INTRANSITIVE CARTESIAN DECOMPOSITIONS PRESERVED BY INNATELY TRANSITIVE PERMUTATION GROUPS

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ABSTRACT. A permutation group is innately transitive if it has a transitive minimal normal subgroup, which is referred to as a plinth. We study the class of finite, innately transitive permutation groups that can be embedded into wreath products in product action. This investigation is carried out by observing that such a wreath product preserves a natural Cartesian decomposition of the underlying set. Previously we classified the possible embeddings in the case where the innately transitive group projects onto a transitive subgroup of the top group. In this article we prove that the transitivity assumption we made in the previous paper was not too restrictive. Indeed, the image of the projection into the top group can only be intransitive when the finite simple group that is involved in the plinth comes from a small list. Even then, the innately transitive group can have at most three orbits on an invariant Cartesian decomposition. A consequence of this result is that if G is an innately transitive subgroup of a wreath product in product action, then the natural projection of G into the top group has at most two orbits.

1. INTRODUCTION

The results presented in this paper play a key rôle in our research program to describe the Cartesian decompositions preserved by an innately transitive permutation group. Recall that a permutation group G is said to be *innately transitive*, if G has a transitive minimal normal subgroup M, which is called a *plinth* of G. Innately transitive groups are investigated in [BamP04]. The aim of our research is to describe certain subgroups of wreath products in product action. We showed in [BPS04] that these subgroups are best understood via studying the natural Cartesian decomposition of the underlying set that is preserved by such a wreath product. In the same paper we demonstrated the scope of this theory by describing transitive simple subgroups and their normalisers in primitive wreath products. Later in [BPS06, PS07] we described those innately transitive subgroups of wreath products in product action that project onto a transitive subgroup of the top group.

Here we consider the remaining case: we describe the innately transitive subgroups of wreath products in product action that project onto an intransitive subgroup of the top group. This amounts to saying that such a group acts intransitively on the corresponding Cartesian decomposition of the underlying set. We show that

this case cannot arise unless the simple direct factors of the plinth of the innately transitive group come from a very small list which consists of finitely many groups and, in addition, two infinite families, each depending on a single (field) parameter (see Theorem 3.1(iv)). Even then, the transitivity assumption can only fail a little: there can only be two orbits. Thus we obtain the following theorem.

Theorem 1.1. *Suppose that Δ is a finite set, $|\Delta| \geqslant 2$, $\ell \geqslant 2$, and W is the wreath product $\mathsf{Sym}\,\Delta \,\mathsf{wr}\, \mathsf{S}_\ell$ acting on Δ^ℓ in product action. Let G be an innately transitive subgroup of W. Then the image of G under the natural projection $W \to \mathsf{S}_\ell$ has at most two orbits on $\{1, \ldots, \ell\}$.*

The proof of Theorem 1.1 is carried out by assuming that G acts intransitively on the underlying natural Cartesian decomposition of Δ^ℓ. Thus we study Cartesian decompositions of sets that are acted upon intransitively by an innately transitive permutation group. Though the above-mentioned Cartesian decomposition of Δ^ℓ is homogeneous, that is, its elements have the same size, we do not restrict our attention to this special case. Instead, we describe innately transitive permutation groups acting intransitively on an arbitrary Cartesian decomposition. The results of this study are collected in Theorem 3.1. In particular this theorem proves that an innately transitive group can have at most three orbits on any Cartesian decomposition it preserves. However, in the case of three orbits, it is proved that the invariant Cartesian decomposition is not homogeneous. Part (iv) of Theorem 3.1 implies Theorem 1.1, and also describes in more detail the embedding in Theorem 1.1.

The organisation of the paper is as follows. First in Section 2 we summarise those results of our previous work on Cartesian decompositions that will be used in this paper. In the next section we build the machinery that is necessary to investigate the scenario of Theorem 1.1. Then we state Theorem 3.1 which, as mentioned above, implies Theorem 1.1. In order to prove our main theorem, we need results about characteristically simple groups, and in Section 4 we study the factorisations of such groups. In Section 5 we prove several results about normalisers of subgroups of characteristically simple groups. Then in Sections 6, 7, and 8 we treat Cartesian systems that are acted upon trivially by a point stabiliser. Finally in Section 9 we prove Theorem 3.1.

Most of our results depend on the correctness of the finite simple group classification. For instance, a lot of information on the factorisations of simple and characteristically simple groups that depend on this classification are used throughout the paper.

The system of notation used in this paper is standard in permutation group theory. Permutations act on the right: if π is a permutation and ω is a point, then the image of ω under π is denoted $\omega\pi$. If G is a group acting on a set Ω, then G^Ω denotes the subgroup of $\mathsf{Sym}\,\Omega$ induced by G. Further, if Γ is a subset of Ω, then G_Γ and $G_{(\Gamma)}$ denote the setwise and the pointwise stabilisers, respectively, in G of Γ.

2. Cartesian decompositions and Cartesian systems

A *Cartesian decomposition* of a set Ω is a set $\{\Gamma_1, \ldots, \Gamma_\ell\}$ of proper partitions of Ω such that

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \quad \text{for all} \quad \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell.$$

This property implies that the following map is a well-defined bijection between Ω and $\Gamma_1 \times \cdots \times \Gamma_\ell$:

$$\omega \mapsto (\gamma_1, \ldots, \gamma_\ell) \text{ where for } i = 1, \ldots, \ell, \ \gamma_i \in \Gamma_i \text{ is chosen so that } \omega \in \gamma_i.$$

Thus the set Ω can naturally be identified with the Cartesian product $\Gamma_1 \times \cdots \times \Gamma_\ell$. The number ℓ is called the *index* of the Cartesian decomposition $\{\Gamma_1, \ldots, \Gamma_\ell\}$. The defining property for a Cartesian decomposition implies that, for each i, the parts of Γ_i all have the same size, though this size may be different for different partitions Γ_i (see, for example, [BPS07, Section 3]). If the parts of Γ_i have the same size for all i, then the Cartesian decomposition is said to be *homogeneous*, and otherwise it is called *inhomogeneous*.

If G is a permutation group acting on Ω, then a Cartesian decomposition $\mathcal{E}$ of Ω is said to be *G-invariant*, if the partitions in $\mathcal{E}$ are permuted by G, and $\mathsf{CD}(G)$ denotes the set of G-invariant Cartesian decompositions of Ω. If $\mathcal{E} \in \mathsf{CD}(G)$ and G acts on $\mathcal{E}$ transitively, then $\mathcal{E}$ is said to be a *transitive G-invariant Cartesian decomposition*; otherwise it is called *intransitive*. The set of transitive G-invariant Cartesian decompositions of Ω is denoted by $\mathsf{CD}_{\mathrm{tr}}(G)$.

The concept of a Cartesian decomposition was introduced by L. G. Kovács in [Kov89b] where it was called a system of product imprimitivity. Kovács suggested that studying $\mathsf{CD}_{\mathrm{tr}}(G)$ (using our terminology) was the appropriate way to identify wreath decompositions for finite primitive permutation groups G. His papers [Kov89b] and [Kov89a] inspired our work.

Suppose that G is an innately transitive subgroup of $\mathsf{Sym}\,\Omega$ with plinth M, and that $\mathcal{E}$ is a G-invariant Cartesian decomposition of Ω. In [BPS04, Proposition 2.1] we proved that each of the Γ_i is an M-invariant partition of Ω. Choose an element ω of Ω and let $\gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell$ be such that $\{\omega\} = \gamma_1 \cap \cdots \cap \gamma_\ell$; set $K_i = M_{\gamma_i}$. Then [BPS04, Lemmas 2.2 and 2.3] imply that the set $\mathcal{K}_\omega(\mathcal{E}) = \{K_1, \ldots, K_\ell\}$ is invariant under conjugation by G_ω, and, in addition,

$$(1) \qquad \bigcap_{i=1}^{\ell} K_i \ = \ M_\omega,$$

$$(2) \qquad K_i \left(\bigcap_{j \neq i} K_j \right) \ = \ M \quad \text{for all} \quad i \in \{1, \ldots, \ell\}.$$

For an arbitrary transitive permutation group M on Ω and a point $\omega \in \Omega$, a set $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ of proper subgroups of M is said to be a *Cartesian system of subgroups with respect to ω* for M, if (1) and (2) hold.

Theorem 2.1 (Theorem 1.4 and Lemma 2.3 of [BPS04]). *Let $G \leqslant \mathsf{Sym}\,\Omega$ be an innately transitive permutation group with plinth M. For a fixed $\omega \in \Omega$ the correspondence $\mathcal{E} \mapsto \mathcal{K}_\omega(\mathcal{E})$ is a bijection between the set of G–invariant Cartesian decompositions of Ω and the set of G_ω–invariant Cartesian systems of subgroups for M with respect to ω. Moreover the G_ω–actions on $\mathcal{E}$ and on $\mathcal{K}_\omega(\mathcal{E})$ are equivalent.*

With $G \leqslant \mathsf{Sym}\,\Omega$, M, and $\omega \in \Omega$ as above, let $\mathcal{K}$ be a G_ω-invariant Cartesian system of subgroups for M with respect to ω. Then Theorem 2.1 implies that $\mathcal{K} = \mathcal{K}_\omega(\mathcal{E})$ for some G-invariant Cartesian decomposition $\mathcal{E}$ of Ω. In fact, $\mathcal{E}$ consists of the M-invariant partitions $\{(\omega^K)^m \mid m \in M\}$ where K runs through the elements of $\mathcal{K}$. This Cartesian decomposition is usually denoted $\mathcal{E}(\mathcal{K})$.

Using this theory we were able to describe in [BPS04] those innately transitive subgroups of wreath products that have a simple plinth. This led to a classification of transitive simple subgroups of wreath products in product action (see [BPS04, Theorem 1.1]). Then in [BPS06, PS07] we extended this classification and described innately transitive subgroups of such wreath products that project onto a transitive subgroup of the top group.

Suppose now that $M = T_1 \times \cdots \times T_k$ where the T_i are groups, and $k \geqslant 1$. For $I \subseteq \{T_1, \ldots, T_k\}$, $\sigma_I : M \to \prod_{T_i \in I} T_i$ denotes the natural projection map. We also write σ_i for $\sigma_{\{T_i\}}$. A subgroup X of M is said to be a *strip* if, for each $i = 1, \ldots, k$, either $\sigma_i(X) = 1$ or $\sigma_i(X) \cong X$. The set of all T_i such that $\sigma_i(X) \neq 1$ is called the *support* of X and is denoted $\operatorname{Supp} X$, and $|\operatorname{Supp} X|$ is called the *length* of X. If $T_m \in \operatorname{Supp} X$, then we also say that X *covers* T_m. Two strips X_1 and X_2 are said to be *disjoint* if $\operatorname{Supp} X_1 \cap \operatorname{Supp} X_2 = \emptyset$. A strip X is said to be *full* if $\sigma_i(X) = T_i$ for all $T_i \in \operatorname{Supp} X$, and it is called *non-trivial* if $|\operatorname{Supp} X| \geqslant 2$. A subgroup K of M is said to be *subdirect* with respect to the direct decomposition $T_1 \times \cdots \times T_k$ if $\sigma_i(K) = T_i$ for all i. If M is a finite, non-abelian, characteristically simple group, then a subgroup K is said to be *subdirect* if it is subdirect with respect to the finest direct decomposition of M (that is, the product decomposition with simple groups as factors).

The importance of strips is highlighted by the following result, which is usually referred to as Scott's Lemma (see the appendix of [Sco80]).

