

SMALL BALL PROBABILITIES FOR GAUSSIAN RANDOM FIELDS AND TENSOR PRODUCTS OF COMPACT OPERATORS

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ABSTRACT. We find the logarithmic L_2 -small ball asymptotics for a large class of zero mean Gaussian fields with covariances having the structure of “tensor product”. The main condition imposed on marginal covariances is the regular behavior of their eigenvalues at infinity that is valid for a multitude of Gaussian random functions including the fractional Brownian sheet, Ornstein – Uhlenbeck sheet, etc. So we get the far-reaching generalizations of well-known results by Csáki (1982) and by Li (1992). Another class of Gaussian fields considered is the class of additive fields studied under the supremum-norm by Chen and Li (2003). Our theorems are based on new results on spectral asymptotics for the tensor products of compact self-adjoint operators in Hilbert space which are of independent interest.

1. INTRODUCTION

The theory of small deviations of random functions is currently in intensive development. In this paper we address small deviations of Gaussian random fields in L_2 -norm.

Suppose we have a Gaussian random field $X(x)$, $x \in \Omega \subset \mathbb{R}^d$, with zero mean and covariance function $G_X(x, y) = EX(x)X(y)$ for $x, y \in \Omega$. Let ρ be a summable nonnegative function on Ω . Set

$$(1.1) \quad \|X\|_\rho = \left(\int_\Omega X^2(x)\rho(x)dx \right)^{1/2}$$

(the subscript ρ will be omitted when $\rho \equiv 1$) and consider

$$Q(X, \rho; \varepsilon) = P\{\|X\|_\rho \leq \varepsilon\}.$$

The problem is to define the behavior of $Q(X, \rho; \varepsilon)$ as $\varepsilon \rightarrow 0$. Note that the case of general weight ρ can be easily reduced to the case $\rho \equiv 1$ replacing X by the Gaussian field $X\sqrt{\rho}$. However, it is more convenient to consider the general case.

The study of the small deviation problem was initiated by Sytaya [Sy] and continued by many scholars (see, for example, [Z1], [Z2], [DH-JS], [I], [Cs], [Li]). The history of the problem and the summary of main results can be found in two excellent reviews [Lf] and [LiS]. These reviews contributed much to the considerable

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progress lately obtained. Latest results on L_2 -small deviation asymptotics for one-dimensional Gaussian random functions related to Brownian motion can be found in [BNO], [GHT]–[GHLT2], [Na1]–[NaNi1]. Some advancements connected to fractional Brownian motion can be found in [Br] and [NaNi2].

Small deviations of Gaussian random functions defined on a set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, are studied much less. We refer to the classical result by Csáki [Cs] who investigated the Brownian sheet $\mathbb{W}_d = \mathbb{W}(x_1, \dots, x_d)$ on the unit cube. His achievements were later extended by Li [Li] to include a larger class of random fields; see also [GL1]. The small deviation asymptotics for the Brownian sheet in the simplest case $d = 2$, as given in [Cs], is as follows:

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln^{-2}(1/\varepsilon^2) \ln P\{\|\mathbb{W}_2\| \leq \varepsilon\} = -\frac{1}{8\pi^2}.$$

In [NaNi2] the logarithmic small deviation asymptotics was described for fractional and ordinary Lévy's random fields on arbitrary domain $\Omega \subset \mathbb{R}^d$.

According to the well-known Karhunen – Loève expansion which is valid also in the multiparameter case, see [Ad], we have in distribution

$$(1.3) \quad \|X\|_\rho^2 = \int_{\Omega} X^2(x) \rho(x) dx = \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where ξ_n , $n \in \mathbb{N}$, are independent standard normal r.v.'s, and $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum_n \lambda_n < \infty$ are the eigenvalues of the integral equation

$$(1.4) \quad \lambda f(x) = \int_{\Omega} G_X(x, y) \sqrt{\rho(x)\rho(y)} f(y) dy, \quad x \in \Omega.$$

Thus we arrive at the equivalent problem of studying the asymptotic behavior as $\varepsilon \rightarrow 0$ of $P\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2\}$. The answer heavily depends on the available information about the eigenvalues λ_n .

In this paper we study the small deviation asymptotics of a vast and important class of Gaussian random fields having the covariance of “tensor product” type. It means that this covariance can be decomposed in a product of “marginal” covariances depending on different arguments. The classical examples of such fields are the Brownian sheet, the Brownian pillow and the Brownian pillow-slip (Kiefer field), see, e.g., [vVW]. Less known examples can be found in [Kh] and [LNN]. The notion of tensor products of Gaussian processes or Gaussian measures was known long ago, see [Ca], [CC] and [Ch] for a more general approach. Such Gaussian fields are also studied in related domains of mathematics, see, e.g., [PW], [RWW], [GHP], [LuP1] and [LuP2].

Suppose we have two Gaussian random functions $X(x)$, $x \in \mathbb{R}^m$, and $Y(y)$, $y \in \mathbb{R}^n$, with zero means and covariances $G_X(x, u)$, $x, u \in \mathbb{R}^m$, and $G_Y(y, v)$, $y, v \in \mathbb{R}^n$, correspondingly. Consider the new Gaussian function $Z(x, y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, which has zero mean and the covariance

$$(1.5) \quad G_Z((x, y), (u, v)) = G_X(x, u)G_Y(y, v).$$

Such Gaussian function obviously exists, and the integral operator with the kernel (1.5) is the tensor product of two “marginal” integral operators with the kernels $G_X(x, u)$ and $G_Y(y, v)$. Therefore we use in the sequel the notation

$$Z = X \otimes Y$$

and we call the Gaussian field Z the *tensor product* of the fields X and Y sometimes using the shortened term *field-product*. The generalization to the multivariate case when obtaining the fields $\bigotimes_{j=1}^d X_j$ is straightforward.

In this way we can construct a large set of Gaussian random functions. The problem of asymptotic behavior for the spectrum of their covariances given we know such behavior for marginal kernels is nontrivial. One often observes for the random fields (see, e.g., [PW] and [Li]) that the eigenvalues of marginal processes have the “pure” power asymptotics, but the asymptotics for the tensor product contain the slowly varying (logarithmic) factor.

It is natural to generalize the problem and to consider the case when the eigenvalues $\lambda_n^{(j)}$ of marginal covariances have the so-called *regular behavior*:

$$(1.6) \quad \lambda_n^{(j)} \sim \frac{\varphi_j(n)}{n^{p_j}}, \quad n \rightarrow \infty,$$

where $p_j > 1$ are called the *marginal exponents* and φ_j are some *slowly varying functions* (SVFs), in particular, the powers of logarithmic function. The assumption of regular behavior is common in recent papers using spectral asymptotics, see, e.g., [BS3], [De], [LuP1] and [LuP2].

Gaussian processes with regular spectral asymptotics can be easily constructed. Let $X(x)$, $x \in [0, 1]$, be the *stationary* Gaussian process with zero mean and covariance

$$G_X(x, y) = g(|x - y|),$$

where g is a smooth function on \mathbb{R}_+ , and

$$g(x) - g(0) = -x^\alpha \varphi(1/x) + O(x^\beta), \quad x \rightarrow 0+,$$

where $0 < \alpha < 2$, $\beta > \alpha$, while φ is a SVF. Then, similarly to [BS1], Appendix 7, one can show that

$$\lambda_n \sim \frac{2 \sin \frac{\pi\alpha}{2} \Gamma(\alpha + 1)}{\pi^{\alpha+1}} \cdot \frac{\varphi(n)}{n^{1+\alpha}}, \quad n \rightarrow \infty.$$

We also note that some simple Gaussian processes with nonregular spectral asymptotics were considered in [Na2].

The problem of small ball behavior for fields-products when the eigenvalues of their factors are decreasing as in (1.6) is considerably more difficult than in the case of purely power decrease. To solve it, we develop a new method. It is based on abstract theorems describing the spectral asymptotics of tensor products and of sums of tensor products for self-adjoint operators in Hilbert space proved below. They contain as a special case the results from [PW], see also [LuP2], Proposition 3.4, where the factors of the tensor product have purely power decrease of eigenvalues.

Two essentially different cases arise in our theorems on spectral asymptotics of tensor products. The first case deals with the equal marginal exponents $p_i \equiv p$ of the factors in the field-product like in the case of a Brownian sheet or a fractional Brownian sheet with same Hurst parameters.

The second one deals with the different marginal exponents of these factors. The situation there is more delicate. The order of logarithmic small ball asymptotics is defined by the “factor” with the *slowest* marginal exponent. However, in order to find the logarithmic small ball constants we usually need the *exact form of the*

spectrum for the “factors” with fast decrease. Similar effects were discovered in the particular case in [Li], Example 2.

The examples of Gaussian processes with exactly known spectrum are rather numerous. We mention here the Brownian motion, the Brownian bridge, the centered Brownian motion, the centered Brownian bridge and some other processes, see §6. If one such process serves as the fast factor, then the logarithmic small ball constant of the slow factor should be multiplied by the value of Riemann zeta-function in a particular point.

It is worth mentioning that the parallel development of small deviation asymptotics for Gaussian fields in the *supremum norm* also came across the substantial difference between fractional Brownian sheets with equal and distinct factors, see, e.g., [MSh], [BL], [KLn], [GL1] and [GL2].

Recently Fill and Torcaso [FT] obtained the *exact* asymptotics of small deviations in L_2 -norm for the Brownian sheet and its integrated counterparts. Their original approach is based on the Mellin transform and is entirely different from our method. It is not clear if their approach makes it possible to obtain small deviation results for a vast class of Gaussian fields-products studied in the present paper when the eigenvalues satisfy (1.6). Moreover, our results are applicable for the weighted L_2 -norm in the case of general (continuous) weights on the unit cube in \mathbb{R}^d . This seems to be beyond the reach of the method using Mellin transform.

Another type of Gaussian fields we consider is the class of *additive processes*, see [CL]. For short, we often call it *fields-sums*. Let $X_j(x_j)$, $1 \leq j \leq d$, $x_j \in [0, 1]^{\ell_j}$, be independent random functions. The additive process $\mathbb{X}(x)$ is defined as

$$(1.7) \quad \mathbb{X}(x) = \mathbb{X}(x_1, \dots, x_d) = \sum_{j=1}^d X_j(x_j), \quad x = (x_1, \dots, x_d) \in [0, 1]^D,$$

where $D = \sum_{j=1}^d \ell_j$.

There are various motivations for the study of additive processes: they are relatively simple approximations to more complicated multiparameter processes and appear, e.g., in the theory of intersections and selfintersections of Brownian processes. Some small deviation results with respect to supremum norm were proved in [CL]. The results in the L_2 -norm obtained below appear for the first time.

The structure of the paper is as follows. First, in §2 we present some auxiliary information about SVFs. The results are partly borrowed from [Se] but some of them seem to be new such as the useful properties of Mellin convolution of SVFs.

Next, in §3 we prove new results on the spectral asymptotics for tensor products of compact self-adjoint operators in Hilbert spaces and for sums of such tensor products. These results are adapted for the applications to small ball asymptotics but are of independent interest.

In §4 we derive the logarithmic small ball asymptotics for Gaussian random functions with regularly varying eigenvalues. In particular, we evaluate the small ball constants for the special rate of eigenvalues decrease, namely

$$\lambda_n \sim C^* \frac{\ln^\sigma(n+1)}{n^p}, \quad \sigma \in \mathbb{R}, \quad p > 1, \quad n \rightarrow \infty.$$

The case $\sigma = 0$ is well known while in the presence of logarithmic factor the results are new. We also obtain a weaker result (namely, the *double logarithmic asymptotics*) for the special case $p = 1$.

