

THE KERNELS OF RADICAL HOMOMORPHISMS AND INTERSECTIONS OF PRIME IDEALS

HUNG LE PHAM

ABSTRACT. We establish a necessary condition for a commutative Banach algebra A so that there exists a homomorphism θ from A into another Banach algebra such that the prime radical of the continuity ideal of θ is not a finite intersection of prime ideals in A . We prove that the prime radical of the continuity ideal of an epimorphism from A onto another Banach algebra (or of a derivation from A into a Banach A -bimodule) is always a finite intersection of prime ideals. Under an additional cardinality condition (and assuming the Continuum Hypothesis), this necessary condition is proved to be sufficient. En route, we give a general result on norming commutative semiprime algebras; extending the class of algebras known to be normable. We characterize those locally compact metrizable spaces Ω for which there exists a homomorphism from $C_0(\Omega)$ into a radical Banach algebra whose kernel is not a finite intersection of prime ideals.

1. INTRODUCTION

Let $\theta : A \rightarrow B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B . The *continuity ideal* of θ is defined to be the ideal

$$\mathcal{I}(\theta) = \{a \in A : \text{the map } b \mapsto \theta(ab), A \rightarrow B, \text{ is continuous}\}.$$

We see that $\mathcal{I}(\theta)$ contains every ideal I in A on which θ is continuous. Let Ω be a locally compact space. In the case where $A = C_0(\Omega)$, it is known that θ is continuous on $\mathcal{I}(\theta)$; however, this is not true in general.

In this paper, we shall give an answer to the following two equivalent questions:

- A Is the kernel of a homomorphism from $C_0(\Omega)$ into a radical Banach algebra always a finite intersection of prime ideals, for each non-compact locally compact space Ω ?
- B Is the continuity ideal $\mathcal{I}(\theta)$ of a homomorphism θ from $C_0(\Omega)$ into a Banach algebra always a finite intersection of prime ideals, for each locally compact space Ω ?

Denoted by $|\cdot|_\Omega$ the uniform norm on Ω . It is a theorem of Kaplansky [19] that, for each (not necessarily complete) algebra norm $\|\cdot\|$ on $C_0(\Omega)$ and each $f \in C_0(\Omega)$, we have $\|f\| \geq |f|_\Omega$. This suggested a conjecture that each algebra norm on $C_0(\Omega)$

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is equivalent to $|\cdot|_{\Omega}$. The latter holds if and only if each homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is continuous; this is known as the problem of Kaplansky.

In the 1970s, this long-standing open problem was resolved in the negative independently by Dales [5] and Esterle [10], [11], [12]. Moreover, they showed that, assuming the Continuum Hypothesis (CH), for each non-compact locally compact space Ω and each non-modular prime ideal P in $\mathcal{C}_0(\Omega)$ with $|\mathcal{C}_0(\Omega)/P| = \mathfrak{c}$, there exists a homomorphism from $\mathcal{C}_0(\Omega)$ into some radical Banach algebra with kernel precisely equal to P . For a full exposition of an improved and stronger form of this theorem, see [6].

Before this problem was resolved, it was proved by Bade and Curtis in [1] that each discontinuous homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B can be decomposed into a sum of a continuous homomorphism and a finite number of discontinuous linear maps, each of which is a homomorphism into the radical of B when restricted to a maximal ideal of $\mathcal{C}_0(\Omega)$ (this result was later improved in [10], [18], and [23]). This important result sheds light on the structure of discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$. We summarize below part of the result that is more relevant to our discussion; here Ω^b denotes the *one-point compactification* of Ω , and the ideals J_p and M_p , for $p \in \Omega^b$, are to be defined in the next section.

Theorem 1.1. *Let Ω be a non-empty locally compact space, and let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B .*

- (i) *The continuity ideal $\mathcal{I}(\theta)$ is the largest ideal of $\mathcal{C}_0(\Omega)$ on which θ is continuous.*
- (ii) *There exists a finite subset $\{p_1, \dots, p_n\}$ of Ω^b such that*

$$\bigcap_{i=1}^n J_{p_i} \subset \mathcal{I}(\theta) \subset \bigcap_{i=1}^n M_{p_i}.$$

- (iii) *There exists a homomorphism μ from $\bigcap_{i=1}^n M_{p_i}$ into $\text{rad } B$ such that $\mathcal{I}(\theta) = \ker \mu$.*
- (iv) *The ideals $\ker \theta$ and $\mathcal{I}(\theta)$ are always intersections of prime ideals.*
- (v) *In the case where B is radical, $\mathcal{I}(\theta) = \ker \theta$.*

From the theorem of Dales and of Esterle, it follows easily that, assuming CH, each ideal which is a finite intersection of non-modular prime ideals in $\mathcal{C}_0(\Omega)$ (with a suitable cardinality condition) is the kernel of a homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra. It was immediately questioned if the converse is true; this is Question A. By Theorem 1.1, (iii) and (v), this is equivalent to Question B. This same theorem, particularly parts (ii) and (iv), also suggests that the answer to the questions should be “yes” for every space Ω . If the answers were “yes”, we would have, in some sense, a complete picture of the structure of homomorphisms from $\mathcal{C}_0(\Omega)$ (at least in the case where $|\mathcal{C}_0(\Omega)| = \mathfrak{c}$).

These questions were first raised in [4], [10]. Question B was partially answered (in the affirmative) in [10] in the cases where Ω is a compact F-space, and where Ω is compact and $\beta(\Omega \setminus \{p\})$ is an F-space for each non-P-point p of Ω (thus implying an affirmative answer for the spaces \mathbb{N} , $\beta\mathbb{N}$, and $\beta\mathbb{N} \setminus \mathbb{N}$). There are further partial answers in [7]: here the authors restrict to ideals of the form J_p . However, the questions were left open for almost all spaces.

To tackle Question A (and B) we first consider a more general version of Question B. We shall reduce the general question of whether or not there exists a homomorphism θ from a given commutative Banach algebra A into another Banach algebra such that the prime radical of $\mathcal{I}(\theta)$ cannot be represented as a finite intersection of prime ideals to an intrinsic question about the structure of the prime ideals of A . We shall then use this reduction to prove that the prime radical of the continuity ideal of an epimorphism from A onto another Banach algebra (or of a derivation from A into a Banach A -bimodule) is always a finite intersection of prime ideals in A (§4).

A little offtrack of our main discussion, and in order to show that it is necessary, in the above reduction, to consider the prime radicals of the continuity ideals, we shall construct homomorphisms of various types between commutative Banach algebras whose continuity ideals are not intersections of prime ideals. In particular, we shall construct a derivation $\mathcal{D} : \mathcal{A} \rightarrow \mathcal{A}'$, where $\mathcal{A} = A(\overline{\mathbb{D}}^2)$, such that the continuity ideal of \mathcal{D} is not an intersection of prime ideals in \mathcal{A} (§5).

In another direction, Esterle [13] generalized the above-mentioned construction to a construction of a homomorphism from an arbitrary commutative algebra A into some radical Banach algebra, with the kernel to be any prescribed non-modular prime ideal P in A such that $|A/P| = \mathfrak{c}$. (See [9, Chapter 5] for a different approach to this result, which will be sketched very briefly at the beginning of §6.) Again, this easily implies the same result for finite intersections of prime ideals. We shall extend the result to intersections of various infinite families of prime ideals, thus enlarging the class of commutative algebras which are known to be normable (§7). The precise statement of this is as follows; we refer the reader to §3 for the definition of ‘pseudo-finiteness’ and of ‘intersection non-redundancy’.

Theorem (CH). *Let A be a commutative algebra, and let I be the intersection of a pseudo-finite family of non-modular prime ideals. Suppose that $|A/I| = \mathfrak{c}$. Then A/I can be embedded into a radical Banach algebra.*

The above extension yields a partial converse of the previously mentioned reduction. In summary, we shall prove the following result (Theorems 4.5 and 7.4).

Theorem. *Let A be a commutative Banach algebra.*

- (i) *Suppose that there exists a homomorphism θ from A into another Banach algebra such that the prime radical of $\mathcal{I}(\theta)$ is not a finite intersection of prime ideals. Then there exists an intersection non-redundant pseudo-finite sequence of non-closed prime ideals in A .*
- (ii) (CH) *Suppose that I is the intersection of an intersection non-redundant pseudo-finite sequence of non-closed prime ideals in A . Suppose further that $|A/I| = \mathfrak{c}$. Then there exists a homomorphism θ from A into another Banach algebra with continuity ideal equal to I – a semiprime ideal, but not a finite intersection of prime ideals in A .*

We shall then apply the above results to Questions A and B. We give the following characterizations (Theorems 8.8 and 9.5; cf. Theorem 8.1).

Theorem (A). *Let Ω be a locally compact metrizable space, and let p denote the point adjoined to Ω to obtain Ω^p .*

- (i) Suppose that $p \notin \partial^{(\infty)}(\Omega^b)$. Then the kernel of each homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra is a finite intersection of non-modular prime ideals.
- (ii) (CH) Suppose that $p \in \partial^{(\infty)}(\Omega^b)$. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra such that the kernel of θ is not a finite intersection of prime ideals.

Theorem (B). Let Ω be a locally compact metrizable space.

- (i) Suppose that Ω^b has finite limit level. Then the continuity ideal $\mathcal{I}(\theta)$ of each homomorphism θ from $\mathcal{C}_0(\Omega)$ into another Banach algebra is a finite intersection of prime ideals.
- (ii) (CH) Suppose that Ω^b has infinite limit level. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra such that the continuity ideal of θ is not a finite intersection of prime ideals.

Thus, for example, assuming CH, there exists a homomorphism θ from $\mathcal{C}_0(\mathbb{R})$ into a radical Banach algebra such that $\ker \theta$ is not a finite intersection of prime ideals. This answers Questions A and B in the negative.

Finally, we shall characterize those locally compact groups (not necessarily metrizable) G such that there exists a homomorphism θ from $\mathcal{C}_0(G)$ into a radical Banach algebra (respectively, a Banach algebra) such that the kernel (respectively, the continuity ideal) of θ is not a finite intersection of prime ideals. We also give an application to prime ideals in Abelian group algebras (§10).

Remark. We note here two other equivalent formulations of Questions A and B.

- C Is the kernel of a homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra always the intersection of a closed ideal and a finite number of prime ideals, for each locally compact space Ω ?

Next, let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra. Then $q(f) = \|\theta(f)\|$ ($f \in \mathcal{C}_0(\Omega)$) is an algebra seminorm on $\mathcal{C}_0(\Omega)$. It is proved in [10] that $\mathcal{I}(\theta)$ is the intersection of the q -closed prime ideals in $\mathcal{C}_0(\Omega)$, and that each chain of such prime ideals are well-ordered (with respect to the inclusion). Questions A, B are also equivalent to the following:

- D Is the set of prime ideals which are closed with respect to a seminorm on $\mathcal{C}_0(\Omega)$ always a finite union of well-ordered chains, for each locally compact space Ω ?

In this paper, we shall work explicitly with Questions A and B only; the corresponding results for C and D can be derived easily.

Remark. In all the constructions mentioned above, the Continuum Hypothesis is indispensable, for it has been proved by Solovay and Woodin that it is relatively consistent with ZFC that all homomorphisms from each algebra $\mathcal{C}_0(\Omega)$ into a Banach algebra are continuous (see [8] for the proof and references).

2. PRELIMINARY DEFINITIONS AND NOTATIONS

In this paper, all the results that require the Continuum Hypothesis will be marked by the abbreviation CH. All our algebras, fields, and integral domains are over the complex field \mathbb{C} .

Let A be a commutative algebra. The (conditional) unitization $A^\#$ of A is defined as the algebra A itself if A is unital, and as A with identity adjoined

otherwise. The identity of $A^\#$ is denoted by either e_A or simply 1. For a subset S of A , the *smallest subalgebra of A containing S* is denoted by $\text{alg } S$.

A prime ideal or semiprime ideal in A must be a proper ideal. However, we consider A itself as a finite intersection of prime ideals (*the intersection of the empty collection of prime ideals*).

Define the *prime radical* \sqrt{I} of an ideal I in A to be the intersection of all prime ideals in A containing I , so that

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

For each ideal I in A and each element $a \in A^\#$, define the *quotient of I by a* to be the ideal

$$I:a = \{b \in A : ab \in I\}.$$

Clearly we have $I \subset I:a$ in each case. Note that, when A is a topological algebra and I is a closed ideal, the ideal $I:a$ is also closed for each $a \in A^\#$.

Let I be an ideal in $A^\#$. A subset S of $A^\#$ is *algebraically independent modulo I* if $p(a_1, \dots, a_n) \notin I$ for each $n \in \mathbb{N}$, each non-zero polynomial $p \in \mathbb{C}[X_1, \dots, X_n]$, and each n -tuple (a_1, \dots, a_n) of distinct elements of S . A *transcendence basis* for $A^\#$ modulo I is a maximal set among all the subsets of $A^\#$ which are algebraically independent modulo I ; such a basis always exists.

Next, we are going to present some standard automatic continuity theory; for details, see [6] or [24].

Let $T : E \rightarrow F$ be a linear operator from a Banach space E into a Banach space F . The *separating space* of T , denoted by $\mathfrak{S}(T)$, is defined as

$$\{v \in F : \text{there exists a sequence } (u_n) \subset E \text{ with } u_n \rightarrow 0 \text{ and } Tu_n \rightarrow v\}.$$

The space $\mathfrak{S}(T)$ is a closed subspace of F , and, by the closed graph theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$. In fact, let $S : F \rightarrow G$ be a bounded linear operator from F into another Banach space G . Then $\mathfrak{S}(ST) = \overline{\mathfrak{S}(T)}$, and ST is continuous if and only if $S(\mathfrak{S}(T)) = \{0\}$; see [6, 5.2.2] or [24, 1.3].

Now let $\theta : A \rightarrow B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B . Then $\mathfrak{S}(\theta)$ is a closed ideal in $\overline{\theta(A)}$. By the previous paragraph, we have

$$\mathcal{I}(\theta) = \{a \in A : \theta(a)\mathfrak{S}(\theta) = \{0\}\}.$$

We have the following famous stability lemma (homomorphism version); see [6, 5.2.7] or [24, 1.6] for the statement and proof of a more general version.

Proposition 2.1 (Stability lemma). *Let θ be a homomorphism from a commutative Banach algebra A into a Banach algebra B . Let $(a_n : n \in \mathbb{N})$ be a sequence in A . Then there is an integer $N \in \mathbb{N}$ such that*

$$\overline{(a_1 \cdots a_n)\mathfrak{S}(\theta)} = \overline{\theta(a_1 \cdots a_m)\mathfrak{S}(\theta)} \quad (m, n \geq N). \quad \square$$

Let A be a Banach algebra, and let M be a Banach A -bimodule. We adopt the convention that the norm on M satisfies

$$\|a \cdot x\| \leq \|a\| \|x\| \quad \text{and} \quad \|x \cdot a\| \leq \|a\| \|x\| \quad (a \in A, x \in M);$$

if this is not the case, we can always replace the given norm on M by another, equivalent, norm such that these inequalities hold.

