TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 1, January 2008, Pages 395–410
S 0002-9947(07)04310-3
Article electronically published on July 23, 2007

COMPOSITION OPERATORS ON HARDY SPACES ON LAVRENTIEV DOMAINS

EVA A. GALLARDO-GUTIÉRREZ, MARÍA J. GONZÁLEZ, AND ARTUR NICOLAU

Abstract. For any simply connected domain Ω, we prove that a *Littlewood type inequality* is necessary for boundedness of composition operators on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$, whenever the symbols are finitely-valent. Moreover, the corresponding "little-oh" condition is also necessary for the compactness. Nevertheless, it is shown that such an inequality is not sufficient for characterizing bounded composition operators even induced by univalent symbols. Furthermore, such inequality is no longer necessary if we drop the extra assumption on the symbol of being finitely-valent. In particular, this solves a question posed by Shapiro and Smith (2003). Finally, we show a striking link between the geometry of the underlying domain Ω and the symbol inducing the composition operator in $\mathcal{H}^p(\Omega)$, and in this sense, we relate both facts characterizing bounded and compact composition operators whenever Ω is a Lavrentiev domain.

1. Introduction and preliminaries

Let Ω be a simply connected domain properly contained in the complex plane $\mathbb{C}$ with locally rectifiable boundary $\partial\Omega$. Let τ be a Riemann map that takes the open unit disc $\mathbb{D}$ onto Ω. For $1 \le p < \infty$, the Hardy space $\mathcal{H}^p(\Omega)$ consists of holomorphic functions F on Ω such that the norm

$$\|F\|_p = \left(\frac{1}{2\pi} \sup_{0<r<1} \int_{\tau(\{|z|=r\})} |F(w)|^p \, |dw| \right)^{1/p}$$

is finite. Here, $|dw|$ denotes the arc-length measure on $\partial\Omega$.

We note that, although this norm depends on the choice of the Riemann map, any other Riemann map induces an equivalent norm on $\mathcal{H}^p(\Omega)$, and therefore, the Hardy space $\mathcal{H}^p(\Omega)$ is well defined. As particular instances, we have the classical Hardy spaces on the unit disc $\mathcal{H}^p(\mathbb{D})$ whenever $\Omega = \mathbb{D}$ and τ is the identity map. For more about these spaces, we refer the reader to Duren's book [5].

If Φ is a holomorphic map on Ω, that takes Ω into itself, then the equation

$$C_\Phi F = F \circ \Phi$$

Received by the editors April 28, 2005 and, in revised form, February 21, 2006.

2000 *Mathematics Subject Classification.* Primary 47B38, 30C85.

Key words and phrases. Composition operator, Lavrentiev domain, Nevanlinna counting function.

The first author was partially supported by Plan Nacional I+D grant no. MTM2006-06431 and Gobierno de Aragón ref. DGA E-64. The second and third authors were partially supported by Plan Nacional I+D grant no. MTM2005-00544 and 2005SGR00774.

defines a composition operator C_Φ on the space $\mathcal{H}(\Omega)$ of all holomorphic functions on Ω. In case that Ω is the open unit disc, Littlewood [8] proved in 1925 that any composition operator C_Φ is bounded on any Hardy space $\mathcal{H}^p(\mathbb{D})$; a result known as *Littlewood Subordination Principle.* In the eighties, Shapiro [11] characterized the compactness of C_Φ on $\mathcal{H}^p(\mathbb{D})$ in terms of the *Nevanlinna counting function* for Φ, which is defined by

$$N_\Phi(w) = \begin{cases} \displaystyle\sum_{z \in \Phi^{-1}\{w\}} \log \frac{1}{|z|} & \text{if} \quad w \in \Phi(\Omega) \setminus \{\Phi(0)\}, \\ 0 & \text{if} \quad w \notin \Phi(\Omega), \end{cases}$$

where $\Phi^{-1}\{w\}$ denotes the sequence of the Φ-preimages of w, with each point written down as many time as its multiplicity.

More recently, Shapiro and Smith [12] have shown that the geometry of the domain Ω plays an important role in the boundedness and compactness of C_Φ on $\mathcal{H}^p(\Omega)$. In particular, they prove that the condition of boundedness for the derivative of the Riemann map τ and its reciprocal actually characterizes the domains Ω for which every composition operator is bounded in $\mathcal{H}^p(\Omega)$. Moreover, $\mathcal{H}^p(\Omega)$ supports a compact composition operator if and only if $\partial\Omega$ has finite one-dimensional Hausdorff measure. They ask for a characterization of boundedness and compactness of composition operators C_Φ along the lines of the results in [11].

The aim of this paper is to relate the geometry of the domain Ω to the fact that boundedness (respectively compactness) of C_Φ on $\mathcal{H}^p(\Omega)$ can be characterized in terms of a Nevanlinna type condition for Φ in Ω. From this point of view, if $\delta(z, \partial\Omega)$ denotes the distance from z to the boundary of Ω, we define the function $\widetilde{N}_{\Phi,\Omega}$ associated to Φ in Ω by

$$\widetilde{N}_{\Phi,\Omega}(w) = \begin{cases} \displaystyle\sum_{z \in \Phi^{-1}\{w\}} \delta(z, \partial\Omega) & \text{if} \quad w \in \Phi(\Omega), \\ 0 & \text{if} \quad w \notin \Phi(\Omega). \end{cases}$$

Observe that when Ω is the unit disc $\mathbb{D}$, the function $\widetilde{N}_{\Phi,\mathbb{D}}$ is closely related to the *Nevanlinna counting function* N_Φ. Actually, *Littlewood's Subordination Principle* is equivalent to the fact that any holomorphic map Φ taking $\mathbb{D}$ into itself and $\Phi(0) = 0$ satisfies

$$N_{\Phi,\mathbb{D}}(w) \le \log \frac{1}{|w|} \qquad (w \in \mathbb{D})$$

(see [11], for instance), or equivalently,

$$\widetilde{N}_{\Phi,\mathbb{D}}(w) \lesssim \delta(w, \partial\mathbb{D}) \qquad (w \in \mathbb{D}). \tag{1}$$

Throughout this work, $a \lesssim b$ denotes that there exists an independent constant C such that $a \le C\,b$. In addition, note that *Shapiro's Compactness Theorem* can be restated by saying that the condition $\widetilde{N}_{\Phi,\mathbb{D}}(w) = \mathbf{o}(\delta(w, \partial\mathbb{D}))$ as $\delta(w, \partial\mathbb{D}) \to 0$ characterizes those symbols Φ taking $\mathbb{D}$ into itself inducing compact composition operators on $\mathcal{H}^p(\mathbb{D})$.

We should mention here that given a simply connected domain Ω and $\varphi : \Omega \to \Omega$ an analytic function, there is a natural way to define the Nevanlinna counting function for φ in Ω in terms of Green's function of Ω with pole at some point in Ω. In fact, this definition fairly generalizes the corresponding one in the unit disc $\mathbb{D}$.

