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ASYMPTOTIC SPECTRAL ANALYSIS OF GROWING REGULAR GRAPHS

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Abstract. We propose the quantum probabilistic techniques to obtain the asymptotic spectral distribution of the adjacency matrix of a growing regular graph. We prove the quantum central limit theorem for the adjacency matrix of a growing regular graph in the vacuum and deformed vacuum states. The condition for the growth is described in terms of simple statistics arising from the stratification of the graph. The asymptotic spectral distribution of the adjacency matrix is obtained from the classical reduction.

Introduction

Spectral analysis of a finite graph has a long history along with algebraic graph theory and combinatorics; see, e.g., Bannai–Ito [4], Biggs [6], Cvetković–Doob–Sachs [9]. As for infinite graphs such as lattices and homogeneous trees, most of the detailed study has been made with harmonic analysis of discrete groups and with probability theory (theory of random walks) tracing back to Kesten [29]; see also Kesten [30], Woess [36], and references cited therein. Most of these works concentrate on the fine structure of the spectrum of a fixed graph, but, in contrast, this paper focuses on a growing family of graphs and its asymptotic spectrum. Our problem is motivated also by the asymptotic combinatorics proposed by Vershik [37] and by the study of evolution of networks, e.g., Dorogovtsev–Mendes [10].

In this paper, employing quantum probabilistic techniques, we shall derive the asymptotic spectral distribution of the adjacency matrix A_ν of a growing graph $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ as a consequence of the quantum central limit theorem. Our basic tool is the quantum decomposition of the adjacency matrix:

$$A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ.$$

The quantum components A_ν^ϵ are noncommutative; however, the quantum decomposition happens to gain a clear insight into the asymptotic nature of A_ν. In fact, equipped with the vacuum or the deformed vacuum state, A_ν and its quantum components become considered as algebraic random variables and the formulation of the (quantum) central limit theorem meets our problem. The main results are

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stated in Theorems 6.1 and 6.2 for the vacuum state and Theorem 7.4 for the deformed vacuum state. It is our achievement that these results are proved under the conditions (A1)–(A4) described in terms of simple statistical data of a growing graph. These conditions are newly formulated and are satisfied by concrete examples in the previous papers (see below). Moreover, the scaling balance for the quantum central limit theorem in the deformed vacuum state is summarized into a single condition (A4) $q\sqrt{\kappa} \sim \gamma$ and the coherent state naturally emerges in the limit. As a result, the former argument according to the situation is extremely simplified. The main results lead to new central limit theorems with potential connections with q-deformed probability theory too.

This paper is organized as follows: In Section 1 the main problem is formulated. In Section 2 the vacuum and deformed vacuum states are defined. In Section 3 we introduce the stratification of a graph and the quantum decomposition of the adjacency matrix. Section 4 assembles some basic results on interacting Fock probability spaces and orthogonal polynomials. In Section 5 we formulate conditions (A1)–(A3) controlling how the graph under consideration grows. Section 6 contains the quantum central limit theorem in the vacuum state and its proof. In Section 7 we introduce condition (A4) and prove the quantum central limit theorem in the deformed vacuum state. Section 8 contains two guiding examples (homogeneous trees and Hamming graphs) for illustrating how our approach is applied to a concrete problem. Section 9 is devoted to a technical result concerning conditions (A1)–(A3).

The quantum probabilistic approach to the spectral analysis of graphs traces back to Hora [18], where the vacuum spectral distribution of the adjacency matrix of a distance-regular graph was derived. The method therein is not based upon the quantum decomposition but requires some classical results. Hashimoto–Obata–Tabei [16] applied the method of quantum decomposition to Hamming graphs and obtained the limit distributions (Gaussian and Poisson distributions) without the combinatorial arguments required in the classical method. Hashimoto [14] applied the same idea to Cayley graphs and developed a general theory. Later on, Hashimoto–Hora–Obata [15] studied limit distributions for distance-regular graphs in general and derived the exponential distributions (Laguerre polynomials) and the geometric distributions (Meixner polynomials) from Johnson graphs. The deformed vacuum states were discussed by Hora [20, 22, 23]; in particular, compound Poisson distributions of exponential and geometric ones were derived from Johnson graphs. A general theory for a growing family of regular graphs was developed by Hora–Obata [24, 25]. Some new examples are studied by Igarashi–Obata [27]. Finally, the asymptotic representation theory of symmetric groups by Kerov [28], Biane [5], Hora [19, 21] and others is interesting also from our aspect since a particular graph arising from Young diagrams is relevant to spectral analysis of symmetric groups.

1. Main problem

Let $\mathcal{G} = (V, E)$ be a graph, where V is a non-empty set of *vertices* and E is a set of *edges*, i.e., $E \subset \{\{x, y\}\,;\, x, y \in V,\, x \neq y\}$. Two vertices $x, y \in V$ are called *adjacent* if $\{x, y\} \in E$, and in this case we write $x \sim y$. A finite sequence $x_0, x_1, \dots, x_n \in V$ is called a *walk* of length n (connecting x_0 and x_n) if $x_i \sim x_{i+1}$ for $i = 0, 1, \dots, n-1$. In a walk some of $x_0, x_1, \dots, x_n$ may occur repeatedly. A graph is called *connected* if any pair of distinct vertices is connected by a walk. The *degree* or *valency* of a vertex $x \in V$ is defined by $\kappa(x) = |\{y \in V\,;\, y \sim x\}|$. A graph is called *locally finite* if $\kappa(x) < \infty$ for all $x \in V$, *uniformly locally finite* if $\sup\{\kappa(x)\,;\, x \in V\} < \infty$, and *regular* if $\kappa(x) \equiv \kappa < \infty$ is a constant.

Convention. Throughout the paper, unless otherwise specified, a graph is always assumed to be connected and locally finite.

The graph structure is fully represented by the adjacency matrix $A = (A_{xy})$ defined by

$$A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise}. \end{cases} \tag{1.1}$$

Let $\mathcal{A}(\mathcal{G})$ be the set of matrices expressible by a polynomial in A with complex coefficients, where the usual matrix operations are performed thanks to the local finiteness. Then, $\mathcal{A}(\mathcal{G})$ becomes a commutative $*$-algebra with identity, which is called the *adjacency algebra* of $\mathcal{G}$. Note that the adjacency matrix is symmetric, i.e., $A = A^*$.

Let $\langle\cdot\rangle$ be a state on $\mathcal{A}(\mathcal{G})$; that is, $a \mapsto \langle a\rangle \in \mathbf{C}$ is a linear functional on $\mathcal{A}(\mathcal{G})$, positive ($\langle a^*a\rangle \ge 0$ for all $a \in \mathcal{A}(\mathcal{G})$), and normalized ($\langle 1_{\mathcal{A}}\rangle = 1$). It follows from Hamburger's theorem that there exists a probability distribution μ on $\mathbf{R}$ such that

$$\langle A^m\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{1.2}$$

Uniqueness of μ does not hold in general (known as the determinate moment problem). We call μ the *spectral distribution* of A in the given state.

Our main interest is to study the spectral distribution of a growing graph. Consider a growing family of graphs

$$\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)}),$$

where a growing parameter ν runs over a directed set (for simplicity, we write $\nu \to \infty$ for the limit). Let A_ν denote the adjacency matrix of $\mathcal{G}^{(\nu)}$ and suppose that each adjacency algebra $\mathcal{A}(\mathcal{G}_\nu)$ is given a state $\langle\cdot\rangle_\nu$. Let μ_ν be a (not necessarily uniquely determined) probability distribution determined as in (1.2). We are interested in the limit of μ_ν as $\nu \to \infty$ with suitable scaling suggested by limit theorems in probability theory. Namely, a natural normalization is given by

$$\frac{A_\nu - \langle A_\nu\rangle_\nu}{\Sigma_\nu(A_\nu)}, \qquad \Sigma_\nu^2(A_\nu) = \langle (A_\nu - \langle A_\nu\rangle_\nu)^2\rangle_\nu. \tag{1.3}$$

(The suffix ν is cumbersome and will occasionally be dropped.) Our goal is to determine a probability distribution μ satisfying

$$\lim_{\nu\to\infty} \left\langle \left(\frac{A_\nu - \langle A_\nu\rangle}{\Sigma(A_\nu)}\right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots. \tag{1.4}$$

The above μ, in general not uniquely determined, is called the *asymptotic spectral distribution* of A_ν in the state $\langle\cdot\rangle_\nu$.

2. Vacuum and deformed vacuum states

In fact, we consider the vacuum state and its deformation. Let $\mathcal{G} = (V, E)$ be a graph. Let $\ell^2(V)$ be the Hilbert space of square-summable functions on V and $C_0(V)$ the dense subspace of functions with finite supports. The inner product of $\ell^2(V)$ is defined by

$$\langle f, g \rangle = \sum_{x \in V} \overline{f(x)}\, g(x), \qquad f, g \in \ell^2(V).$$

For $x \in V$ define a function δ_x by

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\{\delta_x \,;\, x \in V\}$ becomes a complete orthonormal basis of $\ell^2(V)$ and $C_0(V)$ its linear span. The adjacency algebra $\mathcal{A}(\mathcal{G})$ acts in a natural manner on $C_0(V)$.

