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AN ANALOGUE OF THE NOVIKOV CONJECTURE IN COMPLEX ALGEBRAIC GEOMETRY

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Abstract. We introduce an analogue of the Novikov Conjecture on higher signatures in the context of the algebraic geometry of (nonsingular) complex projective varieties. This conjecture asserts that certain "higher Todd genera" are birational invariants. This implies birational invariance of certain extra combinations of Chern classes (beyond just the classical Todd genus) in the case of varieties with large fundamental group (in the topological sense). We prove the conjecture under the assumption of the "strong Novikov Conjecture" for the fundamental group, which is known to be correct for many groups of geometric interest. We also show that, in a certain sense, our conjecture is best possible.

1. Introduction and statement of the conjecture

In recent years, there has been considerable interest in the Novikov Conjecture on higher signatures, in its variants and analogues, and on methods of proof coming from index theory and noncommutative geometry. (See for example the books [8], [9], and [20], and papers such as [16] and [13]. More references to the vast literature may be found in the books just cited.) The classical statement of the Novikov Conjecture is as follows:

Conjecture 1.1 (Novikov)**.** *Let M^n be a closed oriented manifold, with fundamental class $[M] \in H_n(M, \mathbb{Z})$ and total L-class $\mathcal{L}(M) \in H^*(M, \mathbb{Q})$. Let π be a discrete group, let $x \in H^*(B\pi, \mathbb{Q})$, and let $u\colon M \to B\pi$ be a map from M to the classifying space of this group. (In practice, usually $\pi = \pi_1(M)$ and u is the classifying map for the universal cover of M, though it is not necessary that this be the case.) Then the "higher signature" $\langle \mathcal{L}(M) \cup u^*(x), [M] \rangle$ is an oriented homotopy invariant of M. In other words, if $\widehat{M}$ is another closed oriented manifold, and if $h\colon \widehat{M} \to M$ is an orientation-preserving homotopy equivalence, then*

$$\langle \mathcal{L}(M) \cup u^*(x), [M] \rangle = \langle \mathcal{L}(\widehat{M}) \cup h^* \circ u^*(x), [\widehat{M}] \rangle.$$

Since $\mathcal{L}$ is a power series in the Pontrjagin classes, this conjecture implies an additional rigidity for the Pontrjagin classes of manifolds with "large" fundamental group, beyond the constraint of homotopy-invariance of the Hirzebruch signature

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$\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle$. For example, if M is aspherical, the conjecture implies that *all* of the rational Pontrjagin classes of M are homotopy invariants, which is certainly not the case for simply connected manifolds.

We shall in this paper consider an analogue of the Novikov Conjecture in complex algebraic geometry, with higher signatures replaced by higher Todd genera, and with homotopy invariance replaced by birational invariance. I would like to thank Professor Nigel Hitchin of Oxford University for a conversation (in 1998) that led to the following statement:

Conjecture 1.2. *Let V be a nonsingular complex projective variety of complex dimension n, also viewed as a complex manifold. Let $[V] \in H_{2n}(V, \mathbb{Z})$ be its fundamental class and $\mathcal{T}(M) \in H^*(V, \mathbb{Q})$ be its total Todd class in the sense of* [10, Ch. 3]. (*Homology and cohomology are taken in the ordinary sense for the underlying complex manifold of V with its locally compact Hausdorff topology.*) *Let π be a discrete group, let $x \in H^*(B\pi, \mathbb{Q})$, and let $u\colon V \to B\pi$ be a map from V to the classifying space of this group. Then the "higher Todd genus" $\langle \mathcal{T}(V) \cup u^*(x), [V] \rangle$ is a birational invariant of V* (*within the class of nonsingular complex projective varieties*).

Clearly this statement is very closely analogous to Conjecture 1.1. It seems appropriate to use birational maps in this context, as they provide the appropriate notion of weak isomorphism in the category of projective varieties. Conjecture 1.2 again implies an extra rigidity for the Chern classes of V in the case where the fundamental group of V is "large," beyond the birational invariance of the Todd genus $\langle \mathcal{T}(V), [V] \rangle$, which is equal to the arithmetic genus $\sum_{j=1}^{n} (-1)^j \dim H^j(V, \mathcal{O}_V)$ [10, Theorem 20.2.2]. (Here and throughout the paper I mean fundamental group in the usual topologists' sense, not in the sense of algebraic geometry.)

Most of the literature on the Novikov Conjecture (cf. [8], [9], and [20]) does not deal with the statement of Conjecture 1.1 directly. Instead, this statement is reduced, using either surgery theory or index theory, to a statement about injectivity of a certain *assembly map*, which is a property of the group π and which does not involve the manifold M at all. Different formulations are possible, but here we will use:

Definition 1.3 ([25])**.** Let π be a discrete group, and let $C^*(\pi)$ be its group C^*-algebra (either full or reduced; it doesn't matter). Define the **assembly map** for π, $A_\pi\colon K_*(B\pi) \to K_*(C^*(\pi))$, as follows. Let $\mathcal{V}_{B\pi}$ be the **universal flat $C^*(\pi)$-bundle** over $B\pi$, $\mathcal{V}_{B\pi} = E\pi \times_\pi C^*(\pi)$. This is a bundle of rank-one projective (right) $C^*(\pi)$-modules over $B\pi$. As such, it has a class $[\mathcal{V}_{B\pi}]$ in the Grothendieck group $K^0(B\pi; C^*(\pi))$. (See [22] or [26, §1.3].) Then A_π is the Kasparov product (a kind of slant product) with this class. (If there is no compact model for $B\pi$, we interpret the K-homology $K_*(B\pi)$ as $\varinjlim K_*(X)$, as X runs over the finite subcomplexes in a CW model for $B\pi$, and take the inductive limit of the slant products with $[\mathcal{V}_X] \in K^0(X; C^*(\pi)) \cong K_0(C(X) \otimes C^*(\pi))$.)