Nevertheless, this point of view is closely concerned with those Hardy spaces $\mathbb{H}^p(\Omega)$ defined by means of the conformal invariance; that is, if τ is a Riemann map taking $\mathbb{D}$ onto Ω, then $F \in \mathbb{H}^p(\Omega)$ if and only if $F \circ \tau \in \mathcal{H}^p(\mathbb{D})$ for $1 \le p < \infty$. It holds that the spaces $\mathbb{H}^p(\Omega)$ and $\mathcal{H}^p(\Omega)$ coincide if and only if $|\tau'|$ is bounded away from 0 and ∞ (see [5, Chapter 10] for more about this subject). We point out that conformal invariance techniques reduce those questions about boundedness and compactness of composition operators on $\mathbb{H}^p(\Omega)$ to the corresponding ones on the classical Hardy spaces $\mathcal{H}^p(\mathbb{D})$; so already answered.

On the other hand, one of the advantages of considering the function $\widetilde{N}_{\Phi,\,\Omega}$ is that precisely the geometry of the domain Ω plays a fundamental role to determine what symbols Φ induce bounded and compact composition operators on $\mathcal{H}^p(\Omega)$. Roughly speaking, we will show that whenever the boundary of Ω is, in some sense, quasi smooth, bounded and compact composition operators on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$, are completely characterized with a condition similar to that in $\mathcal{H}^p(\mathbb{D})$.

The paper is organized as follows. In Section 2 we show that for any simply connected domain Ω, under the extra hypotheses that Φ is a finitely-valent symbol, the *Littlewood type inequality*

$$\widetilde{N}_{\Phi,\,\Omega}(w) \lesssim \delta(w, \partial\Omega) \qquad (w \in \Omega)$$

is necessary for C_Φ to be bounded in $\mathcal{H}^p(\Omega)$, for any $1 \le p < \infty$. Moreover, we also prove that the corresponding "little-oh" condition is necessary for the compactness of C_Φ on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$.

Nevertheless, we show that the *Littlewood type inequality* does not suffice for characterizing boundedness of composition operators C_Φ, even induced by univalent symbols. In fact, we exhibit a simply connected domain Ω and a composition operator C_Φ such that the inducing symbol satisfies that $\widetilde{N}_{\Phi,\,\Omega}(w) \lesssim \delta(w, \partial\Omega)$ for all $w \in \Omega$ but C_Φ does not take $\mathcal{H}^p(\Omega)$ boundedly into itself.

In Section 3, we show that *Littlewood type inequality* is sufficient, without any extra assumption on the valence of the symbol Φ, if we impose a geometrical condition on the domain: $\partial\Omega$ is a Lavrentiev curve. On the contrary, it is no longer necessary if we drop the extra assumption on Φ of being finitely-valent. To show this, we present an example in Section 4 of an infinite-valent symbol Φ not satisfying *Littlewood type inequality*, but inducing a bounded composition operator on $\mathcal{H}^p(\Omega)$.

The key point of both examples in Sections 2 and 4 is a link between the geometry of the underlying domain Ω and the symbol inducing the composition operator. In this sense, in Section 5, we relate both facts, characterizing boundedness and compactness of composition operators on $\mathcal{H}^p(\Omega)$, whenever Ω is a Lavrentiev domain.

Weighted composition operators. For the sake of completeness, we end this preliminary section relating the composition operator C_Φ acting on $\mathcal{H}^p(\Omega)$ to a weighted composition operator on the Hardy space $\mathcal{H}^p(\mathbb{D})$ (see [12]). Recall that given holomorphic maps φ, ψ on the unit disc $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, the weighted composition operator $W_{\varphi,\psi}$ is defined by

$$W_{\varphi,\psi} f(z) = \psi(z)\, f(\varphi(z)),$$

for any holomorphic map f on the unit disc. Let τ be a fixed Riemann map that takes $\mathbb{D}$ onto Ω. Taking into account that the map $F \to (\tau')^{1/p}(F \circ \tau)$ is a linear isometry which takes $\mathcal{H}^p(\Omega)$ onto $\mathcal{H}^p(\mathbb{D})$, it is not difficult to see that C_Φ is similar to the weighted composition operator induced by $\varphi(z) = \tau^{-1} \circ \Phi \circ \tau$ and $\psi(z) = (\tau'(z)/\tau'(\varphi(z)))^{1/p}$. Observe that φ actually takes $\mathbb{D}$ into itself and ψ is holomorphic on $\mathbb{D}$, since τ' does not vanish on $\mathbb{D}$. Since boundedness and compactness are properties invariant under similarities, we will deal with this particular weighted composition operator. For simplicity of notation, we will denote it by $W_{\varphi,\,p}$.

In addition, we remark here that although the results in this paper are stated for any $1 \le p < \infty$, it is enough to prove them for $p = 2$. This is a consequence of the fact that if C_Φ is bounded (respectively compact) on $\mathcal{H}^p(\Omega)$ for some $1 \le p < \infty$, then it is bounded (respectively compact) for all p (see [3], for instance). Therefore, the corresponding weighted composition operator $W_{\varphi,\,2}$ (for abbreviation W_φ) is given by the formula

$$W_\varphi f = (\tau'(z)/\tau'(\varphi(z)))^{1/2} f(\varphi(z)), \qquad f \in \mathcal{H}^2(\mathbb{D}).$$

Finally, we would like to mention that boundedness and compactness of general weighted composition operators in $\mathcal{H}^p(\mathbb{D})$ were characterized by Contreras and Hernández-Díaz [3] in terms of Carleson conditions on pullback measures. Nevertheless, as mentioned before, we are interested in a different aspect of the subject: the geometry of the underlying domain Ω.

2. Littlewood type inequality and finitely-valent symbols

In this section we begin by showing that *Littlewood type inequality* is necessary for boundedness of composition operators on $\mathcal{H}^p(\Omega)$, $1 \le p < \infty$, whenever the symbol is, at most, of valence finite. Nevertheless, we will provide an example showing that the condition is not sufficient even when univalent symbols are considered.

Theorem 2.1. *Let Ω be a simply connected domain properly contained in $\mathbb{C}$. Let Φ be a holomorphic map, finitely-valent on Ω, such that $\Phi(\Omega) \subset \Omega$. Assume that C_Φ is bounded in $\mathcal{H}^p(\Omega)$, for some $1 \le p < \infty$. Then*

$$\widetilde{N}_{\Phi,\,\Omega}(w) \lesssim \delta(w, \partial\Omega)$$

for any $w \in \Omega$. Moreover, if C_Φ is compact on $\mathcal{H}^p(\Omega)$ for some $1 \le p < \infty$, then

$$\widetilde{N}_{\Phi,\,\Omega}(w) = \mathbf{o}\,(\delta(w, \partial\Omega))$$

as $\delta(w, \partial\Omega)$ tends to zero.

A word about notation. Before proceeding further, we should mention that throughout this paper, we denote $a \approx b$ whenever there exist two positive universal positive constants c and C, such that $c\,b \le a \le C\,b$. In addition, for the sake of simplicity, C will always denote an independent constant, which can be different from one display to another.

Proof of Theorem 2.1. We may restrict ourselves to $p = 2$. First, let us assume that C_Φ is bounded $\mathcal{H}^2(\Omega)$ and the valence of Φ is N, that is, for any $w \in \Omega$, the set $\{z \in \Omega : \, \Phi(z) = w\}$ has at most N elements.