Lemma 2.2. *Let M be a direct product of finitely many non-abelian, finite simple groups and H a subdirect subgroup of M. Then H is a direct product of pairwise disjoint full strips of M.*

Let $M = T_1 \times \cdots \times T_k$ be a finite, non-abelian, characteristically simple group, where $T_1, \ldots, T_k$ are the simple normal subgroups of M, each isomorphic to the same simple group T. If K is a subgroup of M and X is a strip in M such that $K = X \times \sigma_{\{T_1, \ldots, T_k\} \setminus \operatorname{Supp} X}(K)$, then we say that X is *involved* in K. A strip X is said to be involved in a Cartesian system $\mathcal{K}$ for M if X is involved in some element of $\mathcal{K}$. Note that in this case (2) implies that X is involved in a unique element of $\mathcal{K}$.

A non-abelian plinth of an innately transitive group G has the form $M = T_1 \times \cdots \times T_k$ where the T_i are finite, non-abelian, simple groups. Let $\mathcal{E} \in \mathsf{CD}(G)$ and let $\mathcal{K}_\omega(\mathcal{E})$ be a corresponding Cartesian system $\{K_1, \ldots, K_\ell\}$ for M with respect to ω. Then equation (2) implies that, for all $i \leqslant k$ and $j \leqslant \ell$,

$$(3) \qquad \sigma_i(K_j) \left(\bigcap_{j' \neq j} \sigma_i(K_{j'}) \right) = T_i.$$

In particular this means that if $\sigma_i(K_j)$ is a proper subgroup of T_i, then $\sigma_i(K_{j'}) \neq \sigma_i(K_j)$ for all $j' \in \{1, \ldots, \ell\} \setminus \{j\}$. It is thus important to understand the following sets of subgroups:

$$(4) \qquad \mathcal{F}_i(\mathcal{E}, M, \omega) = \{\sigma_i(K_j) \mid j = 1, \ldots, \ell, \ \sigma_i(K_j) \neq T_i\}.$$

From our remarks above, $|\mathcal{F}_i(\mathcal{E}, M, \omega)|$ is the number of indices j such that $\sigma_i(K_j) \neq T_i$. The set $\mathcal{F}_i(\mathcal{E}, M, \omega)$ is independent of i up to isomorphism, in the sense that if $i_1, \ i_2 \in \{1, \ldots, k\}$ and $g \in G_\omega$ are such that $T_{i_1}^g = T_{i_2}$, then $\mathcal{F}_{i_1}(\mathcal{E}, M, \omega)^g =$

$\{L^g \mid L \in \mathcal{F}_{i_1}(\mathcal{E}, M, \omega)\} = \mathcal{F}_{i_2}(\mathcal{E}, M, \omega)$. This argument also shows that the sub-groups in $\mathcal{F}_{i_1}(\mathcal{E}, M, \omega)$ are actually G_ω-conjugate to the subgroups in $\mathcal{F}_{i_2}(\mathcal{E}, M, \omega)$.

The set $\mathsf{CD}_{\mathrm{tr}}(G)$ is further subdivided according to the structure of the subgroups in the corresponding Cartesian systems as follows. The sets $\mathcal{F}_i = \mathcal{F}_i(\mathcal{E}, M, \omega)$ are defined in (4).

$\mathsf{CD}_{\mathrm{S}}(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid \text{the elements of } \mathcal{K}_\omega(\mathcal{E}) \text{ are subdirect subgroups in } M\};$

$\mathsf{CD}_1(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 1 \text{ and } \mathcal{K}_\omega(\mathcal{E}) \text{ involves no non-trivial, full strip}\};$

$\mathsf{CD}_{1\mathrm{S}}(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 1 \text{ and } \mathcal{K}_\omega(\mathcal{E}) \text{ involves non-trivial, full strips}\};$

$\mathsf{CD}_{2\sim}(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 2 \text{ and the}$
$$\mathcal{F}_i \text{ contain two } G_\omega\text{-conjugate subgroups}\};$$

$\mathsf{CD}_{2\nsim}(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 2 \text{ and the}$
$$\text{subgroups in } \mathcal{F}_i \text{ are not } G_\omega\text{-conjugate}\};$$

$\mathsf{CD}_3(G) = \{\mathcal{E} \in \mathsf{CD}_{\mathrm{tr}}(G) \mid |\mathcal{F}_i| = 3\}.$

At first glance, it seems that the definitions of the classes $\mathsf{CD}_{\mathrm{S}}(G)$, $\mathsf{CD}_1(G)$, $\mathsf{CD}_{1\mathrm{S}}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\nsim}(G)$, and $\mathsf{CD}_3(G)$ may depend on the choice of the Cartesian system, and hence on the choice of the point ω. However, the following result, proved in [BPS06, Theorems 6.2 and 6.3], shows that this is not the case, and also implies that these classes form a partition of $\mathsf{CD}_{\mathrm{tr}}(G)$. A finite permutation group is said to be *quasiprimitive* if all its non-trivial normal subgroups are transitive. We also say that a quasiprimitive group has compound diagonal type, if it has a unique minimal normal subgroup, which is non-abelian, and in which a point stabiliser is a non-simple subdirect subgroup.

Theorem 2.3 (6-class Theorem). *If G is a finite, innately transitive permutation group with a non-abelian plinth M, then the classes $\mathsf{CD}_1(G)$, $\mathsf{CD}_{\mathrm{S}}(G)$, $\mathsf{CD}_{1\mathrm{S}}(G)$, $\mathsf{CD}_{2\sim}(G)$, $\mathsf{CD}_{2\nsim}(G)$, and $\mathsf{CD}_3(G)$ are independent of the choice of the point ω used in their definition. They form a partition of $\mathsf{CD}_{\mathrm{tr}}(G)$, and moreover, if M is simple, then $\mathsf{CD}_{\mathrm{tr}}(G) = \mathsf{CD}_{2\sim}(G)$. Suppose, in addition, that T is the common isomorphism type of the simple direct factors of M. Then the following all hold:*

(a) *The group G is quasiprimitive of compound diagonal type if and only if $\mathsf{CD}_{\mathrm{S}}(G) \neq \emptyset$.*

(b) *If $\mathsf{CD}_{1\mathrm{S}}(G) \cup \mathsf{CD}_{2\sim}(G) \neq \emptyset$, then T has a factorisation with two isomorphic, proper subgroups and is isomorphic to one of the groups A_6, M_{12}, $\mathrm{P}\Omega_8^+(q)$, or $\mathsf{Sp}_4(2^a)$ with $a \geqslant 2$. If $\mathcal{E} \in \mathsf{CD}_{1\mathrm{S}}(G) \cup \mathsf{CD}_{2\sim}(G)$, then the subgroups in $\mathcal{F}_i(\mathcal{E}, M, \omega)$ are isomorphic to the groups A and B in the corresponding line of Table 2.*

(c) *If $\mathsf{CD}_{2\nsim}(G) \neq \emptyset$, then T admits a factorisation $T = AB$ with A, B proper subgroups.*

(d) *If $\mathsf{CD}_3(G) \neq \emptyset$, then T is isomorphic to one of the groups $\mathsf{Sp}_{4a}(2)$ with $a \geqslant 2$, $\mathrm{P}\Omega_8^+(3)$, or $\mathsf{Sp}_6(2)$. If $\mathcal{E} \in \mathsf{CD}_3(G)$, then the subgroups in $\mathcal{F}_i(\mathcal{E}, M, \omega)$ are isomorphic to the groups A, B, and C in the corresponding line of Table 3.*

3. Intransitive Cartesian decompositions

In this section we state Theorem 3.1, which can be viewed as a qualitative characterisation of innately transitive groups acting intransitively on a Cartesian decomposition. In particular Theorem 1.1 follows from the first assertion and part (iv)

of this result. Before we can state this theorem, we introduce some notation which will also be used in later parts of this paper.

Suppose that G is an innately transitive permutation group acting on Ω with plinth M and let $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ be a G-invariant Cartesian decomposition of Ω on which G acts intransitively. It follows from [Pra90, Proposition 5.1] that M is non-abelian. Suppose that $\Xi_1, \ldots, \Xi_s$ are the G-orbits on $\mathcal{E}$, and that $\mathcal{K}_\omega(\mathcal{E}) = \{L_1, \ldots, L_\ell\}$ is the Cartesian system of subgroups for M with respect to some fixed $\omega \in \Omega$.

For $i = 1, \ldots, s$ set

$$K_i = \bigcap_{\Gamma_j \in \Xi_i} L_j.$$

For partitions $A_1, \ldots, A_d$ of a set Ω, the infimum $\inf\{A_1, \ldots, A_d\}$ of these partitions is defined as the partition

$$\inf\{A_1, \ldots, A_d\} = \{\gamma_1 \cap \cdots \cap \gamma_d \mid \gamma_1 \in A_1, \ldots, \gamma_d \in A_d\}.$$

The proof that $\inf\{A_1, \ldots, A_d\}$ is a partition of Ω is easy and is left to the reader. We note that $\inf\{A_1, \ldots, A_d\}$ is the coarsest partition that refines each of the A_i, and hence is the infimum of $A_1, \ldots, A_d$ with respect to the natural partial order on the set of partitions of Ω. Let Ω_i denote $\inf \Xi_i$ for $i = 1, \ldots, s$ and let $\bar{\mathcal{E}} = \{\Omega_1, \ldots, \Omega_s\}$. If $\gamma \in \Gamma_j$ for some $\Gamma_j \in \Xi_i$, then γ is a union of blocks from Ω_i and we set

$$\bar{\gamma} = \{\delta \mid \delta \in \Omega_i,\ \delta \subseteq \gamma\}, \quad \bar{\Gamma}_j = \{\bar{\gamma} \mid \gamma \in \Gamma_j\} \quad \text{and} \quad \bar{\Xi}_i = \{\bar{\Gamma}_j \mid \Gamma_j \in \Xi_i\}.$$

It turns out, as shown in the following theorem, that $\bar{\mathcal{E}}$ is a Cartesian decomposition of Ω acted upon trivially by G. Further, each G-invariant partition Ω_i in $\bar{\mathcal{E}}$ admits a G-invariant, transitive Cartesian decomposition, namely $\bar{\Xi}_i$. Thus the study of the original intransitive decomposition $\mathcal{E}$ can be carried out via the study of a G-trivial decomposition and the study of several transitive Cartesian decompositions. This idea is made more explicit in Theorem 3.1. The concepts of full factorisation, full strip factorisation, and strong multiple factorisation occurring in the statement are defined in Section 4.