Then we say that π satisfies the **Strong Novikov Conjecture** (SNC) if A_π is rationally injective.

It was shown by Kasparov ([16], [18], [17]) — see also [13] for a variant of the proof — that the Strong Novikov Conjecture for π implies Conjecture 1.1 (for

the same π, and for arbitrary manifolds M). The SNC also implies [25] the non-existence of metrics of positive scalar curvature on closed spin manifolds for which "higher $\widehat{A}$-genera" are nonzero.

The Strong Novikov Conjecture is weaker than the Baum-Connes Conjecture and is now known to hold for many (overlapping) classes of groups of geometric interest: discrete subgroups of Lie groups [16], groups which act properly on "bolic" spaces [19], groups with finite asymptotic dimension [31], groups with a uniform embedding into a Hilbert space [32], and hyperbolic groups [21].

The main result of this paper will be

Theorem 1.4. *The Strong Novikov Conjecture* (*of Definition* 1.3) *for a group* π *implies Conjecture* 1.2*, for all nonsingular complex projective varieties and for the same group* π*.*

This result is thus formally analogous to Kasparov's. The proof will be given in Sections 2 and 3 of this paper, and will involve mimicking certain aspects of one of the proofs of birational invariance of the Todd genus.

Thus Conjecture 1.2 is true as long as the group π is a discrete subgroup of a Lie group, a group with finite asymptotic dimension, a group with a uniform embedding into a Hilbert space, or a hyperbolic group. This implies some extra birational rigidity statements for Chern classes of certain smooth projective varieties; we will give some concrete examples in Section 5. As we shall see in Section 4, there is also a sense in which Conjecture 1.2 is best possible.

After this paper was written and posted in August, 2005, Jörg Schürmann kindly pointed out to me that Conjecture 1.2, even without assuming the strong Novikov Conjecture, can be deduced from the result in [6, Example 3.3, part 3] that when $f\colon X \to V$ is a morphism of smooth projective varieties and a birational equivalence, then $f_*(\mathcal{T}(X) \cap [X]) = \mathcal{T}(V) \cap [V]$. A similar argument can be found in [4], which points out that the results of [3] can be used to prove the corresponding fact in K-homology, that under the same hypotheses, $f_*([D_X]) = [D_V]$. (The notation in this equation will be explained in Section 2.) This implies the K-homology analogue of Conjecture 1.2, that $u_*([D_V]) \in K_0(B\pi)$ is a birational invariant of V. This fact is somewhat stronger than Conjecture 1.2, since it implies the latter but also gives some torsion information. And finally, Borisov and Libgober [5] have now proven an analogue of Conjecture 1.2 with the Todd genus replaced by the elliptic genus (which includes it as a special case) and with birational invariance replaced by K-invariance. (Two smooth complex projective varieties are said to be K-equivalent — see [28] for a nice survey — if they are birationally equivalent and if they also have "the same c_1," more exactly if they have a common blowup such that the pull-backs of their canonical line bundles to this common blowup are equivalent.)

Thus it is fair to say that this paper is now somewhat superseded. Nevertheless, we have decided to publish it anyway in what is essentially its original form. There are a few reasons for this. First of all, the appearance of [4] and of [5] confirms that our Conjecture 1.2 seems to be of interest. Second, the results of section 4 appear to be new and of independent interest. Finally, the method of proof, and in particular Theorem 3.1, may prove to be useful for other applications.

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2. Method of proof of the main theorem

Before we begin the proof of Theorem 1.4, we first need to clarify precisely what it means. If you look carefully, you will notice that Conjecture 1.2 is not quite as precise as Conjecture 1.1, the reason being that a birational "map" is only defined in the complement of a subvariety of lower dimension. In other words, it is not necessarily a map (in the sense of topologists) at all, only a correspondence. However, fortunately we can appeal to:

Theorem 2.1 (Factorization Theorem [30], [1])**.** *Any birational map*

$$\phi\colon\ \widehat{V} \dashrightarrow V$$

between nonsingular complex projective varieties $\widehat{V}$ and V can be factored into a series of blowings up and blowings down with nonsingular irreducible centers. Furthermore, if ϕ is an isomorphism on an open subset U of V, then the centers of the blowings up and blowings down can be chosen disjoint from U, with all the intermediate varieties projective.

