1. Introduction

The advent of modern algebra owes much to invariant theory. Many of our classical theorems arose from Noether's investigations of a finite group $G \leq \text{Gl}(V)$ acting linearly on an $n$-dimensional vector space $V$ over a field $F$. The action of $G$ induces a natural action on the polynomial ring $F[V] = \text{Sym}(V^*)$. Noether showed that the ring $F[V]^G$ of invariant polynomials is finitely generated as an algebra. We are interested in the case when generators of $F[V]^G$ are algebraically independent: we say that $G$ has a polynomial ring of invariants if $F[V]^G = F[f_1, \ldots, f_n]$ for some homogeneous polynomials $f_i$ called basic invariants. Although there are many choices of basic invariants, their degrees are unique, and thus the integers $\deg f_1 - 1, \ldots, \deg f_n - 1$ depend only on the group. We call these integers the exponents of $G$. When $G$ has a polynomial ring of invariants, we define the Jacobian determinant $J = J(f_1, \ldots, f_n) = \det(\partial f_i/\partial z_j)$. This polynomial is nonzero (see [Ben93, 5.4]) and well-defined up to a nonzero element of $F$ depending on the choice of basic invariants and basis $\{z_j\}$ of $V^*$. In this article, we examine the structure of the Jacobian determinant $J$. No assumption is made on the ground field $F$.

Elements of finite order in $\text{Gl}(V)$ which fix a hyperplane pointwise are called reflections. (We consider the identity a reflection.) For any subgroup $G$ of $\text{Gl}(V)$ and hyperplane $H$ in $V$, define

$$G_H = \{\sigma \in G : \sigma|_H = \text{id}_H\},$$

the pointwise stabilizer of $H$ in $G$. The hyperplanes $H$ for which $G_H$ is nontrivial are called reflecting hyperplanes of $G$. For each hyperplane $H$ in $V$, let $l_H$ in $V^*$ be a linear form with $\ker l_H = H$. 

Received by the editors April 14, 2005 and, in revised form, August 18, 2005.

2000 Mathematics Subject Classification. Primary 13A50, 20F55; Secondary 52C35.

Key words and phrases. Invariant theory, Jacobian determinant, modular, Coxeter group, reflection group, hyperplane arrangement, pointwise stabilizer.

The work of the second author was partially supported by National Security Agency grant MDA904-03-1-0005.

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In Section 2, we examine the set of root vectors of reflections about a common hyperplane. Our analysis shows that every finite subgroup of $\text{Gl}(V)$ which fixes a hyperplane $H$ in $V$ pointwise has a polynomial ring of invariants and the Jacobian determinant is a power of $l_H$.

Our main result (Theorem 3.4) states that if $G \leq \text{Gl}(V)$ is a finite group with a polynomial ring of invariants, its Jacobian determinant factors as a product of linear forms defining the reflecting hyperplanes of $G$. The multiplicity with which each linear form $l_H$ occurs is the sum of the exponents of $G_H$.

This theorem has roots in the rich theory of reflection groups. A finite subgroup of $\text{Gl}(V)$ is a reflection group if it is generated by reflections, and the collection $A$ of its reflecting hyperplanes is called the reflection arrangement. In the nonmodular case, i.e., when the characteristic of $F$ is prime to $|G|$, a well-known theorem of Serre and Shephard, Todd, and Chevalley (see [Smi95, Ch. 7]) states that a finite subgroup of $\text{Gl}(V)$ is a reflection group if and only if it has a polynomial ring of invariants. Steinberg [Ste60] showed in this case that the Jacobian determinant of a set of basic invariants factors into powers of linear forms defining the reflecting hyperplanes:

$$J = \prod_{H \in A} l_H^{\frac{|G_H|}{|G_H| - 1}}$$

(we write $a \doteq b$ to indicate that $a$ and $b$ are equal up to a nonzero constant). In particular, the above factorization holds for all Weyl groups, Coxeter groups, and complex reflection groups (see [OT92, Thm. 6.42]). In the nonmodular case, each $G_H$ is a cyclic group and the only nonzero exponent of $G_H$ is $|G_H| - 1$.

