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UNITARY DUAL OF THE NON-SPLIT INNER FORM OF $Sp(8, F)$

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ABSTRACT. We classify the non-cuspidal part of the unitary dual of the non-quasi-split inner form of $Sp(8, F)$, where F is a non-archimedean field of characteristic zero. We obtain a conjectural description of the discrete L-packets which contain representations of $Sp(4, F)$ and its non-split inner form.

1. INTRODUCTION

We are interested in the classification of the non-cuspidal part of the unitary dual of the non-split inner form of the group $Sp(8, F)$, where F is p-adic field of characteristic zero. We denote this hermitian quaternionic group by $G_2(D, 1)$, where D is a quaternionic division algebra over F. We obtain a complete classification modulo a standard conjecture about the transfer of the Plancherel measure. The analysis of the principal series representations relies mainly on the knowledge of the corresponding Jacquet modules, and in a calculation of those we use the structure of a Ψ–Hopf module on the sum of the Grothendieck groups of the smooth, finite length representations of the hermitian quaternionic groups. We will briefly recall this structure ([20],[5]). The unitary dual of the group $G_2(D, 1)$ has an interesting feature: There is an isolated representation in the unitary dual, and it is a local component of an automorphic representation which lies in the residual spectrum of this group. The consequence of the analysis of the representations which have a cuspidal support on the non-Siegel maximal parabolic subgroup is a conjectural description of the discrete L–packets that contain the representations of $Sp(4, F)$ and its non-split inner form $G_1(D, 1)$.

In the preliminaries we recall the definition of the hermitian quaternionic group and the structure of its Levi subgroups. We also recall the aforementioned structure of the Ψ–Hopf module on the representations. In the second section we analyze the principal series representations, and determine all the subquotients. The case of the principal series representations where the inducing representation of the Levi subgroup $D^* \times D^*$ is of the form $\tau_1 \otimes \tau_2$, for higher dimensional irreducible representations $\tau_i,\ i = 1, 2$ of D^*, is the same as for the non-split inner form of the group $SO(8, F)$, and they are classified in [5]. In the third section we determine all the unitarizable subquotients of these principal series. In the fourth section we calculate the points of reducibility for the representations supported on the Siegel maximal parabolic subgroup and in the fifth section we calculate reducibility points for the representations supported on the non-Sigel maximal parabolic subgroup.

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2. Preliminaries

For an admissible representation σ of any group we consider, we denote by ω_σ its central character (if it exists). We will denote the Steinberg representation of the group G by St_G, and the trivial representation of that group by 1_G. We denote by Δ a basis of the root system on the reductive group G with respect to some maximal split torus (over F). Let θ be a subset of Δ. We denote the corresponding standard parabolic subgroup by P_θ, and the corresponding standard Levi subgroup by M_θ. For an admissible representation σ of the group M_θ, let $\mathrm{Ind}_{P_\theta}^G \sigma$ be the parabolically induced representation (normalized induction). Let w be an element of the Weyl group such that $w(\theta) \subset \Delta$. Then, we will denote by $A_w(\sigma)$ a standard intertwining operator between the representations $\mathrm{Ind}_{P_\theta}^G \sigma$ and $\mathrm{Ind}_{P_{w(\theta)}}^G w(\sigma)$. We denote by $\mathfrak{a}_{\theta,\mathbb{C}}$ the complexified Lie algebra of a standard torus A_θ. Then, $\nu \in \mathfrak{a}_\theta$ defines a character of the Levi subgroup M_θ by the Harish–Chandra homomorphism H_{P_θ}, and a standard intertwining operator for such a "twisted" representation is denoted $A_w(\nu, \sigma)$. The reflection in the Weyl group corresponding to a positive root α is denoted by w_α.

Let F be a non-archimedean local field of characteristic zero, having residual field with q elements. We choose a uniformizer of the field and denote it by $\overline{\omega}$. Let D be a quaternionic algebra, central over F and let τ be an involution, fixing the center of D (involution of the first kind). The division algebra D defines a reductive group G over F as follows. Let

$$V_n = e_1 D \oplus \cdots \oplus e_n D \oplus e_{n+1} D \oplus \cdots \oplus e_{2n} D$$

be a right vector space over D. Fix $\epsilon \in \{1, -1\}$. The relations $(e_i, e_{2n-j+1}) = \delta_{ij}$ for $i = 1, 2, \ldots, n$ define a hermitian form on V_n:

$$(v, v') = \varepsilon \tau((v', v)), v, v' \in V_n, \, \varepsilon \in \{-1, 1\},$$
$$(vx, v'x') = \tau(x)(v, v')x', x, x' \in D.$$

Let $G_n(D, \varepsilon)$ be the group of the isometries of the form $(\cdot, \cdot)$. We are interested in the case $\epsilon = 1$, when the group $G_n(D, 1)$ is the non-split inner form of the group $Sp(4n, F)$. We will fix a maximal F-split torus A_0 of the group $G_n(D, 1)$:

$$A_0(F) = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in F^* \right\}.$$

We denote by s_i an element of the Weyl group which interchanges λ_i and λ_{i+1}, and by c_j the one which interchanges λ_j and λ_j^{-1}, for the element of the torus A_0 of the above form.

For $g = (g_{ij}) \in GL(n, D)$, we denote by g^* an involution $g^* = (\tau(g_{ji}))$, and $g^{-*} = (\tau(g_{ji}))^{-1}$. Then, for an admissible representation π of $GL(n, D)$, we define representation π^* on the same representation space in the following way: $\pi^*(g) = \pi(g^{-*})$. Muić and Savin observed ([12]) that, for an irreducible admissible representation π of $GL(n, D)$, the following holds:

$$\pi^* \cong \widetilde{\pi}.$$

We choose and fix a minimal F–parabolic subgroup of $G_n(D, 1)$ consisting of the upper-triangular matrices, so the standard Levi F–parabolic subgroups are of the following form:

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & A_{n_k} & & & & \\ & & & g & & & \\ & & & & B_{n_k} & & \\ & & & & & \ddots & \\ & & & & & & B_{n_1} \end{pmatrix} : \begin{array}{c} A_{n_i} \in GL(n_i, D), \\ g \in G_r(D, 1) \end{array} \right\},$$

for some integers $n_1, \dots, n_k, r$ such that $\sum n_i + r = n$, and r can be zero. Here, for the matrices B_{n_i} we have $B_{n_i} = J_{n_i} A_{n_i}^{-*} J_{n_i}$, $i = 1, \dots, k$, where J_{n_i} are n_i by n_i matrices with 1's on the opposite diagonal, and zeroes everywhere else.

For admissible representations σ_i, $i = 1, \dots k$, of the groups $GL(n_i, D)$, with $n = n_1 + \cdots + n_k$, a normalized parabolically induced representation $\mathrm{Ind}_P^{GL(n,D)} \sigma_1 \otimes \cdots \otimes \sigma_k$ of the group $GL(n, D)$ will be denoted by $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k$. Here, P is a standard parabolic subgroup with Levi subgroup isomorphic to $GL(n_1, D) \times \cdots \times GL(n_k, D)$.

Analogously, for admissible representations σ_i, $i = 1, \dots, k$, of the groups $GL(n_i, D)$, and an admissible representation τ of the group $G_r(D, 1)$, such that $n = n_1 + \cdots + n_k + r$, a normalized parabolically induced representation $\mathrm{Ind}_P^{G_n(D,1)} \sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau$ of the group $G_n(D, 1)$ will be denoted by $\sigma_1 \times \sigma_2 \cdots \times \sigma_k \rtimes \tau$. The Jacquet module of the representation π of the group $G_n(D, 1)$ with respect to the standard parabolic subgroup P of the above form, will be denoted by $s_{(n_1, n_2, \dots, n_k)}(\pi)$.

We will denote by ν a p–adic norm on F, or the composition of the norm homomorphism from D^* to F^* with the p-adic norm. The standard representations are then of the following form:

$$\sigma_1 \nu^{s_1} \times \sigma_2 \nu^{s_2} \times \cdots \times \sigma_k \nu^{s_k} \rtimes \tau,$$

where the representations σ_i, $i = 1, \dots, k$, are irreducible discrete series representations, the representation τ is irreducible tempered, and s_i, $i = 1, \dots, k$, are real numbers such that $s_1 \geq s_2 \cdots \geq s_k > 0$. The corresponding Langlands quotient is then denoted by $L(\sigma_1 \nu^{s_1}, \sigma_2 \nu^{s_2}, \dots, \sigma_k \nu^{s_k}; \tau)$.

For an irreducible, cuspidal representation σ of the group $GL(n, D)$ there is a discrete series representation σ' of the group $GL(2n, F)$ attached to it by the Jacquet-Langlands correspondence. Depending on whether this representation σ' is cuspidal, or non-cuspidal, we define $\nu_\sigma = \nu$ or $\nu_\sigma = \nu^2$ as a 'distance' in the cuspidal segments ([2]). We will be mainly interested in the case $n = 1$, and in that case, if σ is a character, then $\nu_\sigma = \nu^2$, and if σ is a higher-dimensional representation of D^*, we have $\nu_\sigma = \nu$. When ρ is a cuspidal, irreducible representation of the group $GL(n, D)$, the induced representation $\rho\nu_\rho \times \rho$ of the group $GL(2n, D)$ (which has length equal to two) has a unique irreducible subrepresentation, which is an essentially square integrable representation. We denote it by $\delta(\rho\nu_\rho, \rho)$.

If τ is an irreducible representation of the group D^*, such that the representation $\tau\nu^{s_0} \rtimes 1$ of the group $G_1(D,1)$ reduces for some positive real number s_0, then the representation $\tau\nu^{s_0} \rtimes 1$ has a unique subrepresentation, which is a square integrable, and we denote it by $\delta[\tau\nu^{s_0}; 1]$.

We denote by R_n the Grothendieck group of smooth representations of finite length of the group $GL(n,D)$. Let $R = \bigoplus_{n\geq 0} R_n$. For two finite-length admissible representations π_1 and π_2 of the groups $GL(n_1,D)$ and $GL(n_2,D)$, respectively, we define the multiplication by $m(\pi_1,\pi_2) = \pi_1 \times \pi_2$, and then extend linearly to a mapping $m : R \otimes R \to R$. For a smooth, finite length representation π of $GL(n,D)$, we define

$$m^*(\pi) = \sum_{k=0}^{n} s.s(r_{(k)}(\pi)) \in R \otimes R.$$

Here, $r_{(k)}(\pi)$ denotes the Jacquet module with respect to the standard maximal parabolic subgroup of $GL(n,D)$ with Levi subgroup equal to

$$GL(k,D) \times GL(n-k,D).$$

We extend this comultiplication to a mapping $m^* : R \to R \otimes R$. These two operations define a Hopf algebra structure on R.

Let $R(G_n(D,1)$ denote the Grothendieck group of smooth representations of finite length of the group $G_n(D,1)$, and let $R(G) = \bigoplus_{n\geq 0} R(G_n(D,1))$. Then, $R(G)$ is, by parabolic induction, a module for the algebra R, and the left multiplication by elements of R is, as before, denoted by $\rtimes$. For a smooth, finite length representation σ of the group $G_n(D,1)$ we put

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s(s_{(k)}(\sigma)).$$

We extend μ^* by linearity to $R(G)$. We denote by $s : R \otimes R \to R \otimes R$ a linear map such that $s(\pi_1 \otimes \pi_2) = \pi_2 \otimes \pi_1$ for the representations π_1 and π_2. The ring homomorphism $\Psi : R \to R \otimes R$ is given as the following composition:

$$\Psi = (m \otimes 1) \circ ({}^* \otimes m^*) \circ s \circ m^*.$$

In the previous formula, m is a multiplication, m^* is a comultiplication on R, and * is an involution, defined above for the representations of the $GL(\cdot, D)$–groups. Then, the structure of a Ψ–Hopf module on $R(G)$ is the following ([5],[20]):

$$\mu^*(\pi \rtimes \sigma) = \Psi^*(\pi) \rtimes \mu^*(\sigma).$$

3. The principal series

We shall first analyze the principal series of the form $\pi = \chi_1\nu^\alpha \times \chi_2\nu^\beta \rtimes 1$, where $\chi_i,\ i = 1,2$, are unitary characters of D^*, and α, β are real numbers. It is easy to see (using intertwining operators) that there is a standard representation, unique up to isomorphism, which has the same composition series as the representation π; we denote it by π^s. The following lemma is an easy consequence of the results obtained in [12].

Lemma 3.1. *Let χ be a unitary character of D^*, and $s \in \mathbb{R}$. Then we have the following:*

(i) *If $\chi^2 \neq 1$, the representation $\chi\nu^s \rtimes 1$ of the group $G_1(D,1)$ is irreducible for every $s \in \mathbb{R}$, and then $\chi\nu^s \rtimes 1 = L(\chi\nu^{|s|};1)$ if $s \neq 0$, and the representation $\chi \rtimes 1$ is tempered.*

(ii) *If $\chi^2 = 1$, but $\chi \neq 1$, the representation $\chi\nu^s \rtimes 1$ reduces only for $s = 0$, and in that case, it is a sum of two non-equivalent, tempered representations.*

(iii) *If $\chi = 1$, $\nu^s \rtimes 1$ reduces only for $s = \pm\frac{3}{2}$, and then (in the appropriate Grothendieck group) $\nu^{\pm\frac{3}{2}} \rtimes 1 = L(\nu^{\frac{3}{2}};1) + \delta[\nu^{\frac{3}{2}};1] = 1_{G_1(D,1)} + St_{G_1(D,1)}$.*

Using the factorization of the long intertwining operator ([18], see also [19]), we obtain that the representation $\chi_1\nu^\alpha \times \chi_2\nu^\beta \rtimes 1$ reduces if and only if some of the representations

$$(\chi_1\nu^\alpha)^{\pm 1} \rtimes 1,\ (\chi_2\nu^\beta)^{\pm 1} \rtimes 1,\ (\chi_1\nu^\alpha)^{\pm 1} \times (\chi_2\nu^\beta)^{\pm 1}$$

reduce. So, having in mind the previous lemma and the criterion for the reducibility of the principal series of $GL(2,D)$ mentioned in the previous section, we see that if the representation $\chi\nu^\alpha \times \chi\nu^\beta \rtimes 1$ reduces, then

$$\alpha \ \text{ or } \beta \in \{0, \pm\frac{3}{2}\}, \text{ or } \ \pm\alpha \pm \beta = 2.$$

We describe the reducibility points and the decomposition of this kind of principal series in the next several lemmas.

Proposition 3.2. *Let $\pi = \chi_1\nu^\alpha \times \chi_1\nu^{\alpha+2} \rtimes 1$.*

If $\chi_1^2 \neq 1$, the length of π is two, and

$$\pi = L(\chi_1\nu^{|\alpha+1|}\delta(\nu,\nu^{-1});1) + \chi_1\nu^{\alpha+1}L(\nu,\nu^{-1}) \rtimes 1.$$

The second summand is the Langlands quotient of the standard representation π^s. For $\alpha = -1$ the first summand is a tempered representation.