As a consequence, to make sense of Conjecture 1.2, it suffices to deal with the case of a "blowing down" $\phi\colon \widehat{V} \to V$, which is an actual morphism of varieties, and in particular a smooth map of complex manifolds. We may assume (by Theorem 2.1) that $\widehat{V}$ is obtained from blowing up V along a nonsingular irreducible subvariety Z of (complex) codimension at least 2. In that event, ϕ restricts an isomorphism $(\widehat{V} \smallsetminus \widehat{Z}) \xrightarrow{\cong} (V \smallsetminus Z)$, and $\widehat{Z} = \phi^{-1}(Z)$ is a nonsingular subvariety of $\widehat{V}$ of (complex) codimension 1. Thus we are reduced to showing:

Conjecture 2.2. *Let $\phi\colon \widehat{V} \to V$ be a blowing down of nonsingular complex projective varieties as above, let π be a discrete group, let $x \in H^*(B\pi, \mathbb{Q})$, and let $u\colon V \to B\pi$ be a map from V to the classifying space of this group. Then*

$$\langle \mathcal{T}(V) \cup u^*(x), [V] \rangle = \langle \mathcal{T}(\widehat{V}) \cup \phi^* \circ u^*(x), [\widehat{M}] \rangle.$$

This statement is now precisely analogous to Conjecture 1.1.

Incidentally, it is worth pointing out a fact well known to experts, though not entirely trivial:

Proposition 2.3. *If $\phi\colon \widehat{V} \to V$ is a blowing down of nonsingular complex projective varieties as above, then as a map of complex manifolds, ϕ induces an isomorphism on fundamental groups.*

Proof. Since the open subset $V \smallsetminus Z$ is the complement of Z, which has complex codimension at least 2 (and thus real codimension at least 4), a general position argument shows that the inclusion $\iota\colon V \smallsetminus Z \hookrightarrow V$ induces an isomorphism on π_1. Similarly, since $\widehat{Z}$ has complex codimension 1 (and thus real codimension 2) in $\widehat{V}$, $\widehat{\iota}\colon \widehat{V} \smallsetminus \widehat{Z} \hookrightarrow \widehat{V}$ induces an epimorphism on π_1, though not necessarily (just from general position arguments alone) an isomorphism. Now chase the commutative

diagram

$$\begin{tikzcd} \pi_1(\widehat{V} \smallsetminus \widehat{Z}) \arrow[r, two heads, "\widehat{\iota}_*"] \arrow[d, "\phi_*"', "\cong"] & \pi_1(\widehat{V}) \arrow[d, "\phi_*"] \\ \pi_1(V \smallsetminus Z) \arrow[r, "\iota_*", "\cong"'] & \pi_1(V). \end{tikzcd}$$

Since $\iota_* \circ \phi_*$ is surjective, the map ϕ_* on the right is surjective, i.e., $\phi_*\colon \pi_1(\widehat{V}) \to \pi_1(V)$ is an epimorphism. But since $\iota_* \circ \phi_*$ is also injective and $\widehat{\iota}_*$ is surjective, $\phi_*\colon \pi_1(\widehat{V}) \to \pi_1(V)$ must be an isomorphism. $\square$

Proposition 2.3 at least makes it plausible that there should be birational invariants of projective varieties coming from the cohomology of the fundamental group. In fact, this proposition proves Conjecture 2.2, and thus Conjecture 1.2, in the special case where $u \in H^{2n}(B\pi, \mathbb{Q})$, n being the (complex) dimension of $\widehat{V}$ and V, since the constant term in both $\mathcal{T}(V)$ and $\mathcal{T}(\widehat{V})$ is 1.

Now we can explain the strategy of the proof of Theorem 1.4. Suppose $\phi\colon \widehat{V} \to V$ is a blowing down of nonsingular complex projective varieties as in Proposition 2.3, and fix a map $u\colon V \to B\pi$. Then we can use u to pull back the universal flat $C^*(\pi)$-bundle over $B\pi$, $\mathcal{V}_{B\pi}$ (see Definition 1.3), to a flat $C^*(\pi)$-bundle $\mathcal{V}_V$ over V. Fix Kähler metrics on V and $\widehat{V}$, and let D_V and $D_{\widehat{V}}$ be the associated Dolbeault operators $\bar{\partial} + \bar{\partial}^*$, acting on forms of type $(0, *)$. These operators are odd with respect to the $\mathbb{Z}/2$-grading of the form bundle by parity of degree. A flat connection on $\mathcal{V}_{B\pi}$ pulls back to flat connections on $\mathcal{V}_V$ and on $\phi^*(\mathcal{V}_V) = \mathcal{V}_{\widehat{V}}$. Using these connections, we can make sense of "D_V with coefficients in $\mathcal{V}_V$," say $\mathcal{D}$, and of "$D_{\widehat{V}}$ with coefficients in $\mathcal{V}_{\widehat{V}}$," say $\widehat{\mathcal{D}}$. These are "elliptic operators with coefficients in a C^*-algebra" in the sense of Mishchenko and Fomenko [22, 26]. Again, they are odd with respect to the $\mathbb{Z}/2$-grading of the bundles on which they act. As such, they have "indices" in the sense of [22] which take values in the K-group $K_0(C^*(\pi))$. (The rough idea is that each of these operators should have a kernel which is a $\mathbb{Z}/2$-graded finitely generated projective module over $C^*(\pi)$, and the "index" is the formal difference of the classes of the even and odd parts of the kernel in the Grothendieck group of finitely generated projective modules, i.e., in $K_0(C^*(\pi))$. Strictly speaking, things are slightly more complicated, since the kernel of an elliptic operator in the sense of Mishchenko and Fomenko need not be projective. But the statement above is true after "compact perturbation" of the operator, and one can show that the choice of compact perturbation does not affect the index. See [22] and [26, §1.3.4] for more details.)