Serre [Ser67] showed that in arbitrary characteristic, every finite subgroup of $\text{Gl}(V)$ with a polynomial ring of invariants must be generated by reflections. The converse may fail when the characteristic of $F$ divides the order of $G$ (for example, see [KM97]). Unfortunately, Steinberg’s characterization of the Jacobian determinant in terms of the integers $|G_H|$ no longer holds over arbitrary fields. The stabilizer subgroups $G_H$ may not be cyclic, in which case the integers $|G_H| - 1$ as $H$ runs over all reflecting hyperplanes will usually not sum to the degree of $J$. This is no surprise, as the class of reflections is larger in some sense over an arbitrary field than over a characteristic zero field. The reflections in $\text{Gl}(V)$ not only include diagonalizable reflections (with a single nonidentity eigenvalue), but also transvections, reflections with determinant 1 which cannot be diagonalized. The transvections in $\text{Gl}(V)$ prevent one from developing a theory of reflection groups mirroring that for Coxeter groups or complex reflection groups. (For example, even if a reflection group has a polynomial ring of invariants, the Jacobian $J$ may be invariant or lie in the Hilbert ideal generated by the basic invariants; see Section 4.) If $G$ lacks transvections, then it shares some characteristics with reflection groups over characteristic zero fields, for example, the pointwise stabilizer of any hyperplane in $V$ is cyclic. One can deduce that Steinberg’s description remains valid in this special case (see [Har01]).

Theorem 3.4 implies a statement conjectured by Victor Reiner: if $G$ has a polynomial ring of invariants, then the zero locus of the Jacobian determinant is exactly the union of the reflecting hyperplanes. Reiner, Stanton, and Webb [RSW04] use this corollary in generalizing Springer’s theory of regular numbers in characteristic zero to arbitrary fields.
2. One hyperplane

In this section, we consider finite groups $G \leq \text{Gl}(V)$ that fix a single hyperplane $H$ in $V$ pointwise and we investigate how the geometry of root vectors determines the group structure. It is easy to see that such groups have polynomial invariants in characteristic zero. Landweber and Stong [LS87] prove that the same holds in nonzero characteristic. We give a constructive proof of this fact which also shows that the Jacobian determinant of a set of basic invariants defines the hyperplane.

An explicit description of the basic invariants over fields of prime order can be found in Smith [Smi95, Chap. 8], for example. Unfortunately, parts of the description (at the end of Section 2 in [Smi95]) do not extend to other (finite) fields (see Example 4.1 in the last section).

Let $H$ be a hyperplane in $V$ defined by some linear form $l_H \in V^*$. For any reflection $\tau \in \text{Gl}(V)$ which fixes $H$ pointwise, let $v_\tau$ be the root vector of $\tau$ (with respect to $l_H$) defined by

$$\tau(v) = v + l_H(v) v_\tau$$

for all $v \in V$.

Note that a transvection is a reflection whose root vector lies in its reflecting hyperplane, i.e., $l_H(v_\tau) = 0$ (see for example [NS02], Sect. 6.2). For any set $S$ of reflections, let $\mathcal{R}(S)$ be the corresponding set of root vectors in $V$. (Of course $\mathcal{R}(S)$ depends on our choice of $l_H$.)

If $\text{char}(\mathbb{F}) = 0$, then any group $G$ which fixes a hyperplane pointwise is necessarily cyclic and its order equals the maximal order of a diagonalizable reflection in $G$ (which then generates $G$). When $\text{char}(\mathbb{F}) > 0$, a group $G$ which fixes a hyperplane pointwise is a semidirect product of the normal subgroup $K$ generated by the transvections (the kernel of the determinant map) and a cyclic subgroup generated by a diagonalizable reflection $\sigma$ of maximal order. The next lemma gives the order of $G$ in this case.

**Lemma 2.1.** Assume that $\text{char}(\mathbb{F}) = p > 0$. Suppose $G \leq \text{Gl}(V)$ is a finite group which fixes a hyperplane $H$ in $V$ pointwise. Let $\sigma$ be a diagonalizable reflection in $G$ of maximal order with $c = \det(\sigma)$. Let $K \leq G$ be the subgroup generated by the transvections in $G$. Then

1. The action of $\sigma$ on $K$ by conjugation translates into multiplication by $c$ on $\mathcal{R}(K)$ and thereby endows $\mathcal{R}(K)$ with the structure of an $\mathbb{F}_p(c)$-vector space.
2. $T \subset K$ is a minimal set satisfying $G = \langle T, \sigma \rangle$ if and only if $\mathcal{R}(T) \subset \mathcal{R}(K)$ is a basis for $\mathcal{R}(K)$ over $\mathbb{F}_p(c)$.
3. The group $G$ has order $|c| \cdot |\mathbb{F}_p(c)|^d$, where $d$ is the minimal number of transvections needed to generate $G$ together with $\sigma$.