We then apply the elaborated theory to various concrete examples of Gaussian random fields. We treat in §5 the fields-products with equal marginal exponents and in §6 the fields-products with distinct marginal exponents. The fields-sums are studied in §7.

In §8 we substantially enlarge the class of weights ρ in (1.1) for which the results of §5 are still valid. We prove that instead of the products of univariate weights one can take any continuous weight on the unit cube in \mathbb{R}^d . Finally, in the same section we consider various generalizations to the vector-valued random fields.

A preliminary version of this paper with considerably less general statements was published as a preprint [KNN]. Similar results using a more probabilistic approach were obtained in [GL2].

2. AUXILIARY INFORMATION ON SVFs

We recall that a positive function $\varphi(\tau)$, $\tau > 0$, is called a *slowly varying* function (SVF) at infinity if for any $c > 0$

$$(2.1) \quad \varphi(c\tau)/\varphi(\tau) \rightarrow 1 \quad \text{as } \tau \rightarrow +\infty.$$

Remark 1. Clearly, any smooth positive function φ satisfying $\tau\varphi'(\tau)/\varphi(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$ is slowly varying. This test shows that the functions equal to $\ln^\alpha(\tau)$, $\ln^\alpha(\tau)\ln^{\alpha_1}(\ln(\tau))$, etc. for $\tau \gg 1$ are slowly varying. However, one must keep in mind that SVFs can be less regular (see [BS3]): the SVF $\varphi(\tau) = 2 + \sin(\ln^\alpha(\tau))$, $\alpha < 1$, is bounded and separated from zero, but it has no limit as $\tau \rightarrow +\infty$; the SVF $\varphi(\tau) = 1 + \ln(\ln(\tau))\sin^2(\ln(\ln(\ln(\tau))))$ is unbounded but does not tend to infinity as $\tau \rightarrow +\infty$.

We need some simple properties of SVFs. Their proofs can be found, for example, in [Se].

Proposition 2.1. *Let φ be a SVF. Then the following statements are true:*

1. *The relation (2.1) is uniform with respect to $c \in [a, b]$ for $0 < a < b < +\infty$.*
2. *There exists τ_0 such that*

$$(2.2) \quad \varphi(\tau) = \eta(\tau) \cdot \exp\left(\int_{\tau_0}^{\tau} \frac{\theta(\sigma)}{\sigma} d\sigma\right), \quad \tau > \tau_0.$$

The function $\eta(\tau)$ has a nonzero limit as $\tau \rightarrow +\infty$ while $\theta(\sigma) \rightarrow 0$ as $\sigma \rightarrow +\infty$. Later we assume, without loss of generality, that $\tau_0 = 1$.

3. *The function $\tau \mapsto \tau^p\varphi(\tau)$, $p \neq 0$, is monotone for large τ , and its inverse function is*

$$\tau \mapsto \tau^{1/p}\phi(\tau), \quad p > 0; \quad \tau \mapsto \tau^{1/p}\phi(1/\tau), \quad p < 0,$$

where ϕ is a SVF.

4. *There exists an equivalent SVF $\psi \in \mathcal{C}^2(\mathbb{R}_+)$ (i.e. $\psi(\tau)/\varphi(\tau) \rightarrow 1$ as $\tau \rightarrow +\infty$) such that*

$$(2.3) \quad \tau \cdot (\ln(\psi))'(\tau) \rightarrow 0, \quad \tau^2 \cdot (\ln(\psi))''(\tau) \rightarrow 0, \quad \tau \rightarrow +\infty.$$

5. If

$$(2.4) \quad \int_1^\infty \varphi(\tau) \frac{d\tau}{\tau} < \infty,$$

then $\varphi(\tau) \rightarrow 0$ as $\tau \rightarrow +\infty$.

We define the *Mellin convolution* of two SVFs φ and ψ as

$$(\varphi * \psi)(\tau) = \int_1^\tau \varphi(\sigma) \psi(\tau/\sigma) \frac{d\sigma}{\sigma} = h_{\varphi, \psi}(\tau) + h_{\psi, \varphi}(\tau),$$

where

$$h_{\varphi, \psi}(\tau) = \int_1^{\sqrt{\tau}} \varphi(\sigma) \psi(\tau/\sigma) \frac{d\sigma}{\sigma}.$$

Theorem 2.2. *The following statements are true:*

1. If (2.4) holds, and

$$\int_1^\infty \varphi(\sigma) m_\psi(\sigma) \frac{d\sigma}{\sigma} < \infty,$$

where

$$m_\psi(\sigma) = \sup_{\tau > \sigma^2} \frac{\psi(\tau/\sigma)}{\psi(\tau)},$$

then

$$(2.5_{\varphi\psi}) \quad h_{\varphi, \psi}(\tau) = \psi(\tau) \int_1^\infty \varphi(\sigma) \frac{d\sigma}{\sigma} \cdot (1 + o(1)), \quad \tau \rightarrow +\infty.$$

2. If $\int_1^\infty \varphi(\sigma) \frac{d\sigma}{\sigma} = \infty$, then $\psi(\tau) = o(h_{\varphi, \psi}(\tau))$ as $\tau \rightarrow +\infty$.

3. If $\psi(\tau) = \psi_1(\tau) \cdot (1 + o(1))$, $\tau \rightarrow +\infty$, then

$$h_{\varphi, \psi}(\tau) = h_{\varphi, \psi_1}(\tau) \cdot (1 + o(1)), \quad \tau \rightarrow +\infty.$$

If, in addition, $\int_1^\infty \psi(\sigma) \frac{d\sigma}{\sigma} = \infty$, then

$$h_{\psi, \varphi}(\tau) = h_{\psi_1, \varphi}(\tau) \cdot (1 + o(1)), \quad \tau \rightarrow +\infty.$$

4. $h_{\varphi, \psi}$ is a SVF.

Proof. 1. We have

$$h_{\varphi, \psi}(\tau) = \psi(\tau) \int_1^{\sqrt{\tau}} \varphi(\sigma) \frac{\psi(\tau/\sigma)}{\psi(\tau)} \frac{d\sigma}{\sigma}.$$

By Lebesgue Dominated Convergence Theorem, the statement follows.

2. This statement follows from elementary inequality

$$h_{\varphi, \psi}(\tau) > \int_1^a \varphi(\sigma) \psi(\tau/\sigma) \frac{d\sigma}{\sigma} = \psi(\tau) \int_1^a \varphi(\sigma) \frac{d\sigma}{\sigma} \cdot (1 + o(1)), \quad \tau \rightarrow +\infty.$$

3. The first property is quite elementary. To prove the second one we note that due to 2 for any $a > 1$

$$h_{\psi,\varphi}(\tau) = \int_a^{\sqrt{\tau}} \psi(\sigma)\varphi(\tau/\sigma) \frac{d\sigma}{\sigma} \cdot (1 + o(1)), \quad \tau \rightarrow \infty.$$

Given $\varepsilon > 0$, one can take a so large that $1 - \varepsilon < \psi(\sigma)/\psi_1(\sigma) < 1 + \varepsilon$ for $\sigma > a$, and the statement follows.

4. Due to 3 and to Proposition 2.1, part 4, we can assume ψ smooth. We have

$$\tau h'_{\varphi,\psi}(\tau) = \frac{1}{2} \varphi(\sqrt{\tau})\psi(\sqrt{\tau}) + \int_1^{\sqrt{\tau}} \varphi(\sigma)\psi'(\tau/\sigma) \frac{\tau}{\sigma} \cdot \frac{d\sigma}{\sigma}.$$

For any $a > 1$ and $\tau > a^2$

$$h_{\varphi,\psi}(\tau) > \int_{a^{-1}\sqrt{\tau}}^{\sqrt{\tau}} \varphi(\sigma)\psi(\tau/\sigma) \frac{d\sigma}{\sigma} = \int_{a^{-1}}^1 \varphi(\sqrt{\tau}\sigma)\psi(\sqrt{\tau}/\sigma) \frac{d\sigma}{\sigma}.$$

By Proposition 2.1, part 1, the integrand converges uniformly as $\tau \rightarrow +\infty$, and we obtain

$$h_{\varphi,\psi}(\tau) > \varphi(\sqrt{\tau})\psi(\sqrt{\tau}) \ln(a) \cdot (1 + o(1)).$$

Hence

$$\varphi(\sqrt{\tau})\psi(\sqrt{\tau}) = o(h_{\varphi,\psi}(\tau)), \quad \tau \rightarrow +\infty.$$

Next, due to (2.3) we have

$$\left| \int_1^{\sqrt{\tau}} \varphi(\sigma)\psi'(\tau/\sigma) \frac{\tau}{\sigma} \cdot \frac{d\sigma}{\sigma} \right| \leq \sup_{\lambda \geq \sqrt{\tau}} \left| \frac{\lambda\psi'(\lambda)}{\psi(\lambda)} \right| \cdot h_{\varphi,\psi}(\tau) = o(h_{\varphi,\psi}(\tau)), \quad \tau \rightarrow +\infty.$$

So, $\tau h'_{\varphi,\psi}(\tau) = o(h_{\varphi,\psi}(\tau))$, and 4 follows. □

Remark 2. We underline that the condition (2.4) does not imply (2.5 _{$\varphi\psi$}) without additional assumptions. Also in the general case $h_{\psi,\varphi}(\tau) \neq h_{\psi_1,\varphi}(\tau) \cdot (1 + o(1))$ as $\tau \rightarrow +\infty$.

Example 1. Let $\varphi(\tau) = (1 + \ln(\tau))^{\varkappa_1}$, $\psi(\tau) = (1 + \ln(\tau))^{\varkappa_2}$. One can check by direct calculation that, as $\tau \rightarrow +\infty$,

$$(\varphi * \psi)(\tau) \sim \begin{cases} \mathbf{B}(\varkappa_1 + 1, \varkappa_2 + 1)(1 + \ln(\tau))^{\varkappa_1 + \varkappa_2 + 1}, & \varkappa_1 > -1, \varkappa_2 > -1; \\ \ln(\ln(\tau)) \cdot (1 + \ln(\tau))^{\varkappa_2}, & \varkappa_1 = -1, \varkappa_2 > -1; \\ \frac{1}{|\varkappa_1 + 1|} \cdot (1 + \ln(\tau))^{\varkappa_2}, & \varkappa_1 < -1, \varkappa_1 < \varkappa_2; \\ 2 \ln(\ln(\tau)) \cdot (1 + \ln(\tau))^{-1}, & \varkappa_1 = \varkappa_2 = -1; \\ \frac{2}{|\varkappa_1 + 1|} \cdot (1 + \ln(\tau))^{\varkappa_1}, & \varkappa_1 = \varkappa_2 < -1, \end{cases}$$

where \mathbf{B} is the Euler beta-function. Note that in the last two cases the factor 2 arises because the functions $h_{\varphi,\psi}(\tau)$ and $h_{\psi,\varphi}(\tau)$ have the same asymptotics.

3. SPECTRAL ASYMPTOTICS FOR TENSOR PRODUCTS OF COMPACT SELF-ADJOINT OPERATORS

Let $\mathcal{T} = \mathcal{T}^* \geq 0$ be a compact self-adjoint nonnegative operator in a Hilbert space H ; let $\lambda_n = \lambda_n(\mathcal{T})$ be its positive eigenvalues arranged in a nondecreasing sequence and repeated according to their multiplicity.

In spectral theory it is common and convenient considering the counting function

$$\mathcal{N}(t) = \mathcal{N}(t, \mathcal{T}) = \#\{n : \lambda_n(\mathcal{T}) > t\}.$$

We formulate some of its properties which are necessary for the subsequent arguments. Their proof is quite standard, and we omit it.

