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ON A STOCHASTIC WAVE EQUATION WITH UNILATERAL BOUNDARY CONDITIONS

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ABSTRACT. We prove the existence and uniqueness of solutions to the initial boundary value problem for a one-dimensional wave equation with unilateral boundary conditions and random noise. We also establish the existence of an invariant measure.

§0. INTRODUCTION

In this paper, we discuss a one-dimensional wave equation with unilateral boundary conditions and random noise. The problem is formulated as follows:

$$u_{tt} - u_{xx} + \alpha u_t = \frac{\partial \Phi}{\partial t}, \quad \text{for } (x,t) \in (0,L) \times (0,T), \tag{0.1}$$

$$u_x(0,t) \le 0, \quad u(0,t) \ge 0, \quad u_x(0,t)u(0,t) = 0, \quad \text{for } t \in (0,T), \tag{0.2}$$

$$u(L,t) = 0, \qquad \text{for } t \in (0,T), \tag{0.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \text{for } x \in (0,L). \tag{0.4}$$

This problem is associated with the longitudinal motion of a linear elastic bar against a rigid stationary obstacle. Here u denotes the axial displacement, and the boundary condition (0.2) expresses the condition that the left end of the bar does not stick to the obstacle and the rigid obstacle can exert stress only if the bar is in contact with the obstacle. Φ is a continuous martingale which will be described in detail later, and α is a nonnegative constant. For the case when $\partial_t \Phi$ is replaced by a deterministic function and the damping term αu_t is replaced by a memory term, this initial-boundary value problem has been investigated in [9]. Related deterministic problems were studied in [3], [4], [7], [12], [13] and [15]. However, with a random noise, this is a completely new problem, which is not readily covered by the existing results. For general results on the stochastic evolution equations, the readers are referred to [1]. The nonlinear boundary condition (0.2) combined with a random noise poses a new challenge in stochastic analysis. Our goal is to prove the existence and uniqueness of solutions to (0.1) - (0.4), and to establish the existence of an invariant measure.

For a parabolic equation with unilateral constraint, the existence and uniqueness of solutions can be obtained through the form of a stochastic parabolic variational inequality; see [5], where the penalty method was used. We will also use the penalty

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method which is meaningful from the physical viewpoint. Namely, we approximate the rigid obstacle by elastic obstacles with increasing rigidity and obtain a solution as a limit of the approximate solutions. So our general strategy is fairly standard. However, as the rigidity parameter increases to infinity, we do not have strong convergence of approximate solutions. We only have weak convergence over the sample space. This is in sharp contrast to the parabolic case in [5]. Weak convergence over the sample space is not sufficient to obtain pathwise solutions of our problem. This is the main reason why we will work directly with the above form (0.1) - (0.4) even though the problem can be formulated as a stochastic hyperbolic variational inequality. For a deterministic nonlinear problem, we can derive strong convergence in a larger function class from weak convergence by means of compact imbedding of function spaces to handle nonlinearity. In the case of stochastic problems, weak convergence over the sample space cannot be translated into strong convergence. The nonlinearity is due to the boundary condition (0.2). In particular, weak convergence is not sufficient for a limit function to satisfy the last condition in (0.2). Fortunately, we have some partial strong convergence of the trace of approximate solutions as rigidity tends to infinity. This requires some unusual estimates, which are also essential for the proof of pathwise uniqueness. One of our goals is to address these technical issues. If the noise term is additive, i.e., Φ is independent of the unknown function u, then the technical procedure can be simpler. In this case, we have pathwise convergence of approximate solutions by splitting each approximate solution into two parts. Namely, one part takes care of the random noise, and the other part is a solution of essentially a deterministic problem. This will be shown in Section 5 below.

For asymptotic behavior of solutions to stochastic evolution equations, an invariant measure is an important object. If the probability law of the initial data is the same as an invariant measure, then the probability law of evolving solutions is invariant in time. This corresponds to stationary solutions of deterministic equations. There are well-known results on the existence of invariant measures for general semi-linear evolution equations: see [1] and [2]. But the existence of an invariant measure for (0.1) - (0.3) is a completely new problem which is not covered by these well-known results. Recently, the author [10] obtained a new existence result on the invariant measures of a certain class of evolution equations. It turns out that the above problem (0.1) - (0.3) fits into this class. Our task is to show that all the required assumptions for the result in [10] are satisfied. For this, we need pathwise convergence of approximate solutions to justify necessary conditions for the result in [10]. Hence, we can handle only the case of an additive noise for the existence of an invariant measure.

Finally, we note that a stochastic version of the problem discussed in [12] and [13] is still an open question. When an obstacle is placed in the interior, the uniqueness of the solution is known only under an extra condition of energy conservation. It is not known how such a condition can be modified for the stochastic problem.

§1. Notation and statement of the main results

We will use the following notation:

$$\partial_t u = u_t = \frac{\partial u}{\partial t}, \quad \partial_x u = u_x = \frac{\partial u}{\partial x},$$

and

$$h^+ = \max(h, 0), \qquad h^- = \max(-h, 0).$$

For a real number s, $H^s(0, L)$ denotes the usual Sobolev space of order s over the interval $(0, L)$. If $h \in H^1(0, T)$, then $h(\cdot)^- \in H^1(0, T)$ and

$$\frac{d}{dt}\big(h(\cdot)^-\big) = -\chi\{h(t) < 0\}\frac{d}{dt}h(\cdot) \tag{1.1}$$

where $\chi\{\cdots\}$ is the characteristic function of the set $\{\cdots\}$.

$\big(\Omega, \mathcal{F}, \mathcal{F}_t, P\big)$ is a given stochastic basis, where P is a probability measure, $\mathcal{F}$ is a σ-algebra and $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $\big(\Omega, \mathcal{F}\big)$ such that $\mathcal{F}_0$ contains all P-negligible subsets. $\{B_j(t)\}_{j=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions over $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $E\big(\cdot\big)$ stands for expectation with respect to the probability measure P. In this paper, a stochastic integral is defined in the sense of Ito. When $\mathcal{O}$ is a topological space, $\mathcal{B}(\mathcal{O})$ denotes the Borel σ-algebra over $\mathcal{O}$. When $\mathcal{X}$ is a Banach space, an $\mathcal{X}$-valued function f is said to be $\mathcal{F}$-measurable if $f^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{B}(\mathcal{X})$. This coincides with strong measurability for Bochner integrals when the range of f is separable. When $\mathcal{X}$ is a Banach space, $L^p(\Omega; \mathcal{X})$, $1 \leq p < \infty$, denotes the set of all $\mathcal{X}$-valued strongly measurable functions such that

$$\int_\Omega \|f\|_{\mathcal{X}}^p \, dP < \infty.$$

An $\mathcal{X}$-valued stochastic process $Y(t)$ is said to be progressively measurable if Y restricted to the interval $[0, t]$ is $\mathcal{B}\big([0, t]\big) \otimes \mathcal{F}_t$-measurable for each $t \geq 0$. If $\mathcal{X}$ is a separable Hilbert space, then $L^\infty(0, T; \mathcal{X})$ is the dual of $L^1(0, T; \mathcal{X})$. In this case, let $L^2_*\big(\Omega; L^\infty(0, T; \mathcal{X})\big)$ be the set of all f such that $\langle f, q\rangle$ is $\mathcal{F}$-measurable for every $q \in L^1(0, T; \mathcal{X})$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $L^\infty(0, T; \mathcal{X})$ and $L^1(0, T; \mathcal{X})$, and

$$\int_\Omega \|f(\omega)\|^2_{L^\infty(0,T;\mathcal{X})} dP < \infty.$$

Then, $L^2_*\big(\Omega; L^\infty(0, T; \mathcal{X})\big)$ is the dual of $L^2\big(\Omega; L^1(0, T; \mathcal{X})\big)$; see [14].

Throughout this paper, we suppose that

$$\Phi = \Phi(u)(t) = \sum_{j=1}^{\infty} \int_0^t (\sigma_j u + f_j) dB_j$$

where the f_j's are progressively measurable with respect to $\{\mathcal{F}_t\}$, and

$$\sum_{j=1}^{\infty} \|f_j\|^2_{L^2(\Omega; L^2(0,T; L^2(0,L))} < \infty \tag{1.2}$$

for each $T > 0$. The σ_j's are deterministic functions such that

$$\sum_{j=1}^{\infty} \|\sigma_j\|^2_{L^\infty((0,L)\times(0,T))} < \infty \tag{1.3}$$

for each $T > 0$.

Lemma 1.1. *Suppose that $\{v_k\}_{k=1}^{\infty}$ is a sequence of progressively measurable functions in $L^2\big(\Omega; L^2(0,T; L^2(0,L))\big)$ such that as $k \to \infty$,*

$$v_k \to v \qquad \textit{weakly in } L^2\big(\Omega; L^2(0,T; L^2(0,L))\big),$$
$$\partial_t v_k \to \partial_t v \qquad \textit{weakly in } L^2\big(\Omega; L^2(0,T; L^2(0,L))\big).$$

Then, as $k \to \infty$,

$$\sum_{j=1}^{\infty} \int_0^{(\cdot)} \sigma_j v_k \, dB_j \to \sum_{j=1}^{\infty} \int_0^{(\cdot)} \sigma_j v \, dB_j$$

weakly in $L^2\big(\Omega; L^2(0,T; L^2(0,L))\big)$.

Proof. For any $\epsilon > 0$, there is a positive integer N such that

$$\sum_{j=N}^{\infty} E\bigg(\int_0^T \|\sigma_j v_k\|^2_{L^2(0,L)} dt\bigg) < \epsilon$$

for all $k \geq 1$. Hence, it is enough to show that as $k \to \infty$,

$$\int_0^{(\cdot)} \sigma_j v_k \, dB_j \to \int_0^{(\cdot)} \sigma_j v \, dB_j \tag{1.4}$$

weakly in $L^2\big(\Omega; L^2(0,T; L^2(0,L))\big)$, for each j. Since $\partial_t v_k \in L^2\big(\Omega; L^2(0,T; L^2(0,L))\big)$, we have

$$dv_k = \big(\partial_t v_k\big) dt,$$

which, combined with the Ito calculus (see [8]), yields

$$v_k(t) \int_0^t \sigma_j(s) dB_j(s) = \int_0^t v_k(s) \sigma_j(s) \, dB_j(s) + \int_0^t \bigg(\int_0^s \sigma_j(\eta) dB_j(\eta)\bigg) \partial_s v_k(s) \, ds \tag{1.5}$$

for all $t \in [0,T]$, for almost all $\omega \in \Omega$. Since we have

$$\int_0^{(\cdot)} \sigma_j(s) dB_j(s) \in L^2\big(\Omega; C([0,T]; L^2(0,L))\big),$$

it holds that for any $b \in L^2\big(0,T; L^\infty(0,L)\big)$ and $G \in \mathcal{F}$,

$$\int_G \int_0^T \langle b(t), \, v_k(t) \int_0^t \sigma_j(s) dB_j(s) \rangle dt \, dP \to \int_G \int_0^T \langle b(t), \, v(t) \int_0^t \sigma_j(s) dB_j(s) \rangle dt \, dP$$

as $k \to \infty$. Here $\langle \cdot \, , \, \cdot \rangle$ denotes the duality pairing between $L^1(0,L)$ and $L^\infty(0,L)$.

By changing the order of integration, we see that

$$\begin{aligned} &\int_G \int_0^T \langle b(t), \, \int_0^t \bigg(\int_0^s \sigma_j(\eta) dB_j(\eta)\bigg) \partial_s v_k(s) \, ds \rangle dt \, dP \\ &\quad = \int_G \int_0^T \langle \int_t^T b(s) \, ds, \; \partial_t v_k(t) \int_0^t \sigma_j(s) dB_j(s) \rangle dt \, dP \end{aligned}$$

and hence,

$$\begin{aligned} &\int_G \int_0^T \langle b(t), \, \int_0^t \bigg(\int_0^s \sigma_j(\eta) dB_j(\eta)\bigg) \partial_s v_k(s) \, ds \rangle dt \, dP \\ &\quad \to \int_G \int_0^T \langle b(t), \, \int_0^t \bigg(\int_0^s \sigma_j(\eta) dB_j(\eta)\bigg) \partial_s v(s) \, ds \rangle dt \, dP \end{aligned}$$

as $k \to \infty$. Therefore, it follows from (1.5) that

$$\begin{aligned}(1.6) \qquad &\int_G \int_0^T \langle b(t), \int_0^t v_k(s)\sigma_j(s)dB_j(s)\rangle dt\, dP \\ &\to \int_G \int_0^T \langle b(t), \int_0^t v(s)\sigma_j(s)dB_j(s)\rangle dt\, dP \quad \text{as } k \to \infty\end{aligned}$$

for every $b \in L^2\big(0,T;L^\infty(0,L)\big)$ and $G \in \mathcal{F}$. We note that

$$\int_0^{(\cdot)} \sigma_j\, v_k\, dB_j \quad \text{is bounded in } L^2\big(\Omega; C([0,T];L^2(0,L))\big) \text{ uniformly in } k,$$

and that each function in $L^2\big(\Omega;L^2(0,T;L^2(0,L))\big)$ can be approximated by functions of the form $\sum_{i=1}^N b_i\chi\{G_i\}$, where $1 \le N < \infty$, $b_i \in L^2\big(0,T;L^\infty(0,L)\big)$ and $G_i \in \mathcal{F}$. Consequently, (1.4) follows from (1.6). □

Next we set

$$\Psi(t) = \sum_{j=1}^\infty \int_0^t f_j(s)\, dB_j(s)$$

and recall the following known fact.

Lemma 1.2. *For any $\psi \in C_0^\infty((0,T))$, it holds that*

$$(1.7) \qquad \int_0^T \psi_t(t)\Psi(t)\, dt = -\sum_{j=1}^\infty \int_0^T \psi(t) f_j(t)\, dB_j(t)$$

for almost all $\omega \in \Omega$.

Proof. Choose any $\psi \in C_0^\infty((0,T))$, and set

$$\Psi_m(t) = \sum_{j=1}^m \int_0^t f_j(s)dB_j(s).$$

Let us partition the interval $[0,T]$ as

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad t_k - t_{k-1} = \delta_N = T/N, \quad k = 1,\cdots,N,$$

and define

$$\psi_N(t) = \psi(t_j), \qquad \text{for } t_{j-1} < t \le t_j,\ 1 \le j \le N.$$

Since $\Psi_m \in C\big([0,T];L^2(0,L)\big)$, for almost all ω, and

$$\frac{\psi(t+\delta_N) - \psi(t)}{\delta_N} \to \psi_t(t) \quad \text{uniformly in } t \in [0,T] \text{ as } N \to \infty,$$

we have

$$\begin{aligned}\int_0^T \psi_t(t)\Psi_m(t)dt &= \lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{\psi(t_{k+1}) - \psi(t_k)}{\delta_N} \Psi_m(t_k)\delta_N \\ &= -\lim_{N\to\infty} \sum_{k=1}^{N-1} \psi(t_k)(\Psi_m(t_k) - \Psi_m(t_{k-1})) \\ &= -\lim_{N\to\infty} \sum_{j=1}^m \int_0^T \psi_N(t) f_j(t)dB_j(t),\end{aligned}$$

for almost all ω. In the meantime, as $N \to \infty$,

$$\sum_{j=1}^{m} \int_0^T \psi_N(t) f_j(t) dB_j(t) \to \sum_{j=1}^{m} \int_0^T \psi(t) f_j(t) dB_j(t)$$

in $L^2\big(\Omega; L^2(0,L)\big)$. Hence,

$$\int_0^T \psi_t(t)\Psi_m(t)dt = -\sum_{j=1}^{m} \int_0^T \psi(t) f_j(t) dB_j(t)$$

for almost all $\omega \in \Omega$.