We will transform the Littlewood condition into one easier to handle. Fix $w \in \Omega$ and let $\{z_j\} \subset \Omega$, $1 \le j \le N$, be the sequence (possibly empty) of the Φ-preimages of w. For each $1 \le j \le N$, let ξ_j be in $\mathbb{D}$ such that $\tau(\xi_j) = z_j$. Let ζ also be in $\mathbb{D}$ such that $\tau(\zeta) = w$. Since $\varphi = \tau \circ \Phi \circ \tau^{-1}$, it is clear that $\varphi(\xi_j) = \zeta$ for any $1 \le j \le N$.

Since τ is a conformal mapping, a consequence of Koebe Distortion Theorem asserts that

$$\frac{1}{4}\,|\tau'(u)|\,(1-|u|^2) \leq \delta(\tau(u),\partial\Omega) \leq |\tau'(u)|\,(1-|u|^2) \qquad (u \in \mathbb{D})$$

(see [9, p. 9], for instance). Then, we deduce that

$$\frac{\widetilde{N}_{\Phi,\,\Omega}(w)}{\delta(w,\partial\Omega)} = \frac{\sum_j \delta(z_j,\partial\Omega)}{\delta(w,\partial\Omega)} \leq 4\frac{\sum_j |\tau'(\xi_j)|\,(1-|\xi_j|^2)}{|\tau'(\zeta)|\,(1-|\zeta|^2)}.$$

So, it is enough to show that the quotient in the right hand in the above display remains bounded. Observe that the sum involved is finite since $1 \leq j \leq N$.

For each $1 \leq j \leq N$, let k_{ξ_j} be the reproducing kernel at ξ_j in $\mathcal{H}^2(\mathbb{D})$, that is,

$$k_{\xi_j}(u) = \frac{1}{1-\overline{\xi_j}\,u} \qquad (u \in \mathbb{D}).$$

If W_φ^* denotes the adjoint of W_φ, it is not difficult to see (see [12], for instance) that

$$W_\varphi^* k_{\xi_j}(z) = \overline{\left(\frac{\tau'(\xi_j)}{\tau'(\varphi(\xi_j))}\right)^{1/2}}\, k_{\varphi(\xi_j)}(z).$$

If $K_{\xi_j} = k_{\xi_j}/\|k_{\xi_j}\|$, we deduce that for any $1 \leq j \leq N$ there holds

$$\|W_\varphi^* K_{\xi_j}\|^2 = \left|\frac{\tau'(\xi_j)}{\tau'(\zeta)}\right| \frac{1-|\xi_j|^2}{1-|\zeta|^2} \leq \|W_\varphi^*\|^2,$$

since W_φ^* is a bounded operator by hypotheses. Now, summing in the index j, it follows that

$$\frac{\sum_j |\tau'(\xi_j)|\,(1-|\xi_j|)}{|\tau'(\zeta)|\,(1-|\zeta|)} \leq N\,\|W_\varphi^*\|^2, \tag{2}$$

where N is the valence of Φ. This proves the first half of Theorem 2.1.

Now, assume that C_Φ is compact on $\mathcal{H}^2(\Omega)$. Then, W_φ is compact on $\mathcal{H}^2(\mathbb{D})$. Let $w \in \Omega$ be fixed and $\{\xi_j\}_{j=1}^N \subset \mathbb{D}$ such that $\Phi(\tau(\xi_j)) = w$ for $j = 1,\cdots,N$. As before, we have

$$\frac{\widetilde{N}_{\Phi,\,\Omega}(w)}{\delta(w,\partial\Omega)} \leq C \sum_j \|W_\varphi^* K_{\xi_j}\|^2, \tag{3}$$

for some positive universal constant C. Now, since K_{ξ_j} converges weakly to zero as $|\xi_j| \to 1^-$ (see [4]), and the sum involved is a finite one, we deduce from (3) that

$$\frac{\widetilde{N}_{\Phi,\,\Omega}(w)}{\delta(w,\partial\Omega)} \to 0$$

as $|\xi_j| \to 1^-$ and hence, as $\delta(w,\partial\Omega)$ tends to zero. This completes the proof. $\square$

Remark 2.2. Observe that the constant in the right hand of (2) depends explicitly on the valence of Φ. In fact, this is the crucial point which shows, as we see in the following section, that the proof cannot be generalized to infinite-valent symbols.

2.1. **Littlewood type inequality does not suffice.** Now, we will exhibit a simply connected domain Ω and a holomorphic self-map such that Φ is univalent and satisfies *Littlewood type inequality*, but the composition operator induced by Φ is not bounded in any $\mathcal{H}^p(\Omega)$ any longer.

To describe the required domain Ω, let the boundary $\partial\Omega$ be a heart shaped curve with an inward-pointing cusp in -1, and an outward-pointing cusp in 0 such that a Riemann map $\tau : \mathbb{D} \to \Omega$ with $\tau(-1) = -1$ and $\tau(1) = 0$ behaves in a neighborhood of 1 like $1/\log(1-z)$ (see Figure 1). For the sake of simplicity, we call such an outward-pointing cusp a *logarithmic outward-pointing cusp.*

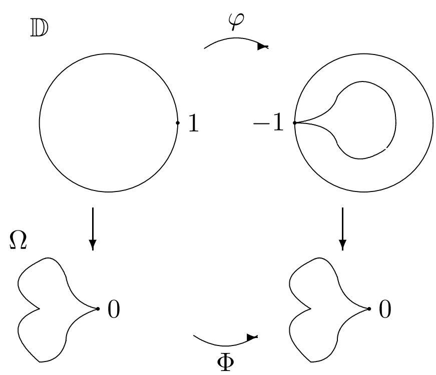

FIGURE 1

Let $\Gamma \subset \overline{\mathbb{D}}$ be a teardrop shaped curve with $\Gamma \cap \partial\mathbb{D} = \{-1\}$ and a logarithmic outward-pointing cusp in -1. Then, a Riemann map φ that takes $\mathbb{D}$ onto the simply connected domain bounded by Γ with $\varphi(1) = -1$ behaves in a neighborhood of 1 like $-1 - 1/\log(1-z)$.

