

PURE SUBRINGS OF REGULAR RINGS ARE PSEUDO-RATIONAL

HANS SCHOUTENS

ABSTRACT. We prove a generalization conjectured by Aschenbrenner and Schoutens (2003) of the Hochster-Roberts-Boutot-Kawamata Theorem: let $R \rightarrow S$ be a pure homomorphism of equicharacteristic zero Noetherian local rings. If S is regular, then R is pseudo-rational, and if R is moreover \mathbb{Q} -Gorenstein, then it is pseudo-log-terminal.

1. INTRODUCTION

Hochster and Roberts showed in [13], using finite characteristic methods, that quotient singularities in characteristic zero are Cohen-Macaulay. This was improved by Boutot in [2] where he shows, using deep vanishing theorems, that they are rational. More precisely, if G is the complexification of a compact Lie group which acts algebraically on an affine smooth scheme X of finite type over \mathbb{C} , then the quotient X/G has rational singularities. In algebraic terms, with $X = \text{Spec } B$, this means that the ring of invariants $A := B^G$ has rational singularities whenever B is regular. (In fact, it suffices that B has at most rational singularities, and there is also a similar result in the analytic category.) When G is finite, Kawamata in [16] showed moreover that X/G has at most log-terminal singularities, and the author showed in [27], using non-standard tight closure, that this remains true for non-finite G , provided X/G is moreover \mathbb{Q} -Gorenstein (a condition that holds automatically if G is finite).

The goal of the present paper is to extend all these results by removing the finite type condition. However, since the notion of rational singularities is defined in terms of a resolution of singularities, which might not be available in such generality, we need to replace it by the notion of pseudo-rationality.

Main Theorem A. *Let $A \rightarrow B$ be a cyclically pure homomorphism of Noetherian rings containing \mathbb{Q} . If B is regular, then A is pseudo-rational.*

Recall that a homomorphism $A \rightarrow B$ is *cyclically pure* if $\mathfrak{a} = \mathfrak{a}B \cap A$ for each ideal \mathfrak{a} in A ; examples are split, pure or faithfully flat homomorphisms. Since the

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inclusion $B^G \subseteq B$ is split (via the so-called Reynolds operator), Boutot's result is therefore just a special case of our first main theorem. Theorem A was conjectured in [1] and proven for algebras of finite type over an algebraically closed field in [26] using canonical big Cohen-Macaulay algebras. The analogue in prime characteristic was proven by Smith in [28], but unlike most applications of tight closure, this proof did not carry over to characteristic zero, the reason being that cyclic purity is not preserved under reduction modulo p . To formulate a corresponding generalization in the \mathbb{Q} -Gorenstein case, we need to make a definition. Call a Noetherian local \mathbb{Q} -Gorenstein ring R *pseudo-log-terminal*, if its canonical cover \tilde{R} (see §7.2) is pseudo-rational. In particular, if we are in a category of local algebras in which 'pseudo-rational' is equivalent with 'rational' (e.g., the category of local algebras essentially of finite type over a field), then 'pseudo-log-terminal' is the same as 'log-terminal' by a result of Kawamata (Theorem 7.3). With this terminology, we get the following generalization, conjectured in [1] and proven for algebras of finite type over an algebraically closed field in [27].

Main Theorem B. *Let $R \rightarrow S$ be a cyclically pure homomorphism of equicharacteristic zero Noetherian local rings with S regular. If R is \mathbb{Q} -Gorenstein, then it is pseudo-log-terminal.*

To prove both theorems, we will transform the argument for finitely generated algebras given in [27] by means of the machinery of faithfully flat Lefschetz hulls introduced in [1]. In that paper, we show that given an equicharacteristic zero Noetherian local ring R , we can find a faithfully flat local R -algebra $\mathfrak{D}(R)$ which is an ultraproduct of rings of prime characteristic (these latter rings are called *approximations* of R , and their ultraproduct is called a *Lefschetz hull* of R). These results enabled us in [1] to generalize the alternative constructions of tight closure and big Cohen-Macaulay algebras from the papers [23, 26, 27] to arbitrary equicharacteristic zero Noetherian local rings. Similar applications, although only implicitly using Lefschetz hulls, can be found in [22, 24].

In the present paper, we will concentrate on one variant coming out of this work, to wit, generic tight closure: an element is in the *generic tight closure* of an ideal if almost all of its approximations belong to the tight closure of the corresponding approximation of the ideal; see §3 for exact definitions. Theorem A will follow from the fact that a generically F -rational ring is pseudo-rational (see Theorem 6.2), where we call a ring (*generically*) F -rational if some ideal generated by a system of parameters is equal to its (generic) tight closure. Smith observes in [28] that F -rationality in prime characteristic is equivalent with the top local cohomology of the ring being Frobenius simple. This enables her to prove that an excellent F -rational Noetherian local ring of prime characteristic is pseudo-rational. We will not use this result directly, but rather the method used to prove it. To this end, we also need Lefschetz hulls for finitely generated algebras over a Noetherian local ring, as such rings appear in the Čech complex that calculates the local cohomology. This is carried out in §2. Therefore, the present proof is entirely self-contained, apart from some material taken from [1].

As for Theorem B, we generalize the notion of an *ultra- F -regular* local ring introduced in [27] as a Noetherian local domain R with the property that for each non-zero c , we can find an ultra-Frobenius \mathbf{F}^ε such that the morphism $x \mapsto c\mathbf{F}^\varepsilon(x)$ is pure (an *ultra-Frobenius* is an ultraproduct of powers of Frobenius; see §2.2 below). We then show that the property of being ultra- F -regular descends under cyclically

pure local homomorphisms (Proposition 7.9) and is preserved under finite extensions which are étale in codimension one (Proposition 7.8). Moreover, we show that an ultra-F-regular local ring is pseudo-rational.

Open questions.

- (1) Does the converse of Theorem 6.2 also hold, that is to say, is pseudo-rational equivalent with generically F-rational? In [26, Theorem 5.11], I gave a proof of this in the finitely generated case which relies on a deep theorem due to Hara: *a local \mathbb{C} -algebra R of finite type has rational singularities if and only if it is of F-rational type*; see [6].
- (2) Does the stronger analogue of Boutot's result also hold, that is to say, can we weaken the assumption in Theorem A that B is only pseudo-rational? In the finitely generated case, a tight closure proof is available if B is moreover Gorenstein ([26, §5.14]), but this again depends on Hara's result.
- (3) In [27], using once again Hara's result, it was shown that for \mathbb{Q} -Gorenstein local domains of finite type over an algebraically closed field, the notions ultra-F-regular and log-terminal are equivalent. Is ultra-F-regular and pseudo-log-terminal the same for \mathbb{Q} -Gorenstein local domains?
- (4) Again, we can weaken in the finite type case [27] the assumption that S is regular to the assumption that it is (pseudo-)log-terminal. Does this also hold in general?
- (5) For local algebras of finite type over a field of characteristic zero, rational and pseudo-rational are the same notions, and so are log-terminal and pseudo-log-terminal. For which other categories of equicharacteristic zero Noetherian local rings is this the case?

2. LEFSCHETZ HULLS

Let S_w be a sequence of rings, where w runs over some infinite set endowed with a non-principal ultrafilter. The *ultraproduct* of this sequence is a ring S_∞ given as the homomorphic image of the product $\prod_w S_w$ modulo the ideal of all sequences which are almost equal to the zero sequence (two sequences (a_w) and (b_w) in the product are said to be *almost equal* if $a_w = b_w$ for almost all w , that is to say, for all w in some member of the ultrafilter). When we want to emphasize the index, we also denote the ultraproduct S_∞ by

$$\operatorname{ulim}_w S_w$$

and similarly, the image of a sequence (a_w) in S_∞ is denoted $\operatorname{ulim}_w a_w$ or simply a_∞ . In case all rings are equal, say $S_w := S$, their ultraproduct is called an *ultrapower* of S . For more details, see [14, §9.5] or [5], or the brief review in [23, §2].