By passing $m \to \infty$, we arrive at (1.7). □

Definition 1.3. A progressively measurable function $u \in L^2\big(\Omega; L^2(0,T; H^1(0,L))\big)$ is said to be a solution of (0.1) - (0.4) if for almost all $\omega \in \Omega$, it satisfies (0.1) in the sense of distributions over $(0,L) \times (0,T)$, (0.2) - (0.3) for almost all $t \in [0,T]$, and (0.4) for almost all $x \in [0,L]$.

Our main results are as follows.

Theorem 1.4. *Let $\alpha \geq 0$, and suppose that u_0 and u_1 are $\mathcal{F}_0$-measurable such that $u_1 \in L^2\big(\Omega; L^2(0,L)\big)$ and $u_0 \in L^2\big(\Omega; H^1(0,L)\big)$ with $u_0(0) \geq 0$ and $u_0(L) = 0$, for almost all $\omega \in \Omega$. For any $T > 0$, there is a pathwise unique solution u of* (0.1) - (0.4) such that u is progressively measurable, and

$$u \in L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big)\Big), \qquad u_t \in L^2\Big(\Omega; C\big([0,T] : L^2(0,L)\big)\Big).$$

Theorem 1.5. *Let $\alpha > 0$, and assume that $\sigma_j \equiv 0$, $j \geq 1$, and that the f_j's are independent of time. Then, there is an invariant measure of* (0.1) - (0.3) over *$H^1(0,L) \times L^2(0,L)$.*

§2. Deterministic equation

In this section, we will present basic facts on the deterministic equation which will be used for the proof of Theorem 1.4.

Lemma 2.1. *Suppose that*

$$v \in L^2\big(0,T; H^1(0,L)\big), \qquad v_t \in L^2\big(0,T; L^2(0,L)\big) \tag{2.1}$$

and that v satisfies

$$v_{tt} - v_{xx} + \alpha v_t = 0 \qquad \text{for } (x,t) \in (0,L) \times (0,T) \tag{2.2}$$

and

$$v(x,0) = 0, \qquad v_t(x,0) = 0 \qquad \text{for } x \in (0,L). \tag{2.3}$$

Then, it holds that for each $0 < \epsilon < T$,

$$v \in C([0,T-\epsilon]; H^1(0,L)), \qquad v_t \in C([0,T-\epsilon]; L^2(0,L)) \tag{2.4}$$

and

$$v_x(0,\cdot), \quad v_x(L,\cdot) \in L^2(0,T-\epsilon), \tag{2.5}$$

$$v_t(0,\cdot), \quad v_t(L,\cdot) \in L^2(0,T-\epsilon). \tag{2.6}$$

This is a special case of Lemma 1.6 of [9]. Here (2.2) and (2.3) are satisfied in the sense of distributions over $(0,L) \times (0,T)$ and $(0,L)$, respectively.

Lemma 2.2. *For given* $q \in L^2(0,T)$ *and* $f \in L^2\big(0,T;L^2(0,L)\big)$, *there is a unique solution* $v \in C\big([0,T];H^1(0,L)\big) \cap C^1\big([0,T];L^2(0,L)\big)$ *of*

$$v_{tt} - v_{xx} + \alpha v_t = f \qquad \text{for } (x,t) \in (0,L) \times (0,T), \tag{2.7}$$

$$v_x(0,t) = q(t), \quad v(L,t) = 0 \qquad \text{for } t \in (0,T), \tag{2.8}$$

$$v(x,0) = 0, \qquad v_t(x,0) = 0 \qquad \text{for } x \in (0,L). \tag{2.9}$$

Furthermore, it holds that

$$\begin{aligned} &\frac{1}{2}\|v_t(t)\|^2_{L^2(0,L)} + \frac{1}{2}\|v_x(t)\|^2_{L^2(0,L)} \\ &= -\int_0^t v_s(0,s)q(s)\,ds - \alpha\int_0^t \|v_s(s)\|^2_{L^2(0,L)}ds + \int_0^t\int_0^L f\,v_s\,dx\,ds \end{aligned} \tag{2.10}$$

for all $t \in [0,T]$,

$$\|v_t(t)\|^2_{L^2(0,L)} + \|v(t)\|^2_{H^1(0,L)} \le M\int_0^t |q(s)|^2 ds + M\int_0^t \|f(s)\|^2_{L^2(0,L)}ds \tag{2.11}$$

for all $t \in [0,T]$, *and*

$$\begin{aligned} &\int_0^t |v_s(0,s)|^2 ds + \int_0^t |q(s)|^2 ds \le M\big(\|v_t(t)\|^2_{L^2(0,L)} + \|v(t)\|^2_{H^1(0,L)}\big) \\ &\quad + M\int_0^t \big(\|v_s(s)\|^2_{L^2(0,L)} + \|v(s)\|^2_{H^1(0,L)}\big)ds + M\int_0^t \|f(s)\|^2_{L^2(0,L)}ds \end{aligned} \tag{2.12}$$

for all $t \in [0,T]$. *Here* M *denotes positive constants independent of* q *and* f.

For the proof, see Proposition 1.5 of [9], where a different boundary condition was used at $x = L$, but the details of the proof are essentially the same. We note that if $q = q_m \in C^3([0,T])$ and $f = f_m \in C^1\big([0,T];L^2(0,L)\big)$, then the corresponding solution v_m satisfies additional regularity such that

$$\begin{aligned} &v_m \in L^\infty\big(0,T;H^2(0,L)\big), \quad \partial_t v_m \in L^\infty\big(0,T;H^1(0,L)\big), \\ &\partial_{tt} v_m \in L^\infty\big(0,T;L^2(0,L)\big). \end{aligned} \tag{2.13}$$

If $q_m \to q$ in $L^2(0,T)$ and $f_m \to f$ in $L^2\big(0,T;L^2(0,L)\big)$, then (2.11) and (2.12) imply that

$$v_m \to v \qquad \text{in } C\big([0,T];H^1(0,L)\big) \cap C^1\big([0,T];L^2(0,L)\big) \tag{2.14}$$

and

$$\partial_t v_m(0,\,\cdot) \to \partial_t v(0,\,\cdot) \qquad \text{in } L^2(0,T). \tag{2.15}$$

Lemma 2.3. *For given* $h \in H^1(0,T)$, *there is a unique solution* $v \in C\big([0,T];H^1(0,L)\big) \cap C^1\big([0,T];L^2(0,L)\big)$ *of*

$$v_{tt} - v_{xx} + \alpha v_t = 0 \qquad \text{for } (x,t) \in (0,L) \times (0,T), \tag{2.16}$$

$$v_x(0,t) = -K\big(v(0,t) + h(t)\big)^-, \quad v(L,t) = 0 \qquad \text{for } t \in (0,T), \tag{2.17}$$

$$v(x,0) = 0, \qquad v_t(x,0) = 0 \qquad \text{for } x \in (0,L) \tag{2.18}$$

where K is a positive constant. Furthermore, if $h(0) \geq 0,$ it holds that

$$\|v_t(t)\|^2_{L^2(0,L)} + \|v(t)\|^2_{H^1(0,L)} + K\big|\big(v(0,t)+h(t)\big)^-\big|^2 + \int_0^t |v_x(0,s)|^2 ds + \int_0^t |v_s(0,s)|^2 ds \leq M \int_0^t |h_s(s)|^2 ds \tag{2.19}$$

for all $t \in [0,T],$ for some constant M independent of K and $h.$ If v_1 and v_2 are solutions corresponding to $h = h_1$ and $h = h_2$, respectively, it holds that

$$\begin{aligned}&\|\partial_t v_1(t) - \partial_t v_2(t)\|^2_{L^2(0,L)} + \|v_1(t) - v_2(t)\|^2_{H^1(0,L)} \\ &+ \int_0^t \big|\partial_s v_1(0,s) - \partial_s v_2(0,s)\big|^2 ds + \int_0^t \big|\partial_x v_1(0,s) - \partial_x v_2(0,s)\big|^2 ds \\ &\leq M_K \int_0^t |h_1(s) - h_2(s)|^2 ds\end{aligned} \tag{2.20}$$

for all $t \in [0,T],$ for some constant M_K independent of h_1 and $h_2.$

Proof. Existence of a solution follows from Lemma 2.2 and an iteration scheme.

For (2.19), we find from (2.10) that

$$\begin{aligned}&\|v_t(t)\|^2_{L^2(0,L)} + \|v_x(t)\|^2_{L^2(0,L)} = 2\int_0^t K\big(v(0,s)+h(s)\big)^-\big(v_s(0,s)+h_s(s)\big)ds \\ &- 2\int_0^t K\big(v(0,s)+h(s)\big)^- h_s(s)\, ds - 2\alpha \int_0^t \|v_s(s)\|^2_{L^2(0,L)} ds\end{aligned} \tag{2.21}$$

for all $t \in [0,T].$ In the meantime, it follows from (2.12) that

$$\begin{aligned}&\int_0^t \Big(K\big(v(0,s)+h(s)\big)^-\Big)^2 ds + \int_0^t |v_s(0,s)|^2 ds \\ &\quad\leq M\big(\|v_t(t)\|^2_{L^2(0,L)} + \|v(t)\|^2_{H^1(0,L)}\big) \\ &\quad+ M\int_0^t \big(\|v_s(s)\|^2_{L^2(0,L)} + \|v(s)\|^2_{H^1(0,L)}\big) ds.\end{aligned} \tag{2.22}$$

Since $v(0,0)+h(0) \geq 0,$ it follows from (1.1) that

$$2\int_0^t K\big(v(0,s)+h(s)\big)^-\big(v_s(0,s)+h_s(s)\big)ds = -K\big|\big(v(0,t)+h(t)\big)^-\big|^2. \tag{2.23}$$

We now use (2.23) and

$$\begin{aligned}&-2\int_0^t K\big(v(0,s)+h(s)\big)^- h_s(s)\, ds \leq \epsilon \int_0^t \Big(K\big(v(0,s)+h(s)\big)^-\Big)^2 ds \\ &\quad+ \frac{1}{\epsilon}\int_0^t |h_s(s)|^2 ds, \quad \text{for each } \epsilon > 0\end{aligned} \tag{2.24}$$

to derive (2.19) from (2.21) and (2.22) with help of the Gronwall inequality.

Next we use (2.11) and (2.12) with $q = -K\big(v_1(0,t)+h_1(t)\big)^- + K\big(v_2(0,t) + h_2(t)\big)^-,$ $v = v_1 - v_2$ and $f \equiv 0.$ By (2.11) and

$$\int_0^t \big|v_1(0,s) - v_2(0,s)\big|^2 ds \leq M\int_0^t \|v_1(s) - v_2(s)\|^2_{H^1(0,L)} ds, \tag{2.25}$$

we can again use the Gronwall inequality to derive

$$\|\partial_t v_1(t) - \partial_t v_2(t)\|^2_{L^2(0,L)} + \|v_1(t) - v_2(t)\|^2_{H^1(0,L)} \le M_K \int_0^t |h_1(s) - h_2(s)|^2 ds \tag{2.26}$$

for all $t \in [0,T]$, for some constant M_K independent of h_1 and h_2.

This, combined with (2.12), yields (2.20). □

§3. Stochastic equation

Let us consider the initial-boundary value problem

$$w_{tt} - w_{xx} + \alpha w_t = f + \frac{\partial}{\partial t}\sum_{j=1}^{\infty}\int_0^t g_j\, dB_j \quad \text{for } (x,t) \in (0,L)\times(0,T), \tag{3.1}$$

$$w_x(0,t) = 0, \quad w(L,t) = 0 \quad \text{for } 0 < t < T, \tag{3.2}$$

$$w(x,0) = u_0(x), \quad w_t(x,0) = u_1(x) \quad \text{for } 0 < x < L. \tag{3.3}$$

Here f and the g_j's are given functions such that they are progressively measurable, $f \in L^2\big(\Omega; L^2(0,T;L^2(0,L))\big)$, for each $T > 0$, and

$$\sum_{j=1}^{\infty} E\big(\|g_j\|^2_{L^2(0,T;L^2(0,L))}\big) < \infty \tag{3.4}$$

for each $T > 0$.

Lemma 3.1. *Suppose that u_0 and u_1 are $\mathcal{F}_0$-measurable such that $u_1 \in L^2\big(\Omega; L^2(0,L)\big)$ and $u_0 \in L^2\big(\Omega; H^1(0,L)\big)$ with $u_0(L) = 0$, for almost all $\omega \in \Omega$. Then, there is a unique solution w of* (3.1) - (3.3) *such that w is progressively measurable and*

$$w \in L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big) \tag{3.5}$$

for each $T > 0$. Furthermore, it holds that

$$\begin{aligned} &E\big(\sup_{s\in[0,t]} \|w(s)\|^2_{H^1(0,L)}\big) + E\big(\sup_{s\in[0,t]} \|w_s(s)\|^2_{L^2(0,L)}\big) + E\big(\int_0^t |w_s(0,s)|^2 ds\big) \\ &\le ME\big(\|u_0\|^2_{H^1(0,L)}\big) + ME\big(\|u_1\|^2_{L^2(0,L)}\big) \\ &\quad + ME\Big(\int_0^t \|f\|^2_{L^2(0,L)} ds\Big) + M\,E\Big(\sum_{j=1}^{\infty}\int_0^t \|g_j\|^2_{L^2(0,L)}\, ds\Big) \end{aligned} \tag{3.6}$$

for all $t \in [0,T]$, for some positive constant M depending only on $T > 0$.

In order to justify manipulations to obtain (3.6), we need more regularity than is indicated by (3.5). Therefore, we first consider the case of more regular data.

Let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal basis for $L^2(0,L)$ such that

$$\begin{cases} -\partial_{xx} e_k = \lambda_k e_k, & \text{for } x \in (0,L), \\ \partial_x e_k(0) = 0, \quad e_k(L) = 0. \end{cases} \tag{3.7}$$

We write

$$\Psi(t) = \sum_{j=1}^{\infty}\int_0^t g_j(s) dB_j(s),$$

and define for each $\nu \geq 1$,

$$u_{0,\nu} = \sum_{k=1}^{\nu} \langle u_0, e_k \rangle e_k, \qquad u_{1,\nu} = \sum_{k=1}^{\nu} \langle u_1, e_k \rangle e_k,$$
$$f_\nu = \sum_{k=1}^{\nu} \langle f,\, e_k \rangle e_k, \qquad g_{j,\nu} = \sum_{k=1}^{\nu} \langle g_j,\, e_k \rangle e_k$$

and

$$\Psi_\nu(t) = \sum_{j=1}^{\infty} \int_0^t g_{j,\nu}(s) dB_j(s).$$

Here, $\langle \cdot\,, \cdot \rangle$ denotes the inner product in $L^2(0, L)$.