The *Banach dual module* of M is the Banach dual space of M , denoted by M' , with the left and right A -module actions on M' being the dual actions of those on M , that is,

$$(\lambda \cdot a)(x) = \lambda(a \cdot x) \quad \text{and} \quad (a \cdot \lambda)(x) = \lambda(x \cdot a) \quad (a \in A, \lambda \in M', x \in M).$$

Then M' is indeed a Banach A -bimodule.

Let D be a derivation from a commutative Banach algebra A into a Banach A -bimodule M . The continuity ideal of D is defined to be

$$\mathcal{I}(D) = \{a \in A : a \cdot \mathfrak{S}(D) = \{0\}\}.$$

Again, we have

$$\mathcal{I}(D) = \{a \in A : \text{the map } b \mapsto D(ab), A \rightarrow M, \text{ is continuous}\}.$$

We see that $\mathcal{I}(D)$ is a closed ideal in A . The continuity ideals of derivations are actually “special cases” of those of homomorphisms. Indeed, define a Banach algebra B to be the direct sum $A \oplus M$ with ℓ^1 -norm, so that

$$\|a \oplus x\| = \|a\| + \|x\| \quad (a \in A, x \in M).$$

The product on B is given by

$$(a \oplus x)(b \oplus y) = ab \oplus (a \cdot y + x \cdot b) \quad (a, b \in A, x, y \in M).$$

Define a homomorphism $\theta : A \rightarrow B$ by setting $\theta(a) = a \oplus Da$. Then $\mathcal{I}(\theta) = \mathcal{I}(D)$.

For a discussion of the theory of the algebras of continuous functions, see [6], [9] or [15]. Here, we just give some facts which are needed in our discussion.

Let Ω be a locally compact space; the convention is that locally compact spaces and compact spaces are Hausdorff. The *Stone-Ćech compactification* of Ω is denoted by $\beta\Omega$. Denote by $\mathcal{C}_c(\Omega)$ the algebra of compactly supported continuous functions on Ω . For each $p \in \Omega$, define

$$\begin{aligned} J_p^\Omega &= \{f \in \mathcal{C}_0(\Omega) : f \text{ is zero on a neighbourhood of } p\}, \\ M_p^\Omega &= \{f \in \mathcal{C}_0(\Omega) : f(p) = 0\}. \end{aligned}$$

We shall omit the superscript Ω and write J_p and M_p when there is no ambiguity. For p being the point (at infinity) adjoined to Ω to obtain Ω^b , we also set

$$J_p = \mathcal{C}_c(\Omega) \quad \text{and} \quad M_p = \mathcal{C}_0(\Omega).$$

For each prime ideal P in $\mathcal{C}_0(\Omega)$, there always exists a unique point $p \in \Omega^b$ such that $J_p \subset P \subset M_p$; we say that P is supported at the point p . It can be seen that P is modular if and only if its support point belongs to Ω .

It is an important fact that, for each prime ideal P in $\mathcal{C}_0(\Omega)$, the set of prime ideals in $\mathcal{C}_0(\Omega)$ which contain P is a chain with respect to the inclusion relation. Thus, for each $n \in \mathbb{N}$, a semiprime ideal in $\mathcal{C}_0(\Omega)$ that contains an intersection of n prime ideals is itself an intersection of n prime ideals.

For each function f continuous on Ω , the zero set of f is denoted by $\mathbf{Z}_\Omega(f)$ or $\mathbf{Z}(f)$. The set of zero sets of continuous functions on Ω is denoted by $\mathbf{Z}[\Omega]$. For each closed subset $Z \subset \Omega$, we have $Z = \mathbf{Z}(f)$ for some function $f \in \mathcal{C}_0(\Omega)$ if and only if $\Omega \setminus Z$ is σ -compact.

A z -filter \mathcal{F} on Ω is a non-empty proper subset of $\mathbf{Z}[\Omega]$ satisfying:

- (i) $Z_1 \cap Z_2$ belongs to \mathcal{F} whenever both Z_1 and Z_2 belong to \mathcal{F} ,
- (ii) if $Z_1 \in \mathcal{F}$, $Z_2 \in \mathbf{Z}[\Omega]$ and $Z_1 \subset Z_2$, then Z_2 also belongs to \mathcal{F} .

Each z -filter \mathcal{F} corresponds to an ideal

$$\{f \in \mathcal{C}(\Omega) : \mathbf{Z}(f) \in \mathcal{F}\},$$

denoted by $\mathbf{Z}_\Omega^{-1}[\mathcal{F}]$ or $\mathbf{Z}^{-1}[\mathcal{F}]$; each such ideal is called a z -ideal. A z -filter \mathcal{F} is a *prime z -filter* if $Z_1 \cup Z_2 \notin \mathcal{F}$ whenever $Z_1, Z_2 \in \mathbf{Z}[\Omega] \setminus \mathcal{F}$. A maximal z -filter is called a *z -ultrafilter*. A z -ultrafilter is always prime.

Finally, let Ω be a compact space. Define $\partial\Omega$ to be *the set of all limit points* of Ω , i.e., $\Omega \setminus \partial\Omega$ is the set of isolated points of Ω . Then $\partial\Omega$ is a compact subset of Ω , called the *derived set* of Ω . Since Ω is compact, $\partial\Omega$ is non-empty unless Ω is finite. We define a non-increasing sequence $(\partial^{(n)}\Omega : n \in \mathbb{Z}^+)$ of compact subsets of Ω as follows:

- (i) put $\partial^{(0)}\Omega = \Omega$;
- (ii) for each $n \in \mathbb{Z}^+$, define $\partial^{(n+1)}\Omega = \partial(\partial^{(n)}\Omega)$.

Define $\partial^{(\infty)}\Omega = \bigcap \{\partial^{(n)}\Omega : n \in \mathbb{Z}^+\}$. By the compactness, either $\partial^{(\infty)}\Omega$ is non-empty or $\partial^{(l)}\Omega$ is empty for some $l \in \mathbb{Z}^+$. In the former case, we say that Ω has *infinite limit level*; in the latter, we say that Ω has *finite limit level*.

3. PSEUDO-FINITE FAMILIES OF PRIME IDEALS

Let $(E_\alpha : \alpha \in S)$ be a family of subsets of a given set E indexed by another set S . The following concepts play an important role in the sequel; we shall see (in §4) how these concepts arise naturally in our problem.

Definition 3.1. The family $(E_\alpha : \alpha \in S)$ is *pseudo-finite* if $a \in E_\alpha$ for all but finitely many $\alpha \in S$ whenever $a \in \bigcup_{\alpha \in S} E_\alpha$.

Certainly, each finite family is pseudo-finite.

Definition 3.2. The family $(E_\alpha : \alpha \in S)$ is *intersection-redundant* if there exists a subset T of S such that $T \neq S$ and such that

$$\bigcap_{\alpha \in T} E_\alpha = \bigcap_{\alpha \in S} E_\alpha.$$

Otherwise, we say that the family is *intersection non-redundant*.

Now let A be a commutative algebra. *Note that*, for each infinite pseudo-finite family of prime ideals $(P_\alpha : \alpha \in S)$ in A , the union $\bigcup_{\alpha \in S} P_\alpha$ must be an ideal and hence it is either prime in A or A itself.

First, we shall present some rather interesting properties which show, to an extent, the similarity between pseudo-finite and finite families of prime ideals.

Lemma 3.3. *Let A be a commutative algebra, and let P be a prime ideal in A . Suppose that $(I_\alpha : \alpha \in S)$ is a pseudo-finite family of ideals in A such that $\bigcap_{\alpha \in S} I_\alpha \subset P$. Then $I_{\alpha_0} \subset P$ for some $\alpha_0 \in S$.*

Proof. Assume toward a contradiction that $I_\alpha \not\subset P$ ($\alpha \in S$). Choose $a_0 \in I_{\alpha_0} \setminus P$ for some $\alpha_0 \in S$. Then, by the pseudo-finiteness, we have $a_0 \in I_\alpha$ for all but finitely many $\alpha \in S$. Let $\alpha_1, \dots, \alpha_n$ be the indices $\alpha \in S$ such that $a_0 \notin I_\alpha$. For each $k \in \mathbb{N}$ with $1 \leq k \leq n$, choose $a_k \in I_{\alpha_k} \setminus P$, and set $a = a_0 a_1 \cdots a_n$. Then $a \in I_\alpha$ ($\alpha \in S$), but $a \notin P$, by the primeness of P ; a contradiction to the fact that $\bigcap_{\alpha \in S} I_\alpha \subset P$. \square

Lemma 3.4. *Let $(P_\alpha : \alpha \in S)$ be a pseudo-finite family of prime ideals in a commutative algebra A . Then the following are equivalent:*

- (a) (P_α) is intersection non-redundant;
- (b) $P_\alpha \not\subset P_\beta$ ($\alpha \neq \beta \in S$);
- (c) $\bigcap_{\beta \neq \alpha} P_\beta \not\subset P_\alpha$ for each $\alpha \in S$.

Proof. This follows from the previous lemma. \square

Lemma 3.5. *Let $(P_\alpha : \alpha \in S)$ be a non-empty pseudo-finite family of prime ideals in a commutative algebra A . Let $(P_\alpha : \alpha \in T)$ be the subfamily of distinct minimal elements of $(P_\alpha : \alpha \in S)$. Then:*

- (i) $(P_\alpha : \alpha \in T)$ is intersection non-redundant, and $\bigcap_{\alpha \in S} P_\alpha = \bigcap_{\alpha \in T} P_\alpha$;
- (ii) the ideal $\bigcap_{\alpha \in S} P_\alpha$ cannot be represented as an intersection of strictly fewer than $|T|$ prime ideals.

In particular, T is non-empty.

Proof. (i) Obviously $(P_\alpha : \alpha \in T)$ satisfies Lemma 3.4(b), so this family is intersection non-redundant. Since $(P_\alpha : \alpha \in S)$ is pseudo-finite, each P_α ($\alpha \in S$) must contain a minimal element P_β for some $\beta \in T$. Thus $\bigcap_{\alpha \in S} P_\alpha = \bigcap_{\alpha \in T} P_\alpha$.

(ii) Set $I = \bigcap_{\alpha \in S} P_\alpha = \bigcap_{\alpha \in T} P_\alpha$. By Lemma 3.4, we can choose

$$a_\alpha \in \left(\bigcap_{\beta \neq \alpha, \beta \in T} P_\beta \right) \setminus P_\alpha \quad \text{for each } \alpha \in T.$$

We see that $a_\alpha a_\beta \in I$ for $\alpha \neq \beta \in T$. It follows easily that I cannot be represented as an intersection of strictly fewer than $|T|$ prime ideals. \square

Lemma 3.6. *Let A be a commutative Banach algebra. Let $(P_\alpha : \alpha \in S)$ be an intersection non-redundant pseudo-finite family of prime ideals in A . Then the number of closed prime ideals in the family (P_α) is finite.*

Proof. Assume towards a contradiction that there exist distinct closed prime ideals P_{α_n} ($n \in \mathbb{N}$) in the family $(P_\alpha : \alpha \in S)$. By Lemma 3.4, for each $n \in \mathbb{N}$, we can choose $a_n \in (\bigcap_{i \neq n} P_{\alpha_i}) \setminus P_{\alpha_n}$; we can further suppose that $\|a_n\| \leq 1/2^n$. Set $a = \sum_{n=2}^{\infty} a_n$. Then $a \in P_{\alpha_1}$. But then, by pseudo-finiteness, we have $a \in P_{\alpha_k}$ for some $k \geq 2$. However, we have $a - a_k = \sum \{a_n : n \notin \{1, k\}\} \in P_{\alpha_k}$, so that $a_k \in P_{\alpha_k}$, contradicting the choice of a_k . \square

Example 3.7. Let S be a non-empty index set, and let X_α be an indeterminate for each $\alpha \in S$. For $n \in \mathbb{N}$, for $\alpha_1, \dots, \alpha_n$ in S , and for $k_1, \dots, k_n \in \mathbb{N}$, the formal product $X_{\alpha_1}^{k_1} \cdots X_{\alpha_n}^{k_n}$ is called a *monomial*. A *formal power series* of indeterminates X_α ($\alpha \in S$) is a formal series of the form $f = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n M_n$, where $\lambda_n \in \mathbb{C}$ and M_n is a monomial (for each n). We identify, add, and multiply two formal power series according to the “standard” rules (similar to those for polynomials). The resulting algebra is called the *formal power series algebra over S* , and is denoted by \mathfrak{F}_S . It is easily seen that \mathfrak{F}_S is an integral domain.

For each $\alpha \in S$, let Q_α be the ideal in \mathfrak{F}_S generated by $\{X_\beta : \beta \in S, \beta \neq \alpha\}$. We can check that Q_α is a prime ideal in \mathfrak{F}_S . Thus $(Q_\alpha : \alpha \in S)$ is a pseudo-finite family of prime ideals, and $(Q_\alpha : \alpha \in S)$ is intersection non-redundant.

Example 3.8. Now let \mathcal{A}_S be the subalgebra of \mathfrak{F}_S that consists of elements f which can be represented as a series $\lambda_0 + \sum_{n=1}^{\infty} \lambda_n M_n$ with

$$\|f\|_1 = \sum_{n=0}^{\infty} |\lambda_n| < \infty,$$

where $(\lambda_n) \subset \mathbb{C}$ ($n \in \mathbb{Z}^+$) and where M_n ($n \in \mathbb{N}$) are distinct monomials. Then $(\mathcal{A}_S, \|\cdot\|_1)$ is a commutative Banach algebra. Set $P_\alpha = \mathcal{A}_S \cap Q_\alpha$. Then $(P_\alpha : \alpha \in S)$ is an intersection non-redundant pseudo-finite family of prime ideals in \mathcal{A}_S . \square

We shall see further examples of commutative Banach algebras with or without an intersection non-redundant pseudo-finite sequence of prime ideals in §§8, 9, 10.

Lemma 3.9. *Let A and \mathcal{A} be commutative algebras such that \mathcal{A} contains A as an ideal. Let $(P_\alpha : \alpha \in S)$ be an intersection non-redundant pseudo-finite family of prime ideals in A . Suppose that $\bigcup_{\alpha \in S} P_\alpha \neq A$. Then \mathcal{A} also has an intersection non-redundant pseudo-finite family $(Q_\alpha : \alpha \in S)$ of prime ideals.*

Proof. For each $\alpha \in S$, define

$$Q_\alpha = \{a \in \mathcal{A} : aA \subset P_\alpha\}.$$

It is standard that Q_α is a prime ideal in \mathcal{A} and $Q_\alpha \cap A = P_\alpha$ (see [6, 1.3.53]).

Let $u \in A$ be such that $u \notin P_\alpha$ ($\alpha \in S$). We see that $Q_\alpha = \{a \in \mathcal{A} : au \in P_\alpha\}$. The pseudo-finiteness of $(Q_\alpha : \alpha \in S)$ then follows. It is easily seen that this family is intersection non-redundant. \square

For the converse of the above lemma, we have the following.

Lemma 3.10. *Let A be a commutative algebra, and let I be an ideal in A . Suppose that each intersection non-redundant pseudo-finite family of prime ideals in each of the algebras A/I and I is finite. Then A itself does not contain an intersection non-redundant pseudo-finite sequence of prime ideals.*

Proof. Assume that there exists an intersection non-redundant pseudo-finite sequence (P_n) of prime ideals in A . Letting E be the subset of \mathbb{N} consists of the numbers n for which $I \not\subset P_n$. Then, for each $n \in E$, we have (again, by [6, 1.3.53])

$$P_n = \{a \in A : aI \subset P_n \cap I\}.$$

So we see that $\{P_n \cap I : n \in E\}$ is an intersection non-redundant pseudo-finite family of prime ideals in I . Thus E must be finite.