2.1. Lefschetz hulls. Let K be an uncountable algebraically closed field of characteristic zero. In [1], we associate to every Noetherian local ring R whose residue field is contained in K , a faithfully flat Lefschetz hull, that is to say, a faithfully flat extension $R \subseteq \mathfrak{D}(R)$ such that $\mathfrak{D}(R)$ is an ultraproduct of prime characteristic (complete) Noetherian local rings R_w . Any sequence of prime characteristic complete Noetherian local rings R_w whose ultraproduct is equal to $\mathfrak{D}(R)$ is called an *approximation* of R . For the extent to which the assignment $R \mapsto \mathfrak{D}(R)$ is functorial, we refer to the cited paper. All we need in the present paper is that if $R \rightarrow S$ is a local homomorphism of Noetherian local rings whose residue field is contained

in K , then there is a homomorphism $\mathfrak{D}(R) \rightarrow \mathfrak{D}(S)$ making the following diagram commute:

$$(1) \quad \begin{array}{ccc} R & \xrightarrow{\quad\quad\quad} & \mathfrak{D}(R) \\ \downarrow & & \downarrow \\ S & \xrightarrow{\quad\quad\quad} & \mathfrak{D}(S). \end{array}$$

For the remainder of this section, R is an equicharacteristic zero Noetherian local ring, R_w an approximation of R and $\mathfrak{D}(R)$ its Lefschetz hull. We always choose K large enough so that it contains all pertinent residue fields, and hence from now on no further reference is made to it. For each w , let \mathbf{F}_w denote the Frobenius on R_w , that is to say the homomorphism given by $x \mapsto x^{p(w)}$, where $p(w)$ is the characteristic of R_w . Given a positive integer e , let eR_w denote the R_w -algebra structure on R_w given by \mathbf{F}_w^e . It follows that $\mathbf{F}_w^e: R_w \rightarrow {}^eR_w$ is R_w -linear.

2.2. Ultra-Frobenius. A *non-standard integer* is an element ε of the ultrapower \mathbb{Z}_∞ of \mathbb{Z} , that is to say, an ultraproduct of integers e_w . If almost all e_w are positive, then we call ε *positive*. For each positive non-standard integer ε , let $\mathbf{F}^\varepsilon: R \rightarrow \mathfrak{D}(R)$ be the ultraproduct of the $\mathbf{F}_w^{e_w}$, that is to say, for $x \in R$ with approximation x_w , we have

$$\mathbf{F}^\varepsilon(x) := \text{ulim}_w \mathbf{F}_w^{e_w}(x_w) \in \mathfrak{D}(R).$$

As in [27], we will call any homomorphism $R \rightarrow \mathfrak{D}(R)$ of the form \mathbf{F}^ε for some ε an *ultra-Frobenius*. If $\varepsilon = 1$, then the corresponding ultra-Frobenius is just the *non-standard Frobenius* introduced in [1].

For each positive non-standard integer ε , we may view $\mathfrak{D}(R)$ as an R -algebra via the homomorphism \mathbf{F}^ε . To denote this algebra structure, we will write ${}^\varepsilon\mathfrak{D}(R)$ (in [27], the alternative notation $(\mathbf{F}^\varepsilon)_*\mathfrak{D}(R)$ was used). In other words, the R -algebra structure on ${}^\varepsilon\mathfrak{D}(R)$ is given by $x \cdot \alpha := \mathbf{F}^\varepsilon(x)\alpha$, for $x \in R$ and $\alpha \in \mathfrak{D}(R)$.

One of the major drawbacks of the functor \mathfrak{D} is its local nature. In particular, since a localization $R \rightarrow R_{\mathfrak{p}}$ is not a local homomorphism, there is no obvious map from $\mathfrak{D}(R)$ to $\mathfrak{D}(R_{\mathfrak{p}})$. Below we will have to deal with localizations of the form R_y , and hence we need a notion of Lefschetz hull for such (non-local) rings as well.

2.3. R -approximations. Let Y be a tuple of indeterminates and let $f \in R[Y]$, say of the form $f = \sum_{\nu \in N} a_\nu Y^\nu$ with $a_\nu \in R$ and N a finite index set. If $a_{\nu w}$ is an approximation of a_ν , for each $\nu \in N$, then we call the sequence of polynomials $f_w := \sum_{\nu \in N} a_{\nu w} Y^\nu$ an *R -approximation* of f .

One checks that any two R -approximations of a polynomial f are almost equal. Similarly, if $I := (f_1, \dots, f_s)$ is an ideal in $R[Y]$ and f_{iw} is an R -approximation of f_i , for each i , then we call the sequence $I_w := (f_{1w}, \dots, f_{sw})R_w[Y]$ an *R -approximation* of I , and if $S = R[Y]/I$, then we call the sequence $S_w := R_w[Y]/I_w$ an *R -approximation* of S .

2.4. Relative hulls. If S is a finitely generated R -algebra and S_w is an R -approximation of S , then the ultraproduct of the S_w is called the *(relative) R -hull* of S and is denoted $\mathfrak{D}_R(S)$.

If $R[Z]/J$ is another presentation of S as an R -algebra, then we have substitution maps $Y \mapsto \mathbf{a}$ and $Z \mapsto \mathbf{b}$ which induce isomorphisms modulo I and J respectively, where \mathbf{a} and \mathbf{b} are tuples of polynomials in the Z and Y -variables respectively. Let \mathbf{a}_w and \mathbf{b}_w be R -approximations of these respective tuples and let J_w be an R -approximation of J . By Los' Theorem the substitutions $Y \mapsto \mathbf{a}_w$ and $Z \mapsto \mathbf{b}_w$ induce for almost all w isomorphisms modulo I_w and J_w respectively. It follows that the ultraproduct of the $R_w[Y]/I_w$ is isomorphic to the ultraproduct of the $R_w[Z]/J_w$, showing that $\mathfrak{D}_R(S)$ is independent from the particular presentation of S and from the particular choice of R -approximations.

Since $\mathfrak{D}_R(S)$ is naturally a $\mathfrak{D}(R)$ -algebra and since by Los' Theorem the tuple Y is algebraically independent over $\mathfrak{D}(R)$, we get a natural $\mathfrak{D}(R)[Y]$ -algebra structure, whence an $R[Y]$ -algebra structure, on $\mathfrak{D}_R(S)$. Under the natural homomorphism $R[Y] \rightarrow \mathfrak{D}_R(S)$, we get $I\mathfrak{D}_R(S) = 0$, so that this induces a homomorphism $S \rightarrow \mathfrak{D}_R(S)$, endowing $\mathfrak{D}_R(S)$ with a canonical S -algebra structure. We can now extend the notion of R -approximation of an element a or an ideal \mathfrak{a} in a finitely generated R -algebra S as follows. Let $S := R[Y]/I$ and choose a polynomial $f \in R[Y]$ and an ideal \mathfrak{A} in $R[Y]$ so that their images in S are respectively a and \mathfrak{a} . Let f_w, \mathfrak{A}_w and S_w be R -approximations of f, \mathfrak{A} and S respectively. We call the image a_w of f_w in S_w (respectively, the ideal $\mathfrak{a}_w := \mathfrak{A}_w S_w$) an *R -approximation* of a (respectively, of \mathfrak{a}). Note that the ultraproduct of the a_w (respectively, of the \mathfrak{a}_w) is equal to the image of a in $\mathfrak{D}_R(S)$ (respectively, equal to the ideal $\mathfrak{a}\mathfrak{D}_R(S)$), showing that any two R -approximations are almost equal.

If $S \rightarrow T$ is an R -algebra homomorphism of finite type, then this extends to an R -algebra homomorphism $\mathfrak{D}_R(S) \rightarrow \mathfrak{D}_R(T)$ giving rise to a commutative diagram

$$(2) \quad \begin{array}{ccc} S & \xrightarrow{\quad} & \mathfrak{D}_R(S) \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad} & \mathfrak{D}_R(T). \end{array}$$

In particular, $\mathfrak{D}_R(\cdot)$ is a functor on the category of finitely generated R -algebras. The argument is the same as in [23, §3.2.4], and we leave the details to the reader.

3. GENERIC TIGHT CLOSURE

One of the tight closure notions introduced in [1] is generic tight closure. In this section, we review the definition and (re)prove some of its main properties. Throughout this section, (R, \mathfrak{m}) will denote an equicharacteristic Noetherian local ring and (R_w, \mathfrak{m}_w) one of its approximations. For generalities on (characteristic p) tight closure, see [15].

3.1. Definition. An element $z \in R$ lies in the *generic tight closure* of an ideal $\mathfrak{a} \subseteq R$, if almost all z_w lie in the tight closure \mathfrak{a}_w^* of \mathfrak{a}_w , where z_w and \mathfrak{a}_w are approximations of z and \mathfrak{a} respectively.

We denote the generic tight closure of an ideal \mathfrak{a} by $\text{cl}_{\text{gen}}(\mathfrak{a})$. One easily checks that

$$(3) \quad \text{cl}_{\text{gen}}(\mathfrak{a}) = (\text{ulim}_w \mathfrak{a}_w^*) \cap R$$

where the contraction is with respect to the canonical embedding $R \rightarrow \mathfrak{D}(R)$. It follows that $\text{cl}_{\text{gen}}(\mathfrak{a})$ is an ideal, containing \mathfrak{a} , with the property that $\text{cl}_{\text{gen}}(\text{cl}_{\text{gen}}(\mathfrak{a})) = \text{cl}_{\text{gen}}(\mathfrak{a})$. We say that an ideal \mathfrak{a} is *generically tightly closed* if $\mathfrak{a} = \text{cl}_{\text{gen}}(\mathfrak{a})$. The proof of the following easy fact is left to the reader.