**Proof.** If $\rho$ and $\tau$ are both reflections about $H$, then the root vector of the product is a linear combination of the root vectors:

$$v_{\rho\tau} = c_\tau v_\rho + v_\tau,$$

where $c_\tau = 1 + l_H(v_\tau)$ is the nonidentity eigenvalue of $\tau$ (if $\tau$ is not a transvection) or the eigenvalue 1 (if $\tau$ is a transvection). In particular, $v_{\rho\tau} = v_\rho + v_\tau$ for all $\rho \in G$, $\tau \in K$. Note that $v_\sigma = -cv_{\sigma^{-1}}$. Fix some transvection $\tau$ in $K$. An easy
computation then shows that
\[ (**) \quad v_{\sigma^{-1} T \sigma} = cv_r \quad \text{and thus} \quad v_{\sigma^{-1} T \sigma^m} = cv_r^m = (mc)v_r \]
for \( m \in \{0, 1, \ldots, p-1\} \).

We claim that the root vector of any element in the subgroup \( \langle \sigma, \tau \rangle \) must lie in the \( \mathbb{F}_p(c) \)-span of \( v_\sigma \) and \( v_\tau \). Indeed, we can write the element as a product of the generators \( \sigma \) and \( \tau \) and use Equation (**) repeatedly. In particular, the root vector of a transvection in \( \langle \sigma, \tau \rangle \) lies on \( H \) and thus must be an \( \mathbb{F}_p(c) \)-multiple of \( v_\tau \) alone (as \( v_\sigma \) lies on \( H \) but \( v_\sigma \) does not). On the other hand, any \( \mathbb{F}_p(c) \)-multiple of \( v_\tau \) is the root vector of some transvection in \( \langle \sigma, \tau \rangle \): if \( e = |c| \) and
\[ v = (m_1 c + m_2 c^2 + \ldots + m_p c^p) \quad \text{for some} \quad m_i \in \{0, 1, \ldots, p-1\}, \]
then \( v \) is the root vector of the transvection
\[ (\sigma^1 T \sigma^{-1})(\sigma^2 T \sigma^{-2}) \ldots (\sigma^e T \sigma^{-e}) \]
in \( \langle \sigma, \tau \rangle \) by Equation (**). Since each transvection about \( H \) is determined by its root vector, the transvections in \( \langle \sigma, \tau \rangle \) correspond bijectively to the \( \mathbb{F}_p(c) \)-multiples of \( v_\tau \).

More generally, if \( \tau_1, \ldots, \tau_k \) are transvections in \( G \), then a similar argument shows that the \( \mathbb{F}_p(c) \)-span of \( v_{\tau_1}, \ldots, v_{\tau_k} \) is the set of root vectors corresponding to the group \( K \cap \langle \sigma, \tau_1, \ldots, \tau_k \rangle \). This proves part (1).

Suppose \( T = \{\tau_1, \ldots, \tau_k\} \) is a minimal subset of \( K \) satisfying \( G = \langle T, \sigma \rangle \) and let \( v_i = v_{\tau_i} \). Then no \( \tau_j \) lies in the group generated by \( \sigma \) and \( \{\tau_i : i \neq j\} \) and hence no \( v_j \) is an \( \mathbb{F}_p(c) \)-linear combination of \( \{v_i : i \neq j\} \). Thus, the root vectors \( v_1, \ldots, v_k \) are linearly independent over \( \mathbb{F}_p(c) \). As \( T \) generates \( G \) together with \( \sigma \), the root vectors \( v_1, \ldots, v_k \) span \( \mathcal{R}(K) \) over \( \mathbb{F}_p(c) \) and hence form a basis. Conversely, if the root vectors of some set \( T \) form an \( \mathbb{F}_p(c) \)-basis of \( \mathcal{R}(K) \), then \( T \) is a minimal subset of \( K \) generating \( G \) together with \( \sigma \), which proves (2).

Finally, if \( \mathcal{R}(T) \) is a basis of \( \mathcal{R}(K) \) over \( \mathbb{F}_p(c) \) for some \( T \subset K \), then
\[ |K| = |\mathcal{R}(K)| = |\text{span}_{\mathbb{F}_p(c)} \mathcal{R}(T)| = |\mathbb{F}_p(c)^{|T|}|, \]
and hence \( |G| = |\sigma| \cdot |K| = |c| \cdot |\mathbb{F}_p(c)^{|T|}|. \]

Recall that a polynomial \( f \in \mathbb{K}[x_1, \ldots, x_r] \) over a field \( \mathbb{K} \) is called additive if it induces an additive homomorphism \( \mathbb{K}^r \to \mathbb{K} \). The following lemma is needed in the proof of Proposition 2.3 to inductively construct basic invariants.

**Lemma 2.2.** Let \( \mathbb{F} \) be a field and let \( A \subset \mathbb{F} \) be a finite additive subgroup. Then the polynomial \( f(X) = \prod_{a \in A} (X + a) \in \mathbb{F}[X] \) is additive.

