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LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS IN WEIGHTED BERGMAN AND HARDY SPACES

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Abstract. Complex linear differential equations of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0 \tag{†}$$

with coefficients in weighted Bergman or Hardy spaces are studied. It is shown, for example, that if the coefficient $a_j(z)$ of (†) belongs to the weighted Bergman space $A_\alpha^{\frac{1}{k-j}}$, where $\alpha \geq 0$, for all $j = 0, \ldots, k-1$, then all solutions are of order of growth at most α, measured according to the Nevanlinna characteristic. In the case when $\alpha = 0$ all solutions are shown to be not only of order of growth zero, but of bounded characteristic. Conversely, if all solutions are of order of growth at most $\alpha \geq 0$, then the coefficient $a_j(z)$ is shown to belong to $A_\alpha^{p_j}$ for all $p_j \in (0, \frac{1}{k-j})$ and $j = 0, \ldots, k-1$.

Analogous results, when the coefficients belong to certain weighted Hardy spaces, are obtained. The non-homogeneous equation associated to (†) is also briefly discussed.

1. Introduction

One way of classifying the growth of the solutions of

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \tag{1.1}$$

where the coefficients are analytic in a complex domain, is by means of Nevanlinna theory [17]. H. Wittich considered the case where the coefficients, and hence the solutions, are entire functions.

Theorem A ([22], Satz 1). *The coefficients $a_0(z), \ldots, a_{k-1}(z)$ in* (1.1) *are polynomials in the complex plane if and only if all solutions of* (1.1) *are entire and of finite order of growth.*

The order of growth of a meromorphic function f in the complex plane is defined by

$$\sigma = \sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ denotes the Nevanlinna characteristic of f.

The growth relation between the coefficients and the solutions of linear differential equations in the complex plane has been studied in more detail, for instance, in [6, 7, 8].

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The first author proved an analogous result to Theorem A in the unit disc D.

Theorem B ([10], Theorem 6.1). *The coefficients* $a_0(z),\dots,a_{k-1}(z)$ *in* (1.1) *are* $\mathcal{H}$*-functions if and only if all solutions of* (1.1) *are analytic in* D *and of finite order of growth.*

A function f, analytic in D, is an $\mathcal{H}$*-function* if there exists a $q\in[0,\infty)$ such that

$$\sup_{z\in D}|f(z)|(1-|z|^2)^q<\infty.$$

The space $A^{-\infty}$, introduced by B. Korenblum [15], coincides with the space of all $\mathcal{H}$-functions. The order of growth of a meromorphic function f in D is defined by

$$\rho=\rho(f)=\limsup_{r\to 1^-}\frac{\log^+T(r,f)}{-\log(1-r)}.$$

The necessary part of Theorem B can also be found in [1] since the space U defined in [1] coincides with the space of all $\mathcal{H}$-functions; see Section 5. Further studies on the growth of analytic solutions of (1.1) in D can be found in [3, 14, 16].

Chr. Pommerenke found a sufficient condition for the coefficient $a(z)$ such that all solutions of

$$f''+a(z)f=0 \tag{1.2}$$

belong to the Nevanlinna class N, the meromorphic functions of bounded characteristic in D.

Theorem C ([18], Theorem 5). *Let the coefficient* $a(z)$ *of* (1.2) *be analytic in* D *satisfying*

$$\int_D|a(z)|^{\frac{1}{2}}\,d\sigma_z<\infty. \tag{1.3}$$

Then all solutions of (1.2) *belong to* N.

The element of the Lebesgue area measure on D is denoted by $d\sigma_z$.

A sufficient condition for the coefficient $a(z)$ implying that all solutions of

$$f^{(k)}+a(z)f=0 \tag{1.4}$$

belong to the Nevanlinna class N was found by the first author.

Theorem D ([10], Theorem 4.5). *Let the coefficient* $a(z)$ *of* (1.4) *be analytic in* D *satisfying*

$$\int_D|a(z)|(1-|z|)^{k-1}\,d\sigma_z<\infty. \tag{1.5}$$

Then all solutions of (1.4) *belong to* N.

For $0<p<\infty$ and $-1<\alpha<\infty$, the *weighted Bergman space* A^p_α consists of those functions f, analytic in D, for which

$$\|f\|_{A^p_\alpha}=\left(\int_D|f(z)|^p(1-|z|^2)^\alpha\,d\sigma_z\right)^{\frac{1}{p}}<\infty.$$

The classical Bergman space A^p is A^p_0. See [5] and [9] for the theory of Bergman spaces.

Theorems C and D may be stated in terms of weighted Bergman spaces.

Theorem C′. *If $a \in A^{\frac{1}{2}}$, then all solutions of* (1.2) *belong to N.*

Theorem D′. *If $a \in A^{1}_{k-1}$, then all solutions of* (1.4) *belong to N.*

The purpose of this paper is to study the growth relation between the coefficients and the solutions of (1.1) in D. The following two problems are studied:

(1) Suppose that, for every $j = 0, \dots, k-1$, the coefficient $a_j(z)$ of (1.1) belongs to a certain analytic function space depending on j. Find the function space or spaces to which all solutions of (1.1) belong.

(2) Suppose that all solutions of (1.1) belong to a certain analytic function space. Find the function space or spaces to which the coefficient $a_j(z)$, $j = 0, \dots, k-1$, of (1.1) belongs.

Problems (1) and (2) above are hereafter referred to as the *direct problem* and the *inverse problem*, respectively.

The main strategy is to first find a suitable set of conditions for the coefficients in (1.1) which force all solutions to belong to a targeted function space. These targeted spaces include the classes N and F, and the ring of all analytic functions of order of growth at most $\alpha \geq 0$. The class F of *non-admissible meromorphic functions* in D consists of those functions f for which

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} < \infty.$$

The second step is to assume conversely that all solutions belong to one of these targeted spaces, and to study what restrictions this induces on the coefficients. Ideally one would return to the same set of conditions where one started from, as is the case in Theorems A and B. The situation is, however, more complex as examples in Section 5 show.

TABLE 1. Summary of some of the main results. Here f denotes the generic solution of (1.1), $\alpha \geq 0$, and $j = 0, \dots, k-1$.

Direct Problem		**Inverse Problem**	
Assumption	*Result*	*Assumption*	*Result*
$a_j \in A^{\frac{1}{k-j}}$	$f \in N$	$f \in N$	$a_j \in \bigcap_{0<p<\frac{1}{k-j}} A^p$
$a_j \in H^{\frac{1}{k-j}}_{k-j}$	$f \in F$	$f \in F$	$a_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{1}{p}}$
$a_j \in A^{\frac{1}{k-j}}_{\alpha}$	$\rho(f) \leq \alpha$	$\rho(f) \leq \alpha$	$a_j \in \bigcap_{0<p<\frac{1}{k-j}} A^p_{\alpha}$
$a_j \in H^{\frac{1}{k-j}}_{(\alpha+1)(k-j)}$	$\rho(f) \leq \alpha$	$\rho(f) \leq \alpha$	$a_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{\alpha+1}{p}}$

For $0 < p \leq \infty$ and $0 \leq q < \infty$, the *weighted Hardy space* H^p_q consists of those functions f, analytic in D, for which

$$\|f\|_{H^p_q} = \sup_{0<r<1} M_p(r,f)(1-r^2)^q < \infty,$$

where

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{\frac{1}{p}},\quad 0<p<\infty,$$

denotes the standard L^p-mean of the restriction of f to the circle of radius r centered at the origin, and

$$M_\infty(r,f)=\max_{0\le\theta\le 2\pi}|f(re^{i\theta})|.$$

The classical *Hardy space* H^p is H_0^p, where $0<p\le\infty$. See [4] for the theory of Hardy spaces.