By analogy of an interacting Fock space (Section 4) we give the following

Definition 2.1. Let $o \in V$ be a fixed origin of a graph $\mathcal{G} = (V, E)$. The vector state on $\mathcal{A}(\mathcal{G})$ defined by

$$\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle, \qquad a \in \mathcal{A}(\mathcal{G}), \tag{2.1}$$

is called the *vacuum state* at $o \in V$.

It is noted that $\langle A^m \rangle_o$ is the number of m-step walks from $o \in V$ to itself. More generally, for $x, y \in V$, we see that $(A^m)_{xy} = \langle \delta_x, A^m \delta_y \rangle$ is the number of m-step walks connecting y and x.

We next define a deformed vacuum state. Let $\partial(x, y)$ be the graph distance; i.e., $\partial(x, y)$ for distinct points $x, y \in V$ is the shortest length of walks connecting them and $\partial(x, x) = 0$ by definition. Given $q \in \mathbf{R}$, consider a linear function $\mathcal{A}(\mathcal{G}) \ni a \to \langle a \rangle_q$ defined by

$$\langle a \rangle_q = \sum_{x \in V} q^{\partial(x,o)} \langle \delta_x, a\delta_o \rangle, \tag{2.2}$$

where the right hand side is, in fact, a finite sum and is well defined. Note that $\langle 1 \rangle_q = 1$ so that $\langle \cdot \rangle_q$ is a normalized linear function. It is convenient to introduce a matrix

$$Q = Q_q = (q^{\partial(x,y)})_{x,y \in V}.$$

For $q = 0$ we tacitly understand Q to be the identity matrix $(0^0 = 1)$. Then (2.2) can be written as

$$\langle a \rangle_q = \langle Q\delta_o, a\delta_o \rangle, \qquad a \in \mathcal{A}(\mathcal{G}). \tag{2.3}$$

Strictly speaking, the right hand side of (2.3) is no more an inner product of $\ell^2(V)$ but is understood to be the canonical sesquilinear form on $C_0(V)^* \times C_0(V)$, where $C_0(V)^*$ is the space of formal linear combinations of δ_x with x running over V. Although the positivity of $\langle \cdot \rangle_q$ is questionable, being slightly free from the strict wording, we give the following

Definition 2.2. The normalized linear function $\langle \cdot \rangle_q$ defined in (2.3) is called a *deformed vacuum state* on $\mathcal{A}(\mathcal{G})$.

Thus, a deformed vacuum state is not necessarily a state. As a simple sufficient condition for the positivity we prove the following.

Proposition 2.3. *A deformed vacuum state $\langle \cdot \rangle_q$ is positive on $\mathcal{A}(\mathcal{G})$ if the following two conditions are fulfilled:*

(Q1) *Q is a positive definite kernel on V, i.e., $\langle f, Qf \rangle \geq 0$ for all $f \in C_0(V)$;*
(Q2) *$QA = AQ$. (Note that the matrix elements of both sides are well defined.)*

Proof. Let $a \in \mathcal{A}(\mathcal{G})$. Since a is a polynomial in A, (Q2) implies that $Qa = aQ$. Then,

$$\langle a^* a \rangle_q = \langle Q\delta_o, a^* a \delta_o \rangle = \langle aQ\delta_o, a\delta_o \rangle = \langle Qa\delta_o, a\delta_o \rangle \geq 0,$$

where the last inequality is by (Q1). □

As for (Q1) we only mention the following two results. The proofs are easy and omitted; for a relevant discussion see Bożejko [7].

Proposition 2.4. *Let $\mathcal{G} = (V, E)$ be a graph with $|V| \geq 2$. In order that $Q = (q^{\partial(x,y)})$ be a positive definite kernel on V it is necessary that $-1 \leq q \leq 1$.*

Proposition 2.5 (Bożejko's quadratic embedding test)**.** *If a graph $\mathcal{G} = (V, E)$ admits a quadratic embedding, i.e., if there is a map F from V into a finite dimensional Euclidean space $\mathbf{R}^N$ such that*

$$\|F(x) - F(y)\|^2 = \partial(x, y), \qquad x, y \in V,$$

then $Q = (q^{\partial(x,y)})$ is positive definite for all $0 \leq q \leq 1$.

As for (Q2) we only mention the following

Proposition 2.6. *Let $\mathcal{G} = (V, E)$ be a graph. If for any pair of vertices $x, y \in V$ there exists an automorphism $\alpha \in \mathrm{Aut}\,(\mathcal{G})$ satisfying $\alpha(x) = y$ and $\alpha(y) = x$, then $QA = AQ$.*

3. Stratification and quantum decomposition

Let $\mathcal{G} = (V, E)$ be a graph. As soon as an origin $o \in V$ is chosen, a natural stratification is introduced:

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V\,;\, \partial(o, x) = n\}. \tag{3.1}$$

If $V_m = \emptyset$ happens for some $m \geq 1$, then $V_n = \emptyset$ for all $n \geq m$. We then define three matrices A^+, A^-, A° by

$$(A^\epsilon)_{yx} = \begin{cases} A_{yx} = 1, & \text{if } y \sim x \text{ and } \partial(o, y) = \partial(o, x) + \epsilon, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in V,$$

where ϵ takes values $+1, -1, 0$ according as $\epsilon = +, -, \circ$ (see Figure 1). Obviously,

$$(A^+)^* = A^-, \qquad (A^\circ)^* = A^\circ,$$

and

$$A = A^+ + A^- + A^\circ. \tag{3.2}$$

This is called the *quantum decomposition* of A and each A^ϵ a *quantum component.* A quantum decomposition depends on the stratification (3.1) and hence on the choice of an origin $o \in V$. The $*$-algebra generated by $\{A^+, A^-, A^\circ\}$ is denoted by $\tilde{\mathcal{A}}(\mathcal{G})$ and is called the *extended adjacency algebra.*

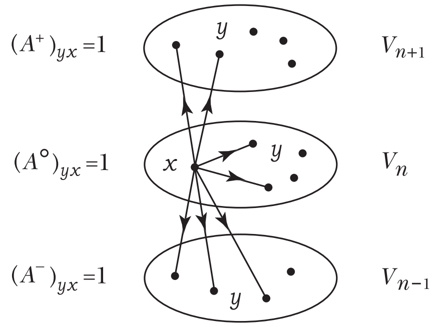

FIGURE 1. Quantum decomposition: $A = A^+ + A^- + A^\circ$

Given a stratification (3.1), we next define an orthonormal set in $\ell^2(V)$. For each $n \geq 0$ with $V_n \neq \emptyset$ we set

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x. \tag{3.3}$$

Let $\Gamma(\mathcal{G}) \subset \ell^2(V)$ be the subspace spanned by $\{\Phi_n\}$. Let us observe that $\Gamma(\mathcal{G})$ is not necessarily kept invariant under the actions of the quantum components. For $x \in V$ and $\epsilon \in \{+, -, \circ\}$ we define

$$\omega_\epsilon(x) = |\{y \in V\,;\, y \sim x,\ \partial(o, y) = \partial(o, x) + \epsilon\}|. \tag{3.4}$$

In other words, $\omega_\epsilon(x)$ is the set of vertices which are adjacent to x and lie in the upper, lower or level stratum according as $\epsilon = +, -, \circ$. Obviously,

$$\kappa(x) = \omega_+(x) + \omega_-(x) + \omega_\circ(x), \qquad x \in V.$$

It follows from the definitions that

$$|V_n|^{1/2} A^+ \Phi_n = \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} \omega_-(y)\, \delta_y,$$

and hence

$$A^+ \Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n+1}} \omega_-(y)\, \delta_y. \tag{3.5}$$

In a similar fashion we obtain

$$A^- \Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n-1}} \omega_+(y)\, \delta_y, \tag{3.6}$$

$$A^\circ \Phi_n = |V_n|^{-1/2} \sum_{y \in V_n} \omega_\circ(y)\, \delta_y. \tag{3.7}$$

It is obvious from (3.5)–(3.7) that $\Gamma(\mathcal{G})$ is invariant under the actions of A^ϵ if and only if $\omega_\epsilon(y)$ is constant on each stratum V_n. Our interest lies in the case where $\Gamma(\mathcal{G})$ is "asymptotically invariant" under A^ϵ.

4. Interacting fock probability space

Definition 4.1. A sequence $\{\omega_n\}$ is called a *Jacobi sequence* if $\{\omega_n\,;\, n = 1, 2, \dots\}$ is an infinite sequence of positive numbers; or if there exists $m_0 \ge 1$ such that $\{\omega_n\,;\, n = 1, 2, \dots, m_0 - 1\}$ is a finite sequence of positive numbers (or an empty sequence if $m_0 = 1$). The former case is called *infinite type* and the latter *finite type.*

Given a Jacobi sequence $\{\omega_n\}$ of infinite or finite type, we consider a Hilbert space of infinite or m_0-dimension with a complete orthonormal basis $\{\Psi_n\,;\, n = 0, 1, 2, \dots\}$. Let Γ be a subspace spanned by this basis and define linear operators $B^\pm$ acting on Γ by

$$B^+\Psi_n = \sqrt{\omega_{n+1}}\,\Psi_{n+1}, \qquad n = 0, 1, 2, \dots; \tag{4.1}$$

$$B^-\Psi_0 = 0, \qquad B^-\Psi_n = \sqrt{\omega_n}\,\Psi_{n-1}, \qquad n = 1, 2, \dots. \tag{4.2}$$

These are called the *creation operator* and *annihilation operator*, respectively. If $\{\omega_n\}$ is of finite type, we understand that $B^+\Psi_{m_0-1} = 0$. Hence it is often convenient to identify a Jacobi sequence of finite type with an infinite sequence by concatenating a zero sequence.

