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MORAVA E-THEORY OF FILTERED COLIMITS

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Abstract. Morava E-theory $E^\vee_{n*}(-)$ is a much-studied theory in algebraic topology, but it is not a homology theory in the usual sense, because it fails to preserve coproducts (resp. filtered homotopy colimits). The object of this paper is to construct a spectral sequence to compute the Morava E-theory of a coproduct (resp. filtered homotopy colimit). The E_2-term of this spectral sequence involves the derived functors of direct sum (resp. filtered colimit) in an appropriate abelian category. We show that there are at most $n-1$ (resp. n) of these derived functors. When $n=1$, we recover the known result that homotopy commutes with an appropriate version of direct sum in the $K(1)$-local stable homotopy category.

Introduction

In algebraic topology, the basic tools for understanding topological spaces are homology theories. However, one theory, known as Morava E-theory, has been the object of much recent study despite the fact that it is not a homology theory. The object of this paper is to measure how far away E is from being a homology theory.

To explain this, fix a prime p and an integer $n > 0$. The Morava E-theory spectrum is the Landweber exact spectrum E with

$$E_* \cong W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}],$$

where $W\mathbb{F}_{p^n}$ is the Witt ring of $\mathbb{F}_{p^n}$, so an unramified extension of $\mathbb{Z}_p$ of degree n, each of the u_i has degree 0, and u has degree 2. Note that E_0 is a complete Noetherian regular local ring with maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$. The letter E will denote this theory throughout the paper.

We have just described the Morava E-theory of a point. For general X, we define $E^\vee_* X = \pi_* L_K(E \wedge X)$, in contrast with $E_* X = \pi_*(E \wedge X)$. Here $K = E/\mathfrak{m}$ is a finite coproduct of copies of $K(n)$, the nth Morava K-theory, and $L_K = L_{K(n)}$ denotes Bousfield localization with respect to K. The reason for defining $E^\vee_* X$ in this way is to better measure K-local phenomena. For example, X is small in the K-local category if and only if $E^\vee_* X$ is finite [HS99, Theorem 8.5].

The cost of introducing $E^\vee_*(-)$, though, is that, because $L_{K(n)}$ does not preserve coproducts, $E^\vee_*(-)$ is not a homology theory. To compute the Morava E-theory of a coproduct, the obvious thing to do is to build a spectral sequence whose E_2 term consists of derived functors of coproduct in the category of E_*-modules. But of course coproducts have no derived functors in any module category. However, Morava E-theory is a sort of topological completion of $E_* X$, so it should take

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values in an appropriate abelian category of completed E_*-modules. In fact, the appropriate category was introduced in [HS99, Appendix 2] and is denoted by $\widehat{\mathcal{M}}$.

The main result of this paper is then the following theorem.

Theorem A. *Suppose $\mathcal{I}$ is either a set or a small filtered category, and X is an $\mathcal{I}$-diagram in an appropriate model category of spectra. Then there is a natural strongly convergent spectral sequence of E_*-modules*

$$E^2_{s,t} = (\operatorname{colim}^s E_*^\vee X_j)_t \Rightarrow E^\vee_{s+t}(\operatorname{hocolim} X_j)$$

with

$$d^r_{s,t}\colon\ E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

Here colim^s *is the sth left derived functor of the filtered colimit in the category $\widehat{\mathcal{M}}$, and* $\operatorname{colim}^s = 0$ *for all $s > n$. If $\mathcal{I}$ is a set, then* $\operatorname{colim}^n = 0$ *as well.*

We note that colim^0 is the colimit in $\widehat{\mathcal{M}}$, which is not just the usual colimit of E_*-modules. By an "appropriate model category" we mean any of the symmetric monoidal model categories that are models for the stable homotopy category, such as S-modules [EKMM97] or symmetric spectra [HSS00]. Note that if $\mathcal{I}$ is a set, so we are just taking the coproduct, an $\mathcal{I}$-diagram of spectra is equivalent to an $\mathcal{I}$-diagram in the homotopy category of spectra, so the model category is unnecessary.

If we apply this in the case $n = 1$, we see that $E_*^\vee(-)$, as a functor to $\widehat{\mathcal{M}}$, commutes with coproducts, so is a homology theory, though it does not commute with filtered homotopy colimits. In fact, in this case homotopy is actually a homology theory to the appropriate version of $\widehat{\mathcal{M}}$, known as the category $\widehat{\mathbf{Ab}}$ of Ext-p-complete abelian groups (which one can think of as the smallest abelian category containing the p-complete abelian groups).

Corollary B. *Suppose $n = 1$ and $\{X_i\}$ is a family of $K = K(1)$-local spectra. Then the natural map*

$$\bigoplus_i^{\widehat{\mathbf{Ab}}} \pi_* X_i \to \pi_* L_K(\bigvee_i X_i)$$

is an isomorphism.

This corollary is not hard to prove directly and was known to Mitchell [Mit05, Section 3.3.1], Mike Hopkins, Neil Strickland, and possibly others. In fact, this paper grew out of conversations the author had with Strickland in the late 1990s. The author thanks him for many helpful discussions over the years. The author also thanks the referee, whose direct approach using the work of Greenlees and May [GM95] replaces the original clumsier approach of the author.

Let us recall the notation we use throughout this paper. The prime number p and the non-negative integer n are fixed throughout the paper. E denotes the Morava E-theory corresponding to p and n, with maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$ in E_*. Also, K denotes $E/\mathfrak{m}$. The symbol L_K denotes Bousfield localization with respect to K, and the symbol $E_*^\vee X$ denotes $\pi_* L_K(E \wedge X)$. The category $\widehat{\mathcal{M}}$ is the category of L-complete E_*-modules, discussed below.