3.2. Lemma. *If $\mathfrak{a} \subseteq R$ is a generically tightly closed ideal, then so is any colon ideal $(\mathfrak{a} :_R \mathfrak{b})$, for $\mathfrak{b} \subseteq R$. \square*

3.3. Theorem. *If R is regular, then every ideal is generically tightly closed.*

Proof. By [1, Theorem 5.2], almost all R_w are regular, and hence all ideals in R_w are tightly closed by [15, Theorem 1.3]. The assertion then follows from (3) and faithful flatness. \square

3.4. Theorem (Persistence). *If $R \rightarrow S$ is a local homomorphism and \mathfrak{a} an ideal in R , then $\text{cl}_{\text{gen}}(\mathfrak{a})S \subseteq \text{cl}_{\text{gen}}(\mathfrak{a}S)$.*

Proof. Immediate from (3) and the fact that persistence holds for each $R_w \rightarrow S_w$, where S_w is an approximation of S (note that R_w is complete, so that [15, Theorem 2.3] applies). \square

3.5. Theorem (Strong Colon Capturing). *Let (x_1, \dots, x_d) be part of a system of parameters of R . For each i , the element x_i is a non-zero divisor modulo $\text{cl}_{\text{gen}}((x_1, \dots, x_{i-1})R)$.*

Proof. By downward induction on i , it suffices to prove the assertion for $i = d$. To this end, suppose $zx_d \in \text{cl}_{\text{gen}}(I)$ with $I := (x_1, \dots, x_{d-1})R$. Let R_w, z_w and x_{dw} be approximations of R, z and x_i respectively and put $I_w := (x_{1w}, \dots, x_{(d-1)w})R_w$. By [1, Corollary 5.3], almost all (x_{1w}, \dots, x_{dw}) are part of a system of parameters in R_w and $z_w x_{dw} \in I_w^*$. Since each R_w is complete, Strong Colon Capturing holds for it, that is to say, x_{dw} is a non-zero divisor modulo I_w^* (see [15, Theorem 3.1A and Lemma 4.1]). Therefore, $z_w \in I_w^*$, whence $z \in \text{cl}_{\text{gen}}(I)$, as we needed to show. \square

3.6. Remark. In particular, the usual Colon Capturing holds, that is to say, for each i , we have an inclusion $((x_1, \dots, x_{i-1})R : x_i) \subseteq \text{cl}_{\text{gen}}((x_1, \dots, x_{i-1})R)$. The same proof can also be used to prove the following stronger version (compare with [15, Theorem 9.2]): let $\mathbb{Z}[X] \rightarrow R$ be given by $X_i \mapsto x_i$ and let $I, J \subseteq \mathbb{Z}[X]$ be ideals. We have an inclusion

$$(4) \quad (\text{cl}_{\text{gen}}(IR) : JR) \subseteq \text{cl}_{\text{gen}}((I : J)R).$$

3.7. Corollary. *If (x_1, \dots, x_d) is part of a system of parameters in R and if $(x_1, \dots, x_d)R$ is generically tightly closed, then so is each $(x_1, \dots, x_i)R$, for $i = 1, \dots, d$. In particular, (x_1, \dots, x_d) is a regular sequence.*

Proof. The last assertion is clear from Colon Capturing and the first assertion. For the first assertion, it suffices to treat the case $i = d - 1$, by downwards induction on i . Let $I := (x_1, \dots, x_{d-1})R$ and let $z \in \text{cl}_{\text{gen}}(I)$. Clearly, $z \in \text{cl}_{\text{gen}}(I + x_d R)$ and this latter ideal is just $I + x_d R$ by hypothesis. Write $z = a + r x_d$, with $a \in I$ and $r \in R$. Therefore, $z - a = r x_d \in \text{cl}_{\text{gen}}(I)$. Since x_d is a non-zero divisor modulo $\text{cl}_{\text{gen}}(I)$

by Theorem 3.5, we get $r \in \text{cl}_{\text{gen}}(I)$. So, we proved that $\text{cl}_{\text{gen}}(I) = I + x_d \text{cl}_{\text{gen}}(I)$. Nakayama’s Lemma then yields that $I = \text{cl}_{\text{gen}}(I)$. \square

3.8. Theorem (Briançon-Skoda). *The generic tight closure of an ideal $\mathfrak{a} \subseteq R$ is contained in its integral closure. If \mathfrak{a} is generated by at most n elements, then the integral closure of \mathfrak{a}^{m+n} is contained in $\text{cl}_{\text{gen}}(\mathfrak{a}^{m+1})$, for each m .*

Proof. Let $z \in \text{cl}_{\text{gen}}(\mathfrak{a})$. In order to show that z is integral over \mathfrak{a} , it suffices by [11, Lemma 2.3] to show that $z \in \mathfrak{a}V$, for each discrete valuation ring V such that $R \rightarrow V$ is a local homomorphism. Now, persistence (Theorem 3.4) yields that z lies in $\text{cl}_{\text{gen}}(\mathfrak{a}V)$, whence, by Theorem 3.3, in $\mathfrak{a}V$.

Assume next that z lies in the integral closure of \mathfrak{a}^{m+n} , for some m and for n the number of generators of \mathfrak{a} . Taking an integral equation witnessing this fact and considering approximations, we see that almost all z_w lie in the integral closure of \mathfrak{a}_w^{m+n} , where z_w and \mathfrak{a}_w are approximations of z and \mathfrak{a} respectively. By the tight closure Briançon-Skoda Theorem (see for instance [15, Theorem 5.7] for an easy proof), almost all z_w lie in the tight closure of \mathfrak{a}_w^{m+1} and the result follows. \square

3.9. Comparison with other tight closure operations. By [1, Theorem 6.21], the generic tight closure of an ideal \mathfrak{a} is contained in its non-standard tight closure, provided R is analytically unramified. This latter condition is imposed to insure the existence of uniform test elements ([1, Proposition 6.20]).

If R is moreover equidimensional and universally catenary, then by [1, Proposition 7.13], the \mathfrak{B} -closure $\mathfrak{a}\mathfrak{B}(R) \cap R$ of \mathfrak{a} is contained in its generic tight closure, with equality if \mathfrak{a} is generated by a system of parameters. Here $\mathfrak{B}(R)$ denotes the canonical big Cohen-Macaulay algebra associated to R from [1, §7]. (In the special case that R is a complete domain with algebraically closed residue field, $\mathfrak{B}(R)$ is obtained as the ultraproduct of the absolute integral closures R_w^+ .)

4. GENERIC F-RATIONALITY

As before, R is an equicharacteristic zero Noetherian local ring and R_w is an approximation of R .

4.1. Definition. We say that R is *generically F-rational*, if there exists a system of parameters \mathbf{x} in R such that $\mathbf{x}R$ is generically tightly closed.

Let us say that R is *\mathfrak{B} -rational*, if there exists a system of parameters \mathbf{x} such that $\mathbf{x}R = \mathbf{x}\mathfrak{B}(R) \cap R$. We will prove below that a ring is generically F-rational if and only if its completion \widehat{R} is. We leave it as an exercise to prove that the same property with ‘ \mathfrak{B} -rational’ instead of ‘generically F-rational’ also holds. Therefore, in view of our discussion in §3.9, a ring is generically F-rational if and only if it is \mathfrak{B} -rational.

4.2. Theorem. *If R is generically F-rational, then it is Cohen-Macaulay.*

Proof. Let \mathbf{x} be a system of parameters in R such that $\mathbf{x}R$ is generically tightly closed. By Corollary 3.7, the sequence \mathbf{x} is regular and hence R is Cohen-Macaulay. \square

4.3. Theorem. *If R is generically F-rational, then any ideal generated by part of a system of parameters is generically tightly closed. In particular, R is normal.*

Proof. By Theorem 4.2, we know that R is Cohen-Macaulay. By Corollary 3.7, it suffices to show that any ideal generated by a system of parameters (y_1, \dots, y_d) is generically tightly closed. Reasoning on the top local cohomology, we can find $t \geq 1$ and $a \in R$ such that $(y_1, \dots, y_d)R = ((x_1^t, \dots, x_d^t)R :_R a)$ (see for instance the proof of [15, Lemma 4.1]). Therefore, if we can show that $(x_1^t, \dots, x_d^t)R$ is generically tightly closed, then so will $(y_1, \dots, y_d)R$ be by Lemma 3.2. Hence we have reduced to the case that $y_i = x_i^t$, for some $t \geq 1$.