If $\chi_1^2 = 1$, then the following hold:

(i) *Assume $\chi_1 = 1$. Then, if $\alpha \notin \{0, \pm\frac{3}{2}\}$, the representation π has length equal to two, and, analogous to the case $\chi_1^2 \neq 1$, we have*

$$\pi = L(\nu^{|\alpha+1|}\delta(\nu,\nu^{-1});1) + \nu^{\alpha+1}L(\nu,\nu^{-1}) \rtimes 1.$$

Moreover, we have the following (in the appropriate Grothendieck group):

(a) *If $\alpha = 0$, then:*

$$\nu^2 \times 1 \rtimes 1 = L(\nu St_{GL(2,D)};1) + L(\nu^2; 1 \rtimes 1).$$

(b) *If $\alpha = \frac{3}{2}$, then:*

$$\nu^{\frac{7}{2}} \times \nu^{\frac{3}{2}} \rtimes 1 = L(\nu^{\frac{7}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{5}{2}} St_{GL(2,D)};1) + L(\nu^{\frac{7}{2}}, \nu^{\frac{3}{2}};1) + St_{G_2(D,1)}.$$

(c) *If $\alpha = -\frac{3}{2}$, then:*

$$\nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1 = L(\nu^{\frac{1}{2}} St_{GL(2,D)};1) + L(\nu^{\frac{1}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{3}{2}}, \nu^{\frac{1}{2}};1) + \pi_1,$$

where π_1 is a square integrable representation.

(ii) *Assume $\chi_1 \neq 1$. If $\alpha \notin \{0, -2\}$, the representation π has length two, and it has the composition series analogous to the case $\chi_1^2 \neq 1$.*

If $\alpha = 0$ (or $\alpha = -2$) we have

$$\chi_1\nu^2 \times \chi_1 \rtimes 1 = L(\chi_1\nu^2; T_1) + L(\chi_1\nu^2; T_2) + 2L(\chi_1\nu St_{GL(2,D)}; 1) + \pi_2 + \pi_3,$$

where $\chi_1 \rtimes 1 = T_1 + T_2$, a sum of two non-equivalent irreducible tempered representations, and π_2 and π_3 are square integrable representations.

Proof. In the following analysis, we extensively use Remark 3.2. and Lemma 3.7. from [21]. The Jacquet modules of the representation involved are

$$\begin{aligned} &s_{(2)}(\chi_1\nu^{\alpha+1}\delta(\nu, \nu^{-1}) \rtimes 1) \\ &\quad = \chi_1\nu^{\alpha+1}\delta(\nu, \nu^{-1}) \otimes 1 + \chi_1^{-1}\nu^{-(\alpha+1)}\delta\widetilde{(\nu, \nu^{-1})} \otimes 1 + \chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2} \otimes 1 \end{aligned}$$

and

$$s_{(1)}(\chi_1\nu^{\alpha+1}\delta(\nu, \nu^{-1}) \rtimes 1) = \chi_1\nu^{\alpha+2} \otimes \chi_1\nu^{\alpha} \rtimes 1 + \chi_1^{-1}\nu^{-\alpha} \otimes \chi_1\nu^{\alpha+2} \rtimes 1.$$

We obtain that the reducibility of the representation $\chi_1\nu^{\alpha+1}\delta(\nu, \nu^{-1}) \rtimes 1$ depends on whether the representations $\chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2}$, $\chi_1\nu^{\alpha} \rtimes 1$ and $\chi_1\nu^{\alpha+2} \rtimes 1$ are irreducible or not. If $\chi_1^2 \neq 1$, all these representations are irreducible, forcing the representation $\chi_1\nu^{\alpha+1}St_{GL(2,D)} \rtimes 1$ and, by the Aubert involution, the representation $\chi_1\nu^{\alpha+1}1_{GL(2,D)} \rtimes 1$, to be irreducible. So we are left to deal with the cases described in the proposition. In the case

$$\nu^2 \times 1 \rtimes 1$$

we apply Proposition 6.3 of [21] and obtain the irreducibility of the representation $\nu St_{GL(2,D)} \rtimes 1$. For $\alpha = -\frac{3}{2}$, we have

$$\begin{aligned} &\nu^{\frac{3}{2}} \times \nu^{\frac{1}{2}} \rtimes 1 \\ &\quad = \nu^{-\frac{1}{2}} St_{GL(2,D)} \rtimes 1 + \nu^{-\frac{1}{2}} 1_{GL(2,D)} \rtimes 1 = \nu^{\frac{1}{2}} \rtimes St_{G_1(D,1)} + \nu^{\frac{1}{2}} \rtimes 1_{G_1(D,1)}. \end{aligned}$$

We analyze the Jacquet module

$$r_{D^* \times G_1(D,1)}(\nu^{\frac{1}{2}} \rtimes St_{G_1(D,1)}) = \nu^{\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \rtimes 1 + \nu^{\frac{1}{2}} \otimes St_{G_1(D,1)} + \nu^{-\frac{1}{2}} \otimes St_{G_1(D,1)}.$$

So, the length of the representation $\nu^{\frac{1}{2}} \rtimes St_{G_1(D,1)}$ is at most three. If we assume it is three, then the representation $\nu^{\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \rtimes 1$ is a Jacquet module of an irreducible subquotient of the representation $\nu^{\frac{1}{2}} \rtimes St_{G_1(D,1)}$. But, by checking all the possibilities for the $r_{GL(2,D)}$–Jacquet module of that subquotient, and then calculating the Jacquet module for the minimal parabolic subgroup, we see that this is impossible. We conclude that in the Grothendieck group we have

$$\nu^{\frac{1}{2}} \rtimes St_{G_1(D,1)} = L(\nu^{\frac{1}{2}}; St_{G_1(D,1)}) + \pi_1,$$

for some irreducible representation π_1. Analyzing the possibilities for the Jacquet module of the representation π_1, we find out that

$$r_{D^* \times G_1(D,1)}(\pi_1) = \nu^{\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \rtimes 1 + \nu^{\frac{1}{2}} \otimes St_{G_1(D,1)}.$$

This forces π_1 to be a square integrable representation. For $\alpha = \frac{3}{2}$ we obtain the representation $\nu^{\frac{7}{2}} \times \nu^{\frac{3}{2}} \rtimes 1$, and by the well-known results (for example [3]), we get that the length of this representation is four, that it has a square integrable subquotient (the Steinberg representation), and the Langlands quotient of this representation is the trivial representation of the group $G_2(D,1)$. The other two subquotients are easily identified. For $\chi_1 \neq 1$, the only more complicated case is $\alpha \in \{0,-2\}$. We denote $\chi_1 \rtimes 1 = T_1 + T_2$, where T_i, $i=1,2$, are non-equivalent, irreducible tempered representations. Then the discussion, which mainly includes the analysis of the Jacquet modules, is very similar to the one in the case of the principal series of the group $G_2(D,-1)$ ([5]). There we faced a similar situation examining the principal series representation $\tau\nu \times \tau \rtimes 1$, where τ is a selfcontragredient representation of D^* of dimension greater than one with a non-trivial central character. □

Proposition 3.3. *We consider a representation* $\pi = \nu^{\frac{3}{2}} \times \chi_2\nu^{\alpha} \rtimes 1$. *If* $\chi_2^2 \neq 1$, *the representation* π *has length equal to two, and we have*

$$\nu^{\frac{3}{2}} \times \chi_2\nu^{\alpha} \rtimes 1 = \chi_2\nu^{\alpha} \rtimes 1_{G_1(D,1)} + \chi_2\nu^{\alpha} \rtimes St_{G_1(D,1)}.$$

The first summand above is the Langlands quotient of the representation π^s, *and the second is* $L(\chi_2\nu^{|\alpha|}; St_{G_1(D,1)})$. *(For* $\alpha = 0$ *this summand is an irreducible tempered representation).*

If $\chi_2^2 = 1$ *we have*

(i) *If we assume that* $\chi_2 = 1$, *then if* $\alpha \notin \{\pm\frac{1}{2}, \pm\frac{7}{2}\}$ *the representation* π *has length equal to two and the composition series equivalent to the case* $\chi_2^2 \neq 1$. *If* $\alpha \in \{\pm\frac{1}{2}, \pm\frac{7}{2}\}$ *we obtain the representations analyzed in the previous proposition.*

(ii) *If we assume that* $\chi_2 \neq 1$, *then if* $\alpha \neq 0$, *the representation* π *has length equal to two, and the composition series is analogous to the case* $\chi_2^2 \neq 1$. *In the case* $\alpha = 0$, *we denote* $\chi_2 \rtimes 1 = T_1 + T_2$, *for* T_1 *and* T_2 *irreducible tempered representations, and we have*

$$\nu^{\frac{3}{2}} \times \chi_2 \rtimes 1 = L(\nu^{\frac{3}{2}}; T_1) + L(\nu^{\frac{3}{2}}; T_2) + T_3' + T_4'.$$

The representations T_3' *and* T_4' *are non-equivalent, irreducible tempered representations such that* $\chi_2 \rtimes St_{G_1(D,1)} = T_3' + T_4'$.

Proof. We consider the following Jacquet modules:

$$r_{D^*\times G_1(D,1)}(\chi_2\nu^{\alpha} \rtimes St_{G_1(D,1)}) = \nu^{\frac{3}{2}} \otimes \chi_2\nu^{\alpha} \rtimes 1 + \chi_2\nu^{\alpha} \otimes St_{G_1(D,1)} + \chi_2^{-1}\nu^{-\alpha} \otimes St_{G_1(D,1)},$$

$$r_{GL(2,D)}(\chi_2\nu^{\alpha} \rtimes St_{G_1(D,1)}) = \chi_2\nu^{\alpha} \times \nu^{\frac{3}{2}} \otimes 1 + \chi_2^{-1}\nu^{-\alpha} \times \nu^{\frac{3}{2}} \otimes 1.$$

Again, by [21], if we assume that the representations $\chi_2\nu^{\alpha} \times \nu^{\frac{3}{2}}$, $\chi_2^{-1}\nu^{-\alpha} \times \nu^{\frac{3}{2}}$ and $\chi_2\nu^{\alpha} \rtimes 1$ are irreducible, then the irreducibility of the representation $\chi_2\nu^{\alpha} \rtimes St_{G_1(D,1)}$ and, by the Aubert involution, of the representation $\chi_2\nu^{\alpha} \rtimes 1_{G_1(D,1)}$, follows. If these condition are violated, then the only representation left to consider, besides the ones considered in the previous proposition, are the following ones:

$$\nu^{\frac{3}{2}} \times \nu^{\frac{3}{2}} \rtimes 1 \text{ and } \chi_2 \times \nu^{\frac{3}{2}} \rtimes 1,$$

with $\chi_2^2 = 1$, $\chi_2 \neq 1$. Consider the first one. We have

$$r_{GL(2,D)}(\nu^{\frac{3}{2}} \rtimes St_{G_1(D,1)}) = \nu^{\frac{3}{2}} \times \nu^{\frac{3}{2}} \otimes 1 + \nu^{-\frac{3}{2}} \times \nu^{\frac{3}{2}} \otimes 1.$$

We conclude that the length of $\nu^{\frac{3}{2}} \rtimes St_{G_1(D,1)}$ is at most two. So, we assume that $\nu^{\frac{3}{2}} \rtimes St_{G_1(D,1)} = L(\nu^{\frac{3}{2}}; St_{G_1(D,1)}) + \sigma$, where σ is some irreducible representation. From the expression for the $GL(2,D)$–Jacquet module we have $r_{GL(2,D)}(\sigma) = \nu^{\frac{3}{2}} \times \nu^{\frac{3}{2}} \otimes 1$. But when we try to reconcile that with the choices for the Jacquet module with respect to the $D^* \times G_1(D,1)$–Levi subgroup, we find that this is impossible. So, $\nu^{\frac{3}{2}} \rtimes St_{G_1(D,1)} = L(\nu^{\frac{3}{2}}; St_{G_1(D,1)})$. In the second case, we have

$$\begin{aligned}\chi_2 \times \nu^{\frac{3}{2}} \rtimes 1 &= \nu^{\frac{3}{2}} \rtimes T_1 + \nu^{\frac{3}{2}} \rtimes T_2 \\ &= \chi_2 \rtimes St_{G_1(D,1)} + \chi_2 \rtimes 1_{G_1(D,1)}.\end{aligned}$$

Because the representations $L(\nu^{\frac{3}{2}}; T_2)$ and $L(\nu^{\frac{3}{2}}; T_1)$ have to be subquotients of $\chi_2 \rtimes 1_{G_1(D,1)}$, the length of $\chi_2 \rtimes 1_{G_1(D,1)}$ is at least two, but by looking at the $GL(2,D)$–Jacquet module of the representation $\chi_2 \rtimes St_{G(D,1)}$ we obtain that the representation $\chi_2 \rtimes St_{G(D,1)}$ has length at most two. So the result follows. $\square$

Proposition 3.4. *For $\chi_1^2 = 1$, $\chi_1 \neq 1$ and unitary character χ_2, we consider the representation $\pi = \chi_2\nu^\alpha \times \chi_1 \rtimes 1$. We also denote $\chi_1 \rtimes 1 = T_1 \oplus T_2$. The following holds:*

(a) *If $\chi_2^2 \neq 1$, the length of the representation π is two, and we have:*

$$\pi = L(\chi_2\nu^{|\alpha|}; T_1) + L(\chi_2\nu^{|\alpha|}; T_2), \text{ for } \alpha \neq 0.$$

For $\alpha = 0$, the representation π is a sum of two non-equivalent irreducible tempered representations.

(b) *If $\chi_2^2 = 1$, then*
   (i) *Assume that $\chi_2 = 1$. Then, if $\alpha \notin \{\pm\frac{3}{2}\}$, the representation π has length equal to two, and the composition series are analogous to the case $\chi_2^2 \neq 1$. If $\alpha = \frac{3}{2}$, this case was covered in the previous proposition.*
   (ii) *Assume that $\chi_2 \neq 1$. If $\alpha \neq 0$, the representation π has length equal to two and the composition series are analogous to the case $\chi_2^2 \neq 1$; if $\alpha = 0$ we have: if $\chi_1 = \chi_2$ the length of π is two, and it is a sum of two non-equivalent tempered representations, and if $\chi_1 \neq \chi_2$, π is a sum of four non-equivalent tempered representations.*

Proof. We have

$$\chi_2\nu^\alpha \times \chi_1 \rtimes 1 = \chi_2\nu^\alpha \rtimes T_1 + \chi_2\nu^\alpha \rtimes T_2.$$

Then

$$r_{D^* \times G_1(D,1)}(\chi_2\nu^\alpha \rtimes T_1) = \chi_1 \otimes \chi_2\nu^\alpha \rtimes 1 + \chi_2\nu^\alpha \otimes T_1 + \chi_2^{-1}\nu^{-\alpha} \otimes T_1.$$

Also

$$r_{GL(2,D)}(\chi_2\nu^\alpha \rtimes T_1) = \chi_2\nu^\alpha \times \chi_1 \otimes 1 + \chi_2^{-1}\nu^{-\alpha} \times \chi_1 \otimes 1.$$

If we assume irreducibility of the representations $\chi_2\nu^\alpha \times \chi_1$, $\chi_2^{-1}\nu^{-\alpha} \times \chi_1$ and $\chi_2\nu^\alpha \rtimes 1$, by [21], it follows that the representation π has length equal to two. Dropping these assumptions, the only new cases left to consider are

$$\chi_1 \times \chi_1 \rtimes 1$$

and

$$\chi_1 \times \chi_2 \rtimes 1, \text{ if } \chi_2^2 = 1,\ \chi_2 \neq 1,\ \chi_2 \neq \chi_1.$$

The former one has length two and the latter four by [6], and all the summands appearing are non-equivalent. ☐

Now we turn to the mixed case. We denote $\pi = \chi\nu^{\alpha} \times \tau\nu^{\beta} \rtimes 1$, where χ is a unitary character of D^*, and τ is a unitary representation of D^* of dimension greater than one. The representation π reduces only if at least one of the representations $\chi\nu^{\alpha} \rtimes 1$ or $\tau\nu^{\beta} \rtimes 1$ reduces. Then we have the following

Proposition 3.5. (i) *If $\tau\nu^{\beta} \rtimes 1$ is irreducible, then the representation $\tau\nu^{\beta} \times \nu^{\frac{3}{2}} \rtimes 1$ has length two and*

$$\pi = \tau\nu^{\beta} \rtimes 1_{G_1(D,1)} + L(\tau\nu^{|\beta|}; St_{G_1(D,1)}),$$

where the first summand is the Langlands quotient of the representation π^s; the second is tempered if $\beta = 0$.