1. Colimits of L-complete modules

For a spectrum X, $E_*^\vee X$ is not just an E_*-module; it is an L-complete E_*-module. To understand $E_*^\vee(-)$, then, we need to first study the category $\widehat{\mathcal{M}}$ of

L-complete E_*-modules, which we do in this section. Our basic reference for this category is [HS99, Appendix A], which is based on [GM92].

1.1. **L-complete modules.** The basic issue is that the completion at $\mathfrak{m}$ functor is not left or right exact on the category of graded E_*-modules. So we replace completion by its 0th left derived functor L_0M. If we have an exact sequence

$$F_1 \to F_0 \to M \to 0,$$

where F_1 and F_0 are free modules, then we should define

$$L_0M = (F_0)^\wedge_{\mathfrak{m}}/(F_1)^\wedge_{\mathfrak{m}}.$$

This definition is not very natural, though, and it turns out to be equivalent to define

$$L_0M = \operatorname{Ext}^n_{E_*}(E_*/\mathfrak{m}^\infty, M).$$

Here $E_*/\mathfrak{m}^\infty$ is defined as usual in algebraic topology. Thus E_*/p^∞ is the quotient $p^{-1}E_*/E_*$, and $E_*/(p^\infty, u_1^\infty)$ is the quotient of $u_1^{-1}(E_*/p^\infty)$ by E_*/p^∞, and we continue in this fashion.

There is a natural surjection $\epsilon_M\colon L_0M \to M^\wedge_{\mathfrak{m}}$, whose kernel is

$$\lim{}^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, M),$$

by [HS99, Theorem A.2]. Unlike completion, L_0 is right exact, and so has left derived functors L_i. In fact, we have

$$L_iM \cong \operatorname{Ext}^{n-i}_{E_*}(E_*/\mathfrak{m}^\infty, M)$$

by [HS99, Theorem A.2(d)], and so, in particular, $L_iM = 0$ for $i > n$.

The natural map $M \to M^\wedge_{\mathfrak{m}}$ factors through a natural map $\eta_M\colon M \to L_0M$, and a module M is called **L-complete** if η_M is an isomorphism. The full subcategory of graded L-complete E_*-modules will be denoted by $\widehat{\mathcal{M}}$. The module $E_*^\vee X$ is always L-complete [HS99, Proposition 8.4].

The category $\widehat{\mathcal{M}}$ is an abelian subcategory of E_*-mod, closed under extensions and inverse limits [HS99, Theorem A.6]. Note that $\widehat{\mathcal{M}}$ contains all the finitely generated E_*-modules, since E_* itself is L-complete and L_0 is right exact. The functor $L_0\colon E_*\text{-mod} \to \widehat{\mathcal{M}}$ is left adjoint and left inverse to the inclusion functor $i\colon \widehat{\mathcal{M}} \to E_*\text{-mod}$, and therefore creates colimits in $\widehat{\mathcal{M}}$. That is, if $F\colon \mathcal{I} \to \widehat{\mathcal{M}}$ is a functor, then the colimit of F in $\widehat{\mathcal{M}}$ is $L_0(\operatorname{colim} iF)$. Thus $\widehat{\mathcal{M}}$ is a bicomplete abelian category.

However, because L_0 is not exact, we cannot expect filtered colimits, or even direct sums, to be exact in $\widehat{\mathcal{M}}$.

1.2. **Derived functors of colimit.** We now measure the failure of filtered colimits and direct sums to be exact in $\widehat{\mathcal{M}}$. Examples of this failure are given in the next section. As left adjoints, of course, direct sums and filtered colimits are right exact. So we should consider the left derived functors of direct sum and filtered colimit. The object of this section, then, is to prove the following theorem.

Theorem 1.1. *Let $\mathcal{I}$ be a set or a small filtered category, and let* colim *denote the colimit functor in $\widehat{\mathcal{M}}$. Then the left derived functors* colim^s *exist for $s \geq 0$ and we have a natural isomorphism*

$$\operatorname{colim}^s \cong L_s \operatorname{colim}_{E_*}$$

where $\operatorname{colim}_{E_*}$ *denotes the usual colimit of E_*-modules.*

Recall that $L_i M \in \widehat{\mathcal{M}}$ for all M, by [HS99, Theorem A.6], so this theorem makes sense. We will also show that $\operatorname{colim}^n = 0$ if $\mathcal{I}$ is a set.

Now, suppose we have a short exact sequence

$$0 \to A \to B \to C \to 0$$

of $\mathcal{I}$-diagrams in $\widehat{\mathcal{M}}$. Then we get a short exact sequence of E_*-modules

$$0 \to \operatorname{colim}_{E_*} A \to \operatorname{colim}_{E_*} B \to \operatorname{colim}_{E_*} C \to 0$$

and so a long exact sequence

$$\begin{aligned} \cdots \to L_{i+1}(\operatorname{colim}_{E_*} C) \to L_i(\operatorname{colim}_{E_*} A) \to L_i(\operatorname{colim}_{E_*} B) \\ \to L_i(\operatorname{colim}_{E_*} C) \to L_{i-1}(\operatorname{colim}_{E_*} A) \to \cdots L_0(\operatorname{colim}_{E_*} C) \to 0. \end{aligned}$$

Since $L_0 \operatorname{colim}_{E_*} \cong \operatorname{colim}$, to prove Theorem 1.1 it suffices to prove that the diagram category has enough projectives and, if P is a projective diagram, then $L_s(\operatorname{colim}_{E_*} P) = 0$ for $s > 0$.