Lemma 3.1. *The following statements are true:*

1. *The relation $\mathcal{N}(t, \mathcal{T}) = O(t^{-1/p})$, $t \rightarrow 0+$, is equivalent to $\lambda_n(\mathcal{T}) = O(n^{-p})$, $n \rightarrow \infty$.*

2. *The asymptotics*

$$(3.1) \quad \mathcal{N}(t, \mathcal{T}) \sim \mathcal{N}_{as}(t) \equiv a \cdot \frac{\varphi(1/t)}{t^{1/p}}, \quad t \rightarrow 0+,$$

(here $p > 0$ while φ is a SVF) is equivalent to the spectral asymptotics

$$\lambda_n(\mathcal{T}) \sim \frac{\psi(n)}{n^p}, \quad n \rightarrow \infty,$$

where ψ is also a SVF.

3. *Let \mathcal{T} have the counting function with the asymptotics (3.1). Then the convergence of the integral $\int_1^\infty \varphi(\tau) \frac{d\tau}{\tau}$ is equivalent to the convergence of the series $\sum_n \lambda_n^{1/p}(\mathcal{T})$.*

Example 2. The asymptotics

$$\mathcal{N}(t, \mathcal{T}) \sim a \frac{\ln^\varkappa(1/t)}{t^{1/p}}, \quad t \rightarrow 0+,$$

with $p > 0$, is equivalent to the spectral asymptotics

$$\lambda_n(\mathcal{T}) \sim a^p p^{p\varkappa} \frac{\ln^{p\varkappa}(n+1)}{n^p}, \quad n \rightarrow \infty.$$

Note that in view of Proposition 2.1, part 4, any SVF arising in an asymptotic formula can be assumed smooth. Moreover, we can take $\eta \equiv 1$ in the representation formula (2.2).

We introduce the functionals

$$\Delta_{p,\varphi}(\mathcal{T}) = \limsup_{t \rightarrow 0+} \frac{t^{1/p}}{\varphi(1/t)} \cdot \mathcal{N}(t, \mathcal{T}), \quad \delta_{p,\varphi}(\mathcal{T}) = \liminf_{t \rightarrow 0+} \frac{t^{1/p}}{\varphi(1/t)} \cdot \mathcal{N}(t, \mathcal{T}).$$

The asymptotics (3.1) means that $\Delta_{p,\varphi}(\mathcal{T}) = \delta_{p,\varphi}(\mathcal{T})$ are finite and are not equal to zero.

Note that due to (2.1) for any positive c we have

$$(3.2) \quad \Delta_{p,\varphi}(c\mathcal{T}) = c^{1/p} \Delta_{p,\varphi}(\mathcal{T}), \quad \delta_{p,\varphi}(c\mathcal{T}) = c^{1/p} \delta_{p,\varphi}(\mathcal{T}).$$

Theorem 3.2. *Let operator \mathcal{T} have the counting function with asymptotics (3.1). Let $\tilde{\mathcal{T}}$ be a compact self-adjoint nonnegative operator in a Hilbert space \tilde{H} with*

$$(3.3) \quad \tilde{\mathcal{N}}(t) \equiv \mathcal{N}(t, \tilde{\mathcal{T}}) = O(t^{-1/\tilde{p}}), \quad t \rightarrow 0+, \quad \tilde{p} > p.$$

Then the operator $\mathcal{T} \otimes \tilde{\mathcal{T}}$ in the space $H \otimes \tilde{H}$ satisfies the relation

$$(3.1) \quad \mathcal{N}_{\otimes}(t) \equiv \mathcal{N}(t, \mathcal{T} \otimes \tilde{\mathcal{T}}) \sim a \cdot \frac{\varphi(1/t)}{t^{1/p}} \sum_n \lambda_n^{1/p}(\tilde{\mathcal{T}}), \quad t \rightarrow 0+.$$

Proof. As the eigenvalues of $\mathcal{T} \otimes \tilde{\mathcal{T}}$ are the products of $\lambda_j = \lambda_j(\mathcal{T})$ and $\tilde{\lambda}_k = \lambda_k(\tilde{\mathcal{T}})$ we have

$$\mathcal{N}_{\otimes}(t) = \#\{k, j : \lambda_j \tilde{\lambda}_k > t\} = \sum_k \#\{j : \lambda_j > t/\tilde{\lambda}_k\} = \sum_k \mathcal{N}(t/\tilde{\lambda}_k).$$

Thus, we obtain

$$\frac{t^{1/p}}{\varphi(1/t)} \cdot \mathcal{N}_{\otimes}(t) = \sum_k \left[\frac{(t/\tilde{\lambda}_k)^{1/p}}{\varphi(t/\tilde{\lambda}_k)} \cdot \mathcal{N}(t/\tilde{\lambda}_k) \right] \cdot \frac{\varphi(\tilde{\lambda}_k/t)}{\varphi(1/t)} \cdot \tilde{\lambda}_k^{1/p}.$$

Due to (3.1) the term in square brackets is uniformly bounded and tends to a as $t \rightarrow 0+$. The second quotient in the sum tends to 1 as φ is a SVF.

Next, by Proposition 2.1, part **3**, for any $\varepsilon > 0$ $\tau^\varepsilon \varphi(\tau)$ increases for $\tau > \tau_0(\varepsilon)$. Hence we have the inequality

$$\frac{\lambda^\varepsilon \varphi(\lambda\tau)}{\varphi(\tau)} = \frac{(\lambda\tau)^\varepsilon \varphi(\lambda\tau)}{\tau^\varepsilon \varphi(\tau)} \leq 1$$

for $\lambda\tau > \tau_0(\varepsilon)$, $\lambda \leq 1$. So for any $\varepsilon > 0$ the estimate

$$\frac{\varphi(\tilde{\lambda}_k/t)}{\varphi(1/t)} \leq C(\varepsilon) \tilde{\lambda}_k^{-\varepsilon}$$

holds uniformly for $t < 1$. Using the estimate (3.3) we get

$$\frac{\varphi(\tilde{\lambda}_k/t)}{\varphi(1/t)} \cdot \tilde{\lambda}_k^{1/p} \leq C_1(\varepsilon) k^{-\tilde{p}(p^{-1}-\varepsilon)}.$$

For sufficiently small ε we observe that $\tilde{p}(p^{-1}-\varepsilon) > 1$. This gives the majorant required in the Lebesgue Theorem. Passing to the limit in the sum we get the asymptotics (3.4). □

Remark 3. If $\tilde{\mathcal{T}}$ is a finite-dimensional operator, then the estimate (3.3) holds true for any $\tilde{p} > 0$ and the sum in (3.4) is finite.

Theorem 3.3. *Let the operator \mathcal{T} be as in Theorem 3.2. Let the counting function of the operator $\tilde{\mathcal{T}}$ in the space \tilde{H} have the analogous asymptotics:*

$$(3.5) \quad \tilde{\mathcal{N}}(t) \sim \tilde{\mathcal{N}}_{as}(t) \equiv \tilde{a} \cdot \frac{\tilde{\varphi}(1/t)}{t^{1/p}}, \quad t \rightarrow 0+.$$

Then for any $\varepsilon > 0$ the distribution function $\mathcal{N}_{\otimes}(t)$ satisfies the following estimates:

$$(3.6) \quad \mathcal{N}_{\otimes}(t) \leq \frac{\alpha_{\pm}(\varepsilon)}{t^{1/p}} \cdot \left[S(t, \varepsilon) + \tilde{S}(t, \varepsilon) + \frac{a\tilde{a}}{p} \int_{\alpha_{\mp}(\varepsilon)/\varepsilon}^{\varepsilon\tau} \varphi\left(\frac{\tau}{\sigma}\right) \tilde{\varphi}(\sigma) \frac{d\sigma}{\sigma} \right]$$

uniformly with respect to $t > 0$ (here τ stands for $\alpha_{\pm}(\varepsilon)t^{-1}$). For $\alpha_{\mp}(\varepsilon)/\varepsilon > \varepsilon\tau$ the integral in (3.6) must be omitted.

In (3.6) $\alpha_{\pm}(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0+$, and the functions S, \tilde{S} have the following asymptotics as $t \rightarrow 0+$:

$$(3.7a) \quad S(t, \varepsilon) \sim a\varphi(1/t) \cdot \sum_{\tilde{\lambda}_k \geq \varepsilon} \tilde{\lambda}_k^{1/p},$$

$$(3.7b) \quad \tilde{S}(t, \varepsilon) \sim \tilde{a}\tilde{\varphi}(1/t) \cdot \left(\sum_{\lambda_k \geq \varepsilon} \lambda_k^{1/p} + a\varphi(1/\varepsilon) \right).$$

Proof. We establish the upper bound for $\mathcal{N}_{\otimes}(t)$. The lower estimate can be proved in the same way. We have

$$t^{1/p}\mathcal{N}_{\otimes}(t) = t^{1/p} \sum_k \mathcal{N}(t/\tilde{\lambda}_k) = t^{1/p} \sum_{\tilde{\lambda}_k < \varepsilon} \mathcal{N}(t/\tilde{\lambda}_k) + S(t, \varepsilon),$$

where

$$S(t, \varepsilon) = t^{1/p} \cdot \sum_{\tilde{\lambda}_k \geq \varepsilon} \mathcal{N}(t/\tilde{\lambda}_k).$$

The asymptotics (3.7a) for the last sum follows from Theorem 3.2.

Denote by $\tilde{\mu}$ the inverse function for $\tilde{\mathcal{N}}_{as}$. Then (3.5) implies $\tilde{\lambda}_k/\tilde{\mu}(k) \rightarrow 1$ as $k \rightarrow \infty$, and we have

$$\alpha_{-}(\varepsilon)\tilde{\mu}(k) \leq \tilde{\lambda}_k \leq \alpha_{+}(\varepsilon)\tilde{\mu}(k)$$

for $\tilde{\lambda}_k < \varepsilon$, with $\alpha_{\pm}(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Using monotonicity of \mathcal{N} we get

$$\sum_{\tilde{\lambda}_k < \varepsilon} \mathcal{N}(t/\tilde{\lambda}_k) \leq \sum_{\tilde{\mu}(k) < \varepsilon\alpha_{-}^{-1}(\varepsilon)} \mathcal{N}(t/\alpha_{+}(\varepsilon)\tilde{\mu}(k)),$$

and by monotonicity of the function $k \mapsto \mathcal{N}(t/\alpha_{+}(\varepsilon)\tilde{\mu}(k))$

$$(3.8) \quad t^{1/p} \sum_{\tilde{\lambda}_k < \varepsilon} \mathcal{N}(t/\tilde{\lambda}_k) \leq t^{1/p} \mathcal{N}\left(\frac{\alpha_{-}(\varepsilon)t}{\alpha_{+}(\varepsilon)\varepsilon}\right) + t^{1/p} \int_0^{\varepsilon\alpha_{-}^{-1}(\varepsilon)} \mathcal{N}(t/\alpha_{+}(\varepsilon)\mu)(-d\mathcal{N}_{as}(\mu)).$$