On the other hand, it is readily seen that $\{P_n/I : n \in \mathbb{N} \setminus E\}$ is an intersection non-redundant pseudo-finite family of prime ideals in A/I . So, again, by the hypothesis, $\mathbb{N} \setminus E$ must be finite, a contradiction. Hence the result holds. \square

Definition 3.11. Let A be a commutative algebra. Let I be an ideal in A , and let (f_n) be a sequence of elements in A . Then I is *extendible* with respect to (f_n) if both the following conditions hold:

- (a) $f_n^k \notin I$ ($n, k \in \mathbb{N}$), and $f_m f_n \in I$ ($m \neq n \in \mathbb{N}$);
- (b) for each $g \in A$, if $g f_{n_0}^{k_0} \in I$ for some $n_0, k_0 \in \mathbb{N}$, then $g f_n \in I$ for all except finitely many $n \in \mathbb{N}$.

Note that, if A has an intersection non-redundant pseudo-finite sequence of prime ideals (P_n) , then, by setting $I = \bigcap_{n=1}^\infty P_n$ and choosing, for each $n \in \mathbb{N}$, an element $f_n \in \bigcap_{i \neq n} P_i \setminus P_n$, the ideal I is extendible with respect to (f_n) . Conversely, we have the following general construction; this construction is crucial in our proof of the main result in §9.

Proposition 3.12. *Let A be a commutative algebra. Suppose that there exist an ideal I and a sequence (f_n) in A such that I is extendible with respect to (f_n) . Then there exists a pseudo-finite sequence of prime ideals (P_n) such that $f_n \in \bigcap_{i \neq n} P_i \setminus P_n$ for each $n \in \mathbb{N}$.*

Proof. We see that the union of a chain of ideals, each of which contains I and is extendible with respect to (f_n) , is also extendible with respect to (f_n) . By Zorn's lemma, we can choose a maximal one among those ideals, and call it J . It is easily seen that the prime radical of J is again extendible with respect to (f_n) . Thus, by the maximality of J , we must have $\sqrt{J} = J$, and so J is semiprime.

For each n , set $J_n = J:f_n$, and set $P = \bigcup_{n=1}^\infty J_n$. By the extensibility of J , we see that whenever $f \in P$ then $f \in J_n$ for all except finitely many $n \in \mathbb{N}$. Thus, in particular, the set P is actually an ideal.

Claim 1: for each $f \in A \setminus P$, we have $J:f = J$. Indeed, since $f \notin P$ and since J is semiprime, we see that $f_n^k \notin J:f$ ($n, k \in \mathbb{N}$). It then follows easily that $J:f$ is extendible with respect to (f_n) . This and the maximality of J imply the claim.

Claim 2: P is either prime in A or is A itself. We have to prove that, whenever $f, g \in A$ are such that $fg \in P$, but $f \notin P$, then $g \in P$. Indeed, let $n_0 \in \mathbb{N}$ be such that $f_{n_0}fg \in J$. Then $f_{n_0}g \in J:f$. The first claim then implies that $f_{n_0}g \in J$, and so $g \in J_{n_0}$.

Now, for each $n \in \mathbb{N}$, define

$$S_n = \{f_n^k : k \in \mathbb{N}\} \cup \{f_n^k f : k \in \mathbb{N} \text{ and } f \in A \setminus P\}.$$

Then, by Claim 2, the set S_n is closed under multiplication. We claim that $S_n \cap J_n = \emptyset$. Indeed, fix $k \in \mathbb{N}$ and $f \in A \setminus P$. Then, since $f_n^{k+1} \notin J$, we see that $f_n^k \notin J_n$, and by Claim 1, we see that $f_n^{k+1}f \notin J$, and so $f_n^k f \notin J_n$.

Since S_n is closed under multiplication and $S_n \cap J_n = \emptyset$, there exists a prime ideal P_n containing J_n such that $S_n \cap P_n = \emptyset$ ($n \in \mathbb{N}$). Then $J_n \subset P_n \subset P$ and $f_n \notin P_n$ ($n \in \mathbb{N}$). The result then follows. (Indeed, it follows from the maximality of J that $P_n = J_n$ for each n .) □

4. GENERAL NECESSARY CONDITION

In this section, we shall consider a homomorphism θ from a commutative Banach algebra A into a Banach algebra B . To simplify the notations, set $\mathfrak{S} = \mathfrak{S}(\theta)$ and $\mathcal{I} = \mathcal{I}(\theta)$.

Let \mathfrak{P} be the collection of all prime ideals in A of the form $\mathcal{I}:a$ for some $a \in A$. Note that $P \supset \mathcal{I}$ for each $P \in \mathfrak{P}$. The following lemma is a modification of the commutative prime kernel theorem due to Sinclair (see [6, Theorem 5.3.15] or [24, Theorem 11.4]); the proof is similar to that of the theorem, and is included here for the sake of completeness.

Lemma 4.1. *Let $a_0 \in A \setminus \sqrt{\mathcal{I}}$. Then there exists a prime ideal $P \in \mathfrak{P}$ such that $a_0 \notin P$. In particular, $\sqrt{\mathcal{I}}$ is the intersection of prime ideals belonging to \mathfrak{P} .*

Proof. By the stability lemma, there exists $N \in \mathbb{N}$ such that

$$(4.1) \quad \overline{\theta(a_0^n)\mathfrak{S}} = \overline{\theta(a_0^N)\mathfrak{S}} \quad (n \geq N).$$

Note that $a_0^N a_0 \notin \mathcal{I}$. We claim that there exists $a_1 \in A$ such that $a_0^N a_1 \notin \mathcal{I}$ and such that, for each $a \in A$, either $\overline{\theta(a_0^N a_1 a)\mathfrak{S}} = \overline{\theta(a_0^N a_1)\mathfrak{S}}$ or $a_0^N a_1 a \in \mathcal{I}$. Indeed, assume the contrary. Then we can choose by induction a sequence (a_n) in A so that, for $n \in \mathbb{N}$, we have

$$\overline{\theta(a_0^N a_1)\mathfrak{S}} \supsetneq \cdots \supsetneq \overline{\theta(a_0^N a_1 \cdots a_{n-1})\mathfrak{S}} \supsetneq \overline{\theta(a_0^N a_1 \cdots a_n)\mathfrak{S}},$$

contradicting the stability lemma. Thus the claim holds.

Define

$$P = \mathcal{I}:a_0^N a_1 = \{a \in A : \theta(a_0^N a_1 a)\mathfrak{S} = \{0\}\}.$$

We have $a_0 \notin P$, for otherwise we would have $a_0^{N+1} a_1 \in \mathcal{I}$, and this implies that $a_0^N a_1 \in \mathcal{I}$ by equation (4.1), a contradiction of the fact that $a_0^N a_1 \notin \mathcal{I}$. We now show that P is a prime ideal. Let $a, b \in A$ with $ab \in P$, but $b \notin P$. Then

$$0 = \overline{\theta(a_0^N a_1 ab)\mathfrak{S}} \supset \theta(a) \overline{\theta(a_0^N a_1 b)\mathfrak{S}} = \theta(a) \overline{\theta(a_0^N a_1)\mathfrak{S}} \supset \theta(a_0^N a_1 a)\mathfrak{S},$$

and so $a \in P$ as required. □

Definition 4.2. A collection \mathcal{G} of subsets of a given set is said to be *FI* if the set $\bigcap \{E : E \in \mathcal{G}\}$ can be represented as an intersection of a finite sub-collection of \mathcal{G} . We say that a collection \mathcal{G} is *NFI* if it has a sub-collection which is not FI.

So being not NFI means that all its sub-collections are FI.

Lemma 4.3. *Suppose that \mathfrak{P} is NFI. Then there exists $\mathcal{G} \subset \mathfrak{P}$ satisfying the following conditions:*

- (i) \mathcal{G} is not FI;
- (ii) for each $a \in A$, the set $\{P \in \mathcal{G} : a \notin P\}$ is either \mathcal{G} itself or is not NFI.

Proof. For each $a \in A \cup \{\mathbf{e}_A\}$, let \mathfrak{P}_a be the set of prime ideals of the form $\mathcal{I}:ab$ for some $b \in A$. We claim that there exists $a_0 \in A \cup \{\mathbf{e}_A\}$ such that \mathfrak{P}_{a_0} is NFI and such that, for each $a \in A$, either $\mathfrak{P}_{a_0 a}$ is not NFI or $\mathcal{I}:a_0 a = \mathcal{I}:a_0$.

Assume the contrary. Then, since $\mathfrak{P}_{\mathbf{e}_A}$ is NFI, by induction, there exists a sequence $(a_n) \subset A$ such that $\mathfrak{P}_{a_1 \cdots a_n}$ is NFI and such that

$$\mathcal{I}:a_1 \cdots a_n \subsetneq \mathcal{I}:a_1 \cdots a_{n+1} \quad (n \in \mathbb{N}).$$

For each $n \in \mathbb{N}$, take $b_n \in \mathcal{I}:a_1 \cdots a_{n+1} \setminus \mathcal{I}:a_1 \cdots a_n$. By the stability lemma, there exists $N \in \mathbb{N}$ such that

$$\overline{\theta(a_1 \cdots a_n)\mathfrak{S}} = \overline{\theta(a_1 \cdots a_{n+1})\mathfrak{S}} \quad (n \geq N).$$

But then

$$\theta(b_n) \overline{\theta(a_1 \cdots a_n)\mathfrak{S}} = \theta(b_n) \overline{\theta(a_1 \cdots a_{n+1})\mathfrak{S}} = 0 \quad (n \geq N),$$

implying that $b_n a_1 \cdots a_n \in \mathcal{I}$ ($n \geq N$). But this contradicts the choice of (b_n) . Hence the claim holds.

Since \mathfrak{P}_{a_0} is NFI, there is a sub-collection \mathcal{G} of \mathfrak{P}_{a_0} which is not FI. Suppose that $a \in A$ and that $\mathcal{G}' = \{P \in \mathcal{G} : a \notin P\}$ is NFI. Then, for each $P \in \mathcal{G}'$, because $a \notin P$ we have $P:a = P$. Thus $\mathcal{G}' \subset \mathfrak{P}_{a_0 a}$, and hence $\mathfrak{P}_{a_0 a}$ is NFI. Therefore, by

the claim, we must have $\mathcal{I}:a_0a = \mathcal{I}:a_0$. We now show that $\mathcal{G}' = \mathcal{G}$. Assume towards a contradiction that $\mathcal{G}' \neq \mathcal{G}$, and let $P \in \mathcal{G} \setminus \mathcal{G}'$, say $P = \mathcal{I}:a_0a_1$ for some $a_1 \in A$. Then, since $a \in P$ we have $a_1 \in \mathcal{I}:a_0a = \mathcal{I}:a_0$, so that $a_0a_1 \in \mathcal{I}$. This implies that $P = A$, a contradiction. This proves that (ii) holds. \square

Lemma 4.4. *Suppose that \mathfrak{P} is NFI. Then there exists a sequence $(P_n) \subset \mathfrak{P}$ which is pseudo-finite and intersection non-redundant.*

Proof. Let \mathcal{G} be the sub-collection of \mathfrak{P} as specified in Lemma 4.3. We claim that, in the partially ordered set (\mathcal{G}, \subset) , every descending chain must be eventually constant.

Assume the contrary. Then there exists a sequence (Q_n) in \mathcal{G} such that

$$Q_n \supsetneq Q_{n+1} \quad (n \in \mathbb{N}).$$

Choose an element $a \in Q_1 \setminus Q_2$. Then $\mathcal{G}' := \{P \in \mathcal{G} : a \notin P\}$ is NFI (since \mathcal{G}' contains $\{Q_n : n > 1\}$, which is not FI), but $\mathcal{G}' \neq \mathcal{G}$ (because $Q_1 \notin \mathcal{G}'$). But this contradicts Lemma 4.3(ii). Thus, in (\mathcal{G}, \subset) , every element must contain a minimal element, and, since \mathcal{G} is not FI, it must have infinitely many minimal elements.

Let $(P_n : n \in \mathbb{N})$ be a sequence of distinct minimal elements of (\mathcal{G}, \subset) . Then obviously we have $P_m \not\subset P_n$ for each $m \neq n$ in \mathbb{N} .

Let $a \in \bigcup_{n=1}^\infty P_n$. By Lemma 4.3(ii), the set $\{P \in \mathcal{G} : a \notin P\}$ is not NFI, and so the set $\{P_n : a \notin P_n\}$ is FI. Let $n_1, \dots, n_k \in \mathbb{N}$ be the indices such that

$$\bigcap_{i=1}^k P_{n_i} = \bigcap \{P_n : a \notin P_n\}.$$

For each $n \in \mathbb{N}$ with $a \notin P_n$, we have $\bigcap_{i=1}^k P_{n_i} \subset P_n$, and, since P_n is a prime ideal, there exists $i \in \{1, \dots, k\}$ such that $P_{n_i} \subset P_n$. Hence $n = n_i$. Thus

$$\{n \in \mathbb{N} : a \notin P_n\} = \{n_1, \dots, n_k\},$$

and so (P_n) is pseudo-finite. The rest then follows from Lemma 3.4. \square

Now we can state the main theorem of this section.

Theorem 4.5. *Let θ be a homomorphism from a commutative Banach algebra A into a Banach algebra B with continuity ideal $\mathcal{I}(\theta) = \mathcal{I}$. Then $\sqrt{\mathcal{I}(\theta)}$ is the intersection of the prime ideals of the form $\mathcal{I}:a$ for some $a \in A$.*

Suppose further that $\sqrt{\mathcal{I}(\theta)}$ is not a finite intersection of prime ideals of the above form. Then there exist non-closed prime ideals $P_n = \mathcal{I}:a_n$, where $a_n \in A$ for each $n \in \mathbb{N}$, such that the sequence $(P_n : n \in \mathbb{N})$ is pseudo-finite and intersection non-redundant, i.e., such that the following conditions hold:

- (a) *if $a \in \bigcup_{i=1}^\infty P_i$, then $a \in P_n$ for all but finitely many $n \in \mathbb{N}$;*
- (b) *$\bigcap_{i \neq n} P_i \not\subset P_n$ ($n \in \mathbb{N}$).*

Proof. This follows in turn from Lemmas 4.1, 4.4, 3.6, and 3.4. \square

Lemma 4.6. *Let I be a closed ideal in a commutative Banach algebra A such that \sqrt{I} is also closed. Then there exists $k \in \mathbb{N}$ such that*

$$\sqrt{I} = \{a \in A : a^k \in I\}.$$

Proof. This is an application of Baire's category theorem. \square

Corollary 4.7. *Let θ be an epimorphism from a commutative Banach algebra A onto a Banach algebra B . Then:*

- (i) $\sqrt{\mathcal{I}(\theta)}$ is a finite intersection of prime ideals;
- (ii) there is a number $k \in \mathbb{N}$ such that $\sqrt{\mathcal{I}(\theta)} = \{a \in A : a^k \in \mathcal{I}(\theta)\}$.