**Proof.** Consider the polynomial
\[ F(X, t) := f(X + t) - f(X) = a_{m-1}(t)X^{m-1} + \ldots + a_1(t)X + a_0(t), \]
where \( t \) is another variable and the \( a_i \) are polynomials in \( t \). Note that \( a_0(t) = F(0, t) = f(t) \) by definition and \( \deg_i(a_i) < m \) for \( i \geq 1 \). But \( F(X, t_0) = 0 \) for all \( t_0 \in A \), since \( f(X + t_0) = f(X) \) by definition of \( f \). Hence, for every \( t_0 \in A \), the coefficients \( a_1(t_0), \ldots, a_{m-1}(t_0) \) are all zero. Thus for \( i \geq 1 \), the polynomial \( a_i(t) \) has at least \( m \) zeroes. Since each \( a_i \) has degree at most \( m - 1 \) in \( t \), it must be
identically zero. This shows that
\[ f(X + t) - f(X) = F(X, t) = a_0(t) = f(t), \]
and so \( f \) is additive.

The reader who is familiar with the Landweber-Stong invariants over prime fields is encouraged to read Example 4.1 in the last section before considering the next proposition and its proof.

**Proposition 2.3.** Let \( H \leq V \) be a hyperplane defined by some \( l_H \in V^* \). Let \( G \leq \text{GL}(V) \) be a finite group fixing \( H \) pointwise. Then

1. The group \( G \) has a polynomial ring of invariants.
2. The Jacobian determinant is \( J = l_H^m \), where \( m \) is the sum of the exponents of \( G \).

**Proof.** The group \( G \) is generated by a diagonalizable reflection \( \sigma \) with eigenvalue \( c \) of order \( e \) together with a minimal set of transvections \( \tau_1, \ldots, \tau_r \) (see Lemma 2.1). For \( k = 1, \ldots, r \), let \( G_k = \langle \sigma, \tau_1, \ldots, \tau_k \rangle \), and let \( G_0 = \langle \sigma \rangle \). Choose a basis \( e_1, \ldots, e_n \) of \( V \) such that \( \sigma \) is in diagonal form and \( e_1, \ldots, e_{n-1} \) span \( H \). Let \( z_1, \ldots, z_n \) be the dual basis of \( V^* \) and rescale \( l_H \) so that \( z_n = l_H \). Consider the case \( \text{char}(F) = p > 0 \).

We prove by induction on \( k \) a stronger statement: \( F[V]^G = F[f_1, \ldots, f_n] \) for some homogeneous polynomials \( f_i \) where \( f_n = z_n^m \), \( J(f_1, \ldots, f_n) = z_n^m \), and for \( i < n \), the degree of \( f_i \) is a \( p \)-power and \( f_i \) is additive as a polynomial in \( F(z_n)[z_1, \ldots, z_{n-1}] \). For \( k = 0 \), these claims are satisfied by setting \( f_n = z_n^e \) and \( f_i = z_i \) for \( i < n \). Note that these are also the basic invariants when the characteristic of \( F \) is zero (as \( G = \langle \sigma \rangle \) in this case).

Let \( k \geq 0 \) and assume that the induction hypothesis holds for the group \( G_k \) with \( F[V]^{G_k} = F[f_1, \ldots, f_n] \). Let \( d_i \) be the degree of each \( f_i \) and let \( \tau = \tau_{k+1} \). By our choice of basis, \( \tau(z_i) = z_i + a_i z_n \) for some \( a_i \in F \) when \( i < n \) and \( \tau(z_n) = z_n \). For \( i < n \), each \( f_i \) is additive over the infinite field \( F(z_n) \), and thus
\[
\tau f_i(z_1, \ldots, z_n) = f_i(z_1 + a_1 z_n, \ldots, z_{n-1} + a_{n-1} z_n, z_n)
\]
\[
= f_i(z_1, \ldots, z_n) + f_i(a_1 z_n, \ldots, a_{n-1} z_n, z_n) = f_i + b_i z_n^{d_i}
\]
for some \( b_i \in F \) (note that the second summand only depends on the variable \( z_n \)). Thus \( b_i = 0 \) exactly when \( f_i \) is invariant under \( \tau \).

Relabel \( f_1, \ldots, f_{n-1} \) so that \( f_1 \) has minimal degree among those \( f_i \) which are not invariant under \( \tau \). Define \( f'_2, \ldots, f'_{n-1} \) by
\[
f'_i = f_i + b'_i f_1^{d_i/d_1} \quad \text{where } b'_i = -b_i / b_1^{d_i/d_1}.
\]
The constants \( b'_i \) are chosen so that each \( f'_i \) is invariant under \( \tau \). The degrees of \( f'_2, \ldots, f'_{n-1} \) are again \( p \)-powers, since \( d_i/d_1 \) is a nonnegative \( p \)-power whenever \( b_i \neq 0 \). Furthermore, \( f'_2, \ldots, f'_{n-1} \) are additive over \( F(z_n) \) as they are the compositions of additive homomorphisms. Define \( f'_n = f_n \). Then \( f'_2, \ldots, f'_n \) are invariant under \( \tau \) and \( \sigma, \tau_1, \ldots, \tau_k \) (as each \( f_i \in F[V]^{G_k} \)). Hence, \( f'_2, \ldots, f'_n \) are invariant under \( G_{k+1} \).