Lemma 3.2. *Fix any $T > 0$ and $\nu \geq 1$. For $u_{0,\nu}$ and $u_{1,\nu}$ defined as above, there is a unique solution w_ν of*

$$\partial_{tt} w_\nu - \partial_{xx} w_\nu + \alpha \partial_t w_\nu = f_\nu + \partial_t \Psi_\nu, \quad \text{for } (x, t) \in (0, L) \times (0, T), \tag{3.8}$$

$$\partial_x w_\nu(0, t) = 0, \qquad w_\nu(L, t) = 0, \qquad \text{for } t \in (0, T), \tag{3.9}$$

$$w_\nu(x, 0) = u_{0,\nu}(x), \qquad \partial_t w_\nu(x, 0) = u_{1,\nu}(x), \qquad \text{for } x \in (0, L) \tag{3.10}$$

such that w_ν is progressively measurable and

$$w_\nu \in L^2\Big(\Omega; C^1\big([0, T]; H^m(0, L)\big)\Big), \qquad \text{for all } m \geq 1. \tag{3.11}$$

Furthermore, it holds that

$$\begin{aligned} &E\Big(\sup_{s\in[0,t]} \|w_\nu(s)\|^2_{H^1(0,L)}\Big) + E\Big(\sup_{s\in[0,t]} \|\partial_s w_\nu(s)\|^2_{L^2(0,L)}\Big) \\ &+ E\Big(\int_0^t |\partial_s w_\nu(0, s)|^2\, ds\Big) \leq ME\big(\|u_{0,\nu}\|^2_{H^1(0,L)}\big) + ME\big(\|u_{1,\nu}\|^2_{L^2(0,L)}\big) \\ &\qquad + ME\Big(\int_0^t \|f_\nu(s)\|^2_{L^2(0,L)} ds\Big) \\ &\qquad + M\,E\Big(\sum_{j=1}^{\infty} \int_0^t \|g_{j,\nu}(s)\|^2_{L^2(0,L)}\, ds\Big) \end{aligned} \tag{3.12}$$

for all $t \in [0, T]$, for some positive constant M depending only on $T > 0$.

Proof. It is easy to see that a solution w_ν can be represented by

$$w_\nu(x, t) = \sum_{k=1}^{\nu} c_k(t) e_k(x) \tag{3.13}$$

where the c_k's satisfy the system of stochastic differential equations

$$\partial_{tt} c_k = -\lambda_k\, c_k - \alpha \partial_t c_k \; + \langle f_\nu + \partial_t \Psi_\nu,\, e_k \rangle, \quad 1 \leq k \leq \nu, \tag{3.14}$$

and the initial conditions

$$c_k(0) = \langle u_0,\, e_k \rangle, \qquad \partial_t c_k(0) = \langle u_1,\, e_k \rangle, \quad 1 \leq k \leq \nu. \tag{3.15}$$

We can write (3.8) as

$$d\big(\partial_t w_\nu\big) = \big(\partial_{xx} w_\nu - \alpha \partial_t w_\nu\big) dt + f_\nu dt + d\Psi_\nu. \tag{3.16}$$

By Ito's rule and integration by parts using (3.9), we have

$$
\begin{aligned}
\|\partial_t w_\nu(t)\|^2_{L^2(0,L)} + \|\partial_x w_\nu(t)\|^2_{L^2(0,L)} &= \|u_{1,\nu}\|^2_{L^2(0,L)} + \|\partial_x u_{0,\nu}\|^2_{L^2(0,L)} \\
&\quad - 2\alpha \int_0^t \|\partial_s w_\nu\|^2_{L^2(0,L)} ds + 2\int_0^t \langle f_\nu,\, \partial_s w_\nu\rangle ds \\
&\quad + 2\sum_{j=1}^\infty \int_0^t \langle \partial_s w_\nu,\, g_{j,\nu}\rangle dB_j(s) + \sum_{j=1}^\infty \int_0^t \|g_{j,\nu}\|^2_{L^2(0,L)} ds
\end{aligned}
\tag{3.17}
$$

for all $t \in [0,T]$, for almost all $\omega \in \Omega$. By the Burkholder-Davis-Gundy inequality,

$$
\begin{aligned}
&E\Big(\sup_{s\in[0,t]}\Big|\sum_{j=1}^\infty \int_0^s \langle \partial_\eta w_\nu, g_{j,\nu}\rangle dB_j(\eta)\Big|\Big) \\
&\le CE\Big(\sum_{j=1}^\infty \int_0^t \big\|\partial_s w_\nu\big\|^2_{L^2(0,L)}\big\|g_{j,\nu}\big\|^2_{L^2(0,L)} ds\Big)^{1/2} \\
&\le \delta E\Big(\sup_{s\in[0,t]}\big\|\partial_s w_\nu(s)\big\|^2_{L^2(0,L)}\Big) + \frac{C^2}{\delta}E\Big(\sum_{j=1}^\infty \int_0^t \big\|g_{j,\nu}\big\|^2_{L^2(0,L)} ds\Big)
\end{aligned}
\tag{3.18}
$$

for all $\delta > 0$, for some positive constant C independent of ν.

Hence, it follows from (3.17) that

$$
\begin{aligned}
&E\Big(\sup_{s\in[0,t]}\|\partial_s w_\nu(s)\|^2_{L^2(0,L)} + \sup_{s\in[0,t]}\|w_\nu(s)\|^2_{H^1(0,L)}\Big) \\
&\le ME\big(\|u_{0,\nu}\|^2_{H^1(0,L)}\big) + ME\big(\|u_{1,\nu}\|^2_{L^2(0,L)}\big) \\
&\quad + ME\Big(\int_0^t \|f_\nu\|_{L^2(0,L)} ds\Big)^2 + M\sum_{j=1}^\infty E\Big(\int_0^t \|g_{j,\nu}\|^2_{L^2(0,L)} ds\Big)
\end{aligned}
\tag{3.19}
$$

for all $t \in [0,T]$, for some positive constant M independent of ν. Next let $\psi \in C^1([0,L])$ such that $\psi(0) = 1$ and $\psi(L) = \psi_x(L) = 0$, and write

$$
d\big(\psi\, \partial_x w_\nu\big) = \big(\psi\, \partial_{xt} w_\nu\big) dt.
\tag{3.20}
$$

By applying Ito's rule to the functional $\langle \partial_t w_\nu,\, \psi\partial_x w_\nu\rangle$, we find

$$
\begin{aligned}
\langle \partial_t w_\nu(t),\, \psi\partial_x w_\nu(t)\rangle - \langle u_{1,\nu},\, \psi\partial_x u_{0,\nu}\rangle &= -\frac{1}{2}\int_0^t |\partial_s w_\nu(0,s)|^2 ds \\
&\quad - \int_0^t \int_0^L \Big(\frac{1}{2}\psi_x(\partial_x w_\nu)^2 + \alpha\psi(\partial_x w_\nu)\,\partial_s w_\nu + \frac{1}{2}\psi_x(\partial_s w_\nu)^2\Big) dx\, ds \\
&\quad + \int_0^t \langle \psi\,\partial_x w_\nu,\, f_\nu\rangle ds + \sum_{j=1}^\infty \int_0^t \langle \psi\,\partial_x w_\nu,\, g_{j,\nu}\rangle dB_j(s)
\end{aligned}
\tag{3.21}
$$

for all $t \in [0,T]$, for almost all $\omega \in \Omega$. By taking the expectation of (3.21) and combining it with (3.19), we obtain (3.12). The argument for the pathwise uniqueness of solutions is the same as that for the deterministic equation. □

Proof of Lemma 3.1. Let $u_0 \in H^1(0,L)$ with $u_0(L) = 0$, and $u_1 \in L^2(0,L)$ be given. We define $u_{0,\nu}$ and $u_{1,\nu}$ as above. Then, as $\nu \to \infty$, $u_{0,\nu} \to u_0$ in $H^1(0,L)$, and $u_{1,\nu} \to u_1$ in $L^2(0,L)$. Let w_ν be the solution of (3.8) - (3.10). We also note that as $\nu \to \infty$,

$$E\Big(\sum_{j=1}^{\infty}\int_0^T \|g_{j,\nu}-g_j\|^2_{L^2(0,L)}dt\Big) \to 0. \tag{3.22}$$

We can estimate $w_{\nu_1} - w_{\nu_2}$ in the same way as for (3.12) and find that as $\nu \to \infty$,

$$w_\nu \to w \qquad \text{in } L^2\big(\Omega; C([0,T]; H^1(0,L))\big) \tag{3.23}$$

and

$$\partial_t w_\nu \to \partial_t w \qquad \text{in } L^2\big(\Omega; C([0,T]; L^2(0,L))\big) \tag{3.24}$$

where w is a solution of

$$\begin{cases} \partial_{tt}w - \partial_{xx}w + \alpha\partial_t w = f + \partial_t\Psi \quad \text{for } (x,t) \in (0,L)\times(0,T), \\ w(L,t) = 0 \qquad \text{for } t \in (0,T), \\ w(x,0) = u_0(x), \qquad \partial_t w(x,0) = u_1(x), \qquad \text{for } x \in (0,L). \end{cases} \tag{3.25}$$

Obviously w is progressively measurable. Next we show that the trace of w_t and w_x at $x = 0$ is well defined and belongs to $L^2\big(\Omega; L^2(0,T)\big)$. For almost all $\omega \in \Omega$, it holds that

$$\partial_{xx}w_\nu = \partial_{tt}w_\nu + \alpha\partial_t w_\nu - f_\nu - \partial_t\Psi_\nu \tag{3.26}$$

and

$$\partial_{xx}w = \partial_{tt}w + \alpha w_t - f - \partial_t\Psi \tag{3.27}$$

in the sense of distributions over $(0,L)\times(0,T)$. Since $\Psi_\nu \to \Psi$ in $L^2\big(\Omega; L^2((0,L)\times(0,T))\big)$ as $\nu \to \infty$, it follows from (3.23) - (3.27) that

$$\partial_{xx}w_\nu \to \partial_{xx}w \qquad \text{in } L^2\big(\Omega; L^2(0,L; H^{-1}(0,T))\big) \text{ as } \nu \to \infty. \tag{3.28}$$

Consequently, as $\nu \to \infty$,

$$\partial_x w_\nu \to \partial_x w \qquad \text{in } L^2\big(\Omega; C([0,L]; H^{-1}(0,T))\big)$$

and

$$\partial_t w_\nu \to \partial_t w \qquad \text{in } L^2\big(\Omega; C([0,L]; H^{-2}(0,T))\big).$$

Thus, it follows that

$$\partial_t w_\nu(0,\cdot) \to \partial_t w(0,\cdot) \qquad \text{in } L^2\big(\Omega; H^{-2}(0,T)\big) \tag{3.29}$$

and

$$\partial_x w_\nu(0,\cdot) \to \partial_x w(0,\cdot) \qquad \text{in } L^2\big(\Omega; H^{-1}(0,T)\big). \tag{3.30}$$

By means of (3.12), it must hold that

$$\partial_t w_\nu(0,\cdot) \to \partial_t w(0,\cdot) \qquad \text{weakly in } L^2\big(\Omega; L^2(0,T)\big). \tag{3.31}$$

For each $\nu \geq 1$, $\partial_t w_\nu(0,\cdot)$ is progressively measurable and hence, $\partial_t w(0,\cdot)$ is also progressively measurable. Since $\partial_x w_\nu(0,\cdot) \equiv 0$, for all ν, we also have

$$\partial_x w(0,\cdot) \equiv 0. \tag{3.32}$$

Hence, w is a solution of (3.1) - (3.3), and (3.6) follows from (3.12), (3.23), (3.24) and (3.31). The pathwise uniqueness of solution of (3.1) - (3.3) follows from the well-known uniqueness result on the deterministic equation. $\square$

Lemma 3.3. *Let f be progressively measurable such that $f \in L^1\big(\Omega; L^1(0,T)\big)$. Suppose that $v \in L^2\bigg(\Omega; C([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L))\bigg)$ is progressively measurable and is a solution of*

$$v_{tt} - v_{xx} + \alpha v_t = \sum_{j=1}^{\infty} \sigma_j v \frac{dB_j}{dt} \quad \textit{for } (x,t) \in (0,L) \times (0,T), \tag{3.33}$$

$$v_x(0,t)v(0,t) \geq -|f(t)| \quad \textit{for } t \in (0,T), \tag{3.34}$$

$$v(L,t) = 0 \qquad \textit{for } t \in (0,T), \tag{3.35}$$

$$v(x,0) = 0, \qquad v_t(x,0) = 0 \qquad \textit{for } x \in (0,L). \tag{3.36}$$

Then, there is some $0 < \delta < T$ and a positive constant M_δ independent of v, the σ_j's and f such that

$$E\big(\sup_{t\in[0,\delta]} |v(0,t)|^2\big) \leq M_\delta E\bigg(\int_0^\delta |f(s)|ds\bigg). \tag{3.37}$$

Proof. Let $\tilde{v} = ve^{\alpha t/2}$. Then, $\tilde{v}$ satisfies the equation

$$v_{tt} - v_{xx} = \frac{\alpha^2}{4} v + \sum_{j=1}^{\infty} \sigma_j v \frac{dB_j}{dt}, \quad \text{for } (x,t) \in (0,L) \times (0,T) \tag{3.38}$$

and (3.34) - (3.36) with $f(t)$ replaced by $f(t)e^{\alpha t}$. Hence, we may consider (3.38) instead of (3.33). Now suppose that v is a solution of (3.34) - (3.36) and (3.38).