The remainder of the paper is organized as follows. In Section 2 the direct problem is studied by using two growth estimates [13] for the solutions of (1.1). For instance, Theorems C and D are generalized for the equation (1.1); see Theorems 2.1 and 2.2 below. The inverse problem is considered in Section 3 by using the standard order reduction procedure combined with integrated logarithmic derivative estimates [12]. In Section 4 the results from Sections 2 and 3 are compared, and it is shown that neither of the generalized conditions (2.1) and (2.2) below corresponding to (1.3) and (1.5) implies the other. In Section 5 a number of examples related to the results proved in Sections 2 and 3 are given. These examples demonstrate, for instance, that the results listed in Table 1 on the direct problem involving weighted Bergman spaces cannot be improved to be "if and only if". Finally, in Section 6, the case of non-homogeneous linear differential equations is briefly discussed.

2. Direct problem

Auxiliary results. For $0<p<\infty$, the *p-characteristic* of a meromorphic function f in D is defined by

$$m_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\log^+|f(re^{i\theta})|\right)^p\,d\theta\right)^{\frac{1}{p}};$$

see, for instance, [21]. The class N^p, which can be considered as a generalized Nevanlinna class, consists of those functions f for which

$$\sup_{0<r<1} m_p(r,f)<\infty.$$

The following two growth estimates for the solutions of (1.1), recently obtained in [13], play a fundamental role in this section. Note that the proof of [10, Lemma 4.6] has been used to write the double integrals in [13, Corollaries 4.2 and 5.3] in terms of area integrals over the disc $D(0,r)=\{z:|z|<r\}$.

Lemma E ([13], Corollary 4.2). *Let f be a solution of (1.1), where $a_0(z),\dots,a_{k-1}(z)$ are analytic in D, and let $1\le p<\infty$. Then there exist a constant $C>0$, depending only on p, k and the initial values of f at the origin, such that*

$$m_p(r,f)^p\le C\left(\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_{D(0,r)}|a_j^{(n)}(z)|^p(1-|z|)^{p(k-j+n-1)}\,d\sigma_z+1\right)$$

for all $0\le r<1$.

Lemma F ([13], Corollary 5.3). *Let f be a solution of (1.1), where $a_0(z),\dots,$ $a_{k-1}(z)$ are analytic in D, and let $1 \le p < \infty$. Then there exists a constant $C>0$, depending only on p, k and the initial values of f at $z_\theta \in D$, where $a_j(z_\theta) \neq 0$ for some $j=0,\dots,k-1$, such that*

$$m_p(r,f)^p \le C\left(\sum_{j=0}^{k-1}\int_{D(0,r)} |a_j(z)|^{\frac{p}{k-j}}\, d\sigma_z + 1\right)$$

for all $0 \le r < 1$.

It is well known that an analytic function f belongs to the weighted Bergman space A^p_α if and only if $f^{(n)}$ belongs to $A^p_{np+\alpha}$. This fact follows by Lemma G, which can be found, for example, in [20, Lemma 3.1].

Lemma G. *Let f be an analytic function in D, and let $0<p<\infty$, $-1<\alpha<\infty$ and $n \in \mathbb{N}$. Then there exist two constants $C_1>0$ and $C_2>0$, depending only on p, α and n, such that*

$$C_1\|f\|_{A^p_\alpha} \le \|f^{(n)}\|_{A^p_{np+\alpha}} + \sum_{j=0}^{n-1}|f^{(j)}(0)| \le C_2\|f\|_{A^p_\alpha}.$$

Coefficients in weighted Bergman spaces. In [3] the direct problem is approached by combining Picard's method of successive approximations with non-integrated logarithmic derivative estimates. Here Lemmas E and F are applied instead.

The first result contains Theorems C and C$'$ as a special case.

Theorem 2.1. *Let $1 \le p < \infty$. If the analytic coefficient a_j belongs to $A^{\frac{p}{k-j}}$, that is, if*

$$\int_D |a_j(z)|^{\frac{p}{k-j}}\, d\sigma_z < \infty, \tag{2.1}$$

for all $j=0,\dots,k-1$, then all solutions of (1.1) belong to N^p.

Proof. The assertion follows by Lemma F and (2.1). □

Theorems D and D$'$ are generalized in the following result.

Theorem 2.2. *Let $1 \le p < \infty$. If the analytic coefficient a_j belongs to $A^p_{p(k-j-1)}$, that is, if*

$$\int_D |a_j(z)|^p(1-|z|^2)^{p(k-j-1)}\, d\sigma_z < \infty, \tag{2.2}$$

for all $j=0,\dots,k-1$, then all solutions of (1.1) belong to N^p.

Proof. An application of Lemma E yields

$$m_p(r,f)^p \le C_1 + C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_D |a_j^{(n)}(z)|^p(1-|z|)^{p(k-j+n-1)}\, d\sigma_z,$$

from which the assertion follows by Lemma G and (2.2). □

Remark. It is shown in Section 4 that conditions (2.1) and (2.2) are not equivalent. Indeed, the spaces $A^p_{p(m-1)}$ and $A^{\frac{p}{m}}$, $m>1$, are not the same by Theorem 4.1.

The following result should be compared with [10, Theorem 6.2] and Theorem B above.

Theorem 2.3. *Let $0<\alpha<\infty$. If the analytic coefficient a_j belongs to $A_\alpha^{\frac{1}{k-j}}$, that is, if*

$$\int_D |a_j(z)|^{\frac{1}{k-j}}(1-|z|^2)^\alpha \, d\sigma_z < \infty, \tag{2.3}$$

for all $j=0,\dots,k-1$, then all solutions of (1.1) *are of order of growth at most α.*

Proof. Lemma F with $p=1$ yields

$$(1-r)^\alpha m(r,f) \le C\left(\sum_{j=0}^{k-1}\int_D |a_j(z)|^{\frac{1}{k-j}}(1-|z|)^\alpha \, d\sigma_z + 1\right),$$

from which the assertion follows by (2.3). □

Note that if (2.3) is satisfied and $\rho(f)=\alpha$ for a solution f of (1.1), then f is of finite type as well.

It is now rather obvious that an analogous result to Theorem 2.3 can be obtained by using Lemma E instead of Lemma F; see Proposition 2.4 below. However, this result turns out to be a consequence of Theorem 2.3; see Section 4.

Proposition 2.4. *Let $0<\alpha<\infty$. If the analytic coefficient a_j belongs to $A^1_{k-j-1+\alpha}$, that is, if*

$$\int_D |a_j(z)|(1-|z|^2)^{k-j-1+\alpha} \, d\sigma_z < \infty,$$

for all $j=0,\dots,k-1$, then all solutions of (1.1) *are of order of growth at most α.*

Coefficients in weighted Hardy spaces. If the coefficients of (1.1) belong to certain weighted Hardy spaces, then all solutions must belong to the Nevanlinna class N, to the class of non-admissible functions F, or to be of finite order.

Proposition 2.5. *Let $a_j \in H_{q_j}^{\frac{1}{k-j}}$, where $q_j \ge 0$, for all $j=0,\dots,k-1$, and denote $\alpha = \max_{0\le j\le k-1}\{\frac{q_j}{k-j}\}-1$.*

1. *If $\alpha<0$, then all solutions of* (1.1) *belong to N.*
2. *If $\alpha=0$, then all solutions of* (1.1) *belong to F.*
3. *If $\alpha>0$, then all solutions of* (1.1) *are of order of growth at most α.*

Proposition 2.5 follows easily by Lemma F.

To make the comparison between results concerning direct and inverse problems easier, the following immediate consequence of Proposition 2.5 is stated.

Corollary 2.6. *Let $\alpha\ge -1$ and let $a_j \in H^{\frac{1}{k-j}}_{(\alpha+1)(k-j)}$ for all $j=0,\dots,k-1$.*

1. *If $\alpha<0$, then all solutions of* (1.1) *belong to N.*
2. *If $\alpha=0$, then all solutions of* (1.1) *belong to F.*
3. *If $\alpha>0$, then all solutions of* (1.1) *are of order of growth at most α.*

Remarks. (1) Part (1) of Proposition 2.5 follows by Theorem 2.1. Indeed, if $m>1$ and $q\ge 0$ are such that $q<m$, it is easy to show that $H_q^{\frac{1}{m}} \subset A^{\frac{1}{m}}$. However, part (2) is of special interest since the class F was not treated earlier.