We take the product over the orbit of \( f_1 \) to produce a polynomial \( f'_1 \) invariant under \( \tau \). Define
\[
h(X) = \prod_{a \in A} (X + a z_n^{d_1}) \in F(z_n)[X],
\]
where \( A = F_p(c) b_1 = \{ b \cdot b_1 | b \in F_p(c) \} \). By Lemma 2.2, \( h(X) \) is additive as a polynomial in \( F(z_n)[X] \). Define \( f'_1 = h(f_1) \in F[z_1, \ldots, z_n] \). Then \( f'_1 \) is additive.
Proposition 2.3. forms defining the reflecting hyperplanes. We begin with an easy consequence of invariants. We show that the Jacobian determinant factors into powers of linear functions of $h$ since both $f_1$ and $z_n$ are invariant. The polynomial $f'_1$ is also invariant under the diagonalizable reflection $\sigma$ since $\sigma(f_1) = f_1$, $\sigma(z_n) = c^{-1}z_n$, and $A$ is closed under multiplication by $\mathbb{F}_p(c)$. (In particular, $f'_1$ is a polynomial in $f_1$ and $f_n$.) Hence, $f'_1, \ldots, f'_n \in \mathbb{F}[V]^{G_{k+1}}$.

We consider each $f'_i$ as a polynomial in $f_1, \ldots, f_n$. By the chain rule,

$$J(f'_1, \ldots, f'_n) = J(f_1, \ldots, f_n) \det \left( \frac{\partial f'_i}{\partial f_j} \right).$$

The matrix $(\partial f'_i/\partial f_j)$ is upper triangular with determinant $\frac{\partial f'_i}{\partial f_j}$. Since $h$ is additive as a polynomial in $\mathbb{F}(z_n)[X]$ by Lemma 2.2, every exponent of $X$ in an expansion of $h$ is a $p$-power (see [Lan02], VI §12, for example). If we expand $f'_1 = h(f_1)$ as polynomial in $f_1$ and $z_n$, every exponent of $f_1$ will thus also be a $p$-power. In particular,

$$\partial f'_i/\partial f_1 = -b_1^{|A|-1} z_n^{d_1(|A|-1)},$$

and hence

$$J(f'_1, \ldots, f'_n) = J(f_1, \ldots, f_n) \cdot z_n^{d_1(|A|-1)}.$$

By the induction hypothesis, $J(f_1, \ldots, f_n)$ is a power of $z_n$ and the exponent of $z_n$ is the sum of the exponents of $G_k$. Substituting this into the last equality shows that assertion (2) holds for $G_{k+1}$ and thus for $G$ by induction. The polynomials $f'_1, \ldots, f'_n$ form a set of basic invariants for $G_{k+1}$ if and only if $J(f'_1, \ldots, f'_n)$ is nonzero and the product of the degrees of the $f'_i$ is the order of the group $G_{k+1}$ (for example, see [Kem96, Prop. 16]). By Lemma 2.1 and the induction hypothesis,

$$\deg f'_1 \cdots \deg f'_n = |\mathbb{F}_p(c)| \deg f_1 \deg f_2 \cdots \deg f_n = |\mathbb{F}_p(c)||G_k| = |G_{k+1}|,$$

and (1) follows.

The proof of Proposition 2.3 shows an interesting fact. The polynomials $f_1, f_2, \ldots, f_n$ in the induction step of the proof form a set of basic invariants for $G_k$. Thus, if we choose basic invariants of $G_k$ wisely, we need only adjust one of them to produce basic invariants for $G_{k+1}$:

Corollary 2.4. Let $G \leq \text{Gl}(V)$ be a finite group which fixes a hyperplane $H$ in $V$ pointwise. Let $G' = \langle G, \tau \rangle$ where $\tau \notin G$ is a transvection about $H$. Then there exist basic invariants $f_1, \ldots, f_n$ for $G$ and an invariant $f'_1$ for $G'$ such that $f'_1, f_2, \ldots, f_n$ form a set of basic invariants for $G'$ (in particular, all basic invariants for $G$ except one can be chosen to be invariant under $G'$).

3. The Jacobian Factors

In this section, we consider a finite group $G \leq \text{Gl}(V)$ with a polynomial ring of invariants. We show that the Jacobian determinant factors into powers of linear forms defining the reflecting hyperplanes. We begin with an easy consequence of Proposition 2.3.