Theorem 3.1. *The number s of G-orbits on $\mathcal{E}$ is at most 3. The partitions Ω_i are G-invariant and the set $\bar{\mathcal{E}} = \{\Omega_1, \ldots, \Omega_s\}$ is a Cartesian decomposition of Ω on which G acts trivially. For $i = 1, \ldots, s$, the subgroup K_i is the stabiliser in M of the block in Ω_i containing ω. Moreover, for $i = 1, \ldots, s$, the M-action on Ω_i is faithful and $\bar{\Xi}_i \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_i})$.*

(i) *If $\bar{\Xi}_i \in \mathsf{CD}_{\mathrm{S}}(G^{\Omega_i})$, for some $i \in \{1, \ldots, s\}$, then $s = 2$. Further, if, say, $\bar{\Xi}_1 \in \mathsf{CD}_{\mathrm{S}}(G^{\Omega_1})$, then (M, K_1, K_2) is a full strip factorisation, and $\bar{\Xi}_2 \in \mathsf{CD}_1(G^{\Omega_2})$.*

(ii) *If $\bar{\Xi}_i \in \mathsf{CD}_{2\not\sim}(G^{\Omega_i})$ for some $i \in \{1, \ldots, s\}$, then $s = 2$, and, for all $j \in \{1, \ldots, k\}$, the group T_j and the subgroups of $\mathcal{F}_j(\mathcal{E}, M, \omega)$ are as in Table 3. If $\bar{\Xi}_1 \in \mathsf{CD}_{2\not\sim}(G^{\Omega_1})$, then $\bar{\Xi}_2 \in \mathsf{CD}_1(G^{\Omega_1})$.*

(iii) *We have, for all $i \in \{1, \ldots, s\}$, that $\bar{\Xi}_i \notin \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_i}) \cup \mathsf{CD}_{2\sim}(G^{\Omega_i}) \cup \mathsf{CD}_3(G^{\Omega_i})$.*

(iv) *If $\mathcal{E}$ is homogeneous, then $s = 2$, $\bar{\Xi}_i \in \mathsf{CD}_1(G^{\Omega_i})$ for $i = 1, 2$, and $(M, \{K_1, K_2\})$ is a full factorisation.*

(v) *If $s = 3$, then $\bar{\bar{\Xi}}_i \in \mathsf{CD}_1(G^{\Omega_i})$ for $i = 1$, 2, 3, and $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation.*

The general part of Theorem 3.1, apart from the bound on s, follows from the following result. The fact that s is at most three will be proved in Theorem 7.1 in Section 7, and the remaining assertions of Theorem 3.1 will be verified at the end of Section 9.

Proposition 3.2. *Let G, M, Ω, ω, $\mathcal{E}$, $\bar{\bar{\Xi}}_1, \ldots, \bar{\bar{\Xi}}_s$, $\Omega_1, \ldots, \Omega_s$, and $K_1, \ldots, K_s$ be as above. Then $\bar{\mathcal{E}} = \{\Omega_1, \ldots, \Omega_s\}$ is a G-invariant Cartesian decomposition of Ω such that the Cartesian system $\mathcal{K}_\omega(\bar{\mathcal{E}})$ coincides with $\{K_1, \ldots, K_s\}$. Moreover, for each i, Ω_i is a G-invariant partition of Ω, K_i is normalised by G_ω, $\bar{\bar{\Xi}}_i \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_i})$, and M acts faithfully on Ω_i.*

Proof. In the first two paragraphs we prove that Ω_1 is a G-invariant partition of Ω, that K_1 is the stabiliser in M of the part of Ω_1 containing ω (and hence G_ω normalises K_1), and that M is faithful on Ω_1. The proofs for the other Ω_i are identical. Suppose that $\Xi_1 = \{\Gamma_1, \ldots, \Gamma_m\}$ and let $\gamma_1 \in \Gamma_1, \ldots, \gamma_m \in \Gamma_m$ be the blocks containing ω. Then, by the definition of Ω_1, $\gamma_1 \cap \cdots \cap \gamma_m \in \Omega_1$. It follows from the definition of the infimum that Ω_1 is a partition of Ω. Let $g \in G$. Then g permutes $\Gamma_1, \ldots, \Gamma_m$ among themselves, and so $\{\gamma_1^g, \ldots, \gamma_m^g\} \cap \Gamma_i$ is a singleton for each $i \in \{1, \ldots, m\}$. Thus $(\gamma_1 \cap \cdots \cap \gamma_m)^g = \gamma_1^g \cap \cdots \cap \gamma_m^g \in \Omega_1$, and so Ω_1 is G-invariant.

Next we prove that K_1 is the stabiliser in M of the element $\gamma_1 \cap \cdots \cap \gamma_m$ in Ω_1. Now $K_1 = L_1 \cap \cdots \cap L_m$ where L_j is the stabiliser in M of γ_j. Hence K_1 stabilises $\gamma_1 \cap \cdots \cap \gamma_m$, the block in Ω_1 that contains ω. Now suppose that some element $g \in M$ stabilises $\gamma_1 \cap \cdots \cap \gamma_m$. The definition of a Cartesian system implies that $\gamma_1 \cap \cdots \cap \gamma_m$ is non-empty. As $\gamma_1, \ldots, \gamma_m$ are blocks of imprimitivity for the M-actions on $\Gamma_1, \ldots, \Gamma_m$, respectively, it follows that g fixes each of $\gamma_1, \ldots, \gamma_m$ setwise. Thus $g \in L_1 \cap \cdots \cap L_m$. Therefore K_1 is the stabiliser in M of $\gamma_1 \cap \cdots \cap \gamma_m$. As Ω_1 is a G-invariant partition of Ω, K_1 is normalised by G_ω. Moreover, since $K_1 \neq M$ it follows that $\bigcap_{g \in G} K_1^g$ is a normal subgroup of G properly contained in M. As M is a minimal normal subgroup of G, this implies that $\bigcap_{g \in G} K_1^g = 1$, and so M acts faithfully on Ω_1.

We now claim that $\bar{\mathcal{E}} = \{\Omega_1, \ldots, \Omega_s\}$ is a Cartesian decomposition of Ω. Let $\delta_1 \in \Omega_1, \ldots, \delta_s \in \Omega_s$. Because of the definition of the Ω_i, there are $\gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell$ such that $\delta_1 \cap \cdots \cap \delta_s = \gamma_1 \cap \cdots \cap \gamma_\ell$. As the Γ_i form a Cartesian decomposition of Ω, we obtain that $|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1$, and so $|\delta_1 \cap \cdots \cap \delta_s| = 1$. Thus $\bar{\mathcal{E}}$ is a Cartesian decomposition of Ω. Since each of the Ω_i is a G-invariant partition of Ω, the group G acts trivially on $\bar{\mathcal{E}}$. Since K_i is the stabiliser in M of the part of Ω_i containing ω, it follows that $\{K_1, \ldots, K_m\}$ is the Cartesian system $\mathcal{K}_\omega(\bar{\mathcal{E}})$.

Finally we prove that $\bar{\bar{\Xi}}_i \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_i})$ for each $i = 1, \ldots, s$, and, as usual, we show this for $i = 1$. Recall that $\Xi_1 = \{\Gamma_1, \ldots, \Gamma_m\}$. Suppose that $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_m$ are as above, and let $\bar{\gamma}_1 \in \bar{\Gamma}_1, \ldots, \bar{\gamma}_m \in \bar{\Gamma}_m$ corresponding to the elements $\gamma_1 \in \Gamma_1, \ldots, \gamma_m \in \Gamma_m$, respectively. Since $\gamma_1 \cap \cdots \cap \gamma_m$ is a block in Ω_1, we have $|\bar{\gamma}_1 \cap \cdots \cap \bar{\gamma}_m| = 1$. This shows that $\bar{\bar{\Xi}}_1$ is a Cartesian decomposition of Ω_1. The G-actions on Ξ_1 and $\bar{\bar{\Xi}}_1$ are naturally equivalent, and, as G is transitive on Ξ_1, we obtain that $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_1})$. $\square$

4. Factorisations of simple and characteristically simple groups

The factorisations of simple and characteristically simple groups play an important rôle in this paper, especially in the proof of Theorem 3.1. Such factorisations were studied earlier in [BP98, PS02]. In this section we summarise and extend the results proved in these papers.

A *group factorisation* is a pair $(G, \{A, B\})$ where G is a group and A, B are subgroups of G such that $AB = G$. In this situation we also say that $\{A, B\}$ is a factorisation of G, and we often write that $G = AB$ is a factorisation. A factorisation is called *non-trivial* if both A and B are proper subgroups. In this paper we only consider non-trivial factorisations.

Let $M = T_1 \times \cdots \times T_k$ be a finite, non-abelian, characteristically simple group where $T_1, \ldots, T_k$ are pairwise isomorphic, simple normal subgroups. Then a factorisation $M = K_1 K_2$ is said to be a *full factorisation* if, for each $i \in \{1, \ldots, k\}$,

(a) the subgroups $\sigma_i(K_1)$, $\sigma_i(K_2)$ are proper subgroups of T_i;
(b) the orders $|\sigma_i(K_1)|$, $|\sigma_i(K_2)|$, and $|T_i|$ are divisible by the same set of primes.

Full factorisations of simple and characteristically simple groups were classified in [BP98] and [PS02], respectively. The following result is a short summary of what we need to know about such factorisations to prove the results in this paper.

Theorem 4.1. *Suppose that $k \geqslant 1$ and $T_1, \ldots, T_k$ are pairwise isomorphic, finite, non-abelian simple groups, and set $M = T_1 \times \cdots \times T_k$. If $M = K_1 K_2$ is a full factorisation, then*

$$\sigma_1(K_j)' \times \cdots \times \sigma_k(K_j)' \leqslant K_j \quad \text{for} \quad j \in \{1, 2\}.$$

Further, for each $i \in \{1, \ldots, k\}$, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ occurs as $(T, \{A, B\})$ in one of the lines of Table 1.

Table 1. Full factorisations $\{A, B\}$ of finite simple groups T

	T	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	M_{11}, $PSL_2(11)$
3	$P\Omega_8^+(q), \; q \geqslant 3$	$\Omega_7(q)$	$\Omega_7(q)$
4	$P\Omega_8^+(2)$	$Sp_6(2)$	A_7, A_8, S_7, S_8, $Sp_6(2)$, $\mathbb{Z}_2^6 \rtimes A_7$, $\mathbb{Z}_2^6 \rtimes A_8$
		A_9	A_8, S_8, $Sp_6(2)$, $\mathbb{Z}_2^6 \rtimes A_7$, A_8, $\mathbb{Z}_2^6 \rtimes A_8$
5	$Sp_4(q), \; q \geqslant 4$ even	$Sp_2(q^2).2$	$Sp_2(q^2).2$, $Sp_2(q^2)$

An important subfamily of full factorisations consists of the factorisations of non-abelian, finite simple groups with two isomorphic subgroups. We will use the extra details about these factorisations given below.

Lemma 4.2. *Let T be a finite simple group and let A, B be proper subgroups of T such that $|A| = |B|$ and $T = AB$. Then the following hold:*

(i) *The isomorphism types of T, A, and B are as in Table 2, and A, B are isomorphic, maximal subgroups of T.*
(ii) *There is an automorphism $\vartheta \in \mathsf{Aut}\, T$ that interchanges A and B.*
(iii) *We have*

$$\mathbb{N}_T\,(A' \cap B') = \mathbb{N}_T\,(A \cap B) = A \cap B \quad \text{and} \quad \mathbb{C}_T(A' \cap B') = \mathbb{C}_T(A \cap B) = 1.$$

TABLE 2. Factorisations of finite simple groups in Lemma 4.2

	T	$A,\ B$
1	A_6	A_5
2	M_{12}	M_{11}
3	$\mathsf{P\Omega}_8^+(q)$	$\Omega_7(q)$
4	$\mathsf{Sp}_4(q),\ q \geqslant 4$ even	$\mathsf{Sp}_2(q^2).2$

Proof. Parts (i) and (ii) were proved in [BPS04, Lemma 5.2], and the same result implies that $A \cap B$ is self-normalising in T. It is not hard to see that the proof of [BPS04, Lemma 5.2] can be used, after minor alteration, to verify that $\mathbb{N}_T(A' \cap B') = A \cap B$. In particular $\mathbb{C}_T(A \cap B) = \mathsf{Z}(A \cap B)$ and $\mathbb{C}_T(A \cap B) = \mathbb{C}_{A \cap B}(A' \cap B')$. We obtain from the [Atlas85] in rows 1–2, from [Kle87, 3.1.1(vi)] in row 3, and from [LPS90, 3.2.1(d)] in row 4 of the table that $A \cap B$ is a centerless group and, in row 4, that $\mathbb{C}_{A \cap B}(A' \cap B') = 1$. $\qquad\square$

Let $M = T_1 \times \cdots \times T_k$ be a finite, non-abelian, characteristically simple group as above. For subgroups $K_1, \ldots, K_\ell$ of M, the pair $(M, \{K_1, \ldots, K_\ell\})$ is said to be a *strong multiple factorisation* if, for all $i \in \{1, \ldots, k\}$ and all pairwise distinct $j_1,\ j_2,\ j_3 \in \{1, \ldots, \ell\}$,

   (a) $\sigma_i(K_1), \ldots, \sigma_i(K_\ell)$ are proper subgroups of T_i; and
   (b) $K_{j_1}(K_{j_2} \cap K_{j_3}) = K_{j_2}(K_{j_1} \cap K_{j_3}) = K_{j_3}(K_{j_1} \cap K_{j_2}) = M$.