(2) A result analogous to Proposition 2.5 can be obtained by using Lemma E instead of Lemma F. Namely, assuming $a_j \in H^1_{q_j}$ and defining α as in Proposition 2.5, the assertions in Proposition 2.5 hold. However, the Hölder inequality shows that $H^1_{q_j} \subset H^{\frac{1}{k-j}}_{q_j}$.

(3) Proposition 2.5(3) should be compared with the following classical result for entire solutions of (1.1); see, for instance, [17, Proposition 7.1]: *If the coefficients* $a_0(z), \dots, a_{k-1}(z)$ *in* (1.1) *are polynomials, then all solutions* f *of* (1.1) *are entire and of order of growth*

$$\sigma(f) \le \max_{0\le j\le k-1} \left\{ \frac{\deg(a_j)}{k-j} \right\} + 1.$$

See [13] for an alternative proof and for further discussion.

If one of the conditions

$$a_j \in \bigcup_{0\le q<\frac{1}{p}+k-j-1} H^p_q, \quad j=0,\dots,k-1, \tag{2.4}$$

and

$$a_j \in \bigcup_{0\le q<\frac{k-j}{p}} H^{\frac{p}{k-j}}_q, \quad j=0,\dots,k-1, \tag{2.5}$$

where $1 \le p < \infty$, is satisfied, then Lemmas E and F, and the inequality

$$M_p(r,g')(1-r) \le 4M_p(\rho,g), \quad \rho=(1+r)/2,$$

which holds for all analytic functions g in D (see [4, p. 80]), imply that all solutions of (1.1) belong to N^p. However, these two results follow by Theorems 2.2 and 2.1. Indeed, it is easy to see that if $a_j \in H^p_{q_j}$, $0 \le q_j < \frac{1}{p}+k-j-1$, then (2.2) holds, and if $a_j \in H^{\frac{p}{k-j}}_{q_j}$, $0 \le q_j < \frac{k-j}{p}$, then (2.1) holds.

3. Inverse problem

Auxiliary results. One of the standard ways to deal with the inverse problem in the complex plane is to combine the order reduction procedure with logarithmic derivative estimates. Here the same line of reasoning is applied in D with integrated logarithmic derivative estimates.

Lemma H ([12], Lemma 3.1(b))**.** *Let* k *and* j *be integers satisfying* $k > j \ge 0$, *and let* α *be a constant satisfying* $0 < \alpha(k-j) < 1$. *Let* f *be a meromorphic function in* D *such that* $f^{(j)}$ *does not vanish identically. Then there exist* $r_0 \in (\frac{1}{2}, 1)$, $C > 0$ *and* $b \in (0,1)$ *such that if* $s(r) = 1-b(1-r)$, *then*

$$\int_0^{2\pi} \left| \frac{f^{(k)}(re^{i\theta})}{f^{(j)}(re^{i\theta})} \right|^\alpha d\theta \le C \left(\frac{1}{1-r} \max\left\{ \log\frac{1}{1-r}, T(s(r),f) \right\} \right)^{\alpha(k-j)}$$

for all $r_0 < r < 1$.

The first step of the order reduction procedure is briefly sketched here; see [10, pp. 38–40] and [17, pp. 55–58] for more details. If $\{f_1, \dots, f_k\}$ is a solution base of (1.1), then the first order reduction of (1.1) results in

$$\nu_1^{(k-1)} + a_{1,k-2}(z)\nu_1^{(k-2)} + \cdots + a_{1,0}(z)\nu_1 = 0, \tag{3.1}$$

where

$$a_{1,j}(z) = a_{j+1}(z) + \sum_{m=1}^{k-j-1} \binom{j+1+m}{m} a_{j+1+m}(z)\frac{f_1^{(m)}(z)}{f_1(z)} \tag{3.2}$$

for $j = 0, \dots, k-2$. Moreover, the meromorphic functions

$$\nu_{1,j}(z) = \frac{d}{dz}\left(\frac{f_{j+1}(z)}{f_1(z)}\right), \quad j = 1, \dots, k-1, \tag{3.3}$$

are linearly independent solutions of (3.1) in D.

The notation above is used in the following two lemmas.

Lemma 3.1. *Let $\alpha \geq 0$. Suppose that all solutions of (1.1) are of order of growth at most α, and that $a_{1,j}(z)$, $j = 0, \dots, k-2$, are the coefficients of (3.1). Denote $p_j = \frac{k}{k-j}p_0$ with $p_0 \in (0, \frac{1}{k})$, and assume that that there exist $r_{1,0}, \dots, r_{1,k-2} \in (0,1)$ such that*

$$\int_{D\setminus D(0,r_{1,j})} |a_{1,j}(z)|^{p_{j+1}}(1-|z|^2)^\alpha \, d\sigma_z < \infty \tag{3.4}$$

for all $j = 0, \dots, k-2$. Then there exist $r_{0,0}, \dots, r_{0,k-1} \in (0,1)$ such that

$$\int_{D\setminus D(0,r_{0,j})} |a_j(z)|^{p_j}(1-|z|^2)^\alpha \, d\sigma_z < \infty \tag{3.5}$$

for all $j = 0, \dots, k-1$.

Proof. Throughout the proof $C > 0$ denotes a constant, which is not necessarily the same at each occurrence.

The identity $a_{1,k-2}(z) = a_{k-1}(z) + k\frac{f_1'(z)}{f_1(z)}$ (note that $a_k(z) \equiv 1$) implies

$$|a_{k-1}(z)|^{p_{k-1}} \leq |a_{1,k-2}(z)|^{p_{k-2+1}} + k\left|\frac{f_1'(z)}{f_1(z)}\right|^{p_{k-1}}. \tag{3.6}$$

Let $0 < \varepsilon < \frac{1}{kp_0} - 1$, so that $(1+\varepsilon)kp_0 < 1$, and define

$$\phi_{\alpha,\varepsilon}(r) = \left(\frac{1}{1-r}\right)^{\alpha+1+\varepsilon}.$$

Since $\rho(f_1) \leq \alpha$ and $p_{k-1} = kp_0$, it follows by Lemma H that there exists an $r_{0,k-1} \in (r_{1,k-2}, 1)$ such that

$$\int_0^{2\pi} \left|\frac{f_1'(re^{i\theta})}{f_1(re^{i\theta})}\right|^{p_{k-1}} d\theta \leq C\phi_{\alpha,\varepsilon}(r)^{kp_0}$$

for all $r_{0,k-1} < r < 1$. Next, multiply both sides of (3.6) by $(1-|z|^2)^\alpha$, then integrate over the annulus $D \setminus D(0, r_{0,k-1})$, and finally use the assumption (3.4) to deduce (3.16) in the case $j = k-1$.

Suppose that the assertion, with corresponding constants $r_{0,k-1}, \dots, r_{0,k-l} \in (0,1)$, is proved for $j = k-1, \dots, k-l$, $l \in \{1, \dots, k-2\}$. Since

$$a_{1,k-(l+2)}(z) = a_{k-(l+1)}(z) + \sum_{m=1}^{l+1} \binom{k-l-1+m}{m} a_{k-(l+1)+m}(z)\frac{f_1^{(m)}(z)}{f_1(z)},$$

it follows that

$$
\begin{aligned}
|a_{k-(l+1)}(z)|^{p_{k-(l+1)}} &\le |a_{1,k-(l+2)}(z)|^{p_{k-(l+1)}} + C\left|\frac{f_1^{(l+1)}(z)}{f_1(z)}\right|^{p_{k-(l+1)}} \\
&\quad + C\sum_{m=1}^{l} |a_{k-(l+1)+m}(z)|^{p_{k-(l+1)}}\left|\frac{f_1^{(m)}(z)}{f_1(z)}\right|^{p_{k-(l+1)}}.
\end{aligned} \tag{3.7}
$$