With this v as a given function, let $\phi \in L^2\bigg(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\bigg)$ be the unique solution of

$$\begin{cases} \phi_{tt} - \phi_{xx} = \frac{\alpha^2}{4} v + \sum_{j=1}^{\infty} \sigma_j v \frac{dB_j}{dt} & \text{for } (x,t) \in (0,L) \times (0,T), \\ \phi_x(0,t) = 0, \quad \phi(L,t) = 0 & \text{for } t \in (0,T), \\ \phi(x,0) = 0, \quad \phi_t(x,0) = 0 & \text{for } x \in (0,L), \end{cases} \tag{3.39}$$

and let $\psi = v - \phi$. Then, $\psi \in C([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L))$, for almost all $\omega \in \Omega$, and ψ satisfies

$$\begin{cases} \psi_{tt} - \psi_{xx} = 0 & \text{for } (x,t) \in (0,L) \times (0,T), \\ \psi(L,t) = 0 & \text{for } t \in (0,T), \\ \psi(x,0) = 0, \qquad \psi_t(x,0) = 0 & \text{for } x \in (0,L). \end{cases} \tag{3.40}$$

There is some $\tilde{\Omega} \subset \Omega$ such that $P\big(\Omega \setminus \tilde{\Omega}\big) = 0$ and for each $\omega \in \tilde{\Omega}$, (3.34) - (3.36) and (3.38) - (3.40) hold. Let us fix any $\omega \in \tilde{\Omega}$. For all (x,t) such that $0 \leq t \leq \min(T, L/2)$, $0 \leq x \leq L/2$, $t \geq x$, it holds that

$$\psi(x,t) + \psi\big(\frac{t-x}{2}, \frac{t-x}{2}\big) = \psi(0, t-x) + \psi\big(\frac{t+x}{2}, \frac{t+x}{2}\big).$$

For this identity, see John [6]. In the meantime, by the domain of dependence,

$$\psi\big(\frac{t-x}{2}, \frac{t-x}{2}\big) = \psi\big(\frac{t+x}{2}, \frac{t+x}{2}\big) = 0.$$

Hence,

$$\psi(x,t)=\psi(0,t-x). \tag{3.41}$$

By the same argument, for all (x,t) such that $L/2\le x\le L,\ 0\le t\le \min(T,L/2)$, $x+t\ge L$,

$$\psi(x,t)=\psi(L,x+t-L). \tag{3.42}$$

Meanwhile, $\psi(L,t)=0,$ for all $0\le t\le T,$ and hence, it follows that, for all (x,t) such that $L/2\le x\le L,\ 0\le t\le \min(T,L/2),\ x+t\ge L,$

$$\psi(x,t)=0. \tag{3.43}$$

Again by the domain of dependence, (3.43) is also valid for all (x,t) such that $0\le x\le L,\ 0\le t\le \min(T,L/2),\ t\le x$ and $t\le L-x$. We see from (3.41) that

$$\psi_t(0,t)=-\psi_x(0,t) \tag{3.44}$$

for almost all $t\in(0,\min(T,L/2))$. Since $\psi_x(0,t)=v_x(0,t)$, it follows that

$$v(0,t)(v_t(0,t)-\phi_t(0,t))=v(0,t)\psi_t(0,t)\le |f(t)| \tag{3.45}$$

for almost all $t\in(0,\min(T,L/2)),$ and thus,

$$|v(0,t)|^2\le \int_0^t|v(0,s)|^2ds+\int_0^t|\phi_s(0,s)|^2ds+2\int_0^t|f(s)|ds \tag{3.46}$$

for all $t\in[0,\min(T,L/2)]$. This is true for each $\omega\in\tilde{\Omega}$. In the meantime, it follows from Lemma 3.1 that

$$\begin{aligned} &E\big(\sup_{s\in[0,t]}\|\phi(s)\|^2_{H^1(0,L)}\big)+E\big(\sup_{s\in[0,t]}\|\phi_s(s)\|^2_{L^2(0,L)}\big)+E\big(\int_0^t|\phi_s(0,s)|^2ds\big)\\ &\quad\le ME\Big(\int_0^t\int_0^L|v|^2\,dx\,ds\Big)\end{aligned} \tag{3.47}$$

and

$$\begin{aligned} E\Big(\int_0^t\int_0^L|v|^2\,dx\,ds\Big)&\le 2E\Big(\int_0^t\int_0^L(|\psi|^2+|\phi|^2)dx\,ds\Big)\\ &\le 2E\Big(\int_0^t\int_0^L|\psi|^2dx\,ds\Big)+2t^2E\Big(\int_0^t\int_0^L|\phi_s|^2dx\,ds\Big)\\ &\le 2E\Big(\int_0^t\int_0^L|\psi|^2dx\,ds\Big)+Mt^2E\Big(\int_0^t\int_0^L|v|^2dx\,ds\Big)\end{aligned} \tag{3.48}$$

for all $t\in[0,T],$ for some positive constant M independent of v. Thus, there is some $0<\delta<\min(T,L/2)$ independent of v and a positive constant M_δ depending on δ such that

$$E\Big(\int_0^t\int_0^L|v|^2dx\,ds\Big)\le M_\delta E\Big(\int_0^t\int_0^L|\psi|^2dx\,ds\Big), \tag{3.49}$$

for all $0\le t\le\delta$. Hence, it follows from (3.47) that

$$E\Big(\int_0^t|\phi_s(0,s)|^2ds\Big)\le M_\delta E\Big(\int_0^t\int_0^L|\psi|^2dx\,ds\Big) \tag{3.50}$$

for all $0 \le t \le \delta$. In the meantime, we derive from (3.41) and (3.43) that

$$
\begin{aligned}
(3.51)\qquad & E\Big(\int_0^t \int_0^L |\psi(x,s)|^2 dx\, ds\Big) = E\Big(\int_0^t \int_0^s |\psi(0,s-x)|^2 dx\, ds\Big) \\
&\le tE\Big(\int_0^t |\psi(0,s)|^2 ds\Big) \\
&\le 2tE\Big(\int_0^t |v(0,s)|^2 ds\Big) + 2tE\Big(\int_0^t |\phi(0,s)|^2 ds\Big) \\
&\le 2tE\Big(\int_0^t |v(0,s)|^2 ds\Big) + 2t^3 E\Big(\int_0^t |\phi_s(0,s)|^2 ds\Big).
\end{aligned}
$$

By taking δ smaller if necessary, we find that for all $0 \le t \le \delta$,

$$
(3.52)\qquad E\Big(\int_0^t |\phi_s(0,s)|^2 ds\Big) \le M_\delta E\Big(\int_0^t |v(0,s)|^2 ds\Big),
$$

which, together with (3.46), yields

$$
(3.53)\qquad E\big(\sup_{s\in[0,t]} |v(0,s)|^2\big) \le M_\delta \int_0^t E\big(|v(0,s)|^2\big) ds + 2E\Big(\int_0^t |f(s)| ds\Big)
$$

for all $0 \le t \le \delta$. By the Gronwall inequality, we have (3.37). $\square$

Remark 3.4. If $f \equiv 0$ in (3.34), then $v(0,t) = 0$ for all $t \in [0,\delta]$, for almost all ω, and thus, $v \equiv 0$ on $[0,\delta]$ for almost all ω. By repetition, we conclude that $v \equiv 0$ on $[0,T]$, for almost all ω.

§4. Proof of Theorem 1.4

Let us choose any $T > 0$. According to Lemma 3.1, let $w \in L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big)$ be a unique solution of

$$
(4.1)\qquad \begin{cases} w_{tt} - w_{xx} + \alpha w_t = \sum_{j=1}^\infty g_j \frac{dB_j}{dt} & \text{for } (x,t) \in (0,L)\times(0,T), \\ w_x(0,t) = 0, \quad w(L,t) = 0 & \text{for } t \in (0,T), \\ w(x,0) = u_0(x), \quad w_t(x,0) = u_1(x) & \text{for } x \in (0,L). \end{cases}
$$

Let v be the unique solution in Lemma 2.3 with

$$
h(t) = w(0,t).
$$

By virtue of (2.20), the mapping

$$
h \mapsto v
$$

is continuous from $L^2(0,T)$ into $C([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L))$. Also, we find that if $h \in L^2\big(\Omega; L^2(0,T)\big)$, then $v \in L^2\big(\Omega; C([0,T]; H^1(0,L)) \cap C^1([0,T]; L^2(0,L))\big)$. At the same time, if h is progressively measurable, so is v. Next we write

$$
u = w + v.
$$

Then, $u \in L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big)$ is a unique solution of

$$\begin{cases} u_{tt} - u_{xx} + \alpha u_t = \sum_{j=1}^{\infty} g_j \frac{dB_j}{dt} \quad \text{for } (x,t) \in (0,L) \times (0,T), \\ u_x(0,t) = -Ku(0,t)^-, \quad u(L,t) = 0 \quad \text{for } t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in (0,L). \end{cases} \tag{4.2}$$

Let u and $\tilde{u}$ be the solutions corresponding to $\{g_j\}_{j=1}^{\infty}$ and $\{\tilde{g}_j\}_{j=1}^{\infty}$, respectively. Then, we can write

$$u = w + v, \qquad \tilde{u} = \tilde{w} + \tilde{v}$$

where w and $\tilde{w}$ are solutions of (4.1) corresponding to $\{g_j\}_{j=1}^{\infty}$ and $\{\tilde{g}_j\}_{j=1}^{\infty}$, respectively, while v and $\tilde{v}$ are solutions in Lemma 2.3 corresponding to $h(t) = w(0,t)$ and $h(t) = \tilde{w}(0,t)$, respectively. By applying (2.20) to $v - \tilde{v}$ and (3.6) to $w - \tilde{w}$, we have

$$\begin{aligned} &E\big(\sup_{s\in[0,t]} \|u(s) - \tilde{u}\|^2_{H^1(0,L)}\big) + E\big(\sup_{s\in[0,t]} \|u_s(s) - \tilde{u}_s(s)\|^2_{L^2(0,L)}\big) \\ &+ E\Big(\int_0^t \big|u_x(0,s) - \tilde{u}_x(0,s)\big|^2\, ds\Big) + E\Big(\int_0^t \big|u_s(0,s) - \tilde{u}_s(0,s)\big|^2\, ds\Big) \\ &\le M_K\, E\Big(\sum_{j=1}^{\infty} \int_0^t \|g_j - \tilde{g}_j\|^2_{L^2(0,L)}\, ds\Big), \qquad \text{for all } t \in [0,T]. \end{aligned} \tag{4.3}$$

Let $u^{(0)} = u_0$, and for given $u^{(m)}$, $m \ge 0$, let $u^{(m+1)}$ be the solution of

$$u_{tt}^{(m+1)} - u_{xx}^{(m+1)} + \alpha u_t^{(m+1)} = \sum_{j=1}^{\infty} \big(f_j + \sigma_j u^{(m)}\big) \frac{dB_j}{dt} \quad \text{for } (x,t) \in (0,L) \times (0,T), \tag{4.4}$$

$$u_x^{(m+1)}(0,t) = -Ku^{(m+1)}(0,t)^-, \quad u^{(m+1)}(L,t) = 0 \quad \text{for } t \in (0,T), \tag{4.5}$$

$$u^{(m+1)}(x,0) = u_0(x), \quad u_t^{(m+1)}(x,0) = u_1(x) \quad \text{for } x \in (0,L). \tag{4.6}$$

As in (4.3), we have

$$\begin{aligned} &E\big(\sup_{s\in[0,t]} \|u^{(m+1)}(s) - u^{(m)}(s)\|^2_{H^1(0,L)}\big) + E\big(\sup_{s\in[0,t]} \|u_s^{(m+1)}(s) - u_s^{(m)}(s)\|^2_{L^2(0,L)}\big) \\ &+ E\Big(\int_0^t \big|u_x^{(m+1)}(0,s) - u_x^{(m)}(0,s)\big|^2\, ds\Big) + E\Big(\int_0^t \big|u_s^{(m+1)}(0,s) - u_s^{(m)}(0,s)\big|^2\, ds\Big) \\ &\le M_K\, E\Big(\int_0^t \|u^{(m)}(s) - u^{(m-1)}(s)\|^2_{L^2(0,L)}\, ds\Big) \end{aligned} \tag{4.7}$$

for all $t \in [0,T]$, and all $m \geq 1$. By induction, we obtain

(4.8)

$$E\big(\sup_{s\in[0,t]} \|u^{(m+1)}(s) - u^{(m)}(s)\|^2_{H^1(0,L)}\big) + E\big(\sup_{s\in[0,t]} \|u_s^{(m+1)}(s) - u_s^{(m)}(s)\|^2_{L^2(0,L)}\big)$$
$$+ E\bigg(\int_0^t \big|u_x^{(m+1)}(0,s) - u_x^{(m)}(0,s)\big|^2 ds\bigg) + E\bigg(\int_0^t \big|u_s^{(m+1)}(0,s) - u_s^{(m)}(0,s)\big|^2 ds\bigg)$$
$$\leq M_K^m t^m/m! \qquad \text{for all } t \in [0,T], \text{ and all } m \geq 1,$$

where M_K stands for a positive constant independent of m. Hence, $\{u^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence in

$$L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big)$$

and each $u^{(m)}$ is progressively measurable. By the same argument as in the proof of Lemma 3.1, the limit u is a solution of

$$u_{tt} - u_{xx} + \alpha u_t = \sum_{j=1}^{\infty} \big(f_j + \sigma_j u\big)\frac{dB_j}{dt} \quad \text{for } (x,t) \in (0,L) \times (0,T), \tag{4.9}$$

$$u_x(0,t) = -Ku(0,t)^-, \quad u(L,t) = 0 \quad \text{for } t \in (0,T), \tag{4.10}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in (0,L) \tag{4.11}$$

such that

$$u \in L^2\big(\Omega; C([0,T]; H^1(0,L))\big),\ u_t \in L^2\big(\Omega; C([0,T]; L^2(0,L))\big), \text{ and } \ u_t(0,\cdot) \in L^2\big(\Omega; L^2(0,T)\big).$$

Taking this u as a given function, let $w \in L^2\Big(C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big)$ be a unique solution of

$$\begin{cases} w_{tt} - w_{xx} + \alpha w_t = \sum_{j=1}^{\infty} (f_j + \sigma_j u)\frac{dB_j}{dt} & \text{for } (x,t) \in (0,L) \times (0,T), \\ w_x(0,t) = 0, \quad w(L,t) = 0 & \text{for } t \in (0,T), \\ w(x,0) = u_0(x), \quad w_t(x,0) = u_1(x) & \text{for } x \in (0,L). \end{cases} \tag{4.12}$$

By virtue of Lemma 2.3, we find that $v = u - w$ is a unique solution of (2.16) - (2.18) for $h(t) = w(0,t)$ such that

$$v \in L^2\Big(\Omega; C\big([0,T]; H^1(0,L)\big) \cap C^1\big([0,T]; L^2(0,L)\big)\Big).$$

We now suppose $u_0(0) \geq 0$. It follows from (2.19) and (3.6) with $f \equiv 0$, $g_j = f_j + \sigma_j u$ for $j \geq 1$, that