The following theorem, combining [BP98, Table V] and [PS02, Theorem 1.7, Corollary 1.8], gives a characterisation of strong multiple factorisations of characteristically simple groups.

Theorem 4.3. *A strong multiple factorisation of a finite characteristically simple group contains exactly three subgroups. If M is a non-abelian, characteristically simple group with simple normal subgroups $T_1, \ldots, T_k$, and $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation, then $\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i$ for $i = 1,\ 2,\ 3$, and, for $i = 1, \ldots, k$, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ occurs as $(T, \{A, B, C\})$ in one of the lines of Table 3. Further, if one of the lines 1–2 of Table 3 is valid, then $\sigma_1(K_i) \times \cdots \times \sigma_k(K_i) = K_i$ for $i = 1,\ 2,\ 3$.*

TABLE 3. Strong multiple factorisations $\{A, B, C\}$ of finite simple groups T

	T	A	B	C
1	$\mathsf{Sp}_{4a}(2),\ a \geqslant 2$	$\mathsf{Sp}_{2a}(4).2$	$\mathsf{O}_{4a}^-(2)$	$\mathsf{O}_{4a}^+(2)$
2	$\mathsf{P\Omega}_8^+(3)$	$\Omega_7(3)$	$\mathbb{Z}_3^6 \rtimes \mathsf{PSL}_4(3)$	$\mathsf{P\Omega}_8^+(2)$
3	$\mathsf{Sp}_6(2)$	$\mathsf{G}_2(2)$	$\mathsf{O}_6^-(2)$	$\mathsf{O}_6^+(2)$
		$\mathsf{G}_2(2)'$	$\mathsf{O}_6^-(2)$	$\mathsf{O}_6^+(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}_6^-(2)'$	$\mathsf{O}_6^+(2)$
		$\mathsf{G}_2(2)$	$\mathsf{O}_6^-(2)$	$\mathsf{O}_6^+(2)'$

The concept of a full strip factorisation is defined for the purposes of this paper. For the characteristically simple group $M = T_1 \times \cdots \times T_k$ and proper subgroups

D, K, the triple (M, D, K) is said to be a *full strip factorisation* if

   (i) $M = DK$;
   (ii) D is a direct product of pairwise disjoint, non-trivial full strips;
   (iii) for all i, $j \in \{1, \ldots, k\}$, $\sigma_i(K)$ is a proper subgroup of T_i and $\sigma_i(K) \cong \sigma_j(K)$.

The following lemma shows that in a full strip factorisation each full strip has length 2.

Lemma 4.4. *If (M, D, K) is a full strip factorisation of a finite, characteristically simple group M, then each non-trivial, full strip involved in D has length 2.*

Proof. Suppose without loss of generality that X is a non-trivial full strip involved in D, covering $T_1, \ldots, T_s$ for some $s \geqslant 2$. We let $I = \{T_1, \ldots, T_s\}$. Then $\sigma_I(D) = X$ and the factorisation $X\sigma_I(K) = T_1 \times \cdots \times T_s$ holds. Then [BPS06, Lemma 4.3] implies that $s \leqslant 3$, and if $s = 3$, then the simple direct factor T of M admits a strong multiple factorisation involving three subgroups isomorphic to the subgroups $\sigma_i(K)$, for $i = 1$, 2, 3. On the other hand, Table 3 shows that finite simple groups do not admit strong multiple factorisations with isomorphic subgroups. This is a contradiction, and hence $s = 2$. $\qquad\square$

The next result provides the link between the concepts of a full factorisation and a full strip factorisation.

Theorem 4.5 (Theorem 1.5 of [PS02]). *Let $M = T_1 \times \cdots \times T_{2k}$ be a characteristically simple group, where the T_i are non-abelian, simple groups, $\varphi_i : T_i \to T_{i+k}$ an isomorphism for $i = 1, \ldots, k$, and set*

$$D = \{(t_1, \ldots, t_k, \varphi_1(t_1), \ldots, \varphi_k(t_k)) \mid t_1 \in T_1, \ldots, t_k \in T_k\}.$$

If (M, D, K) is a full strip factorisation, then $(T_i, \{\sigma_i(K), \varphi_i^{-1}(\sigma_{i+k}(K))\})$ is a factorisation of T_i with isomorphic subgroups for all $i \in \{1, \ldots, k\}$, and $\prod_{i=1}^{2k} \sigma_i(K)' \leqslant K$.

The following useful result from [BP03] shows that in a non-trivial factorisation of a non-abelian characteristically simple group, it is not possible for both factors to be direct products of pairwise disjoint strips.

Lemma 4.6 ([BP03, Lemma 2.2]). *Suppose that $M = T_1 \times \cdots \times T_k$ is a direct product of isomorphic non-abelian, simple groups $T_1, \ldots, T_k$. Suppose that $A_1, \ldots, A_m$ are non-trivial pairwise disjoint strips in M, and so are $B_1, \ldots, B_n$. Then $M \neq (A_1 \times \cdots \times A_m)(B_1 \times \cdots \times B_n)$.*

5. Normalisers in direct products

In this section we collect together some facts about normalisers of subgroups in direct products that will be used in our analysis. It is easy to see that the normaliser in a direct product $G_1 \times \cdots \times G_k$ of a subgroup H is contained in

$$\mathbb{N}_{G_1}(\sigma_1(H)) \times \cdots \times \mathbb{N}_{G_k}(\sigma_k(H)).$$

Moreover, if H is the direct product $\sigma_1(H) \times \cdots \times \sigma_k(H)$, then

$$\mathbb{N}_{G_1 \times \cdots \times G_k}(H) = \mathbb{N}_{G_1}(\sigma_1(H)) \times \cdots \times \mathbb{N}_{G_k}(\sigma_k(H)).$$

The following simple lemma extends this observation to a more general situation.

Lemma 5.1. *Let $G_1, \ldots, G_k$ be groups, set $G = G_1 \times \cdots \times G_k$ and for $i = 1, \ldots, k$ let H_i be a subgroup of G_i. Let H be a subgroup of G such that $H_1 \times \cdots \times H_k \lhd H$, the factor $\mathbb{N}_G (H_1 \times \cdots \times H_k) / (H_1 \times \cdots \times H_k)$ is abelian, and $\mathbb{N}_{G_i} (\sigma_i(H)) = \mathbb{N}_{G_i} (H_i)$. Then*

$$\mathbb{N}_G (H) = \mathbb{N}_G (H_1 \times \cdots \times H_k) = \mathbb{N}_{G_1} (H_1) \times \cdots \times \mathbb{N}_{G_k} (H_k).$$

Proof. As $\mathbb{N}_{G_i} (\sigma_i(H)) = \mathbb{N}_{G_i} (H_i)$, it follows from the remarks above that

$$\mathbb{N}_G (H) \leqslant \prod_{i=1}^{k} \mathbb{N}_{G_i} (H_i) = \mathbb{N}_G (H_1 \times \cdots \times H_k).$$

On the other hand, as $H_1 \times \cdots \times H_k \lhd H$ and $\mathbb{N}_G (H_1 \times \cdots \times H_k) / (H_1 \times \cdots \times H_k)$ is abelian, $\mathbb{N}_G (H_1 \times \cdots \times H_k) \leqslant \mathbb{N}_G (H)$. Therefore equality holds. $\square$

The following lemma, proved in [PS07, Lemma 3.5], determines the normaliser of a strip.

Lemma 5.2. *Let $G_1, \ldots, G_k$ be isomorphic groups, let $\varphi_i : G_1 \to G_i$ be an isomorphism for $i = 2, \ldots, k$, let H_1 be a subgroup of G_1, and let $H = \{(h, \varphi_2(h), \ldots, \varphi_k(h)) \mid t \in H_1\}$ be a non-trivial strip in $G_1 \times \cdots \times G_k$. Then*

$$\mathbb{N}_{G_1 \times \cdots \times G_k} (H) = \{(t, c_2\varphi_2(t), \ldots, c_k\varphi_k(t)) \mid t \in \mathbb{N}_{G_1} (H_1), \ c_i \in \mathbb{C}_{G_i}(\varphi_i(H_1))\}.$$

We use the results above to derive some facts concerning the normalisers of the subgroups that occur in Table 4.

Proposition 5.3. *Suppose that $M = T_1 \times \cdots \times T_k \cong T^k$ is a characteristically simple group and $(M, \{K_1, K_2\})$ is a full factorisation such that, for all i, the pair $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is as $(T, \{A, B\})$ in one of the rows of Table 4.*

    (a) *If T is as in one of rows 1–3 of Table 4, then K_1, K_2, and $K_1 \cap K_2$ are self-normalising in M.*

    (b) *If row 4 of Table 4 is valid, then, for $j = 1$, 2, we have $\mathbb{N}_M (K_j) = \prod_i \sigma_i(K_j)$ and $\mathbb{N}_M (K_1 \cap K_2) = \mathbb{N}_M (K_1) \cap \mathbb{N}_M (K_2)$.*

Proof. (a) In this case the $\sigma_i(K_j)$ are perfect and, by Theorem 4.1, $K_j = \prod_i \sigma_i(K_j)$ for $j = 1$, 2, and Table 4 shows that $\sigma_i(K_j)$ is self-normalising in T_i for all $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Therefore K_1 and K_2 are self-normalising. Further,

$$K_1 \cap K_2 = \prod_{i=1}^{k} \sigma_i(K_1 \cap K_2) = \prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2).$$

Using Lemma 4.2 and the Atlas [Atlas85], we obtain that $\mathbb{N}_{T_i} (\sigma_i(K_1) \cap \sigma_i(K_2)) = \sigma_i(K_1) \cap \sigma_i(K_2)$ for all i. Thus

$$\mathbb{N}_M (K_1 \cap K_2) = \mathbb{N}_M \left(\prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2) \right) = \prod_{i=1}^{k} \mathbb{N}_{T_i} (\sigma_i(K_1) \cap \sigma_i(K_2))$$

$$= \prod_{i=1}^{k} \sigma_i(K_1) \cap \sigma_i(K_2) = \prod_{i=1}^{k} \sigma_i(K_1 \cap K_2) = K_1 \cap K_2.$$

    (b) Now assume that $T \cong \mathsf{Sp}_4(q)$ for some $q \geqslant 4$ even and let $j \in \{1, 2\}$. By Theorem 4.1, $K_j' = \prod_i \sigma_i(K_j)'$, and we can read from Table 4 that $\mathbb{N}_{T_i} (\sigma_i(K_j)') = \mathbb{N}_{T_i} (\sigma_i(K_j)) = \sigma_i(K_j)$ for all $i \in \{1, \ldots, k\}$. As $\mathbb{N}_M (K_j') / K_j'$ is elementary abelian

and $\mathbb{N}_M\left(K'_j\right) \geqslant K_j \geqslant K'_j$, Lemma 5.1 gives $\mathbb{N}_M\left(K'_j\right) = \mathbb{N}_M\left(K_j\right)$. On the other hand,

$$\mathbb{N}_M\left(K'_j\right) = \prod_{i=1}^{k} \mathbb{N}_{T_i}\left(\sigma_i(K_j)'\right) = \prod_{i=1}^{k} \mathbb{N}_{T_i}\left(\sigma_i(K_j)\right) = \prod_{i=1}^{k} \sigma_i(K_j).$$