Define $s_m = \frac{l+1}{m}$ and $q_m = \frac{l+1}{l+1-m}$, where $m = 1, \ldots, l$. Then

$$
\begin{gathered}
s_m > 1, \qquad q_m > 1, \qquad \frac{1}{s_m} + \frac{1}{q_m} = 1, \\
m s_m p_{k-(l+1)} = kp_0 < 1, \qquad q_m p_{k-(l+1)} = p_{k-(l+1)+m}
\end{gathered}
$$

for all $m = 1, \ldots, l$, and the Hölder inequality yields

$$
\begin{aligned}
&\int_0^{2\pi} |a_{k-(l+1)}(re^{i\theta})|^{p_{k-(l+1)}}\, d\theta \\
&\quad \le \int_0^{2\pi} |a_{1,k-(l+2)}(re^{i\theta})|^{p_{k-(l+1)}}\, d\theta + C\int_0^{2\pi} \left|\frac{f_1^{(l+1)}(re^{i\theta})}{f_1(re^{i\theta})}\right|^{p_{k-(l+1)}} d\theta \\
&\quad +C\sum_{m=1}^{l}\left(\int_0^{2\pi} |a_{k-(l+1)+m}(re^{i\theta})|^{q_m p_{k-(l+1)}}\, d\theta\right)^{\frac{1}{q_m}} \\
&\qquad \cdot\left(\int_0^{2\pi}\left|\frac{f_1^{(m)}(re^{i\theta})}{f_1(re^{i\theta})}\right|^{s_m p_{k-(l+1)}} d\theta\right)^{\frac{1}{s_m}}.
\end{aligned}
$$

By Lemma H there exists $r_{0,k-(l+1)} \in (R, 1)$, where

$$
R = \max\{r_{0,k-l}, \ldots, r_{0,k-1}, r_{1,k-(l+2)}\},
$$

such that

$$
\begin{aligned}
&\int_0^{2\pi} |a_{k-(l+1)}(re^{i\theta})|^{p_{k-(l+1)}}\, d\theta \\
&\quad \le \int_0^{2\pi} |a_{1,k-(l+2)}(re^{i\theta})|^{p_{k-(l+2)+1}}\, d\theta + C\phi_{\alpha,\varepsilon}(r)^{(l+1)p_{k-(l+1)}} \\
&\quad + C\sum_{m=1}^{l}\left(\int_0^{2\pi} |a_{k-(l+1)+m}(re^{i\theta})|^{p_{k-(l+1)+m}}\, d\theta\right)^{\frac{1}{q_m}} \phi_{\alpha,\varepsilon}(r)^{mp_{k-(l+1)}}
\end{aligned} \tag{3.8}
$$

for all $r_{0,k-(l+1)} < r < 1$. Next, multiply both sides of (3.7) by

$$
(1-|z|^2)^\alpha = (1-|z|^2)^{\frac{\alpha}{s_m}}(1-|z|^2)^{\frac{\alpha}{q_m}},
$$

then integrate over the annulus $D \setminus D(0, r_{0,k-(l+1)})$ by using (3.8), and finally apply the Hölder inequality again (with the indices s_m and q_m) and use the assumptions to conclude (3.5) in the case $j = k - (l + 1)$.

It has been proved that (3.5) holds for $j = 1, \ldots, k - 1$. Since

$$
a_0(z) = -\frac{f^{(k)}(z)}{f(z)} - a_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} - \cdots - a_1(z)\frac{f'(z)}{f(z)},
$$

it follows that

$$|a_0(z)|^{p_0} \leq \left|\frac{f^{(k)}(z)}{f(z)}\right|^{p_0} + \sum_{m=1}^{k-1} |a_m(z)|^{p_0} \left|\frac{f^{(m)}(z)}{f(z)}\right|^{p_0}. \tag{3.9}$$

Define $u_m = \frac{k}{m}$ and $v_m = \frac{k}{k-m}$, where $m = 1, \ldots, k-1$. Then

$$u_m > 1, \qquad v_m > 1, \qquad \frac{1}{u_m} + \frac{1}{v_m} = 1,$$
$$m u_m p_0 = k p_0 < 1, \qquad v_m p_0 = p_m$$

for all $m = 1, \ldots, k-1$. The same procedure as above, with

$$(1-|z|^2)^\alpha = (1-|z|^2)^{\frac{\alpha}{u_m}} (1-|z|^2)^{\frac{\alpha}{v_m}},$$

yields (3.5) in the case $j = 0$. □

Lemma 3.2. *Let $\phi(r)$ be a continuous increasing function of r such that*

$$\frac{T(r,f)}{1-r} = O(\phi(r)) \tag{3.10}$$

for all solutions f of (1.1). Suppose that $a_{1,j}(z)$, $j = 0, \ldots, k-2$, are the coefficients of (3.1). Denote $p_j = \frac{k}{k-j} p_0$ with $p_0 \in (0, \frac{1}{k})$, and assume that there exist $r_{1,0}, \ldots, r_{1,k-2} \in (0,1)$ such that

$$\int_0^{2\pi} |a_{1,j}(re^{i\theta})|^{p_{j+1}}\, d\theta = O\left(\phi(r)^{p_{j+1}(k-j-1)}\right), \quad r_{1,j} \leq r < 1, \tag{3.11}$$

for all $j = 0, \ldots, k-2$. Then there exist $r_{0,0}, \ldots, r_{0,k-1} \in (0,1)$ such that

$$\int_0^{2\pi} |a_j(re^{i\theta})|^{p_j}\, d\theta = O\left(\phi(r)^{p_j(k-j)}\right), \quad r_{0,j} \leq r < 1, \tag{3.12}$$

for all $j = 0, \ldots, k-1$.

The proof of Lemma 3.2 is almost identical to the proof of Lemma 3.1, and hence it is omitted.

Coefficients in weighted Bergman spaces. The first result in the inverse direction illustrates the sharpness of Theorem 2.1 in the case $p = 1$.

Theorem 3.3. *If all solutions of (1.1) are analytic in D and of order of growth zero, then*

$$\int_D |a_j(z)|^{p_j}\, d\sigma_z < \infty \tag{3.13}$$

for all $j = 0, \ldots, k-1$ and all $p_j \in (0, \frac{1}{k-j})$. In particular, if all solutions of (1.1) belong to N, then (3.13) holds.

Proof. Suppose first that $k = 1$, that is, (1.1) is of the form

$$f' + a_0(z) f = 0. \tag{3.14}$$

Let f be a non-constant solution of (3.14). Since $\rho(f) = 0$,

$$T(r,f) \leq \left(\frac{1}{1-r}\right)^{\varepsilon}$$

for all r close enough to 1. Let $p_0 \in (0,1)$, fix $0 < \varepsilon < \frac{1}{p_0} - 1$, and define

$$\phi_{0,\varepsilon}(r) = \left(\frac{1}{1-r}\right)^{1+\varepsilon}. \tag{3.15}$$

By Lemma H there exists an $r_0 \in (\frac{1}{2}, 1)$ such that

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^{p_0} d\theta \le C\phi_{0,\varepsilon}(r)^{p_0}$$

for all $r_0 < r < 1$. Since $(1+\varepsilon)p_0 < 1$, it follows that

$$\int_D |a_0(z)|^{p_0}\, d\sigma_z = \int_{D(0,r_0)} |a_0(z)|^{p_0}\, d\sigma_z + \int_{D\setminus D(0,r_0)} \left|\frac{f'(z)}{f(z)}\right|^{p_0} d\sigma_z < \infty,$$

and the assertion is proved in the case $k = 1$.

Suppose that $k \ge 2$. Let $p_0 \in (0, \frac{1}{k})$, and define $p_j = \frac{k}{k-j} p_0$ for all $j = 1, \dots, k-1$. Then

$$0 < p_0 < p_1 < \dots < p_{k-1} = kp_0 < 1.$$

Define the function $\phi_{0,\varepsilon}$ as in (3.15), but this time for $0 < \varepsilon < \frac{1}{kp_0} - 1$. Then clearly $(1+\varepsilon)kp_0 < 1$.