(ii) *Assume that $\tau \cong \widetilde{\tau}$ and $\omega_\tau = 1$. Then*

$$\nu^{\frac{3}{2}} \times \tau\nu^{\frac{1}{2}} \rtimes 1 = L(\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + L(\tau\nu^{\frac{1}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1) + \pi_4,$$

where π_4 is a square integrable representation.

(iii) *Assume that $\tau \cong \widetilde{\tau}$ and $\omega_\tau \neq 1$. Then*

$$\nu^{\frac{3}{2}} \times \tau \rtimes 1 = L(\nu^{\frac{3}{2}}; T_1) + L(\nu^{\frac{3}{2}}; T_2) + T_5' + T_6',$$

where $\tau \rtimes 1 = T_1 + T_2$, and T_5' and T_6' are non-equivalent, irreducible tempered representations such that $\tau \rtimes St_{G_1(D,1)} = T_5' + T_6'$.

(iv) *Assume that $\chi^2 = 1$ and $\chi \neq 1$, so that $\chi \rtimes 1 = T_1 + T_2$. Then, if the representation $\tau\nu^{\beta} \rtimes 1$ does not reduce, the representation π has length two, and*

$$\pi = L(\tau\nu^{|\beta|}; T_1) + L(\tau\nu^{|\beta|}; T_2), \quad \text{if } \beta \neq 0.$$

If $\beta = 0$ the representation π is a sum of two non-equivalent tempered representations.

(v) *For $\chi^2 = 1$, $\chi \neq 1$ with $\tau\nu^{\frac{1}{2}} \rtimes 1$ reducible, we have*

$$\tau\nu^{\frac{1}{2}} \times \chi \rtimes 1 = L(\tau\nu^{\frac{1}{2}}; T_1) + L(\tau\nu^{\frac{1}{2}}; T_2) + T_7' + T_8',$$

where T_7' and T_8' are non-equivalent, irreducible tempered representations such that $\chi \rtimes \delta[\tau\nu^{\frac{1}{2}}; 1] = T_7' + T_8'$.

(vi) *For $\chi^2 = 1$, $\chi \neq 1$ with $\tau \rtimes 1$ is reducible, we have*

$$\chi \times \tau \rtimes 1 = T_9' + T_{10}' + T_{11}' + T_{12}',$$

and representations on the right hand side of the above equation are inequivalent, irreducible and tempered.

(vii) *When the representation $\chi\nu^{\alpha} \rtimes 1$ is irreducible and $\tau\nu^{\beta} \rtimes 1$ is reducible, the representation π has length two, and the analysis of the composition series is analogous to the previous cases of reducible $\chi\nu^{\alpha} \rtimes 1$ and irreducible $\tau\nu^{\beta} \rtimes 1$.*

Proof. This is similar to the proof of the previous proposition. We leave the details to the reader. ☐

The groups $G_2(D, 1)$ and $G_2(D, -1)$ have an analogous structure of the Ψ–Hopf module on the Grothendieck group, and in the groups $G_1(D, 1)$ and $G_1(D, -1)$ we have the same reducibilites of the representations $\tau\nu^{\alpha} \rtimes 1$, where $\dim \tau > 1$. So, the composition series of the representations $\tau_1\nu^{\alpha} \times \tau_2\nu^{\beta} \rtimes 1$, where τ_i, $i = 1, 2$, are unitary representations of D^* of dimension greater than one, of the group $G_2(D,1)$, are analogous to the similar representations of the group $G_2(D,-1)$ which were analyzed in [5]. We just note the results, because we need them for the unitarizability questions.

Proposition 3.6. *Let τ_1 denote an irreducible, admissible, unitary representation of D^* of dimension greater than one, and let $\pi = \tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1$. If τ_1 is not a self-dual representation, the representation π has length equal to two, and we have*

$$\pi = L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{\alpha}) \rtimes 1 + L(\nu^{|\alpha+\frac{1}{2}|}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1).$$

The first summand in the previous relation is the Langlands quotient of the representation π^s; the second is tempered for $\alpha = \frac{1}{2}$. Otherwise, we have the following (without loss of generality, we can assume $\alpha \geq -\frac{1}{2}$).

(i) *If $\omega_{\tau_1} = 1$ we have the following:*

$$\tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1$$
$$= \begin{cases} L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu; \tau_1 \rtimes 1), & \text{if } \alpha = 0, \\ L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \pi_4 + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1), & \text{if } \alpha = \frac{1}{2}, \\ L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau\nu_1^{\frac{1}{2}}, \tau\nu_1^{\frac{1}{2}}; 1) + T_1 + T_2, & \text{if } \alpha = -\frac{1}{2}, \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{-\alpha}; 1), & \text{if } \alpha \in (-\frac{1}{2}, 0), \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{\alpha}; 1), & \text{if } \alpha \in \mathbb{R}^+ \setminus \{\frac{1}{2}\}. \end{cases}$$

(ii) *If $\omega_{\tau_1} \neq 1$ and $\tau_1 \rtimes 1 = T_3' + T_4'$, then we have the following:*

$$\tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1$$
$$= \begin{cases} L(\tau_1\nu; T_3') + L(\tau_1\nu; T_4') + 2L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + \pi_5 + \pi_6, & \text{if } \alpha = 0, \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{\alpha}; 1), & \text{if } \alpha > 0, \\ \delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}) \rtimes 1 + L(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{\frac{1}{2}}; 1), & \text{if } \alpha = -\frac{1}{2}, \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{-\alpha}; 1), & \text{if } \alpha \in (-\frac{1}{2}, 0). \end{cases}$$

The representations π_i, $i = 4, 5, 6$, are mutually non-equivalent square–integrable representations, and T_i, $i = 1, 2$, and $\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}) \rtimes 1$ in the second case are mutually non-equivalent tempered (non square-integrable) representations.

Proposition 3.7. *Let τ_2 be a unitary, irreducible self-dual representation of D^* of dimension greater than one, with $\omega_{\tau_2} = 1$, and let τ_1 denote a unitary irreducible representation of D^* of dimension greater than one.*

(a) *If $\tau_1 \not\cong \widetilde{\tau_1}$, then we have the following:*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = \tau_1\nu^{\alpha} \rtimes L(\tau\nu^{\frac{1}{2}}; 1) + L(\tau_1\nu^{|\alpha|}; \delta[\tau\nu^{\frac{1}{2}}; 1]),$$

where the first summand is the Langlands quotient of the representation π^s; the second is tempered for $\alpha = 0$.

(b) *If $\tau_1 \cong \widetilde{\tau_1}$, then we have the following two cases:*

(i) *If $\omega_{\tau_1} = 1$, then we have:*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\tau_1\nu^{|\alpha|}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{|\alpha|}, \tau_2\nu^{\frac{1}{2}}; 1)$$

if $|\alpha| \in \mathbb{R}_0^+ \setminus \{0, \frac{1}{2}, \frac{3}{2}\}$;

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_2\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\frac{1}{2}}, \tau_2\nu^{\frac{1}{2}}; 1) + \pi_7$$

if $|\alpha \models \frac{1}{2}$ *and* $\tau_1 \not\cong \tau_2$;

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau\nu_1^{\frac{1}{2}}, \tau\nu_1^{\frac{1}{2}}; 1) + T_1 + T_2$$

if $|\alpha| = \frac{1}{2}$ *and* $\tau_1 \cong \tau_2$;

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = \tau_1 \rtimes \delta[\tau_2\nu^{\frac{1}{2}}; 1] + L(\tau_2\nu^{\frac{1}{2}}; \tau_1 \rtimes 1)$$

if $\alpha = 0$; *and finally*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \pi_4 + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$$

if $|\alpha| = \frac{3}{2}$ *and* $\tau_1 \cong \tau_2$, *and*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\frac{3}{2}}, \tau_2\nu^{\frac{1}{2}}; 1)$$

if $|\alpha| = \frac{3}{2}$ *and* $\tau_1 \not\cong \tau_2$. *The representation* π_7 *is a square–integrable representation, and* $\tau_1 \rtimes \delta[\tau_2\nu^{\frac{1}{2}}; 1]$ *is an irreducible tempered representation.*

(ii) *If* $\omega_{\tau_1} \neq 1$ *and* $\tau_1 \rtimes 1 = T_3' + T_4'$, *then we have:*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1 = \begin{cases} L(\tau_1\nu^{|\alpha|}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{|\alpha|}, \tau_2\nu^{\frac{1}{2}}; 1) & \text{if } \alpha \neq 0, \\ L(\tau_2\nu^{\frac{1}{2}}; T_3') + L(\tau_2\nu^{\frac{1}{2}}; T_4') + T_5 + T_4 & \text{if } \alpha = 0. \end{cases}$$

The representations T_i, $i = 4, 5$, *are irreducible tempered representations.*

Proposition 3.8. *Let* τ_2 *be a unitary irreducible, self-dual representation of* D^* *such that* $\omega_{\tau_2} \neq 1$, *so that* $\tau_2 \rtimes 1 = T_3' \oplus T_4'$, *and let* τ_1 *be an irreducible unitary representation of* D^*. *Then, we have the following:*

(a) *If* $\alpha \neq 0$, *then we have the following:*

$$\tau_1\nu^{\alpha} \times \tau_2 \rtimes 1 = \begin{cases} L(\tau_1\nu^{\frac{1}{2}}; T_3') + L(\tau_1\nu^{\frac{1}{2}}; T_4') + T_4 + T_5 \text{ if } \alpha = \pm\frac{1}{2},\ \tau_1 \cong \widetilde{\tau_1},\ \omega_{\tau_1} = 1, \\ L(\tau_2\nu; T_3') + L(\tau_2\nu; T_4') + 2L(\nu^{\frac{1}{2}}\delta(\tau_2\nu^{\frac{1}{2}}, \tau_2\nu^{-\frac{1}{2}}); 1) \\ \quad +\pi_5 + \pi_6 \text{ if } \tau_1 \cong \tau_2,\ \alpha = \pm 1, \\ L(\tau_1\nu^{|\alpha|}; T_3') + L(\tau_1\nu^{|\alpha|}; T_4'), \text{ in other cases.} \end{cases}$$

(b) *If* $\alpha = 0$, *then we have the following:*

$$\tau_1 \times \tau_2 \rtimes 1 = \begin{cases} T_6 + T_7 + T_8 + T_9 & \text{if } \tau_1 \cong \widetilde{\tau_1},\ \omega_{\tau_1} \neq 1,\ \tau_1 \not\cong \tau_2, \\ T_{10} + T_{11} & \text{in other cases.} \end{cases}$$

The representations T_i, $i = 6, \ldots, 11$, *are mutually non-equivalent tempered (non-square-integrable) representations.*

3.1. **The summary of all the reducibility points of the principal series representations.** In order to organize the results in a more concise way, we list all the cases discussed in this section. The tempered subquotients appearing in the principal series will be denoted by T_j (or T_j'), and the square integrable ones by π_i, for some i, j.

A) Let $\pi = \chi_1\nu^{\alpha} \times \chi_1\nu^{\alpha+2} \rtimes 1$, where χ_1 is a unitary character of D, and $\alpha \in \mathbb{R}$.

- If $\chi_1^2 \neq 1$, or if $\chi_1 = 1$ and $\alpha \neq \pm\frac{3}{2}$ or if $\chi_1^2 = 1$, $\chi_1 \neq 1$ and $\alpha \notin \{0, -2\}$, π is of the length two and:
$$\pi = L(\chi_1\nu^{|\alpha+1|}\delta(\nu, \nu^{-1}); 1) + \chi_1\nu^{\alpha+1}L(\nu, \nu^{-1}) \rtimes 1.$$
- $\chi_1 = 1$ and $\alpha = \frac{3}{2}$:
$$\pi = L(\nu^{\frac{7}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{5}{2}}St_{GL(2,D)}; 1) + L(\nu^{\frac{7}{2}}, \nu^{\frac{3}{2}}; 1) + St_{G_2(D,1)}.$$
- $\chi_1 = 1$ and $\alpha = -\frac{3}{2}$:
$$\pi = L(\nu^{\frac{1}{2}}St_{GL(2,D)}; 1) + L(\nu^{\frac{1}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{3}{2}}, \nu^{\frac{1}{2}}; 1) + \pi_1.$$
- $\chi_1^2 = 1$, but $\chi_1 \neq 1$ and $\alpha \in \{0, -2\}$:
$$\pi = L(\chi_1\nu^2; T_1) + L(\chi_1\nu^2; T_2) + 2L(\chi_1\nu St_{GL(2,D)}; 1) + \pi_2 + \pi_3,$$
where $\chi_1 \rtimes 1 = T_1 + T_2$.

B) Let $\pi = \nu^{\frac{3}{2}} \times \chi_2\nu^{\alpha} \rtimes 1$, where χ_2 is a unitary character of D^*.

- If $\chi_2^2 \neq 1$, or if $\chi_2 = 1$ and $\alpha \notin \{\pm\frac{1}{2}, \pm\frac{7}{2}\}$, or if $\chi_2^2 = 1$, $\chi_2 \neq 1$ and $\alpha \neq 0$, the representation π is of length two and
$$\pi = \chi_2\nu^{\alpha} \rtimes 1_{G_1(D,1)} + L(\chi_2\nu^{|\alpha|}; St_{G_1(D,1)}).$$
- If $\chi_2^2 = 1$, but $\chi_2 \neq 1$ and $\alpha = 0$, we have
$$\pi = L(\nu^{\frac{3}{2}}; T_1) + L(\nu^{\frac{3}{2}}; T_2) + T_3' + T_4'.$$
- The remaining cases were covered under A).

C) Let $\pi = \chi_2\nu^{\alpha} \times \chi_1 \rtimes 1$, where $\chi_1^2 = 1$, $\chi_1 \neq 1$ and $\chi_1 \rtimes 1 = T_1 + T_2$, and χ_2 is a unitary character.