Let $z \in \text{cl}_{\text{gen}}((x_1^t, \dots, x_d^t)R)$. We need to show that $z \in (x_1^t, \dots, x_d^t)R$. If some zx_i does not lie in $(x_1^t, \dots, x_d^t)R$, we may replace our original z by this new element. Therefore, we may assume that

$$z(x_1, \dots, x_d)R \subseteq (x_1^t, \dots, x_d^t)R.$$

Since (x_1, \dots, x_d) is R -regular, we have

$$((x_1^t, \dots, x_d^t)R : (x_1, \dots, x_d)R) = (x_1^t, \dots, x_d^t, y^{t-1})R,$$

where $y := x_1 \cdots x_d$. In summary, we may assume that $z = uy^{t-1}$, for some $u \in R$. By (4), we then get

$$\begin{aligned} u \in (\text{cl}_{\text{gen}}((x_1^t, \dots, x_d^t)R) : y^{t-1}) &\subseteq \text{cl}_{\text{gen}}(((x_1^t, \dots, x_d^t)R : y^{t-1})) \\ &= \text{cl}_{\text{gen}}((x_1, \dots, x_d)R) = (x_1, \dots, x_d)R. \end{aligned}$$

Therefore, $z = uy^{t-1}$ lies in $(x_1^t, \dots, x_d^t)R$, as we wanted to show.

In order to prove that R is normal, it suffices to show that any height one principal ideal aR is integrally closed. Since the integral closure of aR is contained in $\text{cl}_{\text{gen}}(aR)$ by Theorem 3.8, and since a is part of a system of parameters, the conclusion follows from the first assertion. \square

4.4. Proposition. *A local ring R is generically F -rational if and only if its completion \widehat{R} is. In particular, a generically F -rational ring is analytically unramified.*

Proof. Let \mathbf{x} be a system of parameters in R such that $\mathfrak{n} := \mathbf{x}R$ is generically tightly closed. I claim that $\mathfrak{n}\widehat{R}$ is generically tightly closed, from which it follows that \widehat{R} is generically F -rational. To this end, let $\widehat{z} \in \widehat{R}$ be in the generic tight closure of $\mathfrak{n}\widehat{R}$. Write $\widehat{z} = z + \widehat{w}$ with $z \in R$ and $\widehat{w} \in \mathfrak{n}\widehat{R}$. It follows that $z \in \text{cl}_{\text{gen}}(\mathfrak{n}\widehat{R})$. Let J be the ultraproduct of the \mathfrak{n}_w^* , where \mathfrak{n}_w is an approximation of \mathfrak{n} . Since R_w is also an approximation for \widehat{R} , we get $\text{cl}_{\text{gen}}(\mathfrak{n}\widehat{R}) = J \cap \widehat{R}$ by (3). Hence $z \in J$, and since $J \cap R = \text{cl}_{\text{gen}}(\mathfrak{n}) = \mathfrak{n}$, we get $\widehat{z} = z + \widehat{w} \in \mathfrak{n}\widehat{R}$.

Conversely, suppose \widehat{R} is generically F -rational. Let \mathbf{x} be a system of parameters in R . Let a be in the generic tight closure of $\mathbf{x}R$, whence by persistence (Theorem 3.4), in the generic tight closure of $\mathbf{x}\widehat{R}$. Since \mathbf{x} is a system of parameters in \widehat{R} , the ideal $\mathbf{x}\widehat{R}$ is generically tightly closed by Theorem 4.3. Hence, $a \in \mathbf{x}\widehat{R}$, and therefore, by faithful flatness, $a \in \mathbf{x}R$, proving that R is generically F -rational.

To prove the last assertion, assume R is generically F -rational. Hence so is \widehat{R} by what we just proved. Therefore, \widehat{R} is normal by Theorem 4.3, whence a domain, showing that R is analytically unramified. \square

4.5. Corollary. *If R is generically F -rational, then almost all R_w are Cohen-Macaulay and normal.*

Proof. Since \widehat{R} and R have the same approximations, we may assume by Proposition 4.4 that R is complete. Theorems 4.2 and 4.3 yield that R is normal and

Cohen-Macaulay. By [1, Theorem 5.2], almost all R_w are Cohen-Macaulay. Since R satisfies Serre’s condition (R_1) , so do almost all R_w by [1, Theorem 5.6]. Together with the fact that almost all R_w are Cohen-Macaulay, we get from Serre’s criterion for normality (see for instance [19, Theorem 23.8]) that almost all R_w are normal. \square

4.6. Proposition. *If almost all R_w are F-rational, then R is generically F-rational. The converse holds if R is moreover Gorenstein.*

Proof. Let \mathbf{x} be a system of parameters in R , with approximation \mathbf{x}_w , and let z be in the generic tight closure of $\mathbf{x}R$. By [1, Corollary 5.4], almost all \mathbf{x}_w are systems of parameters in R_w . Hence, by definition of F-rationality, \mathbf{x}_wR_w is tightly closed. Therefore, if z_w is an approximation of z , then $z_w \in \mathbf{x}_wR_w$. Taking ultraproducts, we see that z lies in $\mathbf{x}\mathfrak{D}(R)$ and hence by faithful flatness, in $\mathbf{x}R$, showing that R is generically F-rational.

Suppose next that R is Gorenstein and generically F-rational. Towards a contradiction, assume almost each R_w is not F-rational. If J is the ultraproduct of the $(\mathbf{x}_wR_w)^*$, then this means that $\mathbf{x}\mathfrak{D}(R) \not\subseteq J$. On the other hand, by (3) and our assumption, $J \cap R = \mathbf{x}R$. Put $S := R/\mathbf{x}R$. By [1, §4.9], we have an isomorphism $\mathfrak{D}(S) \cong \mathfrak{D}(R)/\mathbf{x}\mathfrak{D}(R)$ and $\mathfrak{D}(S)$ is an ultrapower of $S \otimes_k K$, where k is the residue field of R and K the algebraically closed field used in the definition of Lefschetz hull. Since S is Gorenstein, so is $S \otimes_k K$, whence also $\mathfrak{D}(S)$, since the Gorenstein property is first order definable (see for instance [21]). Let $a \in R$ be such that its image in S generates the socle of this ring. By faithful flatness, a is a non-zero element in the socle of $\mathfrak{D}(S)$, whence must generate it. Since $J\mathfrak{D}(S) \neq 0$, we must have $a \in J$ whence $a \in J \cap R = \mathbf{x}R$, a contradiction. \square

4.7. Remark. Note that by Smith’s result [28, Theorem 3.1], an F-rational excellent local ring is pseudo-rational; the converse holds by [6]. It follows that if almost all approximations of R are pseudo-rational, then R is generically F-rational, whence pseudo-rational by Theorem 6.2 below. I do not know whether the converse also holds.

Let us call R *weakly generically F-regular*, if each ideal $\mathfrak{a} \subseteq R$ is generically tightly closed. By Theorem 3.3, any regular local ring is weakly generically F-regular. By a similar argument as in the proof of Proposition 4.4, one can show that R is weakly generically F-regular if and only if its completion is. If a ring is weakly generically F-regular, then it is generically F-rational; the converse is true for Gorenstein rings, as we now prove.

4.8. Theorem. *If R is Gorenstein and generically F-rational, then it is weakly generically F-regular.*

Proof. Given an arbitrary ideal $\mathfrak{a} \subseteq R$, we need to show that $\mathfrak{a} = \text{cl}_{\text{gen}}(\mathfrak{a})$. Since \mathfrak{a} is the intersection of \mathfrak{m} -primary ideals, we easily reduce to the case that \mathfrak{a} is \mathfrak{m} -primary. Choose a system of parameters \mathbf{x} such that $\mathbf{x}R \subseteq \mathfrak{a}$. By Theorem 4.3, the ideal $\mathbf{x}R$ is generically tightly closed. Since R is Gorenstein,

$$\mathfrak{a} = (\mathbf{x}R : (\mathbf{x}R : \mathfrak{a}))$$

which is a generically tightly closed ideal by Lemma 3.2. \square

4.9. Proposition. *Let $R \rightarrow S$ be a cyclically pure, local homomorphism between equicharacteristic zero Noetherian local rings. If S is weakly generically F -regular, then so is R .*

Proof. Let $z \in \text{cl}_{\text{gen}}(\mathfrak{a})$, for \mathfrak{a} an ideal in R . By Theorem 3.4, the image of z in S lies in the generic tight closure of $\mathfrak{a}S$, which by assumption is just $\mathfrak{a}S$. Hence $z \in \mathfrak{a}S \cap R = \mathfrak{a}$. \square

4.10. Remark. It is well-known that the localization of an F -rational ring is again F -rational (see [15, Theorem 4.2]; the same property for weakly F -regular rings, though, is still open). However, since Lefschetz hulls are not compatible with localization, I do not know whether the localization of a generically F -rational ring is again generically F -rational.

