod \langle \pi \rangle\},$$
$$E_{6,\mathbb{R}}^{\overline{\mu}} = \sum_{\lambda \in \omega_{\mu}(E)} E_{6,\mathbb{R}}^{\lambda},$$ [2]

and

$$d_1(E) = \sum_{\mu \in \pi \setminus E} E_{6,\mathbb{R}}^{\overline{\mu}}. \tag{6-1}$$

It is known that $(\mathfrak{l}_E, d_1(E))$ is a real form of the PV $(\mathfrak{l}_\theta, d_1(\theta))$ ([Ru-5], Théorème 4.3).

We first focus our attention to the real forms $E_{6,C_2}$ and $E_{6,F_4}$ (cases 1) and 2) respectively). As far as common properties are discussed we just write $E_{6,\mathbb{R}}$ for any of these two forms. The key point here is that the roots $\alpha_1$ and $\alpha_2$ (as in $(3-1)$) are white and are not joined by an arrow to another root in the corresponding Satake diagrams. Then, from the proof of Lemma 3.2.1 in [Ru-5], one obtains that $E_{6,\mathbb{R}}^{\pm\overline{\rho(\alpha_1)}}$, $E_{6,\mathbb{R}}^{\pm\overline{\rho(\alpha_2)}}$, $E_{6,\mathbb{R}}^{\pm\overline{\rho(\delta)}}$ are real forms of $E_6^{\pm\overline{\alpha}_1}$, $E_6^{\pm\overline{\alpha}_2}$, $E_6^{\pm\overline{\delta}}$ respectively (recall that $\delta$ is the highest root of $E_6$). This means that the real forms $E_{6,C_2}$ and $E_{6,F_4}$ are split relatively to the decomposition

$$E_6 = E_6^{-\overline{\delta}} \oplus E_6^{-\overline{\alpha}_1} \oplus E_6^{-\overline{\alpha}_2} \oplus \mathfrak{l}_\theta \oplus E_6^{\overline{\alpha}_1} \oplus E_6^{\overline{\alpha}_2} \oplus E_6^{\overline{\delta}}$$

which is visualized in Figure I.

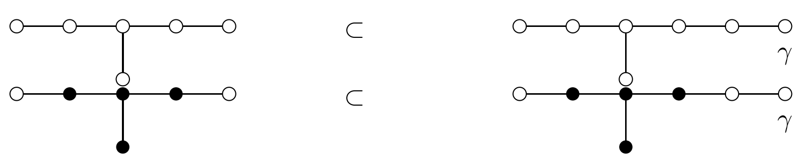
Moreover the elements $H_{\overline{\alpha}_1}, H_{\overline{\alpha}_2}, H_{\overline{\delta}}$ belong to $E_{6,\mathbb{R}}$, and we can choose elements $X_{\pm\overline{\alpha}_1} \in E_{6,\mathbb{R}}^{\pm\overline{\rho(\alpha_1)}}$, $X_{\pm\overline{\alpha}_2} \in E_{6,\mathbb{R}}^{\pm\overline{\rho(\alpha_2)}}$, $X_{\pm\overline{\delta}} \in E_{6,\mathbb{R}}^{\pm\overline{\rho(\delta)}}$ such that the real span of these elements is a split $A_2 \subset E_{6,\mathbb{R}}$. We will denote $A_{2,\mathrm{split}}$ this subalgebra. The centralizer of $A_{2,\mathrm{split}}$ in $E_{6,C_2}$ is a split form of $G_2$ in $\mathfrak{l}'_E \simeq \mathfrak{so}(4,4)$; we will denote it $G_{2,\mathrm{split}}$. The centralizer of $A_{2,\mathrm{split}}$ in $E_{6,F_4}$ is a compact form of $G_2$ in $\mathfrak{l}'_E \simeq \mathfrak{so}(8,\mathbb{R})$; we will denote it $G_{2,\mathrm{c}}$. Of course, as $(A_2, G_2)$ is a dual pair in $E_6$, $(A_{2,\mathrm{split}}, G_{2,\mathrm{split}})$ is a dual pair in $E_{6,C_2}$ and $(A_{2,\mathrm{split}}, G_{2,\mathrm{c}})$ is a dual pair in $E_{6,F_4}$.