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1One can use the fact that $\mathbb{F}_p(c)$ is the splitting field of the polynomial $X^{|A|} - X$ to compute the coefficient; we do not use this coefficient in what follows.
Lemma 3.1. Assume that $G \leq \text{Gl}(V)$ is a finite group with a polynomial ring of invariants. Let $A$ be the reflection arrangement of $G$. Then the Jacobian determinant is divisible by

$$\prod_{H \in A} l_H^{m_H},$$

where each $m_H$ is the sum of the exponents of the pointwise stabilizer $G_H$.

Proof. Fix some reflecting hyperplane $H \in A$. By Proposition 2.3, $G_H$ has a polynomial ring of invariants and

$$J(f_1^H, \ldots, f_n^H) = z_n^{m_H},$$

where $f_1^H, \ldots, f_n^H$ are basic invariants for $G_H$. Let $f_1, \ldots, f_n$ denote basic invariants for $G$. Each $f_i$ is invariant under $G_H$ and hence may be written as a polynomial in the $f_j^H$. Thus $J(f_1^H, \ldots, f_n^H) = l_H^{m_H}$ divides $J(f_1, \ldots, f_n)$ by the chain rule. The claim then follows since the linear forms $l_H$ for different reflecting hyperplanes are pairwise coprime and $\mathbb{F}[V]$ is a unique factorization domain. □

We next verify that we have found all factors of $J$. We compare degrees using the following version of the ramification formula of Benson and Crawley-Boevey (Corollary 3.12.2 in [Ben93]):

Lemma 3.2. Assume that $G \leq \text{Gl}(V)$ is a finite group. Then

$$|G| \psi(\mathbb{F}[V]^G) = \sum_{H \leq V} |G_H| \psi(\mathbb{F}[V]^{G_H})$$

(the sum runs over all hyperplanes in $V$). Here $\psi(M)$ denotes the coefficient of $\frac{1}{(1-t)^{n-1}}$ in the expansion at $t = 1$ of the Poincaré series of a finitely generated $\mathbb{F}[V]^G$-module $M$.

We apply the above lemma and obtain

Lemma 3.3. Assume that $G \leq \text{Gl}(V)$ is a finite group with a polynomial ring of invariants and let $A$ be its reflection arrangement. Let $J$ be the Jacobian determinant of $G$. Then

$$\deg(J) = \sum_{H \in A} m_H,$$

where each $m_H$ is the sum of the exponents of the pointwise stabilizer $G_H$.

Proof. By Proposition 2.3, each $G_H$ has a polynomial ring of invariants. The product of the degrees $d_i^H$ of basic invariants for $G_H$ equals the order of $G_H$ by [Kem96, Prop. 16]. Consequently,

$$\frac{1}{2} \deg(J) = \frac{1}{2} \sum_{i=1}^n (d_i - 1) = |G| \psi(\mathbb{F}[V]^G)$$

$$= \sum_{H} |G_H| \psi(\mathbb{F}[V]^{G_H})$$

by Lemma 3.2

$$= \sum_{H} |G_H| \frac{1}{2|G_H|} \sum_{i=1}^n (d_i^H - 1) = \frac{1}{2} \sum_{H} m_H,$$

which proves the claim. □
The main theorem is a direct consequence.

**Theorem 3.4.** Assume that \( G \leq \text{Gl}(V) \) is a finite group with a polynomial ring of invariants. Then the Jacobian determinant \( J \) factors into a product of powers of linear forms defining the reflecting hyperplanes. In fact,

\[
J = \prod_{H \in A} l_H^{m_H},
\]

where \( m_H \) denotes the sum of the exponents of the pointwise stabilizer \( G_H \).

**Proof.** By Lemma 3.1, the right hand side of the equation divides \( J \). By Lemma 3.3, both sides have the same degree. Thus they are equal up to a scalar. \( \square \)

We immediately obtain

**Corollary 3.5.** Assume that \( G \leq \text{Gl}(V) \) is a finite group with a polynomial ring of invariants. Then the zero set of the Jacobian determinant is the union of all reflecting hyperplanes of \( G \).

**Remark 3.6.** There is a geometric proof of Corollary 3.5 suggested by W. Messing: Since the extension of quotient fields \( \text{Quot}(\mathbb{F}[V])/\text{Quot}(\mathbb{F}[V]^G) \) is separable of degree \( |G| \), the associated quotient morphism \( \pi : V \to V/G \) is an étale covering in a neighborhood of a point \( v \in V \) if and only if \( G \) acts freely on \( v \). By (a variant of) a theorem of Serre, this happens if and only if \( v \) avoids all reflecting hyperplanes. Since \( \mathbb{F}[V]^G \) is a polynomial ring, \( \pi \) is a morphism of affine varieties, and thus it is étale near \( v \) if and only if the Jacobian matrix evaluated at \( v \) is invertible, i.e., has nonzero determinant. See [RSW04] for the full argument.