It is easy to see that the diagram category has enough projectives.

Lemma 1.2. *If $\mathcal{I}$ is a small category, then the category $F(\mathcal{I}, \widehat{\mathcal{M}})$ of functors from $\mathcal{I}$ to $\widehat{\mathcal{M}}$ is a bicomplete abelian category with enough projectives. Furthermore, each projective functor is pointwise projective.*

Proof. It is well known and easy to check that functor categories into abelian categories are abelian. Limits and colimits are taken pointwise. Now $\widehat{\mathcal{M}}$ itself has enough projectives, by [HS99, Corollary A.12]. For an object $i \in \mathcal{I}$, the functor $\operatorname{Ev}_i \colon F(\mathcal{I}, \widehat{\mathcal{M}}) \to \widehat{\mathcal{M}}$ defined by $\operatorname{Ev}_i(X) = X_i$ is exact and has a left adjoint F_i. Here F_i is defined by

$$(F_i M)_j = \coprod_{\mathcal{I}(i,j)} M,$$

where the coproduct is of course taken in $\widehat{\mathcal{M}}$. The reader can check that F_i is left adjoint to Ev_i, and so preserves projectives. Given a diagram X, then, we choose surjections $P_i \to X_i$ for all $i \in \mathcal{I}$. These give maps $F_i P_i \to X$ of diagrams, and the map $\coprod_{i \in \mathcal{I}} F_i P_i \to X$ is then a surjection from a projective, as desired. In particular, if X is itself a projective functor, then X is a retract of $\coprod_{i \in \mathcal{I}} F_i P_i$, and so is pointwise projective. □

Now suppose P is a projective functor. We must show that $L_s \operatorname{colim}_{E_*} P = 0$ for $s > 0$. To see this, we first translate the condition of being projective in $\widehat{\mathcal{M}}$ into a condition that interacts well with ordinary colimits.

Lemma 1.3. *If Q is in $\widehat{\mathcal{M}}$, then Q is projective in $\widehat{\mathcal{M}}$ if and only if*

$$\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, Q) = 0$$

for all $k > 0$ and all $s > 0$.

Proof. If this condition holds, then Q is projective in $\widehat{\mathcal{M}}$ by [HS99, Theorem A.9]. By the same theorem, if Q is projective in $\widehat{\mathcal{M}}$, then $\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}, Q) = 0$ for all $s > 0$. We then prove by induction on k that $\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, Q) = 0$ for all $k > 0$ and all $s > 0$. To do so, we use the short exact sequence

$$0 \to \mathfrak{m}^{k-1}/\mathfrak{m}^k \to E_*/\mathfrak{m}^k \to E_*/\mathfrak{m}^{k-1} \to 0$$

and the fact that $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ is an $E_*/\mathfrak{m}$-vector space. □

Since P is a projective diagram, $P(i)$ is a projective object for all $i \in \mathcal{I}$ by Lemma 1.2. Hence $\operatorname{Tor}_j^{E_*}(E_*/\mathfrak{m}^k, P(i)) = 0$ for all $k > 0$ and all $j > 0$. Since Tor commutes with filtered colimits, we conclude that $\operatorname{Tor}_j^{E_*}(E_*/\mathfrak{m}^k, \operatorname{colim}_{E_*} P) = 0$ for all $j, k > 0$. It follows from [HS99, Theorem A.2(b)] that $L_s(\operatorname{colim}_{E_*} P) = 0$ for $s > 0$, completing the proof of Theorem 1.1.

Theorem 1.1 implies that filtered colimits and direct sums in $\widehat{\mathcal{M}}$ have at most n left derived functors. In fact, though, direct sums have at most $n-1$.

Proposition 1.4. *If $\{M_i\}$ is a family of L-complete modules, then $L_n(\bigoplus M_i) = 0$. In particular, if $n = 1$, then direct sums in $\widehat{\mathcal{M}}$ are exact.*

Proof. Using the embedding of the sum into the product we see that

$$L_n(\bigoplus M_i) = \operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, \bigoplus M_i)$$

is a subobject of

$$\operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, \prod M_i) \cong \prod \operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, M_i) = \prod L_n M_i.$$

Since each M_i is L-complete, $L_n M_i = 0$ by [HS99, Theorem A.6], giving us the desired result. □

1.3. **An example.** In order to convince the reader that these derived functors of the direct sum are really necessary, we give an example of the failure of direct sums to be exact. It is much easier to give an example of the failure of filtered colimits to be exact, so we begin with that case.

Take $n = 1$, and consider the system of short exact sequences below:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p^i"] \arrow[d, equal] & \mathbb{Z}_p \arrow[r] \arrow[d, "\times p"'] & \mathbb{Z}/p^i \arrow[r] \arrow[d, "\times p"] & 0 \\
0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p^{i+1}"'] & \mathbb{Z}_p \arrow[r] & \mathbb{Z}/p^{i+1} \arrow[r] & 0.
\end{tikzcd}$$

With the usual colimit, this gives us the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}/p^\infty \to 0.$$

However, upon applying L_0 we get the sequence

$$0 \to \mathbb{Z}_p \to 0 \to 0 \to 0,$$

which is clearly not exact.