The elements $w_1$, $w_2$ and $w_3$ defined in section 4 can now be viewed as automorphisms of $E_{6,\mathbb{R}}$.

It is also important to notice that there are canonical embeddings

$$E_{6,C_2} \longrightarrow E_{7,A_4},$$
$$E_{6,F_4} \longrightarrow E_{7,E_6\times\mathbb{R}}$$

corresponding to the following embeddings of Satake diagrams:




(where $E_{7,A_4}$ and $E_{7,E_6\times\mathbb{R}}$ denote the real forms of $E_7$ whose maximal compact subalgebra is of type $A_4$ and $E_6 \times \mathbb{R}$ respectively).

[2] $E_{6,\mathbb{R}}^{\overline{\mu}}$ is also the eigenspace of $\mathrm{ad}(\mathfrak{a}_E)$ for the eigenvalue $\overline{\mu} = \mu_{|\mathfrak{a}_E}$.

As the root $\gamma$ is white, the real forms $E_{7,A_4}$ and $E_{7,E_6\times\mathbb{R}}$ split relatively to the decomposition $(3-5)$, and moreover we have

$$E_7^{\overline{\overline{0}}} \cap E_{7,A_4} = E_{6,C_2} \oplus \mathbb{R}H_{\overline{\overline{\gamma}}}, \quad E_7^{\overline{\overline{0}}} \cap E_{7,E_6\times\mathbb{R}} = E_{6,F_4} \oplus \mathbb{R}H_{\overline{\overline{\gamma}}}$$

(this again is a consequence of Théorème 4.3 of [Ru-5]).

Therefore we can choose $X_{\overline{\overline{\gamma}}} \in E_7^{\overline{\overline{\gamma}}} \cap E_{7,A_4}$ (*resp.* $E_7^{\overline{\overline{\gamma}}} \cap E_{7,E_6\times\mathbb{R}}$) and $X_{-\overline{\overline{\gamma}}} \in E_7^{-\overline{\overline{\gamma}}} \cap E_{7,A_4}$ (*resp.* $E_7^{-\overline{\overline{\gamma}}} \cap E_{7,E_6\times\mathbb{R}}$) such that the automorphism

$$A = \exp\operatorname{ad} X_{\overline{\overline{\gamma}}} \exp\operatorname{ad} X_{-\overline{\overline{\gamma}}} \exp\operatorname{ad} X_{\overline{\overline{\gamma}}}$$

restricts to an involution of $E_{6,C_2}$ (*resp.* $E_{6,F_4}$), and also verifies:

$$\begin{aligned} & A(E_{6,C_2}^{\overline{\rho(\alpha_1)}}) = E_{6,C_2}^{-\overline{\rho(\alpha_1)}}, A(E_{6,C_2}^{\overline{\rho(\alpha_2)}}) = E_{6,C_2}^{-\overline{\rho(\alpha_2)}}, A(E_{6,C_2}^{\overline{\rho(\delta)}}) = E_{6,C_2}^{-\overline{\rho(\delta)}} \\ (\textit{resp.} \quad & A(E_{6,F_4}^{\overline{\rho(\alpha_1)}}) = E_{6,F_4}^{-\overline{\rho(\alpha_1)}}, A(E_{6,F_4}^{\overline{\rho(\alpha_2)}}) = E_{6,F_4}^{-\overline{\rho(\alpha_2)}}, A(E_{6,F_4}^{\overline{\rho(\delta)}}) = E_{6,F_4}^{-\overline{\rho(\delta)}}), \\ & A(X_{\overline{\alpha}_1}) = X_{-\overline{\alpha}_1}, A(X_{\overline{\alpha}_2}) = X_{-\overline{\alpha}_2}, A(X_{\overline{\delta}}) = X_{-\overline{\delta}}. \end{aligned}$$

In other words the analogue of Proposition 3.3.1 is true for these two real forms.