Now Theorem 4.1 shows that $K'_1 \cap K'_2 = \prod_i \sigma_i(K'_1 \cap K'_2)$. We also obtain from Theorem 4.1 and Lemma 4.2 that

$$\mathbb{N}_{T_i}\left(\sigma_i(K_1 \cap K_2)\right) = \mathbb{N}_{T_i}\left(\sigma_i(K'_1 \cap K'_2)\right) = \mathbb{N}_{T_i}\left(\sigma_i(K_1)\right) \cap \mathbb{N}_{T_i}\left(\sigma_i(K_2)\right).$$

Thus

$$\mathbb{N}_M\left(K'_1 \cap K'_2\right) = \prod_{i=1}^{k} \mathbb{N}_{T_i}\left(\sigma_i(K'_1 \cap K'_2)\right) = \prod_{i=1}^{k}\left(\mathbb{N}_{T_i}\left(\sigma_i(K_1)\right) \cap \mathbb{N}_{T_i}\left(\sigma_i(K_2)\right)\right)$$

$$= \prod_{i=1}^{k} \mathbb{N}_{T_i}\left(\sigma_i(K_1)\right) \cap \prod_{i=1}^{k} \mathbb{N}_{T_i}\left(\sigma_i(K_2)\right) = \mathbb{N}_M\left(K_1\right) \cap \mathbb{N}_M\left(K_2\right).$$

As $\mathbb{N}_M\left(K'_1 \cap K'_2\right)/(K'_1 \cap K'_2)$ is abelian, and $K'_1 \cap K'_2 \leqslant K_1 \cap K_2 \leqslant \mathbb{N}_M\left(K'_1 \cap K'_2\right)$, Lemma 5.1 implies that $\mathbb{N}_M\left(K_1 \cap K_2\right) = \mathbb{N}_M\left(K'_1 \cap K'_2\right) = \mathbb{N}_M\left(K_1\right) \cap \mathbb{N}_M\left(K_2\right)$.   □

Proposition 5.4. *Let $M = T_1 \times \cdots \times T_{2k} = T^{2k}$, D and K be as in Theorem 4.5, and suppose that $DK = M$.*

  (a) *If T is as in one of the rows 1–3 of Table 2, then K and $K \cap D$ are self-normalising in M.*
  (b) *If T is as in row 4 of Table 2, then $\mathbb{N}_M\left(K\right) = \prod_i \sigma_i(K)$ and $\mathbb{N}_M\left(K \cap D\right) = D \cap \mathbb{N}_M\left(K\right)$.*

Proof. (a) Theorem 4.5 implies that K is the direct product of its projections onto the T_i, and by Table 2 these projections are self-normalising in T. Hence K is self-normalising. Suppose that $X = \{(t, \varphi_1(t)) \mid t \in T_1\}$ is a full strip involved in D where $\varphi_1 : T_1 \to T_{k+1}$ is an isomorphism. Then $\sigma_{\{T_1, T_{k+1}\}}(K \cap D) = \{(t, \varphi_1(t)) \mid t \in \sigma_1(K) \cap \varphi_1^{-1}(\sigma_{k+1}(K))\}$. By Theorem 4.5, $\sigma_1(K)\varphi_1^{-1}(\sigma_{k+1}(K)) = T_1$ is a factorisation with isomorphic subgroups. Hence Lemma 4.2 implies that $\sigma_1(K) \cap \varphi_1^{-1}(\sigma_{k+1}(K))$ is self-normalising in T_1 and that $\mathbb{C}_{T_1}(\sigma_1(K) \cap \varphi_1^{-1}(\sigma_{k+1}(K))) = 1$. Thus Lemma 5.2 yields that $\sigma_{\{T_1, T_{k+1}\}}(K \cap D)$ is self-normalising in $T_1 \times T_{k+1}$. A similar argument shows that $\sigma_{\{T_i, T_{i+k}\}}(K \cap D)$ is self-normalising in $T_i \times T_{i+k}$ for all $i \in \{1, \ldots, k\}$. As $K \cap D = \sigma_{\{T_1, T_{k+1}\}}(K \cap D) \times \cdots \times \sigma_{\{T_k, T_{2k}\}}(K \cap D)$ we obtain that $K \cap D$ is also self-normalising in M.

(b) The argument which was used in part (b) of Proposition 5.3 to compute the normalisers of K_1 and K_2 can be used to verify the claim about $\mathbb{N}_M\left(K\right)$. Using the argument in part (a), it is easy to check that $\mathbb{N}_M\left(D \cap K'\right) = D \cap \mathbb{N}_M\left(K\right)$. Since $\mathbb{N}_M\left(D \cap K'\right)/(D \cap K')$ is abelian, $\mathbb{N}_M\left(D \cap K'\right) \geqslant D \cap K \geqslant D \cap K'$, and $\mathbb{N}_{T_i \times T_{i+k}}\left(\sigma_{\{T_i, T_{i+k}\}}(D \cap K)\right) = \mathbb{N}_{T_i \times T_{i+k}}\left(\sigma_{\{T_i, T_{i+k}\}}(D \cap K')\right)$, we obtain from Lemma 5.1 that $\mathbb{N}_M\left(D \cap K'\right) = \mathbb{N}_M\left(D \cap K\right)$.   □

6. Cartesian systems involving non-trivial strips

We use the notation introduced in Section 2. Let us start with a motivating example.

Example 6.1. Let T be a finite simple group with two proper, isomorphic subgroups A and B, such that $T = AB$. The possibilities for T, A, and B are in Table 2. Suppose that k is even and set $M = T^k$. Let τ be an element of $\mathsf{Aut}\,T$ interchanging A and B; such a τ exists by Lemma 4.2. Consider the following two subgroups of M:

$$\bar{K}_1 = \{(t_1, t_1, \ldots, t_{k/2}, t_{k/2}) \mid t_1, \ldots, t_{k/2} \in T\} \quad \text{and} \quad \bar{K}_2 = (A \times B)^{k/2}.$$

We obtain from Theorem 4.5 that $M = K_1 K_2$. Identify M with $\mathsf{Inn}\,M$ in $\mathsf{Aut}\,M \cong \mathsf{Aut}\,T\,\mathsf{wr}\,\mathsf{S}_k$, and set

$$\bar{G} = M\left(\mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_2\right)\right).$$

We claim that M is a minimal normal subgroup of $\bar{G}$, or equivalently, $\bar{G}$ is transitive by conjugation on the simple direct factors of M. Note that $\sigma_1(\bar{K}_2) = A$ and $\sigma_2(\bar{K}_2) = B$, and the automorphism $(\tau, \tau, 1, \ldots, 1)(1, 2)$ of M interchanges the first two simple direct factors of M, while normalising $\bar{K}_1$ and $\bar{K}_2$. Also the automorphism $(1, 3, \ldots, k-1)(2, 4, \ldots, k)$ of M cyclically permutes the blocks determined by the strips in $\bar{K}_1$, and normalises both $\bar{K}_1$ and $\bar{K}_2$. Therefore the subgroup of $\mathsf{Aut}\,M$ generated by these two outer automorphisms is transitive on the simple direct factors of M, and, in addition, normalises $\bar{K}_1$ and $\bar{K}_2$. Hence M is a minimal normal subgroup of $\bar{G}$ and, since $\mathbb{C}_{\mathsf{Aut}\,M}(M) = 1$, it is the unique such minimal normal subgroup.

Set $G_0 = \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_2\right)$ so that $MG_0 = \bar{G}$. As $\bar{K}_1$ and $\bar{K}_2$ are self-normalising in M, we obtain $M \cap G_0 = \bar{K}_1 \cap \bar{K}_2$. Therefore, by [PS07, Lemma 4.1], the M-action on the coset space $[M : \bar{K}_1 \cap \bar{K}_2]$ can be extended to $\bar{G}$ with point stabiliser G_0. Moreover, $\bar{K}_1$ and $\bar{K}_2$ form a Cartesian system for M acted upon trivially by G_0. Consequently this action of $\bar{G}$ preserves an intransitive $\bar{G}$-invariant Cartesian decomposition, such that one of the subgroups, namely $\bar{K}_1$, in the corresponding Cartesian system $\{\bar{K}_1, \bar{K}_2\}$ is the direct product of disjoint strips.

Our aim in this section is to describe the intransitive, pointwise $\bar{G}$-invariant Cartesian decompositions whose Cartesian systems involve a non-trivial full strip. If K_i involves such a strip for some i, then $\sigma_j(K_i) = T_j$ for some $j \in \{1, \ldots, k\}$. Without loss of generality we may suppose that $1 = i = j$. In this case we obtain the following theorem.

Theorem 6.2. *Let G, M, and $\mathcal{K}$ be as in Section 3, and assume that $\sigma_1(K_1) = T_1$. Then $s = 2$ and (M, K_1, K_2) is a full strip factorisation. In particular, the isomorphism types of T and $\sigma_i(K_2)$ are as in Table 2. Further, $K_2' = \sigma_1(K_2)' \times \cdots \times \sigma_k(K_2)'$, and if T is not as in row 4 of Table 2, then $K_2 = \sigma_1(K_2) \times \cdots \times \sigma_k(K_2)$.*

Proof. Since G_ω normalises K_1 and acts transitively on the T_i, we have that $\sigma_i(K_1) = T_i$ for all i. If $T_i \leqslant K_1$ for some i, then, for the same reason, $T_i \leqslant K_1$ for all i and so $K_1 = M$, which is impossible. Hence, by Scott's Lemma 2.2, K_1 is a direct product of non-trivial full strips. If $\sigma_i(K_j) = T_i$ for some i and some $j \neq 1$, then the same argument shows that K_j is also a direct product of non-trivial, full strips. However, Lemma 4.6 implies that $K_1 K_j \neq M$, which violates the defining properties of Cartesian systems. Thus $\sigma_i(K_j)$ is a proper subgroup of T_i for all i and all $j \geqslant 2$.

Since G_ω normalises K_2, $\sigma_i(K_2) \cong \sigma_j(K_2)$ for all i and j. Thus (M, K_1, K_2) is a full strip factorisation. Lemma 4.4 implies that all strips involved in K_1 have length 2.