The direct sum is exact for $n = 1$ by Proposition 1.4, but it need not be exact for $n = 2$. The following example of this was obtained in joint work with Neil Strickland. Let $n = 2$, let $M_i = E_*/p^i$, and let $f_i\colon M_i \to M_i$ be multiplication by u_1^i. We claim that

$$L_0(\bigoplus_i f_i)\colon L_0(\bigoplus_i M_i) \to L_0(\bigoplus_i M_i)$$

is not injective. To see this, note from [HS99, Theorem A.2(b)] that it suffices to show that

$$\lim_k{}^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \to \lim_k{}^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \tag{1.5}$$

is not injective. One can easily check that

$$\operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, M_i) \cong \mathfrak{m}^{k-i}/\mathfrak{m}^k,$$

where $\mathfrak{m}^{k-i} = E_*$ for $i \geq k$. The map induced by f_i is then visibly 0 on this Tor group. Since Tor commutes with direct sums, it follows that the map of (1.5) is also 0. Hence it suffices to show that

$$\lim_k^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \cong \lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k \neq 0.$$

For this, we note that for fixed i, the tower $\{\mathfrak{m}^{k-i}/\mathfrak{m}^k\}$ is pro-trivial, because every i-fold composite is 0. Hence

$$\lim_k \mathfrak{m}^{k-i}/\mathfrak{m}^k = \lim_k^1 \mathfrak{m}^{k-i}/\mathfrak{m}^k = 0.$$

Hence the map

$$d_i\colon \prod_k \mathfrak{m}^{k-i}/\mathfrak{m}^k \to \prod_k \mathfrak{m}^{k-i}/\mathfrak{m}^k$$

is an isomorphism, where $d_i(x)_k = (x_k - \overline{x_{k+1}})$ and $\overline{x_{k+1}}$ is the image of x_{k+1} in $\mathfrak{m}^{k-i}/\mathfrak{m}^k$. Thus we get the commutative square

$$\begin{array}{ccc} \prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k & \xrightarrow{d} & \prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k \\ \downarrow & & \downarrow \\ \prod_k \prod_i \mathfrak{m}^{k-i}/\mathfrak{m}^k & \xrightarrow{d} & \prod_k \prod_i \mathfrak{m}^{k-i}/\mathfrak{m}^k \end{array}$$

where the vertical arrows are injections, the bottom horizontal arrow is an isomorphism, and the cokernel of the top horizontal map is the group

$$\lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k.$$

Now, consider the element b of $\prod_{ki} \mathfrak{m}^{k-i}/\mathfrak{m}^k$ such that $b_{ki} = 1$ if $k \leq i$, and $b_{ki} = 0$ otherwise. Then $(db)_{ki} = 0$ for $i \geq k+1$, and therefore $db \in \prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k$. Hence db must represent a nontrivial element of

$$\lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k,$$

as required.

2. The spectral sequence

Now that we have an idea of what the E_2-term should be, we construct our spectral sequence. This spectral sequence is a special case of the spectral sequence of Greenlees and May [GM95, Theorem 4.2], so we begin by recalling the Greenlees-May situation. This requires us to work in the triangulated category $\mathcal{D}_E$ of E-modules, so we begin with some preliminaries about $\mathcal{D}_E$.

Recall that the appropriate analogue of a ring in stable homotopy theory is an S-algebra in the sense of [EKMM97]. The derived category $\mathcal{D}_R$ of an S-algebra R is constructed in [EKMM97, Chapters III and VII]. The spectrum E is known to be a commutative S-algebra by [GH04, Corollary 7.6]. It follows that $\mathcal{D}_E$ is a monogenic stable homotopy category in the sense of [HPS97]. This means that it is a closed symmetric monoidal triangulated category such that the unit E is a small weak generator.

As with ordinary rings and modules, there is a forgetful functor from $\mathcal{D}_E$ to the ordinary stable homotopy category $\mathcal{D}_S$. This functor has a left adjoint by Proposition III.4.4 of [EKMM97]. The left adjoint takes X to $E \wedge X$, or, more precisely, an object of $\mathcal{D}_E$ whose underlying spectrum is $E \wedge X$.

Of course, we are interested in $E_*^\vee X = \pi_* L_K(E \wedge X)$. For this, we recall that if M is in $\mathcal{D}_E$, then the Bousfield localization $L_K M$ of the underlying spectrum of M is the underlying spectrum of $L_{E\wedge K}^{\mathcal{D}_E} M$; in particular, the natural map $M \to L_K M$ is (the underlying map of spectra of) a map in $\mathcal{D}_E$. See [EKMM97, Chapter VIII]. Hence we can think of $L_K(E \wedge X)$ as an object of $\mathcal{D}_E$.

On the other hand, we can also take the Bousfield localization $L_K^{\mathcal{D}_E} M$ of an E-module with respect to the E-module $K = E/\mathfrak{m}$. We need to know that the two Bousfield localizations $L_{E\wedge K}^{\mathcal{D}_E}$ and $L_K^{\mathcal{D}_E}$ coincide. We learned the following argument for proving this from Neil Strickland.

Given a commutative S-algebra R, let us denote by $\mathrm{Bous}_R(M)$ the Bousfield class of M as an object of $\mathcal{D}_R$; we can think of this as the class of all N in $\mathcal{D}_R$ such that $M \wedge_R N = 0$, ordered as usual by reverse inclusion.

Lemma 2.1. *Suppose R is a commutative S-algebra and M is an object of $\mathcal{D}_R$. Then* $\mathrm{Bous}_R(R \wedge M) \geq \mathrm{Bous}_R(M)$.