Since the solutions and hence the coefficients are analytic in D, it suffices to show that there exist $r_{0,0}, \dots, r_{0,k-1} \in (0,1)$ such that

$$\int_{D\setminus D(0,r_{0,j})} |a_j(z)|^{p_j}\, d\sigma_z < \infty \tag{3.16}$$

for all $j = 0, \dots, k-1$. The standard order reduction procedure is applied as in the proof of Lemma 3.1 to prove (3.16). After $k-1$ order reduction steps one obtains the differential equation

$$\nu_{k-1}' + a_{k-1,0}(z)\nu_{k-1} = 0, \tag{3.17}$$

where $a_{k-1,0}(z)$ is meromorphic in D, with all solutions being of order of growth zero. By Lemma H there exists an $r_{k-1,0} \in (0,1)$ such that

$$\int_{D\setminus D(0,r_{k-1,0})} |a_{k-1,0}(z)|^{p_{k-1}}\, d\sigma_z = \int_{D\setminus D(0,r_{k-1,0})} \left|\frac{\nu_{k-1}'(z)}{\nu_{k-1}(z)}\right|^{p_{k-1}} d\sigma_z < \infty,$$

and so, by Lemma 3.1, with $\alpha = 0$, and its counterparts in the subsequent order reduction steps, it follows that there exist $r_{0,0}, \dots, r_{0,k-1} \in (0,1)$ such that (3.16) holds for all $j = 0, \dots, k-1$. □

Next, Theorem 3.3 is expressed in terms of the Bergman spaces.

Corollary 3.4. *If all solutions of* (1.1) *are analytic in D and of order of growth zero, then*

$$a_j \in \bigcap_{0<p<\frac{1}{k-j}} A^p \tag{3.18}$$

for all $j = 0, \dots, k-1$. In particular, if all solutions of (1.1) *are analytic in D and belong to N, then* (3.18) *holds.*

The next result illustrates the sharpness of Theorem 2.3, and also provides a natural extension to Theorem 3.3.

Theorem 3.5. *Let $0<\alpha<\infty$. If all solutions of* (1.1) *are analytic in D and of order of growth at most α, then*

$$\int_D |a_j(z)|^{p_j}(1-|z|^2)^\alpha \, d\sigma_z < \infty$$

for all $j=0,\dots,k-1$ and all $p_j \in (0,\frac{1}{k-j})$.

Proof. Since each solution f is of order at most α by the assumption, it follows that, for all r close enough to 1,

$$T(r,f) \le \left(\frac{1}{1-r}\right)^{\frac{\alpha}{kp_0}}, \tag{3.19}$$

where $0<kp_0<1$. The proof of Theorem 3.3 is now followed using inequality (3.19) each time Lemma H is applied. Note that meromorphic functions of order of growth at most α in D form a differential field.

After $k-1$ order reduction steps applied to the differential equation (1.1) one obtains (3.17), where $a_{k-1,0}(z)$ is meromorphic in D, and all solutions of the equation are of order of growth at most α. The assertion follows similarly as in the proof of Theorem 3.3 by applying Lemma 3.1 and its counterparts in subsequent order reduction steps. □

Corollary 3.6. *Let $0<\alpha<\infty$. If all solutions of* (1.1) *are analytic in D and of order of growth at most α, then*

$$a_j \in \bigcap_{0<p<\frac{1}{k-j}} A^p_\alpha$$

for all $j=0,\dots,k-1$.

Coefficients in weighted Hardy spaces. The next two results illustrate the sharpness of Corollary 2.6 (and Proposition 2.5).

Theorem 3.7. *If all solutions of* (1.1) *are analytic in D and of order of growth zero, then*

$$a_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{1}{p}} \tag{3.20}$$

for all $j=0,\dots,k-1$. In particular, if all solutions of (1.1) *are analytic in D and belong to F, then* (3.20) *holds.*

Proof. First note that $H^{p_2}_{(k-j)(1-p_2)+1} \subset H^{p_1}_{(k-j)(1-p_1)+1}$ and $H^{p_2}_{\frac{1}{p_2}} \subset H^{p_1}_{\frac{1}{p_1}}$ for all $0<p_1\le p_2<\frac{1}{k-j}$. Further, since $(k-j)(1-p)+1 \longrightarrow k-j$, as $p \longrightarrow \frac{1}{k-j}$, it follows that

$$\bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{1}{p}} = \bigcap_{0<p<\frac{1}{k-j}} H^p_{(k-j)(1-p)+1}.$$

Therefore it suffices to show that

$$a_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{(k-j)(1-p)+1}, \quad j=0,\dots,k-1. \tag{3.21}$$

Define the constants $p_j = \frac{k}{k-j}p_0$, where $p_0 \in (0, \frac{1}{k})$ and $j = 0, \dots, k-1$, just as in the proof of Theorem 3.3, and choose $\phi(r) = (1-r)^{-1+p_0-\frac{1}{k}}$. Since $kp_0 \in (0,1)$ is arbitrary,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |a_j(re^{i\theta})|^{p_j}\, d\theta\right)^{\frac{1}{p_j}} = O\left(\left(\frac{1}{1-r}\right)^{(k-j)(1-p_j)+1}\right) \tag{3.22}$$

for all $j = 0, \dots, k-1$, and (3.21) follows. Moreover, since (3.12) implies (3.22), to complete the proof it is only needed to check that all assumptions of Lemma 3.2 are satisfied. Relation (3.10) holds because $\rho(f) = 0$. To see that (3.11) also holds, the order reduction can be used to reduce (1.1) into (3.17) by modifying the reasoning at the end of the proof of Theorem 3.3, using Lemma 3.2 in place of Lemma 3.1. □

The case when all solutions of (1.1) are of order at most α, $0 < \alpha < \infty$, can be similarly dealt with Lemma 3.2 by choosing $\phi(r) = (1-r)^{-1-\frac{\alpha}{kp_0}}$.

Theorem 3.8. *Let $0 < \alpha < \infty$. If all solutions of* (1.1) *are analytic in D and of order of growth at most α, then*

$$a_j \in \bigcap_{0<p<\frac{1}{k-j}} H^p_{\frac{\alpha+1}{p}}$$

for all $j = 0, \dots, k-1$.

Although the H^p_q-results concerning the direct problem are weaker than the corresponding A^p_α-results, it is interesting to find that this is not the case with the inverse problem. See Section 4 for more details.

4. Comparison

The results obtained in the previous two sections are now further analyzed. The first result shows that the spaces $A^{\frac{p}{m}}$ and $A^p_{p(m-1)}$ for $m \geq 1$ are not the same unless $m = 1$, and therefore conditions (2.1) and (2.2) are not equivalent.

Theorem 4.1. *Let $m > 1$.*

(1) *If $0 < p < 1$, then $A^p_{p(m-1)} \subsetneq A^{\frac{p}{m}}$.*
(2) *If $2 \leq p < \infty$, then $A^{\frac{p}{m}} \subsetneq A^p_{p(m-1)}$.*
(3) *If $1 \leq p < 2$, then $A^p_{p(m-1)} \not\subset A^{\frac{p}{m}} \not\subset A^p_{p(m-1)}$.*

The proof of Theorem 4.1 involves analytic functions with Hadamard gaps. The function $f(z) = \sum_{k=0}^\infty c_k z^{n_k}$, analytic in D, has *Hadamard gaps*, if $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \geq 0$. The class of analytic functions in D with Hadamard gaps is denoted by HG. The following result characterizes Hadamard gap series in the weighted Bergman spaces; see, for example, [2, Proposition 2.1].