Proof. (i) Assume towards a contradiction that (i) is false. Let $P_n = \mathcal{I}:a_n$ ($n \in \mathbb{N}$) be the prime ideals that were specified in Theorem 4.5. For $n \in \mathbb{N}$, set

$$Q_n = \{b \in B : b\theta(a_n)\mathfrak{S} = \{0\}\}.$$

Then Q_n is closed in B . We see that $P_n = \theta^{-1}(Q_n)$ and that $Q_n = \theta(P_n)$, and so Q_n is a prime ideal in B . It is easily seen that $(Q_n : n \in \mathbb{N})$ is pseudo-finite and intersection non-redundant, contradicting Lemma 3.6. Thus (i) holds.

(ii) Define $\mathcal{J} = \{b \in B : b\mathfrak{S} = \{0\}\}$. Then \mathcal{J} is a closed ideal in B . We see that $\mathcal{I} = \theta^{-1}(\mathcal{J})$ and $\mathcal{J} = \theta(\mathcal{I})$. This and Lemma 4.1 imply that $\sqrt{\mathcal{J}}$ is the intersection of the ideals of the form $\mathcal{J}:b$ for $b \in B$. So $\sqrt{\mathcal{J}}$ is closed in B , since each $\mathcal{J}:b$ is closed. We can then apply Lemma 4.6. \square

Remark. In the case where B is also a semiprime algebra, we see that $\mathcal{I}(\theta)$ is either A or a semiprime ideal, and so $\mathcal{I}(\theta) = \sqrt{\mathcal{I}(\theta)}$. Hence $\mathcal{I}(\theta)$ is a finite intersection of prime ideals. Note, however, that this does not help in answering the (still open) question of whether or not there exists a discontinuous epimorphism from a Banach algebra onto a commutative semiprime Banach algebra; see [2] for some partial results on this question.

We turn to another application. Let $D : A \rightarrow M$ be a derivation from a commutative Banach algebra A into a Banach A -bimodule M . We know that the continuity ideal $\mathcal{I}(D)$ of D is the continuity ideal of a homomorphism from A into some Banach algebra and that $\mathcal{I}(D)$ is closed (see §2). So, for each $a \in A$, the ideal $\mathcal{I}(D):a$ is closed. Thus, we have the following corollary.

Corollary 4.8. *Let D be a derivation from a commutative Banach algebra A into a Banach A -bimodule M . Then:*

- (i) $\sqrt{\mathcal{I}(D)}$ is a finite intersection of closed prime ideals;
- (ii) there is a number $k \in \mathbb{N}$ such that $\sqrt{\mathcal{I}(D)} = \{a \in A : a^k \in \mathcal{I}(D)\}$.

5. CONTINUITY IDEALS WHICH ARE NOT INTERSECTIONS OF PRIME IDEALS

From the previous section, we may ask whether the continuity ideal of a homomorphism is necessarily either a semiprime ideal or the whole algebra itself (see also Theorem 1.1(iv)). In this section, counter-examples to this possibility will be presented. It will be shown that although the continuity ideal of an epimorphism or a derivation is close to being an intersection of prime ideals, as shown by Corollaries 4.7(ii) and 4.8(ii), this is not always the case. The epimorphism we shall construct is in fact an automorphism. By the remark after Corollary 4.7, the domain of this automorphism cannot be a semiprime algebra. However, it is possible to construct a desired derivation from a semisimple algebra. It can even be arranged so that the derivation maps into the dual of the algebra.

Denote by \mathbb{D} the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Define $A(\overline{\mathbb{D}}^n)$ to be the Banach algebra of complex functions of n complex variables continuous on $\overline{\mathbb{D}}^n$ and holomorphic in \mathbb{D}^n , with the uniform norm; we are interested in the cases where $n = 1, 2$.

Theorem 5.1. *Let $\mathcal{A} = A(\overline{\mathbb{D}}^2)$. Then there exists a derivation \mathcal{D} from \mathcal{A} into its dual \mathcal{A}' such that $\mathcal{I}(\mathcal{D})$ is not an intersection of prime ideals in \mathcal{A} .*

Proof. We proceed in two steps.

Step 1: Let A be the disk algebra $A(\overline{\mathbb{D}})$. Denote by $\mathbf{1}$ the function $\mathbf{1}(z) \equiv 1$. By [6, 5.6.81, 5.2.17], there exists a discontinuous derivation $D : A \rightarrow A'$. Then we must have $\mathbf{1} \notin \mathcal{I}(D)$.

Next, we consider $A_1 = A \oplus A$ with ℓ^1 -norm, with pointwise addition, and with product defined by

$$(f_1 \oplus f_2)(g_1 \oplus g_2) = f_1g_1 \oplus (f_1g_2 + f_2g_1) \quad (f_1, f_2, g_1, g_2 \in A).$$

Then A_1 is a commutative Banach algebra. Then, we define left and right actions of A_1 on $A' \oplus A'$ as follows:

$$(\lambda_1 \oplus \lambda_2) \cdot (f_1 \oplus f_2) = (f_1 \oplus f_2) \cdot (\lambda_1 \oplus \lambda_2) = (f_1 \cdot \lambda_1 + f_2 \cdot \lambda_2) \oplus (f_1 \cdot \lambda_2),$$

for each $f_1 \oplus f_2 \in A_1$ and each $\lambda_1 \oplus \lambda_2 \in A' \oplus A'$. We can check that $A' \oplus A'$ with pointwise addition, with the above A_1 actions, and with maximum norm is a Banach A_1 -bimodule. In fact, as Banach A_1 -bimodules, $A' \oplus A'$ is isometrically isomorphic to $(A \oplus A)'$ with the following pairing:

$$\langle \lambda_1 \oplus \lambda_2, f_1 \oplus f_2 \rangle = \lambda_1(f_1) + \lambda_2(f_2) \quad (\lambda_1, \lambda_2 \in A', f_1, f_2 \in A).$$

Define a linear map $D_1 : A_1 \rightarrow A' \oplus A'$ by

$$D_1(f_1 \oplus f_2) = Df_2 \oplus Df_1 \quad (f_1, f_2 \in A).$$

Then it is easy to verify that D_1 is a derivation. We now compute $\mathcal{I}(D_1)$. By definition, $f_1 \oplus f_2 \in \mathcal{I}(D_1)$ if and only if the map

$$g_1 \oplus g_2 \mapsto (f_1 \cdot Dg_2 + f_2 \cdot Dg_1) \oplus (f_1 \cdot Dg_1), \quad A_1 \rightarrow A' \oplus A'$$

is continuous; that is, if and only if $f_1 \oplus f_2 \in \mathcal{I}(D) \oplus \mathcal{I}(D)$. In particular, we have $\{0\} \oplus A \notin \mathcal{I}(D_1)$, since $\mathbf{1} \notin \mathcal{I}(D)$. But $(\{0\} \oplus A)^2 = \{0\}$, and so $\mathcal{I}(D_1)$ is not a semiprime ideal.

Step 2: We now consider D_1 as a derivation from A_1 into A'_1 . For each function $f(Z_1, Z_2) \in \mathcal{A} = A(\overline{\mathbb{D}}^2)$, denote by $\partial_2 f$ the partial derivative of f with respect to the second variable Z_2 . Define a map

$$\Phi : f(Z_1, Z_2) \mapsto f(Z_1, 0) \oplus (\partial_2 f)(Z_1, 0), \quad \mathcal{A} \rightarrow A'_1.$$

Then Φ is an algebra epimorphism from \mathcal{A} onto A'_1 , and, by Cauchy's estimation theorem, Φ is bounded. Consider A_1 as a Banach \mathcal{A} -bimodule with module multiplications defined, for each $f \in \mathcal{A}$ and each $(g_1, g_2) \in A_1$, as

$$f \cdot (g_1 \oplus g_2) = (g_1 \oplus g_2) \cdot f = \Phi(f) \cdot (g_1 \oplus g_2) = (g_1 \oplus g_2) \cdot \Phi(f).$$

Then Φ is also an \mathcal{A} -bimodule epimorphism, and so the adjoint map $\Phi' : A'_1 \rightarrow \mathcal{A}'$ is a Banach \mathcal{A} -bimodule embedding (onto a closed submodule of \mathcal{A}').

Finally, define a map $\mathcal{D} : \mathcal{A} \rightarrow \mathcal{A}'$ by setting $\mathcal{D} = \Phi' \circ D_1 \circ \Phi$. Then \mathcal{D} is a derivation from \mathcal{A} into \mathcal{A}' . We can check that $\mathcal{I}(\mathcal{D}) = \Phi^{-1}[\mathcal{I}(D_1)]$, which is, by Step 1, neither a semiprime ideal in \mathcal{A} nor \mathcal{A} itself.

This complete the proof. □

Thus, there is a homomorphism from the semisimple commutative Banach algebra $A(\overline{\mathbb{D}}^2)$ into another Banach algebra whose continuity ideal \mathcal{I} is not equal to its prime radical $\sqrt{\mathcal{I}}$. We can even find an automorphism with this property;

however, the cost of this (using our construction) is that the underlying algebra is “unnatural”.

Corollary 5.2. *There exists an automorphism θ on a commutative Banach algebra such that $\mathcal{I}(\theta)$ is not an intersection of prime ideals.*

Proof. Let $D : A \rightarrow M$ be any derivation from a commutative Banach algebra A into a commutative Banach A -bimodule M such that $\mathcal{I}(D)$ is a proper ideal but not a semiprime ideal in A . Consider $A \oplus M$ as a commutative Banach algebra as before. Define a map $\theta : A \oplus M \rightarrow A \oplus M$ by

$$\theta(a \oplus x) = a \oplus (Da + x) \quad (a \in A, x \in M).$$

It is easily checked that θ is an automorphism of $A \oplus M$. It follows routinely that $\mathcal{I}(\theta) = \mathcal{I}(D) \oplus M$, which is neither a semiprime ideal in $A \oplus M$ nor $A \oplus M$ itself. \square

6. MAPPINGS INTO ULTRAPOWERS

First, we need some definitions. Let κ be an infinite cardinal. Then \mathbb{C}^κ with pointwise algebraic operations is an algebra. Let \mathcal{U} be a free ultrafilter on κ . Define $M_{\mathcal{U}}$ to be the set of all elements $f \in \mathbb{C}^\kappa$ such that $\{\sigma \in \kappa : f(\sigma) = 0\} \in \mathcal{U}$. Then $M_{\mathcal{U}}$ is a maximal ideal. The field $\mathbb{C}^\kappa/M_{\mathcal{U}}$ is called an *ultrapower*, and is denoted by $\mathbb{C}^\kappa/\mathcal{U}$. An element $f + M_{\mathcal{U}} \in \mathbb{C}^\kappa/\mathcal{U}$ is an *infinitesimal* if

$$\{\sigma \in \kappa : |f(\sigma)| < \varepsilon\} \in \mathcal{U}$$

for each $\varepsilon > 0$. The subalgebra of infinitesimals of $\mathbb{C}^\kappa/\mathcal{U}$ is denoted by $(\mathbb{C}^\kappa/\mathcal{U})^\circ$.

Now let \mathcal{V} be a free ultrafilter on \mathbb{N} . Recall that c_0 is the algebra of the sequences of complex numbers converging to 0 (with pointwise operations). Let $P_{\mathcal{V}}$ be the set of all sequences $(x_n) \in c_0$ such that $\{n \in \mathbb{N} : x_n = 0\} \in \mathcal{V}$. Then $P_{\mathcal{V}}$ is a prime ideal in c_0 , and $c_0/P_{\mathcal{V}}$ is an integral domain which can be naturally identified with a subalgebra of $(\mathbb{C}^{\mathbb{N}}/\mathcal{V})^\circ$. We denote $c_0/P_{\mathcal{V}}$ by c_0/\mathcal{V} . Thus each c_0/\mathcal{V} is indeed a quotient of $\mathcal{C}_0(\mathbb{N})$ by one of its minimal prime ideals.

In their approach to the construction in [13] of an embedding of an arbitrary non-unital integral domain of cardinality \mathfrak{c} into a radical Banach algebra, the authors of [9] used a special case that c_0/\mathcal{V} can be embedded into some radical Banach algebra for any ultrafilter \mathcal{V} on \mathbb{N} (this “classical” case had been constructed by Dales and Esterle in their solution to the conjecture of Kaplansky) – this part requires CH, together with the following theorem (which is [9, Theorem 5.25], and does not require the Continuum Hypothesis).

Theorem 6.1. *Let A be a non-unital integral domain of cardinality \mathfrak{c} . Then there exists an embedding of A into c_0/\mathcal{V} for some free ultrafilter \mathcal{V} on \mathbb{N} . \square*

Our aim in this section is to extend the above result; the construction of embeddings into radical Banach algebras will be extended in the next section. However, it should be noted that much difficulty in both of these extensions is actually hidden behind the results that they extend.

We shall need some basic complex algebraic-geometry results to prove the main theorem of this section. Our references for algebraic geometry will be [20] and [25]. Some conventions: all our varieties will be algebraic and the only topology considered on complex spaces is the Euclidean topology. For a set $S \subset \mathbb{C}[Z_1, Z_2, \dots, Z_n]$, denote by $\mathcal{V}(S)$ the *variety* (i.e., common zero set) of S in \mathbb{C}^n . For each prime ideal

Q in $\mathbb{C}[Z_1, \dots, Z_n]$, the variety $\mathcal{V}(Q)$ is *irreducible*. We shall many times use the following proposition.

Proposition 6.2. *Let V be an irreducible variety of \mathbb{C}^n , and let W be a proper subvariety of V . Then $V \setminus W$ is (relatively) open and dense in V .*

Proof. Clearly, $V \setminus W$ is relatively open in V . The density follows from Theorem 2.11 in [20, Chapter IV]. \square

Notation. For clarity, we shall use X_i, Y_j for variables, x_i, y_j for complex numbers, and a_i, b_j for elements of an algebra. When there is no ambiguity, we shall use boldface characters to denote tuples of elements of the same type; for example, we set

$$\mathbf{X} = (X_1, X_2, \dots, X_m) \quad \text{or} \quad \mathbf{y} = (y_1, \dots, y_n).$$

In the case where $\mathbf{X} = (X_1, \dots, X_m)$, we also denote by $\mathbb{C}_{\mathbf{X}}$ the corresponding space \mathbb{C}^m .

Lemma 6.3. *Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, and let Q be a prime ideal in $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$, where $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$. Suppose that X_1, \dots, X_m are algebraically independent modulo Q . Let V be the variety of Q . Let π be the natural projection $\mathbb{C}_{\mathbf{X}, \mathbf{Y}} \rightarrow \mathbb{C}_{\mathbf{X}}$. Then there exists a dense open subset U of V such that $\pi : U \rightarrow \mathbb{C}_{\mathbf{X}}$ is an open map.*

Proof. Let $\{Y_{j_1}, Y_{j_2}, \dots, Y_{j_k}\}$ be a maximal subset of $\{Y_1, \dots, Y_n\}$ such that

$$\{X_1, \dots, X_m, Y_{j_1}, \dots, Y_{j_k}\}$$

is algebraically independent modulo Q . By renaming Y_{j_t} as X_{m+t} ($1 \leq t \leq k$) and the remaining variables Y_j as Y_1, \dots, Y_{n-k} , we may suppose that the set $\{X_1, \dots, X_m\}$ is indeed a transcendence basis for $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$ modulo Q . If $n = 0$, then $Q = \{0\}$, and so $U = V = \mathbb{C}_{\mathbf{X}}$ is the desired set. From now on, we shall suppose that $n \geq 1$. For each $1 \leq j \leq n$, we choose a polynomial

$$p_j = q_{j,0}Y_j^{k_j} + q_{j,1}Y_j^{k_j-1} + \dots + q_{j,k_j} \in Q$$

with all $q_{j,t} \in \mathbb{C}[\mathbf{X}, Y_1, \dots, Y_{j-1}]$ and $q_{j,0} \notin Q$, and such that p_j is a minimal polynomial of this kind (i.e., with smallest degree in Y_j).