(4.13)
$$E\Big(\sup_{s\in[0,t]} \|u(s)\|^2_{H^1(0,L)}\Big) + E\Big(\sup_{s\in[0,t]} \|\partial_s u(s)\|^2_{L^2(0,L)}\Big) + E\Big(\int_0^t |\partial_x u(0,s)|^2\,ds\Big)$$
$$+ E\Big(\int_0^t |\partial_s u(0,s)|^2 ds\Big) \leq ME\big(\|u_0\|^2_{H^1(0,L)}\big) + ME\big(\|u_1\|^2_{L^2(0,L)}\big)$$
$$+ M\,E\Big(\int_0^t \|u(s)\|^2_{L^2(0,L)}\,ds\Big) + M\sum_{j=1}^\infty E\Big(\int_0^t \|f_j(s)\|^2_{L^2(0,L)} ds\Big)$$

for all $t \in [0,T]$, and thus, by the Gronwall inequality,

(4.14)
$$E\Big(\sup_{t\in[0,T]} \|u(t)\|^2_{H^1(0,L)}\Big) + E\Big(\sup_{t\in[0,T]} \|\partial_t u(t)\|^2_{L^2(0,L)}\Big)$$
$$+ E\Big(\int_0^T |\partial_x u(0,t)|^2\,dt\Big) + E\Big(\int_0^T |\partial_t u(0,t)|^2 dt\Big)$$
$$\leq ME\big(\|u_0\|^2_{H^1(0,L)}\big) + ME\big(\|u_1\|^2_{L^2(0,L)}\big) + M\sum_{j=1}^\infty E\Big(\int_0^T \|f_j(t)\|^2_{L^2(0,L)} dt\Big)$$

where M denotes positive constants independent of K, u_0 and u_1. Next we denote by u_K the solution of (4.9) - (4.11) for each $K > 0$. Since $L^2_*\big(\Omega; L^\infty(0,T; H^n(0,L))\big)$ is the dual of $L^2\big(\Omega; L^1(0,T; H^n(0,L))\big)$, $n = 0,1$, it follows from (4.14) that there is a sequence $\{u_{K_m}\}_{m=1}^\infty$ such that $K_m \uparrow \infty$ as $m \to \infty$, and

$$u_{K_m} \to u \qquad \text{weak star in } L^2_*\big(\Omega; L^\infty(0,T; H^1(0,L))\big), \tag{4.15}$$

$$\partial_t u_{K_m} \to \partial_t u \qquad \text{weak star in } L^2_*\big(\Omega; L^\infty(0,T; L^2(0,L))\big) \tag{4.16}$$

as $m \to \infty$, for some u. Each u_{K_m} satisfies

$$\int_A \int_0^T \Big(\langle \partial_t u_{K_m},\, \partial_t \psi\rangle - \langle \partial_x u_{K_m},\, \partial_x \psi\rangle - \langle \alpha \partial_t u_{K_m},\, \psi\rangle\Big) dt\,dP$$
$$= \int_A \int_0^T \langle \partial_t \psi,\, \sum_{j=1}^\infty \int_0^t \big(f_j + \sigma_j u_{K_m}\big) dB_j\rangle dt\,dP$$

for every $A \in \mathcal{F}$, and every $\psi \in H^1_0\big((0,L)\times(0,T)\big)$. Here $\langle\cdot\,,\,\cdot\rangle$ denotes the inner product in $L^2(0,L)$. Since (4.15) and (4.16) imply that

$$u_{K_m} \to u \qquad \text{weakly in } L^2\big(\Omega; L^2(0,T; H^1(0,L))\big) \text{ as } m \to \infty \tag{4.17}$$

and

$$\partial_t u_{K_m} \to \partial_t u \qquad \text{weakly in } L^2\big(\Omega; L^2(0,T; L^2(0,L))\big) \text{ as } m \to \infty, \tag{4.18}$$

it follows from Lemma 1.1 that

$$\int_A \int_0^T \Big(\langle \partial_t u,\, \partial_t \psi\rangle - \langle \partial_x u,\, \partial_x \psi\rangle - \langle \alpha \partial_t u,\, \psi\rangle\Big) dt\,dP$$
$$= \int_A \int_0^T \langle \partial_t \psi,\, \sum_{j=1}^\infty \int_0^t \big(f_j + \sigma_j u\big) dB_j\rangle dt\,dP$$

for every $A \in \mathcal{F}$, and every $\psi \in H_0^1\big((0,L)\times(0,T)\big)$. Hence, there is $\tilde{\Omega} \subset \Omega$ such that $P(\Omega \setminus \tilde{\Omega}) = 0$ and for each $\omega \in \tilde{\Omega}$,

$$\int_0^T \Big(\langle \partial_t u,\, \partial_t \psi \rangle - \langle \partial_x u,\, \partial_x \psi \rangle - \langle \alpha \partial_t u,\, \psi \rangle \Big) dt = \int_0^T \langle \partial_t \psi,\, \sum_{j=1}^{\infty} \int_0^t \big(f_j + \sigma_j u\big) dB_j \rangle dt \tag{4.19}$$

for all ψ in a countable dense subset of $H_0^1\big((0,L)\times(0,T)\big)$. Consequently, (4.19) holds for every $\psi \in H_0^1\big((0,L)\times(0,T)\big)$, for almost all ω, because $u \in L^2_*\big(\Omega; L^\infty(0,T;H^1(0,L))\big)$ and $u_t \in L^2_*\big(\Omega; L^\infty(0,T;L^2(0,L))\big)$. Therefore, u satisfies (0.1) for almost all ω.

Next we will show that u also satisfies the boundary conditions. (4.15) implies that

$$u_{K_m}(0,\cdot\,) \to u(0,\cdot\,) \qquad \text{weak star in } L^2_*\big(\Omega; L^\infty(0,T)\big) \tag{4.20}$$

and

$$u_{K_m}(L,\cdot\,) \to u(L,\cdot\,) \qquad \text{weak star in } L^2_*\big(\Omega; L^\infty(0,T)\big). \tag{4.21}$$

Thus, $u(L,\cdot) \equiv 0$, for almost all ω. By the same argument as for (3.28), we find that

$$\partial_{xx} u_{K_m} \to \partial_{xx} u \qquad \text{weakly in } L^2\big(\Omega; L^2(0,L;H^{-1}(0,T))\big)$$

and hence,

$$\partial_x u_{K_m}(0,\cdot\,) \to \partial_x u(0,\cdot\,) \qquad \text{weakly in } L^2\big(\Omega; H^{-1}(0,T)\big). \tag{4.22}$$

By virtue of (4.14) and (4.22),

$$\partial_x u_{K_m}(0,\cdot\,) \to \partial_x u(0,\cdot\,) \qquad \text{weakly in } L^2\big(\Omega; L^2(0,T)\big). \tag{4.23}$$

Since

$$\partial_x u_{K_m}(0,\cdot\,) = -K_m u_{K_m}(0,\cdot\,)^-, \tag{4.24}$$

we derive from (4.14) that

$$u_{K_m}(0,\cdot\,)^- \to 0 \qquad \text{strongly in } L^2\big(\Omega; L^2(0,T)\big). \tag{4.25}$$

Choose any nonnegative $\phi \in L^2(\Omega; L^2(0,T))$. Then, it follows from (4.20) that

$$\int_\Omega \int_0^T \Big(u_{K_m}(0,t)^+ - u_{K_m}(0,t)^- \Big) \phi(t)\, dt\, dP \to \int_\Omega \int_0^T u(0,t)\phi(t)\, dt\, dP, \tag{4.26}$$

which, combined with (4.25), yields

$$\int_\Omega \int_0^T u(0,t)\phi(t)\, dt\, dP \geq 0.$$

Also, by (4.23) and (4.24), we have

$$\int_\Omega \int_0^T \partial_x u(0,t)\phi(t)\, dt\, dP \leq 0.$$

Hence, we find that

$$u(0,t)^- = 0 \tag{4.27}$$

and

$$\partial_x u(0,t) \le 0, \tag{4.28}$$

for almost all $t \in [0,T]$, for almost all ω. Next we show that $\{u_{K_m}(0,\cdot)\}_{m=1}^{\infty}$ is strongly convergent in $L^2\big(\Omega; L^2(0,\delta)\big)$ for some $0<\delta<T$. Choose any $K_m < K_n$ and set

$$v_1 = u_{K_m}, \qquad v_2 = u_{K_n}.$$

Then, we can write

$$\partial_x v_1(0,t) - \partial_x v_2(0,t) = -K_m\big(v_1(0,t)^- - v_2(0,t)^-\big) + (K_n - K_m)v_2(0,t)^-$$

and hence, by dropping all nonnegative terms,

$$\begin{aligned}
&\big(\partial_x v_1(0,t) - \partial_x v_2(0,t)\big)\big(v_1(0,t) - v_2(0,t)\big) \\
&= -K_m\big(v_1(0,t)^- - v_2(0,t)^-\big)\big(v_1(0,t)^+ - v_2(0,t)^+\big) \\
&+ K_m\big(v_1(0,t)^- - v_2(0,t)^-\big)^2 + (K_n - K_m)v_2(0,t)^-\big(v_1(0,t)^+ - v_2(0,t)^+\big) \\
&- (K_n - K_m)v_2(0,t)^-\big(v_1(0,t)^- - v_2(0,t)^-\big) \\
&\ge -(K_n - K_m)v_2(0,t)^-\, v_1(0,t)^-.
\end{aligned} \tag{4.29}$$

It follows from (4.14) and (4.24) that

$$E\bigg(\int_0^T |v_1(0,t)^-|^2 dt\bigg) \le \frac{C}{K_m^2} \tag{4.30}$$

and

$$E\bigg(\int_0^T |v_2(0,t)^-|^2 dt\bigg) \le \frac{C}{K_n^2} \tag{4.31}$$

where C stands for the right-hand side of (4.14). Hence, we have

$$\begin{aligned}
&E\bigg(\int_0^T (K_n - K_m)\, v_2(0,t)^-\, v_1(0,t)^-\, dt\bigg) \\
&\qquad \le \bigg(\frac{K_n - K_m}{K_m K_n}\bigg) C.
\end{aligned} \tag{4.32}$$

Applying Lemma 3.3 to $v = v_1 - v_2$ with help from (4.29) and (4.32), there is some $0<\delta<T$ such that $\{u_{K_m}(0,\cdot\,)\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2\big(\Omega; L^2(0,\delta)\big)$. Since

$$\partial_x u_{K_m}(0,\cdot\,) \to \partial_x u(0,\cdot\,) \qquad \text{weakly in } L^2\big(\Omega; L^2(0,T)\big) \ \text{ as } m \to \infty,$$

we find that

$$u_{K_m}(0,\cdot\,)\partial_x u_{K_m}(0,\cdot\,) \to u(0,\cdot\,)\partial_x u(0,\cdot\,) \qquad \text{weakly in } L^1\big(\Omega; L^1(0,\delta)\big).$$

But by (4.14) and (4.24) again, we see that as $m \to \infty$,

$$u_{K_m}(0,\cdot\,)\partial_x u_{K_m}(0,\cdot\,) = K_m\big|u_{K_m}(0,\cdot\,)^-\big|^2 \to 0 \qquad \text{strongly in } L^1\big(\Omega; L^1(0,T)\big).$$

Hence, it holds that

$$u(0,t)\partial_x u(0,t) = 0 \tag{4.33}$$

for almost all $t \in [0,\delta]$, for almost all ω.

Therefore, u is a solution of (0.1) - (0.4) for $0<t<\delta$ such that

$$u \in L^2_*\big(\Omega; L^\infty(0,\delta; H^1(0,L))\big), \qquad u_t \in L^2_*\big(\Omega; L^\infty(0,\delta; L^2(0,L))\big).$$

Here we note that δ is independent of the initial conditions. Again by applying Lemma 2.1 to $u-w$ where w is the solution of (4.12), we find that for almost all ω,

$$u \in C\big([0,\delta]; H^1(0,L)\big), \qquad u_t \in C\big([0,\delta]; L^2(0,L)\big)$$

by taking slightly smaller δ. Since $C\big([0,\delta]; H^1(0,L)\big)$ and $C\big([0,\delta]; L^2(0,L)\big)$ are separable closed subspaces of $L^\infty\big(0,\delta; H^1(0,L)\big)$ and $L^\infty\big(0,\delta; L^2(0,L)\big)$, respectively, we have

$$u \in L^2\big(\Omega; C([0,\delta]; H^1(0,L))\big), \qquad u_t \in L^2\big(\Omega; C([0,\delta]; L^2(0,L))\big). \tag{4.34}$$

See [14, pp. 72 - 73]. For the pathwise uniqueness of solution, we argue as follows. Let v_1 and v_2 be two solutions of (0.1) - (0.4) such that

$$v_i \in L^2\big(\Omega; C([0,T]; H^1(0,L))\big), \qquad \partial_t v_i \in L^2\big(\Omega; C([0,T]; L^2(0,L))\big), \quad i = 1,2,$$

for some $T > 0$. There is a subset $\tilde{\Omega} \subset \Omega$ such that $P\big(\Omega \setminus \tilde{\Omega}\big) = 0$, and for each $\omega \in \tilde{\Omega}$, (0.3) is satisfied by v_1 and v_2, for almost all $t \in [0,T]$. Fix any $\omega \in \tilde{\Omega}$ and t where (0.3) is satisfied by v_1 and v_2. Set $v = v_2 - v_1$. If $v(0,t) = 0$, then $v(0,t)v_x(0,t) = 0$ holds. If $v(0,t) > 0$, then $v_2(0,t) > v_1(0,t) \geq 0$. Hence, $\partial_x v_2(0,t) = 0$, and $v_x(0,t) = \partial_x v_2(0,t) - \partial_x v_1(0,t) \geq 0$. Thus, $v(0,t)v_x(0,t) \geq 0$. By the same argument, if $v(0,t) < 0$, then $v(0,t)v_x(0,t) \geq 0$ again holds. According to Remark 3.4, we conclude that $v \equiv 0$ on $[0,T]$. Finally, since δ in (4.34) is independent of the initial conditions, we can extend the time interval of existence by using the pathwise uniqueness of solution.

§5. Proof of Theorem 1.5

Throughout this section, we assume that $\sigma_j \equiv 0$, for all $j \geq 1$, and that all f_j's are deterministic and independent of time. The proof is based on the following result of [10].

Suppose that $X(t,s;z)$, $0 \leq s \leq t < \infty$ is a pathwise unique solution of a certain stochastic evolution equation such that $X(s,s;z) = z$. We assume the following conditions.

[I] $X(\cdot\,,s;z)$ is a Ξ-valued continuous process adapted to $\{\mathcal{F}_t\}_{t \geq s}$ for each $z \in \Xi$ and $s \geq 0$, where Ξ is a separable Banach space.

We define a function

$$\mathcal{P}(s,z;t,\Gamma) = P\big(X(t,s;z) \in \Gamma\big), \qquad \text{for each } \Gamma \in \mathcal{B}(\Xi),\ 0 \leq s \leq t < \infty,\ z \in \Xi$$

where $\mathcal{B}(\Xi)$ is the Borel σ-algebra of Ξ.

[II] $\mathcal{P}(\cdot,\cdot\,;\,\cdot,\cdot)$ is a time-homogeneous transition probability function. In other words, it satisfies the following conditions.

(i) $\mathcal{P}(s,z;t,\cdot)$ is a probability measure over $\big(\Xi, \mathcal{B}(\Xi)\big)$ for all $z \in \Xi$, and $0 \leq s < t < \infty$.

(ii) $\mathcal{P}(s,\cdot\,;t,\Gamma)$ is $\mathcal{B}(\Xi)$-measurable for all $0 \leq s < t < \infty$ and $\Gamma \in \mathcal{B}(\Xi)$.

(iii) For all $0 \leq s < t < \xi < \infty$ and $\Gamma \in \mathcal{B}(\Xi)$,

$$\mathcal{P}(s,z;\xi,\Gamma) = \int_\Xi \mathcal{P}(s,z;t,dy)\mathcal{P}(t,y;\xi,\Gamma).$$

(iv) $\mathcal{P}(s,\cdot\,;t,\cdot) = \mathcal{P}(s+h,\cdot\,;t+h,\cdot)$ for all $0 \leq s < t < \infty$ and $h > 0$.