**Theorem 6.1.1.** *Let $E_{6,\mathbb{R}}$ denote either $E_{6,C_2}$ or $E_{6,F_4}$. We define a product $\bullet$ on $E_{6,\mathbb{R}}^{\overline{\rho(\alpha_1)}}$ by*

$$\forall x_1, x_2 \in E_6^{\overline{\alpha}_1} \quad x_1 \bullet x_2 = w_2[x_1, w_1 w_2 x_2] = [X_{-\overline{\alpha}_2}, [x_1, [X_{-\overline{\alpha}_1}, [X_{\overline{\alpha}_2}, x_2]]]].$$

*The algebra $(E_{6,C_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ is the split octonion algebra over $\mathbb{R}$, with identity element $X_{\overline{\alpha}_1}$ and with norm $Q_1$ defined by*

$$\forall x \in E_{6,C_2}^{\overline{\rho(\alpha_1)}} \quad Q_1(x) = b(x, Ax).$$

*The group $\mathrm{Aut}(\mathrm{E}_{6,\mathrm{C}_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ of automorphisms of $(E_{6,C_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ is equal to $\exp(G_{2,\mathrm{split}})$ acting on $E_{6,C_2}^{\overline{\rho(\alpha_1)}}$ by the Adjoint action.*

*The algebra $(E_{6,F_4}^{\overline{\rho(\alpha_1)}}, \bullet)$ is the anisotropic octonion algebra over $\mathbb{R}$, with identity element $X_{\overline{\alpha}_1}$ and with norm $Q_1$ defined by*

$$\forall x \in E_{6,F_4}^{\overline{\rho(\alpha_1)}} \quad Q_1(x) = b(x, Ax).$$

*The group $\mathrm{Aut}(\mathrm{E}_{6,\mathrm{F}_4}^{\overline{\rho(\alpha_1)}}, \bullet)$ of automorphisms of $(E_{6,F_4}^{\overline{\rho(\alpha_1)}}, \bullet)$ is equal to $\exp(G_{2,\mathrm{c}})$ acting on $E_{6,F_4}^{\overline{\rho(\alpha_1)}}$ by the Adjoint action.*

*Proof.* From what we have discussed previously in this section we deduce that, as in the complex case, the algebras $(E_{6,C_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ and $(E_{6,F_4}^{\overline{\rho(\alpha_1)}}, \bullet)$ are real composition algebras of dimension 8, with the given composition quadratic forms. From Proposition 2.1.1 we know that there are only two such composition algebras: the split octonions $\mathbb{O}_s$ and the anisotropic octonions $\mathbb{O}_a$. The first one has a composition quadratic form of signature $(4,4)$, the second one has a composition quadratic form of signature $(8,0)$. Therefore, from the list of the $\mathfrak{l}'_E$'s given above, we see that $(E_{6,C_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ is the split octonion algebra, whereas $(E_{6,F_4}^{\overline{\rho(\alpha_1)}}, \bullet)$ is the anisotropic one.