Now, for $1 \leq j \leq n$, let $p \in Q \cap \mathbb{C}[\mathbf{X}, Y_1, \dots, Y_j]$ be arbitrary. Using the division algorithm in $\mathbb{C}[\mathbf{X}, Y_1, \dots, Y_{j-1}][Y_j]$ and the minimality of p_j , we see that, for some $l \in \mathbb{N}$, when dividing $(q_{j,0})^l p$ by p_j the remainder is a polynomial in Y_j with coefficients in $Q \cap \mathbb{C}[\mathbf{X}, Y_1, \dots, Y_{j-1}]$. Applying this inductively and using the fact that $Q \cap \mathbb{C}[\mathbf{X}] = \{0\}$, we obtain, for each $p \in Q$, after multiplying p by some power of $q_{1,0} \cdots q_{n,0}$, a polynomial in the ideal generated by p_1, \dots, p_n . So, if (\mathbf{x}, \mathbf{y}) is not in $\mathcal{V}(q_{1,0} \cdots q_{n,0})$, then it is in $\mathcal{V}(p_1, \dots, p_n)$ if and only if it is in V .

Also, by the minimality of each p_j , we have $\partial p_j / \partial Y_j$ is not in Q ($1 \leq j \leq n$). So

$$\mathcal{V} \left(q_{1,0} \cdots q_{n,0} \frac{\partial p_1}{\partial Y_1} \cdots \frac{\partial p_n}{\partial Y_n} \right)$$

is a variety which does not contain V . Thus, by setting

$$U = V \setminus \mathcal{V} \left(q_{1,0} \cdots q_{n,0} \frac{\partial p_1}{\partial Y_1} \cdots \frac{\partial p_n}{\partial Y_n} \right),$$

we see, by Proposition 6.2, that U is a dense open subset of V . Fix a point $(\mathbf{x}_0, \mathbf{y}_0) \in U$. Then the right $n \times n$ submatrix of the Jacobian

$$\left(\begin{array}{ccc|ccc} \frac{\partial p_1}{\partial X_1} & \cdots & \frac{\partial p_1}{\partial X_m} & \frac{\partial p_1}{\partial Y_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_n}{\partial X_1} & \cdots & \frac{\partial p_n}{\partial X_m} & \cdots & \cdots & \frac{\partial p_n}{\partial Y_n} \end{array} \right)$$

of the map $(p_1, \dots, p_n) : \mathbb{C}_{\mathbf{X}, \mathbf{Y}} \rightarrow \mathbb{C}^n$ at $(\mathbf{x}_0, \mathbf{y}_0)$ is lower triangular, with non-zero diagonal entries. So this submatrix is invertible in an entire neighbourhood Δ (in $\mathbb{C}_{\mathbf{X}, \mathbf{Y}}$) of $(\mathbf{x}_0, \mathbf{y}_0)$. The result then follows from the implicit mapping theorem. \square

Lemma 6.4. *Let A be a non-unital commutative algebra. Let P and P_r be ideals in A which are prime in $A^\#$ and such that $P \supset P_r$ ($r \in \mathbb{N}$). Let $m, n \in \mathbb{Z}^+$ with $m + n \geq 1$, and let $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be finite sequences in A such that $\{a_1, \dots, a_m\}$ is algebraically independent modulo P . Let $\delta > 0$. Then there exist finite sequences of complex numbers*

$$(\mathbf{x}, \mathbf{y}^{(r)}) = (x_1, \dots, x_m, y_1^{(r)}, \dots, y_n^{(r)}) \quad (r \in \mathbb{N})$$

satisfying all the following conditions:

- (i) $p(\mathbf{x}, \mathbf{y}^{(r)}) = 0$ for each $r \in \mathbb{N}$ and each $p \in \mathbb{C}[\mathbf{X}, \mathbf{Y}]$ with $p(\mathbf{a}, \mathbf{b}) \in P_r$;
- (ii) $x_i \neq 0$ ($1 \leq i \leq m$);
- (iii) for each $r \in \mathbb{N}$ and each $1 \leq j \leq n$, we have $y_j^{(r)} \neq 0$ whenever $b_j \notin P_r$;
- (iv) $|x_i| \leq \delta$ and $|y_j^{(r)}| \leq \delta$ ($1 \leq i \leq m, 1 \leq j \leq n, r \in \mathbb{N}$).

Proof. Without loss of generality, we can assume that a_1, \dots, a_m are distinct. Define $Q = \{p \in \mathbb{C}[\mathbf{X}, \mathbf{Y}] : p(\mathbf{a}, \mathbf{b}) \in P\}$, and, for each $r \in \mathbb{N}$, define

$$Q_r = \{p \in \mathbb{C}[\mathbf{X}, \mathbf{Y}] : p(\mathbf{a}, \mathbf{b}) \in P_r\}.$$

Then $Q_r \subset Q$ are prime ideals in $\mathbb{C}[\mathbf{X}, \mathbf{Y}]$. We also see that $\{X_1, \dots, X_m\}$ is algebraically independent modulo Q , and hence modulo Q_r for each r . Let V and V_r be the varieties corresponding to Q and Q_r , respectively. Then $V \subset V_r$. If $m = 0$, then we set $U = V$ and $U_r = V_r$ ($r \in \mathbb{N}$); otherwise, let U (respectively, U_r) be the dense open subset of V (respectively, V_r) specified in Lemma 6.3. For each $r \in \mathbb{N}$, set

$$W_r = \bigcup_{i=1}^m \{(\mathbf{x}, \mathbf{y}) : x_i = 0\} \cup \bigcup_{1 \leq j \leq n, b_j \notin P_r} \{(\mathbf{x}, \mathbf{y}) : y_j = 0\}.$$

Then W_r is a variety which does not contain V_r , and so $V_r \setminus W_r$ is also a dense open subset of V_r because V_r is irreducible. Therefore, $U_r \setminus W_r$ is again a dense open subset of V_r . Set

$$\Delta = \{(\mathbf{x}, \mathbf{y}) : |x_i| < \delta \ (1 \leq i \leq m) \text{ and } |y_j| < \delta \ (1 \leq j \leq n)\}.$$

Finally, set $U'_r = (U_r \setminus W_r) \cap \Delta$ and $U' = U \cap \Delta$. Note that the origin $\mathbf{0}$ is in V and V_r ($r \in \mathbb{N}$). So U' (respectively, U'_r) is a non-empty open subset of V (respectively, V_r).

Case 1: $m = 0$. We have $\mathbf{x} = \emptyset$.

Case 2: $m \geq 1$. We have $\pi(U'_r)$ and $\pi(U'_r)$ ($r \in \mathbb{N}$) are open in $\mathbb{C}_{\mathbf{X}}$ by Lemma 6.3, where $\pi : \mathbb{C}_{\mathbf{X}, \mathbf{Y}} \rightarrow \mathbb{C}_{\mathbf{X}}$ is the natural projection. Now, since $U_r \setminus W_r$ is dense in V_r and $U' \subset V \subset V_r$, we have $\pi(U'_r) \cap \pi(U')$ is dense (and open) in $\pi(U')$. By the Baire category theorem, there exists $\mathbf{x} \in \bigcap_{r=1}^\infty \pi(U'_r)$.

In both cases, for each r , choose $(\mathbf{x}, \mathbf{y}^{(r)}) \in U'_r$. These are the sequences we need. □

We can now prove the following extension of [9, Theorem 24].

Theorem 6.5. *Let A be a commutative algebra. Let $(P_\alpha : \alpha \in S)$ be a pseudo-finite family of non-modular prime ideals in A indexed by a non-empty set S . Then there exist a cardinal κ , a free ultrafilter \mathcal{U} on κ , and, for each $\alpha \in S$, a homomorphism $\theta_\alpha : A \rightarrow (\mathbb{C}^\kappa/\mathcal{U})^\circ$ such that:*

- (a) $\ker \theta_\alpha = P_\alpha$ ($\alpha \in S$), and
- (b) the set $\{\theta_\alpha(a) : \alpha \in S\}$ is finite for each $a \in A$.

Proof. Note that A must be non-unital. Since each P_α is a non-modular prime ideal in A , it is a prime ideal in $A^\#$. If S is finite, set $P = A$; otherwise, set $P = \bigcup_{\alpha \in S} P_\alpha$. In both cases, we have P is a prime ideal in $A^\#$.

Let $\Gamma \subset A$ be a transcendence base for $A^\#$ modulo P . (Note that $\Gamma = \emptyset$ if $P = A$.) Then we have $\text{alg } \Gamma \cap P = \{0\}$. Set $A_0 = \text{alg } \Gamma + P$. Then A_0 is a subalgebra of A . We see that each $a \in A$ satisfies a polynomial of the form $a_0 X^m + a_1 X^{m-1} + \dots + a_m$, with $a_0, \dots, a_{m-1} \in \text{alg } \Gamma + \mathbb{C}$, $a_0 \neq 0$, and $a_m \in A_0$.

Let κ be the set of all tuples of the form $(\delta; F; a_1, \dots, a_m)$, where $\delta > 0$, F is a finite subset of S , and (a_1, \dots, a_m) is a non-empty finite sequence in A . Define a partial order \prec on κ by setting

$$(\delta; F; a_1, a_2, \dots, a_m) \prec (\delta'; F'; a'_1, a'_2, \dots, a'_{m'})$$

if $\delta > \delta'$, $F \subset F'$, and (a_1, \dots, a_m) is a subsequence of $(a'_1, a'_2, \dots, a'_{m'})$. Then (κ, \prec) is a net. Fix an ultrafilter \mathcal{U} on κ majorizing this net.

By Lemma 6.4, for each $w = (\delta; F; a_1, a_2, \dots, a_m) \in \kappa$, we can find sequences $\tau_\alpha(w) = (x_1^{(\alpha)}, x_2^{(\alpha)}, \dots, x_m^{(\alpha)}) \in \mathbb{C}^m$ depending on w (for $\alpha \in F$) such that:

- (i) for each $\alpha \in F$ and each $p \in \mathbb{C}[X_1, \dots, X_m]$ with $p(a_1, \dots, a_m) \in P_\alpha$, we have $p(x_1^{(\alpha)}, \dots, x_m^{(\alpha)}) = 0$;
- (ii) for each $\alpha \in F$ and each $1 \leq k \leq m$ with $a_k \notin P_\alpha$, we have $x_k^{(\alpha)} \neq 0$;
- (iii) $|x_k^{(\alpha)}| \leq \delta$ ($1 \leq k \leq m, \alpha \in F$);
- (iv) for each $1 \leq k \leq m$ with $a_k \in \Gamma$, we have $x_k^{(\alpha)} = x_k^{(\beta)}$ ($\alpha, \beta \in F$).

We then set $\tau_\alpha(w) = \mathbf{0} \in \mathbb{C}^m$ for each $\alpha \in S \setminus F$.

Define $\psi_\alpha : A \rightarrow \mathbb{C}^\kappa$ as follows: for each $a \in A$ and each

$$w = (\delta; F; a_1, \dots, a_m) \in \kappa$$

such that $\alpha \in F$ and such that a is in w , say $a = a_k$, then $\psi_\alpha(a)(w) = \tau_\alpha(w)(k)$; otherwise, $\psi_\alpha(a)(w) = 0$. Note that ψ_α is well-defined, for, if $a_j = a_k$ and $\alpha \in F$, then $a_j - a_k = 0 \in P_\alpha$, so that $\tau_\alpha(w)(j) = \tau_\alpha(w)(k)$ by (i). Define $\theta_\alpha(a)$ to be the equivalence class in $\mathbb{C}^\kappa/\mathcal{U}$ containing $\psi_\alpha(a)$.

Let $\alpha \in S$. By (i), the map θ_α is an algebra homomorphism and $P_\alpha \subset \ker \theta_\alpha$; by combining this with (ii), we see that $\ker \theta_\alpha$ is exactly P_α . By (iii), the image of θ_α is contained in $(\mathbb{C}^\kappa/\mathcal{U})^\circ$. The requirement (b) is trivially satisfied if S is finite. So, for the rest of the proof, we shall assume that S is infinite. By (iv), for each $a \in \text{alg } \Gamma$, the family $(\theta_\alpha(a) : \alpha \in S)$ is constant. Combining this with the pseudo-finiteness of (P_α) , we see that, for each $a \in A_0 = \text{alg } \Gamma + \bigcup P_\alpha$, the set $\{\theta_\alpha(a) : \alpha \in S\}$ is finite. The next lemma will complete the proof. □

Lemma 6.6. *Let $A, (P_\alpha : \alpha \in S), \Gamma,$ and A_0 be as in the previous theorem. Let D be a non-unital integral domain. Suppose that we have homomorphisms $\theta_\alpha : A \rightarrow D$ such that $\ker \theta_\alpha \cap \text{alg } \Gamma = \{0\}$ ($\alpha \in S$), and such that, for each $a \in A_0$, the set $\{\theta_\alpha(a) : \alpha \in S\}$ is finite. Then, for each $a \in A$, the set $\{\theta_\alpha(a) : \alpha \in S\}$ is finite.*

Proof. Obviously, $D^\#$ is also an integral domain. Let $a \in A$. Choose elements $a_0, \dots, a_{m-1} \in \text{alg } \Gamma, x_0, \dots, x_{m-1} \in \mathbb{C},$ and $a_m \in A_0$ with $x_0 + a_0 \neq 0$ such that

$$(x_0 + a_0)a^m + (x_1 + a_1)a^{m-1} + \dots + (x_{m-1} + a_{m-1})a + a_m = 0.$$

Then $\theta_\alpha(a)$ is a root of the following non-zero polynomial:

$$(x_0 + \theta_\alpha(a_0))X^m + (x_1 + \theta_\alpha(a_1))X^{m-1} + \dots + (x_{m-1} + \theta_\alpha(a_{m-1}))X + \theta_\alpha(a_m).$$

When we vary $\alpha \in S$, by the assumption, we will only obtain a finite number of such polynomials. Each such polynomial has at most m roots in $D^\#$. So the set $\{\theta_\alpha(a) : \alpha \in S\}$ is finite. \square

Remark. Let λ be any infinite cardinal. Theorem 6.5 can be generalized by replacing pseudo-finiteness of (P_α) with the condition that “for each $a \in \bigcup_{\alpha \in S} P_\alpha$, the cardinality of $\{\alpha \in S : a \notin P_\alpha\}$ is strictly less than λ ”, and replacing (b) with the requirement that “the cardinality of $\{\theta_\alpha(a) : \alpha \in S\}$ is strictly less than λ for each $a \in A$ ”. The above proof carries over with obvious modifications.