Definition 4.2. For a Jacobi sequence $\{\omega_n\}$, the quadruple $(\Gamma, \{\Psi_n\}, B^+, B^-)$ is called an *interacting Fock space* (of one mode) and is denoted by $\Gamma_{\{\omega_n\}}$.

Definition 4.3. A *Jacobi coefficient* is a pair of sequences $(\{\omega_n\}, \{\alpha_n\})$, where $\{\omega_n\,;\, n = 1, 2, \dots\}$ is an infinite sequence of positive numbers and $\{\alpha_n\,;\, n = 1, 2, \dots\}$ is an infinite sequence of real numbers (infinite type); or there exists $m_0 \ge 1$ such that $\{\omega_n\,;\, n = 1, 2, \dots, m_0 - 1\}$ is a finite sequence of positive numbers (or an empty sequence if $m_0 = 1$) and $\{\alpha_n\,;\, n = 1, 2, \dots, m_0\}$ is a finite sequence of real numbers (finite type). Let $\mathfrak{J}$ be the set of all Jacobi coefficients.

With each Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$, we associate an interacting Fock space $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ and a diagonal operator defined by

$$B^\circ\Psi_n = \alpha_{n+1}\Psi_n, \qquad n = 0, 1, 2, \dots.$$

We often write $B^\circ = \alpha_{N+1}$ with N being the number operator defined by $N\Psi_n = n\Psi_n$ for $n = 0, 1, 2, \dots$. It is easy to see that

$$\langle B^+\Psi_m, \Psi_n\rangle = \langle \Psi_m, B^-\Psi_n\rangle, \qquad \langle B^\circ\Psi_m, \Psi_n\rangle = \langle \Psi_m, B^\circ\Psi_n\rangle.$$

In other words, B^+ and B^- are mutually adjoint and B° is symmetric. The $*$-algebra generated by $\{B^+, B^-, B^\circ\}$ is called the *interacting Fock algebra* associated with a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$ and is considered as an algebraic probability space equipped with the vacuum state, i.e., the vector state corresponding to Ψ_0. The distribution of $B^+ + B^- + B^\circ$ in the vacuum state is fundamental.

Let us recall the orthogonal polynomials. Let $\mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$ denote the set of probability distributions which admit moments of all orders. Given $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbf{R})$, we obtain the orthogonal polynomials $\{P_n(x) = x^n + \dots\}$ by the standard Gram-Schmidt orthogonalization. Then there exists a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ such that

$$\begin{cases} P_0(x) = 1, \\ P_1(x) = x - \alpha_1, \\ xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n = 1, 2, \dots. \end{cases} \tag{4.3}$$

We call $(\{\omega_n\}, \{\alpha_n\})$ the Jacobi coefficient of μ. Note that the map $\mathfrak{P}_{\rm fm}(\mathbf{R}) \to \mathfrak{J}$ is surjective but not injective. If the counterimage of a Jacobi coefficient consists of a single distribution μ, we say that *μ is the solution of a determinate moment problem*. If μ has a compact support, it is the solution of a determinate moment problem. In particular, if μ is a finite sum of δ-measures, or equivalently if the associated Jacobi coefficient is of finite type, μ is the solution of a determinate moment problem. Carleman's condition for μ being the solution of a determinate moment problem is that $\sum_{n=1}^{\infty} \omega_n^{-1/2} = +\infty$; see Shohat–Tamarkin [35, Section 2.17].

Theorem 4.4. *Given $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$, let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ and B° be the associated interacting Fock space and diagonal operator, respectively. Then there exists a probability distribution $\mu \in \mathfrak{P}_{\rm fm}(\mathbf{R})$ such that*

$$\langle \Psi_0, (B^+ + B^- + B^\circ)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots .$$

Moreover, the Jacobi coefficient of μ is $(\{\omega_n\}, \{\alpha_n\})$.

The first half is due to Hamburger's theorem (see, e.g., [35]). The second half is by comparison of the actions of B^ϵ and the three-term recurrence relation (4.3); see Accardi–Bożejko [2]. By definition the correspondence between probability measures $\mu \in \mathfrak{P}_{\rm fm}(\mathbf{R})$ and Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ is indirect. A more direct correspondence is given by the Stieltjes transform.

Theorem 4.5. *Let $(\{\omega_n\}, \{\alpha_n\})$ be a Jacobi coefficient and μ be an associated probability distribution. If μ is the solution of a determinate moment problem, the Stieltjes transform admits a continued fraction expansion:*

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x} = \frac{1}{z - \alpha_1} \, \frac{}{-} \, \frac{\omega_1}{z - \alpha_2} \, \frac{}{-} \, \frac{\omega_2}{z - \alpha_3} \, \frac{}{-} \, \frac{\omega_3}{z - \alpha_4} \, \frac{}{-} \, \cdots, \tag{4.4}$$

which converges in $\{\operatorname{Im} z \neq 0\}$.

As a direct application to the spectral analysis of a graph, we only mention the following

Proposition 4.6. *Let $\mathcal{G} = (V, E)$ be a graph with a fixed origin $o \in V$. Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition of the adjacency matrix. If $\Gamma(\mathcal{G})$ is invariant under the actions of the quantum components A^ϵ, then $(\Gamma(\mathcal{G}), \{\Phi_n\}, A^+, A^-)$ becomes an interacting Fock space and A° a diagonal operator.*

The proof is straightforward from (3.5)–(3.7). A (finite or infinite) distance-regular graph satisfies the conditions in Proposition 4.6; for a relevant discussion see also Section 8.

5. Growing regular graphs

Let $\mathcal{G} = (V, E)$ be a graph with a fixed origin $o \in V$. As before, consider the stratification $V = \bigcup_{n=0}^{\infty} V_n$ and define

$$\omega_\epsilon(x) = |\{y \in V \,;\, y \sim x,\ \partial(o, y) = \partial(o, x) + \epsilon\}|, \qquad x \in V, \quad \epsilon \in \{+, -, \circ\}.$$

The statistics of $\omega_\epsilon(x)$ play a crucial role. We define

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}\omega_\epsilon(x),$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}\{\omega_\epsilon(x) - M(\omega_\epsilon|V_n)\}^2,$$

$$L(\omega_\epsilon|V_n) = \max\{\omega_\epsilon(x)\,;\, x\in V_n\}.$$

Namely, $M(\omega_\epsilon|V_n)$ is the mean value of $\omega_\epsilon(x)$ when x runs over V_n, and $\Sigma^2(\omega_\epsilon|V_n)$ is its variance.

Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph, where the growing parameter ν runs over an infinite directed set. The degree of $\mathcal{G}^{(\nu)}$ is denoted by $\kappa(\nu)$. For each graph $\mathcal{G}^{(\nu)}$ we fix an origin $o_\nu \in V^{(\nu)}$ and consider as usual the stratification:

$$V^{(\nu)} = \bigcup_{n=0}^{\infty} V_n^{(\nu)}, \qquad V_n^{(\nu)} = \{y\in V^{(\nu)};\, \partial(o,y) = n\}. \tag{5.1}$$

($V_n^{(\nu)} = \emptyset$ may occur.) Then, for $n = 0, 1, 2, \dots$ we define a unit vector in $\ell^2(V^{(\nu)})$ by

$$\Phi_n^{(\nu)} = |V_n^{(\nu)}|^{-1/2}\sum_{x\in V_n^{(\nu)}}\delta_x. \tag{5.2}$$

Let $\Gamma(\mathcal{G}^{(\nu)})$ denote the linear span of $\{\Phi_0^{(\nu)}, \Phi_1^{(\nu)}, \dots\}$. Let A_ν denote the adjacency matrix of $\mathcal{G}^{(\nu)}$. According to the stratification (5.1) we have a *quantum decomposition*:

$$A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ. \tag{5.3}$$

We do not assume that $\Gamma(\mathcal{G}^{(\nu)})$ is invariant under the actions of quantum components A_ν^ϵ, but we need asymptotic invariance. This requirement is fulfilled by natural conditions on how the graph grows.

For a growing regular graph $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ we consider:

(A1) $\lim_\nu \kappa(\nu) = \infty$;

(A2) for each $n = 1, 2, \dots$ there exists a limit

$$\omega_n = \lim_\nu M(\omega_-|V_n^{(\nu)}) < \infty \tag{5.4}$$

and

$$\lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \tag{5.5}$$

$$\sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty; \tag{5.6}$$

(A3) for each $n = 0, 1, 2, \dots$ there exists a limit

$$\alpha_{n+1} = \lim_\nu M\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}}\,\middle|\, V_n^{(\nu)}\right) = \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty \tag{5.7}$$

and

$$\lim_{\nu} \Sigma^2\left(\frac{\omega_\circ}{\sqrt{\kappa(\nu)}}\,\middle|\, V_n^{(\nu)}\right) = \lim_{\nu} \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \tag{5.8}$$

$$\sup_{\nu} \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \tag{5.9}$$

Remark 5.1. Condition (A2) for $n = 1$ and (A3) for $n = 0$ are always satisfied. Obviously, $\omega_1 = 1$ and $\alpha_1 = 0$.