Here $R \wedge M$ is the free R-module on the underlying spectrum of M. Since $(R\wedge M)\wedge_R N \cong M \wedge N$ as spectra, this lemma is analogous to the obvious algebraic fact that $M \otimes N = 0$ implies $M \otimes_R N = 0$.

Proof. It suffices to show that $M \wedge N = 0$ implies that $M \wedge_R N = 0$. We can assume that M and N are cofibrant as R-modules in the model structure of [EKMM97, Chapter VII]. Then $M \wedge N$ is a cofibrant $R\wedge R$-module, and $M \wedge_R N$ is the image of $M \wedge N$ under the left Quillen functor induced by the map $R \wedge R \to R$ of commutative S-algebras. If $M \wedge N = 0$ in $\mathcal{D}_R$, then $M \wedge N$ is contractible as a spectrum, and so is trivially cofibrant as an $R \wedge R$-module. Hence its image $M \wedge_R N$ is trivially cofibrant as an R-module. □

This is in general all one can say. However, in our situation, equality holds.

Proposition 2.2. $\mathrm{Bous}_E(E \wedge K) = \mathrm{Bous}_E(K)$.

Proof. It suffices to show that $K \wedge_E X = 0$ implies that $K \wedge X = 0$. Let $I = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ be an ideal such that S/I exists as a spectrum (see [HS98] or [HS99, Corollary 4.14]). Then $E/I = E \wedge S/I$, but E/I is also in the thick subcategory of $\mathcal{D}_E$ generated by K. Hence

$$S/I \wedge X = E/I \wedge_E X = 0.$$

Applying K_* and using the Kunneth theorem, we see that $K_* X = 0$, as required. □

The Greenlees-May spectral sequence of [GM95, Theorem 4.2] specializes to give the following spectral sequence.

Theorem 2.3. *For $M \in \mathcal{D}_E$, there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s \pi_* M)_t \Rightarrow \pi_{s+t} L_K^{\mathcal{D}_E} M,$$

where $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

This theorem follows because $K = E/\mathfrak{m}$ is the Greenlees-May $K_1(J)$ for $J = \mathfrak{m}$. We need Proposition 2.2 to recognize the underlying spectrum of $L_K^{\mathcal{D}_E} M$ as $L_K M$.

We now prove Theorem A, which we restate for the convenience of the reader.

Theorem 2.4. *Suppose $\mathcal{I}$ is either a set or a small filtered category, and X is an $\mathcal{I}$-diagram in an appropriate model category of spectra. Then there is a natural strongly convergent spectral sequence of E_*-modules*

$$E^2_{s,t} = (\operatorname{colim}^s E_*^\vee X_j)_t \Rightarrow E^\vee_{s+t}(\operatorname{hocolim} X_j)$$

with

$$d^r_{s,t}\colon\ E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

Here colim^s *is the sth left derived functor of the filtered colimit in the category* $\widehat{\mathcal{M}}$*, and* $\operatorname{colim}^s = 0$ *for all* $s > n$*. If* $\mathcal{I}$ *is a set, then* $\operatorname{colim}^n = 0$ *as well.*

To prove this theorem, we will need two facts about the homotopy colimit (both of which are obvious for the coproduct). The first is that the derived functors of left Quillen functors preserve homotopy colimits; that is, if $F\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor between model categories, and X is an $\mathcal{I}$-diagram in $\mathcal{M}$, then there is a natural isomorphism

$$(LF)(\operatorname{hocolim} X) \to \operatorname{hocolim} FX$$

in the homotopy category of $\mathcal{N}$. This is a basic formal property of homotopy colimits, and so should be well known. In fact, however, homotopy colimits are difficult and there are several possible different foundations for the theory. The author prefers the theory described in [DHKS04]; in particular, the fact we need is in paragraph 20.4 of that book.

We also need to know that

$$\pi_* \operatorname{hocolim} X \cong \operatorname{colim}_{E_*} \pi_* X$$

for $\mathcal{I}$-diagrams in $\mathcal{D}_E$. This depends both on $\mathcal{I}$ being filtered (or a set), and on the model category $\mathcal{D}_E$ being especially nice. There does not seem to be a reference for this fact in the literature, so we will prove it in an appendix.

Proof. Assuming the two facts given above about homotopy colimits, we simply apply the Greenlees-May spectral sequence to the object $\operatorname{hocolim} L_K(E \wedge X_i)$ of $\mathcal{D}_E$, where the homotopy colimit is taken in $\mathcal{D}_E$. For this to make sense, we need the functors $E \wedge (-)$ and L_K to be left Quillen functors defined on the model category level. To make $E \wedge (-)$ a left Quillen functor, we need our model category of spectra to be good enough that we get induced model categories of modules over an arbitrary monoid; this is guaranteed by the monoid axiom of [SS00], which is satisfied in symmetric spectra and S-modules. To make L_K a left Quillen functor, we need a theory of Bousfield localization at the model category level as in [Hir03] in general, and [EKMM97, Chapter VIII] for S-modules. Note that, at the model category level, L_K is just the identity functor, since all we are doing is changing model structures on the same underlying category.

The E_2-term of the spectral sequence is

$$E^2_{s,t} = (L_s[\pi_* \operatorname{hocolim} L_K(E \wedge X_i)])_t.$$

Since homotopy commutes with the homotopy colimit, this is the same as

$$(L_s[\operatorname{colim}_{E_*} \pi_* L_K(E \wedge X_i)])_t \cong (\operatorname{colim}^s E_*^\vee X_i)_t,$$

in view of Theorem 1.1.