- If $\chi_2^2 \neq 1$, or if $\chi_2 = 1$ and $\alpha \neq \pm\frac{3}{2}$, or if $\chi_2^2 = 1$, $\chi_2 \neq 1$ and $\alpha \neq 0$, or if $\chi_2 = \chi_1$ and $\alpha = 0$ we have
$$\pi = L(\chi_2\nu^{|\alpha|}; T_1) + L(\chi_2\nu^{|\alpha|}; T_2).$$
- If $\chi_2^2 = 1$, $\chi_2 \neq 1$, $\chi_2 \neq \chi_1$ and $\alpha = 0$ the representation π is a sum of four non-equivalent tempered representations.

D) Let $\pi = \tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1$, where τ_1 is an irreducible, unitarizable representation, and $\dim \tau_1 > 1$. If $\tau_1 \cong \widetilde{\tau_1}$ we can assume $\alpha \geq -\frac{1}{2}$.

- If $\tau_1 \ncong \widetilde{\tau_1}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $\alpha \in (-\frac{1}{2}, +\infty) \setminus \{\frac{1}{2}\}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} \neq 1$ and $\alpha \in [-\frac{1}{2}, +\infty) \setminus \{0\}$, the length of π is two and we have
$$\pi = L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{\alpha}) \rtimes 1 + L(\nu^{|\alpha+\frac{1}{2}|}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1).$$
- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $\alpha = \frac{1}{2}$:
$$\pi = L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \pi_4 + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1).$$

- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $\alpha = -\frac{1}{2}$:

$$\pi = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau\nu_1^{\frac{1}{2}}, \tau\nu_1^{\frac{1}{2}}; 1) + T_5 + T_6.$$

- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} \neq 1$ and $\alpha = 0$ with $\tau_1 \rtimes 1 = T_3' + T_4'$, we have

$$\pi = L(\tau_1\nu; T_3') + L(\tau_1\nu; T_4') + 2L(\nu^{\frac{1}{2}}\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + \pi_5 + \pi_6.$$

E) Let $\pi = \tau_1\nu^{\alpha} \times \tau_2\nu^{\frac{1}{2}} \rtimes 1$, where $\tau_2 \cong \widetilde{\tau_2}$, $\omega_{\tau_2} = 1$, and τ_1 is an irreducible unitarizable representation; $\dim \tau_i > 1$, $i = 1, 2$.

- If $\tau_1 \not\cong \widetilde{\tau_1}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $|\alpha| \in \mathbb{R}_0^+ \setminus \{\frac{1}{2}, \frac{3}{2}\}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$, $\tau_1 \not\cong \tau_2$ and $|\alpha| = \frac{3}{2}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} \neq 1$ and $\alpha \neq 0$, the length of π is two and we have

$$\pi = \tau_1\nu^{\alpha} \rtimes L(\tau\nu^{\frac{1}{2}}; 1) + L(\tau_1\nu^{|\alpha|}; \delta[\tau\nu^{\frac{1}{2}}; 1]).$$

- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$, $\tau_1 \not\cong \tau_2$ and $|\alpha \models \frac{1}{2}$:

$$\pi = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_2\nu^{\frac{1}{2}}; 1]) + L(\tau_2\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\frac{1}{2}}, \tau_2\nu^{\frac{1}{2}}; 1) + \pi_7.$$

- If $\tau_1 \cong \tau_2$ and $|\alpha| = \frac{1}{2}$:

$$\pi = L(\tau_1\nu^{\frac{1}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau\nu_1^{\frac{1}{2}}, \tau\nu_1^{\frac{1}{2}}; 1) + T_1 + T_2.$$

- If $\tau_1 \cong \tau_2$ and $|\alpha| = \frac{3}{2}$:

$$\pi = L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + L(\tau_1\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + \pi_4 + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1).$$

- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} \neq 1$ and $\alpha = 0$, with $\tau_1 \rtimes 1 = T_3' + T_4'$, we have

$$\pi = L(\tau_2\nu^{\frac{1}{2}}; T_3') + L(\tau_2\nu^{\frac{1}{2}}; T_4') + T_5 + T_6.$$

F) Let $\pi = \tau_1\nu^{\alpha} \times \tau_2 \rtimes 1$, such that $\dim \tau_i > 1$, $i = 1, 2$, $\tau_2 \cong \widetilde{\tau_2}$, $\omega_{\tau_2} \neq 1$, with $\tau_2 \rtimes 1 = T_3' + T_4'$. Assume $\alpha > 0$.

- If $\tau_1 \not\cong \widetilde{\tau_1}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $|\alpha| \neq \frac{1}{2}$, or if $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} \neq 1$ and $\tau_1 \not\cong \tau_2$, or if $\tau_1 \cong \tau_2$ and $|\alpha| \neq 1$, we have:

$$\pi = L(\tau_1\nu^{|\alpha|}; T_3') + L(\tau_1\nu^{|\alpha|}; T_4').$$

- If $\tau_1 \cong \widetilde{\tau_1}$, $\omega_{\tau_1} = 1$ and $\alpha = \pm\frac{1}{2}$:

$$\pi = L(\tau_1\nu^{\frac{1}{2}}; T_3') + L(\tau_1\nu^{\frac{1}{2}}; T_4') + T_4 + T_5.$$

- The remaining cases were covered in D).

Assume $\alpha = 0$. The representation π is a sum of four non-equivalent tempered representations if $\tau_1 \rtimes 1$ reduces and $\tau_1 \not\cong \tau_2$. Otherwise, π is a sum of two tempered representations.

G) Let $\pi = \chi\nu^{\alpha} \times \tau\nu^{\beta} \rtimes 1$, with χ a unitary character and $\dim \tau > 1$.

- If $\tau\nu^{\beta} \rtimes 1$ does not reduce, and $\chi\nu^{\alpha} = \nu^{\pm\frac{3}{2}}$, we have:

$$\pi = \tau\nu^{\beta} \rtimes 1_{G_1(D,1)} + L(\tau\nu^{|\beta|}; St_{G_1(D,1)}).$$

- If $\tau\nu^{\beta} \rtimes 1$ does not reduce, and $\chi \rtimes 1 = T_1 + T_2$, for $\alpha = 0$ we have:

$$\pi = L(\tau\nu^{|\beta|}; T_1) + L(\tau\nu^{|\beta|}; T_2).$$

- If $\tau\nu^{\frac{1}{2}} \rtimes 1$ reduces, and $\chi\nu^{\alpha} \rtimes 1$ does not, for $\beta = \pm\frac{1}{2}$ the representation π is of the length two and:

$$\pi = \chi\nu^{\alpha} \rtimes L(\tau\nu^{\frac{1}{2}}; 1) + L(\chi\nu^{\alpha}; \delta[\tau\nu^{\frac{1}{2}}; 1]).$$

- If $\tau\nu^{\frac{1}{2}} \rtimes 1$ reduces, $\chi = 1$ and $\alpha = \pm\frac{3}{2}$ (with $\beta = \pm\frac{1}{2}$):
$$\pi = L(\nu^{\frac{3}{2}}; \delta[\tau\nu^{\frac{1}{2}}; 1]) + L(\tau\nu^{\frac{1}{2}}; St_{G_1(D,1)}) + L(\nu^{\frac{3}{2}}, \tau\nu^{\frac{1}{2}}; 1) + \pi_4.$$
- If $\tau\nu^{\frac{1}{2}} \rtimes 1$ reduces, and $\chi \rtimes 1 = T_1 + T_2$ (with $\alpha = 0$ and $\beta = \pm\frac{1}{2}$), we have
$$\pi = L(\tau\nu^{\frac{1}{2}}; T_1) + L(\tau\nu^{\frac{1}{2}}; T_2) + T_7' + T_8'.$$
- If $\tau \rtimes 1 = T_1' + T_2'$, and $\chi\nu^{\alpha} \rtimes 1$ does not, π is of the length two and (with $\beta = 0$):
$$\pi = L(\chi\nu^{|\alpha|}; T_1') + L(\chi\nu^{|\alpha|}; T_2').$$
- If $\tau \rtimes 1 = T_1' + T_2'$, and $\chi = 1,\ \alpha = \pm\frac{3}{2}$ (with $\beta = 0$):
$$\pi = L(\nu^{\frac{3}{2}}; T_1') + L(\nu^{\frac{3}{2}}; T_2') + T_5' + T_6'.$$
- If $\tau \rtimes 1 = T_1' + T_2'$ and $\chi \rtimes 1 = T_1 + T_2$ (with $\alpha = \beta = 0$), π is a sum of four tempered representations.

4. Unitary subquotients of the principal series

By χ_1 and χ_2 we denote the unitary characters of the group D^*. Let $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. We are interested in the unitarizability of the subquotients of the representation π, and we can assume (and do, throughout this section) that $s_1 \geq s_2 \geq 0$. These subquotients were identified in the previous sections.

Proposition 4.1. *Assume that $\chi_1^2 \neq 1$, $\chi_2^2 \neq 1$. Then*

(i) *If $\chi_1 \neq \chi_2^{\pm 1}$, or $\chi_1 = \chi_2$, then the representation π has an irreducible hermitian subquotient if and only if $s_1 = s_2 = 0$, and then this subquotient is a tempered representation.*

(ii) *If $\chi_1 = \chi_2^{-1}$, then, for $s_2 = 0$, the representation π has a hermitian subquotient only if $s_1 = 0$ (and then it is tempered), and for $s_2 > 0$, the hermitian subquotients occur only if $s_1 = s_2$. Then, for each $s_1 > 0$, all the subquotients of the representation π are hermitian. For $s_1 \in (0,1)$ and $s_1 > 1$ we have $\pi = \chi_1\nu^{s_1} \times \chi_1^{-1}\nu^{s_1} \rtimes 1 = L(\chi_1\nu^{s_1}, \chi_1^{-1}\nu^{s_1}; 1)$, and this is unitarizable for $s_1 \in (0,1)$. If $s_1 = 1$, in the appropriate Grothendieck group we have*
$$\pi = \chi_1\delta(\nu, \nu^{-1}) \rtimes 1 + L(\chi_1\nu, \chi_1^{-1}\nu; 1),$$
where the first subquotient is a tempered representation and the second is a unitarizable non-tempered representation.

Proof. The first case follows from the criterion for hermiticity of the Langlands quotient, due to Knapp-Zuckerman, and for the second, we just observe that, for $s_1 \in (0,1)$, the representation $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1}$ is in the complementary series of the group $GL(2, D)$. $\square$

Proposition 4.2. *Assume that $\chi_1^2 = 1$ and $\chi_2^2 \neq 1$. Then the representation π has a hermitian subquotient only when $s_2 = 0$; in that case all the subquotients are hermitian. Keeping this assumption, we have: if $s_1 = 0$, then if $\chi_1 \neq 1$ the representation π is a sum of two non-equivalent irreducible tempered representations, and if $\chi_1 = 1$ the representation π is an irreducible tempered representation. If $s_1 > 0$, then: if $\chi_1 \neq 1$ the representation π is irreducible non-unitarizable, and if $\chi_1 = 1$, π is unitarizable and irreducible for $s \in (0, \frac{3}{2})$, irreducible non-unitarizable for $s > \frac{3}{2}$, and for $s = \frac{3}{2}$ the representation π is (in the Grothendieck group) a sum of a tempered and a non-tempered unitarizable representation. Analogously, it* *follows that in the case $\chi_1^2 \neq 1$, $\chi_2^2 = 1$, the hermitian subquotients exist only when $s_1 = s_2 = 0$, and the representation π is then an irreducible tempered representation if $\chi_2 = 1$, or a sum of two irreducible tempered representations if $\chi_2 \neq 1$.*

Proof. This is standard. We just comment on the case $\chi_1 = 1$ and $\chi_2^2 \neq 1$ with $s_2 = 0$. The standard intertwining operators $A_{w_{2\alpha+\beta}}(s_1) : \nu^{s_1} \times \chi_2 \rtimes 1 \to \nu^{-s_1} \times \chi_2 \rtimes 1$ converge for $s_1 > 0$. These operators, for $s_1 > \frac{3}{2}$, define a continuous family, indexed by s_1, of the non-degenerate hermitian forms on the compact picture X of the representation $1 \times \chi_2 \rtimes 1$ in the following way:

$$(f_1, f_2)_{s_1} = \int_K \langle f_{1,s_1(k)}, A_{w_{2\alpha+\beta}}(s_1) f_{2,s_1} \rangle \mathrm{d}k.$$

Here, f_1 and f_2 belong to the space X, and f_{i,s_1}, $i = 1, 2$, denote the corresponding holomorphic sections. The indexing set is connected, so, from the unitarizability of one hermitian form would follow the unitarizability of all of them. Because of the unboundedness of the matrix coefficients when $s_1 \to \infty$, each hermitian form is non-unitarizable. In the same way, we can prove unitarizability of the representations for $s_1 \in [0, \frac{3}{2})$, but this time, we must normalize the standard intertwining operators $A_{w_{2\alpha+\beta}}(s_1)$ because, for $s_1 = 0$, this operator has a pole. The unitarizability of all the subquotients at the end of the complementary series follows from the well-known result of Miličić. □

Proposition 4.3. *Assume that $\chi_1^2 = \chi_2^2 = 1$, and $\chi_1 \neq \chi_2$. Then all the subquotients of the representation π are hermitian.*

(i) *Assume that $\chi_1 = 1$ and $\chi_2 \neq 1$. The representation π has a unitarizable subquotient only if $s_1 \in [0, \frac{3}{2}]$ and $s_2 = 0$ (and then all of them are unitarizable). The tempered irreducible subquotients occur for $s_1 = 0$ and for $s_1 = \frac{3}{2}$.*

(ii) *Assume that $\chi_1 \neq 1$ and $\chi_2 \neq 1$. The unitarizable subquotients appear only for $s_1 = s_2 = 0$, and then π is a sum of four non-equivalent tempered representations.*

(iii) *If $\chi_1 \neq 1$ and $\chi_2 = 1$, then, under given conditions ($s_1 \geq s_2 \geq 0$), the unitarizable subquotients appear only for $s_1 = s_2 = 0$, and then the representation π is a sum of two non-equivalent irreducible tempered representations.*

Proposition 4.4. (i) *Assume that $\chi_1 = \chi_2 = 1$. With notation as in Figure 1 we have: the unitarizable subquotients appear only for (s_1, s_2) from the closure of the region i and for the point $(s_1, s_2) = (\frac{7}{2}, \frac{3}{2})$. In the latter case the representation π has two unitarizable subquotients: the trivial representation and the Steinberg representation.*

(ii) *Assume that $\chi_1^2 = 1$, $\chi_1 \neq 1$, $\chi_1 = \chi_2$. Then the representation π has a unitarizable subquotient only for $s_1 + s_2 \leq 2$ (and then all of them are unitarizable).*