Let us consider $\Phi = \tau \circ \varphi \circ \tau^{-1}$. Observe that Φ actually takes Ω into itself. Under the conditions stated above, we have the following

Proposition 2.3. *C_Φ is not bounded on $\mathcal{H}^p(\Omega)$ for any $1 \le p < \infty$. Nevertheless,*

$$\widetilde{N}_{\Phi,\Omega}(w) \lesssim \delta(w, \partial\Omega) \qquad \textit{for all } w \in \Omega.$$

Proof. Again, we restrict ourselves to $p = 2$. First, we show that C_Φ is not bounded $\mathcal{H}^2(\Omega)$. It suffices to prove that the corresponding weighted composition operator

$$W_\varphi f = (\tau'(z)/\tau'(\varphi(z)))^{1/2} f(\varphi(z)), \qquad f \in \mathcal{H}^2(\mathbb{D}),$$

is not bounded on $\mathcal{H}^2(\mathbb{D})$. To this purpose, we show that $W_\varphi 1 \notin \mathcal{H}^2(\mathbb{D})$, i.e., the integral

$$\int_0^{2\pi} \left| \frac{\tau'(e^{i\theta})}{\tau'(\varphi(e^{i\theta}))} \right| d\theta \tag{4}$$

is not convergent. To check this, first we note that τ can be extended to a homeomorphism on $\overline{\mathbb{D}}$ mapping the boundary $\partial\mathbb{D}$ onto $\partial\Omega$. Moreover, since $\partial\Omega$ is Dini-smooth and has an inward-pointing cusp in -1, $\tau(z)$ behaves in a neighborhood of -1 like $(z+1)^2 - 1$ (see [9, Theorem 3.9]). By assumption, there is a neighborhood of 1 so that $\tau(z)$ behaves like $1/\log(1-z)$ and $\varphi(z)$ behaves like $-1-1/\log(1-z)$. Hence, in a neighborhood of 1, we deduce that $|\tau'(z)|$ behaves like

$$\left|\frac{1}{(1-z)\log^2(1-z)}\right|,$$

and since $\varphi(1)=-1$, we have that $|\tau'(\varphi((z)))|$ behaves like

$$2|\varphi(z)+1| \approx 2/|\log(1-z)|.$$

Then, we deduce that the integral

$$\int_0^{2\pi} \left| \frac{\frac{1}{(1-e^{i\theta})\log^2(1-e^{i\theta})}}{\frac{2}{\log(1-e^{i\theta})}} \right| \, d\theta$$

diverges, and therefore, W_φ is not bounded.

Now, our task is to show that $\widetilde{N}_{\Phi,\Omega}(w) \lesssim \delta(w,\partial\Omega)$ for all $w \in \Omega$. Since Φ is univalent, we are reduced to proving that

$$\delta(\Phi^{-1}(w),\partial\Omega) \lesssim \delta(w,\partial\Omega) \qquad (w \in \Omega). \tag{5}$$

Let $w \in \Phi(\Omega)$ and $z \in \mathbb{D}$ be the τ-preimage of $\Phi^{-1}(w)$. It is clear that $\varphi(z)$ is the τ-preimage of w since $\Phi = \tau \circ \varphi \circ \tau^{-1}$. Once again, as a consequence of Koebe Distortion Theorem, we deduce that

$$\frac{\delta(\Phi^{-1}(w),\partial\Omega)}{\delta(w,\partial\Omega)} \le 4\,\frac{|\tau'(z)|\,(1-|z|^2)}{|\tau'(\varphi(z))|\,(1-|\varphi(z)|^2)}, \tag{6}$$

and therefore, it is enough to show that the quotient in the right hand in (6) remains bounded for $z \in \mathbb{D}$.

In order to check this, we observe that it is enough to deal with just those points z whose image $\varphi(z)$ is close to the boundary $\partial\mathbb{D}$. Because of the choice of φ, it suffices to prove that such a quotient remains bounded when z is close to 1. Once again, using the behavior of φ and τ around 1 and taking into account that $\varphi(\mathbb{D})$ is contained in a Stolz angle with vertex at -1, we deduce that there exist a neighborhood U of 1 and a positive constant C such that

$$1-|\varphi(z)|^2 \ge C\,|1+\varphi(z)|$$

for $z \in U$, and therefore a little computation shows that

$$|\tau'(z)|\,(1-|z|^2)/|\tau'(\varphi(z))|\,(1-|\varphi(z)|^2)$$

remains bounded when z is close to 1, which completes the proof. $\square$

Remark 2.4. Observe that the composition operator induced by φ is not only bounded, but also compact on the Hardy space $\mathcal{H}^2(\mathbb{D})$. This is a consequence of the fact that $\varphi(\mathbb{D})$ is contained in a nontangential approach region near -1 (see [4, Ch. 3] for the details to this argument). Nevertheless, C_Φ does not inherit from C_φ even the property of being bounded.

3. Littlewood type inequality and geometry of the domain Ω

In this section, we show that the *Littlewood type inequality* suffices, without any extra assumption on the valence of the symbol Φ, if we impose a geometrical condition on the domain. Roughly speaking, if we restrict ourselves to domains Ω where the boundary $\partial\Omega$ is, somehow, *smooth*, we show that C_Φ is bounded on $\mathcal{H}^p(\Omega)$ for any $1 \le p < \infty$ if Φ satisfies the *Littlewood type inequality.*

We begin by recalling that a *Lavrentiev curve* Γ in the complex plane $\mathbb{C}$ is a rectifiable Jordan curve such that there exists a constant $M > 0$ satisfying

$$\text{length}(\Gamma(a,b)) \le M|a-b| \qquad \text{for } a,\, b \in \Gamma,$$

where $\Gamma(a,b)$ is the shorter arc of Γ between a and b. In other words, the length of the arc between a and b is comparable to the chord joining a and b. Lavrentiev curves are also called *chord-arc* curves. The inner domain of a closed Lavrentiev curve is called a *Lavrentiev domain.* For more about the subject, we refer the reader to Pommerenke's book ([9, chapter 7]). We are in a position to state our result:

Theorem 3.1. *Let Ω be a Lavrentiev domain properly contained in $\mathbb{C}$. Let Φ be any holomorphic self-map of Ω so that*

$$\widetilde{N}_{\Phi,\,\Omega}(w) \lesssim \delta(w,\partial\Omega)$$

for all $w \in \Omega$. Then C_Φ is bounded on $\mathcal{H}^p(\Omega)$ for any $1 \le p < \infty$. Furthermore, if

$$\widetilde{N}_{\Phi,\,\Omega}(w) = \mathbf{o}(\delta(w,\partial\Omega))$$

as $\delta(w,\partial\Omega) \to 0$, then C_Φ is compact on $\mathcal{H}^p(\Omega)$ for any $1 \le p < \infty$.

Remark 3.2. Observe that we are not assuming any extra condition on the valence of the symbol Φ in the statement of Theorem 3.1.

Proof of Theorem 3.1. We may consider $p = 2$. We begin by proving the first part of the theorem. To this purpose, we observe that since $\partial\Omega$ is rectifiable, the norm in $\mathcal{H}^2(\Omega)$ is equivalent to the one performed by the integral

$$\|F\| = \left(\int_{\partial\Omega} |F(z)|^2 \, |dz| \right)^{1/2} \qquad (F \in \mathcal{H}^2(\Omega))$$

(see [5, Chapter 10]). So, in order to prove that C_Φ is bounded on $\mathcal{H}^2(\Omega)$, it is enough to show that there exists a positive constant C such that

$$\int_{\partial\Omega} |F \circ \Phi(z)|^2 \, |dz| \le C \int_{\partial\Omega} |F(z)|^2 \, |dz| \qquad (F \in \mathcal{H}^2(\Omega)).$$

Let us fix $F \in \mathcal{H}^2(\Omega)$. Without loss of generality, we may assume that $0 \in \Omega$. The key point of the proof relies on the fact that, since $\partial\Omega$ is a Lavrentiev curve, given any holomorphic function F on Ω, it holds that

$$\int_{\partial\Omega} |F(z)|^2 \, |dz| \approx |F(0)|^2 + \int_{\Omega} |F'(z)|^2 \delta(z,\partial\Omega)\, dm(z) \tag{7}$$

(see [1] and [7]). Here m denotes the Lebesgue measure in the complex plane.