Theorem 6.7. *Let A be a commutative algebra, and let $(P_\alpha : \alpha \in S)$ be a pseudo-finite family of non-modular prime ideals in A indexed by a non-empty set S . Suppose that*

$$\left| A / \bigcap_{\alpha \in S} P_\alpha \right| = \mathfrak{c}.$$

Then there exist a free ultrafilter \mathcal{V} on \mathbb{N} and homomorphisms $\varphi_\alpha : A \rightarrow c_0/\mathcal{V}$ for each $\alpha \in S$ such that:

- (a) $\ker \varphi_\alpha = P_\alpha$ ($\alpha \in S$), and
- (b) the set $\{\varphi_\alpha(a) : \alpha \in S\}$ is finite for each $a \in A$.

Proof. First we find homomorphisms θ_α from A into $(\mathbb{C}^\kappa/\mathcal{U})^\circ$ for some ultrapower $\mathbb{C}^\kappa/\mathcal{U}$, as in Theorem 6.5. Let B be the subalgebra of $(\mathbb{C}^\kappa/\mathcal{U})^\circ$ generated by all the images of θ_α ($\alpha \in S$). Then B is a non-unital integral domain. We also have

$$|B| = \left| \bigcup_{\alpha \in S} \theta_\alpha(A) \right| = \left| \bigcup_{a \in A} \{\theta_\alpha(a) : \alpha \in S\} \right| = \mathfrak{c};$$

since, for each $b \in a + \bigcap_{\alpha \in S} P_\alpha$, we have

$$\{\theta_\alpha(b) : \alpha \in S\} = \{\theta_\alpha(a) : \alpha \in S\}.$$

Let $\psi : B \rightarrow c_0/\mathcal{V}$ be the embedding guaranteed by Theorem 6.1. We can then set $\varphi_\alpha = \psi \circ \theta_\alpha$ for each $\alpha \in S$. \square

Remark. For the objective of proving Theorems 7.2 and 7.4 in the next section, we only need the fact that the algebra B in the previous proof is a non-unital integral domain of cardinality \mathfrak{c} , and so can be embedded into some radical Banach algebra. However, by using Theorem 6.1, we can replace B by c_0/\mathcal{V} , which is more “concrete”, so that the result is (at least theoretically) “nicer”.

7. NORMING COMMUTATIVE SEMIPRIME ALGEBRAS
AND THE CONVERSE OF THEOREM 4.5

We now aim to generalize the result in [13] mentioned earlier on embedding non-unital integral domains of cardinality \mathfrak{c} into (radical) Banach algebras. Specifically, we seek to construct an algebra norm on $A/\bigcap P_\alpha$, given a commutative algebra A and a family of non-modular prime ideals P_α indexed by a set S (we are mostly interested in the case where the index set S is infinite). An “obvious” approach is to choose, assuming the Continuum Hypothesis as well as some “minor” cardinality condition, for each $\alpha \in S$, an algebra seminorm p_α on A such that $\ker p_\alpha = P_\alpha$ (by the mentioned result), and then attempt to define an algebra seminorm p on A with kernel $\bigcap P_\alpha$ by setting

$$p(a) = \sup_{\alpha \in S} p_\alpha(a) \quad (a \in A).$$

Unfortunately, unless the index set S is a finite set, the above supremum can well be infinite. (In fact, when S is infinite, for each $a \notin \bigcup P_\alpha$, there exists a family $\{p_\alpha : \alpha \in S\}$ such that $p(a) = \infty$.) However, the finiteness condition can be relaxed, yet this approach is still applicable if we can find a “nice” family of seminorms. Indeed, when the family $\{P_\alpha : \alpha \in S\}$ is pseudo-finite, because of the results in §6, a family of seminorms $\{p_\alpha : \alpha \in S\}$ can be chosen such that $\{p_\alpha(a) : \alpha \in S\}$ is only a finite set for each $a \in A$. Before going into details, we need to recall some definitions.

For the definition of *universal* algebras, see [6, Definition 5.7.8]. The important fact that we need is the existence of universal radical Banach algebras. For example, the integral domain $L^1(\mathbb{R}^+, \omega)$ is universal for each radical weight ω bounded near the origin [6, Theorem 5.7.25]. Indeed, the class of universal commutative radical Banach algebras has been characterized in [14] (see also [6, Theorem 5.7.28]). We also need the following theorem from [13] (see also [6, Theorem 5.7.11]); actually, we only need the particular case of $A = c_0/\mathcal{V}$.

Theorem 7.1 (CH). *Let A be a non-unital integral domain of cardinality \mathfrak{c} , and let L be a universal algebra. Then A can be embedded into L . \square*

Let B be a Banach algebra, and let S be an indexing set. Define $\ell^\infty(S, B)$ to be the Banach algebra of all bounded families $(b_\alpha : \alpha \in S)$ in B under pointwise algebraic operations and the supremum norm.

Theorem 7.2 (CH). *Let A be a commutative algebra, and let $(P_\alpha : \alpha \in S)$ be a pseudo-finite family of non-modular prime ideals in A such that*

$$\left| A / \bigcap_{\alpha \in S} P_\alpha \right| = \mathfrak{c}.$$

Then there exist a commutative radical Banach algebra R and a homomorphism $\theta : A \rightarrow R$ with $\ker \theta = \bigcap_{\alpha \in S} P_\alpha$.

Proof. Let $\varphi_\alpha : A \rightarrow c_0/\mathcal{V}$ ($\alpha \in S$) be the homomorphisms specified in Theorem 6.7 (see also the Remark after that theorem). Fix a universal commutative radical Banach algebra R_0 . By Theorem 7.1, there is an embedding $\psi : c_0/\mathcal{V} \rightarrow R_0$. Consider the following map:

$$\theta : A \rightarrow \prod_{\alpha \in S} R_0, \quad a \mapsto ((\psi \circ \varphi_\alpha)(a) : \alpha \in S).$$

Then θ is a homomorphism with kernel $\bigcap P_\alpha$. We see, by Theorem 6.7, that the image of θ is in $\ell^\infty(S, R_0)$, and indeed is in its radical R . Thus $\theta : A \rightarrow R$ is the required homomorphism. \square

Corollary 7.3 (CH). *Let A be a commutative semiprime algebra with $|A| = \mathfrak{c}$. Suppose that $\{0\}$ is the intersection of a pseudo-finite family of non-modular prime ideals. Then A is normable.*

When the family of prime ideals contains modular ones, the problem cannot be dealt with as smoothly. This is because a unital normed algebra must have a character, but there are unital algebras without any character. The situation becomes more complicated when we have infinitely many modular prime ideals.

The next result is the desired converse of Theorem 4.5.

Theorem 7.4 (CH). *Let A be a commutative Banach algebra. Suppose that $I = \bigcap_{\alpha \in S} P_\alpha$, where $(P_\alpha : \alpha \in S)$ is a pseudo-finite family of non-closed prime ideals in A satisfying*

$$|A/I| = \mathfrak{c}.$$

Then there exists a commutative Banach algebra B and a homomorphism $\theta : A \rightarrow B$ with $\mathcal{I}(\theta) = \ker \theta = I$. In the case where all the P_α are non-modular, we can choose B to be radical.

In the case where the family $(P_\alpha : \alpha \in S)$ is intersection non-redundant, the ideal I cannot be represented as an intersection of strictly fewer than $|S|$ prime ideals in A .

Proof. Fix a universal commutative radical Banach algebra R_0 such that R_0 is an integral domain. Note that, for each $\alpha \in S$, either P_α is non-modular or P_α is modular and contained in the kernel of a character of A . So, in both cases, by Theorem 7.1, A/P_α can be embedded in $R_0^\#$ (and even in R_0 if P_α is non-modular). Therefore, for each finite number of prime ideals P_{α_k} ($1 \leq k \leq n$), there is a homomorphism from A into $\prod_{k=1}^n R_0^\#$ with kernel $\bigcap_{k=1}^n P_{\alpha_k}$. We now consider the case where S is infinite. In this case, we see that $P = \bigcup_{\alpha \in S} P_\alpha$ is either a prime ideal in A , or A itself.

Case 1: $P = A$. For each P_α , there is a homomorphism θ_α from A into $R_0^\#$ (into R_0 if P_α is non-modular) with kernel P_α . Define

$$\theta : A \rightarrow \prod_{\alpha \in S} R_0^\#, \quad a \mapsto (\theta_\alpha(a)).$$

It is easily seen that the image of θ is contained in $\ell^\infty(S, R_0^\#)$, even in the radical of $\ell^\infty(S, R_0)$ if all P_α are non-modular.

Case 2: P is a non-modular prime ideal in A . Then each P_α is non-modular. So, by Theorem 7.2, there exists a homomorphism $\theta : A \rightarrow R$ with $\ker \theta = I$, where R is the radical of $\ell^\infty(S, R_0)$.

Case 3: P is a modular prime ideal in A . Let $u \in A$ be a modular identity for P . Choose $a \notin P$. Then $(1-u)a \in P$, so $(1-u)a \in P_\alpha$ for all except finitely many $\alpha \in S$. For these exceptional indices α we can proceed as in the first paragraph of the proof, so that we can suppose that $(1-u)a \in P_\alpha$ ($\alpha \in S$). Since each P_α is a prime ideal, we see that u is a (common) modular identity for all the P_α . Thus u is a modular identity for I . Let M be a maximal modular ideal in A containing P . Then M has codimension 1, that is $A = \mathbb{C}u \oplus M$. Since P and each P_α are

non-modular in M , by either Cases 1 or 2 above, there is a homomorphism from M into $\ell^\infty(S, R_0)$ with kernel I . We can then extend this to a homomorphism from A into $\ell^\infty(S, R_0^\#)$ by sending u to $\mathbf{1} = (1, 1, \dots)$.

Thus, in each case, we can find a homomorphism

$$\theta : A \rightarrow \ell^\infty(S, R_0^\#), \quad a \mapsto (\theta_\alpha(a) : \alpha \in S),$$

where $\theta_\alpha : A \rightarrow R_0^\#$ is a homomorphism with kernel P_α ($n \in \mathbb{N}$). Fix $a \in \mathcal{I}(\theta)$ and $\alpha \in S$. Then a is also in the continuity ideal of θ_α . The map θ_α has kernel P_α , which is not closed, and so θ_α is discontinuous. Therefore $\mathfrak{S}(\theta_\alpha) \neq \{0\}$. Since $\theta_\alpha(a)\mathfrak{S}(\theta_\alpha) = \{0\}$ and $R_0^\#$ is an integral domain, we must have $\theta_\alpha(a) = 0$. Thus, $\mathcal{I}(\theta) \subset P_\alpha$ ($\alpha \in S$), and hence $\mathcal{I}(\theta) \subset I = \ker \theta \subset \mathcal{I}(\theta)$.

The last assertion follows from Lemma 3.5. □

Remark. Let A be a commutative algebra, and let $(P_\alpha : \alpha \in S)$ be a family of prime ideals in A which is pseudo-finite and intersection non-redundant. Suppose that we have

$$\left| A / \bigcap_{\alpha \in S} P_\alpha \right| = \mathfrak{c}.$$

Then we must have $|S| \leq \mathfrak{c}$, and with the Continuum Hypothesis there are three possibilities left for the set S : either S is a finite set or $|S| = \aleph_0$ or $|S| = \aleph_1$.

Example 7.5 (Example 3.7 revisited, with CH). Let \mathfrak{F}_S and Q_α ($\alpha \in S$) as in Example 3.7. Suppose that $|S|$ is either \aleph_0 or \aleph_1 . Then $|\mathfrak{F}_S| = \mathfrak{c}$. Thus, by Corollary 7.3, the semiprime algebra $\mathfrak{F}_S / \bigcap_{\alpha \in S} Q_\alpha$, which is a priori without any obvious algebra norm, is normable.

Example 7.6 (Example 3.8 revisited, with CH). Let \mathcal{A}_S and P_α ($\alpha \in S$) be as in Example 3.8. Suppose that $|S| \leq \mathfrak{c}$. Applying Theorem 7.4, we can find a homomorphism from the commutative Banach algebra \mathcal{A}_S into another Banach algebra whose continuity ideal is $\bigcap_{\alpha \in S} P_\alpha$, which is an intersection of $|S|$ prime ideals but not fewer. Thus, we have an example of a homomorphism whose continuity ideal is a semiprime ideal but not a finite intersection of prime ideals, in the case where $|S|$ is either \aleph_0 or \aleph_1 .

8. THE CASE $\mathcal{C}_0(\Omega)$: POSITIVE RESULTS

In this section, we shall consider two classes of spaces Ω for which Questions A and B have a positive answer. First, applying the results of previous sections, we have the following reduction.

Theorem 8.1. *Let Ω be a non-compact locally compact space.*

- (i) *Let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R . Then either $\ker \theta$ is a finite intersection of non-modular prime ideals in $\mathcal{C}_0(\Omega)$, or $\mathcal{C}_0(\Omega)$ contains an intersection non-redundant pseudo-finite sequence of non-modular prime ideals.*
- (ii) (CH) *Suppose that (P_n) is an intersection non-redundant pseudo-finite sequence of non-modular prime ideals in $\mathcal{C}_0(\Omega)$. Suppose further that*

$$\left| \mathcal{C}_0(\Omega) / \bigcap_{n=1}^\infty P_n \right| = \mathfrak{c}.$$

Then there exists a homomorphism from $\mathcal{C}_0(\Omega)$ into a commutative radical Banach algebra with kernel $\bigcap_{n=1}^{\infty} P_n$, which is not a finite intersection of prime ideals.

Proof. (i) Since θ maps into a radical algebra, we see that $\ker \theta: f$ is non-modular for each $f \in \mathcal{C}_0(\Omega)$. The result then follows from Theorems 4.5 and 1.1.

(ii) This is a special case of Theorem 7.2. \square

Remark.

1. A similar result for Question B follows from Theorems 4.5, 1.1(iv), and 7.4.

2. Let I be the kernel (respectively, the continuity ideal) of a homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra (respectively, a Banach algebra). Denote by \mathfrak{P} the set of prime ideals of the form $I:f$ for some $f \in \mathcal{C}_0(\Omega)$. With a little more effort, we can show, in this case, that each sequence of distinct prime ideals in \mathfrak{P} contains an intersection non-redundant pseudo-finite subsequence. So, although \mathfrak{P} is not always finite (as we shall see), it does have a “compactness type” property.

Lemma 8.2. *Let Ω be a locally compact space. Let $(P_\alpha : \alpha \in S)$ be a pseudo-finite family of prime ideals in $\mathcal{C}_0(\Omega)$. Then $\bigcup_{\alpha \in S} P_\alpha \neq \mathcal{C}_0(\Omega)$. Furthermore, if S is infinite, then $\bigcup_{\alpha \in S} P_\alpha$ is a prime ideal.*

Proof. Suppose first that S is infinite. Choose distinct elements α_n ($n \in \mathbb{N}$) in S . For each $n \in \mathbb{N}$, choose $f_n \notin P_{\alpha_n}$; we can further suppose that

$$0 \leq f_n(p) \leq \frac{1}{2^n} \quad (p \in \Omega).$$

Set $f = \sum_{n=1}^{\infty} f_n$. Then $f \in \mathcal{C}_0(\Omega)$, but $f \notin P_{\alpha_n}$ because $f \geq f_n$ ($n \in \mathbb{N}$). By pseudo-finiteness of (P_α) , we must have $f \notin \bigcup_{\alpha \in S} P_\alpha$. Hence $\bigcup_{\alpha \in S} P_\alpha \neq \mathcal{C}_0(\Omega)$.