Remark 5.2. If $\mathcal{G}^{(\nu)}$ is a finite graph, $V_n^{(\nu)} = \emptyset$ happens for large n and $M(\omega_\epsilon|V_n^{(\nu)})$ is not defined for all n. This causes, however, no trouble in defining infinite sequences $\{\omega_n\}$ and $\{\alpha_n\}$. In fact, as is proved in Proposition 5.3 below, conditions (A1), (5.6) and (5.9) imply that for each $n \ge 1$ there exists ν_n such that $V_n^{(\nu)} \neq \emptyset$ for all $\nu \ge \nu_n$.

The meaning of (A1) is clear. Condition (A2) means that, in each stratum most of the vertices have the same number of downward edges, and the fluctuation of that number tends to zero as the graph grows. Condition (A3) is for edges lying in each stratum. The number of such edges may increase as the graph grows, but the growth rate is bounded by $\kappa(\nu)^{1/2}$. Roughly speaking, conditions (A1), (5.6) and (5.9) control the growth rate of the number of edges sprouting from a "generic" vertex $x \in V_n^{(\nu)}$ as follows:

$$\omega_+(x) = O(\kappa(\nu)), \qquad \omega_\circ(x) = O(\kappa(\nu)^{1/2}), \qquad \omega_-(x) = O(1).$$

Proposition 5.3. *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3). Then, $(\{\omega_n\}, \{\alpha_n\})$ defined in these conditions is a Jacobi coefficient of infinite type.*

Proof. It follows from (A1) that there exists ν_1 such that $V_1^{(\nu)} \neq \emptyset$ for all $\nu > \nu_1$. Take $x \in V_1^{(\nu)}$ and consider the obvious equality:

$$\frac{\omega_+(x)}{\kappa} + \frac{\omega_-(x)}{\kappa} + \frac{\omega_\circ(x)}{\kappa} = 1.$$

By (5.6) and (5.9) there exists $\nu_2 > \nu_1$ such that the first term is positive for all $\nu > \nu_2$; namely, $V_2^{(\nu)} \neq \emptyset$. By induction, we can find $\nu_1 < \nu_2 < \cdots < \nu_n < \dots$ such that $V_n^{(\nu)} \neq \emptyset$ for all $\nu > \nu_n$, $n = 1, 2, \dots$. Then, for any $x \in V_n^{(\nu)}$ we have $\omega_-(x) \ge 1$; hence $M(\omega_-|V_n^{(\nu)}) \ge 1$. Consequently, $\omega_n \ge 1$ for all n. $\square$

Moreover, we have the following noteworthy consequence. The proof is deferred to the Appendix.

Proposition 5.4. *The notation and assumptions being the same as in Proposition 5.3, the Jacobi sequence $\{\omega_n\}$ defined therein consists of positive integers.*

6. Quantum central limit theorem in the vacuum state

The main result in this section is stated in the following

Theorem 6.1 (QCLT in the vacuum state)**.** *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying conditions (A1)–(A3) and A_ν its adjacency matrix. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\}$ and B° the* *diagonal operator defined by* $\{\alpha_n\}$, *where* $\{\omega_n\}$ *and* $\{\alpha_n\}$ *are given in conditions (A1)–(A3). Then we have*

$$\lim_{\nu} \frac{A_{\nu}^{\epsilon}}{\sqrt{\kappa(\nu)}} = B^{\epsilon}, \qquad \epsilon \in \{+,-,\circ\}, \tag{6.1}$$

in the sense of stochastic convergence with respect to the vacuum states, i.e.,

$$\lim_{\nu} \left\langle \Phi_0^{(\nu)}, \frac{A_{\nu}^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_{\nu}^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_0^{(\nu)} \right\rangle = \langle \Psi_0, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_0 \rangle, \tag{6.2}$$

for any $\epsilon_1, \dots, \epsilon_m \in \{+,-,\circ\}$ *and* $m = 1, 2, \dots$.

As a classical reduction we immediately obtain the following

Theorem 6.2 (CLT in the vacuum state)**.** *The notation and assumptions being the same as in Theorem* 6.1, *let* μ *be a probability distribution of which the Jacobi coefficient is* $(\{\omega_n\},\{\alpha_n\})$. *Then it holds that*

$$\lim_{\nu} \left\langle \Phi_0^{(\nu)}, \left(\frac{A_{\nu}}{\sqrt{\kappa(\nu)}} \right)^m \Phi_0^{(\nu)} \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots.$$

In other words, μ *is the asymptotic spectral distribution of* A_{ν} *in the vacuum state.*

In fact, we prove the following result, which is more general than Theorem 6.1, and which will be used in the next section too.

Theorem 6.3 (QCLT in a general form)**.** *The notation and assumptions being the same as in Theorem* 6.1, *we have*

$$\lim_{\nu} \left\langle \Phi_j^{(\nu)}, \frac{A_{\nu}^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_{\nu}^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_j, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_n \rangle, \tag{6.3}$$

for any $\epsilon_1, \dots, \epsilon_m \in \{+,-,\circ\}$, $m = 1, 2, \dots,$ *and* $j, n = 0, 1, 2, \dots$.

For (6.3) we need to study

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} \Phi_n.$$

The explicit actions of the quantum components A^{ϵ} are given in (3.5)–(3.7). They are rephrased by using the mean values $M(\omega_{\epsilon}|V_n)$ as follows: for $n = 0, 1, 2, \dots$,

$$A^{+}\Phi_n = M(\omega_-|V_{n+1}) \left(\frac{|V_{n+1}|}{|V_n|} \right)^{1/2} \Phi_{n+1} + \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_{n+1}} (\omega_-(y) - M(\omega_-|V_{n+1}))\delta_y, \tag{6.4}$$

$$A^{-}\Phi_n = M(\omega_+|V_{n-1}) \left(\frac{|V_{n-1}|}{|V_n|} \right)^{1/2} \Phi_{n-1} + \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_{n-1}} (\omega_+(y) - M(\omega_+|V_{n-1}))\delta_y, \tag{6.5}$$

$$A^{\circ}\Phi_n = M(\omega_\circ|V_n)\Phi_n + \frac{1}{\sqrt{|V_n|}} \sum_{y \in V_n} (\omega_\circ(y) - M(\omega_\circ|V_n))\delta_y, \tag{6.6}$$

where we understand that $A^-\Phi_0 = 0$ for the second formula. We want to unify the above three formulae (6.4)–(6.6). We set

$$\gamma_n^+ = M(\omega_-|V_n)\left(\frac{|V_n|}{\kappa|V_{n-1}|}\right)^{1/2}, \quad n = 1, 2, \dots, \tag{6.7}$$

$$\gamma_n^- = M(\omega_+|V_n)\left(\frac{|V_n|}{\kappa|V_{n+1}|}\right)^{1/2}, \quad n = 0, 1, 2, \dots, \tag{6.8}$$

$$\gamma_n^\circ = \frac{M(\omega_\circ|V_n)}{\sqrt{\kappa}}, \quad n = 0, 1, 2, \dots, \tag{6.9}$$

and

$$\begin{aligned} S_n^+ &= \frac{1}{\sqrt{\kappa|V_{n-1}|}} \sum_{y \in V_n} (\omega_-(y) - M(\omega_-|V_n))\delta_y, \quad n = 1, 2, \dots, \\ S_n^- &= \frac{1}{\sqrt{\kappa|V_{n+1}|}} \sum_{y \in V_n} (\omega_+(y) - M(\omega_+|V_n))\delta_y, \quad n = 0, 1, 2, \dots, \\ S_n^\circ &= \frac{1}{\sqrt{\kappa|V_n|}} \sum_{y \in V_n} (\omega_\circ(y) - M(\omega_\circ|V_n))\delta_y, \quad n = 0, 1, 2, \dots. \end{aligned}$$

Then by definition

$$S_0^- = S_0^\circ = 0.$$

We tacitly set

$$S_0^+ = \gamma_{-1}^-\Phi_{-1} = S_{-1}^- = 0.$$

With this notation (6.4)–(6.6) are unified in the following form:

$$\frac{A^\epsilon}{\sqrt{\kappa}}\Phi_n = \gamma_{n+\epsilon}^\epsilon \Phi_{n+\epsilon} + S_{n+\epsilon}^\epsilon, \qquad \epsilon \in \{+, -, \circ\}, \quad n = 0, 1, 2, \dots. \tag{6.10}$$

Then its repeated action is expressible in a concise form:

$$\begin{aligned} &\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}}\Phi_n \\ &\quad = \gamma_{n+\epsilon_1}^{\epsilon_1}\gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m}\Phi_{n+\epsilon_1+\cdots+\epsilon_m} \\ &\qquad + \sum_{k=1}^m \underbrace{\gamma_{n+\epsilon_1}^{\epsilon_1} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_{k-1}}^{\epsilon_{k-1}}}_{(k-1)\text{ times}} \underbrace{\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa}}}_{(m-k)\text{ times}} S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k}. \end{aligned} \tag{6.11}$$

For the estimate of the second term of (6.11) we define

$$M_{n,q}^- = \max\left\{\prod_{j=1}^q L(\omega_-|V_{k_j})\,;\, 1 \le k_1, k_2, \dots, k_q \le n\right\}, \tag{6.12}$$

$$M_{n,q}^\circ = \max\left\{\prod_{j=1}^q \frac{L(\omega_\circ|V_{k_j})}{\sqrt{\kappa}}\,;\, 1 \le k_1, k_2, \dots, k_q \le n\right\}, \tag{6.13}$$

$$M_{n,0}^- = M_{n,0}^\circ = 1, \tag{6.14}$$

where $n, q = 1, 2, \dots$.