The two real forms $E_{6,C_2}^{\overline{\rho(\alpha_1)}}$ and $E_{6,F_4}^{\overline{\rho(\alpha_1)}}$ of $E_6^{\overline{\alpha}_1}$ define two real structures on the algebraic group $\mathrm{Aut}(E_6^{\overline{\alpha}_1}, \bullet) = \exp(G_2)$ ([S-V], Prop. 2.4.6 p.35), and the corresponding groups of real points are the groups $\mathrm{Aut}(\mathrm{E}_{6,\mathrm{C}_2}^{\overline{\rho(\alpha_1)}}, \bullet)$ and $\mathrm{Aut}(\mathrm{E}_{6,\mathrm{F}_4}^{\overline{\rho(\alpha_1)}}, \bullet)$, respectively (see the introduction of Chapter 2 in [S-V]). On the other hand we obviously have $\exp(G_{2,\mathrm{split}}) \subset \mathrm{Aut}(\overline{\mathrm{E}^{\rho(\alpha_1)}_{6,\mathrm{C}_2}}, \bullet)$ and $\exp(G_{2,\mathrm{c}}) \subset \mathrm{Aut}(\overline{\mathrm{E}^{\rho(\alpha_1)}_{6,\mathrm{F}_4}}, \bullet)$. It is also well known that there is only one (connected) algebraic group of type $G_2$ since the roots of $G_2$ span the full lattice of weights, in other words $\exp(G_2)$ is of simply connected type. By Theorem 35.3 in [Hu], we know that the group of real points of an algebraic group of simply connected type is connected in the usual topology. Therefore we obtain $\exp(G_{2,\mathrm{split}}) = \mathrm{Aut}(\overline{\mathrm{E}^{\rho(\alpha_1)}_{6,\mathrm{C}_2}}, \bullet)$ and $\exp(G_{2,\mathrm{c}}) = \mathrm{Aut}(\overline{\mathrm{E}^{\rho(\alpha_1)}_{6,\mathrm{F}_4}}, \bullet)$. $\square$

We now consider the real forms $E_{6,A_1\times A_5}$ (case 3) in Figure III) and $E_{6,D_5\times\mathbb{R}}$ (case 4)). Again, as far as common properties are discussed, we just write $E_{6,\mathbb{R}}$ for any of these two forms. From Figure IV we know that the corresponding real forms $\mathfrak{l}'_E$ of $\mathfrak{l}'_\theta$ are respectively $\mathfrak{so}(5,3)$ and $\mathfrak{so}(7,1)$.

Let $G_\mathbb{R}$ be the analytic subgroup of $G$ with Lie algebra $E_{6,\mathbb{R}}$ and let $L_E$ (resp. $L'_E$) denote the analytic subgroup of $G_\mathbb{R}$ with Lie algebras $\mathfrak{l}_E$ (resp. $\mathfrak{l}'_E$). The subgroups $L'_E$ are real forms of $\mathbf{Spin(8)}$. We denote them by $\mathbf{Spin(5,3)}$ and $\mathbf{Spin(7,1)}$ in case 3) and 4) respectively (as in the complex case, this is justified because these groups are effectively isomorphic to the groups $\mathrm{Spin}(5,3)$ and $\mathrm{Spin}(7,1)$, respectively).

Let $\mathcal{L}$ be a semi-simple complex Lie algebra and let $\mathcal{L}_\mathbb{R}$ be a real form of $\mathcal{L}$. Recall from Iwahori [Iw], that the finite dimensional complex irreducible representations $(\mathcal{L}, V)$ split into two families:

– either $(\mathcal{L}_\mathbb{R}, V)$ is $\mathbb{R}-$ irreducible

– or $(\mathcal{L}_\mathbb{R}, V)$ is $\mathbb{R}-$ reducible and then $V = V_\mathbb{R} + iV_\mathbb{R}$ where $V_\mathbb{R}$ is a real form of $V$ which is invariant and $\mathbb{R}-$ irreducible under $\mathcal{L}_\mathbb{R}$.

It is well known that the groups $\mathrm{Spin}(5,3)$ and $\mathrm{Spin}(7,1)$ have three 8–dimensional $\mathbb{C}$–irreducible representations, namely the two non-equivalent Spin representations and the so-called vectorial representation which is the only 8–dimensional representation which factors through $\mathrm{SO}(5,3)$ and $\mathrm{SO}(7,1)$, respectively. The vectorial representation is also the only 8–dimensional representation which is $\mathbb{R}$–reducible.