The case where S is finite can be proved similarly. \square

We have seen that our questions can be translated into an algebraic question of the structure of the algebra in consideration (subject to the Continuum Hypothesis in one of the directions). We now explain how Question A (and B) on $\mathcal{C}_0(\Omega)$ depends purely on the topological property of the underlying space Ω .

Note that, for a pseudo-finite sequence of prime z -filters (\mathcal{F}_n) , the union $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is again a prime z -filter, and also that $(\mathbf{Z}^{-1}[\mathcal{F}_n])$ is a pseudo-finite sequence of prime z -ideals in $\mathcal{C}(\Omega)$.

Lemma 8.3. *Let Ω be a locally compact space. Let (\mathcal{F}_n) be a pseudo-finite sequence of prime z -filters on Ω . Then the following are equivalent:*

- (a) (\mathcal{F}_n) is intersection non-redundant;
- (b) $\mathcal{F}_m \not\subset \mathcal{F}_n$ ($m \neq n \in \mathbb{N}$);
- (c) $\bigcap_{n \neq m} \mathcal{F}_n \not\subset \mathcal{F}_m$ for each $m \in \mathbb{N}$.

Proof. The proof is the same as that of Lemma 3.4. \square

By Lemmas 3.9, 3.10, and 8.2, we see that the question of the existence of an intersection non-redundant pseudo-finite sequence of prime ideals for $\mathcal{C}_0(\Omega)$ is equivalent to the same question for $\mathcal{C}(\Omega^b)$. Thus, we can pass freely between Ω and the compact space Ω^b .

Let Ω be a locally compact space, and let E be a closed subset of Ω . We define

$$I_E = \{f \in \mathcal{C}_0(\Omega) : E \subset \mathbf{Z}(f)\}.$$

Lemma 8.4. *Let Ω be a compact space. Then the following are equivalent:*

- (a) *there exists an intersection non-redundant pseudo-finite sequence of prime ideals (P_n) in $\mathcal{C}(\Omega)$;*
- (b) *there exists an intersection non-redundant pseudo-finite sequence of prime z -filters (\mathcal{F}_n) on Ω ;*
- (c) *there exists an intersection non-redundant pseudo-finite sequence of prime z -ideals (Q_n) in $\mathcal{C}(\Omega)$.*

Proof. Obviously (b) \Leftrightarrow (c) \Rightarrow (a).

Now, we shall prove that (a) \Rightarrow (b). For each $n \in \mathbb{N}$, set

$$\mathcal{F}_n = \{Z \in \mathbf{Z}[\Omega] : I_Z \subset P_n\}.$$

Then \mathcal{F}_n is a prime z -filter (by [9, Theorem 4.22]).

Let $Z \in \mathbf{Z}[\Omega]$. Suppose that $Z \notin \mathcal{F}_{n_i}$, for some (increasing) sequence (n_i) . For each $i \in \mathbb{N}$, choose $f_i \in I_Z \setminus P_{n_i}$; we can further suppose that $0 \leq f_i \leq 1/2^i$. Define $f = \sum_{i=1}^\infty f_i$. Then $f \in I_Z$, but $f \notin P_{n_i}$, since $f \geq f_i$ ($i \in \mathbb{N}$). By the pseudo-finiteness of (P_n) , we must have $f \notin P_n$ ($n \in \mathbb{N}$). So $Z \notin \mathcal{F}_n$ ($n \in \mathbb{N}$). This proves the pseudo-finiteness of (\mathcal{F}_n) .

Now assume toward a contradiction that $\mathcal{F}_{n_0} \subset \mathcal{F}_{m_0}$ for some $n_0 \neq m_0$. Then

$$Q := \mathbf{Z}^{-1}[\mathcal{F}_{n_0}] \subset \mathbf{Z}^{-1}[\mathcal{F}_{m_0}] \subset P_{m_0}.$$

Also we have $Q \subset P_{n_0}$. Since Q is a prime ideal in $\mathcal{C}(\Omega)$, it follows that either $P_{m_0} \subset P_{n_0}$ or $P_{n_0} \subset P_{m_0}$, contradicting the intersection non-redundancy of (P_n) . Hence $\mathcal{F}_n \not\subset \mathcal{F}_m$ ($n \neq m \in \mathbb{N}$). By Lemma 8.3, we conclude that (\mathcal{F}_n) is intersection non-redundant. □

Let Ω be a locally compact space. Suppose that (P_n) is a pseudo-finite sequence of prime ideals in $\mathcal{C}_0(\Omega)$. Then $P = \bigcup_{n=1}^\infty P_n$ is again a prime ideal. Let $p \in \Omega^b$ be the unique support point for P . Then p is also the support point for each P_n , and so we have

$$J_p \subset P_n \subset P \subset M_p \quad (n \in \mathbb{N}).$$

Proposition 8.5. *Let Ω be a locally compact space satisfying one of the following conditions:*

- (i) *$\beta\Omega$ is an F -space, or*
- (ii) *$\beta(\Omega \setminus \{p\})$ is an F -space for each non- P -point p in Ω^b .*

Then $\mathcal{C}_0(\Omega)$ contains no intersection non-redundant pseudo-finite sequence of prime ideals.

Proof. Assume toward a contradiction that $\mathcal{C}_0(\Omega)$ contains an intersection non-redundant pseudo-finite sequence of prime ideals (P_n) .

(i) Since $\mathcal{C}_0(\Omega)$ is an ideal in $\mathcal{C}(\beta\Omega)$, by Lemmas 3.9 and 8.2, we can find an intersection non-redundant pseudo-finite sequence (Q_n) of prime ideals in $\mathcal{C}(\beta\Omega)$. As in the paragraph preceding this proposition, we see that Q_n ($n \in \mathbb{N}$) contains $J_p^{\beta\Omega}$ for some $p \in \beta\Omega$. Since $\beta\Omega$ is an F -space, it follows that $J_p^{\beta\Omega}$ is a prime ideal (see, for example, [6, Proposition 4.2.18]), and so (Q_n) , ordered by inclusion, must be a chain. This contradicts the intersection non-redundancy of (Q_n) .

(ii) Again, we have seen that, for some $p \in \Omega^b$,

$$J_p^\Omega \subset P_n \subset M_p^\Omega \quad (n \in \mathbb{N}).$$

If p is a P-point in Ω^b , then $J_p^\Omega = M_p^\Omega$. So $P_n = J_p^\Omega$ ($n \in \mathbb{N}$), contradicting the intersection non-redundancy of (P_n) . Hence p is a non-P-point in Ω^b . The ideal M_p^Ω can be naturally identified with $\mathcal{C}_0(\Omega \setminus \{p\})$. The desired contradiction then follows from (i). \square

Combining Theorems 1.1(iv), 4.5, and Proposition 8.5, we obtain the following theorem of [10].

Theorem 8.6. *Let Ω be a locally compact space satisfying the hypothesis of Proposition 8.5. Then the continuity ideal of each homomorphism from $\mathcal{C}_0(\Omega)$ into another Banach algebra is always a finite intersection of prime ideals.*

Next, we consider another class of locally compact spaces.

Proposition 8.7. *Let Ω be a compact space. Suppose that $p \in \Omega \setminus \partial^{(\infty)}\Omega$. Then $\mathcal{C}(\Omega)$ does not contain an intersection non-redundant pseudo-finite sequence of prime ideals such that each P_n is supported at the point p .*

Proof. In the case where Ω has a finite limit level, let $l \in \mathbb{N}$ be such that $\partial^{(l)}\Omega = \emptyset$. For each $j \in \mathbb{Z}$ with $0 \leq j \leq l$, set $I_j = I_{\partial^{(j)}\Omega}$. Note that $I_0 = \{0\}$ and $I_l = \mathcal{C}_0(\Omega)$. We see that, for each j , I_j/I_{j-1} is isomorphic to $\mathcal{C}_0(U_j)$, where $U_j = \partial^{(j-1)}\Omega \setminus \partial^{(j)}\Omega$ is discrete and so satisfies condition (i) of Proposition 8.5. Thus each algebra I_j/I_{j-1} does not contain an intersection non-redundant pseudo-finite sequence of prime ideals, and, applying Lemma 3.10 inductively, we deduce that neither does $I_l = \mathcal{C}_0(\Omega)$. The result holds in this case.

In general, assume toward a contradiction that $\mathcal{C}(\Omega)$ contains an intersection non-redundant pseudo-finite sequence (P_n) of prime ideals such that each ideal is supported at p . Let K be a compact neighbourhood of p such that $K \cap \partial^{(\infty)}\Omega = \emptyset$. Then, since $I_K \subset J_p \subset P_n$ ($n \in \mathbb{N}$), we see that $\mathcal{C}(\Omega)/I_K$, which is isomorphic to $\mathcal{C}(K)$, contains an intersection non-redundant pseudo-finite sequence of prime ideals. However, K has a finite limit level, thus contradicting the previously proved case. \square

Theorem 8.8. *Let Ω be a locally compact space. Denote by p the point adjoined to Ω to obtain Ω^b .*

- (i) *Suppose that $p \notin \partial^{(\infty)}(\Omega^b)$. Then the kernel of each homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra is a finite intersection of non-modular prime ideals.*
- (ii) *Suppose that Ω^b has finite limit level (i.e., $\partial^{(\infty)}(\Omega^b) = \emptyset$). Then each homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is continuous on a finite intersection of prime ideals.*

Proof. This follows from Theorem 8.1(i) and Proposition 8.7; see also the remark after Theorem 8.1. \square

9. THE CASE $\mathcal{C}_0(\Omega)$: NEGATIVE RESULTS

In this section, we show that, in the class of locally compact metrizable spaces, the only spaces for which the answer to Questions A and B is positive are those which satisfy the hypothesis of Theorem 8.8.

First, we define two ‘‘prototype’’ spaces Ξ_0 and Ξ_1 . Denote by ∞ the point adjoined to \mathbb{N} to obtain its one-point compactification \mathbb{N}^b . The product space $(\mathbb{N}^b)^\mathbb{N}$

is a compact metrizable space. For abbreviation, we shall write (n_1, n_2, \dots, n_k) instead of $(n_1, n_2, \dots, n_k, \infty, \infty, \dots)$, where $n_1, \dots, n_k \in \mathbb{N}^b$. Define Ξ_0 to be the subset of $(\mathbb{N}^b)^\mathbb{N}$ of all elements of the form (n_1, n_2, \dots, n_k) , where $k \in \mathbb{Z}^+$ and $n_1, n_2, \dots, n_k \in \mathbb{N}$ are such that $k \leq n_1 < n_2 < \dots < n_k$. Define Ξ_1 to be the subset of $(\mathbb{N}^b)^\mathbb{N}$ of all elements that can be written in the (not necessarily unique) abbreviated form as (n_1, n_2, \dots, n_k) , where $k \in \mathbb{Z}^+$ and $n_1, n_2, \dots, n_k \in \mathbb{N}^b$ are such that $n_i \geq k$ ($1 \leq i \leq k$).

- Lemma 9.1.** (i) *The sets $\Xi_0 \subset \Xi_1$ are countable compact subspaces of $(\mathbb{N}^b)^\mathbb{N}$.*
 (ii) *There exists a continuous embedding of Ξ_1 into Ξ_0 ; it follows automatically that the point (∞, ∞, \dots) is mapped to itself.*

Proof. Part (i) is routine.

We now prove part (ii). First, we see that the space Ξ_0 is homeomorphic to the subspace

$$\Delta := \{0\} \cup \left\{ \sum_{i=1}^k 2^{-n_i} : k, n_1, n_2, \dots, n_k \in \mathbb{N} \text{ and } k \leq n_1 < \dots < n_k \right\}$$

of $[0, 1]$. Next, for each $i \in \mathbb{N}$, choose an increasing sequence $(r_{ij} : j \in \mathbb{N})$ in \mathbb{N} such that these sequences are pairwise disjoint; furthermore, we can suppose that $r_{ij} \geq j$ ($i, j \in \mathbb{N}$). For convenience, for each $i \in \mathbb{N}$, set $r_{i\infty} = \infty$. Also, our convention is that $2^{-\infty} = 0$. Define a map τ from Ξ_1 into $[0, 1]$ as follows: for each $(n_1, n_2, \dots) \in \Xi_1$, set

$$\tau(n_1, n_2, \dots) = \sum_{i=1}^{\infty} 2^{-r_{in_i}}.$$

We see that τ is well-defined, injective and continuous. It can be checked that τ actually maps Ξ_1 into Δ . The result then follows. □

Lemma 9.2. *Let Ω be a locally compact metrizable space. Suppose that there exists a point $p \in \partial^{(\infty)}(\Omega^b)$. Then there exists a continuous embedding $\iota : \Xi_1 \hookrightarrow \Omega^b$ such that $\iota(\infty, \infty, \dots) = p$.*

Remark. It follows that Ξ_1 is homeomorphic to the subspace $\iota(\Xi_1)$ of Ω^b .

Proof of Lemma 9.2. Fix a metric on Ω . In this proof, we shall denote by B_x an open ball in Ω centered at a point $x \in \Omega$ with some positive radius, whose exact value is not important for our purpose. We first aim to embed Ξ_0 into Ω^b .

We claim that there exists a sequence $(\iota_1(n))$ of distinct elements in $\Omega \setminus \{p\}$ such that $\lim_{n \rightarrow \infty} \iota_1(n) = p$ and that $\iota_1(n) \in \partial^{(n-1)}(\Omega^b)$ ($n \in \mathbb{N}$). The proof of this is divided into the three following cases:

Case 1: $p \in \Omega$. Since Ω is metrizable and $p \in \partial^{(\infty)}(\Omega^b)$, this case is obvious.

Case 2: p is the point adjoined to Ω to obtain Ω^b , and p is a limit point of $\partial^{(\infty)}(\Omega^b)$. Then, since $\Omega \cap \partial^{(\infty)}(\Omega^b)$ is non-compact and metrizable, it is not sequentially compact. Choose a sequence $(\iota_1(n))$ with no convergent subsequences in $\Omega \cap \partial^{(\infty)}(\Omega^b)$. Then $\lim_{n \rightarrow \infty} \iota_1(n) = p$ in Ω^b .

Case 3: p is the point adjoined to Ω to obtain Ω^b , and p is an isolated point in $\partial^{(\infty)}(\Omega^b)$. Choose a compact neighbourhood K of p such that $K \cap \partial^{(\infty)}(\Omega^b) = \{p\}$. We see that $\partial^{(\infty)}K = \{p\}$. For each $n \in \mathbb{N}$, we choose an element

$$\iota_1(n) \in \partial^{(n-1)}K \setminus \{p\} \subset \Omega \cap \partial^{(n-1)}(\Omega^b).$$

Then each limit point of $\{\iota_1(n) : n \in \mathbb{N}\}$ must be in $\bigcap \partial^{(n)}K = \partial^{(\infty)}K$. Therefore we deduce that $\lim_{n \rightarrow \infty} \iota_1(n) = p$.

Thus the claim holds.