With equivalence (7) at hand, the rest of the proof basically follows the lines of the corresponding proof in $\mathcal{H}^2(\mathbb{D})$ (see [4, Chapter 3], for instance), although a careful analysis is required. First, we deduce that

$$\begin{aligned}\int_{\partial\Omega} |F\circ\Phi(z)|^2\,|dz| &\approx |F(\Phi(0))|^2 + \int_{\Omega} |(F\circ\Phi)'(z)|^2\,\delta(z,\partial\Omega)\,dm(z)\\ &= |F(\Phi(0))|^2 + \int_{\Phi(\Omega)} |F'(w)|^2 \left(\sum_{\Phi(z)=w} \delta(z,\partial\Omega)\right) dm(w)\\ &= |F(\Phi(0))|^2 + \int_{\Omega} |F'(w)|^2\,\widetilde{N}_{\Phi,\Omega}(w)\,dm(w), \qquad (8)\end{aligned}$$

where in the second line the change of variables $\Phi(z) = w$ has been performed. Now, we proceed to show that both terms in the sum in (8) are bounded in terms of $\|F\|^2_{\mathcal{H}^2(\Omega)}$. On one hand, we have that

$$F(\Phi(0)) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{F(\xi)}{\xi - \Phi(0)}\,d\xi$$

(see [5, Chapter 10]). Then, by mean of Cauchy-Schwarz inequality, we deduce

$$|F(\Phi(0))|^2 \lesssim \|F\|^2_{\mathcal{H}^2(\Omega)} \int_{\partial\Omega} \frac{|d\xi|}{|\xi - \Phi(0)|^2}. \qquad (9)$$

The key point now is that $\partial\Omega$ is a Lavrentiev curve, and therefore, it holds that for any $w \in \mathbb{C}\setminus\partial\Omega$

$$\int_{\partial\Omega} \frac{|d\zeta|}{|\zeta - w|^2} \approx \frac{1}{\delta(w,\partial\Omega)}$$

(see [1] and [7]). Thus, from (9) it follows that

$$|F(\Phi(0))|^2 \lesssim \frac{\|F\|^2_{\mathcal{H}^2(\Omega)}}{\delta(\Phi(0),\partial\Omega)}. \qquad (10)$$

On the other hand, since $\widetilde{N}_{\Phi,\Omega}(w) \lesssim \delta(w,\partial\Omega)$ for all $w \in \Omega$ by hypotheses, we have that

$$\begin{aligned}\int_{\Omega} |F'(w)|^2\,\widetilde{N}_{\Phi,\Omega}(w)\,dm(w) &\lesssim \int_{\Omega} |F'(w)|^2 \delta(w,\partial\Omega)\,dm(w)\\ &\lesssim \|F\|^2_{\mathcal{H}^2(\Omega)}. \qquad (11)\end{aligned}$$

From (10) and (11), the desired result follows.

Now, assume that $\widetilde{N}_{\Phi,\Omega}(w) = \mathbf{o}(\delta(w,\partial\Omega))$ as $\delta(w,\partial\Omega) \to 0$. Note that a sequence $\{F_n\} \subset \mathcal{H}^2(\Omega)$ converges weakly to zero if it is bounded in $\mathcal{H}^2(\Omega)$ and $F_n \to 0$ uniformly on compact subsets of Ω. Therefore, relation (7) along with an argument entirely similar to that one used for the corresponding result in $\mathcal{H}^2(\mathbb{D})$ (see [10, Chapter 10], for instance), yields that C_Φ is compact on $\mathcal{H}^2(\Omega)$. This completes the proof of Theorem 3.1. □

We note that Theorem 3.1 along with Theorem 2.1 provides, as an immediate consequence, the characterization of boundedness (resp. compactness) of composition operators induced by finite-valent symbols in terms of a big-oh (resp. little-oh) condition which involves the *Littlewood type inequality.* We state it as a corollary:

Corollary 3.3. *Let Ω be a Lavrentiev domain properly contained in $\mathbb{C}$ and Φ a finite-valent holomorphic self-map of Ω. Then C_Φ is bounded on $\mathcal{H}^p(\Omega)$ for some $1 \le p < \infty$ if and only if*

$$\widetilde{N}_{\Phi,\Omega}(w) \lesssim \delta(w,\partial\Omega)$$

for all $w \in \Omega$. Furthermore, C_Φ is compact on $\mathcal{H}^p(\Omega)$ for some $1 \le p < \infty$ if and only if

$$\widetilde{N}_{\Phi,\Omega}(w) = \mathbf{o}(\delta(w, \partial\Omega))$$

as $\delta(w, \partial\Omega)$ tends to 0.

Remark 3.4. We point out that the geometrical assumption on the domain Ω in Corollary 3.3 is necessary. In fact, this is clear from the example of the heart-shaped domain Ω showed in the previous section, since it is not a Lavrentiev domain.

4. Littlewood type inequality, Lavrentiev domains and infinitely-valent symbols

In this section, we show that *Littlewood type inequality* is not necessary for boundedness of composition operators induced by infinitely-valent symbols, even assuming that Ω is a Lavrentiev domain. In particular, neither Theorem 2.1 nor Corollary 3.3 holds if we consider infinitely-valent symbols. We state the result:

Proposition 4.1. *There exist a simply connected Lavrentiev domain Ω and a holomorphic infinitely-valent self-map Φ on Ω so that C_Φ is bounded on $\mathcal{H}^p(\Omega)$, for any $1 \le p < \infty$, and Φ does not satisfy Littlewood type inequality.*

Proof. Let $B(z)$ be the Blaschke product in $\mathbb{D}$ whose sequence of zeros is $\{1-\frac{1}{j^2}\}_{j\ge1}$, and $\varphi(z) = B(z)/2$, that is:

$$\varphi(z) = \frac{1}{2}\prod_{j=1}^{\infty} \frac{-z + (1 - 1/j^2)}{1 - (1 - 1/j^2)z} \qquad (z \in \mathbb{D}).$$

It is clear that φ takes the unit disc $\mathbb{D}$ into itself. Let $\tau(z) = 1 - (1 - z)^{1/4}$ for $z \in \mathbb{D}$. Then, $\tau(\mathbb{D})$ is a teardrop-shaped domain, symmetric about the real axis whose boundary meets $\partial\mathbb{D}$ just at 1, where it makes an angle of $\pi/8$ radians with the unit interval (see Figure 2).