**Theorem 6.1.2.** *Recall that $E_{6,\mathbb{R}}$ is either $E_{6,A_1\times A_5}$ or $E_{6,D_5\times\mathbb{R}}$. Recall also that in the first case we have $L'_E = \mathbf{Spin(5,3)}$ and in the second case we have $L'_E = \mathbf{Spin(7,1)}$. We denote by $\sigma$ the conjugation of $E_6$ relatively to $E_{6,\mathbb{R}}$.*

*1) The complex representations $(L'_E, E_6^{\overline{\alpha}_1})$ and $(L'_E, E_6^{\overline{\alpha}_2})$ are the two* Spin *representations of $L'_E$. They are $\mathbb{R}$–irreducible and non-$\mathbb{C}$–equivalent. However the conjugation $\sigma$ restricts to a $\mathbb{R}$–linear isomorphism between $E_6^{\overline{\alpha}_1}$ and $E_6^{\overline{\alpha}_2}$ which is $L'_E$–equivariant.*

*2) The complex representation $(L'_E, E_6^{\overline{\delta}})$ is $\mathbb{R}$–reducible and $(L'_E, \overline{E^{\rho(\delta)}_{6,\mathbb{R}}})$ is the real vectorial representation of $L'_E$ .*

*3) The real form $d_1(E)$ has been defined in $(6-1)$. We have $d_1(E) = \{x + \sigma(x), x \in E_6^{\overline{\alpha}_1}\} = \{y + \sigma(y), y \in E_6^{\overline{\alpha}_2}\}$. The real representation $(L'_E, d_1(E))$ is $\mathbb{R}$–isomorphic to $(L'_E, E_6^{\overline{\alpha}_1})$ and $(L'_E, E_6^{\overline{\alpha}_2})$ and is therefore $\mathbb{R}$–irreducible. Morever the $\mathbb{R}$–bilinear skew-symmetric mapping:*

$$\begin{aligned} d_1(E) \times d_1(E) &\longrightarrow \overline{E^{\rho(\delta)}_{6,\mathbb{R}}}, \\ (X,Y) &\longmapsto [X,Y] \end{aligned}$$

*is $L'_E$–equivariant and non-zero. Therefore the real vectorial representation occurs in the exterior square $\Lambda^2(Spin)$ of the Spin representation of the groups* $\mathbf{Spin(5,3)}$ *and* $\mathbf{Spin(7,1)}$.

*Proof.* Denote by $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weights of $D_4$ relative to the basis $\beta_1, \beta_2, \beta_3, \beta_4$ (the roots are numbered as in $(3-1)$). It is easy to see that $\omega_1, \omega_4, \omega_3$ are the highest weight of the representations $(D_4, E_6^{\overline{\alpha}_1})$, $(D_4, E_6^{\overline{\alpha}_2})$ and $(D_4, E_6^{\overline{\delta}})$ relatively to $-\Psi$, respectively (see [Ru-2]). We have seen in section 3 that these three representations integrate to the group $\mathbf{Spin(8)}$ and hence to the group $L'_E = \mathbf{Spin(5,3)}$ or $\mathbf{Spin(7,1)}$. As these three representations correspond to distinct highest weights, they are not $\mathbb{C}$–equivalent.

Using Proposition 3.1.1 and Théorème 3.1.2 in [Ru-5] one obtains that in both cases the representations $(\mathfrak{l}'_E, E_6^{\overline{\alpha}_1})$ and $(\mathfrak{l}'_E, E_6^{\overline{\alpha}_2})$ are $\mathbb{R}$–irreducible. Hence $(L'_E, E_6^{\overline{\alpha}_1})$ and $(L'_E, E_6^{\overline{\alpha}_2})$ are $\mathbb{R}$–irreducible.