Lemma 6.4. *Let $m, n = 1, 2, \dots$. If $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$ satisfy*

$$n + \epsilon_1 \geq 0, \quad n + \epsilon_1 + \epsilon_2 \geq 0, \quad \dots, \quad n + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_m \geq 0, \tag{6.15}$$

then, denoting respectively by p, q and r the numbers of $+$, $-$ and $\circ$ in $\{\epsilon_1, \dots, \epsilon_m\}$, we have

$$\begin{aligned} &\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \right| \\ &\qquad \leq \Sigma(\omega_-|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}}, \end{aligned} \tag{6.16}$$

$$\begin{aligned} &\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^- \right\rangle \right| \\ &\qquad \leq \{\Sigma(\omega_-|V_n) + \Sigma(\omega_\circ|V_n)\} M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n+1}|}}, \end{aligned} \tag{6.17}$$

$$\begin{aligned} &\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^\circ \right\rangle \right| \\ &\qquad \leq \Sigma(\omega_\circ|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} \sqrt{|V_n|}}{\sqrt{|V_{n+p-q}|}}. \end{aligned} \tag{6.18}$$

Proof. Note first that

$$\begin{aligned} &\frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \\ &\qquad = \frac{1}{\sqrt{\kappa |V_{n-1}|}} \sum_{y \in V_n} (\omega_-(y) - M(\omega_-|V_n)) \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} \delta_y \\ &\qquad = \frac{\kappa^{-m/2}}{\sqrt{\kappa |V_{n-1}|}} \sum_{y \in V_n} (\omega_-(y) - M(\omega_-|V_n)) A^{\epsilon_m} \dots A^{\epsilon_1} \delta_y. \end{aligned} \tag{6.19}$$

We use a new notation. For $y, z \in V$ and $\epsilon \in \{+, -, \circ\}$ we write $y \xrightarrow{\epsilon} z$ if $z \sim y$ and $\partial(z, o) = \partial(y, o) + \epsilon$. For $y, z \in V$ we put

$$\begin{aligned} &w(y; \epsilon_1, \dots, \epsilon_m; z) \\ &\qquad = |\{(z_1, \dots, z_{m-1}) \in V^{m-1}; \, y \xrightarrow{\epsilon_1} z_1 \xrightarrow{\epsilon_2} z_2 \cdots \xrightarrow{\epsilon_{m-1}} z_{m-1} \xrightarrow{\epsilon_m} z\}|. \end{aligned}$$

This counts the walks from y to z along edges with directions $\epsilon_1, \dots, \epsilon_m$. Then (6.19) becomes

$$= \frac{\kappa^{-m/2}}{\sqrt{\kappa |V_{n-1}|}} \sum_{y \in V_n} \sum_{z \in V_{n+p-q}} (\omega_-(y) - M(\omega_-|V_n)) w(y; \epsilon_1, \dots, \epsilon_m; z) \delta_z.$$

Therefore,

$$\left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \tag{6.20}$$
$$= \frac{1}{\sqrt{|V_{n+p-q}|}} \frac{\kappa^{-m/2}}{\sqrt{\kappa |V_{n-1}|}}$$
$$\times \sum_{y \in V_n} \sum_{z \in V_{n+p-q}} (\omega_-(y) - M(\omega_-|V_n)) w(y; \epsilon_1, \dots, \epsilon_m; z).$$

Let $y \in V_n$ be fixed. Then

$$\sum_{z \in V_{n+p-q}} w(y; \epsilon_1, \dots, \epsilon_m; z) \tag{6.21}$$

coincides with the number of walks from y to a certain point in V_{n+p-q} along m edges with directions $\epsilon_1, \dots, \epsilon_m$ in order. Consider an intermediate point $\xi \in V_k$ in such a walk. The number of edges from ξ with $-$ direction is bounded by $L(\omega_-|V_k)$, with $\circ$ direction by $L(\omega_\circ|V_k)$, and with $+$ direction by κ. Given $(\epsilon_1, \dots, \epsilon_m)$, the $+$, $-$ and $\circ$ directions appear p, q and r times, respectively, and the intermediate point ξ lies in $V_0 \cup V_1 \cup \dots \cup V_{n+p}$. Hence by (6.12) and (6.13) we obtain

$$\sum_{z \in V_{n+p-q}} w(y; \epsilon_1, \dots, \epsilon_m; z) \le \kappa^{p+\frac{r}{2}} M^-_{n+p,q} M^\circ_{n+p,r}.$$

Noting that the right hand side is independent of $y \in V_n$, we estimate (6.20) as follows:

$$\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \right|$$
$$\le \frac{\kappa^{p+\frac{r}{2}} M^-_{n+p,q} M^\circ_{n+p,r}}{\sqrt{|V_{n+p-q}|}} \frac{\kappa^{-m/2}}{\sqrt{\kappa |V_{n-1}|}} \sum_{y \in V_n} |\omega_-(y) - M(\omega_-|V_n)|$$
$$\le \frac{\kappa^{p+\frac{r}{2}-\frac{m}{2}-\frac{1}{2}} M^-_{n+p,q} M^\circ_{n+p,r}}{\sqrt{|V_{n+p-q}||V_{n-1}|}} \left(\sum_{y \in V_n} |\omega_-(y) - M(\omega_-|V_n)|^2 \right)^{1/2} |V_n|^{1/2}$$
$$= \Sigma(\omega_-|V_n) M^-_{n+p,q} M^\circ_{n+p,r} \frac{\kappa^{p+\frac{r-m-1}{2}} |V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}}.$$

This proves inequality (6.16). The proofs of (6.17) and (6.18) are similar. □

Lemma 6.5. *Let $\mathcal{G} = (V, E)$ be a regular graph with degree κ. Fix an origin $o \in V$ and consider the stratification as usual. Then, for any $n = 1, 2, \dots$ with $V_n \neq \emptyset$ we have*

$$|V_n| = \kappa^n \prod_{j=1}^{n} M(\omega_-|V_j)^{-1} \prod_{j=0}^{n-1} \left(1 - \frac{M(\omega_-|V_j)}{\kappa} - \frac{M(\omega_\circ|V_j)}{\kappa} \right). \tag{6.22}$$

Proof. By counting the edges connecting two strata V_n and V_{n-1}, we have

$$\begin{aligned}\kappa|V_{n-1}| &= \sum_{x\in V_{n-1}} \{\omega_+(x)+\omega_-(x)+\omega_\circ(x)\} \\ &= \sum_{y\in V_n}\omega_-(y) + \sum_{x\in V_{n-1}}\omega_-(x) + \sum_{x\in V_{n-1}}\omega_\circ(x) \\ &= M(\omega_-|V_n)|V_n| + M(\omega_-|V_{n-1})|V_{n-1}| + M(\omega_\circ|V_{n-1})|V_{n-1}|.\end{aligned}$$

Hence,

$$|V_n| = \kappa M(\omega_-|V_n)^{-1}|V_{n-1}|\left(1-\frac{M(\omega_-|V_{n-1})}{\kappa}-\frac{M(\omega_\circ|V_{n-1})}{\kappa}\right). \tag{6.23}$$

Noting that $V_n \neq \emptyset$ implies $V_{n-1}\neq\emptyset$, $\dots$, $V_1\neq\emptyset$, we obtain (6.22) by repeated application of (6.23). □

Lemma 6.6. *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph satisfying (A1)–(A3). Then, for any $n = 1, 2, \dots$ we have*

$$\lim_\nu \frac{|V_n^{(\nu)}|}{\kappa(\nu)^n} = \frac{1}{\omega_n\dots\omega_1}. \tag{6.24}$$

Proof. By Lemma 6.5 we have

$$\lim_\nu \frac{|V_n^{(\nu)}|}{\kappa(\nu)^n} = \lim_\nu \prod_{j=1}^n M(\omega_-|V_j^{(\nu)})^{-1}\prod_{j=0}^{n-1}\left(1-\frac{M(\omega_-|V_j^{(\nu)})}{\kappa(\nu)}-\frac{M(\omega_\circ|V_j^{(\nu)})}{\kappa(\nu)}\right).$$

The first product converges to $(\omega_n\dots\omega_1)^{-1}$ by (A2) and the second one to 1 by (A3) so that (6.24) follows. □

Proof of Theorem 6.3. Let $\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph as stated therein. Given $\epsilon_1,\dots,\epsilon_m \in \{+,-,\circ\}$, $m = 1,2,\dots$, and $n, j = 0,1,2,\dots$ we consider

$$\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}}\,\Phi_n^{(\nu)}\right\rangle. \tag{6.25}$$

Let p, q, r be the numbers of $+, -, \circ$ appearing in $\{\epsilon_1,\dots,\epsilon_m\}$, respectively. In view of the up-down action of A^ϵ we see easily that (6.25) is zero unless (6.15) and $j = n+p-q$ hold. On the other hand, in that case it follows by the definition of an interacting Fock space $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ that

$$\langle \Psi_j, B^{\epsilon_m}\dots B^{\epsilon_1}\Psi_n\rangle = 0.$$

We have thus proved (6.3) for the case where (6.15) or $j = n+p-q$ is not fulfilled.