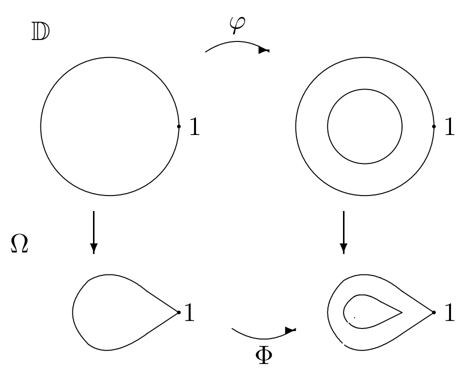

Figure 2

Let Ω be the domain $\tau(\mathbb{D})$ and $\Phi = \tau \circ \varphi \circ \tau^{-1}$. It is clear that Φ is a holomorphic self-map of Ω. Moreover, we claim that C_Φ is bounded on $\mathcal{H}^2(\Omega)$. To check this, it is enough to show that the corresponding weighted composition operator W_φ, defined in our case by:

$$W_\varphi f(z) = \left(\frac{1-\varphi(z)}{1-z}\right)^{3/8} f\circ\varphi(z),$$

for $f \in \mathcal{H}^2(\mathbb{D})$ and $z \in \mathbb{D}$, is bounded in $\mathcal{H}^2(\mathbb{D})$. Since φ has sup-norm strictly less than one, $f\circ\varphi \in \mathcal{H}^\infty(\mathbb{D})$. This along with the fact that $(1-z)^{-3/8} \in \mathcal{H}^2(\mathbb{D})$ yields that that W_φ is bounded on $\mathcal{H}^2(\mathbb{D})$. In general, it holds that whenever φ has sup-norm strictly less than one, W_φ is bounded on $\mathcal{H}^2(\mathbb{D})$ if and only if $\tau' \in \mathcal{H}^1(\mathbb{D})$.

Now, we proceed to show that Φ does not satisfy the *Littlewood type inequality*, that is, $\widetilde{N}_{\Phi,\,\Omega}(w)$ is not bounded by $\delta(w,\Omega)$ for some $w\in\Omega$.

Let us consider $w=\tau(0)=0$. Observe that $\{z_j=\tau(1-1/j^2)\}_{j\geq 1}$ is the sequence of the Φ-preimages of w. We claim that the series

$$\widetilde{N}_{\Phi,\,\Omega}(0)=\sum_{j=1}^{\infty}\delta(z_j,\partial\Omega)$$

is not convergent. In fact, by the Koebe Distortion Theorem, this is equivalent to showing that

$$\sum_{j=1}^{\infty}|\tau'(1-1/j^2)|\,\frac{1}{j^2}$$

diverges, which is the case since $\tau'(z)=1/4(1-z)^{3/4}$. □

5. Boundedness and compactness of composition operators on Lavrentiev domains: The complete characterization

In this section, we characterize boundedness and compactness of composition operators on $\mathcal{H}^p(\Omega)$ for any $1\leq p<\infty$, whenever Ω is a Lavrentiev domain. As we have shown in the previous section, boundedness of composition operators induced by general symbols cannot be characterized on $\mathcal{H}^p(\Omega)$ even when Ω is a Lavrentiev domain, in terms of the pointwise condition stated by the *Littlewood type inequality.* Nevertheless, we show that it is possible to get such a characterization if we consider *Littlewood type inequality* "in means".

Before stating our result, we recall some facts. Let Ω be a simply connected domain with $\partial\Omega$ locally rectifiable. A *Carleson disc in* Ω is any disc centered at a point in the boundary $\partial\Omega$. A positive measure μ on Ω is called a *Carleson measure in* Ω if

$$\|\mu\| = \sup_{\xi\in\partial\Omega,\ r>0}\frac{\mu(B(\xi,r))}{r}<\infty,$$

for any Carleson disc $B(\xi,r)$ in Ω (see [14]). We call Ω a Carleson domain if there exists a positive constant C such that for any Carleson measure μ in Ω

$$\int_\Omega |f(z)|\,d\mu(z)\leq C\|\mu\|\int_{\partial\Omega}|f(z)|\,|dz|,$$

for any $f\in\mathcal{H}^1(\Omega)$. It is a well known theorem due to Carleson [2] that the unit disc $\mathbb{D}$ is a Carleson domain. A characterization of Carleson domains was given by Zinsmeister [14], showing, in particular, that Lavrentiev domains are Carleson domains.

Let us suppose that Ω is a Lavrentiev domain. Then $\partial\Omega$ is a Lavrentiev curve, and such curves are characterized as the images of the unit circle under bilipschitz maps from $\mathbb{C}$ onto $\mathbb{C}$. Recall that h is a bilipschitz map of a set A into $\mathbb{C}$ if there exists a positive constant c such that

$$c^{-1}\,|z_1 - z_2| \leq |h(z_1) - h(z_2)| \leq c\,|z_1 - z_2|, \tag{12}$$

for z_1, $z_2 \in A$. The smallest constant c in (12) is called the Lavrentiev constant of $\partial\Omega$ (see [9, chapter 7], for instance).

In what follows, for any set E and any integrable function f, we denote the integral mean of f over E by

$$⨍_E f\,dm = \frac{1}{m(E)} \int_E f\,dm.$$

We are in a position to state our result:

Theorem 5.1. *Let Ω be a Lavrentiev domain and Φ a holomorphic map on Ω, taking Ω into itself. Then C_Φ is bounded on $\mathcal{H}^p(\Omega)$ for any $1 \leq p < \infty$ if and only if for any Carleson disc $B(\xi, r)$ in Ω*

$$⨍_{B(\xi,r)} \widetilde{N}_{\Phi,\,\Omega}\,dm \leq C\,r,$$

where C is a positive constant just depending on the Lavrentiev constant of $\partial\Omega$. Furthermore, C_Φ is compact on $\mathcal{H}^p(\Omega)$ for any $1 \leq p < \infty$ if and only if for any Carleson disc $B(\xi, r)$ in Ω

$$\lim_{r\to 0} \frac{1}{r} ⨍_{B(\xi,r)} \widetilde{N}_{\Phi,\,\Omega}\,dm = 0.$$

Proof. Assume that C_Φ is bounded on $\mathcal{H}^p(\Omega)$, for $1 \leq p < \infty$. As usual, we may restrict ourselves to $p = 2$.

Let us fix $B(\xi_0, r)$ a Carleson disc in Ω. Since $\partial\Omega$ is a Lavrentiev curve, there exists $w_0^* \in \mathbb{C} \setminus \Omega$ such that $|w_0^* - \xi_0| \approx \delta(w_0^*, \partial\Omega) \approx r$. We claim that the analytic function on the domain Ω

$$K_{w_0^*}(w) = \frac{(\delta(w_0^*, \partial\Omega))^{1/2}}{w - w_0^*} \qquad (w \in \Omega),$$

belongs to $\mathcal{H}^2(\Omega)$. So, let us check that

$$\int_{\partial\Omega} |K_{w_0^*}(w)|^2\,|dw|$$

remains bounded. But this holds as a consequence of the following result (also used in the proof of Theorem 3.1): if Γ is a Lavrentiev curve, then for any $w \in \mathbb{C} \setminus \Gamma$

$$\int_\Gamma \frac{|d\zeta|}{|\zeta - w|^2} \approx \frac{1}{\delta(w, \Gamma)}$$

(see [1] and [7]). Therefore, $K_{w_0^*} \in \mathcal{H}^2(\Omega)$ and, in particular, $\|K_{w_0^*}\|_{\mathcal{H}^2(\Omega)} \approx 1$. Since C_Φ is bounded on $\mathcal{H}^2(\Omega)$, clearly it follows that

$$\|C_\Phi K_{w_0^*}\|_{\mathcal{H}^2(\Omega)} \leq C, \tag{13}$$

for some constant C, independent of w_0^*.