For $H \in \mathfrak{h}$ and $\mu \in \mathfrak{h}^*$, we define as usual a new linear form $\mu^\sigma$ by

$$\mu^\sigma(H) = \overline{\mu(\sigma(H))}.$$

Then for any root $\alpha \in R$ we have $\sigma(E_6^\alpha) = E_6^{\alpha^\sigma}$ and if $\alpha \in R_0$, then $\alpha^\sigma = -\alpha$. As $\alpha_1$ and $\alpha_2$ are in both cases white roots in the Satake diagram which are joined by an arrow, one knows from general facts that $\alpha_1^\sigma = \alpha_2 \bmod R_0$ and $\alpha_2^\sigma = \alpha_1 \bmod R_0$. Therefore $\sigma : E_6^{\overline{\alpha}_1} \longrightarrow E_6^{\overline{\alpha}_2}$ is a $\mathbb{R}$–linear isomorphism which is $L'_E$–equivariant. The first assertion is now proved.

From the proof of Théorème 4.3. in [Ru-5] one knows that the space $d_2(E) = [d_1(E), d_1(E)]$ is a real form of $d_2(\theta) = E_6^{\overline{\delta}}$ which of course is $L'_E$–invariant. Therefore $(L'_E, E_6^{\overline{\delta}})$ is $\mathbb{R}$–reducible. From what we have seen above before the statement of the theorem, we obtain that $(L'_E, d_2(E))$ is the real vectorial representation of $L'_E$. Moreover as $d_2(E) = E_6^{\overline{\delta}} \cap E_{6,\mathbb{R}} = E_{6,\mathbb{R}}^{\overline{\rho(\delta)}}$, the proof of the second assertion is now complete.

The real subspace $\{x + \sigma(x), x \in E_6^{\overline{\alpha}_1}\} = \{y + \sigma(y), y \in E_6^{\overline{\alpha}_2}\}$ of $E_{6,\mathbb{R}}$ is a real form of $d_1(\theta)$. Therefore it is equal to $d_1(E)$. The mappings

$$\begin{aligned} E_6^{\overline{\alpha}_1} &\longrightarrow d_1(E), & \qquad E_6^{\overline{\alpha}_2} &\longrightarrow d_1(E), \\ x &\longmapsto (x + \sigma(x)), & y &\longmapsto (y + \sigma(y)) \end{aligned}$$

are $L'_E$–equivariant $\mathbb{R}$–linear isomorphisms. Hence the representation $(L'_E, d_1(E))$ is $\mathbb{R}$–irreducible.

The proposed skew-symmetric $\mathbb{R}$–bilinear mapping $(X, Y) \longmapsto [X, Y]$ from $d_1(E) \times d_1(E) \longrightarrow E_{6,\mathbb{R}}^{\overline{\rho(\delta)}}$ is of course $L'_E$–equivariant and non-zero. Therefore there exists an $L'_E$-equivariant linear surjective mapping

$$\Phi : \Lambda^2(E_6^{\overline{\alpha}_1}) \longrightarrow E_{6,\mathbb{R}}^{\overline{\rho(\delta)}}.$$

Hence the real vectorial representation $E_{6,\mathbb{R}}^{\overline{\rho(\delta)}}$ occurs in $\Lambda^2(E_6^{\overline{\alpha}_1})$. The same is true for $\Lambda^2(E_6^{\overline{\alpha}_2})$. □

*Remark* 6.1.3. a) Notice that for the last two real forms $E_{6,A_1\times A_5}$ and $E_{6,D_5\times\mathbb{R}}$ concerned by the preceding theorem, the spaces $E_6^{\overline{\alpha}_1}$ and $E_6^{\overline{\alpha}_2}$ are not defined over $\mathbb{R}$, whereas the $L'_E$–invariant quadratic form on $E_{6,\mathbb{R}}^{\overline{\rho(\delta)}} = E_6^{\overline{\alpha}_1} \cap E_{6,\mathbb{R}}$ is of signature $(5,3)$ and $(7,1)$, respectively. Therefore it is hopeless to define a real octonion product on these spaces. In some sense one could say that the triality between these three spaces is not defined over $\mathbb{R}$.