The next Briançon-Skoda type theorem was proven first in [18] for pseudo-rational local rings. Since we will show in the next section that a generically F -rational local ring is pseudo-rational, this version generalizes their result.

4.11. Theorem. *If R is a d -dimensional generically F -rational local ring, then the integral closure of \mathfrak{a}^{m+d} is contained in \mathfrak{a}^{m+1} , for all m and all ideals $\mathfrak{a} \subseteq R$.*

Proof. We follow the argument in [26, Theorem 6.4], where the special case that R is of finite type over an algebraically closed field is proven. Let a be an element of the integral closure of \mathfrak{a}^{m+d} . Assume first that \mathfrak{a} is generated by a system of parameters. Therefore, a lies in $\text{cl}_{\text{gen}}(\mathfrak{a}^{m+1})$, by Theorem 3.8, whence in \mathfrak{a}^{m+1} , by Lemma 4.12 below. This proves the assertion for parameter ideals. Assume next that \mathfrak{a} is merely \mathfrak{m} -primary, where \mathfrak{m} is the maximal ideal of R . In that case, \mathfrak{a} admits a reduction I generated by a system of parameters. Since I^{m+d} is then a reduction of \mathfrak{a}^{m+d} , we get that a lies in the integral closure of I^{m+d} , whence in I^{m+1} , by the first case, and, therefore, ultimately in \mathfrak{a}^{m+1} , also establishing this case. For arbitrary \mathfrak{a} , write \mathfrak{a} as the intersection of all $\mathfrak{a} + \mathfrak{m}^n$ and use the previous case. \square

4.12. Lemma. *If (R, \mathfrak{m}) is a generically F -rational local ring, (x_1, \dots, x_d) a system of parameters and J an \mathfrak{m} -primary ideal generated by monomials in the x_i , then J is generically tightly closed.*

Proof. By [4], we can write J as the intersection of ideals of the form $(x_1^{e_1}, \dots, x_d^{e_d})R$, for some non-zero e_i . Each such ideal is generically tightly closed by Theorem 4.3, whence so is J . \square

5. LOCAL COHOMOLOGY

Before we turn to pseudo-rationality, we must say something about local and sheaf cohomology and their respective ultraproducts. For our purposes, local cohomology is most conveniently approached via Čech cohomology, which we quickly review. Let \mathfrak{a} be an ideal in a Noetherian ring S and choose a tuple $\mathbf{x} := (x_1, \dots, x_d)$ so that \mathfrak{a} and $\mathbf{x}S$ have the same radical. For each $i \leq d$, define

$$C^i(\mathbf{x}; S) := \bigoplus_{1 \leq l_1 < l_2 < \dots < l_i \leq d} S_{x_{l_1} x_{l_2} \dots x_{l_i}}$$

(with the convention that $C^0(\mathbf{x}; S) = S$). The $C^i(\mathbf{x}; S)$ are the modules appearing in a complex $C^\bullet(\mathbf{x}; S)$, called the *algebraic Čech complex* with respect to \mathbf{x} , where

the differential $C^i(\mathbf{x}; S) \rightarrow C^{i+1}(\mathbf{x}; S)$ is given by the inclusion maps among the localizations, with the choice of an appropriate sign to make $C^\bullet(\mathbf{x}; S)$ a complex (see [3, §3.5] for more details). The cohomology of this complex is called the *local cohomology* of S with respect to \mathfrak{a} and is denoted $H_{\mathfrak{a}}^\bullet(S)$. One shows that $H_{\mathfrak{a}}^\bullet(S)$ only depends on the radical of \mathfrak{a} and, in particular, is independent from the choice of d -tuple \mathbf{x} . We will be mainly interested in the top cohomology group $H_{\mathfrak{a}}^d(S)$, and we use the following notation. Since $H_{\mathfrak{a}}^d(S)$ is a homomorphic image of $C^d(\mathbf{x}; S) = S_{x_1 \dots x_d}$, an arbitrary element is the image of a fraction $\frac{a}{(x_1 \dots x_d)^n}$, and we will denote this image by $[\frac{a}{(x_1 \dots x_d)^n}]_S$.

Local cohomology and sheaf cohomology. Let Y be a scheme and Z a closed subset of Y . The collection of those global sections in $H^0(Y, \mathcal{O}_Y)$ whose support is contained in Z is denoted $H_Z^0(Y)$ and is called the *global sections with support in Z* . The derived functors $H_Z^i(Y)$ of the left-exact functor H_Z^0 are called the *cohomology with support in Z* . The cohomology groups with support are connected to the usual sheaf cohomology via an exact sequence

$$(5) \quad \dots \rightarrow H^{i-1}(Y, \mathcal{O}_Y) \xrightarrow{\rho^{i-1}} H^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) \xrightarrow{\partial^i} H_Z^i(Y) \rightarrow H^i(Y, \mathcal{O}_Y) \rightarrow \dots$$

where ∂^i are the connecting morphisms (see for instance [7, Corollary 1.9]).

For (quasi-)projective schemes, we also have a relationship between local cohomology and sheaf cohomology as follows. Let R be a Noetherian ring. A *standard graded R -algebra* is a Noetherian graded ring

$$S = \bigoplus_{n \geq 0} [S]_n$$

such that $R = [S]_0$ and S is (finitely) generated as an R -algebra by $[S]_1$. The irrelevant ideal of S will be denoted by $S^+ := \bigoplus_{n > 0} [S]_n$. Let $Y := \text{Proj } S$ be the projective scheme over $\text{Spec } R$ defined by S and let Z be a closed subset of Y , defined by some homogeneous ideal $\mathfrak{a} \subseteq S$. For each $i \geq 2$, we have

$$(6) \quad H^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) \cong [H_{\mathfrak{a}}^i(S)]_0.$$

Local ultracohomology. For the remainder of this section, R is an equicharacteristic zero Noetherian local ring and S is a finitely generated R -algebra. Let \mathfrak{a} be an ideal in S and let \mathbf{x} be a d -tuple in S such that \mathfrak{a} and $\mathbf{x}S$ have the same radical. Note that each module in the algebraic Čech complex $C^\bullet(\mathbf{x}; S)$ is a finitely generated R -algebra, whence admits an R -hull. The *non-standard algebraic Čech complex* $C_\infty^\bullet(\mathbf{x}; S)$ over S with respect to \mathbf{x} is by definition the complex whose i th module is $\mathfrak{D}_R(C^i(\mathbf{x}; S))$ and for which the differentials are induced by the differentials on $C^\bullet(\mathbf{x}; S)$. The *local ultracohomology* of S with respect to \mathfrak{a} is by definition the cohomology of the non-standard algebraic Čech complex $C_\infty^\bullet(\mathbf{x}; S)$ and is denoted $\text{UH}_{\mathfrak{a}}^\bullet(S)$.

Without proof, we state that $\text{UH}_{\mathfrak{a}}^\bullet(S)$ is independent from the choice of a d -tuple \mathbf{x} . By (2), the canonical homomorphisms $C^i(\mathbf{x}; S) \rightarrow \mathfrak{D}_R(C^i(\mathbf{x}; S))$ give rise to a map of complexes $C^\bullet(\mathbf{x}; S) \rightarrow C_\infty^\bullet(\mathbf{x}; S)$, and hence for each $i \leq d$, we get a natural morphism

$$j_{\mathfrak{a}}^i: H_{\mathfrak{a}}^i(S) \rightarrow \text{UH}_{\mathfrak{a}}^i(S).$$

Let S_w , \mathfrak{a}_w and \mathbf{x}_w be R -approximations of S , \mathfrak{a} and \mathbf{x} respectively. Since we can calculate the local cohomology $H_{\mathfrak{a}_w}^\bullet(S_w)$ with aid of the algebraic Čech complex of

\mathbf{x}_w and since taking ultraproducts commutes with cohomology, we get

$$(7) \quad \mathrm{UH}_{\mathfrak{a}}^i(S) \cong \operatorname{ulim}_w \mathrm{H}_{\mathfrak{a}_w}^i(S_w)$$

for each i . In particular, if $\varphi: S \rightarrow T$ is an R -algebra homomorphism of finite type, then the diagram

$$(8) \quad \begin{array}{ccc} \mathrm{H}_{\mathfrak{a}}^i(S) & \xrightarrow{j_{\mathfrak{a}}^i} & \mathrm{UH}_{\mathfrak{a}}^i(S) \\ \mathrm{H}_{\mathfrak{a}}^i(\varphi) \downarrow & & \downarrow \mathrm{UH}_{\mathfrak{a}}^i(\varphi) \\ \mathrm{H}_{\mathfrak{a}}^i(T) & \xrightarrow{j_{\mathfrak{a}T}^i} & \mathrm{UH}_{\mathfrak{a}}^i(T) \end{array}$$

commutes for each i , where the vertical arrows are the natural maps.