Theorem I. *Let $0 < p < \infty$, $-1 < \alpha < \infty$, and let $f(z) = \sum_{k=0}^\infty c_k z^{n_k} \in HG$. Then $f \in A^p_\alpha$ if and only if $\sum_{k=0}^\infty n_k^{-(\alpha+1)}|c_k|^p < \infty$.*

Proof of Theorem 4.1. (1) If $0 < p < 1$, then, by the Hölder inequality,

$$\int_D |f(z)|^{\frac{p}{m}}\, d\sigma_z \leq \left(\int_D |f(z)|^p\,(1-|z|^2)^{p(m-1)}\, d\sigma_z\right)^{\frac{1}{m}}$$
$$\cdot \left(\int_D (1-|z|^2)^{-p}\, d\sigma_z\right)^{\frac{m-1}{m}},$$

and it follows that $A^p_{p(m-1)} \subset A^{\frac{p}{m}}$. To see that the inclusion is strict, let $c > 0$, and define

$$f_1(z) = \sum_{n=1}^{\infty} n^{\frac{m}{p}(n-1-c)} z^{n^n}, \quad z \in D.$$

Then it is easy to see that $f_1 \in HG$; see, for example, [19, Lemma 2.1.1]. Moreover,

$$\sum_{n=1}^{\infty} n^{-n} \left(n^{\frac{m}{p}(n-1-c)}\right)^{\frac{p}{m}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+c}} < \infty,$$

and therefore $f_1 \in A^{\frac{p}{m}}$ by Theorem I. But

$$\sum_{n=1}^{\infty} n^{-n(p(m-1)+1)} \left(n^{\frac{m}{p}(n-1-c)}\right)^{p} = \sum_{n=1}^{\infty} n^{n(m-1)(1-p)-m(1+c)} = \infty$$

for $0 < p < 1$ and $1 < m < \infty$, and hence $f_1 \notin A^p_{p(m-1)}$ by Theorem I.

(2) If $2 \leq p < \infty$, then $A^{\frac{p}{m}} \subset A^p_{2(m-1)} \subset A^p_{p(m-1)}$ by [2, Theorem 1.3]. To see that the inclusion $A^{\frac{p}{m}} \subset A^p_{p(m-1)}$ is strict, define

$$f_2(z) = \sum_{n=1}^{\infty} n^{\frac{m}{p}(n-1)} z^{n^n}, \quad z \in D.$$

Then

$$\sum_{n=1}^{\infty} n^{-n} \left(n^{\frac{m}{p}(n-1)}\right)^{\frac{p}{m}} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

and therefore $f_2 \notin A^{\frac{p}{m}}$ by Theorem I. But

$$\sum_{n=1}^{\infty} n^{-n(p(m-1)+1)} \left(n^{\frac{m}{p}(n-1)}\right)^{p} = \sum_{n=1}^{\infty} \frac{1}{n^{n(m-1)(p-1)+m}} < \infty$$

for $1 \leq p < \infty$, and hence $f_2 \in A^p_{p(m-1)}$ by Theorem I.

(3) If $1 \leq p < 2$, then the function f_2 defined above shows the first relation. To see the second one, let $\varepsilon > 0$, and, for a fixed branch, define $f_3(z) = (1-z)^{1-m-\frac{2}{p}-\varepsilon}$, where $0 < \varepsilon < \frac{1}{p}(2-p)(m-1)$. Then, by [4, p. 65],

$$\begin{aligned}\int_D |f_3(z)|^{\frac{p}{m}}\, d\sigma_z &\leq \int_0^1 \left(\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{\frac{p}{m}(m+\frac{2}{p}+\varepsilon-1)}}\right) dr \\ &\leq C \int_0^1 \frac{dr}{(1-r)^{p+\frac{2}{m}+\frac{p\varepsilon}{m}-\frac{p}{m}-1}} < \infty,\end{aligned}$$

and thus $f_3 \in A^{\frac{p}{m}}$. A similar reasoning shows that $f_3 \notin A^p_{p(m-1)}$. □

Theorem 4.1 shows that Theorem 2.2 is better than Theorem 2.1 when $2 \leq p < \infty$ in the sense that it gives a weaker condition for the coefficients $a_j(z)$, $j = 0, \ldots, k-1$, still implying that $f \in N^p$. If, on the other hand, $1 \leq p < 2$, neither of these theorems is essentially better than the other, and therefore the following consequence of Theorems 2.1 and 2.2 is worth stating.

Corollary 4.2. *Let* $1 \le p < 2$. *If*

$$a_j \in A^{\frac{p}{k-j}} \bigcup A^p_{p(k-j-1)}$$

for all $j = 0, \dots, k-1$, *then all solutions of* (1.1) *belong to* N^p.

The following proposition shows that Theorem 2.3 is better than Proposition 2.4 in the case $k \ge 2$.

Proposition 4.3. *Let* $1 < m < \infty$ *and* $0 < \alpha < \infty$. *Then*

$$A^1_{m+\alpha-1} \subsetneq A^{\frac{1}{m}}_{\alpha}.$$

Proof. By the Hölder inequality,

$$\int_D |f(z)|^{\frac{1}{m}} (1-|z|^2)^\alpha \, d\sigma_z \le \left(\frac{\pi}{\alpha}\right)^{\frac{m-1}{m}} \left(\int_D |f(z)|(1-|z|^2)^{m+\alpha-1} \, d\sigma_z\right)^{\frac{1}{m}},$$

and therefore $A^1_{m+\alpha-1} \subset A^{\frac{1}{m}}_{\alpha}$. The function $g(z) = (1-z)^{-\frac{1}{2}(m+1)(\alpha+2)}$ shows that the inclusion is strict. □

Finally it is shown that Theorems 3.7 and 3.8 imply Corollaries 3.4 and 3.6, respectively.

Proposition 4.4. *Let* $1 \le m < \infty$ *and* $0 < \alpha < \infty$. *Then*

$$\bigcap_{0<p<\frac{1}{m}} H^p_{\frac{1}{p}} = \bigcap_{0<p<\frac{1}{m}} H^p_{m(1-p)+1} \subset \bigcap_{0<p<\frac{1}{m}} A^p \tag{4.1}$$

and

$$\bigcap_{0<p<\frac{1}{m}} H^p_{\frac{\alpha+1}{p}} = \bigcap_{0<p<\frac{1}{m}} H^p_{m+\frac{\alpha}{p}} \subset \bigcap_{0<p<\frac{1}{m}} A^p_\alpha. \tag{4.2}$$

Proof. A similar reasoning as in the beginning of the proof of Theorem 3.7 shows the equalities in (4.1) and (4.2). To see the inclusion in (4.1), let $0 < p < \frac{1}{m}$ and $f \in H^p_{m(1-p)+1}$. Then

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \le \frac{C}{(1-r)^{(m(1-p)+1)p}}, \quad 0 \le r < 1,$$

for some constant $C > 0$. Since $(m(1-p)+1)p < 1$, it follows that $f \in A^p$, proving the inclusion in (4.1). The inclusion in (4.2) can be proved in a similar manner. □

5. Examples

Solutions in the Nevanlinna class. The first example shows that the conditions in Theorem C′ and Theorem 2.1 are not necessary.

Example 5.1. The functions

$$f_j(z) = (1-z)\exp\left((-1)^{j+1}\frac{1+z}{1-z}\right), \quad j = 1, 2,$$

are linearly independent solutions of (1.2), where $a(z) = -4(1-z)^{-4}$. Since $f_1 \in N$ and $f_2 \in H^\infty$, all solutions belong to the Nevanlinna class N, yet

$$\begin{aligned}\int_{D(0,r)} |a(z)|^{\frac{1}{2}}\, d\sigma_z &= \int_0^r \frac{2s}{1-s^2}\left(\int_0^{2\pi} \frac{1-s^2}{|1-se^{i\theta}|^2}\, d\theta\right) ds \\ &= 2\pi \log\frac{1}{1-r^2},\end{aligned} \tag{5.1}$$

and therefore $a \notin A^{\frac{1}{2}}$. Further, the functions $f_1 f_2$, f_1^2 and f_2^2 are linearly independent solutions of the equation

$$f''' + a_1(z)f' + a_0(z)f = 0,$$

where $a_1(z) = 4a(z)$ and $a_0(z) = 2a'(z)$. Therefore all solutions belong to N, but $a_1 \notin A^{\frac{1}{2}}$ by (5.1), while $a_0 \in A^{\frac{1}{3}}$.