b) Notice also that for the real forms $E_{6,C_2}$ and $E_{6,F_4}$ which give rise to the real octonions, the definition of octonion product $\bullet$ in Theorem 6.1.1 needs to embed these real forms of $E_6$ into real forms of $E_7$. Curiously enough the Satake diagrams of $E_{6,A_1\times A_5}$ and $E_{6,D_5\times\mathbb{R}}$ (see Figure III) are not subdiagrams of any Satake diagram of $E_7$.

## Acknowledgments

The author thanks Marcus J. Slupinski and Robert J. Stanton for many useful conversations concerning this paper.

## References


[All-Fer-1] H.P ALLEN, J.C. FERRAR — *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. London Math. Soc. 9 No. 1 (1977),1-35. MR0444729 (56:3079)

[All-Fer-2] H.P ALLEN, J.C. FERRAR — *Jordan algebras and exceptional subalgebras of the exceptional algebra $E_6$*, Pacific J. Math. 32 (1970), 283-297 . MR0262308 (41:6917)

[Al-1] B.N. ALLISON — *A construction of Lie algebras from J-ternary algebras*, Americ. J. Math., 98 (1976), 285-294. MR0430010 (55:3018)

[Al-2] B.N. ALLISON — *A class of nonassociative algebras with involutions containing the class of Jordan algebras*, Math. Ann., 237 (1978), no 2, 133-156. MR507909 (81h:17003)

[Al-3] B.N. ALLISON — *Models of isotropic simple Lie algebras*, Comm. Algebra., 7 (1979), no 17, 1835-1875. MR547712 (81d:17005)

[Ba] J. BAEZ — *The Octonions*, Bul. Amer. Mat. Soc.(NS), 39 (2002) Number 2, 145-205. MR1886087 (2003f:17003)

[B-S] C.H. BARTON – A. SUDBERY — *Magic squares of Lie Algebras*, arXiv:math.RA/0001083.

[Bro-1] R. B. BROWN — *A new type of associative algebras*, Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 947-949. MR0158913 (28:2135)

[Bro-2] R. B. BROWN — *Groups of Type $E_7$*, J. Reine Angew. Math. 236 (1969), 79-102. MR0248185 (40:1439)

[Che] C. CHEVALLEY — *The Algebraic Theory of Spinors*, Columbia University Press, 1954 MR0060497 (15:678d)

[Fau] J.R. FAULKNER — *A construction of Lie algebras from a class of ternary algebras*, Trans. Amer. Math. Soc., 155 (1971),397-408. MR0294424 (45:3494)

[Fer] J.C. FERRAR — *Lie algebras of type $E_6$*, J. Algebra 13 (1969), 57-72, — *Lie algebras of type $E_6$ II*, J. Algebra 52 (1978) No. 1 , 201-209. MR0263881 (41:8480)

[Freu] H. FREUDENTHAL — *Lie groups and the foundations of geometry*, Adv. Math. 1 (1964), 145-190. MR0170974 (30:1208)

[Gar] S. GARIBALDI — *Structurable algebras and groups of type $E_6$ and $E_7$* , J. Algebra 236, No.2 (2001), 651-691. MR1813495 (2001m:20069)

[Hei] W. HEIN — *A construction of Lie algebras by triple systems*, Trans. Amer. Math. Soc., 205 (1975), 79-95. MR0393153 (52:13963)

[Hu] J.E. HUMPHREYS — *Linear Algebraic Groups*, Graduate Texts in Math. 21. Springer, Berlin, New York, 1975. MR0396773 (53:633)

[Ig] J-I. IGUSA — *A Classification of Spinors up to dimension twelve*, Amer. J. Math. 92 (1970) 997-1028. MR0277558 (43:3291)

[Iw] N. IWAHORI — *On real irreducible representation of Lie algebras*, Nagoya. Math. J. vol. 14 (1959), 59-83. MR0102534 (21:1325)

[Jac] N. JACOBSON — *Exceptional Lie Algebras*, Marcel Dekker, New York, 1971. MR0284482 (44:1707)

[Ka] I.L. KANTOR — *Models of exceptional Lie algebras*, Soviet. Math. Doklady, 14 (1973),254-258.