Sheaf ultracohomology. Assume moreover that S is a standard graded R -algebra and \mathfrak{a} is homogeneous. By an argument similar to the one in [27, §2.9], almost all S_w are standard graded R_w -algebras and almost all \mathfrak{a}_w are homogeneous. For each non-standard integer $n_{\infty} := \operatorname{ulim}_w n_w$ we define the *degree n_{∞} part* of $\mathfrak{D}_R(S)$ as

$$[\mathfrak{D}_R(S)]_{n_{\infty}} := \operatorname{ulim}_w [S_w]_{n_w}.$$

If we apply this to each term in the algebraic Čech complex for \mathfrak{a} and take cohomology, we get the degree n_{∞} part of the non-standard local cohomology groups $\mathrm{UH}_{\mathfrak{a}}^i(S)$, and by (7) this is also equal to the ultraproduct of the degree n_w parts of the local cohomology of the approximations. In view of isomorphism (6), we define for $i = 2, \dots, d$ the *sheaf ultracohomology* of $Y - Z$ as

$$\mathrm{UH}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) := [\mathrm{UH}_{\mathfrak{a}}^i(S)]_0.$$

It follows from (6) and (7) that

$$\mathrm{UH}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) = \operatorname{ulim}_w \mathrm{H}^{i-1}(Y_w - Z_w, \mathcal{O}_{Y_w - Z_w}),$$

where $Z_w := V(\mathfrak{a}_w)$. The natural map $j_{\mathfrak{a}}^i: \mathrm{H}_{\mathfrak{a}}^i(S) \rightarrow \mathrm{UH}_{\mathfrak{a}}^i(S)$ induces in degree zero a map

$$u_{Y-Z}^{i-1}: \mathrm{H}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) \rightarrow \mathrm{UH}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}).$$

The restriction maps induce a diagram

$$(9) \quad \begin{array}{ccc} \mathrm{H}^{i-1}(Y, \mathcal{O}_Y) & \xrightarrow{\rho^{i-1}} & \mathrm{H}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) \\ u_Y^{i-1} \downarrow & & \downarrow u_{Y-Z}^{i-1} \\ \mathrm{UH}^{i-1}(Y, \mathcal{O}_Y) & \xrightarrow{\rho_{\infty}^{i-1}} & \mathrm{UH}^{i-1}(Y - Z, \mathcal{O}_{Y-Z}) \end{array}$$

where ρ_{∞}^{i-1} is the ultraproduct of the restriction maps

$$\rho_w^{i-1}: \mathrm{H}^{i-1}(Y_w, \mathcal{O}_{Y_w}) \rightarrow \mathrm{H}^{i-1}(Y_w - Z_w, \mathcal{O}_{Y_w - Z_w}).$$

Making the appropriate identifications between local cohomology and sheaf cohomology given by (6), diagram (9) is the degree zero part of

$$(10) \quad \begin{array}{ccc} H_{S^+}^i(S) & \xrightarrow{r^i} & H_{\mathfrak{a}}^i(S) \\ \downarrow j_{S^+}^i & & \downarrow j_{\mathfrak{a}}^i \\ UH_{S^+}^i(S) & \xrightarrow{r_{\infty}^i} & UH_{\mathfrak{a}}^i(S) \end{array}$$

where r_{∞}^i is the ultraproduct of the natural maps

$$r_w^i: H_{S_w^+}^i(S_w) \rightarrow H_{\mathfrak{a}_w}^i(S_w).$$

It is easy to check that (10) commutes, whence so does (9).

6. PSEUDO-RATIONALITY

The notion of pseudo-rationality was introduced by Lipman and Teissier to extend the notion of rational singularities to a situation where there is not necessarily a resolution of singularities available.

6.1. Pseudo-rationality. A Noetherian local ring (R, \mathfrak{m}) is called *pseudo-rational*, if it is analytically unramified, normal, Cohen-Macaulay and for any projective birational map $f: Y \rightarrow \text{Spec } R$ with Y normal, the canonical epimorphism between the top cohomology groups $\delta: H_{\mathfrak{m}}^d(R) \rightarrow H_Z^d(Y)$ is injective, where Z is the closed fiber $f^{-1}(\mathfrak{m})$ and d the dimension of R (see (11) below for the definition of δ). Moreover, if $\text{Spec } R$ admits a desingularization $Y \rightarrow \text{Spec } R$, then it suffices to check the above condition for just this one Y (see [18, §2, Remark (a) and Example (b)]). From this, one can show using Matlis duality, that if R is essentially of finite type over a field of characteristic zero, then R is pseudo-rational if and only if it has rational singularities. A Noetherian ring A is called *pseudo-rational*, if $A_{\mathfrak{p}}$ is pseudo-rational for every prime ideal \mathfrak{p} in A .

The key ingredient in proving Theorems A and B is the following result linking generic tight closure with pseudo-rationality, analogous to Smith's characterization [28] in prime characteristic.

6.2. Theorem. *If an equicharacteristic zero Noetherian local ring R is generically F -rational, then it is pseudo-rational.*

Proof. By Theorems 4.2 and 4.3 and Proposition 4.4, we know that R is analytically unramified, Cohen-Macaulay and normal. Let $X := \text{Spec } R$ and let $f: Y = \text{Proj } S \rightarrow X$ be a projective birational map with Y normal. In particular, S is a standard graded R -algebra. Let $i: R \rightarrow S$ be the embedding identifying R with $[S]_0$, let \mathfrak{m} be the maximal ideal of R and let $Z := V(\mathfrak{m}S)$ be the closed fiber of f . The image of the canonical map $H_{\mathfrak{m}}^d(i): H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}S}^d(S)$ lies entirely in degree zero, whence in view of (6), induces a morphism $\gamma^d: H_{\mathfrak{m}}^d(R) \rightarrow H^{d-1}(Y - Z, \mathcal{O}_{Y-Z})$.

Combining this with the tail of the exact sequence (5) and with (9) gives a commutative diagram

$$(11) \quad \begin{array}{ccccc} & & \mathbb{H}_{\mathfrak{m}}^d(R) & & \\ & & \downarrow \gamma^d & \searrow \delta & \\ \mathbb{H}^{d-1}(Y, \mathcal{O}_Y) & \xrightarrow{\rho^{d-1}} & \mathbb{H}^{d-1}(Y-Z, \mathcal{O}_{Y-Z}) & \xrightarrow{\partial^d} & \mathbb{H}_Z^d(Y) \\ \downarrow u_Y^{d-1} & & \downarrow u_{Y-Z}^{d-1} & & \\ \text{UH}^{d-1}(Y, \mathcal{O}_Y) & \xrightarrow{\rho_{\infty}^{d-1}} & \text{UH}^{d-1}(Y-Z, \mathcal{O}_{Y-Z}) & & \end{array}$$

in which the middle row is exact.

Let \mathbf{x} be a system of parameters in R such that $\mathbf{x}R$ is generically tightly closed. Note that the algebraic Čech complex of \mathbf{x} over R (respectively, over S) calculates the local cohomology of \mathfrak{m} (respectively, of $\mathfrak{m}S$). We need to show that the kernel of δ is zero, hence suppose the contrary. In particular, it must contain a non-zero element of the form $[\frac{a}{y}]_R$, with $a \in R$ and where y is the product of the entries in \mathbf{x} . From the exactness of (11), we see that $\delta([\frac{a}{y}]_R) = 0$ means that $\gamma^d([\frac{a}{y}]_R)$ lies in the image of ρ^{d-1} . Under the isomorphism $\mathbb{H}^{d-1}(Y-Z, \mathcal{O}_{Y-Z}) \cong [\mathbb{H}_{\mathfrak{m}S}^d(S)]_0$ from (6), we may identify $\gamma^d([\frac{a}{y}]_R)$ with $[\frac{a}{y}]_S$. Since the square in (11) commutes, $u_{Y-Z}^{d-1}([\frac{a}{y}]_S)$ lies in the image of ρ_{∞}^{d-1} .