Next, we apply the index theorem of Mishchenko and Fomenko [22] to this situation. Exactly as in [25] (the only difference being that here we are using the Dolbeault operator; there we were using the signature operator), the result is that

$$\begin{aligned} \operatorname{Ind}(\mathcal{D}) = \langle [D_V], [\mathcal{V}_V] \rangle &= \langle [D_V], u^*[\mathcal{V}_{B\pi}] \rangle \\ &= \langle u_*[D_V], [\mathcal{V}_{B\pi}] \rangle = A_\pi([V \xrightarrow{u} B\pi]) \in K_0(C^*(\pi)), \end{aligned} \tag{2.1}$$

where A_π is the assembly map. Here $[D_V] \in K_0(V)$ is the K-homology class of the Dolbeault operator D_V in the sense of Kasparov ([14] and [15]), $[\mathcal{V}_V]$ is the class of the C^*-bundle $[\mathcal{V}_V]$ in the Grothendieck group $K^0(V; C^*(\pi)) \cong K_0(C(V) \otimes C^*(\pi))$ of such bundles, $\langle \ , \ \rangle$ is the pairing between K-homology and K-cohomology,

and $[V \xrightarrow{u} B\pi] \in K_0(B\pi)$ is the K-homology class defined by the map u from the complex manifold V into $B\pi$. (This K-homology class is the image of the class of $V \xrightarrow{u} B\pi$ in complex bordism $\Omega^U_{2n}(B\pi)$ under the map $\Omega^U_{2n}(B\pi) \to K_{2n}(B\pi) \cong K_0(B\pi)$ given by the map of spectra $\mathbf{MU} \to \mathbf{K}$ defined by the usual K-theory orientation of complex manifolds; see [27, Ch. VII].)

If we now rationalize, applying the Chern character Ch (which is an isomorphism from rationalized K-cohomology and K-homology to ordinary rational cohomology and homology) to equation (2.1) and recalling the fact that $\mathrm{Ch}([D_V])$ is Poincaré dual to the Todd class (see for example [2, §4]), we obtain:

$$\begin{aligned}\mathrm{Ind}(\mathcal{D}) \otimes \mathbb{Q} &= \langle \mathcal{T}(V) \cup \mathrm{Ch}[\mathcal{V}_V], [M]\rangle = \langle \mathcal{T}(V) \cup u^* \mathrm{Ch}[\mathcal{V}_{B\pi}], [M]\rangle \\ &= \langle \mathcal{T}(V) \cap [M], u^* \mathrm{Ch}[\mathcal{V}_{B\pi}]\rangle \in K_0(C^*(\pi)) \otimes \mathbb{Q}.\end{aligned} \tag{2.2}$$

The Strong Novikov Conjecture implies that all the even-dimensional rational cohomology of π can be obtained from the class $\mathrm{Ch}[\mathcal{V}_{B\pi}] \in H^*(B\pi, K_0(C^*(\pi)) \otimes \mathbb{Q})$ (via homomorphisms $K_0(C^*(\pi)) \otimes \mathbb{Q} \to \mathbb{Q}$), and thus that all higher Todd genera (in the sense of Conjecture 1.2) can be obtained from $\mathrm{Ind}(\mathcal{D})$. So to prove Theorem 1.4, it will suffice to prove the following slightly stronger statement:

Theorem 2.4. *Let $\phi\colon \widehat{V} \to V$ be a blowing down of nonsingular n-dimensional complex projective varieties as above, and let π be a discrete group. Let $\mathcal{D}$ and $\widehat{\mathcal{D}}$ be the $C^*(\pi)$-linear Dolbeault operators on V and on $\widehat{V}$ as above. Then* $\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\widehat{\mathcal{D}})$ *in $K_0(C^*(\pi))$.*

We will prove Theorem 2.4 by reducing it to equality of certain spaces of holomorphic forms on $\widehat{V}$ and V, using the fact that these manifolds agree in the complement of the lower-dimensional varieties $\widehat{Z}$ and Z.

3. Technical details

The proof of Theorem 2.4 proceeds in several steps. Some of these steps have to do with technicalities of Mishchenko-Fomenko index theory; others would be required just to prove that $\mathrm{Ind}(D_V) = \mathrm{Ind}(D_{\widehat{V}})$, and thus to deduce the classical result that the arithmetic genus is birationally invariant.