We now show that $s = 2$. Suppose on the contrary that $s \geqslant 3$. Let X be a strip in K_1 whose support is, without loss of generality, $\{T_1, T_2\}$. Then $X = \{(t, \alpha(t)) \mid t \in T_1\}$ for some isomorphism $\alpha : T_1 \to T_2$. For $i \geqslant 2$, $\sigma_1(K_i)$ and $\sigma_2(K_i)$ are proper subgroups of T_1 and T_2, respectively, and it follows from Theorem 4.5 that $(T_1, \{\sigma_1(K_i), \alpha^{-1}(\sigma_2(K_i))\})$ is a factorisation with isomorphic subgroups. As K_2 and K_3 are normalised by G_ω, so is their intersection $K_2 \cap K_3$. Hence

$$\sigma_1(K_2 \cap K_3) \cong \sigma_2(K_2 \cap K_3).$$

Since $K_1(K_2 \cap K_3) = M$ and $\sigma_{\{T_1, T_2\}}(K_1)$ is the full strip X, we obtain from [PS02, Lemma 2.1] that $(T_1, \{\sigma_1(K_2 \cap K_3), \alpha^{-1}(\sigma_2(K_2 \cap K_3))\})$ is also a full factorisation with isomorphic subgroups. In such factorisations the subgroups involved are maximal subgroups of T_1 (see Table 2), and so $\sigma_1(K_2 \cap K_3)$ and $\alpha^{-1}(\sigma_2(K_2 \cap K_3))$ are maximal subgroups of T_1. However, $\sigma_1(K_2 \cap K_3) \leqslant \sigma_1(K_2) \cap \sigma_1(K_3)$, which, as $\sigma_1(K_2)$ and $\sigma_1(K_3)$ are proper subgroups of T_1, implies that $\sigma_1(K_2 \cap K_3)$, $\sigma_1(K_2)$, and $\sigma_1(K_3)$ coincide. Hence $\sigma_1(K_2 K_3) = \sigma_1(K_2)\sigma_1(K_3) < T_1$, which is a contradiction, as $K_2 K_3 = M$. Thus $s = 2$. The rest of the theorem follows from Theorem 4.5 and from the fact that the subgroups A and B in rows 1–3 of Table 2 are perfect. □

If K_1 is a subdirect subgroup of M, then we prove that $\mathbb{C}_G(M)$ is small, in fact, in most cases $\mathbb{C}_G(M) = 1$ and G is quasiprimitive. If G is a permutation group with a unique minimal normal subgroup M, such that M is transitive, then G is quasiprimitive. Moreover if M is not simple, a point stabiliser in M is non-trivial and is not a subdirect subgroup of M, then G is said to have quasiprimitive type PA; see [BP03].

Proposition 6.3. *Let G, M, and $\mathcal{K}$ be as in Section 3. Assume that $\sigma_1(K_1) = T_1$. If the group T is as in rows 1–3 of Table 2, then $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) = 1$, and in particular G is quasiprimitive of type PA. If T is as in row 4, then $\mathbb{N}_M(M_\omega) = K_1 \cap \mathbb{N}_M(K_2)$, and*

$$\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) \cong (K_1 \cap \mathbb{N}_M(K_2))/(K_1 \cap K_2) \cong \mathbb{N}_M(K_2)/K_2.$$

In particular $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M)$ is an elementary abelian 2-group of rank at most $k/2$, and all minimal normal subgroups of G different from M are elementary abelian 2-groups.

Proof. By Theorem 6.2, $s = 2$, and so, $M_\omega = K_1 \cap K_2$. Note that by [DM96, Theorem 4.2A]

$$\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) \cong \mathbb{N}_M(M_\omega)/M_\omega = \mathbb{N}_M(K_1 \cap K_2)/(K_1 \cap K_2).$$

If T is as in one of the rows 1–3 of Table 2, then Proposition 5.4 implies that $K_1 \cap K_2$ is self-normalising in M, and hence $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) = 1$. This implies that M is the unique minimal normal subgroup in G, and so G is quasiprimitive. As K_1 involves a non-trivial full strip, $k \geqslant 2$. Moreover, it follows from Table 2 that $M_\omega \neq 1$ and M_ω is not a subdirect subgroup of M. Thus G has quasiprimitive type PA. If T is as in row 5, then, again by Proposition 5.4, we only have to prove that $(K_1 \cap \mathbb{N}_M(K_2))/(K_1 \cap K_2)$ and $\mathbb{N}_M(K_2)/K_2$ are isomorphic.

Recall that $K_1 K_2 = M$, and so $\mathbb{N}_M(K_2) = (K_1 \cap \mathbb{N}_M(K_2))K_2$. By the second isomorphism theorem, $\mathbb{N}_M(K_2)/K_2 \cong (K_1 \cap \mathbb{N}_M(K_2))/(K_1 \cap K_2)$ under the isomorphism $\psi(xK_2) = x'(K_1 \cap K_2)$ where $x = x'x'' \in \mathbb{N}_M(K_2)$ with $x' \in K_1 \cap \mathbb{N}_M(K_2)$, $x'' \in K_2$. A proof that ψ is well-defined and is an isomorphism can be found in most group theory textbooks; see for instance [Hup67, 3.12 Satz]. □

7. Bounding the number of orbits
in an intransitive Cartesian system

We apply the results of the last section to prove that $s \leqslant 3$.

Theorem 7.1. *The index s of the Cartesian system $\mathcal{K}$ in Theorem 3.1 is at most 3. Further, if $s = 3$, then $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation. Hence, in this case, for all i, the factorisation $(T_i, \{\sigma_i(K_1), \sigma_i(K_2), \sigma_i(K_3)\})$ is in Table 3. Moreover if T is as in the first two rows, then*

$$K_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i) \quad for \quad i = 1, \, 2, \, 3,$$

while if T is as in the third row, then

$$\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i \leqslant \sigma_1(K_i) \times \cdots \times \sigma_k(K_i) \quad for \quad i = 1, \, 2, \, 3.$$

Proof. If $\sigma_i(K_j) = T_i$ for some i and j, then, by Theorem 6.2, we have $s = 2$. Therefore we may assume without loss of generality that all projections $\sigma_i(K_j)$ are proper in T_i. Then $K_1, \ldots, K_s$ form a strong multiple factorisation of M. Thus, by Theorem 4.3, $s \leqslant 3$.

If $s = 3$, then, by Theorem 6.2, $\sigma_i(K_j) < T_i$ for all i and j. Thus, by (2), $\{K_1, K_2, K_3\}$ is a strong multiple factorisation of M, and T_i, $\sigma_i(K_1)$, $\sigma_i(K_2)$, $\sigma_i(K_3)$ are as in Table 3. The assertions about the K_i follow from Theorem 4.3. □

A generic example with $s = 3$ can easily be constructed as follows.

Example 7.2. Let A, B, C be maximal subgroups of a finite simple group T forming a strong multiple factorisation of T, and let $\bar{K}_1 = A^k$, $\bar{K}_2 = B^k$, $\bar{K}_3 = C^k$ be the corresponding subgroups of $M = T^k$. Then $(T, \{A, B, C\})$ and $(M, \{\bar{K}_1, \bar{K}_2, \bar{K}_3\})$ are strong multiple factorisations. Identify M with $\mathsf{Inn}\, M$ in $\mathsf{Aut}\, M$, and let

$$\bar{G} = M \left(\mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_2\right) \cap \mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_3\right) \right).$$

Since the cyclic subgroup of $\mathsf{Aut}\, M$ generated by the automorphism

$$\tau : (x_1, \ldots, x_k) \mapsto (x_k, x_1, \ldots, x_{k-1})$$

is transitive on the simple direct factors of M and normalises $\bar{K}_1$, $\bar{K}_2$, and $\bar{K}_3$, we have that M is a minimal normal subgroup of $\bar{G}$. Moreover, since $\mathbb{C}_{\mathsf{Aut}\, M}(M) = 1$, M is the unique minimal normal subgroup of $\bar{G}$.

If $G_0 = \mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_2\right) \cap \mathbb{N}_{\mathsf{Aut}\, M}\left(\bar{K}_3\right)$, then $MG_0 = \bar{G}$ and, since $\bar{K}_1$, $\bar{K}_2$, and $\bar{K}_3$ are self-normalising in M, $M \cap G_0 = \bar{K}_1 \cap \bar{K}_2 \cap \bar{K}_3$. Therefore, by [PS07, Lemma 4.1], the M-action on the coset space $[M : \bar{K}_1 \cap \bar{K}_2 \cap \bar{K}_3]$ can be extended to $\bar{G}$ with point stabiliser G_0. Moreover, $\{\bar{K}_1, \bar{K}_2, \bar{K}_3\}$ is a Cartesian system for M acted upon trivially by G_0. Consequently this action of $\bar{G}$ preserves an intransitive $\bar{G}$-invariant Cartesian decomposition given by the Cartesian system $\{\bar{K}_1, \bar{K}_2, \bar{K}_3\}$.

The defining properties of $\mathcal{K}$ give us some useful constraints on T. For instance if the K_i involve no non-trivial, full strips, then $T_i = \sigma_i(K_j)\sigma_i(K_m)$ for all i, j, m such that $j \neq m$. In particular T has a proper factorisation, and so, for example, $T \not\cong \mathsf{PSU}_{2d+1}(q)$ unless $(d, q) \in \{(1, 3), (1, 5), (4, 2)\}$. Many sporadic simple groups can also be excluded. See the tables in [LPS90].

In general it is difficult to give a complete description of Cartesian decompositions that involve no strips. However we can give such a description when the initial intransitive Cartesian decomposition $\mathcal{E}$ is homogeneous, This leads to the proof of

Theorem 1.1. Describing the remaining case would be more difficult than finding all factorisations of finite simple groups, as demonstrated by the following generic example.

Example 7.3. Let T be a finite simple group, $k \geqslant 1$, and set $M = T^k$. Let $\{A, B\}$ be a non-trivial factorisation of the group T and set $K_1 = A^k$, $K_2 = B^k$. Then clearly $K_1 K_2 = T^k$, and the base group T^k is the unique minimal normal subgroup of $G = T \operatorname{wr} \mathsf{S}_k$. Consider the coset action of G on $\Omega = [G : G_0]$ where $G_0 = (A \cap B) \operatorname{wr} \mathsf{S}_k$. Then $K_1 \cap K_2 = (A \cap B)^k = M \cap G_0$, and K_1, K_2 are normalised by G_0, so they give rise to a G-invariant intransitive Cartesian decomposition of Ω with index 2.

The example above shows that a detailed description of all Cartesian decompositions preserved by an innately transitive group would first require determining all factorisations of finite simple groups. But even assuming that such a classification is available, determining the relevant factorisations of characteristically simple groups is still a difficult task. In the cases that we investigate in the remainder of this paper the required factorisations of the T_i were readily available. The subgroups of these factorisations were almost simple or perfect, which made possible an explicit description of the occurring factorisations of M.

8. Intransitive homogeneous Cartesian decompositions

The aim of this section is to describe homogeneous, intransitive Cartesian decompositions preserved by an innately transitive group. Such Cartesian decompositions need to be studied if we want to investigate embeddings of innately transitive groups in wreath products in product action. First we note, using the notation introduced for Theorem 3.1, that $|\Gamma_i| = |\Gamma_j|$ for all i, j. Then, for each $i \in \{1, \ldots, \ell\}$, $m = |\Gamma_i|$ (independent of i), and there is an integer ℓ_i such that $|\Omega_i| = |M : K_i| = |\Gamma_1|^{\ell_i} = m^{\ell_i}$ for all $i \in \{1, \ldots, s\}$.