The first term in (3.8) is $O(\varepsilon^{1/p}\varphi(1/t))$. Therefore, adding it to the term $S(t, \varepsilon)$, we obtain the term $\alpha_{+}(\varepsilon)S(t, \varepsilon)$ with $\alpha_{+}(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. Further,

$$-d\mathcal{N}_{as}(\mu) = \tilde{a} \cdot \frac{\tilde{\varphi}(1/\mu)}{\mu^{1+1/p}} \cdot \left(\frac{1}{p} + \tilde{\theta}(1/\mu)\right) d\mu,$$

where $\tilde{\theta}$ is the function from the formula (2.2) for SVF $\tilde{\varphi}$. Since $\tilde{\theta}(1/\mu) \rightarrow 0$ as $\mu \rightarrow 0$, we can estimate the integral term in (3.8) by

$$(3.9) \quad \alpha_{+}(\varepsilon) \frac{\tilde{a}}{p} t^{1/p} \int_0^{\varepsilon\alpha_{-}^{-1}(\varepsilon)} \mathcal{N}(t/\alpha_{+}(\varepsilon)\mu) \frac{\tilde{\varphi}(1/\mu)}{\mu^{1+1/p}} d\mu.$$

Splitting the integral in (3.9) and changing variables we obtain

$$(3.10) \quad \alpha_{+}(\varepsilon) \frac{\tilde{a}}{p} \cdot \int_{\varepsilon}^{\infty} \mathcal{N}(s) \frac{\tilde{\varphi}(\alpha_{+}(\varepsilon)s/t)}{s^{1-1/p}} ds + \alpha_{+}(\varepsilon) \frac{\tilde{a}}{p} t^{1/p} \int_{\alpha_{-}(\varepsilon)/\varepsilon}^{\varepsilon\tau} \mathcal{N}(\sigma/\tau) \frac{\tilde{\varphi}(\sigma)}{\sigma^{1-1/p}} d\sigma.$$

The first integral in (3.10) gives us the term $\tilde{S}(t, \varepsilon)$. By Proposition 2.1, part 1, $\frac{\tilde{\varphi}(\alpha_+(\varepsilon)s/t)}{\tilde{\varphi}(1/t)} \rightarrow 1$ as $t \rightarrow 0$ uniformly with respect to $s \in [\varepsilon, \|T\|]$ (for $s > \|T\|$ the integrand equals 0), and we have

$$\tilde{S}(t, \varepsilon) \sim \tilde{a}\tilde{\varphi}(1/t) \cdot \int_{\varepsilon}^{\infty} \mathcal{N}(s) ds^{1/p}.$$

Integrating by parts we arrive at (3.7b).

Due to (3.1) we can estimate \mathcal{N} in the second integral by $\alpha_+(\varepsilon)\mathcal{N}_{as}$ that gives the integral term in (3.6). □

Theorem 3.4. *Let operators \mathcal{T} and $\tilde{\mathcal{T}}$ be as in Theorem 3.3.*

1. *If*

$$(3.11) \quad \int_1^{\infty} \varphi(\tau) \frac{d\tau}{\tau} = \int_1^{\infty} \tilde{\varphi}(\tau) \frac{d\tau}{\tau} = \infty,$$

then

$$(3.12) \quad \mathcal{N}_{\otimes}(t) \sim \frac{\phi(1/t)}{t^{1/p}}, \quad t \rightarrow 0,$$

where

$$(3.13) \quad \phi(s) = \frac{a\tilde{a}}{p}(\varphi * \tilde{\varphi})(s).$$

2. *If*

$$(3.14) \quad \int_1^{\infty} \varphi(\tau) \frac{d\tau}{\tau} < \infty, \quad \int_1^{\infty} \tilde{\varphi}(\tau) \frac{d\tau}{\tau} = \infty,$$

and the asymptotic formula (2.5 $_{\tilde{\varphi}\varphi}$) is fulfilled, then (3.12) holds true with

$$(3.15) \quad \phi(s) = \tilde{a}\tilde{\varphi}(s) \sum_n \lambda_n^{1/p} + \frac{a\tilde{a}}{p} h_{\tilde{\varphi},\varphi}(s).$$

3. *If*

$$\int_1^{\infty} \varphi(\tau) \frac{d\tau}{\tau} < \infty, \quad \int_1^{\infty} \tilde{\varphi}(\tau) \frac{d\tau}{\tau} < \infty,$$

and the asymptotic formulas (2.5 $_{\varphi\tilde{\varphi}}$), (2.5 $_{\tilde{\varphi}\varphi}$) are fulfilled, then (3.12) holds true with

$$(3.16) \quad \phi(s) = a\varphi(s) \sum_n \tilde{\lambda}_n^{1/p} + \tilde{a}\tilde{\varphi}(s) \sum_n \lambda_n^{1/p}.$$

Proof. **1.** Fix arbitrary $\varepsilon > 0$ and consider the estimates (3.6). Due to relation (3.11) and to Theorem 2.2, part **2**, we have

$$S(t, \varepsilon) = o((\varphi * \tilde{\varphi})(1/t)), \quad \tilde{S}(t, \varepsilon) = o((\varphi * \tilde{\varphi})(1/t)), \quad t \rightarrow 0+.$$

Further, the integral in the right-hand side of (3.6) differs from the convolution $(\varphi * \tilde{\varphi})(\tau)$ by the integrals

$$\int_{\varepsilon\tau}^{\tau} \varphi\left(\frac{\tau}{\sigma}\right)\tilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} = \int_1^{1/\varepsilon} \tilde{\varphi}\left(\frac{\tau}{\sigma}\right)\varphi(\sigma)\frac{d\sigma}{\sigma} \sim \tilde{\varphi}(\tau) \int_1^{1/\varepsilon} \varphi(\sigma)\frac{d\sigma}{\sigma} = o((\varphi * \tilde{\varphi})(\tau)), \quad \tau \rightarrow \infty,$$

$$\int_1^{\alpha_{\mp}/\varepsilon} \varphi\left(\frac{\tau}{\sigma}\right)\tilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} \sim \varphi(\tau) \int_1^{\alpha_{\mp}/\varepsilon} \tilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} = o((\varphi * \tilde{\varphi})(\tau)), \quad \tau \rightarrow \infty,$$

(we recall that $\tau = \alpha_{\pm}(\varepsilon)t^{-1}$).

Due to Theorem 2.2, part 4, $(\varphi * \tilde{\varphi})(\tau) \sim (\varphi * \tilde{\varphi})(1/t)$, and hence

$$\Delta_{p,\phi}(\mathcal{T} \otimes \tilde{\mathcal{T}}) \leq \alpha_+(\varepsilon)\frac{a\tilde{a}}{p}, \quad \delta_{p,\phi}(\mathcal{T} \otimes \tilde{\mathcal{T}}) \geq \alpha_-(\varepsilon)\frac{a\tilde{a}}{p},$$

where ϕ is defined in (3.13). Taking into account that $\alpha_{\pm}(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$, we arrive at (3.12).

2. Fix arbitrary $\varepsilon > 0$ and consider the estimates (3.6). Due to relation (3.14) and Theorem 2.2, part 2, we have

$$S(t, \varepsilon) = o(h_{\tilde{\varphi},\varphi}(1/t)), \quad t \rightarrow 0+.$$

Further, we rewrite the integral in (3.6) as follows:

$$\int_{\alpha_{\mp}(\varepsilon)/\varepsilon}^{\varepsilon\tau} \varphi\left(\frac{\tau}{\sigma}\right)\tilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} = h_{\tilde{\varphi},\varphi}(\tau) - \int_1^{\alpha_{\mp}(\varepsilon)/\varepsilon} \varphi\left(\frac{\tau}{\sigma}\right)\tilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} + h_{\varphi,\tilde{\varphi}}(\tau) - \int_1^{1/\varepsilon} \varphi(\sigma)\tilde{\varphi}\left(\frac{\tau}{\sigma}\right)\frac{d\sigma}{\sigma}.$$

Due to (3.14) and to Theorem 2.2, part 2, the first integral in the right-hand side is $O(\varphi(\tau)) = o(h_{\tilde{\varphi},\varphi}(\tau))$ as $\tau \rightarrow \infty$, while (2.5 $_{\varphi\tilde{\varphi}}$) implies

$$h_{\varphi,\tilde{\varphi}}(\tau) - \int_1^{1/\varepsilon} \varphi(\sigma)\tilde{\varphi}\left(\frac{\tau}{\sigma}\right)\frac{d\sigma}{\sigma} \sim \tilde{\varphi}(\tau) \int_{1/\varepsilon}^{\infty} \varphi(\sigma)\frac{d\sigma}{\sigma}, \quad \tau \rightarrow \infty.$$

Finally, (3.7b) and Proposition 2.1, part 5, give us

$$\tilde{S}(t, \varepsilon) \sim \tilde{a}\tilde{\varphi}(1/t) \cdot \left(\sum_n \lambda_n^{1/p} + \nu(\varepsilon)\right), \quad t \rightarrow 0,$$

where $\nu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Substituting all these estimates in (3.6) we obtain

$$\mathcal{N}_{\otimes}(t) \leq \alpha_{\pm}(\varepsilon)\frac{\phi(\tau)}{t^{1/p}},$$

where ϕ is defined in (3.15). Due to Theorem 2.2, part 4, $\phi(\tau) \sim \phi(1/t)$, and hence

$$\Delta_{p,\phi}(\mathcal{T} \otimes \tilde{\mathcal{T}}) \leq \alpha_+(\varepsilon), \quad \delta_{p,\phi}(\mathcal{T} \otimes \tilde{\mathcal{T}}) \geq \alpha_-(\varepsilon).$$

Passing to the limit as $\varepsilon \rightarrow 0$ we arrive at (3.12). The proof of part 3 is similar. \square

Example 3. Let

$$\mathcal{N}(t) \sim a \cdot \frac{\ln^{\kappa_1}(1/t)}{t^{1/p}}, \quad \tilde{\mathcal{N}}(t) \sim \tilde{a} \cdot \frac{\ln^{\kappa_2}(1/t)}{t^{1/p}}, \quad t \rightarrow 0+.$$

Then the asymptotic formula (3.12) holds true with

$$\phi(s) = \begin{cases} \frac{a\tilde{a}}{p} \cdot \mathbf{B}(\varkappa_1 + 1, \varkappa_2 + 1) \ln^{\varkappa_1 + \varkappa_2 + 1}(s), & \varkappa_1 > -1, \varkappa_2 > -1; \\ \frac{a\tilde{a}}{p} \cdot \ln(\ln(s)) \cdot \ln^{\varkappa_2}(s), & \varkappa_1 = -1, \varkappa_2 > -1; \\ \frac{2a\tilde{a}}{p} \cdot \ln(\ln(s)) \ln^{-1}(s), & \varkappa_1 = \varkappa_2 = -1; \\ \tilde{a} \sum_n \lambda_n^{1/p} \cdot \ln^{\varkappa_2}(s), & \varkappa_1 < -1, \varkappa_1 < \varkappa_2; \\ (a \sum_n \tilde{\lambda}_n^{1/p} + \tilde{a} \sum_n \lambda_n^{1/p}) \cdot \ln^{\varkappa_1}(s), & \varkappa_1 = \varkappa_2 < -1. \end{cases}$$

The particular case $\varkappa_1 = 0, \varkappa_2 = 0$ can be extracted from [PW], see also [LuP2].

Later we use the following inequalities for the functionals Δ, δ :

Proposition 3.5. *Let \mathcal{Q} and \mathcal{Q}_1 be compact operators and let φ be a SVF. Then*

$$(3.17a) \quad |\Delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q}|) - \Delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q}_1|)| \leq \Delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q} - \mathcal{Q}_1|);$$

$$(3.17b) \quad |\delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q}|) - \delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q}_1|)| \leq \Delta_{p,\varphi}^{\frac{p}{p+1}}(|\mathcal{Q} - \mathcal{Q}_1|)$$

(we recall that $|\mathcal{Q}| = \sqrt{\mathcal{Q}^* \mathcal{Q}}$, see, e.g., [RS], §VI.4).