[Ka-Sko] I.L. KANTOR, I. M. SKOPETS — *Some Results on Freudenthal Triple Systems*, Sel. Math. Sov. vol. 2. No. 3. (1982), 293-305 (originally published in Trudy Sem. Vektor. Tenzor. Anal 18 (1978) 250-263).

[Ki] T. KIMURA — *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs, 215. American Mathematical Society, Providence, RI (2003). MR1944442 (2003k:11180)

[Ko] M. KOECHER — *Imbedding of Jordan algebras into Lie algebras I*, Americ. J. Math., 89 (1967),787-815. MR0214631 (35:5480)

[L-M] J.M. LANDSBERG – L. MANIVEL — *Triality, exceptional Lie algebras and Deligne dimension formulas* , Adv. Math. 171 (2002), no. 1, 59–85. MR1933384 (2003i:17012)

[Lo] O. LOOS — *Jordan Pairs*, Lecture Notes in Math., vol. 460, Springer-Verlag, New York (1975) . MR0444721 (56:3071)

[M-R-S] I. MULLER – H. RUBENTHALER – G. SCHIFFMANN — *Structure des espaces préhomogènes associés à certaines algèbres de Lie graduées*, Math. Ann. 274 (1986), 95–123. MR834108 (88e:17025)

[Mey] K. MEYBERG — *Zur Konstruktion von Lie-Algebren aus Jordan Triple-Systemen*, Manuscripta math. 3, (1970), 115-132 . MR0285576 (44:2794)

[Ru-1] H. RUBENTHALER — *Construction de certaines sous-algèbres remarquables dans les algèbres de Lie semi-simples*, J. Algebra 81 (1983) No. 1, 268-278. MR696139 (84g:17010)

[Ru-2] H. RUBENTHALER — *Algèbres de Lie et espaces préhomogènes*, Travaux en Cours, Hermann, Paris (1992).

[Ru-3] H. RUBENTHALER — *Les paires duales dans les algèbres de Lie réductives*, Astérisque No. 219 (1994), 121 pp. MR1264015 (95b:17010)

[Ru-4] H. RUBENTHALER — *Non Parabolic Prehomogeneous Vector Spaces and Exceptional Lie Algebras*, J. Algebra 281 (2004) 366-394. MR2091977 (2005k:20110)

[Ru-5] H. RUBENTHALER — *Formes réelles des espaces préhomogènes irréductibles réguliers*, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 1, 1–38. MR840712 (87k:17010)

[Sa-Ki] M. SATO – T. KIMURA — *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. 65 (1977), 1–155. MR0430336 (55:3341)

[S-V] T. A. SPRINGER – F. D. VELDKAMP — *Octonions, Jordan Algebras and Exceptional Groups*, Springer Monographs in Mathematics, Springer, Berlin (2000) MR1763974 (2001f:17006)

[Ti-1] J. TITS — Une classe d'algèbres de Lie en relation avec les algèbres de Jordan, Indag. Math. 24 (1962), 530-535. MR0146231 (26:3753)

[Ti-2] J. TITS — Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles, Indag. Math. 28 (1966), 223-237. MR0219578 (36:2658)

[Wa] G. WARNER — Harmonic Analysis on Semi-Simple Lie Groups I, Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972. MR0498999 (58:16979)

[Yam] K. YAMAGUTI — *On algebras of totally geodesic spaces (Lie triple systems)*, J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 107-113 . MR0100046 (20:6482)

Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René Descartes, 67084 Strasbourg Cedex, France