Theorem 8.1. *Let G, M, $\mathcal{E}$, and $\mathcal{K}$ be as in Section 3. Assuming that $\mathcal{E}$ is homogeneous, we have $\sigma_i(K_j) < T_i$ for all i and j. Further, in this case, $s = 2$ and $(M, \{K_1, K_2\})$ is a full factorisation.*

Proof. Let us first prove that $\sigma_i(K_j) < T_i$ for all i and j. Suppose without loss of generality that $\sigma_1(K_1) = T_1$. Then Theorem 6.2 implies that $s = 2$, K_1 is the direct product of strips of length 2, and $K_2' = \sigma_1(K_2)' \times \cdots \times \sigma_k(K_2)' \leqslant K_2$. Recall that there exist non-negative integers m, ℓ_1, ℓ_2 such that $[M : K_1] = m^{\ell_1}$ and $[M : K_2] = m^{\ell_2}$. Since $|K_1| \cong |T|^{k/2}$ we have $[M : K_1] = |T|^{k/2}$, and so all primes that divide $|T|$ will also divide m. Since $K_2' \leqslant K_2$ and K_2' is the direct product of its projections $\sigma_i(K_2)'$, it follows that $|M : K_2'| = |T_1 : \sigma_1(K_2)'|^k$, and so all prime divisors p of $|T|$ divide $|T_1 : \sigma_1(K_2)'|$. This is not the case if $T \cong \mathsf{A}_6$ or $T \cong \mathsf{M}_{12}$ (take $p = 5$ in both cases). If $T \cong \mathsf{P\Omega}_8^+(q)$ and $\sigma_1(K_2) \cong \Omega_7(q)$, then

$$|T| = \frac{1}{d^2} q^{12}(q^6 - 1)(q^4 - 1)^2(q^2 - 1) \quad \text{and} \quad |T_1 : \sigma_1(K_2)'| = \frac{1}{d} q^3(q^4 - 1)$$

where $d = (2, q - 1)$. By Zsigmondy's Theorem (see [LPS90, §2.4]), there exists a prime r dividing $q^6 - 1$ and not dividing $q^4 - 1$, whence r divides $|T|$ but not $|T_1 : \sigma_1(K_2)'|$. When $T \cong \mathsf{Sp}_4(q)$ with q even, $q \geqslant 4$, then $\sigma_1(K_2)' \cong \mathsf{Sp}_2(q^2)$, so

$$|T| = q^4(q^4 - 1)(q^2 - 1) \quad \text{and} \quad |T_1 : \sigma_1(K_2)'| = q^2(q^2 - 1).$$

Using Zsigmondy's theorem we find that $q^4 - 1$ has a prime divisor r that does not divide $q^2 - 1$. Thus r divides $|T|$ but not $|T_1 : \sigma_1(K_2)'|$. Therefore $\sigma_i(K_j) < T_i$ for all i and j.

For all distinct i, $j \in \{1, \ldots, s\}$ we have $M = K_i K_j$, and hence $m^{\ell_i} = |M : K_i| = |K_j : K_i \cap K_j|$ divides $|K_j|$. It follows that every prime divisor of m divides $|K_j|$. Let p be a prime divisor or $|T|$. Then p divides $|M|$. Since $|M : K_j| = m^{\ell_j}$, either p divides $|K_j|$ or p divides m, and in the latter case we also obtain that p divides $|K_j|$. By Proposition 3.2, G_ω normalises K_j, and since $G = MG_\omega$, G_ω acts transitively by conjugation on $\{T_1, \ldots, T_k\}$. It follows that, for $1 \leqslant i \leqslant k$, the projections $\sigma_i(K_j)$ are pairwise isomorphic, proper subgroups of T_i. Thus, since p divides $|K_j|$, we deduce that p divides $\sigma_i(K_j)$, for each i. Hence each prime divisor of $|T|$ divides $|\sigma_i(K_j)|$ for all i and j. Set $Q_j = \sigma_1(K_j)$ for $j = 1, \ldots, s$.

If $s \geqslant 3$, then, since $\mathcal{K}$ is a Cartesian system, $(T_1, \{Q_1, \ldots, Q_s\})$ is a strong multiple factorisation (see the paragraph before Theorem 4.3). Moreover, since $|T|$, $|Q_i|$, $|Q_j|$ are divisible by the same primes, $(T_1, \{Q_i, Q_j\})$ is a full factorisation for all $i \neq j$. Comparing Tables 1 and 3, we find that no strong multiple factorisation of a finite simple group exists in which any two of the subgroups form a full factorisation. Hence we obtain that $s = 2$ and $(M, \{K_1, K_2\})$ is a full factorisation. $\square$

TABLE 4. The table for Theorem 8.2

	T	A	B
1	A_6	A_5	A_5
2	M_{12}	M_{11}	M_{11}, $\mathsf{PSL}_2(11)$
3	$\mathsf{P\Omega}_8^+(q)$	$\Omega_7(q)$	$\Omega_7(q)$
4	$\mathsf{Sp}_4(q)$, $q \geqslant 4$ and q even	$\mathsf{Sp}_4(q^2).2$	$\mathsf{Sp}_2(q^2).2$

Theorem 8.2. *Let G, M, $T_1, \ldots, T_k$, $\mathcal{E}$, and $\mathcal{K}$ be as in Section 3. If $\mathcal{E}$ is homogeneous, then, for all $i \in \{1, \ldots, k\}$, $(T_i, \{\sigma_i(K_1), \sigma_i(K_2)\})$ is a factorisation $(T, \{A, B\})$ as in one of the lines of Table 4. If T is as in rows 1–3, then $K_i = \sigma_1(K_i) \times \cdots \times \sigma_k(K_i)$ for $i = 1$, 2. Moreover in this case $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) = 1$, and in particular G is quasiprimitive. If T is as in row 4, then*
(5)
$$\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' \leqslant K_i \leqslant \sigma_1(K_i) \times \cdots \times \sigma_k(K_i) = \mathbb{N}_M(K_i) \quad for \quad i = 1, \; 2,$$
and
$$\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) \cong (\mathbb{N}_M(K_1) \cap \mathbb{N}_M(K_2))/(K_1 \cap K_2).$$
In particular $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M)$ is an elementary abelian 2-group of rank at most k, and all minimal normal subgroups of G different from M are elementary abelian 2-groups.

Proof. Set $A = \sigma_1(K_1)$ and $B = \sigma_1(K_2)$, so that, by Theorem 8.1, $T_1 = AB$ is a full factorisation. We have to eliminate all full factorisations of T which are not contained in Table 4. These involve the group $T = \mathsf{Sp}_4(q)$ or $\mathsf{P\Omega}_8^+(2)$, and we consider these families separately.

Suppose first that $T \cong \mathsf{Sp}_4(q)$ with q even, $q \geqslant 4$. If A and B are isomorphic to $\mathsf{Sp}_2(q^2).2$, then line 4 of Table 4 is valid. Suppose that A, say, is isomorphic

to $\mathsf{Sp}_2(q^2)$. Then K_1 is isomorphic to $A^k \cong (\mathsf{Sp}_2(q^2))^k$. As the factorisation $K_1 K_2 = M$ holds, we must have, for all i, that $\sigma_i(K_2) \cong \mathsf{Sp}_2(q^2) \cdot 2$, and hence $|K_1| < |K_2|$. For a positive integer n and a prime p let n_p denote the exponent of the largest p-power dividing n. Recall that there is an integer m such that $|M : K_i| = m^{\ell_i}$ for $i = 1,\ 2$. For any odd prime p we have $|M : K_1|_p = |M : K_2|_p$, which implies that $\ell_1 = \ell_2$ and so $|K_1| = |K_2|$, a contradiction.

Suppose now that $T \cong \mathsf{P\Omega}_8^+(2)$. By Theorem 4.1 $\sigma_1(K_i)' \times \cdots \times \sigma_k(K_i)' = K_i'$. We read from Table 1 that in every case $|K_i : K_i'|$ is a 2-power, and $|T_i : \sigma_i(K_j)|_5 = 1$. Therefore $|M : K_i|_5 = k$ for $i = 1,\ 2$, and so $\ell_1 = \ell_2$. This forces $|A| = |B|$, and inspection of Table 1 yields that $A \cong B \cong \mathsf{Sp}_6(2) \cong \Omega_7(2)$. Therefore, line 3 of Table 4 holds with $q = 2$.

Suppose that one of rows 1–3 of Table 4 is valid. The groups A and B in these rows are perfect, and so we only have to show that $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M) = 1$. By [DM96, Theorem 4.2A],

$$(6) \qquad \mathbb{C}_{\mathsf{Sym}\,\Omega}(M) \cong \mathbb{N}_M\left(M_\omega\right)/M_\omega = \mathbb{N}_M\left(K_1 \cap K_2\right)/(K_1 \cap K_2).$$

It follows, however, from Proposition 5.3 that in this case $K_1 \cap K_2$ is self-normalising in M, and so M is the unique minimal normal subgroup of G. Thus G is quasiprimitive. Suppose now that row 4 of Table 4 is valid. Then (5) follows from Theorem 4.1 and Proposition 5.3. By Proposition 5.3

$$\mathbb{N}_M\left(K_1 \cap K_2\right) = \mathbb{N}_M\left(K_1\right) \cap \mathbb{N}_M\left(K_2\right).$$

As $(\mathbb{N}_M\left(K_1\right) \cap \mathbb{N}_M\left(K_2\right))/(K_1' \cap K_2')$ is an elementary abelian group of order 2^k, by (6), so is $\mathbb{C}_{\mathsf{Sym}\,\Omega}(M)$, and so all minimal normal subgroups of G different from M are also elementary abelian groups of order at most 2^k. $\qquad\qquad\square$

Finally in this section we show how to construct examples.

Example 8.3. Let T be a finite simple group with a non-trivial factorisation $T = AB$, where T, A, and B are as in Table 4. Set $\bar{K}_1 = A^k$ and $\bar{K}_2 = B^k$. Identify M with $\mathsf{Inn}\,M$ in $\mathsf{Aut}\,M$, and let

$$\bar{G} = M\left(\mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_2\right)\right).$$

Since the cyclic subgroup of $\mathsf{Aut}\,M$ generated by the automorphism

$$\tau : (x_1, \ldots, x_k) \mapsto (x_k, x_1, \ldots, x_{k-1})$$

is transitive on the set of simple direct factors of M and normalises $\bar{K}_1$, $\bar{K}_2$, we have that M is a minimal normal subgroup of $\bar{G}$. Moreover, since $\mathbb{C}_{\mathsf{Aut}\,M}(M) = 1$, we have that M is the unique minimal normal subgroup of $\bar{G}$.

If $G_0 = \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_1\right) \cap \mathbb{N}_{\mathsf{Aut}\,M}\left(\bar{K}_2\right)$, then $MG_0 = \bar{G}$. As $\bar{K}_1$ and $\bar{K}_2$ are self-normalising in M, $M \cap G_0 = \bar{K}_1 \cap \bar{K}_2$. Therefore, by [PS07, Lemma 4.1] the M-action on the coset space $[M : \bar{K}_1 \cap \bar{K}_2]$ can be extended to $\bar{G}$ with point stabiliser G_0. Moreover, $\bar{K}_1$ and $\bar{K}_2$ form a Cartesian system for M acted upon trivially by G_0. Consequently this action of $\bar{G}$ preserves an intransitive $\bar{G}$-invariant Cartesian decomposition given by the Cartesian system $\{\bar{K}_1, \bar{K}_2\}$.

9. THE PROOF OF THEOREM 3.1

In this section we prove Theorem 3.1 working with the notation introduced in Section 3.

Lemma 9.1. *Let* $T_1, \ldots, T_k$, $K_1, \ldots, K_s$, $\bar{\bar{\Xi}}_1, \ldots, \bar{\bar{\Xi}}_s$, *and* $\Omega_1, \ldots, \Omega_s$ *be as in Section 3. If, for some* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, s\}$, $\sigma_i(K_j)$ *is a proper maximal subgroup of* T_i, *then* $\bar{\bar{\Xi}}_j \in \mathsf{CD}_1(G^{\Omega_j})$.