If \mathcal{A} is a bounded operator, then

$$(3.18) \quad \Delta_{p,\varphi}(|\mathcal{A}\mathcal{Q}|) \leq \|\mathcal{A}\|^{1/p} \cdot \Delta_{p,\varphi}(|\mathcal{Q}|).$$

Proof. The inequality (3.17a) is a triangle inequality for the functional $\Delta_{p,\varphi}^{\frac{p}{p+1}}$. The inequalities (3.17a), (3.17b) strengthen the well-known Weyl’s theorem ([We], see also [BS1], Lemma 1.17). They mean that the functionals $\Delta_{p,\varphi}$ and $\delta_{p,\varphi}$ are continuous in a certain sense. These inequalities are well-known for $\varphi = 1$ (see [RSSh], Subsection 11.1; the proofs can be found also in [BS1], Lemma 1.18). The general case was treated in [BS3], Lemma 3.1.

The inequality (3.18) is a simple consequence of variational principle (see, e.g., [BS2], Sec.10.2). As $|\mathcal{A}\mathcal{Q}| \leq \|\mathcal{A}\| \cdot |\mathcal{Q}|$, we have

$$\mathcal{N}(t, |\mathcal{A}\mathcal{Q}|) \leq \mathcal{N}(t, \|\mathcal{A}\| \cdot |\mathcal{Q}|) \implies \Delta_{p,\varphi}(|\mathcal{A}\mathcal{Q}|) \leq \Delta_{p,\varphi}(\|\mathcal{A}\| \cdot |\mathcal{Q}|),$$

and formula (3.2) yields (3.18). □

The next theorem deals with the sum of tensor products of compact operators. Let $\mathcal{T}_1, \mathcal{T}_2$ and $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$ be compact self-adjoint nonnegative operators in Hilbert spaces H and \tilde{H} respectively. We consider the operators $\mathcal{S}_1 = \mathcal{T}_1 \otimes \tilde{\mathcal{T}}_1$ and $\mathcal{S}_2 = \mathcal{T}_2 \otimes \tilde{\mathcal{T}}_2$ in the space $H \otimes \tilde{H}$.

Theorem 3.6. *The following statements are true:*

1. *Let φ be a SVF. If, as $t \rightarrow 0+$,*

$$\mathcal{N}(t, \mathcal{S}_1) \sim at^{-1/p} \varphi(1/t), \quad \mathcal{N}(t, \mathcal{S}_2) = o(\mathcal{N}(t, \mathcal{S}_1)),$$

then

$$\mathcal{N}(t, \mathcal{S}_1 + \mathcal{S}_2) \sim \mathcal{N}(t, \mathcal{S}_1).$$

2. *Let φ_1 and $\tilde{\varphi}_2$ be SVFs. If, as $t \rightarrow 0+$,*

$$\mathcal{N}(t, \mathcal{T}_1) \sim at^{-1/p} \varphi_1(1/t), \quad \mathcal{N}(t, \tilde{\mathcal{T}}_2) \sim \tilde{a}t^{-1/p} \tilde{\varphi}_2(1/t),$$

and the operators $\tilde{\mathcal{T}}_1, \mathcal{T}_2$ are finite-dimensional, then

$$(3.19) \quad \mathcal{N}(t, \mathcal{S}_1 + \mathcal{S}_2) \sim \mathcal{N}(t, \mathcal{S}_1) + \mathcal{N}(t, \mathcal{S}_2) \sim t^{-1/p} \varphi(1/t), \quad t \rightarrow 0+,$$

where

$$\phi(s) = a\varphi_1(s) \sum_n \lambda_n^{1/p}(\tilde{\mathcal{T}}_1) + \tilde{a}\tilde{\varphi}_2(s) \sum_n \lambda_n^{1/p}(\mathcal{T}_2).$$

3. Let φ be a SVF. If, as $t \rightarrow 0+$,

$$(3.2) \quad \mathcal{N}(t, \mathcal{T}_1) \sim at^{-1/p}\varphi(1/t), \quad \mathcal{N}(t, \tilde{\mathcal{T}}_2) \sim \tilde{a}t^{-1/p}\varphi(1/t),$$

$$(3.20) \quad \mathcal{N}(t, \tilde{\mathcal{T}}_1) = O(t^{-1/\tilde{p}}), \quad \mathcal{N}(t, \mathcal{T}_2) = O(t^{-1/\tilde{p}}), \quad \tilde{p} > p,$$

then

$$\mathcal{N}(t, \mathcal{S}_1 + \mathcal{S}_2) \sim \mathcal{N}(t, \mathcal{S}_1) + \mathcal{N}(t, \mathcal{S}_2) \sim At^{-1/p}\varphi(1/t), \quad t \rightarrow 0+,$$

where

$$A = a \sum_n \lambda_n^{1/p}(\tilde{\mathcal{T}}_1) + \tilde{a} \sum_n \lambda_n^{1/p}(\mathcal{T}_2).$$

Proof. 1. This statement is a variant of Weyl's theorem. Actually it is a simple corollary of Proposition 3.5: as $\Delta_{p,\varphi}(\mathcal{S}_2) = 0$, we have using (3.17a), (3.17b)

$$\Delta_{p,\varphi}(\mathcal{S}_1 + \mathcal{S}_2) = \Delta_{p,\varphi}(\mathcal{S}_1), \quad \delta_{p,\varphi}(\mathcal{S}_1 + \mathcal{S}_2) = \delta_{p,\varphi}(\mathcal{S}_1),$$

and the statement follows.

In particular, we see that for any finite-dimensional operator $\mathcal{K} = \mathcal{K}^* \geq 0$

$$(3.21) \quad \mathcal{N}(t, \mathcal{T} + \mathcal{K}) \sim \mathcal{N}(t, \mathcal{T}), \quad t \rightarrow 0+.$$

2. Let P be the orthogonal projector in H onto the subspace $\ker(\mathcal{T}_2)$. Correspondingly, denote by \tilde{P} the orthogonal projector in \tilde{H} onto $\ker(\tilde{\mathcal{T}}_1)$. Put

$$\mathcal{R}_1 = (P\mathcal{T}_1P) \otimes \tilde{\mathcal{T}}_1, \quad \mathcal{R}_2 = \mathcal{T}_2 \otimes (\tilde{P}\tilde{\mathcal{T}}_2\tilde{P}).$$

Then $\mathcal{S}_1 - \mathcal{R}_1 = ((I - P)\mathcal{T}_1 + P\mathcal{T}_1(I - P)) \otimes \tilde{\mathcal{T}}_1$, and therefore

$$\text{rank}(\mathcal{S}_1 - \mathcal{R}_1) \leq 2 \cdot \text{rank}(I - P) \cdot \text{rank}\tilde{\mathcal{T}}_1 < \infty.$$

Similarly, $\text{rank}(\mathcal{S}_2 - \mathcal{R}_2) < \infty$, and by (3.21)

$$\mathcal{N}(t, \mathcal{S}_i) \sim \mathcal{N}(t, \mathcal{R}_i), \quad t \rightarrow 0+, \quad i = 1, 2.$$

By construction the operators \mathcal{R}_1 and \mathcal{R}_2 are orthogonal, i.e. $\mathcal{R}_1\mathcal{R}_2 = \mathcal{R}_2\mathcal{R}_1 = 0$. Hence due to the variational principle we get

$$\mathcal{N}(t, \mathcal{R}_1 + \mathcal{R}_2) = \mathcal{N}(t, \mathcal{R}_1) + \mathcal{N}(t, \mathcal{R}_2).$$

As $(\mathcal{S}_1 + \mathcal{S}_2) - (\mathcal{R}_1 + \mathcal{R}_2)$ is also the finite-dimensional operator, we have

$$\mathcal{N}(t, \mathcal{S}_1 + \mathcal{S}_2) \sim \mathcal{N}(t, \mathcal{R}_1 + \mathcal{R}_2) \sim \mathcal{N}(t, \mathcal{R}_1) + \mathcal{N}(t, \mathcal{R}_2) \sim \mathcal{N}(t, \mathcal{S}_1) + \mathcal{N}(t, \mathcal{S}_2),$$

and (3.19) is proved.

3. Consider the sequences \mathcal{P}_j and $\tilde{\mathcal{P}}_j$ of orthogonal projectors onto the first j eigenvectors of \mathcal{T}_2 and $\tilde{\mathcal{T}}_1$, respectively. Put

$$\mathcal{Q}_{1j} = \mathcal{T}_1 \otimes (\tilde{\mathcal{P}}_j\tilde{\mathcal{T}}_1\tilde{\mathcal{P}}_j), \quad \mathcal{Q}_{2j} = (\mathcal{P}_j\mathcal{T}_2\mathcal{P}_j) \otimes \tilde{\mathcal{T}}_2.$$

According to (3.19) we have

$$(3.22) \quad \begin{aligned} \Delta_{p,\varphi}(\mathcal{Q}_{1j} + \mathcal{Q}_{2j}) &= \Delta_{p,\varphi}(\mathcal{Q}_{1j}) + \Delta_{p,\varphi}(\mathcal{Q}_{2j}); \\ \delta_{p,\varphi}(\mathcal{Q}_{1j} + \mathcal{Q}_{2j}) &= \delta_{p,\varphi}(\mathcal{Q}_{1j}) + \delta_{p,\varphi}(\mathcal{Q}_{2j}). \end{aligned}$$

By Theorem 3.2

$$\Delta_{p,\varphi}(\mathcal{S}_1 - \mathcal{Q}_{1j}) = a \sum_n \lambda_n^{1/p}(\tilde{\mathcal{T}}_1 - \tilde{\mathcal{P}}_j\tilde{\mathcal{T}}_1\tilde{\mathcal{P}}_j) = a \sum_{n>j} \lambda_n^{1/p}(\tilde{\mathcal{T}}_1)^{1/p} \rightarrow 0, \quad j \rightarrow \infty.$$

Using (3.17a), (3.17b) we have, as $j \rightarrow \infty$,

$$\Delta_{p,\varphi}(\mathcal{Q}_{ij}) \rightarrow \Delta_{p,\varphi}(\mathcal{S}_i), \quad \delta_{p,\varphi}(\mathcal{Q}_{ij}) \rightarrow \delta_{p,\varphi}(\mathcal{S}_i), \quad i = 1, 2.$$

By the triangle inequality (3.17a)

$$\Delta_{p,\varphi}^{\frac{p}{p+1}}((\mathcal{S}_1 + \mathcal{S}_2) - (\mathcal{Q}_{1j} + \mathcal{Q}_{2j})) \leq \Delta_{p,\varphi}^{\frac{p}{p+1}}(\mathcal{S}_1 - \mathcal{Q}_{1j}) + \Delta_{p,\varphi}^{\frac{p}{p+1}}(\mathcal{S}_2 - \mathcal{Q}_{2j}),$$

so that by (3.17a), (3.17b)

$$\Delta_{p,\varphi}(\mathcal{Q}_{1j} + \mathcal{Q}_{2j}) \rightarrow \Delta_{p,\varphi}(\mathcal{S}_1 + \mathcal{S}_2), \quad \delta_{p,\varphi}(\mathcal{Q}_{1j} + \mathcal{Q}_{2j}) \rightarrow \delta_{p,\varphi}(\mathcal{S}_1 + \mathcal{S}_2).$$

Passage to the limit in (3.22) completes the proof. □

Remark 4. It is easy to see that the condition (3.20) can be weakened. For instance, one can assume that

$$\mathcal{N}(t, \tilde{\mathcal{T}}_1) = o(t^{-1/p}\varphi(1/t)), \quad \mathcal{N}(t, \mathcal{T}_2) = o(t^{-1/p}\varphi(1/t)), \quad t \rightarrow 0+,$$

and both tensor products \mathcal{S}_1 and \mathcal{S}_2 satisfy the relations (3.12) and (3.16).