4. **Examples**

We first give an example to illustrate Lemma 2.1 and Proposition 2.3 and also to clarify what goes wrong with the proofs of Theorems 8.2.14 and 8.2.19 in [Smir95] when the ground field does not have prime order. We then give examples illustrating Theorem 3.4.

**Example 4.1.** Consider the group \( G \leq \text{Gl}_n(\mathbb{F}) \) generated by the matrices

\[
A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

where \( \mathbb{F} = \mathbb{F}_p(a, b, c) \) for some nonzero \( a, b, c \) in \( \mathbb{F} \). The group \( G \) fixes the hyperplane defined by the equation \( z_3 = 0 \). Lemma 2.1 is transparent in this example: The set \( \mathcal{R}(G) \) of root vectors of \( G \) is just the \( \mathbb{F}_p \)-span of \((a, 0, 0), (0, b, 0)\) and \((c, c, 0)\) in \( \mathbb{F}^3 \). The minimum number of vectors needed to span \( \mathcal{R}(G) \) is exactly the minimum number of group elements needed to generate \( G \). Thus, \( G \) can be generated by \( d \) elements and no fewer if and only if the dimension of \( \mathcal{R}(G) \) over \( \mathbb{F}_p \) is \( d \).

Assume \( A, B, \) and \( C \) form a minimum generating set for \( G \). (In other words, either \( a \) and \( c \) are independent over \( \mathbb{F}_p \) or \( b \) and \( c \) are independent over \( \mathbb{F}_p \).) The
group $G$ has a polynomial ring of invariants with basic invariants
\[
f_1 = (z_1^p - a^{p-1} z_1 z_3^{p-1})^p - c^{p-1} (a^{p-1} - c^{p-1}) (z_1^p - a^{p-1} z_1 z_3^{p-1}) z_3^{p(p-1)},
\]
\[
f_2 = (z_2^p - b^{p-1} z_2 z_3^{p-1}) - \left( \frac{b^{p-1} - c^{p-1}}{a^{p-1} - c^{p-1}} \right) (z_1^p - a^{p-1} z_1 z_3^{p-1}),
\]
\[f_3 = z_3.
\]
These basic invariants are given by the proof of Proposition 2.3.

In the special case where $a = b = 1$ and $c$ is algebraic over $\mathbb{F}_p$, but $c \notin \mathbb{F}_p$, the group $G$ is defined over the finite field $\mathbb{F}_p(c) = \mathbb{F}$, yet Theorem 8.2.14 and the proof of Theorem 8.2.19 in [Smi95] do not describe the group and basic invariants.

**Example 4.2.** Let $F$ be the finite field $\mathbb{F}_q$ and let $A$ be the set of all hyperplanes in $V = \mathbb{F}^n$. For each hyperplane $H$, choose some $l_H \in V^*$ with $\ker l_H = H$, and let $Q$ be the product of all these linear forms: $Q = \prod_{H \in A} l_H$. Then $Q$ has degree $|A| = |V|/|V^*| = (q^n - 1)/(q - 1)$.

The group $G = \text{Gl}_n(\mathbb{F})$ is generated by reflections about all hyperplanes in $V$. The invariants of $G$ form a polynomial ring: $\mathbb{F}[V]^G = \mathbb{F}[f_1, \ldots, f_n]$, where $f_{i+1}$ is the Dickson polynomial
\[
d_{n.i} = \sum_{W \leq V \atop \text{codim} W = 1} \prod_{v \in V^* \atop v|W \neq 1} v
\]
of degree $q^n - q^i$ (for example, see [Ben93, Prop. 8.1.3]).

Note that $f_1 = d_{n,0} = Q^{q-1}$.