[III] If $\|z\|_\Xi \leq M$, then

$$E\big(\|X(t,0;z)\|_\Xi\big) \leq C_M, \quad \text{for all } t \geq 0,$$

for some positive constant C_M.

[IV] There is a separable Banach space Υ such that $\Xi \subset \Upsilon$, the imbedding $\Xi \to \Upsilon$ is continuous, and each closed ball of finite radius in Ξ is a compact subset of Υ. Furthermore, for each bounded continuous function ψ on Ξ, there is a sequence of continuous functions $\{\psi_k\}_{k=1}^\infty$ on Υ such that ψ_k is bounded uniformly in k and

$$\lim_{k\to\infty} \psi_k(y) = \psi(y), \quad \text{for each } y \in \Xi.$$

[V] For each fixed $0 \le t < \infty$, and each fixed closed ball S of finite radius in Ξ, if $\{z_n\}_{n=1}^\infty$ is a sequence in S such that

$$z_n \to z \qquad \text{in } \Upsilon,$$

then

$$E\big(\phi(X(t,0;z_n))\big) \to E\big(\phi(X(t,0;z))\big),$$

for every bounded continuous function ϕ on Υ.

Theorem ([10])**.** *Under the assumptions* [I] - [V], *there is an invariant measure for the above process $X(\cdot)$. In other words, there is a probability measure μ on Ξ such that*

$$\int_\Xi E\big(\psi(X(t,0;z))\big)\mu(dz) = \int_\Xi \psi(z)\,\mu(dz)$$

for all $t \ge 0$, and every bounded continuous function ψ on Ξ.

For the proof of Theorem 1.5, it is enough to establish the above assumptions [I] - [V].

We will first prove pathwise convergence of approximate solutions, where the assumption $\sigma_j \equiv 0$, $j \ge 1$, is essentially used.

Lemma 5.1. *Let u_K be the solution of* (4.9) - (4.11) *with $\sigma_j \equiv 0$, $j \ge 1$. Then, for each sequence $\{u_{K_m}\}_{m=1}^\infty$, it holds that for almost all ω,*

$$u_{K_m} \to u \qquad \textit{weak star in } L^\infty(0,T;H^1(0,L)), \tag{5.1}$$

$$\partial_t u_{K_m} \to \partial_t u \qquad \textit{weak star in } L^\infty(0,T;L^2(0,L)) \tag{5.2}$$

as $K_m \uparrow \infty$, for each $T > 0$, where u is the solution in Theorem 1.4.

Proof. Choose any $T > 0$, and let w be the solution of (4.12) with $\sigma_j \equiv 0$, $j \ge 1$. Then, there is some $\tilde\Omega \subset \Omega$ such that $P(\Omega \setminus \tilde\Omega) = 0$, and for each $\omega \in \tilde\Omega$,

$$w \in C\big([0,T];H^1(0,L)\big) \cap C^1\big([0,T];L^2(0,L)\big), \quad w(0,\cdot) \in H^1(0,T),$$

$$\Phi = \sum_{j=1}^\infty f_j\, B_j(\cdot) \in C([0,T];L^2(0,L))$$

and w satisfies (4.12).

Let v_K be the solution of (2.16) - (2.18) with $h(t) = w(0,t)$. Then, according to the construction of the solution u_K of (4.9) - (4.11) with $\sigma_j \equiv 0$, $j \ge 1$,

$$u_K = w + v_K, \qquad \text{for each } \omega \in \tilde\Omega.$$

It follows from (2.19) that

$$\begin{aligned} &\|\partial_t v_K(t)\|^2_{L^2(0,L)} + \|v_K(t)\|^2_{H^1(0,L)} + \int_0^t |\partial_x v_K(0,s)|^2 ds \\ &\le M \int_0^t |w_s(0,s)|^2 ds \end{aligned} \tag{5.3}$$

for all $t \in [0,T]$, for each $\omega \in \tilde{\Omega}$. Here M is a constant independent of K and ω. Now fix any $\omega \in \tilde{\Omega}$. By (5.3), there is a sequence $\{v_{K_m}\}_{m=1}^{\infty}$ such that

$$v_{K_m} \to v_\omega \qquad \text{weak star in } L^\infty\big(0,T;H^1(0,L)\big), \tag{5.4}$$

$$\partial_t v_{K_m} \to \partial_t v_\omega \qquad \text{weak star in } L^\infty\big(0,T;L^2(0,L)\big) \tag{5.5}$$

for some function v_ω. This v_ω must satisfy (2.16), (2.18) and

$$\begin{cases} v_\omega(0,t) + w(0,t) \geq 0, \\ \partial_x v_\omega(0,t) \leq 0, \\ \partial_x v_\omega(0,t)\big(v_\omega(0,t)+w(0,t)\big) = 0, \\ v_\omega(L,t) = 0 \end{cases} \tag{5.6}$$

for almost all $t \in [0,T]$. Here we note that (5.4) and (5.5) imply that for any $\eta > 0$,

$$v_{K_m} \to v_\omega \qquad \text{strongly in } C\big([0,T];H^{1-\eta}(0,L)\big) \tag{5.7}$$

and hence,

$$v_{K_m}(0,\cdot\,) \to v_\omega(0,\cdot\,) \qquad \text{strongly in } C\big([0,T]\big).$$

For (5.7), see Lemma 1.7 of [9]. Meanwhile, by (5.3),

$$\partial_x v_{K_m}(0,\cdot\,)\big(v_{K_m}(0,\cdot\,)+w(0,\cdot\,)\big) = K_m\big|\big(v_{K_m}(0,\cdot\,)+w(0,\cdot\,)\big)^-\big|^2 \to 0 \quad \text{in } L^1(0,T),$$

which yields the third property in (5.6). Next it follows from Lemma 2.1 that

$$v_\omega \in C\big([0,T);H^1(0,L)\big) \cap C^1\big([0,T);L^2(0,L)\big). \tag{5.8}$$

Suppose $\tilde{v}_\omega$ is a limit of another sequence. Then, by the same argument as in Section 4, $v_\omega - \tilde{v}_\omega$ satisfies the boundary condition

$$\big(v_\omega(0,t) - \tilde{v}_\omega(0,t)\big)\big(\partial_x v_\omega(0,t) - \partial_x \tilde{v}_\omega(0,t)\big) \geq 0$$

for almost all $t \in [0,T]$, for each $\omega \in \tilde{\Omega}$. Again by Remark 3.4, we conclude that $v_\omega \equiv \tilde{v}_\omega$. Hence, every sequence converges to the same limit. We now fix any sequence $\{v_{K_m}\}_{m=1}^{\infty}$. For each $\omega \in \tilde{\Omega}$, we define v_ω as the limit of this sequence. Since $C\big([0,T-\epsilon];H^1(0,L)\big)$ is a separable closed subspace of $L^\infty\big(0,T-\epsilon;H^1(0,L)\big)$, for each $0 < \epsilon < T$, it follows from (5.4) and (5.8) that $v = v_\omega$ is $C\big([0,T-\epsilon];H^1(0,L)\big)$-valued strongly measurable over $\big(\Omega,\mathcal{F}_{T-\epsilon}\big)$, which implies that v is also progressively measurable. Similarly, $\partial_t v$ is $C\big([0,T-\epsilon];L^2(0,L)\big)$-valued strongly measurable over $\big(\Omega,\mathcal{F}_{T-\epsilon}\big)$. By virtue of (5.4), (5.5) and Fatou's lemma, it holds that for each $0 < \epsilon < T$,

$$\begin{aligned} &E\bigg(\|v+w\|^2_{C([0,T-\epsilon];H^1(0,L))} + \|v_t + w_t\|^2_{C([0,T-\epsilon];L^2(0,L))}\bigg) \\ &\leq E\bigg(\varliminf_{m\to\infty}\bigg(\|v_{K_m}+w\|^2_{L^\infty(0,T;H^1(0,L))} + \|\partial_t v_{K_m} + w_t\|^2_{L^\infty(0,T;H^1(0,L))}\bigg)\bigg) \\ &\leq \varliminf_{m\to\infty} E\bigg(\|v_{K_m}+w\|^2_{L^\infty(0,T;H^1(0,L))} + \|\partial_t v_{K_m} + w_t\|^2_{L^\infty(0,T;H^1(0,L))}\bigg), \end{aligned}$$

which, together with (5.3), yields

$$v + w \in L^2\big(\Omega; C([0,T-\epsilon];H^1(0,L))\big), \quad v_t + w_t \in L^2\big(\Omega; C([0,T-\epsilon];L^2(0,L))\big).$$

Obviously, $v+w$ is a solution of (0.1) - (0.4) on the interval $[0,T-\epsilon]$, for each $0<\epsilon<T$. By the pathwise uniqueness of the solution, this coincides with the solution in Theorem 1.4, and the proof of Lemma 5.1 is complete. □

We next establish the following estimate.

Lemma 5.2. *Let* u *be the solution in* Theorem 1.4 *under the conditions of* Theorem 1.5. *It holds that*

$$E\bigg(\|u(t)\|^2_{H^1(0,L)}+\|u_t(t)\|^2_{L^2(0,L)}\bigg)\le M$$

for all $t\ge 0$, *for some constant* $M>0$.

Proof. Fix any $T>0$ and $K=k$. Let w be the solution of (4.12) with $\sigma_j\equiv 0$, $j\ge 1$, and let v_k be the solution in Lemma 2.3 with $h=w(0,\cdot)$. We write $u_k=v_k+w$. As above, let us define

$$u_{0,\nu}=\sum_{k=1}^{\nu}\langle u_0,e_k\rangle e_k,\qquad u_{1,\nu}=\sum_{k=1}^{\nu}\langle u_1,e_k\rangle e_k,$$
$$f_{j,\nu}=\sum_{k=1}^{\nu}\langle f_j,e_k\rangle e_k,\qquad \Phi_\nu=\sum_{j=1}^{\infty}f_{j,\nu}\,B_j(\cdot).$$

Let w_ν be the solution in Lemma 3.2 with $f_\nu+\partial_t\Psi_\nu$ replaced by $\partial_t\Phi_\nu$. Then, as $\nu\to\infty$,

$$w_\nu\to w\qquad \text{in } L^2\bigg(\Omega;C\big([0,T];H^1(0,L)\big)\cap C^1\big([0,T];L^2(0,L)\big)\bigg),\tag{5.9}$$

$$w_\nu(0,\,\cdot\,)\to w(0,\,\cdot\,)\qquad \text{in } L^2(\Omega;H^1(0,T)).\tag{5.10}$$

After extending $-k\big(v_k(0,t)+w(0,t)\big)^-$ to be zero for $t\notin[0,T]$, we define

$$q_{k,\nu}(t)=-\int_{-\infty}^{\infty}k\bigg(v_k(0,t-s)+w(0,t-s)\bigg)^-\rho_\nu(s)\,ds$$

where $\rho_\nu(t)=\nu\rho(\nu t)$, $\rho\in C_0^\infty\big((0,1)\big)$ such that $\|\rho\|_{L^1(0,1)}=1$.

Then, $q_{k,\nu}\in C_0^\infty(R)$ is adapted to $\{\mathcal{F}_t\}$ and, as $\nu\to\infty$,

$$q_{k,\nu}\to -k\big(v_k(0,\,\cdot\,)+w(0,\,\cdot\,)\big)^-\qquad \text{in } L^2\big(\Omega;L^2(0,T)\big).\tag{5.11}$$

Let $v_{k,\nu}$ be the solution in Lemma 2.2 with $q=q_{k,\nu}$ and $f\equiv 0$, satisfying the additional regularity (2.13). Then, as $\nu\to\infty$,

$$v_{k,\nu}\to v_k\qquad \text{in } L^2\bigg(\Omega;C\big([0,T];H^1(0,L)\big)\cap C^1\big([0,T];L^2(0,L)\big)\bigg)\tag{5.12}$$

and

$$\partial_t v_{k,\nu}(0,\cdot\,)\to\partial_t v_k(0,\cdot\,)\qquad \text{in } L^2\big(\Omega;L^2(0,L)\big).\tag{5.13}$$

We may write

$$d(\partial_t w_\nu+\partial_t v_{k,\nu})=(\partial_{xx}w_\nu+\partial_{xx}v_{k,\nu})dt-\alpha(\partial_t w_\nu+\partial_t v_{k,\nu})dt+d\Phi_\nu$$

and

$$d\big(w_\nu+v_{k,\nu}\big)=\big(\partial_t w_\nu+\partial_t v_{k,\nu}\big)dt.$$

By applying Ito's rule to the functionals

$$\langle\partial_t w_\nu+\partial_t v_{k,\nu},\,\partial_t w_\nu+\partial_t v_{k,\nu}\rangle$$

and

$$\langle \partial_t w_\nu + \partial_t v_{k,\nu},\, w_\nu + v_{k,\nu} \rangle,$$

we have for each $\nu \geq 1$,

$$
\begin{aligned}
(5.14)\qquad & \|\partial_t w_\nu(t) + \partial_t v_{k,\nu}(t)\|^2_{L^2(0,L)} + \|\partial_x w_\nu(t) + \partial_x v_{k,\nu}(t)\|^2_{L^2(0,L)} \\
& = \|\partial_x u_{0,\nu}\|^2_{L^2(0,L)} + \|u_{1,\nu}\|^2_{L^2(0,L)} \\
& - 2\int_0^t q_{k,\nu}(s)\big(\partial_s v_{k,\nu}(0,s) + \partial_s w_\nu(0,s)\big) ds \\
& - 2\alpha \int_0^t \|\partial_s w_\nu(s) + \partial_s v_{k,\nu}(s)\|^2_{L^2(0,L)} ds \\
& + 2\sum_{j=1}^\infty \int_0^t \langle \partial_s w_\nu(s) + \partial_s v_{k,\nu}(s),\; f_{j,\nu}\rangle dB_j(s) + \sum_{j=1}^\infty t\, \|f_{j,\nu}\|^2_{L^2(0,L)}
\end{aligned}
$$

and

$$
\begin{aligned}
(5.15)\qquad & \epsilon\langle \partial_t w_\nu(t) + \partial_t v_{k,\nu}(t),\, w_\nu(t) + v_{k,\nu}(t)\rangle - \epsilon\langle u_{0,\nu},\, u_{1,\nu}\rangle \\
& + \frac{\epsilon\alpha}{2}\|w_\nu(t) + v_{k,\nu}(t)\|^2_{L^2(0,L)} - \frac{\epsilon\alpha}{2}\|u_{0,\nu}\|^2_{L^2(0,L)} \\
& = -\epsilon \int_0^t q_{k,\nu}(s)\big(v_{k,\nu}(0,s) + w_\nu(0,s)\big) ds \\
& - \epsilon \int_0^t \|\partial_x w_\nu(s) + \partial_x v_{k,\nu}(s)\|^2_{L^2(0,L)} ds \\
& + \epsilon \int_0^t \|\partial_s w_\nu(s) + \partial_s v_{k,\nu}(s)\|^2_{L^2(0,L)} ds \\
& + \epsilon \sum_{j=1}^\infty \int_0^t \langle w_\nu(s) + v_{k,\nu}(s), f_{j,\nu}\rangle dB_j(s)
\end{aligned}
$$