4. SMALL BALL ASYMPTOTICS AND THE RATE OF EIGENVALUES DECREASE

In this section we connect the given asymptotic behavior of eigenvalues λ_n from (1.4) with the logarithmic rate of decrease for small deviations. The case $\lambda_n = Cn^{-p}$, $p > 1$, is now a classical one and is known from [DH-JS], [I], [Z1], [Z2]; see also [DLL], where the *exact* asymptotics is obtained. The generalization for the case $\lambda_n = Cn^{-p}(1 + o(1))$, $n \rightarrow \infty$, can be found in [NaNi2], Proposition 2.1, and in [Fa].

As explained in the introduction, we consider the general case when

$$(4.1) \quad \lambda_n = C^* \frac{\psi(n)}{n^p} (1 + o(1)), \quad n \rightarrow \infty,$$

where $p > 1$, and the positive function ψ is twice differentiable and satisfies (2.3). We will use the following result which is a concretization of Corollary 3.2 in [DLL] under our conditions.

Lemma 4.1. *Denote, for $t, u \geq 0$ and $p > 1$*

$$\Phi(t) = t^{-p}\psi(t), \quad f(t) = (1 + 2t)^{-1/2},$$

$$I_0(u) = \int_1^\infty \ln f(u\Phi(t))dt, \quad I_1(u) = uI_0'(u), \quad I_2(u) = u^2I_2''(u).$$

Then, as $r \rightarrow 0$,

$$(4.2) \quad P\left\{ \sum_{n=1}^\infty \Phi(n)\xi_n^2 \leq r \right\} \sim C_\Phi \sqrt{\frac{f(u\Phi(1))}{I_2(u)}} \cdot \exp(I_0(u) + ur),$$

where the constant C_Φ can be expressed in terms of Φ while $u = u(r)$ is any function satisfying

$$(4.3) \quad \lim_{r \rightarrow 0} \frac{I_1(u) + ur}{\sqrt{I_2(u)}} = 0.$$

Note that (2.3) provides logarithmic convexity of Φ for large values of argument; so all regularity conditions assumed in [DLL] are fulfilled.

All the quantities given above require long but relatively simple calculations. We begin by the asymptotic analysis of $I_1(u)$ as $u \rightarrow \infty$ because this integral plays the key role in the sequel. In our case we clearly have

$$I_1(u) = -u \int_1^\infty \frac{\psi(t) dt}{t^p + 2u\psi(t)}.$$

Changing the integration to $(0, \infty)$ and making a substitution

$$(4.4) \quad t = z\Phi^{-1}(1/u) \equiv zu^{1/p}\varphi(u)$$

we get

$$\begin{aligned} I_1(u) &= -u^{1/p}\varphi(u) \cdot \int_0^\infty \frac{dz}{2 + z^p \frac{(\varphi(u))^p}{\psi(zu^{1/p}\varphi(u))}} + O(1) \\ &= -u^{1/p}\varphi(u) \cdot \int_0^\infty \frac{dz}{2 + z^p \frac{\psi(u^{1/p}\varphi(u))}{\psi(zu^{1/p}\varphi(u))}} + O(1), \quad u \rightarrow \infty. \end{aligned}$$

By Proposition 2.1, part **3**, for any $\varepsilon > 0$ the quotient $\psi(s)/s^\varepsilon$ decreases for large s . Hence for $z > 1, s > 1$ we have

$$\frac{\psi(s)}{\psi(zs)} = \frac{1}{z^\varepsilon} \cdot \frac{\psi(s)}{s^\varepsilon} \bigg/ \frac{(zs)^\varepsilon}{\psi(zs)} \geq \frac{C(\varepsilon)}{z^\varepsilon}.$$

This estimate gives us the majorant required in the Lebesgue Theorem. Passing to the limit in the integral we obtain as $u \rightarrow \infty$

$$I_1(u) \sim -u^{1/p}\varphi(u) \cdot \int_0^\infty \frac{dz}{2 + z^p} = -u^{1/p}\varphi(u) \cdot \frac{2^{1/p}\pi}{2p \sin \frac{\pi}{p}},$$

where the last equality follows from [GR], formula 3.241.2. Analogously we get

$$I_2(u) = 2u^2 \int_1^\infty \frac{(\psi(t))^2 dt}{(t^p + 2u\psi(t))^2} \sim -cI_1(u), \quad u \rightarrow \infty.$$

Now note that

$$\left(\frac{I_1(u)}{u}\right)' = \frac{I_2(u)}{u^2} > 0, \quad u > 0.$$

Hence by the implicit function theorem the equation $I_1(u) + ur = 0$ has for sufficiently small r the unique solution $u(r)$ such that $u(r) \rightarrow \infty$ as $r \rightarrow 0$. Moreover the relation

$$r = -\frac{I_1(u)}{u} \sim \frac{2^{1/p}\pi}{2p \sin \frac{\pi}{p}} \cdot u^{-\frac{p-1}{p}} \varphi(u) \equiv \tilde{\Phi}(u), \quad u \rightarrow \infty,$$

yields

$$(4.5) \quad u(r) \sim \tilde{\Phi}^{-1}(r) \equiv r^{-\frac{p}{p-1}} \zeta(1/r), \quad r \rightarrow 0.$$

As $u(r)$ trivially satisfies (4.3) we can apply the formula (4.2) with $u = u(r)$. Note that in contrast to the case $\psi \equiv \text{const}$ the replacement of $u(r)$ by one-term asymptotics (4.5) breaks the relation (4.3). Therefore it is not fit to extract the exact small ball asymptotics from (4.2). Happily, it fits for the logarithmic asymptotics. Namely, we have

$$I_0(u) = \frac{1}{2} \ln(1 + 2u\Phi(1)) + u \int_1^\infty \frac{-p\psi(t) + t\psi'(t)}{t^p + 2u\psi(t)} dt.$$

Changing the integration to $(0, \infty)$, making the substitution (4.4) and taking into account the first relation in (2.3), we get

$$I_0(u) = I_1(u) \cdot (p + o(1)), \quad u \rightarrow \infty.$$

Therefore, as $r \rightarrow 0$ and $u = u(r)$,

$$\ln P \left\{ \sum_{n=1}^\infty \Phi(n)\xi_n^2 \leq r \right\} \sim I_0(u) + ur \sim -(p-1)ur \sim -(p-1)r^{-\frac{1}{p-1}}\zeta(1/r),$$

where the SVF ζ is defined in (4.5).

Now we are able to formulate the following useful statement.

Theorem 4.2. *Let the eigenvalues λ_n from (1.4) have the asymptotics (4.1). Then, as $\varepsilon \rightarrow 0$,*

$$\ln P\{\|X\|_\rho \leq \varepsilon\} \sim -(p-1)(C^*)^{\frac{1}{p-1}}\varepsilon^{-\frac{2}{p-1}}\zeta(\varepsilon^{-2}),$$

where the SVF ζ is defined in (4.5).

For $\psi \equiv \text{const}$ this statement is known, see, e.g., [NaNi2] or [Fa]. The proof in the general case is very similar to [NaNi2], Proposition 2.1, and we omit it.

Example 4. Let $\psi(s) = \ln^\sigma(s+1)$, $\sigma \in \mathbb{R}$. The calculation using (4.4) and (4.5) yields

$$\begin{aligned} \varphi(u) &\sim p^{-\sigma/p} \cdot \ln^{\sigma/p}(u), \quad u \rightarrow \infty; \\ \zeta(1/r) &\sim \frac{1}{2} \left(\frac{\pi}{p \sin \frac{\pi}{p}} \right)^{\frac{p}{p-1}} \left(\frac{\ln(1/r)}{p-1} \right)^{\frac{\sigma}{p-1}}, \quad r \rightarrow 0, \end{aligned}$$

and we arrive at

Proposition 4.3. *Let the eigenvalues λ_n from (1.4) have the asymptotics*

$$\lambda_n \sim C^* \frac{\ln^\sigma(n+1)}{n^p}, \quad n \rightarrow \infty, \quad \sigma \in \mathbb{R}, \quad p > 1, \quad C^* > 0.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{2}{p-1}} \ln^{-\frac{\sigma}{p-1}}(1/\varepsilon) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\} = -(C^*)^{\frac{1}{p-1}} \left(\frac{p-1}{2} \right)^{1-\frac{\sigma}{p-1}} \left(\frac{\pi}{p \sin \frac{\pi}{p}} \right)^{\frac{p}{p-1}}.$$

Let us briefly examine a special case $p = 1$. In this case, if SVF ψ satisfies the natural condition (2.4), the statement of Lemma 4.1 also holds true. However, one can check that $r = -I_1(u)/u$ is a slowly varying function of u . Hence, in contrast with the case $p > 1$, its one-term asymptotics does not imply the one-term asymptotics of the inverse function $u(r)$. Therefore we cannot obtain the explicit form of logarithmic asymptotics. For the reason of place we restrict ourselves to one example of weaker asymptotics.

Proposition 4.4. *Let the eigenvalues λ_n from (1.4) have the asymptotics*

$$\lambda_n \sim \frac{C^*}{n \ln^\sigma(n+1)}, \quad n \rightarrow \infty, \quad \sigma > 1, \quad C^* > 0.$$

Then

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{2}{\sigma-1}} \cdot \ln \left| \ln P\{ \|X\|_\rho \leq \varepsilon \} \right| = \left(\frac{C^*}{\sigma-1} \right)^{\frac{1}{\sigma-1}}.$$

Proof. Similarly to Theorem 4.2 we can assume that λ_n are exactly $\frac{C^*}{n \ln^\sigma(n+1)}$. Applying Lemma 4.1, we calculate

$$r = -\frac{I_1(u)}{u} = \int_1^\infty \frac{C^* dt}{t \ln^\sigma(t+1) + 2u C^*}.$$

Define $z = z(u)$ such that

$$(4.6) \quad z(\ln(z))^s = u, \quad \sigma - 1 < s < \sigma.$$

Then, as $u \rightarrow \infty$,

$$\int_z^\infty \frac{C^* dt}{t \ln^\sigma(t+1) + 2u C^*} \sim \int_z^\infty \frac{C^*}{t \ln^\sigma(t)} \cdot \frac{dt}{1 + \frac{2u C^*}{t \ln^\sigma(t)}} \sim \int_z^\infty \frac{C^* dt}{t \ln^\sigma(t)} = \frac{C^*}{(\sigma-1) \ln^{\sigma-1}(z)},$$

while

$$\int_1^z \frac{C^* dt}{t \ln^\sigma(t+1) + 2u C^*} \leq \frac{z}{2u} = o\left(\frac{1}{\ln^{\sigma-1}(z)}\right).$$

Thus

$$(4.7) \quad r \sim \frac{C^*}{(\sigma-1) \ln^{\sigma-1}(z)} \sim \frac{C^*}{(\sigma-1) \ln^{\sigma-1}(u)}.$$

Next, as $u \rightarrow \infty$, we calculate

$$\begin{aligned} \ln P\left\{ \sum_{n=1}^\infty \lambda_n \xi_n^2 \leq r \right\} &= I_0(u) - I_1(u) + O(\ln(u)) = \\ &= -u \int_1^\infty \frac{t}{t+1} \cdot \frac{C^* \sigma dt}{\ln(t+1)(t \ln^\sigma(t+1) + 2u C^*)} + O(\ln(u)). \end{aligned}$$