The spectral sequence will converge to

$$\pi_* L_K(\operatorname{hocolim} L_K(E \wedge X_i)).$$

We have a natural map of diagrams from $E \wedge X_i$ to $L_K(E \wedge X_i)$ that induces a map

$$\operatorname{hocolim}(E \wedge X_i) \to \operatorname{hocolim} L_K(E \wedge X_i).$$

We claim this map is a K-equivalence. Indeed, using the fact that homotopy and smashing with K commute with homotopy colimits (smashing with K does so because it is the total derived functor of a left Quillen functor), we get isomorphisms

$$\pi_*(K \wedge_E \operatorname{hocolim}(E \wedge X_i)) \cong \operatorname{colim} \pi_*(K \wedge_E (E \wedge X_i))$$

and

$$\pi_*(K \wedge_E \operatorname{hocolim} L_K(E \wedge X_i)) \cong \operatorname{colim} \pi_*(K \wedge_E L_K(E \wedge X_i)).$$

Thus our spectral sequence converges to

$$\pi_* L_K(\operatorname{hocolim}(E \wedge X_i)) \cong \pi_* L_K(E \wedge \operatorname{hocolim} X_i) = E_*^\vee(\operatorname{hocolim} X_i),$$

as required. □

This line of proof for Theorem 2.4 was suggested by the referee, who also points out that if we apply the Greenlees-May spectral sequence to $\operatorname{hocolim}(E \wedge X_i)$, then we get a spectral sequence also converging to $E_*^\vee(\operatorname{hocolim} X_i)$, but whose E_2-term is

$$E^2_{s,t} \cong L_s(\operatorname{colim}_{E_*} E_* X_i)_t.$$

The simplest case of Theorem 2.4 is when each $E_*^\vee X_i$ is the completion of a free module; such modules are called **pro-free** and are the projective objects of $\widehat{\mathcal{M}}$ by [HS99, Theorem A.9].

Corollary 2.5. *Suppose $\mathcal{I}$ is either a set or a small filtered category, and X is an $\mathcal{I}$-diagram in an appropriate model category of spectra such that $E_*^\vee X_i$ is pro-free for all i. Then the natural map*

$$\operatorname{colim} E_*^\vee X_j \to E_*^\vee(\operatorname{hocolim} X_j)$$

is an isomorphism, where the colimit is taken in $\widehat{\mathcal{M}}$.

Proof. It suffices to show that $L_s(\operatorname{colim}_{E_*} E_*^\vee X_i) = 0$ for $s > 0$. This is proved using the same argument that was used to prove Theorem 1.1, immediately following Lemma 1.3 . □

3. The $K(1)$-local stable homotopy category

One of the difficulties in understanding the K-local stable homotopy category is that L_K is not smashing, so that homotopy groups do not commute with coproducts in the K-local category. We have just seen, however, that $E_*^\vee(-)$ preserves coproducts when $n = 1$, as a functor to the L-complete category. It seems reasonable, then, to ask whether homotopy groups also commute with coproducts in the K-local category when $n = 1$, as a functor to the L-complete category. We will see in this section that the answer is yes, proving Corollary B. As mentioned in the

introduction, this result was certainly known to Hopkins, Mitchell, Strickland, and probably others.

Now, when $n = 1$, $E_* \cong \mathbb{Z}_p[u, u^{-1}]$, and L_0 is the 0th left derived functor of p-completion. Of course, the homotopy of a K-local spectrum will not be an E_*-module, but it will be a $\mathbb{Z}_p$-module. We will therefore also use L_0 to denote the 0th left derived functor of p-completion on the category of $\mathbb{Z}_p$-modules. We first point out that this apparent clash of notation is in fact euphonic.

Lemma 3.1. *Suppose $n = 1$ and M is a graded E_*-module. Then*

$$(L_0 M)_k \cong L_0 M_k$$

where the second L_0 is taken in the category of $\mathbb{Z}_p$-modules.

Proof. We have the short exact sequences

$$0 \to \lim{}^1 \operatorname{Tor}^{E_*}(E_*/p^i, M) \to L_0 M \to M_p^\vee \to 0$$

and

$$0 \to \lim{}^1 \operatorname{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^i, M_k) \to L_0 M_k \to (M_k)_p^\vee \to 0,$$

from [HS99, Theorem A.2(b)]. One can easily check that

$$(\lim{}^1 \operatorname{Tor}^{E_*}(E_*/p^i, M))_k \cong \lim{}^1 \operatorname{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^i, M_k)$$

and

$$(M_p^\vee)_k \cong (M_k)_p^\vee,$$

from which the result follows. □

Now we note that $\pi_* X$ is L-complete when X is K-local.

Lemma 3.2. *If $n = 1$ and X is K-local, then $\pi_k X$ is an L-complete $\mathbb{Z}_p$-module for all k.*

Proof. It is well known that $X = \operatorname{holim}(X/p^i)$; see [HS99, Proposition 7.10] for example. Now $\pi_*(X/p^i)$ is bounded p-torsion, so in particular is p-complete and so L-complete. It follows from the Milnor exact sequence and [HS99, Theorem A.5] that $\pi_* X$ is L-complete. □

We can now prove the desired theorem.