Now we consider the case where both (6.15) and $j = n+p-q$ are fulfilled. Using (6.11), we obtain

$$\begin{aligned}&\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}}\,\Phi_n^{(\nu)}\right\rangle \\ &\quad= \gamma_{n+\epsilon_1}^{\epsilon_1}\gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2}\cdots\gamma_{n+\epsilon_1+\dots+\epsilon_m}^{\epsilon_m} \\ &\qquad+\sum_{k=1}^m \gamma_{n+\epsilon_1}^{\epsilon_1}\cdots\gamma_{n+\epsilon_1+\dots+\epsilon_{k-1}}^{\epsilon_{k-1}}\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A_\nu^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}}\,S_{n+\epsilon_1+\dots+\epsilon_k}^{\epsilon_k}\right\rangle.\end{aligned} \tag{6.26}$$

Recall that the coefficient γ_n^ϵ depends on ν. The explicit expressions of γ_n^ϵ being given in (6.7)–(6.9), with the help of Lemma 6.6 and conditions (A1)–(A3) we arrive at

$$\lim_\nu \gamma_n^+ = \lim_\nu \sqrt{M(\omega_-|V_n)} = \sqrt{\omega_n}, \tag{6.27}$$

$$\lim_\nu \gamma_n^- = \lim_\nu \{\kappa - M(\omega_-|V_n) - M(\omega_\circ|V_n)\}\frac{\sqrt{M(\omega_-|V_{n+1})}}{\kappa} = \sqrt{\omega_{n+1}}, \tag{6.28}$$

$$\lim_\nu \gamma_n^\circ = \alpha_{n+1}. \tag{6.29}$$

Then by the definition of B^ϵ we obtain

$$\lim_\nu \gamma_{n+\epsilon_1}^{\epsilon_1}\gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2}\cdots\gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m} = \langle \Psi_j, B^{\epsilon_m}\dots B^{\epsilon_1}\Psi_n\rangle.$$

Thus, for (6.3) it is sufficient to show that the second term of (6.26) vanishes in the limit. Since it is a finite sum, we need only to show that

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}}S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k}\right\rangle = 0. \tag{6.30}$$

For this it is sufficient to show that the right hand sides of (6.16)–(6.18) in Lemma 6.4 vanish in the limit, i.e.,

$$\lim_\nu \Sigma(\omega_-|V_n)M_{n+p,q}^- M_{n+p,r}^\circ \frac{\kappa^{p+\frac{r-m-1}{2}}|V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}} = 0, \tag{6.31}$$

$$\lim_\nu \{\Sigma(\omega_-|V_n)+\Sigma(\omega_\circ|V_n)\}M_{n+p,q}^- M_{n+p,r}^\circ \frac{\kappa^{p+\frac{r-m-1}{2}}|V_n|}{\sqrt{|V_{n+p-q}||V_{n+1}|}} = 0, \tag{6.32}$$

$$\lim_\nu \Sigma(\omega_\circ|V_n)M_{n+p,q}^- M_{n+p,r}^\circ \frac{\kappa^{p+\frac{r-m-1}{2}}\sqrt{|V_n|}}{\sqrt{|V_{n+p-q}|}} = 0. \tag{6.33}$$

First note that $M_{n+p,q}^- M_{n+p,r}^\circ$ converges to a finite limit, as is seen from (6.12)–(6.14) and conditions (A1)–(A3). On the other hand, we see by Lemma 6.6 that

$$\frac{\kappa^{p+\frac{r-m-1}{2}}|V_n|}{\sqrt{|V_{n+p-q}||V_{n-1}|}} = O(1), \qquad \frac{\kappa^{p+\frac{r-m-1}{2}}|V_n|}{\sqrt{|V_{n+p-q}||V_{n+1}|}} = O(\kappa^{-1}),$$

$$\frac{\kappa^{p+\frac{r-m-1}{2}}\sqrt{|V_n|}}{\sqrt{|V_{n+p-q}|}} = O(\kappa^{-1/2}).$$

Then, again by (A2) and (A3), we may see (6.31)–(6.33) with no difficulty. Thus the proof is complete. □

7. Quantum central limit theorem in the deformed vacuum state

In this section we focus on the deformed vacuum state introduced in Section 2. Let $\mathcal{G} = (V, E)$ be a regular graph with degree κ and A its adjacency matrix. Given a deformed vacuum state $\langle\cdot\rangle_q$ on $\mathcal{A}(\mathcal{G})$, by a simple computation we obtain

$$\langle A\rangle_q = \kappa q, \tag{7.1}$$

$$\Sigma_q^2(A) = \kappa(1-q)(1+q+qM(\omega_\circ|V_1)). \tag{7.2}$$

For normalization we need $\Sigma_q^2(A) > 0$ so that we assume that

$$-\frac{1}{1 + M(\omega_\circ|V_1)} < q < 1. \tag{7.3}$$

(Here we do not assume the positivity of $\langle\cdot\rangle_q$.) Then the normalized adjacency matrix becomes

$$\frac{A - \langle A\rangle_q}{\Sigma_q(A)} = \frac{\tilde{A}^+}{\Sigma_q(A)} + \frac{\tilde{A}^-}{\Sigma_q(A)} + \frac{\tilde{A}^\circ}{\Sigma_q(A)}, \tag{7.4}$$

where

$$\tilde{A}^\pm = A^\pm, \quad \tilde{A}^\circ = A^\circ - \langle A\rangle_q,$$

and

$$A = A^+ + A^- + A^\circ$$

is the quantum decomposition.

Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph and A_ν the adjacency matrix. Suppose that for each ν the adjacency algebra $\mathcal{A}(\mathcal{G}^{(\nu)})$ is given a deformed vacuum state with parameter $q = q(\nu)$ satisfying (7.3). We consider the normalization of A_ν as in (7.4). Then we are interested in

$$\lim_\nu \left\langle \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)} \right\rangle_q = \lim_\nu \left\langle Q\delta_o, \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)} \delta_o \right\rangle. \tag{7.5}$$

We consider the following condition:

(A4) $\lim_\nu q(\nu) = 0$ and there exists a limit

$$\gamma = \lim_\nu q(\nu)\sqrt{\kappa(\nu)}. \tag{7.6}$$

As will be seen below, this is a unique scaling balance which yields a meaningful limit for (7.5).

Lemma 7.1. *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph and keep the notation as above. Under conditions (A1)–(A4) with $\gamma > -1/\alpha_2$ we have*

$$\lim_\nu \frac{\Sigma_q^2(A_\nu)}{\kappa(\nu)} = 1 + \gamma\alpha_2, \tag{7.7}$$

$$\lim_\nu \frac{\langle A_\nu\rangle_q}{\Sigma_q(A_\nu)} = \frac{\gamma}{\sqrt{1 + \gamma\alpha_2}}, \tag{7.8}$$

$$\lim_\nu q^n \sqrt{|V_n|} = \frac{\gamma^n}{\sqrt{\omega_n \dots \omega_1}}, \tag{7.9}$$

where $\{\omega_n\}$, $\{\alpha_n\}$ and γ are defined in (A2)–(A4).

Proof. It follows from (7.2) that

$$\begin{aligned}\frac{\Sigma_q^2(A)}{\kappa} &= (1-q)(1 + q + qM(\omega_\circ|V_1)) \\ &= (1-q)\left(1 + q + q\sqrt{\kappa}\,\frac{M(\omega_\circ|V_1)}{\sqrt{\kappa}}\right).\end{aligned}$$

Then, (7.7) follows from condition (A3) and (7.6). By (7.1) and (7.7) we have

$$\lim_\nu \frac{\langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} = \lim_\nu \frac{\kappa q}{\sqrt{\kappa(1+\gamma\alpha_2)}} = \frac{\gamma}{\sqrt{1+\gamma\alpha_2}},$$

which proves (7.8). Finally, using Lemma 6.6, we have

$$\lim_\nu q^{2n}|V_n| = \lim_\nu q^{2n}\kappa^n \prod_{k=1}^n M(\omega_-|V_k)^{-1} = \frac{\gamma^{2n}}{\omega_n \dots \omega_1},$$

from which (7.9) follows. □

Lemma 7.2. *The notation and assumptions being the same as in Lemma* 7.1*, let* $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ *be the interacting Fock space associated with* $\{\omega_n\}$ *and* B° *the diagonal operator associated with* $\{\alpha_n\}$*. Define*

$$\tilde{B}^\pm = \frac{B^\pm}{\sqrt{1+\gamma\alpha_2}}, \qquad \tilde{B}^\circ = \frac{B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}}.$$

Then we have

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_j, \tilde{B}^{\epsilon_m} \dots \tilde{B}^{\epsilon_1} \Psi_n \rangle, \tag{7.10}$$

for any $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$*,* $m = 1, 2, \dots,$ *and* $j, n = 0, 1, 2, \dots$.