Taking $z = z(u)$ from (4.6), we have

$$\int_z^\infty \frac{t}{t+1} \cdot \frac{C^* \sigma dt}{\ln(t+1)(t \ln^\sigma(t+1) + 2u C^*)} \sim \int_z^\infty \frac{C^* \sigma}{t \ln^{\sigma+1}(t)} \cdot \frac{dt}{1 + \frac{2u C^*}{t \ln^\sigma(t)}} \sim \frac{C^*}{\ln^\sigma(z)},$$

while

$$\begin{aligned} \int_1^z \frac{t}{t+1} \cdot \frac{C^* \sigma dt}{\ln(t+1)(t \ln^\sigma(t+1) + 2u C^*)} &\leq \\ &\leq \int_1^z \frac{\sigma dt}{2u \ln(t+1)} \sim \frac{\sigma z}{2u \ln(z)} = o\left(\frac{1}{\ln^\sigma(z)}\right). \end{aligned}$$

Hence

$$\ln P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\} \sim -\frac{C^* u}{\ln^\sigma(z)} \sim -\frac{C^* u}{\ln^\sigma(u)}.$$

Due to (4.7), we arrive at

$$\ln \left| \ln P \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\} \right| \sim \ln(u) \sim \left(\frac{C^*}{(\sigma - 1)r} \right)^{\frac{1}{\sigma-1}}, \quad r \rightarrow 0.$$

Replacing r by ε^2 completes the proof. □

5. FIELDS-PRODUCTS WITH EQUAL MARGINAL EXPONENTS

In this section we deal with the fields of the form

$$(5.1) \quad X(x) = \otimes_{j=1}^d X_j(x_j), \quad x = (x_1, \dots, x_d) \in [0, 1]^d,$$

where X_j are independent zero mean Gaussian processes with *equal* marginal exponents p_j in the asymptotic representation (1.6). Also we consider weight functions of the form

$$(5.2) \quad \rho(x) = \prod_{j=1}^d \rho_j(x_j).$$

In §8 we examine a more general class of weights.

We start with the fractional Brownian sheet $\mathbb{W}_d^H(x)$. Its marginal processes (factors) X_j are the fractional Brownian motions W^H ($H \in]0, 1[$ is the Hurst parameter), and the corresponding covariance function is

$$G_{\mathbb{W}_d^H}(x, y) = \frac{1}{2^d} \prod_{j=1}^d (x_j^{2H} + y_j^{2H} - |x_j - y_j|^{2H}).$$

Proposition 5.1. *Let $0 < H < 1$ and let ρ be a summable nonnegative function on $[0, 1]^d$ of the form (5.2). Denote $J_h = \int_{[0,1]^d} \rho(x)^{1/h} dx$. Then*

$$(5.3) \quad \begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \ln^{-(d-1)\frac{2H+1}{2H}}(1/\varepsilon) \cdot \ln P\{\|\mathbb{W}_d^H\|_\rho \leq \varepsilon\} \\ &= \frac{-J_{2H+1} H^{2-d} \pi^{1-d}}{(d-1)!(2H+1) \sin(\frac{\pi}{2H+1})} \left(\frac{J_{2H+1} (\Gamma(2H+1) \sin(\pi H))^d}{(d-1)!(\pi H)^{d-1} (2H+1) \sin(\frac{\pi}{2H+1})} \right)^{\frac{1}{2H}}. \end{aligned}$$

Proof. For $d = 1$ the statement coincides with Theorem 3.1 in [NaNi2]. It follows from the proof of this theorem that marginal “one-dimensional” operators corresponding to the marginal covariances have the one-term spectral asymptotics

$$(5.4) \quad \lambda_n^{(j)} \sim C^{(j)} n^{-(2H+1)}, \quad j = 1, \dots, d,$$

where $C^{(j)} = (J_{2H+1}^{(j)} / \pi)^{2H+1} \Gamma(2H+1) \sin(\pi H)$, while $J_h^{(j)} = \int_0^1 \rho_j(t)^{1/h} dt$.

Using Example 3 successively $d - 1$ times and then applying Example 2, we obtain that

$$(5.5) \quad \lambda_n \sim \frac{\prod_{j=1}^d C^{(j)}}{(d-1)!^{2H+1}} \cdot \frac{\ln^{(d-1)(2H+1)}(n+1)}{n^{2H+1}}, \quad n \rightarrow \infty.$$

Since

$$\prod_{j=1}^d C^{(j)} = J_{2H+1}^{2H+1} \left(\frac{\Gamma(2H+1) \sin(\pi H)}{\pi^{2H+1}} \right)^d,$$

then, using Proposition 4.3 with $p = 2H + 1$ and $\sigma = (d - 1)(2H + 1)$, we arrive at (5.3). □

Remark 5. Formula (5.5) in the case $\rho \equiv 1$ was obtained in [LuP2].

As an example we consider the simplest case of Brownian sheet when $H = 1/2$. Formula (5.3) gives now

$$(5.6) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln^{-2(d-1)}(1/\varepsilon) \cdot \ln P\{\|\mathbb{W}_d\|_\rho \leq \varepsilon\} = -\frac{2^{2d-5} \left(\int_{[0,1]^d} \rho(x)^{1/2} dx\right)^2}{(d-1)!^2 \pi^{2d-2}}.$$

Formula (5.6) (for $\rho \equiv 1$) is equivalent to one obtained in [Cs], see also [Li], Example 3. Note that instead of the norming factor $\ln(1/\varepsilon^2)$ we use the factor $\ln(1/\varepsilon)$ that explains the seeming difference in constants (for example, when $d = 2$ we obtain the constant $-1/(2\pi^2)$ while (1.2) gives $-1/(8\pi^2)$).

Remark 6. Note, analogously to Remark 3.1 in [NaNi2], that the replacement of any factor in (5.1) by the fractional Brownian bridge $B^H(x_j) = W^H(x_j) - x_j W^H(1)$, by the centered fractional Brownian motion $\overline{W}^H(x_j) = W^H(x_j) - \int_0^1 W^H(s) ds$ or by similar process does not influence on the relation (5.3). For example, the fractional Brownian pillow $\mathbb{B}_d^H(x) = \bigotimes_{j=1}^d B^H(x_j)$ satisfies, as $\varepsilon \rightarrow 0$, the relation

$$\ln P\{\|\mathbb{B}_d^H\|_\rho \leq \varepsilon\} \sim \ln P\{\|\mathbb{W}_d^H\|_\rho \leq \varepsilon\}.$$

The same is true in more general examples. Let us consider the ‘‘pinned down’’ fractional Brownian sheet $\mathbb{W}_d^H(x)$ with covariance

$$(5.7) \quad G_{\mathbb{W}_d^H}(x, y) = G_{\mathbb{W}_d^H}(x, y) - G_{\mathbb{W}_d^H}(x, \mathbf{1})G_{\mathbb{W}_d^H}(\mathbf{1}, y), \quad x, y \in [0, 1]^d$$

(here $\mathbf{1} = (1, 1, \dots, 1)$). For $H = 1/2$ this Gaussian field arises in the study of multidimensional goodness-of-fit tests, see [Ni] and [vVW] for similar constructions.

The covariance (5.7) differs from $G_{\mathbb{W}_d^H}(x, y)$ by the finite-dimensional term. By (3.21) this term does not influence the one-term eigenvalues asymptotics, so we have, as $\varepsilon \rightarrow 0$,

$$\ln P\{\|\mathbb{W}_d^H\|_\rho \leq \varepsilon\} \sim \ln P\{\|\mathbb{W}_d^H\|_\rho \leq \varepsilon\}.$$

Now we consider the *isotropically integrated* fractional Brownian sheet

$$(\mathbb{W}_d^H)_m(x) = \bigotimes_{j=1}^d (W^H)_m(x_j),$$

where

$$(W^H)_m(t) \equiv (W^H)_m^{[\beta_1, \dots, \beta_m]}(t) = (-1)^{\beta_1 + \dots + \beta_m} \underbrace{\int_{\beta_m}^t \dots \int_{\beta_1}^{t_1}}_m W^H(s) ds dt_1 \dots$$

(here any β_k equals either zero or one, $t \in [0, 1]$; for various j the multi-indices $[\beta_1, \dots, \beta_m]$ can differ). See also [GHT], [NaNi1] and [NaNi2] for various forms of integrated processes.

Proposition 5.2. *Let $m \in \mathbb{N}$. Then, with the same notation as in Proposition 5.1,*

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/(m+H)} \ln^{-(d-1)\frac{2(m+H)+1}{2(m+H)}} (1/\varepsilon) \cdot \ln P\{\|(\mathbb{W}_d^H)_m\|_\rho \leq \varepsilon\} \\ &= -\frac{J_{2(m+H)+1}(m+H)^{2-d}}{(d-1)! \pi^{d-1} (2(m+H)+1) \sin\left(\frac{\pi}{2(m+H)+1}\right)} \\ &\quad \cdot \left(\frac{J_{2(m+H)+1}(\Gamma(2H+1) \sin(\pi H))^d}{(d-1)! (\pi(m+H))^{d-1} (2(m+H)+1) \sin\left(\frac{\pi}{2(m+H)+1}\right)} \right)^{\frac{1}{2(m+H)}}. \end{aligned}$$

Proof. This statement can be proved in the same way as Proposition 5.1 using Theorem 3.2 in [NaNi2]. □

The next example deals with the fractional Ornstein – Uhlenbeck sheet $\mathbb{U}_{d,(\alpha)}^H(x)$ (here $\alpha \in \mathbb{R}_+^d$). Its marginal processes X_j are fractional Ornstein – Uhlenbeck processes $U_{(\alpha_j)}^H$. The corresponding covariance function is

$$G_{\mathbb{U}_{d,(\alpha)}^H}(x, y) = \exp\left(-\sum_{j=1}^d \alpha_j |x_j - y_j|^{2H}\right), \quad 0 < H < 1.$$

In the probabilistic literature there exist different definitions of fractional Ornstein – Uhlenbeck process, here we use the definition from [LuP1]. See the discussion on other possible definitions in [NaNi2], §4.

The isotropically integrated fractional Ornstein – Uhlenbeck sheet $(\mathbb{U}_{d,(\alpha)}^H)_m(x)$ is defined similarly to the isotropically integrated fractional Brownian sheet.