The second example is a modification of Example 2 in [18]. It shows that condition (1.3) is the best possible in the sense that the exponent $\frac{1}{2}$ cannot be replaced by a smaller number.

Example 5.2. Let $g(z) = 1-z$ and $h(z) = \exp\left(\frac{i}{1-z}\right)$. Then $f_1(z) = g(z)h(z)$ and $f_2(z) = \frac{g(z)}{h(z)}$ are linearly independent solutions of (1.2), where $a(z) = (1-z)^{-4}$. By [4, p. 65], for $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 0$ such that

$$\int_D |a(z)|^{\frac{1}{2}-\varepsilon}\, d\sigma_z \le C\int_0^1 \frac{dr}{(1-r)^{1-4\varepsilon}} < \infty.$$

Since

$$\operatorname{Re}\left(\frac{i}{1-z}\right) = -\frac{r\sin\theta}{1-2r\cos\theta + r^2}, \qquad z = re^{i\theta},$$

is non-negative for $\pi \le \theta \le 2\pi$, it follows that

$$m(r,h) = -\frac{1}{2\pi}\int_\pi^{2\pi} \frac{r\sin\theta}{1-2r\cos\theta+r^2}\, d\theta = \frac{1}{2\pi}\log\frac{1+r}{1-r},$$

and so $h \notin N$. By the first fundamental theorem of Nevanlinna theory, it follows that $\frac{1}{h} \notin N$. As $g \in H^\infty$, neither f_1 nor f_2 belongs to N.

The third example addresses the question of whether it is still possible to determine when all solutions of (1.2) belong to the Nevanlinna class, even though condition (1.3) fails to be satisfied.

Example 5.3. For a fixed $C \in \mathbb{C}$, consider the equation

$$f'' + a(z)f = 0, \quad a(z) = -\frac{C}{(1-z)^4}, \tag{5.2}$$

where $C \neq 0$. (Otherwise the normalized fundamental solution base of (5.2) is just $\{1, z\}$.) By a similar computation as in Example 5.1,

$$\int_{D(0,r)} |a(z)|^{\frac{1}{2}}\, d\sigma_z = |C|^{\frac{1}{2}}\pi\log\frac{1}{1-r^2},$$

and hence all solutions of (5.2) belong to F by Lemma F. However, Examples 5.1 and 5.2 show that the solutions of (5.2) can either be in N or in $F \setminus N$ depending on the choice of C. Therefore equations of the type (5.2) are extremal for the class

N in the sense of Theorem C′. It is next shown that the values of C such that the general solution of equation (5.2) belongs to N are precisely those in $[0,\infty)$.

For $C \neq 0$, let $h(z) = \exp(\frac{\sqrt{C}}{2}\frac{1+z}{1-z})$, where the branch of the square root is fixed such that $\sqrt{C} = |C|^{\frac{1}{2}}\exp(i\frac{\arg(C)}{2})$, and $\arg(C) \in [-\pi,\pi)$. Then $f_1(z) = (1-z)h(z)$ and $f_2(z) = \frac{1-z}{h(z)}$ are linearly independent solutions of (5.2). If $h \in N$ (resp. $h \in F \setminus N$), then, by the first fundamental theorem of Nevanlinna theory, $\frac{1}{h} \in N$ (resp. $\frac{1}{h} \in F \setminus N$).

If $\arg(C) = 0$, then

$$\mathrm{Re}\left(\frac{\sqrt{C}}{2}\frac{1+z}{1-z}\right) = \frac{\sqrt{C}}{2}\cdot\frac{1-r^2}{1-2r\cos\theta+r^2}, \quad z = re^{i\theta},$$

is non-negative for $0 \le \theta \le 2\pi$, and it follows that $h \in N$ and $\frac{1}{h} \in H^\infty$. Therefore $f_1 \in N$ and $f_2 \in H^\infty$, hence all solutions of (5.2) belong to N.

If $\arg(C) \neq 0$, denote $d = \arg(C)/2$. Then

$$m(r,h) = \frac{|C|^{\frac{1}{2}}}{4\pi}\int_{G(r,C)} \frac{(1-r^2)\cos d - 2r\sin d\sin\theta}{1-2r\cos\theta+r^2}\,d\theta,$$

where $G(r,C) = \{\theta \in [-\pi,\pi) : (1-r^2)\cos d - 2r\sin d\sin\theta \ge 0\}$. Let $\chi_{G(r,C)}$ denote the characteristic function of $G(r,C)$. Then, by Fatou's lemma,

$$\begin{aligned}\lim_{r\to1^-} m(r,h) &\ge \liminf_{r\to1^-}\frac{|C|^{\frac{1}{2}}}{4\pi}\int_{G(r,C)} \frac{(1-r^2)\cos d - 2r\sin d\sin\theta}{1-2r\cos\theta+r^2}\,d\theta \\ &\ge \frac{|C|^{\frac{1}{2}}}{4\pi}\int_{-\pi}^{\pi}\liminf_{r\to1^-}\frac{(1-r^2)\cos d - 2r\sin d\sin\theta}{1-2r\cos\theta+r^2}\chi_{G(r,C)}(\theta)\,d\theta \\ &= \frac{|C|^{\frac{1}{2}}}{4\pi}|\sin d|\int_0^{\pi}\frac{\sin\theta}{1-\cos\theta}\,d\theta = \infty,\end{aligned}$$

and it follows that $f_1, f_2 \in F \setminus N$. Moreover, if $C_1C_2 \neq 0$, then the zero sequence $\{z_n\}$ of the linear combination $f = C_1f_1 + C_2f_2$ does not satisfy the Blaschke condition $\sum_n(1-|z_n|) < \infty$, and hence $f \in F \setminus N$. See [11, Example 3.6] for a similar discussion.

Solutions of finite order of growth. The next example shows that the space U defined in [1] coincides with the space of all $\mathcal{H}$-functions, and therefore [1, Theorem 1] follows by Theorem B.

Example 5.4. For $p > 0$, let U_p denote the space of all analytic functions f in D for which

$$\int_0^{2\pi}\int_0^r |f(se^{i\theta})|\,dsd\theta = O\left(\frac{1}{(1-r)^p}\right).$$

It is shown that

$$U = \bigcup_{p>0} U_p = \bigcup_{p>0} H_p^\infty, \tag{5.3}$$

where U is the space defined in [1]. Assume first that $f \in U_p$. By the mean value property of analytic functions, the inequality

$$|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi}|f(z+re^{i\theta})|\,d\theta$$

holds for $0 < r < 1 - |z|$. Multiplying both sides by r and integrating, it follows that

$$\frac{1}{2}\left(\frac{1-|z|}{2}\right)^2 |f(z)| \leq \frac{1}{2\pi}\int_0^{\frac{1-|z|}{2}}\int_0^{2\pi} |f(z+re^{i\theta})|rdrd\theta$$
$$\leq \frac{1}{2\pi}\int_0^{\frac{1+|z|}{2}}\int_0^{2\pi} |f(re^{i\theta})|drd\theta,$$

thus

$$|f(z)|(1-|z|)^2 \leq \frac{4}{\pi}\int_0^{\frac{1+|z|}{2}}\int_0^{2\pi} |f(re^{i\theta})|drd\theta = O\left(\frac{1}{(1-|z|)^p}\right),$$

and therefore $f \in H^\infty_{p+2}$. Moreover, if $f \in H^\infty_{p+2}$, then

$$\int_0^{2\pi}\int_0^r |f(se^{i\theta})|dsd\theta = O\left(\int_0^r \frac{ds}{(1-s)^{p+2}}\right) = O\left(\frac{1}{(1-|z|)^{p+1}}\right),$$

that is, $f \in U_{p+1}$. Thus $U_p \subset H^\infty_{p+2} \subset U_{p+1}$, and so (5.3) follows.