Theorem 3.1. *Let M be a compact complex manifold, let A be a unital C^*-algebra, and let $\mathcal{V}$ be an A-vector bundle which has a holomorphic structure. Then $H^0(M, \mathcal{O}_{\mathcal{V}})$, the space of holomorphic sections of $\mathcal{V}$, is finitely generated and projective as an A-module.*

Proof. We know that the Dolbeault operator $D_M = \bar\partial + \bar\partial^*$ is elliptic since M is compact, and thus $D_{\mathcal{V}}$, or D_M with coefficients in $\mathcal{V}$ (defined with respect to some choice of connection on $\mathcal{V}$), is elliptic in the sense of Mishchenko-Fomenko. Thus the kernel of $D_{\mathcal{V}}$, and hence $H^0(M, \mathcal{O}_{\mathcal{V}})$ (which is the intersection of that kernel with the 0-forms with values in $\mathcal{V}$) is a submodule of a finitely generated projective A-module. It suffices to show that $H^0(M, \mathcal{O}_{\mathcal{V}})$ is A-projective. This follows from the existence of a "reproducing kernel," which in turn follows from the Cauchy integral formula (see [24, Ch. 1, Proposition 2]). (Locally, projection from the Hilbert A-module of L^2-sections of $\mathcal{V}$ to the holomorphic sections is given by the same formula as projection from L^2-functions on a ball in $\mathbb{C}^n$ to the Bergman space of L^2-holomorphic functions, except that everything is now A-linear.) Existence of the projection proves that $H^0(M, \mathcal{O}_{\mathcal{V}})$ is a Hilbert A-module. Since it is also

a submodule of a finitely generated and projective A-module, it is itself finitely generated and projective. □

Theorem 3.2. *Let M be a compact Kähler manifold, let π be a countable discrete group, let $u\colon M \to B\pi$, let $A = C^*(\pi)$ be a unital C^*-algebra, and let $\mathcal{V} = \mathcal{V}_M$ be the associated flat A-bundle as above. Then the A-index of the Dolbeault operator on M with coefficients in $\mathcal{V}$ is equal to $\sum_{j=i}^{n}(-1)^j \left[H^0(M, \Omega^j_{\mathcal{V}})\right]$, the alternating sum of the classes of the spaces of holomorphic j-forms with coefficients in $\mathcal{V}$, the sum being computed in $K_0(A)$. (The terms in the sum are well-defined in $K_0(A)$ because of Theorem* 3.1.)

Proof. By repeated application of Theorem 3.1, each $H^0(M, \Omega^j_{\mathcal{V}})$ is finitely generated and projective as an A-module. On the other hand, complex conjugation exchanges ∂ and $\bar{\partial}$ and preserves the bundle $\mathcal{V}$ (which is just the complexification of the universal flat bundle for the *real* group C^*-algebra). Because of the Kähler condition, it sends $H^0(M, \Omega^j_{\mathcal{V}})$ to the space of harmonic $(0, j)$-forms with values in $\mathcal{V}$ (see [29, Theorem 4.7], the proof of [29, Theorem 5.1], and [10, §15.7]), and thus this latter space is also finitely generated and A-projective. In other words, compact perturbation of the operation is not needed in defining the Mishchenko-Fomenko A-index of the Dolbeault operator on M with coefficients in $\mathcal{V}$, and thus the latter agrees with the alternating sum $\sum_{j=i}^{n}(-1)^j \left[H^0(M, \Omega^j_{\mathcal{V}})\right]$. □

Theorems 3.1 and 3.2 are the analytical steps in the proof of Theorem 2.4. To conclude the proof, there is one more algebraic geometry or complex analysis part of the proof, namely:

Theorem 3.3. *Let $\phi\colon \widehat{V} \to V$ be a blowing down of nonsingular n-dimensional complex projective varieties as Section* 2 *above, and let π be a discrete group. Then for each j, pull-back of forms induces an isomorphism $H^0(V, \Omega^j_{\mathcal{V}_V}) \xrightarrow{\cong} H^0(\widehat{V}, \Omega^j_{\mathcal{V}_{\widehat{V}}})$.*

Proof. (Compare, for example, [11, §5.4] and [23, Proposition 6.16].) Recall that ϕ is an isomorphism from $\widehat{V} \smallsetminus \widehat{Z}$ to $V \smallsetminus Z$. Furthermore, any holomorphic form on V (resp., $\widehat{V}$) is determined by its restriction to the dense subset $V \smallsetminus Z$ (resp., $\widehat{V} \smallsetminus \widehat{Z}$). So ϕ^* is certainly injective. (If $\phi^*\omega = 0$, then $\phi^*\omega|_{\widehat{V}\smallsetminus\widehat{Z}} = 0$, hence $\omega|_{V\smallsetminus Z} = 0$, hence $\omega = 0$.) So we need only show that ϕ^* is surjective. Suppose η is a holomorphic j-form on $\widehat{V}$ with values in $\mathcal{V}_{\widehat{V}}$. We need to show that it comes from such a form on V. Since ϕ is an isomorphism from $\widehat{V} \smallsetminus \widehat{Z}$ to $V \smallsetminus Z$ and ϕ induces an isomorphism on π_1 (Proposition 2.3), we may identify $\eta_{\widehat{V}\smallsetminus\widehat{Z}}$ with a holomorphic j-form on $V \smallsetminus Z$ with values in $\mathcal{V}_V$. We just need to show that it extends to all of V. (Then the pull-back of the extension will be a holomorphic j-form that agrees with η on a dense set, and hence is equal to η.) But η is given to be holomorphic on the complement in V of the subvariety Z, and it is locally bounded since it came from a continuous form on the compact manifold $\widehat{V}$. So it extends by the Riemann Extension Theorem [24, Ch. 4, Proposition 2] or by related extension theorems such as Hartogs' Theorem (see the rest of [24, Ch. 4]). □

Now Theorem 2.4, and thus Theorem 1.4, follows from Theorems 3.2 and 3.3.