Theorem 3.3. *Suppose $n = 1$ and $\{X_i\}$ is a family of K-local spectra. Then the natural map*

$$L_0(\bigoplus_i \pi_* X_i) \to \pi_* L_K(\bigvee_i X_i)$$

is an isomorphism.

Here L_0 denotes the 0th derived functor of p-completion in the category of $\mathbb{Z}_p$-modules.

Proof. We will show that the collection $\mathcal{D}$ of all spectra F such that the natural map

$$\Psi_F \colon\ L_0(\bigoplus_i F_*^\vee X_i) \to F_*^\vee(\bigvee X_i)$$

is an isomorphism is a thick subcategory. Here $F_*^\vee X = \pi_* L_K(F \wedge X)$.

It is clear that $Y \in \mathcal{D}$ if and only if $\Sigma Y \in \mathcal{D}$. To see that $\mathcal{D}$ is closed under retracts, simply note that if Y is a retract of Z, then Ψ_Y is a retract of Ψ_Z. Now suppose that

$$W \to Y \to Z \to \Sigma W$$

is an exact triangle, and $W, Z \in \mathcal{D}$. We have the exact sequences

$$\cdots \to W_n^\vee X_i \to Y_n^\vee X_i \to Z_n^\vee X_i \to W_{n-1}^\vee X_i \to \cdots$$

for all i. Since direct sums are exact in $\widehat{\mathcal{M}}$ when $n = 1$, we get the following exact sequence:

$$\cdots \to L_0(\bigoplus_i W_n^\vee X_i) \to L_0(\bigoplus_i Y_n^\vee X_i) \to L_0(\bigoplus_i Z_n^\vee X_i) \to \cdots . \tag{3.4}$$

On the other hand, we have an exact triangle

$$L_K(W \wedge \bigvee X_i) \to L_K(Y \wedge \bigvee X_i) \to L_K(Z \wedge \bigvee X_i) \to \Sigma L_K(W \wedge \vee X_i).$$

This gives rise to the exact sequence below:

$$\cdots \to W_n^\vee(\bigvee X_i) \to Y_n^\vee(\bigvee X_i) \to Z_n^\vee(\bigvee X_i) \to \cdots .$$

There is a map from the exact sequence (3.4) to this one, which is an isomorphism on two out of every three terms. Hence it is also an isomorphism on the third term, so $Y \in \mathcal{D}$, as required.

We now know that $\mathcal{D}$ is a thick subcategory. We know that E is in $\mathcal{D}$ by Theorem A and Lemma 3.1. Since $L_K S^0$ is in the thick subcategory generated by E [HS99, Theorem 8.9], we see that $L_K S^0 \in \mathcal{D}$, proving the theorem. □

Corollary 3.5. *Suppose we have a sequence*

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots$$

of $K(1)$-local spectra with sequential colimit X. Then we have a short exact sequence

$$0 \to L_0(\operatorname{colim}_n \pi_s X_n) \to \pi_s L_{K(1)} X \to L_1(\operatorname{colim}_n \pi_{s-1} X_n) \to 0.$$

Here $L_1 A = \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_p, A)$ for an abelian group A. This corollary is proved in the same way as the Milnor exact sequence for the homotopy of an inverse limit. A nice example of this corollary is the sequence

$$L_{K_1} M(p) \to L_{K(1)} M(p^2) \to L_{K(1)} M(p^3) \to \cdots$$

It is well known that $L_{K(1)}(\operatorname{colim} L_{K(1)} M(p^n)) = \Sigma L_{K(1)} S^0$, but this is always slightly mysterious since $L_{K(1)} M(p^n)$ is torsion. In terms of homotopy groups, though, we have

$$\pi_0 \operatorname{colim} L_{K(1)} M(p^n) = \mathbb{Q}/\mathbb{Z}_p,$$

and so

$$L_1(\pi_0 \operatorname{colim} L_{K(1)} M(p^n)) = \mathbb{Z}_p,$$

contributing to $\pi_1 L_{K(1)}(\operatorname{colim} L_{K(1)} M(p^n))$.

4. Appendix: Homotopy colimits

The object of this appendix is to prove that homotopy commutes with homotopy colimits in nice enough model categories, such as the category of modules over an S-algebra R. We will need some model category theory; basic definitions can be found in [Hir03] or [Hov99]. First of all, we will be working with a cofibrantly generated model category $\mathcal{M}$. In this setting, it is easy to construct homotopy colimits. Given a small category $\mathcal{I}$, there is a model structure on the diagram category $\mathcal{M}^{\mathcal{I}}$ in which weak equivalences and fibrations of diagrams are simply objectwise weak equivalences and fibrations. The colimit functor is a left Quillen functor, and the homotopy colimit is simply the left derived functor of colimit.

Now recall that an object A in a category $\mathcal{C}$ is called **finitely presented** with respect to a subcategory $\mathcal{D}$ if the natural map

$$\operatorname{colim} \mathcal{C}(A, X_i) \to \mathcal{C}(A, \operatorname{colim} X_i)$$

is an isomorphism for all filtered diagrams $\{X_i\}$ in $\mathcal{D}$. For example, the argument of [Hov99, Proposition 2.4.2] shows that compact topological spaces are finitely presented with respect to the closed T_1 inclusions, and in particular, with respect to the Serre cofibrations. It then follows, with some work, that the domains and codomains of the generating cofibrations and trivial cofibrations of S-modules $\mathcal{M}_S$ or R-modules $\mathcal{M}_R$ for R an S-algebra are also finitely presented with respect to the cofibrations in their model structures. See, for example, Lemma 2.3 of [EKMM97].