If the coefficient $a(z)$ of (1.2) belongs to the weighted Bergman space $A^{\frac{1}{2}}_\alpha$, $\alpha \geq 0$, then all solutions are of order of growth at most α by Theorem 2.3. Conversely, if all solutions of (1.2) are analytic and of order of growth at most α, then $a(z)$ need not belong to $A^{\frac{1}{2}}_\alpha$, as is seen next.

Example 5.5. Let $\alpha \geq 0$. For a fixed branch, the functions

$$f_j(z) = (1-z)^{\frac{\alpha+2}{2}} \exp\left((-1)^j\left(\frac{1}{1-z}\right)^{\alpha+1}\right), \quad j = 1, 2,$$

are linearly independent solutions of (1.2), where

$$a(z) = -\frac{1}{4}\frac{\alpha(\alpha+2)}{(1-z)^2} - \frac{(\alpha+1)^2}{(1-z)^{2\alpha+4}}.$$

Since $\rho(f_1) = \rho(f_2) = \alpha$, all non-trivial solutions are of order of growth at most α. However, there are constants $C_1, C_2 > 0$, depending only on α, such that

$$\int_{D(0,r)} |a(z)|^{\frac{1}{2}}(1-|z|^2)^\alpha\, d\sigma_z \geq (\alpha+1)\int_{D(0,r)} \frac{(1-|z|^2)^\alpha}{|1-z|^{\alpha+2}}\, d\sigma_z - C_2$$
$$\geq C_1\int_0^r \frac{ds}{1-s} - C_2$$
$$= C_1 \log\frac{1}{1-r} - C_2, \quad 0 < r < 1,$$

and it follows that $a \notin A^{\frac{1}{2}}_\alpha$.

6. Non-homogeneous equations

Consider the non-homogeneous differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = H(z) \tag{6.1}$$

and the associated homogeneous differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \tag{6.2}$$

where $a_0(z),\dots,a_{k-1}(z)$ and $H(z)$ are analytic in D. If $\{f_1,\dots,f_k\}$ is a fundamental system of solutions of (6.2), then all solutions of (6.1) are of the form

$$f = C_1 f_1 + \cdots + C_k f_k + f_p, \tag{6.3}$$

where $C_1,\dots,C_k \in \mathbb{C}$ are arbitrary and f_p is the particular solution of (6.1). Moreover, by [17, p. 145],

$$f_p = B_1 f_1 + \cdots + B_k f_k, \tag{6.4}$$

where $B_1,\dots,B_k$ are analytic functions in D such that

$$B_j' = H G_j(f_1,\dots,f_k) e^{\Phi}, \quad j = 1,\dots,k, \tag{6.5}$$

where each $G_j(f_1,\dots,f_k)$ is a differential polynomial in $f_1,\dots,f_k$ with constant coefficients, and $\Phi(z)$ is a primitive function of $a_{k-1}(z)$.

Direct problem. The first result is a generalization of Theorem 2.3 for non-homogeneous equations.

Proposition 6.1. *Let $0 \le \alpha < \infty$. Suppose that $a_j \in A_\alpha^{\frac{1}{k-j}}$ for all $j = 0,\dots,k-1$, and that $H(z)$ is analytic in D and of order of growth at most α. Then all solutions of* (6.1) *are of order of growth at most α.*

Proof. Let $\{f_1,\dots,f_k\}$ be a fundamental system of solutions of (6.2). By Theorem 2.3 (or by Theorem 2.1 with $p = 1$ if $\alpha = 0$), the functions $f_1,\dots,f_k$ are of order of growth at most α. By (6.3) it is enough to show that f_p in (6.4) is of order of growth at most α. Moreover, since $\rho(g) = \rho(g')$ for any analytic function g in D, it suffices to show that the functions $B_1',\dots,B_k'$ in (6.5) are of order of growth at most α. The only non-trivial step is to show that $\rho(e^{\Phi}) \le \alpha$, where $\Phi(z)$ is a primitive function of $a_{k-1}(z)$. Since

$$T(r, e^{\Phi}) = m(r, e^{\Phi}) \le \frac{1}{2\pi}\int_0^{2\pi} |\Phi(re^{i\theta})|\, d\theta, \tag{6.6}$$

it is enough to show that $\Phi \in H_\alpha^1$. Now, for $z = re^{i\theta} \in D$,

$$\Phi(z) = \int_0^z a_{k-1}(\zeta)\, d\zeta + \Phi(0), \tag{6.7}$$

where the integration is taken along the line segment $[0,z]$. Hence $\zeta = se^{i\theta}$ and $d\zeta = e^{i\theta} ds$, where $0 \le s \le r$. Further, by (6.7) and [10, Lemma 4.6], it follows that

$$\begin{aligned}(1-r^2)^\alpha \int_0^{2\pi} |\Phi(re^{i\theta})|\, d\theta &\le (1-r^2)^\alpha \int_0^{2\pi}\int_0^r |a_{k-1}(se^{i\theta})|\, ds d\theta + o(1)\\ &\le C\int_D |a_{k-1}(z)|(1-|z|^2)^\alpha\, d\sigma_z + o(1)\end{aligned}$$

for some constant $C > 0$. Since $a_{k-1} \in A_\alpha^1$ by the assumption, $\Phi \in H_\alpha^1$, and the assertion follows. □

A simple modification of the proof of Proposition 6.1 shows that an analogous generalization of Proposition 2.4 for non-homogeneous equations hold. While generalizing Corollary 2.6(3), the essential step is to show that the primitive function

$\Phi(z)$ of $a_{k-1} \in H^1_{\alpha+1}$ satisfies $\rho(e^\Phi) \le \alpha$, where $\alpha > 0$. By (6.6) it is only needed to show that $\Phi \in H^1_\alpha$. But this follows by

$$\begin{aligned}(1-r^2)^\alpha \int_0^{2\pi} |\Phi(re^{i\theta})|\, d\theta &\le (1-r^2)^\alpha \int_0^{2\pi}\int_0^r |a_{k-1}(re^{i\theta})|\, ds\, d\theta + o(1)\\ &\le (1-r^2)^\alpha \int_0^r \frac{C}{(1-s^2)^{\alpha+1}}\, ds + o(1) \le C,\end{aligned}$$

where (6.7) has been used.

Next, the problem of when all solutions of (6.1) belong to a given function space, denoted by FS, is discussed. The reasoning in the proof of Proposition 6.1 works, provided that FS has the following properties:

(1) If $f, g \in FS$, then $f + g \in FS$.
(2) If $f, g \in FS$, then $fg \in FS$.
(3) If $f \in FS$, then $f' \in FS$.

All function spaces satisfy property (1). Moreover, note that:

- If $FS = H^\infty$, then (2) holds while (3) does not.
- If $FS = H^p$, $0 < p < \infty$, then neither (2) nor (3) holds.
- If $FS = N$, then (2) holds while (3) does not; see [4, p. 106] for the requirement (3).
- If $FS = F$, then both (2) and (3) hold; see [3, Lemma 5.3] for the requirement (3).

To generalize Corollary 2.6(2) for the non-homogeneous equations, the essential step is, as earlier, to prove that the primitive function $\Phi(z)$ of $a_{k-1} \in H^1_1$ satisfies $e^\Phi \in F$. This is, however, clearly true.

Inverse problem. This section is completed by considering the inverse problem when all solutions either are of order of growth at most α or belong to F.

(i) Suppose that all solutions of (6.1) are analytic in D and of order at most α. Using the notation introduced in the beginning of the present section, the functions $f_1, \ldots, f_k, f_p$ are analytic in D and of order at most α. It remains to show that $H(z)$ is of order at most α. Suppose on the contrary that $\rho(H) > \alpha$. Then, as the coefficients $a_0(z), \ldots, a_{k-1}(z)$ of (6.1) are $\mathcal{H}$-functions, and hence of order zero, elementary facts from Nevanlinna theory applied to (6.1) show that $\rho(f_p) > \alpha$, which is a contradiction.
(ii) Suppose then that $f_1, \ldots, f_k, f_p \in F$. Noting that $\mathcal{H}$-functions are non-admissible and that F is a differential field, a reasoning similar to the one in (i) above shows that $H \in F$.

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