Proof. We prove (i). Unitarizability of the subquotients of the representation π for (s_1, s_2) from the closure of the region i follows from unitarizability of the representations $\nu^{s_1} \times \nu^{-s_1}$ of the group $GL(2, D)$ for $s_1 \in [0, 1)$. For the point $A = (\frac{7}{2}, \frac{3}{2})$, we recall a standard fact about the composition series of the representation π which is induced from the modular character of the minimal parabolic subgroup ([3]): it has a length four and only two of the subquotients, the Steinberg and the trivial

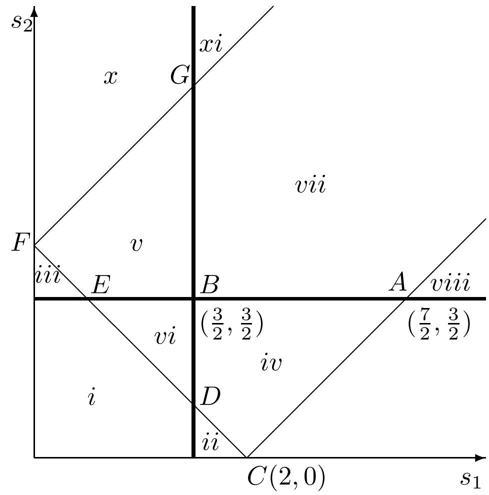

FIGURE 1. $\nu^{s_1} \times \nu^{s_2} \rtimes 1$

representation, are unitarizable. This implies the non-unitarizability of the representation π for the open regions iv and v. Now consider a family of the standard intertwining operators

$$A_{w_{\alpha+\beta}}(s) : \nu^{s+2} \times \nu^{s} \rtimes 1 \to \nu^{-s} \times \nu^{-s-2} \rtimes 1,$$

for $s \in (0, \frac{3}{2})$. It is easy to see that $A_{w_{\alpha+\beta}}(s)|_{St_{GL(2,D)}\nu^{s+1}\rtimes 1} \neq 0$, so this family of operators induces a family of the non-degenerate hermitian forms, indexed by $s \in (0, \frac{3}{2})$, on the compact picture of the representation $St_{GL(2,D)} \rtimes 1$. The representation $St_{GL(2,D)}\nu^{\frac{5}{2}} \rtimes 1$ has a non-unitarizable subquotient, so we conclude that among the representations $St_{GL(2,D)}\nu^{s+1} \rtimes 1$, for $s \in (0, \frac{3}{2})$, there are no unitarizable ones. Analogously, if we consider a quotient intertwining operator

$$\overline{A}_{w_{\alpha+\beta}}(s) : \nu^{s+1}1_{GL(2,D)} \rtimes 1 \to \nu^{-s-1}1_{GL(2,D)} \rtimes 1,$$

for $s \in (0, \frac{3}{2})$, we obtain nonunitarizability of the representations $\nu^{s+1}1_{GL(2,D)} \rtimes 1$. By using Proposition 6.3 from [21], we obtain the irreducibility of the representation $St_{GL(2,D)}\nu^{s+1} \rtimes 1$ for $s = 0$. In this way, we obtain the non-unitarizability of the representations $St_{GL(2,D)}\nu^{s+1} \rtimes 1$ for $s \in [0, \frac{3}{2})$. After applying the Aubert involution on the representation $St_{GL(2,D)}\nu \rtimes 1$, it follows that the representation $\nu 1_{GL(2,D)} \rtimes 1$ is irreducible. We normalize the operators $A_{w_{\alpha+\beta}}(s)$ to remove the pole for $s = 0$ and then pass to the quotient operators, and we analogously obtain the non-unitarizability of the representation $\nu 1_{GL(2,D)} \rtimes 1$. This gives non-unitarizability of all the subquotients on the segment $[C, A)$, and consequently, on the open regions ii, iii and iv, and on the interval (C, D). As for the point $B = (\frac{3}{2}, \frac{3}{2})$, the corresponding representation π has length equal to two, and one of the irreducible subquotients is $\nu^{\frac{3}{2}} \rtimes St_{G_1(D,1)}$. The representations $\nu^{s} \rtimes St_{G_1(D,1)}$ are non-unitarizable for $s \in (\frac{3}{2}, \frac{7}{2})$, because they are irreducible and the representation $\nu^{\frac{7}{2}} \rtimes St_{G_1(D,1)}$ has a non-unitarizable subquotient. So we have, for $s \in [\frac{3}{2}, \frac{7}{2})$, a non-degenerate family of the hermitian forms generated by the standard intertwining operators $A(s_1) : \nu^{s_1} \rtimes St_{G_1(D,1)} \to \nu^{-s_1} \rtimes St_{G_1(D,1)}$.

These operators are, for $s_1 \in (\frac{3}{2}, \frac{7}{2})$, the restrictions of the standard operators $A_{w_{2\alpha+\beta}}(s_1) : \nu^{s_1} \times \nu^{\frac{3}{2}} \rtimes 1 \to \nu^{-s_1} \times \nu^{\frac{3}{2}} \rtimes 1$. Using the similar arguments as before, i.e. by eliminating the pole of the operator $A_{w_{2\alpha+\beta}}$ and then, by passing to the quotient, we obtain the non-unitarizability of the representation $\nu^s \rtimes 1_{G_1(D,1)}$ for $s \in [\frac{3}{2}, \frac{7}{2})$. Also, for $s \in (\frac{1}{2}, \frac{3}{2})$, the non-unitarizability of the representation $\nu^s \rtimes 1_{G_1(D,1)} = L(\nu^{\frac{3}{2}}, \nu^s; 1)$ follows from the existence of the non-degenerate hermitian form on this representation generated by the action of the long intertwining operator $A_{w_0}(\frac{3}{2}, s)$ on the quotient of the representation $\nu^{\frac{3}{2}} \times \nu^s \rtimes 1$, and from the non-unitarizability of the representation $\nu^{\frac{3}{2}} \rtimes 1_{G_1(D,1)}$. This gives non-unitarizability on the open region *vi*. The proof of (ii) is left to the reader. □

Now, we consider the subquotients of the principal series $\tau_1\nu^{s_1} \times \tau_2\nu^{s_2} \rtimes 1$, for $\dim \tau_i > 1$, $i = 1, 2$. Again we denote $\pi = \tau_1\nu^{s_1} \times \tau_2\nu^{s_2} \rtimes 1$, and assume that $s_1 \geq s_2 \geq 0$. As already mentioned, the irreducible subquotients of these representations are the same as in the case of the group $G_2(D, -1)$. Also, the unitarizable subquotients are the same as for the group $G_2(D, -1)$ ([5]), but the arguments for the (non-)unitarizability for some of them are different because we were not able to establish a direct (non-conjectural) transfer of the Plancherel measure $\mu(s, \delta)$ to the Plancherel measure $\mu(s, \delta')$. In this case, δ denotes a discrete series representation of the group $D^* \times G_1(D, 1)$ and δ' (one of the) discrete series representations of $GL(2, F) \times Sp(4, F)$ corresponding to δ by the Langlands correspondence. So, we discuss the case where the arguments for $G_2(D, 1)$ and $G_2(D, -1)$ differ.

We now recall the definition of the Plancherel measure for a general reductive group. If $\nu \in a_\theta$, and σ is a discrete series representation of M_θ (notation as in the Preliminaires), up to a factor which depends only on the normalization of the measures on the reductive group G and its parabolic subgroup P_θ, we have:

$$A_{w^{-1}}({}^w\sigma, {}^w\nu) A_w(\sigma, \nu) = \mu_w(\sigma, \nu)^{-1}.$$

In the above relation, the representation ${}^w\sigma$ of the group $M_{w(\theta)}$ is defined by ${}^w\sigma(m) = \sigma(w^{-1}mw)$. We define ${}^w\nu$ analogously. If there is no subscript below μ, we assume that w equals the longest element of the Weyl group.

Proposition 4.5. *Assume that $\tau_1 \cong \tau_2$ and τ_1 selfcontragredient with $\omega_{\tau_1} = 1$. In Figure 2, considering the open regions, we have the unitarizable subquotients appearing only on the region I, where we have the non-tempered representations. On the boundaries, we have a square integrable subquotient for $(s_1, s_2) = \left(\frac{3}{2}, \frac{1}{2}\right)$, and the tempered subquotients for $(s_1, s_2) \in \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0, 0)\right\}$. On the boundary of the region I all the appearing subquotients are unitarizable and the Langlands quotient $L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$ is unitarizable.*

Proof. The only non-trivial thing is deciding on the unitarizability of the subquotients of the representation $\tau_1\nu^{\frac{3}{2}} \times \tau_1\nu^{\frac{1}{2}} \rtimes 1$. We have:

$$\begin{aligned} &\tau_1\nu^{\frac{3}{2}} \times \tau_1\nu^{\frac{1}{2}} \rtimes 1 \\ &\qquad = L(\nu\delta(\tau_1\nu^{\frac{1}{2}}, \tau_1\nu^{-\frac{1}{2}}); 1) + \pi_4 + L(\tau\nu^{\frac{3}{2}}; \delta[\tau_1\nu^{\frac{1}{2}}; 1]) + L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1). \end{aligned}$$

The unitarizability of the representation $L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$ is proved using global methods, modulo a technical condition. This is an isolated unitary representation in the unitary dual.

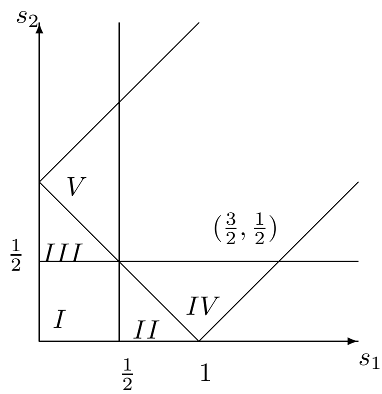

FIGURE 2. $\tau\nu^{s_1} \times \tau\nu^{s_2} \rtimes 1$

Lemma 4.6. *The representation* $L(\tau_1\nu^{\frac{3}{2}}, \tau_1\nu^{\frac{1}{2}}; 1)$ *is unitarizable.*

Proof of Lemma 4.6. Let k be a number field with the following property: there exist two places of k, v_1 and v_2, such that $k_{v_i} \cong F$, $i = 1, 2$, and a division algebra $\mathbf{D}$ over k such that it ramifies only at v_i, $i = 1, 2$. Then $\mathbf{D} \otimes_k F \cong D$. By A_k we denote the ring of adeles of k. Let $\sigma = \bigotimes_v \sigma_v$ be an automorphic cuspidal representation of $\mathbf{D}^*(A_k)$ with the trivial central character, such that $\sigma_{v_i} \cong \tau_1$, $i = 1, 2$. We assume that the representation σ is realized on the space of automorphic functions V on $\mathbf{D}^*(A_k)$. Let $\sigma' = \bigotimes_v \sigma'_v$ denote a cuspidal automorphic representation of $GL(2, A_k)$ which is a lift of the representation σ, i.e. $\sigma'_v \cong \sigma_v, \forall v \notin \{v_1, v_2\}$ and $\sigma'_{v_1} \cong \sigma'_{v_2} \cong \tau'_1$, a representation which is a Jacquet-Langlands lift of τ_1. We recall that the condition on the central character forces σ (and, consequently σ') to be self-contragredient. Let $G_2(\mathbf{D}, 1)(A_k)$ be a group of points in the adeles of the hermitian quaternionic group. Then $G_2(\mathbf{D}, 1)(k_v) \cong Sp(8, k_v)$, $v \notin \{v_1, v_2\}$, and $G_2(\mathbf{D}, 1)(k_{v_i}) \cong G_2(D, 1)$, $i = 1, 2$. Let $P(A_k) = M(A_k)U(A_k)$ denote a standard upper triangular parabolic subgroup with the Levi subgroup $M(A_k)$ isomorphic to $\mathbf{D}^*(A_k) \times \mathbf{D}^*(A_k)$, and let A_M be the center of M. Also, let $X(A_M) = \mathrm{Hom}_k(A_M, GL(1))$, and $a_M = X(A_M) \otimes \mathbb{R}$. We denote by A_+ a set of all $a \in A_M(A_k)$ such that $a_v = 1$ for v finite, and $\chi(a_v) = a$, where a is a positive number independent of v infinite, for all $\chi \in X(A_M)$. We fix a maximal compact subgroup $K = \prod K_v$ in a usual way.

For each (s_1, s_2), there is an induced representation

$$\pi(s_1, s_2) = \mathrm{Ind}_{\mathbf{D}^*(A_k) \times \mathbf{D}^*(A_k)}^{G_2(\mathbf{D},1)(A_k)} \sigma\nu^{s_1} \otimes \sigma\nu^{s_2}.$$

Here ν denotes a product of the local ν's over all the places. These representations form a fibre bundle of representations and the sections are constructed as follows ([9]): Denote by $\mathcal{H}$ the space of functions f on $G_2(\mathbf{D}, 1)(A_k)$ satisfying the following conditions:

(i) $f(u\gamma ag) = f(g)$ for $u \in U(A_k)$, $\gamma \in P(k)$, $a \in A_+$;
(ii) f is K–finite, and for each $k \in K$ the function

$$m \mapsto f(mk)$$

belongs to the space $V \otimes V$ of functions on $\mathbf{D}^*(A_k) \times \mathbf{D}^*(A_k)$.