Theorem 4.1. *Suppose $\mathcal{M}$ is a pointed simplicial cofibrantly generated model category in which the domains and codomains of the generating cofibrations are cofibrant, and the domains and codomains of the generating trivial cofibrations are finitely presented with respect to the cofibrations. For any cofibrant A such that A and $A \times I$ are finitely presented with respect to the cofibrations, and for any filtered diagram $\{X_i\}$, the natural map*

$$\operatorname{colim} \operatorname{ho} \mathcal{M}(A, X_i) \to \operatorname{ho} \mathcal{M}(A, \operatorname{hocolim} X_i)$$

is an isomorphism. In particular, homotopy commutes with filtered homotopy colimits of R-modules for any S-algebra R.

Before proving this theorem, we explain why we require that the domains and codomains of the generating cofibrations be cofibrant.

Lemma 4.2. *Suppose $\mathcal{M}$ is a cofibrantly generated model category in which the domains and codomains of the generating cofibrations are cofibrant, and $\mathcal{I}$ is a small category. Then for any cofibrant object X of $\mathcal{M}^{\mathcal{I}}$, the structure maps $X_i \to X_j$ are cofibrations.*

Proof. The generating cofibrations of $\mathcal{M}^{\mathcal{I}}$ are the maps $F_i f$, where $f\colon K \to L$ is a generating cofibration of $\mathcal{M}$ and $i \in \mathcal{I}$. We can write X as a retract of a transfinite composition

$$* = X^0 \xrightarrow{\sigma^0} X^1 \xrightarrow{\sigma^1} \cdots$$

where each map σ^α is a pushout of a map of the form $F_i f$. We prove by transfinite induction on α that the structure maps of X^α are cofibrations. For the successor

ordinal step, given $j \to k$ in $\mathcal{I}$, we have a map from the pushout square

$$\begin{tikzcd} (F_iK)_j \arrow[r, "F_if"] \arrow[d] & (F_iL)_j \arrow[d] \\ X_j^\alpha \arrow[r] & X_j^{\alpha+1} \end{tikzcd}$$

to the pushout square below:

$$\begin{tikzcd} (F_iK)_k \arrow[r, "F_if"] \arrow[d] & (F_iL)_k \arrow[d] \\ X_k^\alpha \arrow[r] & X_k^{\alpha+1}. \end{tikzcd}$$

This map is a cofibration in all of the corners except possibly the lower right corners, by induction and the fact that K and L are cofibrant. Furthermore, the map

$$(F_iK)_k \amalg_{(F_iK)_j} (F_iL)_j \to (F_iL)_k$$

is a coproduct of some copies of L with a coproduct of copies of $f\colon K \to L$, so is a cofibration. Lemma 7.2.15 of [Hir03] then implies that $X_j^{\alpha+1} \to X_k^{\alpha+1}$ is a cofibration.

Now let β be a limit ordinal. We claim that the map of β-sequences

$$\begin{tikzcd} X_j^0 \arrow[r] \arrow[d] & X_j^1 \arrow[r] \arrow[d] & \cdots \\ X_k^0 \arrow[r] & X_k^1 \arrow[r] & \cdots, \end{tikzcd}$$

which is a degreewise cofibration by the induction hypothesis, is in fact a cofibration in the model structure on β-sequences. This model structure is a special case of the model structure on diagrams and is described in [Hov99, Section 5.1]. In particular, to show this map is a cofibration, we must show that the map

$$q\colon\ X_k^\alpha \amalg_{X_j^\alpha} X_j^{\alpha+1} \to X_k^{\alpha+1}$$

is a cofibration. Since $X^\alpha \to X^{\alpha+1}$ is a pushout of F_if for some generating cofibration $f\colon\ K \to L$ and some $i \in \mathcal{I}$, we see that q is isomorphic to the map

$$X_k^\alpha \amalg_{(F_iK)_k} [(F_iK)_k \amalg_{(F_iK)_j} (F_iL)_j] \to X_k^\alpha \amalg_{(F_iK)_k} (F_iL)_k.$$

This map is a cofibration by another application of [Hir03, Lemma 7.2.15]. Thus $X_j^* \to X_k^*$ is a cofibration of β-sequences. Since the colimit functor is a left Quillen functor, we conclude that the map $X_j^\beta \to X_k^\beta$ is a cofibration, completing the limit ordinal step of the induction. $\square$

Proof of Theorem 4.1. To compute $\operatorname{ho}\mathcal{M}(A, \operatorname{hocolim} X_i)$, we can assume X is cofibrant and fibrant in $\mathcal{M}^{\mathcal{I}}$. This means that every X_i is fibrant and that $\operatorname{hocolim} X_i \cong \operatorname{colim} X_i$ in $\operatorname{ho}\mathcal{M}$. Because the domains and codomains of the generating trivial cofibrations are finitely presented with respect to the cofibrations, it follows, using Lemma 4.2, that $\operatorname{colim} X_i$ is fibrant. Hence

$$\operatorname{ho}\mathcal{M}(A, \operatorname{hocolim} X_i) \cong \mathcal{M}(A, \operatorname{colim} X_i)/(\sim),$$

where $\sim$ denotes the (left or right) homotopy relation. Since both A and $A \times I$ are finitely presented with respect to the cofibrations, this is in turn isomorphic to

$$\operatorname{colim} \mathcal{M}(A, X_i)/(\sim) \cong \operatorname{colim} \operatorname{ho} \mathcal{M}(A, X_i),$$

as required. □

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