Now, the key point of the proof relies on the fact that given any holomorphic function F on Ω, it holds that

$$\int_{\partial\Omega} |F(w)|^2 \, |dw| \approx \int_{\Omega} |F'(w)|^2 \delta(w, \partial\Omega) \, dm(w) + |F(0)|^2. \tag{14}$$

Actually, this follows just because $\partial\Omega$ is a Lavrentiev curve (see [1] and [7]). In particular, for $F = C_\Phi K_{w_0^*}$ in (14) we get

$$\begin{aligned} \int_{\partial\Omega} |C_\Phi K_{w_0^*}(w)|^2 \, |dw| &\gtrsim \int_{\Omega} \frac{\delta(w_0^*, \partial\Omega)}{|\Phi(w) - w_0^*|^4} \, |\Phi'(w)|^2 \delta(w, \partial\Omega) \, dm(w) \\ &= \int_{\Phi(\Omega)} \frac{\delta(w_0^*, \partial\Omega)}{|\zeta - w_0^*|^4} \left(\sum_{\Phi(w)=\zeta} \delta(w, \partial\Omega) \right) dm(\zeta) \\ &= \int_{\Omega} \frac{\delta(w_0^*, \partial\Omega)}{|\zeta - w_0^*|^4} \, \widetilde{N}_{\Phi,\Omega}(\zeta) \, dm(\zeta) \\ &\geq \int_{B(\xi_0, r)\cap\Omega} \frac{\delta(w_0^*, \partial\Omega)}{|\zeta - w_0^*|^4} \, \widetilde{N}_{\Phi,\Omega}(\zeta) \, dm(\zeta), \end{aligned} \tag{15}$$

where in the second line the change of variables $\Phi(w) = \zeta$ has been performed. Now, observe that for any $\zeta \in B(\xi_0, r)$ it holds that $|\zeta - w_0^*| \approx \delta(w_0^*, \partial\Omega) \approx r$ because of our choice of w_0^*. Thus, from (15) and the fact that $m(B(\xi_0, r)) \approx r^2$ we get

$$\|C_\Phi K_{w_0^*}\|^2_{\mathcal{H}^2(\Omega)} \gtrsim \frac{1}{r} \fint_{B(\xi_0, r)} \widetilde{N}_{\Phi,\Omega}(\zeta) \, dm(\zeta),$$

which along with (13) shows the first half of the statement of the theorem.

Now, let us assume that for any Carleson disc $B(\xi, r)$ in Ω

$$\fint_{B(\xi, r)} \widetilde{N}_{\Phi,\,\Omega} \, dm \leq C\, r,$$

where C is a positive constant just depending on the Lavrentiev constant of $\partial\Omega$. In order to prove that C_Φ is bounded in $\mathcal{H}^2(\Omega)$, it is enough to show that there exists a positive constant C such that

$$\int_{\partial\Omega} |f \circ \Phi(w)|^2 \, |dw| \leq C \int_{\partial\Omega} |f(w)|^2 \, |dw| \qquad (f \in \mathcal{H}^2(\Omega)). \tag{16}$$

Let μ be the pull-back measure of the arc-length measure $|dw|$ under Φ, that is, μ is the measure defined on the subsets E of Ω as follows:

$$\mu(E) = \text{ length } (\Phi^{-1}(E) \cap \partial\Omega).$$

Then, expression (16) can be rewritten as

$$\int_{\Omega} |f(w)|^2 \, d\mu(w) \leq C \int_{\partial\Omega} |f(w)|^2 \, |dw|, \tag{17}$$

for any $f \in \mathcal{H}^2(\Omega)$. Since Ω is a Carleson domain, (17) holds if μ is a Carleson measure on Ω. We claim the following

Claim: *μ is a Carleson measure on Ω if*

$$\sup_{w^* \in \mathbb{C}\setminus\overline{\Omega}} \int_{\Omega} \frac{\delta(w^*, \partial\Omega)}{|\zeta - w^*|^2} \, d\mu(\zeta) < \infty. \tag{18}$$

Assume that the claim is already proved. Then, in order to prove that C_Φ is bounded on $\mathcal{H}^2(\Omega)$ it is enough to show that condition (18) is satisfied.

Let us fix $w_0^* \in \mathbb{C} \setminus \overline{\Omega}$. Since $\partial\Omega$ is a Lavrentiev curve, we deduce that

$$\begin{aligned}
\int_\Omega \frac{\delta(w_0^*, \partial\Omega)}{|w - w_0^*|^2}\, d\mu(w) &= \int_{\partial\Omega} \frac{\delta(w_0^*, \partial\Omega)}{|\Phi(\zeta) - w_0^*|^2}\, |d\zeta| \\
&\approx \int_\Omega \frac{\delta(w_0^*, \partial\Omega)}{|\Phi(\zeta) - w_0^*|^4}\, |\Phi'(\zeta)|^2\, \delta(\eta, \partial\Omega)\, dm(\zeta) \\
&= \int_\Omega \frac{\delta(w_0^*, \partial\Omega)}{|w - w_0^*|^4}\, \widetilde{N}_{\Phi,\Omega}(w)\, dm(w), \qquad (19)
\end{aligned}$$

where the change of variables $\Phi(\zeta) = w$ has been accomplished. Let $w_0 \in \Omega$ be a point such that $\delta(w_0, \partial\Omega) \approx \delta(w_0^*, \partial\Omega)$. Observe that such a point exists since $\partial\Omega$ is a Lavrentiev curve. Let $D_0 = B(w_0, \delta(w_0, \partial\Omega))$ and for each $n \geq 1$,

$$D_n = \left\{ w \in \Omega :\ 2^{n-1}\, \delta(w_0, \partial\Omega) < |w - w_0| < 2^n\, \delta(w_0, \partial\Omega) \right\}.$$

Observe that $\Omega = \bigcup_n D_n$, so the expression in (19) is equivalent to

$$\begin{aligned}
\int_{\bigcup_n D_n} \frac{\delta(w_0^*, \partial\Omega)}{|w - w_0^*|^4}\, \widetilde{N}_{\Phi,\Omega}(w)\, dm(w) &= \sum_{n=0}^{\infty} \int_{D_n} \frac{\delta(w_0^*, \partial\Omega)}{|w - w_0^*|^4}\, \widetilde{N}_{\Phi,\Omega}(w)\, dm(w) \\
&\approx \sum_{n=0}^{\infty} \frac{1}{2^{4n}\, (\delta(w_0, \partial\Omega))^3} \int_{D_n} \widetilde{N}_{\Phi,\Omega}(w)\, dm(w) \qquad (20)
\end{aligned}$$

because for any $w \in D_n$, it holds that $|w - w_0^*| \approx 2^n\, \delta(w_0, \partial\Omega)$. In addition, note that

$$\begin{aligned}
\frac{1}{(2^n\, \delta(w_0, \partial\Omega))^2} \int_{D_n} \widetilde{N}_{\Phi,\Omega}(w)\, dm(w) &\lesssim \int_{\{w \in \Omega : |w - w_0| < 2^n\, \delta(w_0, \partial\Omega)\}} \widetilde{N}_{\Phi,\Omega}(w)\, dm(w) \\
&\approx 2^n \delta(w_0, \partial\Omega),
\end{aligned}$$

since $\{w \in \Omega : |w - w_0| < 2^n\, \delta(w_0, \partial\Omega)\}$ is contained in a Carleson disc in Ω of comparable radius. Therefore, we deduce that the series in (20) is uniformly bounded by a constant not depending on w_0. Thus, from (19) and (20) we have that

$$\sup_{w^* \in \mathbb{C} \setminus \overline{\Omega}} \int_\Omega \frac{\delta(w^*, \partial\Omega)}{|\zeta - w^*|^2}\, d\mu(\zeta) < \infty,$$

which along with (17) yields the boundedness of C_Φ as soon as the Claim is proved.