Now we choose a sequence of pairwise disjoint balls $(B_{\iota_1(n)})$ such that each $B_{\iota_1(n)}$ has radius at most 2^{-n} and does not contain p . We observe that if, for each $n \in \mathbb{N}$, we choose $x_n \in B_{\iota_1(n)}$, then we still have $\lim x_n = p$.

Let $k \in \mathbb{N}$. Assuming that, for each $n_1, n_2, \dots, n_k \in \mathbb{N}$ with $n_1 \geq k$, we already have a ball

$$B_{\iota_1(n_1, \dots, n_k)} \text{ with } \iota_1(n_1, \dots, n_k) \in \partial^{(n_1-k)}\Omega.$$

We continue the construction inductively as follows. If $n_1 > k$, then, for each $n \in \mathbb{N}$, we can choose a ball $B_{\iota_1(n_1, \dots, n_k, n)}$ contained in $B_{\iota_1(n_1, \dots, n_k)}$, not containing $\iota_1(n_1, \dots, n_k)$, and whose center is of distance less than 2^{-n} from the point $\iota_1(n_1, \dots, n_k)$, such that all these new balls are disjoint; furthermore, we can assume that all the new points

$$\iota_1(n_1, \dots, n_k, n) \in \partial^{(n_1-k-1)}\Omega \quad (n \in \mathbb{N}).$$

Finally, set $\iota_1(\infty, \infty, \dots) = p$.

Obviously, Ξ_0 is contained in the domain of definition of ι_1 . Consider the restriction $\iota_2 : \Xi_0 \rightarrow \Omega$ of ι_1 . We see that ι_2 is injective and continuous. The result then follows by applying Lemma 9.1. \square

Let Ω be a compact space, and let \mathcal{F} be a prime z -filter on Ω . Then we say that \mathcal{F} is supported at a point p if $p \in Z$ for each $Z \in \mathcal{F}$. There always exists a unique support point for each prime z -filter (on a compact space).

Lemma 9.3. *There exists an intersection non-redundant pseudo-finite sequence of prime z -filters on Ξ_1 such that each z -filter is supported at the point (∞, ∞, \dots) .*

Proof. For each $n \in \mathbb{N}$, define $f_n \in \mathcal{C}(\Xi_1)$ by setting $f_n(j_1, j_2, \dots) = 2^{-j_n}$ for each $(j_1, j_2, \dots) \in \Xi_1$. Thus

$$\mathbf{Z}(f_n) = \{(j_1, j_2, \dots) \in \Xi_1 : j_n = \infty\}.$$

Let \mathcal{F} to be the z -filter generated by all $\mathbf{Z}(f_m f_n)$ ($m, n \in \mathbb{N}, m \neq n$). Then define $I = \mathbf{Z}^{-1}[\mathcal{F}]$. We claim that I is extendible with respect to (f_n) (see Definition 3.11).

Obviously, $f_m f_n \in I$ ($m \neq n$). We see that $\mathbf{Z}(f_n^k) = \mathbf{Z}(f_n) \notin \mathcal{F}$, and so $f_n^k \notin I$ ($n, k \in \mathbb{N}$). Now suppose that $g \in \mathcal{C}(\Xi_1)$ such that $g f_{n_0}^{k_0} \in I$ for some $n_0, k_0 \in \mathbb{N}$. Then $\mathbf{Z}(g) \cup \mathbf{Z}(f_{n_0}) \in \mathcal{F}$. Therefore, there exist $n_1, n_2, \dots, n_k \in \mathbb{N} \setminus \{n_0\}$ such that

$$\mathbf{Z}(g) \cup \mathbf{Z}(f_{n_0}) \supset \bigcap_{i=1}^k \mathbf{Z}(f_{n_i}).$$

This and the closedness of $\mathbf{Z}(g)$ imply that

$$\mathbf{Z}(g) \supset \overline{\bigcap_{i=1}^k \mathbf{Z}(f_{n_i})} \setminus \mathbf{Z}(f_{n_0}) \supset \bigcap_{i=1}^N \mathbf{Z}(f_i),$$

where $N = \max\{n_0, n_1, \dots, n_k\}$. So, for each $n \geq N + 1$, we have $g f_n \in I$. Hence the claim holds.

The proof is completed by applying Proposition 3.12 and Lemma 8.4. \square

Remark. We can prove a result on constructing pseudo-finite sequence of prime z -filters similar to Proposition 3.12, and then use it directly in the previous proof, so that we do not need to refer to Lemma 8.4.

Recall that when p is the point adjoined to Ω to obtain Ω^b , a prime ideal P in $\mathcal{C}_0(\Omega)$ is supported at p means that P is non-modular.

Proposition 9.4. *Let Ω be a locally compact metrizable space. Suppose that $p \in \partial^{(\infty)}(\Omega^b)$. Then there exists an intersection non-redundant pseudo-finite sequence (\mathcal{F}_n) of prime z -filters on Ω^b , where each z -filter is supported at p . Moreover, by setting $P_n = \mathcal{C}_0(\Omega) \cap \mathcal{Z}_{\Omega^b}^{-1}[\mathcal{F}_n]$, we obtain an intersection non-redundant pseudo-finite sequence of prime ideals in $\mathcal{C}_0(\Omega)$, each of which is supported at p , such that*

$$\left| \mathcal{C}_0(\Omega) / \bigcap_{n=1}^{\infty} P_n \right| = \mathfrak{c}.$$

Proof. This follows from Lemmas 9.2, 9.3, 8.4 and the fact that $|\mathcal{C}(\Xi_1)| = \mathfrak{c}$. □

The next theorem and its corollaries provide the negative answer to Questions A and B.

Theorem 9.5 (CH). *Let Ω be a locally compact metrizable space. Denote by p the point adjoined to Ω to obtain Ω^b .*

- (i) *Suppose that $p \in \partial^{(\infty)}(\Omega^b)$. Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra such that the kernel of θ is not a finite intersection of prime ideals.*
- (ii) *Suppose that Ω^b has an infinite limit level (i.e., $\partial^{(\infty)}(\Omega^b) \neq \emptyset$). Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra such that the continuity ideal of θ is not a finite intersection of prime ideals.*

Proof. This follows from Proposition 9.4 and Theorem 7.4. □

Note that each uncountable compact metrizable space Ω has an infinite limit level. For example, this is the case for the closed unit interval $[0, 1]$.

Corollary 9.6 (CH). *There exists a homomorphism θ from $\mathcal{C}_0(\mathbb{R})$ into a radical Banach algebra such that the kernel of θ is not a finite intersection of prime ideals.*

There are many countable compact metrizable spaces with infinite limit level. We note down the following specific example. Recall that in the proof of Lemma 9.1, we define the following countable compact subset of $[0, 1]$:

$$\Delta = \{0\} \cup \left\{ \sum_{i=1}^k 2^{-n_i} : k, n_1, n_2, \dots, n_k \in \mathbb{N} \text{ and } k \leq n_1 < \dots < n_k \right\}.$$

Corollary 9.7 (CH). *There exists a homomorphism θ from $\mathcal{C}_0(\Delta \setminus \{0\})$ into a radical Banach algebra such that the kernel of θ is not a finite intersection of prime ideals.*

10. LOCALLY COMPACT GROUPS

In this final section, we consider the algebra $\mathcal{C}_0(G)$ for a locally compact group G .

Lemma 10.1. *Let G be a non-discrete locally compact group satisfying either one of the following conditions:*

- (i) G is Abelian, or
- (ii) G is not totally disconnected.

Then G contains a non-discrete metrizable closed subgroup.

Proof. For the case where G is both compact and Abelian, see [22, Theorem 7].

In the case where G is totally disconnected and Abelian, by [16, Theorem 7.7], G contains a compact open subgroup. The result then follows from the previous case.

We now suppose that G is not totally disconnected. Let G_1 be the connected component of the identity e of G . Then G_1 is a non-discrete closed normal subgroup of G . By Iwasawa structure theorem ([21, p. 118]), G_1 contains a closed subgroup homeomorphic to \mathbb{R}^n for some $n \in \mathbb{Z}^+$ and a compact subgroup K such that G_1 is homeomorphic to $\mathbb{R}^n \times K$. If $n > 0$, then we are done.

Now suppose that $n = 0$, i.e., G_1 is compact. (In this case, for the purpose of proving the next lemma, we note that G_1 is contained in an open compact subgroup of G .) Let G_2 be a maximal connected Abelian subgroup of G_1 . Then G_2 is closed, and, by [17, Theorem 9.32], G_2 is maximal among the Abelian subgroup of G_1 . Therefore $G_2 \neq \{e\}$. The result, again, follows from the first case. \square

Lemma 10.2. *Let G be a non-discrete locally compact group. Then G contains a metrizable closed subset Ω such that Ω has no isolated point. If, furthermore, G is non-compact, then Ω can be further required to be non-compact either.*

Proof. For the first assertion, by Lemma 10.1(ii), we only need to consider the case where G is totally disconnected. In this case, by [16, Theorems 7.7 and 9.15], G contains a compact open subgroup, where each such subgroup is homeomorphic to a space $\{0, 1\}^\kappa$ for some infinite cardinal κ , and, in turn, each such space contains a copy of the Cantor set.

For the second assertion, we see that, in each case where our previously constructed set Ω is compact, Ω is contained in a compact open subgroup, say H , of G . In these cases, since G is non-compact, we can choose a sequence (x_n) in G such that $x_n H$ ($n \in \mathbb{N}$) are distinct elements of G/H ; we can then replace Ω by $\bigcup_{n=1}^\infty x_n \Omega$. \square

Proposition 10.3. *Let G be a non-discrete locally compact group. Then there exists an intersection non-redundant pseudo-finite sequence of prime z -filters (\mathcal{F}_n) on G^\flat . Furthermore, if G is non-compact, we can require each \mathcal{F}_n to be supported at the adjoined point (the point at infinity).*

Proof. This follows from Proposition 9.4 and Lemma 10.2. \square

Theorem 10.4. *Let G be a non-discrete locally compact group. Then:*

- (i) *there exists an intersection non-redundant pseudo-finite sequence of prime ideals (P_n) in $\mathcal{C}_0(G)$ with $|\mathcal{C}_0(G)/\bigcap_{n=1}^\infty P_n| = \mathfrak{c}$;*
- (ii) (CH) *there exists a homomorphism θ from $\mathcal{C}_0(G)$ into another Banach algebra such that the continuity ideal of θ is not a finite intersection of prime ideals.*

Proof. These follow from Proposition 10.3 and Theorem 7.4. \square

Theorem 10.5. *Let G be a non-discrete non-compact locally compact group. Then:*

- (i) *there exists an intersection non-redundant pseudo-finite sequence of non-modular prime ideals (P_n) in $\mathcal{C}_0(G)$ with $|\mathcal{C}_0(G)/\bigcap_{n=1}^\infty P_n| = \mathfrak{c}$;*
- (ii) (CH) *there exists a homomorphism θ from $\mathcal{C}_0(G)$ into a radical Banach algebra such that the kernel of θ is not a finite intersection of prime ideals.*

Proof. These follow from Proposition 10.3 and Theorem 8.1. □

For each locally compact Abelian group G , denote by \widehat{G} its dual group. For each $f \in L^1(G)$, denote by \widehat{f} the Fourier transform of f . Then

$$f \mapsto \widehat{f}, \quad L^1(G) \rightarrow \mathcal{C}_0(\widehat{G}),$$

is a continuous homomorphism.

Lemma 10.6. *Let G be a locally compact Abelian group. Let E be a closed subset of \widehat{G} and let K be a σ -compact subset of \widehat{G} such that $E \cap K = \emptyset$. Then there exists $f \in L^1(G)$ such that $\widehat{f} = 0$ on E but \widehat{f} never vanishes on K .*

Proof. Let $K = \bigcup_{n=1}^\infty K_n$, where each K_n is compact. By [6, Lemma 4.5.16], for each $n \in \mathbb{N}$, we can find $f_n \in L^1(G)$ such that $\widehat{f}_n = 0$ on E , such that \widehat{f}_n never vanishes on K_n , and such that $\widehat{f}_n \geq 0$ on \widehat{G} . We can further suppose that $\|f_n\|_1 \leq 2^{-n}$. Set $f = \sum_{n=1}^\infty f_n$. Then $f \in L^1(G)$ is the desired function. □

Theorem 10.7. *Let G be a non-compact locally compact Abelian group. Then:*

- (i) *there exists an intersection non-redundant pseudo-finite sequence of prime ideals (P_n) in $L^1(G)$ with $|L^1(G)/\bigcap_{n=1}^\infty P_n| = \mathfrak{c}$;*
- (ii) (CH) *there exists a homomorphism θ from $L^1(G)$ into another Banach algebra such that $\mathcal{I}(\theta) = \ker \theta$ is a semiprime ideal but not a finite intersection of prime ideals.*

Proof. Since G is non-compact, the dual \widehat{G} is non-discrete. Let (\mathcal{F}_n) be an intersection non-redundant pseudo-finite sequence of prime z -filters on \widehat{G}^b as specified in Proposition 10.3. For each $n \in \mathbb{N}$, set $Q_n = \mathcal{C}_0(\widehat{G}) \cap \mathbf{Z}_{\widehat{G}^b}^{-1}[\mathcal{F}_n]$. Then (Q_n) is an intersection non-redundant pseudo-finite sequence of prime ideals in $\mathcal{C}_0(\widehat{G})$.

Denote by p the point adjoined to \widehat{G} to obtain \widehat{G}^b . We see that, for each $n \in \mathbb{N}$,

$$Q_n = \left\{ h \in \mathcal{C}_0(\widehat{G}) : \mathbf{Z}_{\widehat{G}}(h) \cup \{p\} \in \mathcal{F}_n \right\}.$$

Define, for each $n \in \mathbb{N}$,

$$P_n = \left\{ f \in L^1(G) : \widehat{f} \in Q_n \right\} = \left\{ f \in L^1(G) : \mathbf{Z}_{\widehat{G}}(\widehat{f}) \cup \{p\} \in \mathcal{F}_n \right\};$$

we obtain a pseudo-finite sequence (P_n) of prime ideals in $L^1(G)$.

For each $n \in \mathbb{N}$, choose $g_n \in \bigcap_{i \neq n} Q_i \setminus Q_n$. By Lemma 10.6, there exists $f_n \in L^1(G)$ such that $\mathbf{Z}_{\widehat{G}}(\widehat{f}_n) = \mathbf{Z}_{\widehat{G}}(g_n)$. Then we see that $f_n \in \bigcap_{i \neq n} P_i \setminus P_n$ ($n \in \mathbb{N}$). Hence (P_n) is intersection non-redundant. The cardinality statement is obvious from the construction.

Part (ii) then follows from Theorem 7.4. □

Remark. In the preceding proof, by Lemma 10.6, if the ideal Q_n is non-modular, then so is the ideal P_n . Therefore, if, furthermore, the topology on G is non-discrete, then we can further require each P_n to be non-modular and the range of the homomorphism θ to be radical.

For each discrete space G , there exists no intersection non-redundant pseudo-finite sequence of prime ideals in $\mathcal{C}_0(G)$. However, this does not answer the following.

Question 10.8. Does there exist an intersection non-redundant pseudo-finite sequence of prime ideals in $L^1(G)$, for each compact Abelian group G ?

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, ENGLAND

E-mail address: `hung@maths.leeds.ac.uk`

Current address: Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2E1

E-mail address: `hlpham@math.ualberta.ca`