4. A "converse theorem"

Surgery theory can be used to prove that the signature is essentially the only Pontrjagin number for closed oriented manifolds (characteristic number of the form $\langle f(p_1, p_2, \cdots), [M]\rangle$), with f a polynomial function of the rational Pontrjagin classes $p_1, p_2, \cdots$, that is a homotopy invariant. In fact, the result can be improved for nonsimply connected manifolds, and the higher signatures are essentially the only more general characteristic numbers of the form $\langle f(p_1, p_2, \cdots) \cup u^*(x), [M]\rangle$ that can be homotopy invariant for all $x \in H^*(B\pi_1(M), \mathbb{Q})$, i.e., are the only characteristic functions of the form $u_*(f(p_1, p_2, \cdots) \cap [M]) \in H_*(B\pi_1(M), \mathbb{Q})$ that can be homotopy invariant. A precise theorem to this effect may be found, for instance, in [7, Theorem 6.5] or in [20, §1.2, Remark 1.9]. What Davis's theorem (*loc. cit.*) really says is that one has freedom to vary $\mathcal{L}(M)$ more or less arbitrarily within the oriented homotopy class of M, as long as one doesn't change $u_*(\mathcal{L}(M) \cap [M]) \in H_*(B\pi_1(M), \mathbb{Q})$. But the rational Pontrjagin classes can be recovered from $\mathcal{L}$, so this implies that $p_1, p_2, \cdots$ can be varied more or less arbitrarily within the oriented homotopy class of M, as long as one doesn't change $u_*(\mathcal{L}(M) \cap [M]) \in H_*(B\pi_1(M), \mathbb{Q})$. Another version of a very similar result is the following:

Theorem 4.1. *If $\xi\colon \Omega_k(B\pi)\otimes\mathbb{Q} \to \mathbb{Q}$ is a linear functional on the rational oriented bordism of a group π, and if $\xi([M \xrightarrow{u} B\pi]) = \xi([\widehat{M} \xrightarrow{u\circ h} B\pi])$ whenever $h\colon \widehat{M} \to M$ is an orientation-preserving homotopy equivalence, then ξ is given by a higher signature, i.e.,*

$$\xi([M \xrightarrow{u} B\pi]) = \langle u_*(\mathcal{L}(M) \cap [M]), x\rangle$$

for some $x \in H^*(B\pi, \mathbb{Q})$.

Proof. We know that

$$\Omega_k(B\pi) \otimes \mathbb{Q} \cong \bigoplus_{0\le j\le \lfloor \frac{k}{4}\rfloor} H_{k-4j}(B\pi, \mathbb{Q}) \otimes_{\mathbb{Q}} (\Omega_{4j}\otimes_{\mathbb{Q}}).$$

Thus ξ can be written as a sum $\sum_j x_j \otimes f_j$, where $x_j \in H^{k-4j}(B\pi, \mathbb{Q})$ and f_j is a linear functional on $\Omega_{4j}\otimes\mathbb{Q}$. The splitting of $\Omega_k(B\pi)\otimes\mathbb{Q}$ corresponds geometrically to a splitting of any bordism class into a sum of classes with representatives of the form $M^{k-4j} \times N^{4j} \xrightarrow{u_j} B\pi$, where u_j factors through M^{k-4j} and $M^{k-4j} \xrightarrow{u_j} B\pi$ represents a class in $H_{k-4j}(B\pi, \mathbb{Q})$. So homotopy invariance of ξ means that each f_j must be homotopy invariant on $\Omega_{4j} \otimes \mathbb{Q}$. However, it is shown in [12] that the differences $[N] - [N']$, where N and N' are related by an orientation-preserving homotopy equivalence, generate an ideal I_* of codimension 1 in $\Omega_* \otimes \mathbb{Q}$, with the quotient $(\Omega_* \otimes \mathbb{Q})/I_*$ detected by the signature. So f_j must be a multiple of the signature. Absorbing the constant factor into x_j, the Hirzebruch Signature Theorem proves that

$$\xi([M \xrightarrow{u} B\pi]) = \langle u_*(\mathcal{L}(M) \cap [M]), x\rangle, \quad \text{with } x = \sum x_j.$$

□

Because of Theorem 4.1, the Novikov Conjecture asserts that every functional on $\Omega_*(B\pi) \otimes \mathbb{Q}$ that *could be* homotopy invariant *is* homotopy invariant. The analogy which we proposed in Section 1, which involves replacing closed oriented manifolds by nonsingular complex projective varieties, Pontrjagin classes by Chern classes,

the L-class by the Todd class, and homotopy equivalence by rational equivalence, motivates the following analogues of this result:

Theorem 4.2 (cf. [12]). *The differences $[N]-[N']$, where N and N' are birationally equivalent nonsingular complex projective varieties, generate an ideal I_* of codimension 1 in the complex cobordism ring $\Omega_*^U \otimes \mathbb{Q}$, with the quotient $(\Omega_*^U \otimes \mathbb{Q})/I_*$ detected by the Todd genus.*

Proof. Clearly these differences generate an ideal, since if $N \overset{h}{\dashrightarrow} N'$ is a birational map, so is $N \times P \overset{h\times \mathrm{id}}{\dashrightarrow} N' \times P$, for any nonsingular complex projective variety P, and also all of $\Omega_*^U \otimes \mathbb{Q}$ ($* > 0$) is represented by nonsingular complex projective varieties [27, p. 130]. In fact one can take the complex projective spaces as generators for $\Omega_*^U \otimes \mathbb{Q}$. It's nontrivial that one has inverses within the monoid generated by the nonsingular projective varieties, but it's not known if one can choose the varieties to be connected. Fortunately this last point doesn't concern us. In any event, it suffices to observe that we have a $\mathbb{Q}$-basis for $\Omega_{2k}^U \otimes \mathbb{Q}$ consisting of all products $\mathbb{CP}^{j_1} \times \mathbb{CP}^{j_2} \times \cdots \times \mathbb{CP}^{j_r}$ with $j_1 \geq j_2 \geq \cdots \geq j_r$ and $j_1 + j_2 + \cdots + j_r = k$ [27, page 111]. But all of these varieties are in the same birational equivalence class, since they all have the same function field, namely a purely transcendental extension $\mathbb{C}(x_1, \cdots, x_k)$ of $\mathbb{C}$ of transcendence degree k. (To put things another way, all the above products of projective spaces are clearly projective completions of the same affine k-space.) Thus I_{2k} has codimension 1 in $\Omega_{2k}^U \otimes \mathbb{Q}$, and the quotient is detected by the Todd genus, since all of these products of projective spaces have Todd genus 1. That completes the proof. □

Theorem 4.3. *Suppose π is a group such that for all k, $H_{2k}(B\pi, \mathbb{Q})$ is spanned by the classes of maps from k-dimensional nonsingular projective varieties into $B\pi$. (Just as an example, this is certainly the case if π is free abelian, since then all even homology of π is generated by k-dimensional abelian varieties.) If $\xi\colon \Omega_{2k}^U(B\pi) \otimes \mathbb{Q} \to \mathbb{Q}$ is a linear functional on the rational complex bordism of a group π, and if $\xi([M \xrightarrow{u} B\pi]) = \xi([\widehat{M} \xrightarrow{u\circ h} B\pi])$ whenever $h\colon \widehat{M} \to M$ is a(n everywhere defined) birational equivalence of nonsingular complex projective varieties of complex dimension k, then ξ is given by a higher Todd genus, i.e.,*

$$\xi([M \xrightarrow{u} B\pi]) = \langle u_*(\mathcal{T}(M) \cap [M]), x\rangle$$

for some $x \in H^*(B\pi, \mathbb{Q})$.

Proof. We can prove this the same way we proved Theorem 4.1, with Theorem 4.2 substituting for Kahn's Theorem [12]. Only one point requires clarification: the extra hypothesis on π that $H_{2k}(B\pi, \mathbb{Q})$ is spanned by the classes of maps from k-dimensional nonsingular projective varieties into $B\pi$. This ensures that $\Omega_{2k}^U(B\pi) \otimes \mathbb{Q}$ is spanned by classes of maps $M^{2k-2j} \times N^{2j} \xrightarrow{u_j} B\pi$, where u_j factors through M^{2k-2j}, $M^{2k-2j} \xrightarrow{u_j} B\pi$ represents a class in $H_{2k-2j}(B\pi, \mathbb{Q})$, and M and N are *smooth projective varieties* of complex dimensions $k-j$ and j, respectively. (Otherwise we would just know that we could choose M to be stably almost complex, which isn't good enough for our purposes.) The rest of the proof is the same as before. □

5. Examples and concluding remarks

To conclude, we mention a few examples where the hypotheses of Theorems 1.4 and 4.3 apply, and explicate exactly what these theorems say in these special cases. We also mention some open problems.

Example 5.1. The simplest case of our theory is when π is free abelian. Since the first Betti number of a compact Kähler manifold has to be even, it is natural to start with the case $\pi = \mathbb{Z}^2$, which of course is the fundamental group of an elliptic curve E. The hypotheses of Theorems 1.4 and 4.3 certainly apply; in fact, essentially all known ways of proving SNC apply in this case. Putting Theorems 1.4 and 4.3 together, we conclude the following:

(1) Let V be a smooth projective variety of complex dimension $n \geq 1$ with a homomorphism $u\colon \pi_1(V) \to \mathbb{Z}^2$. Then $\langle \mathcal{T}(V) \cup u^*(x), [V]\rangle$ is a birational invariant of V, for x the usual generator of $H^2(B\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z}$. (This is the only nontrivial higher Todd genus in this case.)