Proof of the Claim. Let us assume that the condition (18) holds and let $B(\xi_0, r)$ be a Carleson disc in Ω. Since $\partial\Omega$ is Lavrentiev, we may take $w_0^* \in B(\xi_0, r) \cap (\mathbb{C} \setminus \overline{\Omega})$ such that $\delta(w_0^*, \partial\Omega) \approx r$. Then

$$\begin{aligned}
\int_\Omega \frac{\delta(w_0^*, \partial\Omega)}{|\zeta - w_0^*|^2}\, d\mu(\zeta) &\geq \int_{\Omega \cap B(\xi_0, r)} \frac{\delta(w_0^*, \partial\Omega)}{|\zeta - w_0^*|^2}\, d\mu(\zeta) \\
&\approx \frac{\mu(B(\xi_0, r))}{\delta(w_0^*, \partial\Omega)},
\end{aligned}$$

where the second line follows because $|\zeta - w_0^*| \approx \delta(w_0^*, \partial\Omega)$ for any $\zeta \in B(\xi_0, r)$. This proves the claim, and therefore the first half of the theorem is proved.

Now, we deal with the compact part of the theorem. First, let us assume that C_Φ is compact on on $\mathcal{H}^p(\Omega)$, for $1 \le p < \infty$. As before, we can fix $p = 2$.

For $w^* \in \mathbb{C} \setminus \overline{\Omega}$, let K_{w^*} be the analytic function on Ω defined by:

$$K_{w^*}(\zeta) = \frac{(\delta(w^*, \partial\Omega))^{1/2}}{\zeta - w^*} \qquad (\zeta \in \Omega).$$

As we proved at the beginning, $K_{w^*} \in \mathcal{H}^2(\Omega)$. Moreover, note that K_{w^*} converges weakly to zero as $\delta(w^*, \partial\Omega) \to 0$. Since C_Φ is compact, we deduce $\|C_\Phi K_{w^*}\|_{\mathcal{H}^2(\Omega)} \to 0$ as $\delta(w^*, \partial\Omega) \to 0$. Proceeding with the estimates as we did before to prove the boundedness of C_Φ, we obtain the desired result.

Now, assume that for any Carleson disc $B(\xi, r)$ in Ω it holds that

$$\lim_{r\to 0} \frac{1}{r} \fint_{B(\xi,r)} \widetilde{N}_{\Phi,\Omega}\, dm = 0.$$

In order to prove that C_Φ is compact on $\mathcal{H}^2(\Omega)$, it is enough to show that for any $\{F_n\} \subset \mathcal{H}^2(\Omega)$ sequence of functions weakly convergent to zero,

$$\|C_\Phi F_n\|^2 = \int_{\partial\Omega} |F_n(\Phi(w))|^2\, |dw| \to 0, \qquad \text{as } n \to \infty.$$

But this is equivalent to

$$\int_\Omega |F_n(w)|^2\, d\mu(w) \to 0, \qquad \text{as } n \to \infty,$$

where μ denotes as before the pull-back measure under Φ of the arc-length measure on $\partial\Omega$. Since Ω is a Lavrentiev domain, it is enough to prove that for any Carleson disc $B(\xi, r)$ in Ω,

$$\lim_{r\to 0} \frac{\mu(B(\xi, r))}{r} = 0. \tag{21}$$

A similar argument to that used to prove the Claim above, yields that (21) holds if

$$\int_\Omega \frac{\delta(w^*, \partial\Omega)}{|\zeta - w^*|^2}\, d\mu(\zeta) \to 0 \qquad \text{as } \delta(w^*, \partial\Omega) \to 0,$$

which can be straightforward verified proceeding as before. This completes the proof of the theorem. □

Remark 5.2. We point out that Theorem 3.1 could be drawn from Theorem 5.1 since the condition stated by the *Littlewood type inequality* implies the corresponding one "in means". Nevertheless, in this way, it is shown that the role played by the assumption on the domain Ω to be Lavrentiev. In fact, we conjecture that such an assumption is also necessary in order to get the characterization of bounded composition operators in terms of the *Littlewood type inequality* "in means".

Acknowledgments

The authors thank the referee for helpful comments and suggestions which improved the final form of the paper.

References

[1] J. Bruna and M. J. González, *L^2 estimates on chord-arc curves*, Pacific J. Math. **190**, No. 2 (1999), 225-233. MR1722889 (2001a:30006)

[2] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80**, (1958), 921-930. MR0117349 (22:8129)

[3] M. D. Contreras and A. G. Hernández-Díaz, *Weighted composition operators on Hardy spaces*, J. Math. Anal. App. **263** (2001), 224-233. MR1864316 (2002j:47045)

[4] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995. MR1397026 (97i:47056)

[5] P. L. Duren, *Theory of $\mathcal{H}^p$ spaces*, Academic Press, New York, 1970. MR0268655 (42:3552)

[6] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981. MR628971 (83g:30037)

[7] D. Jerison and C. Kenig, *Hardy spaces, A^∞, and singular integrals on chord-arc domains*, Math. Scand., **50** (1982) 221-247. MR672926 (84k:30037)

[8] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925) 481-519.

[9] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992. MR1217706 (95b:30008)

[10] J. H. Shapiro, *Composition 0perators and Classical Function Theory*, Springer-Verlag, Berlin, 1993. MR1237406 (94k:47049)

[11] J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math., **125** (1987), 375-404. MR881273 (88c:47058)

[12] J. H. Shapiro and W. Smith, *Hardy spaces that support no compact composition operators*, J. Functional Analysis, **205** (2003), 62-89. MR2020208 (2004h:30041)

[13] J. H. Shapiro and P. D. Taylor, *Compact, Nuclear, and Hilbert-Schmidt Composition Operators on $\mathcal{H}^2$*, Indiana Univ. Math. J. **23** (1973), 471-496. MR0326472 (48:4816)

[14] M. Zinsmeister, *Les domaines de Carleson*, Michigan Math. J., **36** (1989), 213-220. MR1000525 (90i:30054)

Departamento de Matemáticas, Universidad de Zaragoza, Plaza San Francisco s/n, 50009 Zaragoza, Spain

E-mail address: eva@unizar.es

Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain

E-mail address: majose.gonzalez@uca.es

Departamento de Matemáticas, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain

E-mail address: artur@mat.uab.es