We extend the Harish-Chandra H_P function to the entire $G_2(\mathbf{D},1)(A_k)$ in the way that it is K–invariant. Then for each (s_1,s_2), the representation of $G_2(\mathbf{D},1)(A_k)$ (or of the appropriate Hecke algebra) on the space of functions of the form

$$g \mapsto f(g)\mathrm{exp}\langle H_P(g), s+\rho\rangle,\ f \in \mathcal{H},$$

is equivalent to $\pi(s_1,s_2)$. We form the corresponding Eisenstein series:

$$E(g,s_1,s_2,f) = \sum_{P(k)\backslash G_2(\mathbf{D},1)(k)} f(\gamma g)\mathrm{exp}\langle H_P(\gamma g), s+\rho\rangle.$$

It converges absolutely for s such that the real part of s is in the positive Weyl chamber shifted by the half-sum of the roots corresponding to the parabolic subgroup P (the one which has a Levi subgroup isomorphic to $\mathbf{D}^*(A_k)\times\mathbf{D}^*(A_k)$). The poles of the Eisenstein series coincide with the poles of its constant term (along P), which is given by

$$E_P(g,s_1,s_2,f) = \sum_{w\in W}[T(w,s_1,s_2)f](g)\mathrm{exp}\langle H_P(g), w(s_1,s_2)+\rho\rangle,$$

where the sum is over an absolute Weyl group of $G_2(\mathbf{D},1)$. For each w, $T(w,s_1,s_2)$ is an intertwining operator from $\mathcal{H}$ to $\mathcal{H}$ defined by

$$\begin{aligned}[T(w,s_1,s_2)f](g)&\mathrm{exp}\langle H(g), w(s_1,s_2)+\rho\rangle\\ &= \int f(w^{-1}ug)\mathrm{exp}\langle H(w^{-1}ug), s+\rho\rangle \mathrm{d}u,\end{aligned}$$

and the integral is over

$$U(k)\cap wU(k)w^{-1}\setminus U(A_k)\cap wU(A_k)w^{-1}.$$

We can identify these global intertwining operators with

$$T(w,s_1,s_2) = \bigotimes_v T_v(w,s_1,s_2),$$

where $T_v(w,s_1,s_2)$ are analogously defined local intertwining operators. Let S denote a finite set of places of k which includes the archimedean places, v_1, v_2, and all the ramified places, i.e. $v\notin S \Rightarrow \sigma_v\nu^{s_1}\times\sigma_v\nu^{s_2}\rtimes 1$ is a spherical representation. In that case, let f_v denote the unique K_v–invariant function, normalized with $f(e_v)=1$, and $\tilde{f}_v$ an analogous function in the representation space of $T_v(w,s_1,s_2)(\sigma_v\nu^{s_1}\times\sigma_v\nu^{s_2}\rtimes 1)$. Let $f=\bigotimes_v f_v$ be a function in the representation space of the induced representation, such that for $v\notin S$, f_v is the function fixed above. The elements s and c_2 of the Weyl group were defined in the Preliminaries, where s was denoted by s_1. We now use s to avoid confusion with the complex numbers s_1, s_2. Then we have the following expressions for the global intertwining operators :

$$T(c_2,s_1,s_2)f=(\otimes_{v\in S}T_v(c_2,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)\times\frac{L_S(s_2,\sigma)}{L_S(1+s_2,\sigma)}\frac{L_S(2s_2,1)}{L_S(1+2s_2,1)},$$

$$T(s,s_1,s_2)f=(\otimes_{v\in S}T_v(s,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)\times\frac{L_S(s_1-s_2,\sigma\times\sigma)}{L_S(1+s_1-s_2,\sigma\times\sigma)},$$

$$(sc_2,s_1,s_2)f=(\otimes_{v\in S}T_v(sc_2,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)$$
$$\times\frac{L_S(s_2,\sigma)}{L_S(1+s_2,\sigma)}\frac{L_S(2s_2,1)}{L_S(1+2s_2,1)}\frac{L_S(s_1+s_2,\sigma\times\sigma)}{L_S(1+s_1+s_2,\sigma\times\sigma)},$$

$$T(c_2sc_2,s_1,s_2)f=(\otimes_{v\in S}T_v(c_2sc_2,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)$$
$$\times\frac{L_S(s_2,\sigma)}{L_S(1+s_2,\sigma)}\frac{L_S(2s_2,1)}{L_S(1+2s_2,1)}\frac{L_S(s_1+s_2,\sigma\times\sigma)}{L_S(1+s_1+s_2,\sigma\times\sigma)}\frac{L_S(s_1,\sigma)}{L_S(1+s_1,\sigma)}\frac{L_S(2s_1,1)}{L_S(1+2s_1,1)},$$

$$T(sc_2sc_2,s_1,s_2)f=(\otimes_{v\in S}T_v(sc_2sc_2,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)$$
$$\times\frac{L_S(s_2,\sigma)}{L_S(1+s_2,\sigma)}\frac{L_S(2s_2,1)}{L_S(1+2s_2,1)}\frac{L_S(s_1+s_2,\sigma\times\sigma)}{L_S(1+s_1+s_2,\sigma\times\sigma)}\frac{L_S(s_1,\sigma)}{L_S(1+s_1,\sigma)}\frac{L_S(2s_1,1)}{L_S(1+2s_1,1)}$$
$$\times\frac{L_S(s_1-s_2,\sigma\times\sigma)}{L_S(1+s_1-s_2,\sigma\times\sigma)},$$

$$T(c_2s,s_1,s_2)=(\otimes._{v\in S}T_v(c_2s,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)$$
$$\times\frac{L_S(s_1-s_2,\sigma\times\sigma)}{L_S(1+s_1-s_2,\sigma\times\sigma)}\times\frac{L_S(s_1,\sigma)}{L_S(1+s_1,\sigma)}\frac{L_S(2s_1,1)}{L_S(1+2s_1,1)},$$

$$T(sc_2s,s_1,s_2)=(\otimes_{v\in S}T_v(sc_2s,s_1,s_2)f_v\otimes_{v\notin S}\tilde{f}_v)$$
$$\times\frac{L_S(s_1-s_2,\sigma\times\sigma)}{L_S(1+s_1-s_2,\sigma\times\sigma)}\times\frac{L_S(s_1,\sigma)}{L_S(1+s_1,\sigma)}\frac{L_S(2s_1,1)}{L_S(1+2s_1,1)}\times\frac{L_S(s_1+s_2,\sigma\times\sigma)}{L_S(1+s_1+s_2,\sigma\times\sigma)}.$$

The symbol $L_S(\cdot)$ denotes a partial L function obtained as a product of local L –functions over all the places except the ones in S. Note that, in the previous formulas, we have the partial standard L– function, the partial Hecke and Rankin-Selberg L–functions. We want to study a behavior of the intertwining operators near the point $(s_1,s_2)=(\frac{3}{2},\frac{1}{2})$. We analyze the local intertwining operators for $v\in S$. For $v\in\{v_1,v_2\}$, we have a standard intertwining operator

$$T_v(w,\frac{3}{2},\frac{1}{2})=A_w(\frac{3}{2},\frac{1}{2})$$

(using the previous notation), which acts on the standard representation $\tau_1\nu^{\frac{3}{2}}\times\tau_1\nu^{\frac{1}{2}}\rtimes 1$, and as such, is holomorphic near that point. If $v\in S\setminus\{v_1,v_2\}$, observe that $\sigma_v\cong\sigma'_v$ is a local component of the automorphic cuspidal representation σ' of $GL(2,A_k)$. This forces the unitary representation σ_v to be of the following two kinds:

(i) σ_v is a tempered representation,
(ii) σ_v is a complementary series representation, $\sigma_v\cong\chi_v\nu^s\times\chi_v\nu^{-s}$, for $s\in(0,\frac{1}{2})$, and some unitary character χ_v.

In the first case, $\sigma_v\nu^{\frac{3}{2}}\times\sigma_v\nu^{\frac{1}{2}}\rtimes 1$ is a standard representation, and in the second case, we have

$$\sigma_v\nu^{\frac{3}{2}}\times\sigma_v\nu^{\frac{1}{2}}\rtimes 1\cong\chi_v\nu^{s+\frac{3}{2}}\times\chi_v\nu^{-s+\frac{3}{2}}\times\chi_v\nu^{s+\frac{1}{2}}\times\chi_v\nu^{-s+\frac{1}{2}}\rtimes 1,$$

and the right-hand side is a standard representation of $Sp(8,k_v)$, so the operators $T_v(w,\frac{3}{2},\frac{1}{2})$, with $w \in W \subset W(Sp(8,k_v))$, are holomorphic. We conclude that all the possible poles of the global intertwining operators come from the poles of the partial L–functions. Now, we use the fact that, for the above L–functions, the global and the partial L–functions have the same poles for $\mathrm{Re}s \geq 1$ ([8]). We calculate the iterated residues of the partial L–functions for $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$. We see immediately that the iterated residues are non-trivial only for the partial L–functions which appear in the global intertwining operator associated with the longest element in the Weyl group, and then only if $L_S(\frac{1}{2},\sigma') \neq 0$, which is equivalent to $L(\frac{1}{2},\sigma') \neq 0$. We now identify the local components of the representations appearing in these residues. For $v \notin \{v_1,v_2\}$ we have: if σ_v is a tempered representation of $GL(2,k_v)$, then because c_2sc_2s is the longest element in the relative Weyl group W_M, where $M \cong GL(2,k_v) \times GL(2,k_v)$, we have

$$T_v(c_2sc_2s,\frac{3}{2},\frac{1}{2})(\sigma_v\nu^{\frac{3}{2}} \times \sigma_v\nu^{\frac{1}{2}} \rtimes 1) = L(\sigma_v\nu^{\frac{3}{2}},\sigma_v\nu^{\frac{1}{2}};1).$$

If $\sigma_v \cong \chi_v\nu^s \times \chi_v\nu^{-s}$, the representation

$$\chi_v\nu^{s+\frac{3}{2}} \times \chi_v\nu^{-s+\frac{3}{2}} \times \chi_v\nu^{s+\frac{1}{2}} \times \chi_v\nu^{-s+\frac{1}{2}} \rtimes 1$$

is a standard representation, but c_2sc_2s is not the longest element in the absolute Weyl group of $Sp(8,k_v)$, and the operator acts in the following way:

$$\begin{aligned}T_v(c_2sc_2s,\frac{3}{2},\frac{1}{2}) : \chi_v\nu^{s+\frac{3}{2}} \times \chi_v\nu^{-s+\frac{3}{2}} \times \chi_v\nu^{s+\frac{1}{2}} \times \chi_v\nu^{-s+\frac{1}{2}} \rtimes 1 \\ \to \chi_v\nu^{s-\frac{3}{2}} \times \chi_v\nu^{-s-\frac{3}{2}} \times \chi_v\nu^{s-\frac{1}{2}} \times \chi_v\nu^{-s-\frac{1}{2}} \rtimes 1.\end{aligned}$$

But

$$\chi_v\nu^{s-\frac{3}{2}} \times \chi_v\nu^{-s-\frac{3}{2}} \cong \chi_v\nu^{-s-\frac{3}{2}} \times \chi_v\nu^{s-\frac{3}{2}}$$

and

$$\chi_v\nu^{s-\frac{1}{2}} \times \chi_v\nu^{-s-\frac{1}{2}} \cong \chi_v\nu^{-s-\frac{1}{2}} \times \chi_v\nu^{s-\frac{1}{2}},$$

so the image of the action of this operator is the same as the image of the action of the long intertwining operator, i.e., equal to $L(\chi_v\nu^{s+\frac{3}{2}},\chi_v\nu^{-s+\frac{3}{2}},\chi_v\nu^{s+\frac{1}{2}},\chi_v\nu^{-s+\frac{1}{2}};1)$. For $v \in \{v_1,v_2\}$,

$$T_v(sc_2sc_2,\frac{3}{2},\frac{1}{2})(\tau_1\nu^{\frac{3}{2}} \times \tau_1\nu^{\frac{1}{2}} \rtimes 1) = L(\tau_1\nu^{\frac{3}{2}},\tau_1\nu^{\frac{1}{2}};1).$$

We can conclude that in the residual spectrum there is a representation whose local components are Langlands quotients described above; especially, at the places v_1 and v_2 this representation has $L(\tau_1\nu^{\frac{3}{2}},\tau_1\nu^{\frac{1}{2}};1)$ as a local component. This proves that the representation $L(\tau_1\nu^{\frac{3}{2}},\tau_1\nu^{\frac{1}{2}};1)$ is unitarizable, provided $L(\frac{1}{2},\sigma') \neq 0$. But, by the result of Waldspurger ([24], Théorème 5) for a cuspidal automorphic representation of the group $GL(2,A_k)$ with the trivial central character, such as σ', and fixed finite set of places V, there exists a quadratic character χ such that $\chi_v = 1$ for $v \in V$ and $L(\sigma' \otimes \chi,\frac{1}{2}) \neq 0$, provided $\epsilon(\sigma',\frac{1}{2}) = 1$. So, if we take $v_1,v_2 \in V$, we can twist σ' by χ, but the local components involved remain the same at v_1 and v_2. But we can do that provided $\epsilon(\sigma',\frac{1}{2}) = 1$. □

We now prove the non-unitarizability of the representations $L(\nu\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)$ and $L(\tau\nu^{\frac{3}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])$. We do that in the following way: For the first one we will calculate the order of the pole of the Plancherel measure $\mu(s,\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}}))$ for

$s=1$, and then use this in the calculation of the Jantzen filtrations. This will give us the non-unitarizability. By $A(s)$ we denote the standard intertwining operator

$$A(s):\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\nu^s \rtimes 1 \to \delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\nu^{-s} \rtimes 1.$$

We will prove that the Plancherel measure has a simple pole for $s=1$ and that $A(s)$ has no pole for $s=-1$. We denote it by $\delta=\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})$. We use the aforementioned result ([12]) which states that $\mu(s,\delta)=\mu(s,\delta')$. Here, the representation δ' is a discrete series representation of the group $GL(4,F)$, which is the Jacquet-Langlands lift of the representation δ. The Plancherel measure $\mu(s,\delta)$ is with respect to the group $G_2(D,1)$, and the measure $\mu(s,\delta')$ is with respect with its split form, namely the group $Sp(8,F)$. The representation δ' is generic, so we can apply the results from [16] and [15] to compute the Plancherel measure in terms of γ–factors. Up to an exponential factor, we have

$$\mu(s,\delta')=\frac{\gamma(s,\delta',\rho_4,\psi)}{\gamma(1+s,\delta',\rho_4,\psi)}\frac{\gamma(2s,\delta',\Lambda^2\rho_4,\psi)}{\gamma(1+2s,\delta',\Lambda^2\rho_4,\psi)}.$$

Now, using the multiplicativity of γ–factors, we obtain

$$\mu(s,\delta')=\frac{(1-q^{-1-2s})(1-q^{1-2s})(1-q^{1+2s})(1-q^{-1+2s})(1-q^{-2rs})(1-q^{2rs})}{(1-q^{2s})(1-q^{-2s})(1-q^{-2+2s})(1-q^{-2-2s})(1-q^{-r+2rs})(1-q^{-r-2rs})},$$

where r is some integer. So, the Plancherel measure has a simple pole for $s=1$. Then consider the intertwining operator

$$A_{w_{\alpha+\beta}}(s):\tau_1\nu^{s+\frac{1}{2}}\times\tau_1\nu^{s-\frac{1}{2}}\rtimes 1\to\tau_1\nu^{-s+\frac{1}{2}}\times\tau_1\nu^{-s-\frac{1}{2}}\rtimes 1.$$

The poles of the operator $A(s)$ are among the poles of the operator $A_{w_{\alpha+\beta}}(s)$ because $A_{w_{\alpha+\beta}}(s)|_{\delta\nu^s\rtimes 1}=A(s)$. But, by using the factorization of the operator $A_{w_{\alpha+\beta}}(s)$ ([14]), we see that it has no poles for $s=-1$. Let X denote the compact picture of the representation $\delta\nu^s\rtimes 1$. We will consider the Jantzen filtration of the space X, for $s\in[0,1]$. For $s\in(0,1)$, the representations $\delta\nu^s\rtimes 1$ are irreducible, and the interval $(0,1)$ parameterizes a non-degenerate family of the hermitian forms on the compact picture X. For $s=0$ $A(s)$ is holomorphic, and, normalized, generates the intertwining algebra of the representation $\delta\rtimes 1=T_1+T_2$. The operator $A(0)$ endows the space of this representation with a hermitian form which is of a different sign on each of the T_i's. This gives us the non-unitarizability of $\delta\nu^s\rtimes 1$ for $s\in(0,1)$. By the theory of Jantzen filtrations ([22]), for $s=1$ we consider a filtration

$$X=X_1^0\supset X_1^1\supset\cdots\supset 0.$$

The spaces X_1^i are $G_2(D,1)$–invariant spaces, and each of them is a radical of the certain hermitian form defined on the previous space ([22]). For $s=1$, the representation X_1^0 is a compact picture of the standard representation $\delta\nu\rtimes 1$, so, by the results from [1], the space X_1^1 is a compact picture of π_4, which is a square integrable representation. We will prove that $X_1^2=\{0\}$, i.e. that a hermitian form defined on X_1^1 by

$$\langle v,v'\rangle_1=\lim_{s\to 1}\int_K\langle v(k),\frac{1}{s-1}A(s)v'_s(k)\rangle \mathrm{d}k$$

is non-degenerate, so its radical, namely X_1^2, is trivial. As before, v_s denotes the corresponding holomorphic section. Because of the simplicity of the pole of the

Plancherel measure for $s=1$, we have

$$A(-s)\frac{1}{s-1}A(s)=h(s)\mathrm{id},$$

where h is a holomorphic function in the neighborhood of $s=1$, and $h(1)\neq 0$. From this follows that for a non-null $v'\in X$ such that $v'_1\in\pi_4$ we have $\lim_{s\to 1}\frac{1}{s-1}A(s)v'_s\notin L(\delta\nu,1)$. Now, we can choose $v\in X_1^1$ such that

$$\langle v,v'\rangle_1=\lim_{s\to 1}\int_K\langle v(k),\frac{1}{s-1}A(s)v'_s(k)\rangle \mathrm{d}k\neq 0.$$

If we denote the signature of $L(\delta\nu,1)$ by (p_0,q_0) (by [22]), we can obtain the signature of $\delta\nu^s\rtimes 1$ for $s>1$ and for $s<1$ in terms of signatures (p_0,q_0) and (p_1,q_1) (which is signature of X_1^1, i.e. of π_4, so $p_1=0$, or $q_1=0$). We have just proved that $(p_i,q_i)=(0,0)$, $i\geq 2$. In more detail, the signature of the representation $\delta\nu^s\rtimes 1$ for $s>1$ is (p_0+p_1,q_0+q_1), and for $s<1$ is (p_0+q_1,p_1+q_0). But if we assume $p_1=0$ or, the same, if we assume $q_1=0$, and knowing that both for $s>1$ and for $s<1$ we have the non-unitarizable representations, we conclude $p_0\neq 0$ and $q_0\neq 0$, which is equivalent to the non-unitarizability of the representation $L(\delta\nu,1)$.