Fix some hyperplane $H$ in $V$ and let $G_H$ be its pointwise stabilizer in $G$. In an appropriate basis (with $z_n = l_H$), $\mathbb{F}[V]^{G_H} = \mathbb{F}[u_1, \ldots, u_n]$ where
\[
u_i = z_i^q - z_n^{q-1} z_i \quad \text{for} \quad i < n \quad \text{and} \quad u_n = z_n^{q-1}
\]
(see [LS87]). Note that the sum of the exponents of $G_H$ is
\[m_H = (n-1)(q-1) + (q - 2) = n(q-1) - 1.
\]
Since each $f_i$ lies in $\mathbb{F}[V]^{G_H}$, each $f_i = h_i(u_1, \ldots, u_n)$ for some $h_i \in \mathbb{F}[V]$. Then
\[
\frac{\partial f_i}{\partial z_k} = \sum_j \left( \frac{\partial h_i}{\partial u_j} \right) \frac{\partial u_j}{\partial z_k}
\]
is divisible by $z_n^{q-2}$ if $k = n$ and divisible by $z_n^{q-1}$ otherwise. Hence, $J = \det (\frac{\partial f_i}{\partial z_k})$ is divisible by $z_n^{m_H}$. As $H$ was arbitrary, $\prod_{H \in A} l_H^{m_H}$ divides $J$. But one may check that
\[\deg J = \deg Q \cdot (n(q-1) - 1),
\]
and thus
\[J = Q^{n(q-1)-1} = \prod_{H \in A} l_H^{m_H}.
\]
Alternatively, one can use the description of the Dickson invariants in terms of Vandermonde-like determinants given in [Wil83, Prop. 1.3] to verify that $z_n^{n(q-1)-1}$ divides $J$:
\[
d_{n,k} = \Delta_k \Delta^{-1} \quad \text{where} \quad \left\{
\begin{array}{l}
\Delta_k = \det (z_j^{q^i})_{j=0, \ldots, n, \ i \neq k}, \\
\Delta = \det (z_j^{q^i})_{j=1, \ldots, n-1, \ i = 1, \ldots, n}.
\end{array}
\right.
\]
Apply the quotient rule to \( \partial / \partial z_i (d_{n,k}) \) and expand \( \partial / \partial z_i (\Delta_k) \) and \( \partial / \partial z_i (\Delta) \) about the \( i \)-th row. If \( i \neq n \), then \( z_{n}^{i+1} \) divides \( \partial / \partial z_i (\Delta_k) \) and \( \partial / \partial z_i (\Delta) \) and hence \( z_{n}^{i-1} \) divides \( \partial / \partial z_i (d_{n,k}) \). If \( i = n \), expand \( \Delta_k \) and \( \Delta \) about the \( n \)-th row and cancel terms to see that \( z_{n}^{i-2} \) divides \( \partial / \partial z_i (d_{n,j}) \). The last column of this Jacobian matrix is divisible by \( z_{n}^{i-2} \) and the other columns are each divisible by \( z_{n}^{i-1} \).

The Jacobian of the Dickson invariants was also examined by K. Kuhnigk in her Doktorarbeit ([Kuh03]).

**Example 4.3.** Let \( G = \text{Sl}_n (\mathbb{F}_q) \). As in the last example, let \( F = \mathbb{F}_q \) and \( Q = \prod_{H \in \mathcal{A}} l_H \), where \( \mathcal{A} \) is the set of all hyperplanes in \( V = \mathbb{F}^n \). Every reflection in \( G \) is a transvection and

\[
F[V]^G = F[f_1, f_2, \ldots, f_n],
\]

where \( f_1 = Q \) and \( f_{i+1} \) is the Dickson invariant \( d_{n,i} \) (for \( i \geq 1 \)). Fix some hyperplane \( H \) in \( V \) and let \( G_H \) be its pointwise stabilizer. In an appropriate basis (with \( z_n = l_H \)), \( F[V]^G_H = F[u_1, \ldots, u_n] \) where \( u_i = z_i^n - z_i^{n-1} z_n \) for \( i < n \) and \( u_n = z_n \). Note that the sum of the exponents of \( G_H \) is \( m_H = (n-1)(q-1) \). Since each \( f_i \) lies in \( F[V]^G_H \), \( f_i = h_i (u_1, \ldots, u_n) \) for some \( h_i \in F[V] \). Then

\[
\partial f_i / \partial z_k = \sum_j (\partial h_i / \partial z_j) (\partial u_j / \partial z_k)
\]

is divisible by \( z_{n}^{i-1} \) if \( k \neq n \). Hence, \( J = \det (\partial f_i / \partial z_k) \) is divisible by \( z_{n}^{(n-1)(q-1)} \) and thus \( Q^{m_H} = \prod_{H \in \mathcal{A}} l_H^{m_H} \) divides \( J \). But

\[
\deg J = (q^n - 1)(n-1) = \deg Q(n-1)(q-1),
\]

and so

\[
J = Q^{(n-1)(q-1)} = \prod_{H \in \mathcal{A}} l_H^{m_H}.
\]

Note that for both groups \( \text{Gl}_n (\mathbb{F}_q) \) and \( \text{Sl}_n (\mathbb{F}_q) \), the Jacobian determinant \( J \) lies in the Hilbert ideal generated by the basic invariants: The image of \( J \) in the coinvariant algebra \( F[V]/(f_1, \ldots, f_n) \) is zero in both cases.

**Acknowledgments**

We owe thanks to Peter Müller and Larry Smith for valuable comments on early versions of this article, and to Victor Reiner for helpful discussions.

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