Proof. This follows directly from Theorem 6.3 by changing constant factors with the help of (7.7) and (7.8) in Lemma 7.1. □

We are now in a position to study (7.5). In view of the up-down actions of $\tilde{A}_\nu^\epsilon$ we see that

$$\left\langle Q\delta_o, \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)} \delta_o \right\rangle = \sum_{n=0}^m q^n \sqrt{|V_n|} \left\langle \Phi_n, \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A)} \Phi_0 \right\rangle. \tag{7.11}$$

Namely, although $Q\delta_o$ is an infinite (formal) sum:

$$Q\delta_o = \sum_{x\in V} q^{\partial(o,x)} \delta_x = \sum_{n=0}^\infty q^n \sqrt{|V_n|}\, \Phi_n,$$

only the partial sum up to $n = m$ (independent of ν) contributes to the inner product in the left hand side of (7.11). It then follows immediately from Lemmata 7.1 and 7.2 that

$$\lim_\nu \left\langle Q\delta_o, \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A)} \delta_o \right\rangle = \sum_{n=0}^m \frac{\gamma^n}{\sqrt{\omega_n \dots \omega_1}} \langle \Psi_n, \tilde{B}^{\epsilon_m} \dots \tilde{B}^{\epsilon_1} \Psi_0 \rangle. \tag{7.12}$$

The next definition is useful.

Definition 7.3. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be an interacting Fock space associated with a Jacobi sequence $\{\omega_n\}$. A generalized vector defined by

$$\Omega_\gamma = \Psi_0 + \sum_{n=1}^\infty \frac{\gamma^n}{\sqrt{\omega_n \dots \omega_1}} \Psi_n, \qquad \gamma \in \mathbf{C}, \tag{7.13}$$

is called a *coherent vector*. A normalized linear function $a \mapsto \langle \Omega_\gamma, a\Psi_0 \rangle$, where a runs over the $*$-algebra generated by $B^\pm$ and a diagonal operator, is called a *coherent state* (disregarding the positivity).

If $\{\omega_n\}$ is of infinite type, the coherent vector Ω_γ is a generalized eigenvector of B^-, i.e., $B^-\Omega_\gamma = \gamma\Omega_\gamma$. Obviously, the right hand side of (7.12) coincides with

$$\langle \Omega_\gamma, \tilde{B}^{\epsilon_m} \dots \tilde{B}^{\epsilon_1} \Psi_0 \rangle.$$

Summing up,

Theorem 7.4 (QCLT in the deformed vacuum state)**.** *Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing regular graph with a fixed origin $o_\nu \in V^{(\nu)}$ and $A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ$ the quantum decomposition of its adjacency matrix. Let each adjacency algebra $\mathcal{A}(\mathcal{G}^{(\nu)})$ be given a deformed vacuum state with $q = q(\nu)$. Define*

$$\tilde{A}_\nu^\pm = A_\nu^\pm, \qquad \tilde{A}_\nu^\circ = A_\nu^\circ - \langle A_\nu \rangle_q.$$

Assume that conditions (A1)–(A4) are satisfied with $\gamma > -1/\alpha_2$. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\}$ and B° the diagonal operator associated with $\{\alpha_n\}$. Define

$$\tilde{B}^\pm = \frac{B^\pm}{\sqrt{1+\gamma\alpha_2}}, \qquad \tilde{B}^\circ = \frac{B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}}.$$

Then for any $\epsilon_1, \dots, \epsilon_m \in \{+, -, \circ\}$ and $m = 1, 2, \dots,$ we have

$$\lim_\nu \left\langle \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A)} \right\rangle_q = \langle \Omega_\gamma, \tilde{B}^{\epsilon_m} \dots \tilde{B}^{\epsilon_1} \Psi_0 \rangle. \tag{7.14}$$

In particular,

$$\lim_\nu \left\langle \left(\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} \right)^m \right\rangle_q = \left\langle \Omega_\gamma, \left(\frac{B^+ + B^- + B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}} \right)^m \Psi_0 \right\rangle, \tag{7.15}$$

for any $m = 1, 2, \dots$.

If the deformed vacuum state $\langle \cdot \rangle_q$ on $\mathcal{A}(\mathcal{G}^{(\nu)})$ is positive, then there exists a probability distribution μ such that

$$\lim_\nu \left\langle \left(\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} \right)^m \right\rangle_q = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 0, 1, 2, \dots.$$

The above μ is called the asymptotic spectral distribution in a deformed vacuum state, and to find this μ is our main question; see Section 1. In that case, we see from (7.15) that the coherent state in the right hand side is also positive on the algebra generated by $B^+ + B^- + B^\circ$. Thus, our question is reduced to finding a probability distribution μ satisfying the identity:

$$\int_{-\infty}^{+\infty} x^m \mu(dx) = \left\langle \Omega_\gamma, \left(\frac{B^+ + B^- + B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}} \right)^m \Psi_0 \right\rangle, \qquad m = 0, 1, 2, \dots.$$

This question is to be solved within the theory of interacting Fock spaces and orthogonal polynomials. Simple examples are found in the next section.

8. Examples and comments

We discuss two guiding examples of distance-regular graphs (see, e.g., Bannai–Ito [4] for the fundamentals) to illustrate how our approach is applied to a concrete problem.

Let $\mathcal{G}^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a (finite or infinite) growing distance-regular graph with intersection numbers $p_{ij}^k(\nu)$. Since $\Gamma(\mathcal{G}^{(\nu)})$ is invariant under the quantum components A_ν^ϵ, the conditions (A1)–(A3) are reduced to much simpler forms:

(DA1) $\lim_\nu \kappa(\nu) = \lim_\nu p_{11}^0(\nu) = \infty$;

(DA2) for each $n = 1, 2, \dots$ the limit $\omega_n = \lim_\nu p_{1,n-1}^n(\nu) < \infty$ exists;

(DA3) for each $n = 0, 1, 2, \dots$ the limit $\alpha_{n+1} = \lim_\nu \dfrac{p_{1,n}^n(\nu)}{\sqrt{p_{11}^0(\nu)}} < \infty$ exists.

It is also noteworthy that $AQ = QA$ holds for any distance-regular graph. Hence, the deformed vacuum state $\langle\cdot\rangle_q$ is positive whenever Q is a positive definite kernel on V; see Proposition 2.3.

Example 8.1 (Homogeneous trees)**.** A homogeneous tree with degree $\kappa = 2, 3, \dots$, denoted by $\mathcal{T}_\kappa$, is a distance-regular graph with intersection numbers:

$$p_{11}^0 = \kappa, \qquad p_{1,n-1}^n = 1, \qquad p_{1,n}^n = 0,$$

from which we see that conditions (DA1)–(DA3) are fulfilled with $\{\omega_n \equiv 1\}$ and $\{\alpha_n \equiv 0\}$. Hence the limit is described in terms of the free Fock space $\Gamma_{\text{free}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$. In particular, for $m = 1, 2, \dots$ we have

$$\begin{aligned}\lim_{\kappa\to\infty} \left\langle \delta_o, \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m \delta_o \right\rangle &= \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle \\ &= \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx;\end{aligned}$$

that is, the spectral distribution in the vacuum state is the Wigner semicircle law. We know, from a different aspect, that this is a prototype of the free central limit theorem due to Voiculescu [38]. The spectral distribution of $\mathcal{T}_\kappa$ in the vacuum state (for each κ) is also obtained by our method, and Kesten's result [29] is reproduced. As for the deformed vacuum state, we first recall that Q is a positive definite kernel on $\mathcal{T}_\kappa$ for all $-1 \le q \le 1$, which is due to Haagerup [12]; see also Bożejko [7]. We often call $\langle\cdot\rangle_q$ the *Haagerup state*. Thus, as a direct consequence of Theorem 7.4, for any $\gamma \in \mathbf{R}$ we have

$$\lim_{\substack{\kappa\to\infty \\ q\to 0 \\ q\sqrt{\kappa}\to\gamma}} \left\langle \left(\frac{A_\nu - \langle A\rangle_q}{\Sigma_q(A_\nu)} \right)^m \right\rangle_q = \langle \Omega_\gamma, (B^+ + B^- - \gamma)^m \Psi_0 \rangle, \quad m = 1, 2, \dots, \tag{8.1}$$

where Ω_γ is the coherent vector. To obtain the asymptotic spectral distribution $\mu = \mu_\gamma$ we need to manipulate the right hand side of (8.1). In fact, employing the following interesting formula for the free Fock space:

$$\langle \Omega_\gamma, (B^+ + B^- - \gamma)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- - \gamma B^+ B^-)^m \Psi_0 \rangle, \quad m = 1, 2, \dots,$$

we see that μ_γ is an affine transformation of the free Poisson distribution; for the definition see, e.g., Hiai–Petz [17]. We also remark that μ_γ was first obtained by Hashimoto [13] by means of a Fourier transform and Bessel functions.