The proof of the non-unitarizability of $L(\tau_1\nu^{\frac{3}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])$ follows the same pattern: First, we will prove that the Plancherel measure $\mu(s_1,\delta[\tau_1\nu^{\frac{1}{2}};1])$ has a simple pole at $s_1=\frac{3}{2}$. Observe that, for the longest element w_0 from the Weyl group, we have $w_0=w_\beta w_\alpha w_\beta w_\alpha=w_\beta w_{2\alpha+\beta}=w_{2\alpha+\beta}w_\beta$. So, for the standard intertwining operator

$$A_{w_0}(s_1,s_2,\tau_1\otimes\tau_1):\tau_1\nu^{s_1}\times\tau_1\nu^{s_2}\rtimes 1\to\tau_1\nu^{-s_1}\times\tau_1\nu^{-s_2}\rtimes 1$$

the following holds:

$$A_{w_0}(s_1,s_2,\tau_1\otimes\tau_1)=A_{w_\beta}(-s_1,s_2,\tau_1\otimes\tau_1)A_{w_{2\alpha+\beta}}(s_1,s_2,\tau_1\otimes\tau_1).$$

From the definition of the Plancherel measure then follows

$$\mu^{-1}_{w_\beta}(-s_1,s_2,\tau_1\otimes\tau_1)\mu^{-1}_{w_{2\alpha+\beta}}(s_1,s_2,\tau_1\otimes\tau_1)=\mu(s_1,s_2,\tau_1\otimes\tau_1)^{-1}. \tag{1}$$

Now, we apply a result of Heiermann ([7]), which says that in the case of the induction from a cuspidal representation such that the obtained representation has a square integrable subquotient, the corresponding Plancherel measure has a pole of the order equal to the corank of the Levi subgroup. So, the right hand side of (1) has a zero of order 2 for $(s_1,s_2)=(\frac{3}{2},\frac{1}{2})$. The operator A_{w_β} is induced from the standard intertwining operator on $G_1(D,1)$, and so $\mu_{w_\beta}(-\frac{3}{2},*,\tau_1\otimes\tau_1)$ has a pole for $s_2=\frac{1}{2}$. This means that $\mu_{w_{2\alpha+\beta}}(s_1,\frac{1}{2},\tau_1\otimes\tau_1)=\mu(s_1,\delta[\tau_1\nu^{\frac{1}{2}};1])$ has a simple pole for $s_1=\frac{3}{2}$. Now, analogously as for the representation $L(\nu\delta(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}});1)$, using the Jantzen filtrations, we prove the non-unitarizability of the representation $L(\tau_1\nu^{\frac{3}{2}};\delta[\tau_1\nu^{\frac{1}{2}};1])$. The only Langlands quotient left to settle is

$$L(\tau_1\nu;\tau_1\rtimes 1)=\nu^{\frac{1}{2}}L(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1.$$

We obtain the hermiticity of the representations $\pi_s=\nu^sL(\tau_1\nu^{\frac{1}{2}},\tau_1\nu^{-\frac{1}{2}})\rtimes 1$ for $s\in(0,1)$ using the action of the long intertwining operator acting on the space $\tau_1\nu^{s+\frac{1}{2}}\times\tau_1\nu^{s-\frac{1}{2}}\rtimes 1$. But unitarity of the representation π_s for $s=\frac{1}{2}$ would imply the unitarizability of all the subquotients for $s=1$, which contradicts what we have just proved. $\square$

As for the unitary subquotients for the mixed case

$$\pi = \chi\nu^{\alpha} \times \tau\nu^{\beta} \rtimes 1,\ \alpha, \beta \in \mathbb{R}_0^+,$$

of the principal series, we have the following proposition.

Proposition 4.7. *Let* $\pi = \chi\nu^{\alpha} \times \tau\nu^{\beta} \rtimes 1,\ \alpha, \beta \in \mathbb{R}_0^+,\ \dim\chi = 1,\ \dim\tau > 1.$ *Then the following holds:*

(i) *If none of the representations* χ, τ *is self-contragredient, the representation* π *has a unitarizable subquotient only when* $\alpha = \beta = 0$*; then* π *is an irreducible tempered representation.*

(ii) *If* $\chi^2 = 1$ *and* $\tau \ncong \widetilde{\tau}$*, the unitarizable subquotients appear if and only if* $\beta = 0$ *and* $\alpha \in [0, \alpha_0]$*, where* $\alpha = \alpha_0$ *is a unique non-negative point of reducibility of the representation* $\chi\nu^{\alpha} \rtimes 1$*. Then all the appearing subquotients are unitarizable.*

(iii) *If* $\chi^2 \neq 1$ *and* $\tau \cong \widetilde{\tau}$*, the unitarizable subquotients appear if and only if* $\alpha = 0$ *and* $\beta \in [0, \beta_0]$*, where* $\beta = \beta_0$ *is the unique non-negative point of reducibility of the representation* $\tau\nu^{\beta} \rtimes 1$*. Then all the appearing subquotients are unitarizable.*

(iv) *If both of the representations are self-contragredient, the unitarizable subquotients appear if and only if* $\alpha \in [0, \alpha_0]$ *and* $\beta \in [0, \beta_0]$*, where* α_0 *and* β_0 *are the reducibility points of the corresponding representations. Then all the subquotients of the representation* π *are unitarizable.*

The points of the reducibility α_0 *and* β_0 *are determined in Lemma* 3.1*, and the irreducible subqoutients appearing here are described in Proposition* 3.5*.*

Proof. This is straightforward from the criterion for the hermiticity of the Langland's quotient. ☐

5. The Siegel case

We are investigating the reducibility points and possible unitarizable subquotients of the representation

$$\sigma\nu^{s} \rtimes 1,\ s \in \mathbb{R}_0^+,$$

where σ is an irreducible cuspidal representation of $GL(2, D)$. By σ' we denote the Jacquet-Langlands lift of σ, and this is a discrete series representation of $GL(4, F)$ ([2]). Although the Jacquet-Langlands lift is between the discrete series representations of $GL(n, D)$ and the discrete series representations of $GL(2n, F)$, in the case $n = 2$, σ' is actually a cuspidal representation, too ([2]). If $\sigma \cong \widetilde{\sigma}$, then $\sigma' \cong \widetilde{\sigma'}$ also holds. But when σ is a cuspidal representation, the zeroes and poles of the Plancherel measure completely determine the reducibility points of the representation $\sigma\nu^{s} \rtimes 1$. If $\sigma \ncong \widetilde{\sigma}$, it does not reduce for any $s \in \mathbb{R}$. So, if $\sigma \cong \widetilde{\sigma}$, by ([17]), there is a unique non-negative $s = s_0$ such that the representation $\sigma\nu^{s} \rtimes 1$ reduces. We have the following characterization of the reducibility points in terms of the Plancherel measure:

$$\mu(s, \sigma) \neq 0, \text{ for } s = 0 \iff \sigma \rtimes 1 \text{ reduces},$$
$$\mu(s, \sigma) = \infty, \text{ for } s = s_0 > 0 \iff \sigma\nu^{s_0} \rtimes 1 \text{ reduces}.$$

By [12], we know that

$$\mu(s, \sigma) = \mu(s, \sigma').$$

(We already used that in a calculation in the principal series case.) So, in order to find a reducibility point of the representation $\sigma\nu^s \rtimes 1$, we have to analyze the zeroes and the poles of the Plancherel measure $\mu(s, \sigma')$. But, the representation σ' is a generic (cuspidal) representation of the group $GL(4, F)$, and we can use the results of ([16]) to calculate it in terms of L–functions.

Theorem 5.1. (i) *If $\sigma \ncong \widetilde{\sigma}$, the representation $\sigma\nu^s \rtimes 1$ has a unitarizable (even hermitian) subquotient only for $s = 0$, and then it is an irreducible tempered representation.*

(ii) *If $\sigma \cong \widetilde{\sigma}$, the representation $\sigma\nu^s \rtimes 1$ reduces only for $s = 0$ or only for $s = \frac{1}{2}$. In more words: The representation $\sigma \rtimes 1$ reduces if and only if $L(s, \sigma', \Lambda^2\rho_4)$ does not have a pole for $s = 0$. If it has a pole for $s = 0$, then the representation $\sigma\nu^s \rtimes 1$ reduces for $s = \frac{1}{2}$. If the reducibility point is $s = 0$, the representation $\sigma\nu^s \rtimes 1$ has a unitarizable subquotient only for $s = 0$; then $\sigma \rtimes 1$ is a sum of two non-equivalent tempered representations. If $\sigma\nu^s \rtimes 1$ reduces for $s = \frac{1}{2}$, only for $s \in [0, \frac{1}{2}]$ do the unitarizable subquotients appear. Then $\sigma\nu^s \rtimes 1 = L(\sigma\nu^s; 1)$ for $s \in (0, \frac{1}{2})$, and $\sigma\nu^{\frac{1}{2}} \rtimes 1 = L(\sigma\nu^{\frac{1}{2}}; 1) + \pi_7$. Here both of the subquotients are unitarizable, and π_7 is a square integrable representation.*

Proof. The reducibility points (when $\sigma \cong \widetilde{\sigma}$) follow directly from a calculation of the Plancherel measure (a calculation similar to the one done in the previous section). All the subquotients of the representation $\sigma\nu^s \rtimes 1$ are hermitian, and for s greater of the reducibility point, there is a family of the hermitian forms on the representations $\sigma\nu^s \rtimes 1$, which is indexed by an unbounded, connected interval, so if for some s from this interval the representation is unitarizable, it would have to be unitarizable for every s from this interval. This is impossible, because the matrix coefficients of these representations become unbounded for s large enough, as we can see from their asymptotics, given by the asymptotics of the Jacquet module coefficients. $\square$

6. The non-Siegel case of the maximal parabolic subgroup

Let σ be an irreducible representation of D^* and ρ an irreducible cuspidal representation of $G_1(D, 1)$. We will consider the representation

$$\pi_s = \sigma\nu^s \rtimes \rho,$$

for $s \in \mathbb{R}$. Similarly to the Siegel case, if $\sigma \ncong \widetilde{\sigma}$, the representation $\sigma\nu^s \rtimes \rho$ never reduces. Now, we assume $\sigma \cong \widetilde{\sigma}$ and $s \geq 0$, and again, there exists a unique $s_0 \geq 0$ such that $\sigma\nu^{s_0} \rtimes \rho$ reduces ([17]). The following characterization of the reducibility points in terms of the Plancherel measure is analogous to the one in the Siegel case

$$\mu(s, \sigma \otimes \rho) \neq 0, \text{ for } s = 0 \iff \sigma \rtimes \rho \text{ reduces},$$

$$\mu(s, \sigma \otimes \rho) = \infty, \text{ for } s = s_0 > 0 \iff \sigma\nu^{s_0} \rtimes \rho \text{ reduces}.$$

We were unable to transfer directly the Plancherel measure from $G_2(D, 1)$ to $Sp(8, F)$ (as we did in [5] for $G_2(D, -1)$), so we have to rely on two standard conjectures in the harmonic analysis on the quasi-split p–adic group: the first one describes the space of the stable distributions on p-adic group in terms of the stable tempered characters, and assuming it is true, Shahidi ([16]) proved that the Plancherel measure of a discrete series representation of the Levi subgroup depends only on the (discrete series) L-packet. The second one claims that every tempered L–packet is generic (for example, [16]). Also, we assume ([16]) that the Plancherel measures are the same for inner forms.

So, for a cuspidal representation ρ of $G_1(D,1)$, let the representation ρ' be a (conjectural) corresponding generic discrete series representation, and, as usual, σ' denotes the Jacquet-Langlands lift of a cuspidal representation σ of D^*.