Proof. The group G_ω is transitive by conjugation on the set $\{T_1, \ldots, T_k\}$, and, by Proposition 3.2, each of the K_j is normalised by G_ω. Thus it suffices to prove that if $\sigma_1(K_1)$ is a proper maximal subgroup of T_1, then $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_1(G^{\Omega_1})$. Assume without loss of generality that $\Xi_1 = \{\Gamma_1, \ldots, \Gamma_m\}$. By Proposition 3.2, the G-action on Ω_1 is equivalent to the G-action on $[M : K_1]$, and $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_1})$. Thus if $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{\mathrm{S}}(G^{\Omega_1})$, then $\sigma_1(K_1) = T_1$. If $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{2\sim}(G^{\Omega_1}) \cup \mathsf{CD}_{2\not\sim}(G^{\Omega_1}) \cup \mathsf{CD}_3(G^{\Omega_1})$, then there are distinct $j_1, j_2 \in \{1, \ldots, m\}$ such that $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2}) < T_1$, $\sigma_1(L_{j_1})\sigma_1(L_{j_2}) = T_1$, and $\sigma_1(K_1) \leqslant \sigma_1(L_{j_1}) \cap \sigma_1(L_{j_2})$. Hence $\sigma_1(K_1)$ is not a maximal subgroup of T_1.

Suppose finally that $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_1})$. Then, by [BPS06, Theorem 6.1], we may assume without loss of generality that there is a full strip X of length 2 involved in L_1 covering T_1 and T_2, and there are indices $j_1, j_2 \in \{2, \ldots, m\}$ such that $\sigma_1(L_{j_1}) < T_1$, $\sigma_2(L_{j_2}) < T_2$. Let $\alpha : T_1 \to T_2$ be the isomorphism such that $X = \{(t, \alpha(t)) \mid t \in T_1\}$. It follows from [PS02, Lemma 2.1] that $\sigma_1(L_{j_1})\alpha^{-1}(\sigma_2(L_{j_2})) = T_1$. In particular $\sigma_1(L_{j_1})$ and $\alpha^{-1}(\sigma_2(L_{j_2}))$ are distinct subgroups of T_1. On the other hand, $\sigma_1(K_1) \leqslant \sigma_1(L_1 \cap L_{j_1} \cap L_{j_2}) \leqslant \sigma_1(L_{j_1}) \cap \alpha^{-1}(\sigma_2(L_{j_2}))$. Hence $\sigma_1(K_1)$ cannot be a maximal subgroup of T_1. Therefore the only remaining possibility is that $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_1(G^{\Omega_1})$. $\qquad\square$

Recall that $\{L_1, \ldots, L_\ell\}$ is the original Cartesian system corresponding to the intransitive Cartesian decomposition $\mathcal{E}$. The following lemma is an easy consequence of [BPS04, Lemma 3.1].

Lemma 9.2. *Let* $L_1, \ldots, L_\ell$ *be as in Theorem* 3.1, *and suppose that* $I_1, \ldots, I_m$ *are pairwise disjoint subsets of* $\{1, \ldots, \ell\}$, *and, for* $i = 1, \ldots, m$, *set* $Q_i = \bigcap_{j \in I_i} L_j$. *Then*

$$Q_i \left(\bigcap_{j \neq i} Q_j \right) = M \quad \text{for all} \quad i \in \{1, \ldots, m\}.$$

Proof of Theorem 3.1. By Proposition 3.2, the partitions Ω_i are G-invariant and $\bar{\mathcal{E}} = \{\Omega_1, \ldots, \Omega_s\}$ is a Cartesian decomposition of Ω on which G acts trivially. By the same result, for $i = 1, \ldots, s$, the subgroup K_i is the stabiliser in M of the block in Ω_i containing ω, $\bar{\bar{\Xi}}_i \in \mathsf{CD}_{\mathrm{tr}}(G^{\Omega_i})$, and M is faithful on Ω_i. It follows from Theorem 7.1 that the number s of G-orbits on $\mathcal{E}$ is at most 3.

We prove the rest of Theorem 3.1 part by part.

(i) Suppose first, without loss of generality, that $\bar{\bar{\Xi}}_1 \in \mathsf{CD}_{\mathrm{S}}(G^{\Omega_1})$. Then by Theorem 2.3(a), K_1 is a subdirect subgroup of M, and it follows from Theorem 6.2 that $s = 2$ and that (M, K_1, K_2) is a full strip factorisation. In particular, $\sigma_i(K_2)$ is a maximal subgroup of T_i for all i, and hence Lemma 9.1 implies that $\bar{\bar{\Xi}}_2 \in \mathsf{CD}_1(G^{\Omega_2})$, as required.

(ii) Next assume without loss of generality that $\bar{\Xi}_1 \in \mathsf{CD}_{2\not\sim}(G^{\Omega_1})$, and that $\Xi_1 = \{\Gamma_1, \ldots, \Gamma_m\}$. Note that there are j_1, $j_2 \in \{1, \ldots, m\}$ such that $\sigma_1(K_{j_1})$, $\sigma_1(K_{j_2}) < T_1$. If $\sigma_1(K_2) = T_1$, then, by Theorem 6.2, $\bar{\Xi}_2 \in \mathsf{CD}_\mathrm{S}(G^{\Omega_2})$, and so part (i) implies that $\bar{\Xi}_1 \in \mathsf{CD}_1(G^{\Omega_1})$, which is a contradiction. Hence $\sigma_1(K_2) < T_1$. If $s \geqslant 3$, then the same argument shows that $\sigma_1(K_3) < T_1$ and, by Lemma 9.2, $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(K_2)$, $\sigma_1(K_3)$ form a strong multiple factorisation of the finite simple group T_1. As, by Theorem 4.3, such a factorisation has at most 3 subgroups, this yields that $s = 2$. Similarly, if there are two indices j_3, $j_4 \in \{m+1, \ldots, \ell\}$ such that $\sigma_1(L_{j_3})$, $\sigma_1(L_{j_4}) < T_1$, then the subgroups $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(L_{j_3})$, $\sigma_1(L_{j_4})$ form a strong multiple factorisation of T_1. This again is a contradiction, and so $\bar{\Xi}_2 \in \mathsf{CD}_1(G^{\Omega_2}) \cup \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_2})$. Thus there is a unique index $j_3 \in \{m+1, \ldots, \ell\}$ such that $\sigma_1(L_{j_3}) < T_1$. The subgroups $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(L_{j_3})$ form a strong multiple factorisation of T_1, and so T_1 and these subgroups are as in Table 3. If $\bar{\Xi}_2 \in \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_2})$, then, by Theorem 2.3(b), T_1 must also be as in Table 2, and so $T_1 \cong \mathsf{P}\Omega_8^+(3)$. Further, $\sigma_1(L_{j_3})$, and hence $\sigma_1(K_2)$, must be a maximal subgroup of T_1. This, however, cannot be the case if $\bar{\Xi}_2 \in \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_2})$, by Lemma 9.1. Thus all the assertions in part (ii) hold.

(iii) Suppose without loss of generality that $\bar{\Xi}_1 \in \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_1}) \cup \mathsf{CD}_{2\sim}(G^{\Omega_1}) \cup \mathsf{CD}_3(G^{\Omega_1})$ and that $\Xi_1 = \{\Gamma_1, \ldots, \Gamma_m\}$. It follows from part (i) that $\bar{\Xi}_i \notin \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_i})$ for all $i \in \{2, \ldots, s\}$. Thus for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, s\}$ the projection $\sigma_i(K_j)$ is proper in T_i. If $\bar{\Xi}_1 \in \mathsf{CD}_3(G^{\Omega_1})$, then there are pairwise distinct indices j_1, j_2, $j_3 \in \{1, \ldots, m\}$ such that $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(L_{j_3}) < T_1$. By Lemma 9.2, the subgroups $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(L_{j_3})$, $\sigma_1(K_2)$ form a strong multiple factorisation of T_1, which is a contradiction, by Theorem 4.3. Thus $\bar{\Xi}_1 \notin \mathsf{CD}_3(G^{\Omega_1})$.

Suppose next that $\bar{\Xi}_1 \in \mathsf{CD}_{2\sim}(G^{\Omega_1})$. Then there are distinct indices j_1, $j_2 \in \{1, \ldots, m\}$ such that $\sigma_1(L_{j_1})$ and $\sigma_1(L_{j_2})$ are proper isomorphic subgroups of T_1. On the other hand, as $\sigma_1(K_2) < T_1$, the subgroups $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(K_2)$ form a strong multiple factorisation of T_1. By Table 3 such a factorisation cannot contain two isomorphic subgroups, and so this is a contradiction. Thus $\bar{\Xi}_1$ cannot be an element of $\mathsf{CD}_{2\sim}(G^{\Omega_1})$.

Suppose finally that $\bar{\Xi}_1 \in \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_1})$. Then, by [BPS06, Theorem 6.1], we may assume without loss of generality that there is a full strip X of length 2 involved in L_1 covering T_1 and T_2, and there are indices j_1, $j_2 \in \{2, \ldots, m\}$ such that $\sigma_1(L_{j_1}) < T_1$, $\sigma_2(L_{j_2}) < T_2$. Let $\alpha : T_1 \to T_2$ be the isomorphism such that $X = \{(t, \alpha(t)) \mid t \in T_1\}$. It follows from [PS02, Lemma 2.1] that $\sigma_1(L_{j_1})\alpha^{-1}(\sigma_2(L_{j_2})) = T_1$. Theorem 6.2 and part (i) implies that $\sigma_1(K_2) < T_1$. As $(L_1 \cap L_{j_1} \cap L_{j_2})K_2 = M$ and

$$\sigma_1(L_1 \cap L_{j_1} \cap L_{j_2}) \leqslant \sigma_1(L_{j_1}) \cap \alpha^{-1}(\sigma_2(L_{j_2})),$$

we obtain that $(\sigma_1(L_{j_1}) \cap \alpha^{-1}(\sigma_2(L_{j_2})))\sigma_1(K_2) = T_1$. Then [BP98, Lemma 4.3(iii)] implies that $(T_1, \{\sigma_1(L_{j_1}), \alpha^{-1}(\sigma_2(L_{j_2})), \sigma_1(K_2)\})$ is a strong multiple factorisation. By Table 3 distinct subgroups in such a factorisation cannot be isomorphic. This is a contradiction, and so $\bar{\Xi}_1 \notin \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_1})$.

(iv) Suppose that $\mathcal{E}$ is homogeneous. Then it follows from Theorem 8.1 that G has exactly 2 orbits on $\mathcal{E}$, and so $s = 2$. The same result implies that K_1, K_2 form a full factorisation of M, and that $\sigma_i(K_j)$ is a maximal subgroup of T_i, for each i and j. Thus Lemma 9.1 gives $\bar{\Xi}_i \in \mathsf{CD}_1(G^{\Omega_i})$ for $i = 1$, 2.

(v) Finally suppose that $s = 3$. By part (i) $\bar{\Xi}_i \notin \mathsf{CD}_\mathrm{S}(G^{\Omega_i})$ and, by part (iii), $\bar{\Xi}_i \notin \mathsf{CD}_{1\mathrm{S}}(G^{\Omega_i})$ for $i = 1$, 2, 3. If $\bar{\Xi}_i \notin \mathsf{CD}_1(G^{\Omega_i})$ for some i, then there must be

4 pairwise distinct indices j_1, j_2, j_3, $j_4 \in \{1, \ldots, \ell\}$ such that $\sigma_1(L_{j_1})$, $\sigma_1(L_{j_2})$, $\sigma_1(L_{j_3})$, $\sigma_1(L_{j_4})$ are proper subgroups of T_1. By (3), these subgroups form a strong multiple factorisation of T_1, which is a contradiction, by Theorem 4.3. Thus $\bar{\bar{\Xi}}_i \in$ $\mathsf{CD}_1(G^{\Omega_i})$ for $i = 1$, 2, 3. It also follows from Theorem 7.1 that $(M, \{K_1, K_2, K_3\})$ is a strong multiple factorisation. $\square$

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