Example 8.2 (Hamming graphs)**.** Let $d \ge 1$, $N \ge 1$ be a pair of integers and set

$$V = \{x = (\xi_1, \dots, \xi_d)\,;\, \xi_i \in \{1, 2, \dots, N\}\}, \quad E = \{\{x, y\}\,;\, x, y \in V,\ \partial(x, y) = 1\},$$

where $\partial(x, y)$ is the Hamming distance. The graph (V, E) is called the *Hamming graph* and denoted by $H(d, N)$. Among the intersection numbers we note that

$$p_{11}^0 = d(N-1), \qquad p_{1,n-1}^n = n, \qquad p_{1,n}^n = n(N-2).$$

In order that conditions (DA1)–(DA3) are fulfilled, for the growing parameter $\nu = (d, N)$ we take the following limit:

$$d \to \infty, \quad \frac{N}{d} \to \tau \in [0, \infty). \tag{8.2}$$

In that case we have $\{\omega_n = n\}$ and $\{\alpha_n = \sqrt{\tau}\,(n-1)\}$. Therefore the limit is described by means of the Boson Fock space $\Gamma_{\mathrm{Boson}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ and the diagonal operator B° associated with $\{\alpha_n\}$. Note also that $B^\circ = \sqrt{\tau}\,N = \sqrt{\tau}\,B^+B^-$, where N is the number operator. Thus, for the asymptotic spectral distribution in the vacuum state we have

$$\lim_{(8.2)} \left\langle \Phi_0, \left(\frac{A_{d,N}}{\sqrt{\kappa_{d,N}}}\right)^m \Phi_0 \right\rangle = \langle \Psi_0, (B^+ + B^- + \sqrt{\tau}\,B^+B^-)^m \Psi_0 \rangle, \quad m = 1, 2, \dots .$$

The right hand side is the m-th moment of the standard Gaussian distribution for $\tau = 0$ or of an affine transformation of the Poisson distribution with parameter τ^{-1} for $\tau > 0$; see Hashimoto–Obata–Tabei [16] for an explicit description. Note next that Q is a positive definite kernel on V for $0 \le q \le 1$ since a Hamming graph admits a quadratic embedding [20]. Hence the deformed vacuum state $\langle \cdot \rangle_q$ is positive for $0 \le q \le 1$. As a direct consequence from Theorem 7.4, we have

$$\lim_{\substack{(8.2)\\ q\sqrt{\kappa}\to\gamma}} \left\langle \left(\frac{A_{d,N} - \langle A_{d,N}\rangle_q}{\Sigma_q(A_{N,d})}\right)^m \right\rangle_q = \left\langle \Omega_\gamma, \left(\frac{B^+ + B^- + \sqrt{\tau}\,B^+B^- - \gamma}{\sqrt{1+\gamma\sqrt{\tau}}}\right)^m \Psi_0 \right\rangle,$$

where $\gamma \ge 0$ and Ω_γ is the coherent vector of the Boson Fock space. The asymptotic spectral distribution is computed by transforming the right hand side to an expression in terms of the vacuum state. In fact, this can be done by using the formulae:

$$\langle \Omega_\gamma, (B^+ + B^-)^m \Psi_0 \rangle = \langle \Psi_0, (B^+ + B^- + \gamma)^m \Psi_0 \rangle,$$

$$\begin{aligned} &\langle \Omega_\gamma, \{(B^+ + \lambda)(B^- + \lambda)\}^m \Psi_0 \rangle \\ &\qquad = \big\langle \Psi_0, \big\{\big(B^+ + \sqrt{\lambda(\lambda+\gamma)}\,\big)\big(B^- + \sqrt{\lambda(\lambda+\gamma)}\,\big)\big\}^m \Psi_0 \big\rangle, \quad \lambda \ge 0. \end{aligned}$$

As a result, the asymptotic spectral distribution in a deformed vacuum state is the standard Gaussian distribution or an affine transformation of the Poisson distribution, depending on the parameters τ and γ; see Hora [20] for the explicit expressions.

We wish to emphasize that the argument in the above examples is much simpler and clearer than that in the previous papers. Moreover, conditions (A1)–(A4) give a clear insight into a growing regular graph from the viewpoint of spectral analysis.

9. Appendix: Rephrasing conditions (A1)–(A3)

Proposition 9.1. *In the conditions (A1)–(A3), we may replace* (5.4), (5.5) *with a single condition: for each $n = 1, 2, \dots$ there exists a constant number ω_n independent of ν such that*

$$\lim_{\nu} \frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) = \omega_n\}|}{|V_n^{(\nu)}|} = 1. \tag{9.1}$$

Proof. Throughout the proof, $n = 1, 2, \dots$ is fixed arbitrarily. We first prove that (9.1) implies (5.4) and (5.5). Divide $V_n^{(\nu)}$ into two parts:

$$U_{\mathrm{reg}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, \omega_-(x) = \omega_n\}, \qquad U_{\mathrm{sing}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, \omega_-(x) \neq \omega_n\},$$

where the index n is omitted for simplicity. The average of $\omega_-(x)$ is given by

$$\begin{aligned} M(\omega_-|V_n^{(\nu)}) &= \frac{1}{|V_n^{(\nu)}|}\Bigg(\sum_{x\in U_{\mathrm{reg}}^{(\nu)}} \omega_-(x) + \sum_{x\in U_{\mathrm{sing}}^{(\nu)}} \omega_-(x) \Bigg) \\ &= \frac{|U_{\mathrm{reg}}^{(\nu)}|}{|V_n^{(\nu)}|}\,\omega_n + \frac{1}{|V_n^{(\nu)}|} \sum_{x\in U_{\mathrm{sing}}^{(\nu)}} \omega_-(x). \end{aligned}$$

In view of (5.6) we set

$$W_n = \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

Then $\omega_-(x) \le W_n$ for $x \in V_n^{(\nu)}$ and we obtain

$$|M(\omega_-|V_n^{(\nu)}) - \omega_n| \le \Bigg(1 - \frac{|U_{\mathrm{reg}}^{(\nu)}|}{|V_n^{(\nu)}|}\Bigg)\omega_n + \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|}\,W_n \le \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|}\,(\omega_n + W_n).$$

Since

$$\lim_\nu \frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|} = 0, \tag{9.2}$$

by (9.1), we obtain

$$\lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n, \tag{9.3}$$

which proves (5.4). We next consider the variance. By Minkowski's inequality, we obtain

$$\begin{aligned} &\Sigma(\omega_-|V_n^{(\nu)}) \\ &= \Bigg\{ \frac{1}{|V_n^{(\nu)}|} \sum_{x\in V_n^{(\nu)}} (\omega_-(x) - M(\omega_-|V_n^{(\nu)}))^2 \Bigg\}^{1/2} \\ &\le \Bigg\{ \frac{1}{|V_n^{(\nu)}|} \sum_{x\in V_n^{(\nu)}} (\omega_-(x) - \omega_n)^2 \Bigg\}^{1/2} + \Bigg\{ \frac{1}{|V_n^{(\nu)}|} \sum_{x\in V_n^{(\nu)}} (\omega_n - M(\omega_-|V_n^{(\nu)}))^2 \Bigg\}^{1/2} \\ &= \Bigg\{ \frac{1}{|V_n^{(\nu)}|} \sum_{x\in U_{\mathrm{sing}}^{(\nu)}} (\omega_-(x) - \omega_n)^2 \Bigg\}^{1/2} + \big|\omega_n - M(\omega_-|V_n^{(\nu)})\big|. \end{aligned}$$

Since $|\omega_-(x) - \omega_n| \le \omega_-(x) + \omega_n \le W_n + \omega_n$ for $x \in V_n^{(\nu)}$, we have

$$\Sigma(\omega_-|V_n^{(\nu)}) \le \Bigg(\frac{|U_{\mathrm{sing}}^{(\nu)}|}{|V_n^{(\nu)}|} \Bigg)^{1/2} (W_n + \omega_n) + \big|\omega_n - M(\omega_-|V_n^{(\nu)})\big|,$$

and hence (5.5) follows by (9.2) and (9.3).

We next show that (9.1) is derived from (5.4) and (5.5). By (5.4), for any $\epsilon > 0$ there exists ν_0 such that

$$|M(\omega_-|V_n^{(\nu)}) - \omega_n| < \epsilon, \quad \nu \ge \nu_0.$$

If $x \in V_n^{(\nu)}$ satisfies $|\omega_-(x) - \omega_n| \ge 2\epsilon$, we have

$$\big|\omega_-(x) - M(\omega_-|V_n^{(\nu)})\big| \ge |\omega_-(x) - \omega_n| - |\omega_n - M(\omega_-|V_n^{(\nu)})| \ge \epsilon.$$

Hence

$$\frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|} \le \frac{|\{x \in V_n^{(\nu)}\,;\, \big|\omega_-(x) - M(\omega_-|V_n^{(\nu)})\big| \ge \epsilon\}|}{|V_n^{(\nu)}|}.$$

By Chebyshev's inequality and (5.5) we have

$$\frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|} \le \frac{\Sigma^2(\omega_-|V_n^{(\nu)})}{\epsilon^2} \to 0, \qquad \nu \to \infty. \tag{9.4}$$

We prove that ω_n is an integer. Suppose otherwise. Then, since $\omega_-(x)$ is always an integer, we can choose a sufficiently small $\epsilon > 0$ such that

$$V_n^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \ge 2\epsilon\}.$$

But this contradicts (9.4) and hence ω_n is an integer. Since $\omega_-(x)$ and ω_n are all integers, we may choose a sufficiently small $\epsilon > 0$ such that

$$\frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \ne \omega_n\}|}{|V_n^{(\nu)}|} = \frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \ge 2\epsilon\}|}{|V_n^{(\nu)}|}. \tag{9.5}$$

As is shown in (9.4), the right hand side of (9.5) tends to 0 as $\nu \to \infty$. Therefore

$$\lim_{\nu} \frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \ne \omega_n\}|}{|V_n^{(\nu)}|} = 0$$

and (9.1) follows. □

Combining Proposition 5.3, we see that $\{\omega_n\}$ is necessarily an infinite sequence of positive integers (Proposition 5.4).

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