(2) *Every* birationally invariant linear functional on $\Omega^U_{2n}(B\mathbb{Z}^2) \otimes \mathbb{Q}$ is a linear combination of the Todd genus (or arithmetic genus) and the higher Todd genus of (1).

For example, suppose we consider the case of surfaces ($n = 2$). In this case, the Todd genus is $\langle \frac{1}{12}(c_1(V)^2 + c_2(V)), [V]\rangle$, but we see that $\langle c_1 \cup u^*(x), [V]\rangle$ is also a birational invariant. In particular, suppose V is birationally equivalent to $E \times \mathbb{P}^1$ (in which case we can choose u to be an isomorphism on π_1). Now $H^*(E \times \mathbb{P}^1, \mathbb{Z})$ is the tensor product of an exterior algebra on generators x_1 and x_2 (each of degree 1) and a truncated polynomial ring on a generator y in degree 2, and we can choose $u^*(x) = x_1 \cup x_2$, and the Chern classes of $E \times \mathbb{P}^1$ are $c_1 = 2y$, $c_2 = 0$. So V has vanishing Todd genus, hence $c_1(V)^2 + c_2(V) = 0$, and

$$\langle c_1(V) \cup u^*(x), [V]\rangle = \langle c_1(E \times \mathbb{P}^1) \cup u^*(x), [E \times \mathbb{P}^1]\rangle = \langle x_1 \cup x_2 \cup 2y, [E \times \mathbb{P}^1]\rangle = 2.$$

This implies (among other things) that $c_1(V)$ has to be nonzero and even, which we wouldn't know just from the vanishing of the Todd genus. For example, consider the case where V is $E \times \mathbb{P}^1$ with one point blown up. Topologically, such a V is a connected sum $(T^2 \times S^2)\#\overline{\mathbb{CP}^2}$, which has Euler characteristic 1 and signature -1. So $\langle c_2(V), [V]\rangle = 1$ (the top Chern class is also the Euler class) and $\langle c_1(V)^2, [V]\rangle = -1$, which we could deduce in many ways, but more interestingly, $\langle c_1(V) \cup x_1 \cup x_2, [V]\rangle = 2$. (In fact, since the cohomology ring of V is the same as for $E \times \mathbb{P}^1$, but with one additional generator z in degree 2 with $z^2 = -x_1 \cup x_2 \cup y$, $z \cup x_1 = 0$, $z \cup x_2 = 0$, $z \cup y = 0$, we find that $c_2(V) = -z^2$, $c_1(V) = 2y + z$, which does satisfy the above conditions.)

Example 5.2. For another example, suppose V_1 and V_2 are birationally equivalent and both of them are aspherical, i.e., they are each homotopy equivalent to $B\pi$, where π is the common fundamental group. We fix the isomorphism $\pi_1(V_1) \to \pi_1(V_2) \to \pi$ determined by the birational equivalence; this fixes an identification of both cohomology rings $H^*(V_1, \mathbb{Z})$ and $H^*(V_2, \mathbb{Z})$ with the group cohomology ring $H^*(\pi, \mathbb{Z})$. We also assume that SNC holds for π; this is the case, for instance, if either V_1 or V_2 is locally symmetric, since then π is a discrete subgroup of a Lie group and [16] applies. (In fact, [19] also applies in this case.) The Novikov Conjecture implies that $\mathcal{L}(V_1) = \mathcal{L}(V_2)$, and thus V_1 and V_2 have the same rational Pontrjagin classes. But what about the Chern classes? Some constraints on them

are implied by the relationship between the Chern and Pontrjagin classes, that if (formally) $1+c_1+c_2+\cdots=\prod(1+x_i)$, then $1+p_1+p_2+\cdots=\prod(1+x_i^2)$. But this by itself does not determine the Chern classes. (For example, we have $p_1=c_1^2-2c_2$, and in the case of an algebraic surface, c_2 is determined by the Euler characteristic, so we know c_1^2, but not necessarily c_1 itself.) Conjecture 1.2, which in this case is an actual theorem, thanks to Theorem 1.4, says that we also have $\mathcal{T}(V_1)=\mathcal{T}(V_2)$. Thus, for example, $c_1(V_1)=c_1(V_2)$, which is an extra piece of information. I am not sure if it is possible to deduce that all of the Chern classes of V_1 and V_2 must agree, but this seems plausible, and it certainly holds if $\dim_{\mathbb{C}} V\leq 3$. For example, if $\dim_{\mathbb{C}} V_1=\dim_{\mathbb{C}} V_2=3$, then $c_3(V_1)=c_3(V_2)$ since the Euler characteristics agree, and as we have seen, $c_1(V_1)=c_1(V_2)$ since the part of $\mathcal{T}$ in degree 2 must match up, and then $c_2(V_1)=c_2(V_2)$ since $2c_2=c_1^2-p_1$.

Remark 5.3. An obvious open problem is to find a purely algebraic analogue of these results, preferably one which would also apply in characteristic p. This may require a new notion of "fundamental group," since the usual algebraic fundamental group only takes finite coverings into account, and if π is finite, its rational cohomology vanishes, so that the higher Todd genera do not tell us any more than the Todd genus itself.

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