Let (R_w, \mathfrak{m}_w) be an approximation of (R, \mathfrak{m}) . By Corollary 4.5, almost all R_w are Cohen-Macaulay and normal, whence in particular domains. Let S_w be an R -approximation of S , put $X_w := \text{Spec}(R_w)$ and $Y_w := \text{Proj}(S_w)$, and let $Z_w := V(\mathfrak{m}_w S_w)$ be the closed fiber of $Y_w \rightarrow X_w$. Let a_w and \mathbf{x}_w be approximations of a and \mathbf{x} respectively, and put y_w equal to the product of all the entries in \mathbf{x}_w . By definition, $u_{Y-Z}^{d-1}([\frac{a}{y}]_S)$ is the ultraproduct of the $[\frac{a_w}{y_w}]_{S_w}$. Hence by Łos' Theorem, almost each $[\frac{a_w}{y_w}]_{S_w}$ lies in the image of

$$\rho_w^{d-1}: \mathbb{H}^{d-1}(Y_w, \mathcal{O}_{Y_w}) \rightarrow \mathbb{H}^{d-1}(Y_w - Z_w, \mathcal{O}_{Y_w - Z_w})$$

since ρ_{∞}^{d-1} is the ultraproduct of the ρ_w^{d-1} . By the same argument as above, we have for each w , an exact diagram

$$(12) \quad \begin{array}{ccccc} & & \mathbb{H}_{\mathfrak{m}_w}^d(R_w) & & \\ & & \downarrow \gamma_w^d & \searrow \delta_w & \\ \mathbb{H}^{d-1}(Y_w, \mathcal{O}_{Y_w}) & \xrightarrow{\rho_w^{d-1}} & \mathbb{H}^{d-1}(Y_w - Z_w, \mathcal{O}_{Y_w - Z_w}) & \xrightarrow{\partial_w^d} & \mathbb{H}_{Z_w}^d(Y_w). \end{array}$$

By reversing the above arguments, this diagram then shows that almost each $[\frac{a_w}{y_w}]_{R_w}$ lies in the kernel L_w of δ_w . Let us briefly recall the argument from [28] regarding how for a fixed w this implies that a_w lies in the tight closure of $\mathbf{x}_w R_w$. Namely, since the Frobenius \mathbf{F}_w acts on the local cohomology groups, the kernel L_w is invariant under its action by functoriality. Hence

$$(13) \quad \mathbf{F}_w^m([\frac{a_w}{y_w}]_{R_w}) = [\frac{\mathbf{F}_w^m(a_w)}{\mathbf{F}_w^m(y_w)}]_{R_w} \in L_w.$$

Since L_w is a proper subgroup of $H_{\mathfrak{m}_w}^d(R_w)$ (note that δ_w^d is non-zero), the Matlis dual of L_w is a proper homomorphic image of the canonical module ω_{R_w} . Since the canonical module has rank one, the Matlis dual of L_w has torsion, whence so does L_w itself. Hence for some non-zero $c_w \in R_w$ we have $c_w L_w = 0$. Together with (13), this yields

$$\left[\frac{c_w \mathbf{F}_w^m(a_w)}{\mathbf{F}_w^m(y_w)}\right]_{R_w} = 0$$

for each m . Since almost each R_w is Cohen-Macaulay, we get

$$c_w \mathbf{F}_w^m(a_w) \in \mathbf{F}_w^m(\mathbf{x}_w)R_w,$$

for all m , proving our claim that a_w lies in the tight closure of $\mathbf{x}_w R_w$. Since this holds for almost all w , we conclude that a lies in the generic tight closure of $\mathbf{x}R$, which, by assumption, is just $\mathbf{x}R$. However, this means that $[\frac{a}{y}]_R$ is zero, contradiction. \square

Proof of Theorem A. Since all properties localize, we may assume that A and B are moreover local and that $A \rightarrow B$ is a local homomorphism. Since B is weakly generically F-regular by Theorem 3.3, so is A , by Proposition 4.9. Therefore, A is pseudo-rational by Theorem 6.2. \square

7. ULTRA-F-REGULAR RINGS AND LOG-TERMINAL SINGULARITIES

In this section, we extend the argument from [27] in order to prove Theorem B.

7.1. \mathbb{Q} -Gorenstein singularities. Let R be an equicharacteristic zero Noetherian local domain and put $X := \text{Spec } R$. We say that R is \mathbb{Q} -Gorenstein if it is normal and some positive multiple of the canonical divisor K_X is Cartier; the least such positive multiple is called the *index* of R . If R is the homomorphic image of an excellent regular local ring (which is for instance the case if R is complete), then X admits an *embedded resolution of singularities* $f: Y \rightarrow X$ by [9]. If E_i are the irreducible components of the exceptional locus of f , then the canonical divisor K_Y is numerically equivalent to $f^*(K_X) + \sum a_i E_i$ (as \mathbb{Q} -divisors), for some $a_i \in \mathbb{Q}$. The rational number a_i is called the *discrepancy* of X along E_i ; see [17, Definition 2.22]. If all $a_i > -1$, we call R *log-terminal* (in case we only have a weak inequality, we call R *log-canonical*).

7.2. Canonical cover. Recall the construction of the canonical cover of a \mathbb{Q} -Gorenstein local ring R due to Kawamata. If r is the index of R , then $\mathcal{O}_X(rK_X) \cong \mathcal{O}_X$, where $X := \text{Spec } R$ and K_X the canonical divisor of X . This isomorphism induces an R -algebra structure on

$$\tilde{R} := H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \cdots \oplus \mathcal{O}_X((r-1)K_X)),$$

which is called the *canonical cover* of R ; see [16]. An important property for our purposes is that $R \rightarrow \tilde{R}$ is étale in codimension one (see for instance [29, 4.12]). We also use the following result proven by Kawamata in [16, Proposition 1.7]:

7.3. Theorem. *Let R be a homomorphic image of an equicharacteristic zero, excellent regular local ring. If R is \mathbb{Q} -Gorenstein, then it has log-terminal singularities if and only if its canonical cover is rational.*

7.4. Definition. Inspired by Kawamata’s result, we can now give a resolution-free variant of log-terminal singularities. We call a Noetherian local domain *pseudo-log-terminal* if it is \mathbb{Q} -Gorenstein and its canonical cover is pseudo-rational.

In the remainder of this section, R is an equicharacteristic zero Noetherian local ring and R_w is an approximation of R .

7.5. Ultra-F-regularity. We say that R is *ultra-F-regular*, if it is a domain and for each non-zero $c \in R$, we can find an ultra-Frobenius \mathbf{F}^ε such that the R -module morphism

$$(14) \quad R \rightarrow {}^\varepsilon\mathfrak{D}(R) : x \mapsto c\mathbf{F}^\varepsilon(x)$$

is pure. Note that in order for (14) to be R -linear, we need to view $\mathfrak{D}(R)$ as an R -algebra via \mathbf{F}^ε , that is to say, the target must be taken to be ${}^\varepsilon\mathfrak{D}(R)$ (see §2.2). Since $\mathfrak{D}(R) = \mathfrak{D}(\widehat{R})$, an analytically unramified local ring R is ultra-F-regular if and only if its completion \widehat{R} is.

Over normal domains, purity and cyclical purity are the same by [10, Theorem 2.6]. Hence for R normal, the purity of (14) is equivalent to the weaker condition that for every $x \in R$ and every ideal $I \subseteq R$, we have

$$(15) \quad c\mathbf{F}^\varepsilon(x) \in \mathbf{F}^\varepsilon(I)\mathfrak{D}(R) \quad \text{implies} \quad x \in I.$$

One can show that if R is moreover analytically unramified, then either condition entails normality, and hence in that case, they are equivalent (this follows for instance from the discussion below and the Briançon-Skoda property of generic tight closure).

7.6. Proposition. *If R is regular, then it is ultra-F-regular.*

Proof. By the above discussion, we need only verify the weaker condition (15). In fact, we will show that for any c , we may take $\varepsilon = 1$ in (15). Indeed, assume $c\mathbf{F}(x) \in \mathbf{F}(I)\mathfrak{D}(R)$. Since \mathbf{F} preserves regular sequences, ${}^1\mathfrak{D}(R)$ is a balanced big Cohen-Macaulay R -algebra whence is flat by [25, Theorem IV.1] or [12, Lemma 2.1(d)]. Hence

$$c \in (\mathbf{F}(I)\mathfrak{D}(R) : \mathbf{F}(x)) = \mathbf{F}(I : x)\mathfrak{D}(R).$$

Suppose $x \notin I$. Since $(I : x)$ then lies in the maximal ideal of R , its image under \mathbf{F} lies in the ideal of infinitesimals of $\mathfrak{D}(R)$. Hence $\mathbf{F}(I : x)\mathfrak{D}(R) \cap R = (0)$, contradicting that $c \neq 0$. □

7.7. Theorem. *If R is analytically unramified and ultra-F-regular, then it is weakly generically F-regular, whence in particular pseudo-rational.*

Proof. The last assertion follows from the first by Theorem 6.2. Since all properties are invariant under completion, we may assume that R is complete. Let I be an ideal in R and $x \in \text{cl}_{\text{gen}}(I)$. We want to show that $x \in I$. By [1, Proposition 6.24], there exists $c \in R$ such that almost all c_w are test elements in R_w , where c_w and R_w are approximations of c and R respectively. Let x_w and I_w be approximations of x and I respectively, so that almost all $x_w \in I_w^*$. Hence, for almost all w and all e , we have

$$(16) \quad c_w \mathbf{F}_w^e(x_w) \in \mathbf{F}_w^e(I_w)R_w.$$

By assumption, there is an ultra-Frobenius \mathbf{F}^ε so that $x \mapsto c\mathbf{F}^\varepsilon(x)$ is pure whence cyclically pure, that is to say, so that (15) holds. Let ε be the ultraproduct of

integers e_w . Taking e equal to e_w in (16) and taking ultraproducts shows that $c\mathbf{F}^\varepsilon(x) \in \mathbf{F}^\varepsilon(I)\mathfrak{D}(R)$. Therefore, from (15) we get $x \in I$, as we wanted to show. \square

7.8. Proposition. *Let $R \subseteq S$ be a finite extension of Noetherian local domains which is étale in codimension one. Let c be a non-zero element of R and \mathbf{F}^ε an ultra-Frobenius. If $R \rightarrow {}^\varepsilon\mathfrak{D}(R): x \mapsto c\mathbf{F}^\varepsilon(x)$ is pure, then so is its base change $S \rightarrow {}^\varepsilon\mathfrak{D}(S): x \mapsto c\mathbf{F}^\varepsilon(x)$.*

In particular, if R is ultra-F-regular, then so is S .