for all $t \in [0,T]$, for almost all ω. Here ϵ is a positive constant, which will be determined later. By means of (5.9) - (5.13), we can extract a subsequence $\big\{w_{\nu_m} + v_{k,\nu_m}\big\}_{m=1}^\infty$ for pathwise convergence and pass $\nu_m \to \infty$ in (5.14) and (5.15) to arrive at

$$
\begin{aligned}
(5.16)\qquad & \|\partial_t u_k(t)\|^2_{L^2(0,L)} + \|\partial_x u_k(t)\|^2_{L^2(0,L)} = \|\partial_x u_0\|^2_{L^2(0,L)} + \|u_1\|^2_{L^2(0,L)} \\
& + 2\int_0^t k u_k(0,s)^- \partial_s u_k(0,s) ds - 2\alpha \int_0^t \|\partial_s u_k(s)\|^2_{L^2(0,L)} ds \\
& + 2\sum_{j=1}^\infty \int_0^t \langle \partial_s u_k,\, f_j\rangle dB_j(s) + \sum_{j=1}^\infty t\, \|f_j\|^2_{L^2(0,L)}
\end{aligned}
$$

and

$$
\begin{aligned}
(5.17)\qquad & \epsilon\langle \partial_t u_k(t), u_k(t)\rangle - \epsilon\langle u_0, u_1\rangle + \frac{\epsilon\alpha}{2}\|u_k(t)\|^2_{L^2(0,L)} - \frac{\epsilon\alpha}{2}\|u_0\|^2_{L^2(0,L)} \\
& = \epsilon \int_0^t k u_k(0,s)^- u_k(0,s) ds - \epsilon \int_0^t \|\partial_x u_k(s)\|^2_{L^2(0,L)} ds \\
& + \epsilon \int_0^t \|\partial_s u_k(s)\|^2_{L^2(0,L)} ds + \epsilon \sum_{j=1}^\infty \int_0^t \langle u_k(s),\, f_j\rangle dB_j(s)
\end{aligned}
$$

for all $t \in [0,T]$, for almost all $\omega \in \Omega$. Since $T > 0$ could be chosen arbitrarily, (5.16) and (5.17) are valid for all $t \geq 0$, for almost all ω.

Next, by means of (1.1) and $u_0(0) \geq 0$, we find that

$$\int_0^t k u_k(0,s)^- \partial_s u_k(0,s)ds = -\frac{k}{2}\big|u_k(0,t)^-\big|^2. \tag{5.18}$$

Let us set

$$Q_k(t) = \|\partial_t u_k(t)\|^2_{L^2(0,L)} + \|\partial_x u_k(t)\|^2_{L^2(0,L)} + \epsilon\langle\partial_t u_k(t), u_k(t)\rangle + \frac{\epsilon\alpha}{2}\|u_k(t)\|^2_{L^2(0,L)}$$

and choose sufficiently small $0 < \epsilon < \alpha$ such that

$$c_1\big(\|\partial_t u_k(t)\|^2_{L^2(0,L)}+\|u_k(t)\|^2_{H^1(0,L)}\big) \leq Q_k(t) \leq c_2\big(\|\partial_t u_k(t)\|^2_{L^2(0,L)}+\|u_k(t)\|^2_{H^1(0,L)}\big)$$

for some positive constants c_1, c_2 independent of $u_k(t)$. We now set

$$\mathcal{R}_k(t) = Q_k(t) + k|u_k(0,t)^-|^2.$$

It follows that for any $0 \leq t_1 < t_2$,

$$\begin{aligned}
E(\mathcal{R}_k(t_2)) - E(\mathcal{R}_k(t_1)) &= -(2\alpha-\epsilon)\int_{t_1}^{t_2} E\big(\|\partial_t u_k(t)\|^2_{L^2(0,L)}\big)dt \\
&- \epsilon\int_{t_1}^{t_2} E\big(k|u_k(0,t)^-|^2\big)dt - \epsilon\int_{t_1}^{t_2} E\big(\|\partial_x u_k(t)\|^2_{L^2(0,L)}\big)dt \\
&+ \sum_{j=1}^{\infty}(t_2-t_1)\|f_j\|^2_{L^2(0,L)}.
\end{aligned} \tag{5.19}$$

Hence, there is some positive constant c independent of k such that

$$\frac{d}{dt}E(\mathcal{R}_k(t)) \leq -c\,E(\mathcal{R}_k(t)) + \sum_{j=1}^{\infty}\|f_j\|^2_{L^2(0,L)}$$

for all $t > 0$, which implies that

$$E(\mathcal{R}_k(t)) \leq M, \qquad \text{for all } k \geq 1, \text{ and all } t \geq 0. \tag{5.20}$$

Meanwhile, according to Lemma 5.1, there is $\tilde{\Omega} \subset \Omega$ such that $P(\Omega\setminus\tilde{\Omega}) = 0$ and for each $\omega \in \tilde{\Omega}$,

$$u_k \to u \qquad \text{weak star in } L^\infty\big(0,T;H^1(0,L)\big), \text{ as } k \to \infty, \tag{5.21}$$

$$\partial_t u_k \to u_t \qquad \text{weak star in } L^\infty\big(0,T;L^2(0,L)\big), \text{ as } k \to \infty, \tag{5.22}$$

$$\Phi \in C\big([0,T];L^2(0,L)\big) \tag{5.23}$$

and

(5.24)

u and all u_k's satisfy (0.1) in the sense of distributions over $(0,L)\times(0,T)$,

for all $T > 0$. It follows that for each $\omega \in \tilde{\Omega}$,

$$\partial_t\big(\partial_t u_k - \Phi\big) \to \partial_t\big(u_t - \Phi\big) \qquad \text{weak star in } L^\infty(0,T;H^{-1}(0,L)), \tag{5.25}$$

as $k \to \infty$. Thus, for each $\omega \in \tilde{\Omega}$, it holds that

$$u_k \to u \qquad \text{in } C([0,T];H^{1-\eta}(0,L)) \tag{5.26}$$

and

$$\partial_t u_k \to u_t \qquad \text{in } C([0,T];H^{-\eta}(0,L)) \tag{5.27}$$

for any $\eta > 0,$ for all $T > 0$. See Lemma 1.7 of [9]. Consequently, for all $t \geq 0$, and all $\omega \in \tilde{\Omega}$,

$$u_k(t) \to u(t) \qquad \text{in } H^{1-\eta}(0,L) \tag{5.28}$$

and

$$\partial_t u_k(t) \to u_t(t) \qquad \text{in } H^{-\eta}(0,L), \tag{5.29}$$

as $k \to \infty$. Now fix any $t \geq 0$. Define

$$\mathcal{S}_k(t) = \|u_k(t)\|^2_{H^1(0,L)} + \|\partial_t u_k(t)\|^2_{L^2(0,L)}.$$

By Fatou's lemma, (5.20) yields

$$E\left(\varliminf_{k\to\infty} \mathcal{S}_k(t)\right) \leq M \tag{5.30}$$

where M is a positive constant independent of t. This implies that there is some $\Omega_t \subset \Omega$ such that $P(\Omega \setminus \Omega_t) = 0$ and, for each $\omega \in \Omega_t$,

$$\varliminf_{k\to\infty} \mathcal{S}_k(t) < \infty,$$

which, combined with (5.28) and (5.29), implies that for each fixed $\omega \in \tilde{\Omega} \cap \Omega_t$, there is a subsequence $\{u_{k_m}\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} \mathcal{S}_{k_m}(t) = \varliminf_{k\to\infty} \mathcal{S}_k(t) < \infty,$$

$$u_{k_m}(t) \to u(t) \qquad \text{weakly in } H^1(0,L)$$

and

$$\partial_t u_{k_m}(t) \to u_t(t) \qquad \text{weakly in } L^2(0,L).$$

Consequently,

$$\|u(t)\|^2_{H^1(0,L)} + \|u_t(t)\|^2_{L^2(0,L)} \leq \varliminf_{k\to\infty} \mathcal{S}_k(t).$$

This is true for each $\omega \in \Omega_t \cap \tilde{\Omega}$. Hence, we have

$$E\big(\|u(t)\|^2_{H^1(0,L)} + \|u_t(t)\|^2_{L^2(0,L)}\big) \leq M$$

for all $t \geq 0$. This proves the lemma. □

Next we define

$$\mathcal{Y} = \big\{z = (z_1, z_2) \quad | \quad z_1 \in H^1(0,L),\, z_1(0) \geq 0,\, z_1(L) = 0, \quad z_2 \in L^2(0,L)\big\}.$$

$\mathcal{Y}$ is a closed subset of $H^1(0,L) \times L^2(0,L)$, and thus, $\mathcal{B}(\mathcal{Y}) \subset \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)$. For each $z \in \mathcal{Y}$, we write

$$X(t; s, z) = (u(t), u_t(t)), \qquad \text{for } t \geq s \geq 0 \tag{5.31}$$

where u is the solution of (0.1) - (0.4) with the initial condition $(u(s), u_t(s)) = z$. We then write

$$\mathcal{P}(s, z; t, \Gamma) = P\big\{X(t; s, z) \in \Gamma\big\}$$

for each $\Gamma \in \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)$ and $t \geq s \geq 0$.

Lemma 5.3. *For each* $t > s \geq 0$, *and* $z \in \mathcal{Y}$, $\mathcal{P}(s, z; t, \cdot)$ *is a probability measure over* $\Big(H^1(0,L) \times L^2(0,L), \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)\Big)$ *supported on* $\mathcal{Y}$. *For each* $t > s \geq 0$ *and* $\Gamma \in \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)$, $\mathcal{P}(s, \cdot\,; t, \Gamma)$ *is* $\mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)$-*measurable.*

Proof. The first assertion is obvious from Theorem 1.4. We will prove the second assertion. Choose any arbitrary sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathcal{Y}$ such that $z_n \to z$ in $H^1(0,L) \times L^2(0,L)$, as $n \to \infty$. Let w_n be the solution of (4.12) with $\sigma_j \equiv 0$, $j \geq 1$, and the initial condition $(w_n(s), \partial_t w_n(s)) = z_n$ for each $n \geq 1$. Let $v_{k,n}$ be the solution in Lemma 2.3 with $K = k$, $h = w_n(0, \cdot)$ and the initial condition $v_{k,n}(\cdot\,, s) = 0$, $\partial_t v_{k,n}(\cdot\,, s) = 0$. We write

$$X_k(\cdot\,, s; z_n) = \big(w_n + v_{k,n},\ \partial_t w_n + \partial_t v_{k,n}\big).$$

Then, for each $t \geq s \geq 0$,

$$X_k(t, s; z_n) \to X_k(t, s; z) \qquad \text{in } L^2\big(\Omega; H^1(0,L) \times L^2(0,L)\big) \tag{5.32}$$

as $n \to \infty$. We define a function space $H^*(0,L)$ by

$$H^*(0,L) = \left\{ \sum_{k=1}^{\infty} d_k e_k \;\middle|\; \sum_{k=1}^{\infty} \frac{|d_k|^2}{\lambda_k} < \infty \right\}$$

where the λ_k's and e_k's are the same as in (3.7).

Let ψ be a bounded continuous function on $L^2(0,L) \times H^*(0,L)$. Then, it is also a bounded continuous function on $H^1(0,L) \times L^2(0,L)$. It follows from (5.32) that there is a subsequence $\big\{X_k(t, s; z_{n_m})\big\}_{m=1}^{\infty}$ that converges to $X_k(t, s; z)$ in $H^1(0,L) \times L^2(0,L)$, for almost all ω. This implies that

$$\int_\Omega \psi(X_k(t, s; z_n))dP \to \int_\Omega \psi(X_k(t, s; z))dP$$

as $n \to \infty$, and hence,

$$\int_\Omega \psi(X_k(t, s; z))dP \quad \text{is continuous in } z \in \mathcal{Y}.$$

Meanwhile, it follows from (5.26) and (5.27) that for almost all $\omega \in \Omega$,

$$X_k(t, s; z) \to X(t, s; z) \qquad \text{in } L^2(0,L) \times H^*(0,L)$$

as $k \to \infty$, because $H^{-\eta}(0,L)$ is imbedded into $H^*(0,L)$ for $0 \leq \eta \leq 1/2$. Hence,

$$\int_\Omega \psi(X(t, s; z))dP \quad \text{is } \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)\text{-measurable in } z. \tag{5.33}$$

Next let ϕ be a bounded continuous function on $H^1(0,L) \times L^2(0,L)$. We define ϕ_m by

$$\phi_m(z) = \phi\big((\Pi_m z_1, \Pi_m z_2)\big)$$

where $z = (z_1, z_2) \in L^2(0,L) \times H^*(0,L)$ and Π_m is the projection onto the subspace spanned by $\{e_1, \cdots, e_m\}$. Then, for each m, ϕ_m is a bounded continuous function on $L^2(0,L) \times H^*(0,L)$ and for each $y \in \mathcal{Y}$,

$$\phi_m(y) \to \phi(y) \qquad \text{as } m \to \infty. \tag{5.34}$$

Thus, by (5.33),

$$\int_\Omega \phi_m(X(t, s; z))dP \quad \text{is } \mathcal{B}\big(H^1(0,L) \times L^2(0,L)\big)\text{-measurable in } z$$

and, by (5.34),

$$\int_\Omega \phi(X(t,s;z))dP \quad \text{is } \mathcal{B}\big(H^1(0,L)\times L^2(0,L)\big)\text{-measurable in } z.$$

Thus, $\mathcal{P}(s,z;t,\Gamma)$ is $\mathcal{B}\big(H^1(0,L)\times L^2(0,L)\big)$-measurable in z, for each $t>s\geq 0$ and each $\Gamma\in\mathcal{B}\big(H^1(0,L)\times L^2(0,L)\big)$.