Conjecture 6.1.

$$\mu(s,\sigma\otimes\rho)=\mu(s,\sigma'\otimes\rho').$$

Theorem 6.1. *Let σ denote an irreducible representation of D^*, and let ρ be an irreducible cuspidal representation of $G_1(D,1)$. Let ρ' denote a (conjectural) generic discrete series representation of $Sp(4,F)$ which is (one of) the Jacquet-Langlands lift of the representation ρ, and let σ' denote the Jacquet-Langlands lift of the representation σ. Assume that Conjecture 6.1 holds. Let $\pi_s=\sigma\nu^s\rtimes\rho$, $s\in\mathbb{R}$. Then, the representation ρ' is a cuspidal representation of $Sp(4,F)$, or has a cuspidal support on $F^*\times SL(2,F)$, and the following holds:*

If $\sigma\not\cong\widetilde{\sigma}$, the representation π_s never reduces, and is unitarizable only for $s=0$, and then it is a tempered representation. Assume now that $\sigma\cong\widetilde{\sigma}$ and $s\geq 0$. Then, one of the following holds:

- (i) *If $\dim\sigma>1$ and $\omega_\sigma(\overline{\varpi})=1$, then the representation π_s reduces for $s=\frac{1}{2}$.*
- (ii) *If $\dim\sigma>1$ and $\omega_\sigma(\overline{\varpi})=-1$, then*
  - (a) *If ρ' is cuspidal, π_s reduces for $s=1$ if $L(0,\sigma'\times\widetilde{\rho'})=\infty$, or for $s=0$ if $L(0,\sigma'\times\widetilde{\rho'})\neq\infty$.*
  - (b) *If $\rho'\hookrightarrow\chi_1\nu\rtimes\pi$ (for some unitary character χ_1 of F^* and a cuspidal representation π of $SL(2,F)$), the representation π_s reduces for $s=1$ if $L(0,\sigma'\times\widetilde{\pi})=\infty$, or for $s=0$ if $L(0,\sigma'\times\widetilde{\pi})\neq\infty$.*
- (iii) *If $\dim\sigma=1$, i.e. $\sigma=\chi$, a quadratic character of D^*, then*
  - (a) *If ρ' is cuspidal, the representation π_s reduces for $s=\frac{1}{2}$ if $L(0,\chi\times\rho')\neq\infty$ or for $s=\frac{3}{2}$ if $L(0,\chi\times\rho')=\infty$.*
  - (b) *If $\rho'\hookrightarrow\chi_1\nu\times\pi$, the representation π_s reduces for $s=\frac{5}{2}$ if $\chi=\chi_1$, for $s=\frac{1}{2}$ if $\chi\neq\chi_1$ but $\chi\rtimes\pi$ reduces, or for $s=\frac{3}{2}$ if $\chi\neq\chi_1$ and $\chi\nu\rtimes\pi$ reduces.*

If the representation π_s reduces for $s_0=0$, the unitarizable subquotients appear only for $s=0$, and then π_0 is a sum of two non-equivalent tempered representations. If $s_0>0$, the unitarizable subquotients appear if and only if $s\in[0,s_0]$, and then all of them are unitarizable. In that case, for $s=0$, π_0 is irreducible tempered, for $s\in(0,s_0)$, $\pi_s=L(\sigma\nu^s;1)$, and $\pi_{s_0}=L(\sigma\nu^{s_0};1)+\pi_8$, where π_8 is an irreducible square integrable representation.

Proof. Let $P=MN$ be a standard maximal parabolic subgroup of the group $Sp(8,F)$, such that $M\cong GL(2,F)\times Sp(4,F)$. By ${}^LM$ we denote a corresponding Levi subgroup in the dual group of $Sp(8,F)$, i.e. in $SO(9,\mathbb{C})$. Let ψ_F denote a non-trivial, additive character of F. The representation of the group ${}^LM\cong GL(2,\mathbb{C})\times SO(5,\mathbb{C})$ on ${}^Ln$, the Lie algebra of ${}^LN$, decomposes as

$$\rho_2\otimes\rho_5+\Lambda^2\rho_2.$$

Here, ρ_2 denotes the standard representation of $GL(2,\mathbb{C})$ and ρ_5 denotes the standard representation of $SO(5,\mathbb{C})$. By [16] and [15], we have

$$\mu(s,\sigma'\otimes\rho') = \frac{\gamma(s,\sigma'\times\rho',\psi_F)}{\gamma(1+s,\sigma'\times\rho',\psi_F)}\frac{\gamma(2s,\sigma',\Lambda^2\rho_2,\psi_F)}{\gamma(1+2s,\sigma',\Lambda^2\rho_2,\psi_F)}. \tag{2}$$

Assume that the representation σ' is cuspidal, i.e. $\dim\sigma > 1$. Then, when $\omega_{\sigma'}(\overline{\omega}) = 1$, the relation (2) becomes

$$\mu(s,\sigma'\otimes\rho') = \frac{L(1-s,\sigma'\times\widetilde{\rho'})}{L(s,\sigma'\times\rho')}\frac{L(1+s,\sigma'\times\rho')}{L(-s,\sigma'\times\widetilde{\rho'})}\frac{(1-q^{-2s})}{(1-q^{-1+2s})}\frac{(1-q^{2s})}{(1-q^{-1-2s})}. \tag{3}$$

The L–functions $L(z,\sigma'\times\rho')$ and $L(z,\sigma'\times\widetilde{\rho'})$ are holomorphic for $\mathrm{Re}z>0$ ([11]), so it follows that $L(1-s,\sigma'\times\widetilde{\rho'})$ and $L(1+s,\sigma'\times\rho')$ are holomorphic near $s=0$. This means that $\mu(0,\sigma'\otimes\rho')=0$, so the representation $\sigma'\rtimes\rho'$ is irreducible, and $\sigma\rtimes\rho$ is irreducible. On the other hand, for $s>0$ the quotient $\frac{L(1+s,\sigma'\times\rho')}{L(s,\sigma'\times\rho')}\neq 0$ is holomorphic. The right hand side of the relation (3) has a pole for $s=\frac{1}{2}$, unless $L(-\frac{1}{2},\sigma'\times\widetilde{\rho'})=\infty$. But, from Lemma 5.3 of [11], we can conclude that, in the case of cuspidal ρ', the only possible pole of $L(s,\sigma'\times\widetilde{\rho'})$ is $s=0$. So, for $s=\frac{1}{2}$ measure $\mu(s,\sigma'\otimes\rho')$ has a pole, and this pole must be unique, so $L(0,\sigma'\times\widetilde{\rho'})\neq\infty$, and, for a cuspidal representation ρ', we have reducibility for $s=\frac{1}{2}$. We now consider the case of a non-cuspidal representation ρ'.

If ρ' has support on the non-Siegel maximal parabolic subgroup, i.e. $\rho'\hookrightarrow\chi_1\nu\rtimes\pi$, for a quadratic character χ_1 and a cuspidal representation π of $SL(2,F)$, we have

$$\begin{aligned}\gamma(s,\sigma'\times\rho',\psi_F) = \gamma(s,\sigma'\times\chi_1\nu,\psi_F)\gamma(s,\sigma'\times\pi,\psi_F)\gamma(s,\sigma'\times\chi_1\nu^{-1},\psi_F)\\ = \gamma(s,\sigma'\times\pi,\psi_F).\end{aligned}$$

Then we have

$$\frac{\gamma(s,\sigma'\times\pi,\psi_F)}{\gamma(1+s,\sigma'\times\pi,\psi_F)} = \frac{L(1+s,\sigma'\times\pi,\psi_F)}{L(s,\sigma'\times\pi,\psi_F)}\frac{L(1-s,\sigma'\times\widetilde{\pi},\psi_F)}{L(-s,\sigma'\times\widetilde{\pi},\psi_F)}.$$

From Lemma 5.3 of [11] we again conclude that $L(-\frac{1}{2},\sigma'\times\widetilde{\pi})\neq\infty$, so again we have reducibility of $\sigma'\nu^s\rtimes\rho'$ for $s=\frac{1}{2}$.

We will show that there is no need to consider the case of ρ' having support on the minimal parabolic subgroup, or on the Siegel parabolic subgroup.

Now assume that, for σ' still cuspidal, $\omega_{\sigma'}(\overline{\omega})=-1$. Then the reducibility points are completely determined by the L-function $L(s,\sigma'\times\rho')$. From the previous discussion we can conclude:

If ρ' is cuspidal, we can have $L(0,\sigma'\times\widetilde{\rho'})=\infty$. So, if $L(0,\sigma'\times\widetilde{\rho'})=\infty$, then $\sigma'\rtimes\rho'$ is irreducible, and $\sigma'\nu\rtimes\rho'$ is reducible. If $L(0,\sigma'\times\widetilde{\rho'})\neq\infty$ (then also $L(0,\sigma'\times\rho')\neq\infty$) the representation $\sigma'\rtimes\rho'$ is reducible.

If ρ' has cuspidal support on $F^*\times SL(2,F)$, i.e. $\rho'\hookrightarrow\chi_1\nu\rtimes\pi$, we have

$$\gamma(s,\sigma'\times\rho',\psi_F)=\gamma(s,\sigma'\times\pi,\psi_F),$$

and the discussion is the same as in the previous case of cuspidal ρ'.

If we assume that ρ' has a cuspidal support on the Siegel parabolic subgroup, i.e. $\rho'\hookrightarrow\pi\nu^{\frac{1}{2}}\rtimes 1$, for selfcontragredient cuspidal representation π of $GL(2,F)$, we have $\gamma(s,\sigma'\times\rho',\psi_F)=\gamma(s,\sigma'\times\pi\nu^{\frac{1}{2}})\gamma(s,\sigma'\times\pi\nu^{-\frac{1}{2}})=1$, because we can take

$\sigma' \not\cong \pi$. Then the right-hand side of (3) has no poles or zeroes, which is impossible, so ρ' cannot have a cuspidal support on the Siegel maximal parabolic.

Now assume that σ' is not cuspidal, i.e. $\sigma = \chi$ as a representation of D^* and $\sigma' = \chi St_{GL(2,F)}$. Using the multiplicativity of γ–factors, the relation (2) now becomes

$$\mu(s, \chi St_{GL(2,F)} \otimes \rho') = \frac{L(\frac{3}{2}-s, \chi \times \widetilde{\rho'})}{L(s-\frac{1}{2}, \chi \times \rho')} \frac{L(\frac{3}{2}+s, \chi \times \rho')}{L(-\frac{1}{2}-s, \chi \times \widetilde{\rho'})} \frac{(1-q^{-2s})}{(1-q^{-1+2s})} \frac{(1-q^{2s})}{(1-q^{-1-2s})}.$$

Assume now that ρ' is supported on the minimal parabolic subgroup, i.e. ρ' is the Steinberg representation or $\rho' \hookrightarrow \nu\xi \times \xi \rtimes 1$, where ξ is a character of order two ([13]). Then, in the second case, from the previous relation it follows that

$$\mu(s, \chi St_{GL(2,F)} \otimes \rho') = \frac{\gamma(s-\frac{1}{2}, \chi \times \xi, \psi_F)^2}{\gamma(s+\frac{5}{2}, \chi \times \xi, \psi_F)} \frac{\gamma(s-\frac{3}{2}, \chi \times \xi, \psi_F)}{\gamma(s+\frac{3}{2}, \chi \times \xi, \psi_F)^2} \times \frac{(1-q^{-2s})}{(1-q^{-1+2s})} \frac{(1-q^{2s})}{(1-q^{-1-2s})}.$$

The right-hand side of the previous relation is equal to

$$\frac{L(\frac{3}{2}-s, \chi \times \xi)^2}{L(-\frac{1}{2}+s, \chi \times \xi)^2} \frac{L(\frac{5}{2}-s, \chi \times \xi)}{L(-\frac{3}{2}+s, \chi \times \xi)} \times \frac{L(\frac{5}{2}+s, \chi \times \xi)}{L(-\frac{3}{2}-s, \chi \times \xi)} \frac{L(\frac{3}{2}+s, \chi \times \xi)^2}{L(-\frac{1}{2}-s, \chi \times \xi)^2} \frac{(1-q^{-2s})}{(1-q^{-1+2s})} \frac{(1-q^{2s})}{(1-q^{-1-2s})}.$$

If we assume that $\chi = \xi$, this expression has poles in $s = \frac{3}{2}$ and $s = \frac{5}{2}$, but this is impossible, so such a discrete series ρ' cannot correspond to a cuspidal representation of $G_1(D,1)$. Analogously, if we assume $\rho' = St_{Sp(4,F)}$ and $\chi = 1$ it follows that the expression for the Plancherel measure has poles for $s = \frac{1}{2}$ and for $s = \frac{7}{2}$, which is also impossible. So, we can conclude that the discrete series representations of $Sp(4,F)$ supported on the minimal parabolic subgroup are not split counterparts of the cuspidal representations of $G_1(D,1)$.

Assume now that ρ' has support on the non-Siegel maximal parabolic subgroup, i.e. $\rho' \hookrightarrow \chi_1\nu \rtimes \pi$. Then

$$\mu(s, \chi St_{GL(2,F)} \otimes \rho') = \frac{\gamma(s-\frac{3}{2}, \chi \times \chi_1, \psi_F)}{\gamma(s+\frac{3}{2}, \chi \times \pi, \psi_F)} \frac{\gamma(s-\frac{1}{2}, \chi \times \pi, \psi_F)}{\gamma(s+\frac{5}{2}, \chi \times \chi_1, \psi_F)} \times \frac{(1-q^{-2s})}{(1-q^{-1+2s})} \frac{(1-q^{2s})}{(1-q^{-1-2s})}.$$

If we assume that $\chi = \chi_1$, we obtain that the Plancherel measure has a unique positive pole for $s = \frac{5}{2}$. On the other hand, if $\chi \neq \chi_1$ and $\chi \rtimes \pi$ is reducible (i.e. $\chi = 1$, or $\chi^2 = 1$, but $\chi \notin (F^*/F_\pi^*)^\wedge$ in the notation of [13]) we obtain that $\chi St_{GL(2,F)}\nu^{\frac{1}{2}} \rtimes \rho'$ is reducible. If $\chi \neq \chi_1$, but $\chi\nu \rtimes \pi$ is reducible (meaning $\chi \in (F^*/F_\pi^*)^\wedge$), we obtain that $\chi St_{GL(2,F)}\nu^{\frac{3}{2}} \rtimes \rho'$ is reducible.

If we assume that ρ' is cuspidal, then $\chi St_{GL(2,F)}\nu^s \rtimes \rho'$ reduces for $s = \frac{1}{2}$ if $L(0, \chi \times \rho') = \overline{L(0, \chi \rtimes \widetilde{\rho'})} \neq \infty$ ([16]) or for $s = \frac{3}{2}$ if $L(0, \chi \times \rho') = \infty$.

Now, when the reducibility points are established, analogously as in the Siegel case, we see that the unitarizable representations occur only if $s = 0$, if the reducibility point of the representation π_s is $s_0 = 0$, otherwise they occur for the complementary series ($s \in (0, s_0)$) and at the end of the complementary series ($s = s_0$) where a square integrable subrepresentation occurs. $\square$

Remark. The description of the cuspidal support of the representation ρ' obtained this way coincides with the Langlands correspondence. In ([23]) is explained a conjectural way in which the discrete series representations of an inner class of a rational form simultaneously "fill" the L–packets. The inner class of a rational form, in our case, consists only of the groups $Sp(4, F)$ and $G_1(D, 1)$. Each packet should contain a discrete series representation of $Sp(4, F)$, and the number of the representations in it is governed by the centralizer of an admissible homomorphism parameterizing the L-packet (for example, [10]). The discrete series representations of $Sp(4, F)$ with the cuspidal support on $GL(2, F)$ are in the same packet with discrete series representation of $G_1(D, 1)$ with cuspidal support on D^*, and the representations of $GL(2, F)$ and D^* are corresponding by the Jacquet-Langlands correspondence. The discrete series representations of $Sp(4, F)$ with the cuspidal support on the minimal parabolic subgroup are the Steinberg representations and two non-equivalent discrete series subrepresentations of the representation $\xi\nu \times \xi \rtimes 1$, for each character ξ of order two ([13]). Examining the centralizer of the parameter, it follows that in each of these L-packets we have two representations, so the Steinberg representation of the group $Sp(4, F)$ is with the Steinberg representation of $G_1(D)$, and other pairs of discrete series of $Sp(4, F)$ exhaust the whole L–packet. This means that no cuspidal representation of $G_1(D)$ can appear in these L–packets. A different approach can be found in [4].

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