Proof. Let $R \subseteq S$ be an arbitrary finite extension of d -dimensional Noetherian local domains and fix a non-zero element $c \in R$ and an ultra-Frobenius \mathbf{F}^ε . Let \mathfrak{n} be the maximal ideal of S and ω_S its canonical module. I claim that if $R \subseteq S$ is étale, then

$$(17) \quad {}^\varepsilon\mathfrak{D}(S) \cong S \otimes_R {}^\varepsilon\mathfrak{D}(R).$$

Assuming the claim, let $R \subseteq S$ now only be étale in codimension one. It follows from the claim that the supports of the kernel and the cokernel of the natural map $S \otimes_R {}^\varepsilon\mathfrak{D}(R) \rightarrow {}^\varepsilon\mathfrak{D}(S)$ have codimension at least two. Hence the same is true for the base change

$$\omega_S \otimes_R {}^\varepsilon\mathfrak{D}(R) \rightarrow \omega_S \otimes_S {}^\varepsilon\mathfrak{D}(S).$$

Applying the top local cohomology functor $H_{\mathfrak{n}}^d$, we get from the long exact sequence of local cohomology and Grothendieck Vanishing, an isomorphism

$$(18) \quad H_{\mathfrak{n}}^d(\omega_S \otimes_R {}^\varepsilon\mathfrak{D}(R)) \cong H_{\mathfrak{n}}^d(\omega_S \otimes_S {}^\varepsilon\mathfrak{D}(S)).$$

Recall that by Grothendieck duality, $H_{\mathfrak{n}}^d(\omega_S)$ is the injective hull E of the residue field of S .

Let $c_{\varepsilon,R}$ denote the R -linear morphism $R \rightarrow {}^\varepsilon\mathfrak{D}(R): x \mapsto c\mathbf{F}^\varepsilon(x)$. For an arbitrary R -module M , let $c_{\varepsilon,R,M}: M \rightarrow M \otimes_R {}^\varepsilon\mathfrak{D}(R)$ be the base change of $c_{\varepsilon,R}$ over M . In particular, we have a commutative diagram

$$\begin{array}{ccc} \omega_S & \xrightarrow{c_{\varepsilon,R,\omega_S}} & \omega_S \otimes_R {}^\varepsilon\mathfrak{D}(R) \\ \parallel & & \downarrow \\ \omega_S & \xrightarrow{c_{\varepsilon,S,\omega_S}} & \omega_S \otimes_S {}^\varepsilon\mathfrak{D}(S). \end{array}$$

Taking top local cohomology yields the outer square in the following commutative diagram:

$$(19) \quad \begin{array}{ccccc} E = H_{\mathfrak{n}}^d(\omega_S) & \xrightarrow{c_{\varepsilon,R,E}} & E \otimes_R {}^\varepsilon\mathfrak{D}(R) & \longrightarrow & H_{\mathfrak{n}}^d(\omega_S \otimes_R {}^\varepsilon\mathfrak{D}(R)) \\ \parallel & & \downarrow & & \downarrow \cong \\ E = H_{\mathfrak{n}}^d(\omega_S) & \xrightarrow{c_{\varepsilon,S,E}} & E \otimes_S {}^\varepsilon\mathfrak{D}(S) & \longrightarrow & H_{\mathfrak{n}}^d(\omega_S \otimes_S {}^\varepsilon\mathfrak{D}(S)) \end{array}$$

where the isomorphism at the right comes from (18). Since $c_{\varepsilon,R}$ is pure, so is its base change $c_{\varepsilon,R,\omega_S}$. Purity is preserved when taking cohomology, so that the top composite map in (19) is pure, whence so is the bottom composite map, since it is isomorphic to it. Since $c_{\varepsilon,S,E}$ is a factor of this map, it is itself pure, whence in particular injective. By [12, Lemma 2.1(e)], to verify the purity of $c_{\varepsilon,S}$, one only needs to show that its base change $c_{\varepsilon,S,E}$ over E is injective, and this is exactly what we just showed.

To prove the claim (17), observe that if $R \rightarrow S$ is étale with approximation $R_w \rightarrow S_w$, then almost all of these are étale. Indeed, by [20, Corollary 3.16], we can write S as $R[X]/I$, with $X = (X_1, \dots, X_n)$ and $I = (f_1, \dots, f_n)R[X]$, such that the Jacobian $J(f_1, \dots, f_n)$ is a unit in R , and by Los' Theorem, this property is preserved for almost all approximations. Quite generally, if $C \rightarrow D$ is an étale extension of characteristic p domains, then we have for each e an isomorphism ${}^e D \cong D \otimes_C {}^e C$ (see for instance [15, p. 50] or the proof of [29, Theorem 4.15]). Applied to the current situation, we get ${}^e S_w \cong S_w \otimes_{R_w} {}^e R_w$ (see [15, p. 50]). Therefore, applied with $e =: e_w$, where e_w is an approximation of ε , we get after taking ultraproducts,

$${}^\varepsilon \mathfrak{D}(S) \cong \mathfrak{D}(S) \otimes_{\mathfrak{D}(R)} {}^\varepsilon \mathfrak{D}(R) \cong S \otimes_R {}^\varepsilon \mathfrak{D}(R)$$

as required, where we used the isomorphism $\mathfrak{D}(S) \cong S \otimes_R \mathfrak{D}(R)$, which holds by [1, §4.10.4], since $R \rightarrow S$ is finite.

To prove the last assertion, we have to show that we can find for each non-zero $c \in S$ an ultra-Frobenius \mathbf{F}^ε such that $c_{\varepsilon,S}$ is pure. However, if we can do this for some non-zero multiple of c , then we can also do this for c , and hence, since S is finite over R , we may assume without loss of generality that $c \in R$. Since R is ultra-F-regular, we can therefore find an ultra-Frobenius \mathbf{F}^ε such that $c_{\varepsilon,R}$ is pure, and hence by the first assertion, so then is $c_{\varepsilon,S}$, proving that S is ultra-F-regular. \square

7.9. Proposition. *Let $R \rightarrow S$ be a cyclically pure homomorphism of equicharacteristic zero Noetherian local rings. If S is ultra-F-regular and analytically unramified, then so is R .*

Proof. Since $\widehat{R} \rightarrow \widehat{S}$ is again cyclically pure by [1, Lemma 6.7], we may assume without loss of generality that S is complete. Let $c \in R$ be non-zero and let \mathbf{F}^ε be an ultra-Frobenius for which the S -module morphism

$$(20) \quad c_{\varepsilon,S}: S \rightarrow {}^\varepsilon \mathfrak{D}(S): x \mapsto c\mathbf{F}^\varepsilon(x)$$

is pure. We want to show that the same is true upon replacing S by R , that is to say, that $c_{\varepsilon,R}$ is pure. Since S is weakly generically F-regular by Theorem 7.7, so is R by Proposition 4.9. Hence R is in particular normal by Theorem 4.3, so that it suffices to verify (15). Let $x \in R$ and $I \subseteq R$ be such that $c\mathbf{F}^\varepsilon(x) \in \mathbf{F}^\varepsilon(I)\mathfrak{D}(R)$. Therefore, x belongs to IS by (20), whence to $IS \cap R = I$ by cyclical purity. \square

Note that in the proof, the condition that S is analytically unramified was only used to get the normality of R .

Proof of Theorem B. Proposition 7.6 yields that S is ultra-F-regular, whence so is R by Proposition 7.9. Let \tilde{R} be the canonical cover of R . By Proposition 7.8, \tilde{R} is also ultra-F-regular, whence pseudo-rational by Theorem 7.7. \square

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DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NEW YORK 10016

E-mail address: hschoutens@citytech.cuny.edu