□

Lemma 5.4. *For each $t\geq 0$, $s\geq 0$, $z\in\mathcal{Y}$ and $\Gamma\in\mathcal{B}\big(H^1(0,L)\times L^2(0,L)\big)$, it holds that*

$$\mathcal{P}(0,z;t,\Gamma)=\mathcal{P}(s,z;t+s,\Gamma).$$

Proof. Fix any $s>0$ and $z=(u_0,u_1)\in\mathcal{Y}$. Let us write

$$B_j^*(t)=B_j(t+s)-B_j(s)$$

and

$$\Phi^*(t)=\Phi(t+s)-\Phi(s)=\sum_{j=1}^{\infty} f_j\, B_j^*(t).$$

Choose any $T>0$, and let w^* be the solution of

$$\begin{cases} w^*_{tt}-w^*_{xx}+\alpha w^*_t=\partial_t\Phi^* & \text{for } (x,t)\in(0,L)\times(0,T),\\ w^*_x(0,t)=0, \qquad w^*(L,t)=0 & \text{for } t\in(0,T),\\ w^*(x,0)=u_0(x), \qquad w^*_t(x,0)=u_1(x) & \text{for } x\in(0,L). \end{cases} \tag{5.35}$$

Let w be the solution of (4.12). Then, w and w^* have the same distribution in the sense that

$$P\big((w,w_t)\in\mathcal{G}\big)=P\big((w^*,w^*_t)\in\mathcal{G}\big)$$

for every $\mathcal{G}\in\mathcal{B}\Big(C\big([0,T];H^1(0,L)\times L^2(0,L)\big)\Big)$. Let v_k and v_k^* be the solutions in Lemma 2.3 with $K=k$, and $h=w(0,\cdot)$, $h=w^*(0,\cdot)$, respectively. The mapping

$$(w,\, w_t)\mapsto h\mapsto (v_k,\, \partial_t v_k)$$

is continuous from $C\big([0,T];H^1(0,L)\times L^2(0,L)\big)$ into itself. Therefore, it holds that

$$P\Big(\big(w(t)+v_k(t),w_t(t)+\partial_t v_k(t)\big)\in\Gamma\Big)=P\Big(\big(w^*(t)+v_k^*(t),w_t^*(t)+\partial_t v_k^*(t)\big)\in\Gamma\Big) \tag{5.36}$$

for every $\Gamma\in\mathcal{B}\big(H^1(0,L)\times L^2(0,L)\big)$, and every $t\in[0,T]$. Let ϕ be a continuous bounded function on $H^1(0,L)\times L^2(0,L)$. As above, let $\{\phi_n\}_{n=1}^\infty$ be a sequence of uniformly bounded functions on $H^1(0,L)\times L^2(0,L)$ which are continuous with respect to the norm of $L^2(0,L)\times H^*(0,L)$ such that

$$\phi_n(y)\to\phi(y) \qquad \text{for each } y\in\mathcal{Y}. \tag{5.37}$$

It follows from Lemma 5.1 that for almost all $\omega\in\Omega$,

$$X_k=(w+v_k,\ \partial_t w+\partial_t v_k)\to X=(w+v,\ w_t+v_t)$$

weak star in $L^\infty(0,T;H^1(0,L)\times L^2(0,L))$, as $k\to\infty$, and

$$X_k^*=(w^*+v_k^*,\ \partial_t w^*+\partial_t v_k^*)\to X^*=(w^*+v^*,\ w_t^*+v_t^*)$$

weak star in $L^\infty\big(0,T;H^1(0,L)\times L^2(0,L)\big)$. In the meantime, we can write

$$\partial_t\big(w_t+\partial_t v_k-\Phi\big)=w_{xx}+\partial_{xx}v_k-\alpha(w_t+\partial_t v_k)$$

and

$$\partial_t\big(w_t^*+\partial_t v_k^*-\Phi^*\big)=w_{xx}^*+\partial_{xx}v_k^*-\alpha(w_t^*+\partial_t v_k^*).$$

As above, it follows from (5.26) and (5.27) that for almost all $\omega\in\Omega$,

$$X_k=(w+v_k,\ \partial_t w+\partial_t v_k)\to X=(w+v,\ w_t+v_t)$$

strongly in $C\big([0,T];L^2(0,L)\times H^*(0,L)\big)$, as $k\to\infty$, and

$$X_k^*=(w^*+v_k^*,\ \partial_t w^*+\partial_t v_k^*)\to X^*=(w^*+v^*,\ w_t^*+v_t^*)$$

strongly in $C\big([0,T];L^2(0,L)\times H^*(0,L)\big)$. Hence, for each $n\ge 1$, and each $t\in[0,T]$,

$$\int_\Omega \phi_n(X_k(t))dP\to\int_\Omega \phi_n(X(t))dP$$

and

$$\int_\Omega \phi_n(X_k^*(t))dP\to\int_\Omega \phi_n(X^*(t))dP,$$

as $k\to\infty$. But, by (5.36), we see that for each $n\ge 1$,

$$\int_\Omega \phi_n(X_k(t))dP=\int_\Omega \phi_n(X_k^*(t))dP.$$

Thus, for all $n\ge 1$,

$$\int_\Omega \phi_n(X(t))dP=\int_\Omega \phi_n(X^*(t))dP.$$

By passing $n\to\infty$, we use (5.37) to arrive at

$$\int_\Omega \phi(X(t))dP=\int_\Omega \phi(X^*(t))dP,$$

which completes the proof of Lemma 5.4. □

Next let $\{z_n\}_{n=1}^\infty$ be a sequence in $\mathcal{Y}$ such that it is bounded with respect to the norm of $H^1(0,L)\times L^2(0,L)$, and

$$z_n\to z\qquad\text{in } L^2(0,L)\times H^*(0,L).$$

This implies that $z\in\mathcal{Y}$. Let us write $X_n=\big(u_n,\ \partial_t u_n\big)$ where u_n is the solution of (0.1) - (0.4) if the initial condition is $X_n(0)=z_n$, and write $X=(u,\ u_t)$ if the initial condition is $X(0)=z$.

Lemma 5.5. *For any $T>0$, and any bounded continuous function ϕ on $L^2(0,L)\times H^*(0,L)$,*

$$\int_\Omega \phi\big(X_n(T)\big)dP\to\int_\Omega \phi\big(X(T)\big)dP\qquad \textit{as } n\to\infty.$$

Proof. Fix any $T>0$. Since the sequence $\{z_n\}_{n=1}^\infty$ is bounded in $\mathcal{Y}$, it follows from (4.14) - (4.16) and (4.23) that

$$E\Big(\sup_{0\le t\le T}\|X_n(t)\|^2_{H^1(0,L)\times L^2(0,L)}\Big)+E\Big(\|\partial_x u_n(0,\cdot)\|^2_{L^2(0,T)}\Big)\le M \tag{5.38}$$

for all $n \geq 1$, for some positive constant M. Choose any $\epsilon > 0$. Then there is some constant $K > 0$ independent of n such that

$$P\Big\{ \sup_{0 \leq t \leq T} \|X_n(t)\|_{H^1(0,L)\times L^2(0,L)} \geq K \Big\} + P\Big\{ \|\partial_x u_n(0,\cdot)\|_{L^2(0,T)} \geq K \Big\} < \epsilon. \tag{5.39}$$

Choose any bounded continuous function ϕ on $L^2(0,L) \times H^*(0,L)$. We assume $\phi \geq 0$, and let

$$C = \sup_{z \in L^2(0,L)\times H^*(0,L)} \phi(z), \tag{5.40}$$

$$\mathcal{A}_{n,K} = \Big\{ \omega \,\Big|\, \sup_{0\leq t\leq T} \|X_n(t)\|_{H^1(0,L)\times L^2(0,L)} \leq K,\ \|\partial_x u_n(0,\cdot)\|_{L^2(0,T)} \leq K \Big\}$$

and

$$\Xi_n = \phi(X_n(T))\,\chi\{\mathcal{A}_{n,K}\}$$

where $\chi\{\cdot\}$ is the characteristic function. We can also define a subset $\Omega^\dagger \subset \Omega$ such that $P(\Omega \setminus \Omega^\dagger) = 0$ and for each fixed $\omega \in \Omega^\dagger$,

$$\begin{cases} X(\omega) \text{ and } X_n(\omega),\ n \geq 1, \text{ belong to } C\big([0,T]; H^1(0,L) \times L^2(0,L)\big), \\ \partial_x u(0,\cdot) \text{ and } \partial_x u_n(0,\cdot),\ n \geq 1, \text{ belong to } L^2(0,T), \\ \Phi(\omega) \in C\big([0,T]; L^2(0,L)\big), \\ u \text{ and } u_n,\ n \geq 1, \text{ satisfy (0.1) in the sense of distributions}, \\ u \text{ and } u_n,\ n \geq 1, \text{ satisfy (0.2) for almost all } t \in [0,T]. \end{cases}$$

We will show that for all $\omega \in \Omega^\dagger$,

$$0 \leq \overline{\lim_{n\to\infty}}\, \Xi_n(\omega) \leq \phi\big(X(T,\omega)\big). \tag{5.41}$$

If $\overline{\lim}_{n\to\infty} \Xi_n(\omega) = 0$, then it holds because $\phi \geq 0$. Suppose $\overline{\lim}_{n\to\infty} \Xi_n(\omega) > 0$, for some $\omega \in \Omega^\dagger$. Then, there is a subsequence $\{X_{n_k}(\omega)\}_{k=1}^\infty$ such that

$$\overline{\lim_{n\to\infty}}\, \Xi_n(\omega) = \lim_{k\to\infty} \Xi_{n_k}(\omega) = \lim_{k\to\infty} \phi(X_{n_k}(T,\omega)), \tag{5.42}$$

$$\sup_{0\leq t\leq T} \|X_{n_k}(t,\omega)\|_{H^1(0,L)\times L^2(0,L)} \leq K, \tag{5.43}$$

$$\|\partial_x u_{n_k}(0,\cdot)\|_{L^2(0,T)} \leq K, \tag{5.44}$$

$$u_{n_k} \to u^* \qquad \text{weak star in } L^\infty\big(0,T; H^1(0,L)\big), \tag{5.45}$$

$$\partial_t u_{n_k} \to \partial_t u^* \qquad \text{weak star in } L^\infty\big(0,T; L^2(0,L)\big), \tag{5.46}$$

and

$$\partial_x u_{n_k}(0,\cdot) \to \partial_x u^*(0,\cdot) \qquad \text{weakly in } L^2(0,T) \tag{5.47}$$

where u^* satisfies (0.1) in the sense of distributions for this fixed ω. As above, it follows from (0.1), (5.45) and (5.46) that

$$u_{n_k} \to u^* \qquad \text{strongly in } C\big([0,T]; H^{1-\eta}(0,L)\big) \tag{5.48}$$

and

$$\partial_t u_{n_k} \to \partial_t u^* \qquad \text{strongly in } C\big([0,T]; H^{-\eta}(0,L)\big) \tag{5.49}$$

for every $\eta > 0$. Hence, u^* satisfies (0.2) - (0.4). Since $\sigma_j \equiv 0$, $j \geq 1$, we can apply the uniqueness of solution of (0.1) - (0.4) for this fixed ω to conclude that

$$X(\omega) = \big(u^*,\, \partial_t u^*\big). \tag{5.50}$$

By (5.48) and (5.49), we see that

$$X_{n_k}(T,\omega) \to X(T,\omega) \qquad \text{in } L^2(0,L)\times H^*(0,L).$$

Hence,

$$\phi(X_{n_k}(T,\omega)) \to \phi(X(T,\omega)), \qquad \text{as } k\to\infty$$

and thus,

$$\overline{\lim_{n\to\infty}}\, \Xi_n(\omega) = \phi(X(T,\omega))$$

and the inequality holds. Consequently,

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_\Omega \Xi_n(\omega)dP &\le \int_\Omega \overline{\lim_{n\to\infty}}\, \Xi_n(\omega)dP \\ &\le \int_\Omega \phi(X(T,\omega))dP.\end{aligned}$$

Hence, by (5.39) and (5.40),

$$\overline{\lim_{n\to\infty}} \int_\Omega \phi\big(X_n(T,\omega)\big)dP - C\epsilon \le \int_\Omega \phi\big(X(T,\omega)\big)dP.$$

Next let

$$\Theta_n = \phi(X_n(T))\,\chi\{\mathcal{A}_{n,K}\} \vee C\big(1-\chi\{\mathcal{A}_{n,K}\}\big)$$

where C is the same constant as in (5.40). We will show that for all $\omega\in\Omega^\dagger$,

$$\phi(X(T,\omega)) \le \underline{\lim}_{n\to\infty}\, \Theta_n(\omega). \tag{5.51}$$

If $\underline{\lim}_{n\to\infty}\Theta_n(\omega) = C,$ then it is true because of the definition of C. Suppose that $\underline{\lim}_{n\to\infty}\Theta_n(\omega) < C,$ for some $\omega\in\Omega^\dagger$. Then, as above, there is a subsequence $\{\Theta_{n_k}\}_{k=1}^\infty$ such that

$$\underline{\lim}_{n\to\infty}\, \Theta_n(\omega) = \lim_{k\to\infty} \Theta_{n_k}(\omega) = \lim_{k\to\infty} \phi(X_{n_k}(T,\omega)) \tag{5.52}$$

and (5.43) - (5.49) hold for this ω. By the same argument as above, as $k\to\infty$,

$$X_{n_k}(T,\omega) \to X(T,\omega) \qquad \text{in } L^2(0,L)\times H^*(0,L)$$

and hence,

$$\underline{\lim}_{n\to\infty}\, \Theta_n(\omega) = \lim_{k\to\infty} \phi(X_{n_k}(T,\omega)) = \phi(X(T,\omega)).$$

Thus, the inequality holds for all $\omega\in\Omega^\dagger$. So it follows from (5.39) and (5.40) that

$$\begin{aligned}\int_\Omega \phi(X(T,\omega))dP &\le \int_\Omega \underline{\lim}_{n\to\infty}\, \Theta_n(\omega)dP \\ &\le \underline{\lim}_{n\to\infty} \int_\Omega \Theta_n(\omega)dP \le \underline{\lim}_{n\to\infty} \int_\Omega \phi(X_n(T,\omega))dP + C\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \int_\Omega \phi(X_n(T,\omega))dP = \int_\Omega \phi(X(T,\omega))dP.$$

We can drop the assumption that $\phi \ge 0,$ by writing $\phi = \phi^+ - \phi^-$. $\square$

We now check the conditions [I] - [V] for the existence of an invariant measure. Here we take $\Xi = H^1(0,L) \times L^2(0,L)$ and $\Upsilon = L^2(0,L) \times H^*(0,L)$. $X(t,s;z)$ is defined by (5.31) for $t \geq s \geq 0$ and $z \in \mathcal{Y}$. Even though $X(t,s;z)$ is defined only for $z \in \mathcal{Y}$, this is not a restriction because $X(t,s;z) \in \mathcal{Y}$ for all $t \geq s \geq 0$, for almost all ω, if $z \in \mathcal{Y}$. Accordingly, the condition on $\{\psi_k\}_{k=1}^{\infty}$ in [IV] is relaxed by

$$\lim_{k\to\infty} \psi_k(y) = \psi(y), \qquad \text{for all } y \in \mathcal{Y},$$

and the sequence $\{z_n\}_{n=1}^{\infty}$ in [V] is chosen from $\mathcal{S} \cap \mathcal{Y}$. [I] follows from Theorem 1.4 and [II] is verified by Lemmas 5.3 and 5.4. [III] is verified by Lemma 5.2. [V] is verified by Lemma 5.5. It remains to verify [IV]. Let ψ be a bounded continuous function on Ξ. We define ψ_k by

$$\psi_k(y) = \psi\big((\Pi_k y_1, \Pi_k y_2)\big), \qquad \text{for each } y = (y_1, y_2) \in \Upsilon,$$

where Π_k is the projection onto the subspace spanned by $\{e_1, e_2, \cdots, e_k\}$. Then, ψ_k satisfies the required property. This completes the proof of Theorem 1.5.

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