Proposition 5.3. *Let $0 < H < 1$, $m \in \mathbb{N} \cup \{0\}$ and let ρ be a summable nonnegative function on $[0, 1]^d$ of the form (5.2). Then, as $\varepsilon \rightarrow 0$,*

$$\ln P\{\|(\mathbb{U}_{d,(\alpha)}^H)_m\|_\rho \leq \varepsilon\} \sim (2^d \alpha_1 \dots \alpha_d)^{\frac{1}{2(m+H)}} \ln P\{\|(\mathbb{W}_d^H)_m\|_\rho \leq \varepsilon\}.$$

Proof. This statement can be proved in the same way as Proposition 5.1 using Theorem 4.2 from [NaNi2]. □

We can also consider multiparameter marginal processes. For example, let Ω_j , $j = 1, \dots, d$ be bounded domains in \mathbb{R}^ℓ with $0 \in \overline{\Omega_j}$. Set $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_d \subset \mathbb{R}^{d\ell}$. Consider the Gaussian function on Ω

$$\mathbb{L}_{d,\ell}^H(x) = \bigotimes_{j=1}^d \mathcal{L}_\ell^H(x_j),$$

where $\mathcal{L}_\ell^H(x_j)$, $0 < H < 1$, is Lévy’s fractional Brownian motion on $\overline{\Omega_j}$ with the covariance

$$G_{\mathcal{L}_\ell^H}(x_j, y_j) = \frac{1}{2} (\|x_j\|^{2H} + \|y_j\|^{2H} - \|x_j - y_j\|^{2H}), \quad x_j, y_j \in \overline{\Omega_j}.$$

Proposition 5.4. *Let $0 < H < 1$ and let ρ be a summable nonnegative function on Ω of the form (5.2). Denote $J_h = \int_\Omega \rho(x)^{1/h} dx$. Then*

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \varepsilon^{\ell/H} \ln^{-(d-1)\frac{2H+\ell}{2H}} (1/\varepsilon) \cdot \ln P\{\|\mathbb{L}_{d,\ell}^H\|_\rho \leq \varepsilon\} \\ &= -H \cdot \left(\frac{2}{H\Gamma(\ell/2)} \right)^d \cdot \left(\frac{\Gamma(\ell/2 + H)}{\Gamma(\ell/2)\Gamma(1-H)\pi^H} \right)^{\frac{\ell d}{2H}} \cdot \left(\frac{J_{\frac{2H+\ell}{\ell}} \pi H (2H + \ell)^{-1}}{(d-1)! \sin\left(\frac{\pi \ell}{2H+\ell}\right)} \right)^{\frac{2H+\ell}{2H}}. \end{aligned}$$

Proof. This statement can be proved in the same way as Proposition 5.1 using Theorem 3.3 in [NaNi2]. \square

Finally, we can consider the fields-products corresponding to essentially different marginal processes. We restrict ourselves to only one example. Let $H \in]0, 1[$. On $[0, 1]^3$ consider the Gaussian field $W^{H/2}(x_1) \otimes \mathcal{L}_2^H(x')$, where $x' = (x_2, x_3) \in [0, 1]^2$.

Proposition 5.5. *With the same notations as in Proposition 5.1,*

$$(5.8) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{2/H} \ln^{-\frac{H+1}{H}}(1/\varepsilon) \cdot \ln P\{\|W^{H/2} \otimes \mathcal{L}_2^H\|_\rho \leq \varepsilon\} \\ = -\frac{J_{H+1}}{\pi(H+1) \sin(\frac{\pi}{H+1})} \left(\frac{J_{H+1} H (\Gamma(H+1))^3 \sin(\pi H/2) \sin(\pi H)}{4\pi(H+1) \sin(\frac{\pi}{H+1})} \right)^{\frac{1}{H}}.$$

Proof. The spectral asymptotics of marginal operators was established in Theorems 3.1 and 3.3 in [NaNi2]. Applying Example 3, we obtain

$$\lambda_n \sim \left(\frac{J_{H+1}}{\pi^2} \right)^{H+1} (\Gamma(1+H))^3 \sin(\pi H/2) \sin(\pi H) \cdot \frac{\ln^{H+1}(n+1)}{n^{H+1}}.$$

By Proposition 4.3 we immediately arrive at (5.8). \square

For $H = 1/2$ and $\rho \equiv 1$ that corresponds to the tensor product of classical non-fractional Lévy's Brownian motion (the so-called isotropic Brownian field) \mathcal{L}_2 and fractional Brownian motion $W^{1/4}$ the result reads

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^4 \ln^{-3}(1/\varepsilon) \cdot \ln P\{\|\mathcal{L}_2 \otimes W^{1/4}\| \leq \varepsilon\} = -\frac{1}{10368\sqrt{3}}.$$

6. FIELDS-PRODUCTS WITH DISTINCT MARGINAL EXPONENTS

As before we deal with the fields of the form (5.1) but in this section the marginal processes can have different power rates of eigenvalues decrease.

We call X_j a *slow factor* if it has the least exponent p_j in (1.5) and call it a *fast factor* otherwise. Our Theorem 3.2 shows that the rate of eigenvalues decrease for the tensor product is determined only by slow factors. However, in order to calculate the constant in the one-term asymptotics we need rather precise information concerning the fast factors spectra. Using Theorem 4.2 we obtain the following statement.

Lemma 6.1. *Let $\lambda_n^{(X)}$ and $\lambda_n^{(Y)}$ be the eigenvalues of the integral equation (1.4) corresponding to the zero mean Gaussian processes X and Y on $[0, 1]$. Suppose the eigenvalues $\lambda_n^{(X)}$ satisfy the asymptotics (4.1) while $\lambda_n^{(Y)} = O(n^{-\tilde{p}})$, $n \rightarrow \infty$, with $\tilde{p} > p$. Then, as $\varepsilon \rightarrow 0$,*

$$(6.1) \quad \ln P\{\|\sqrt{\rho}X \otimes Y\| \leq \varepsilon\} \sim \left(\sum_n (\lambda_n^{(Y)})^{1/p} \right)^{\frac{p}{p-1}} \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}.$$

Remark 7. Formula (6.1) in a particular case was obtained in [Li], Example 2.

Certainly one can obtain the logarithmic small ball asymptotics for the tensor product with arbitrary number of fast factors applying Lemma 6.1 repeatedly.

Corollary 6.2. *Let the Gaussian random functions X and Y_j , $j = 1, \dots, k$, satisfy the conditions of Lemma 6.1. Then, as $\varepsilon \rightarrow 0$,*

$$(6.2) \quad \ln P\{\|\sqrt{\rho}X \otimes \left(\bigotimes_{j=1}^k Y_j\right)\| \leq \varepsilon\} \sim \prod_{j=1}^k \left(\sum_n (\lambda_n^{(Y_j)})^{1/p}\right)^{\frac{p}{p-1}} \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}.$$

Unfortunately, even if we know exact expression for the eigenvalues of the fast factors, the trace-type sums in (6.1) and (6.2) usually cannot be written out *explicitly*. Consider the following examples. Let B be the standard Brownian bridge. The eigenvalues of Anderson – Darling process $B(t)/\sqrt{t(1-t)}$, $t \in [0, 1]$, see [AnD], are $\lambda_n = (n^2 + n)^{-1}$, $n \geq 1$. The processes $t^\theta W(t)$ and $t^\theta B(t)$, $\theta > -1$, $t \in [0, 1]$, have the eigenvalues which can be expressed in terms of Bessel functions roots, see [Li], Theorem 6, [DM], Theorem 1.3, and [Na1], Theorem 3.3. In these examples the trace-type sums in (6.1) and (6.2) do not admit explicit expressions.

We give below a number of cases when these sums *can* be expressed in terms of Riemann zeta-function. For any Gaussian process $X(t)$, $t \in [0, 1]$, we introduce the *centered process* $\bar{X}(t) = X(t) - \int_0^1 X(s) ds$ and the so-called *on-line centered process* $\hat{X}(t) = X(t) - t^{-1} \int_0^t X(s) ds$, see [KlB].

Proposition 6.3. *Let the eigenvalues λ_n of the Gaussian random function X from (1.4) have the asymptotics (4.1). If $p \in]1, 2[$ then, as $\varepsilon \rightarrow 0$,*

$$\begin{aligned} \ln P\{\|\sqrt{\rho}X \otimes B\| \leq \varepsilon\} &\sim \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}; \\ \ln P\{\|\sqrt{\rho}X \otimes \bar{B}\| \leq \varepsilon\} &\sim 2^{-\frac{2-p}{p-1}} \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}; \\ \ln P\{\|\sqrt{\rho}X \otimes W\| \leq \varepsilon\} &\sim (2^{\frac{2}{p}} - 1)^{\frac{p}{p-1}} \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}; \\ \ln P\{\|\sqrt{\rho}X \otimes \bar{W}\| \leq \varepsilon\} &\sim \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}; \\ \ln P\{\|\sqrt{\rho}X \otimes \widehat{W}\| \leq \varepsilon\} &\sim \pi^{-\frac{2}{p-1}} \zeta^{\frac{p}{p-1}}(2/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}. \end{aligned}$$

If $p \in]1, 4[$, then, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \ln P\{\|\sqrt{\rho}X \otimes (\bar{W})_1\| \leq \varepsilon\} &\sim \pi^{-\frac{4}{p-1}} \zeta^{\frac{p}{p-1}}(4/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}; \\ \ln P\{\|\sqrt{\rho}X \otimes \bar{B}_1\| \leq \varepsilon\} &\sim \pi^{-\frac{4}{p-1}} \zeta^{\frac{p}{p-1}}(4/p) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}. \end{aligned}$$

If $m \in \mathbb{N}$ and $p \in]1, 2(m+1)[$, then, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \ln P\{\|\sqrt{\rho}X \otimes (W)_m^{[1,0,1,\dots]}\| \leq \varepsilon\} \\ \sim (2^{\frac{2(m+1)}{p}} - 1)^{\frac{p}{p-1}} \pi^{-\frac{2(m+1)}{p-1}} \zeta^{\frac{p}{p-1}}\left(\frac{2(m+1)}{p}\right) \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}. \end{aligned}$$

Proof. Due to requirements on p , it is easy to see that the second factor in all cases satisfies the condition of Lemma 6.1. Therefore one needs only to derive $\sum_n (\lambda_n^{(Y)})^{1/p}$.

It is well known that $\lambda_n^{(B)} = (\pi n)^{-2}$ and $\lambda_n^{(W)} = (\pi(n-1/2))^{-2}$. The spectrum of the centered Wiener process $\lambda_n^{(\bar{W})} = (\pi n)^{-2}$ can be extracted from [D-MY], see also direct calculations in [BNO]. It was proved in [Wa] that the centered Brownian bridge has double spectrum $\lambda_{2n-1}^{(\bar{B})} = \lambda_{2n}^{(\bar{B})} = (2\pi n)^{-2}$.

Integrated processes were intensively studied in recent years. It was discovered in [BNO] that $\lambda_n^{((\overline{W})_1)} = (\pi n)^{-4}$, while in [HN] there was found that $\lambda_n^{(\overline{B}_1)} = (\pi n)^{-4}$. The eigenvalues of so-called *Euler integrated* Wiener process were derived in [CH]: $\lambda_n^{((W)_m^{[1,0,1,\dots]})} = (\pi(n - 1/2))^{-2(m+1)}$.

As for online centered Wiener process, its covariance $G_{\overline{W}}(t, s) = s^2/(3t) \wedge t^2/(3s)$ was derived in [KLB]. It is easy to check that $G_{\overline{W}}$ is the Green function of the boundary value problem

$$(6.3) \quad -y'' + \frac{2y}{t^2} = \mu y; \quad y(0) = (y' + y)(1) = 0.$$

With regard to the first boundary condition, we see from [GR], formula 8.491.5, that

$$y(t) = c\sqrt{t}J_{3/2}(\sqrt{\mu t}) = c_1 \left(\frac{\sin(\sqrt{\mu t})}{\sqrt{\mu t}} - \cos(\sqrt{\mu t}) \right).$$

The second boundary condition in (6.3) gives $\mu_n = (\pi n)^2$ and hence $\lambda_n^{(\overline{W})} = (\pi n)^{-2}$.

Substituting these eigenvalues into (6.1) we obtain all given formulas. □

Let us illustrate the Proposition 6.3 by two particular examples.

1. Consider the Gaussian field $W^H \otimes \overline{B}_1$. Using the asymptotics obtained in [NaNi2], Theorem 3.1, and applying Proposition 6.3 with $p = 2H + 1$, we get, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \ln P\{|\sqrt{\rho}W^H \otimes \overline{B}_1| \leq \varepsilon\} &\sim -\pi^{-\frac{2}{H}}\zeta^{\frac{2H+1}{2H}} \left(\frac{4}{2H+1}\right) \\ &\cdot \frac{J_{2H+1}H}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \left(\frac{J_{2H+1}\Gamma(2H+1)\sin(\pi H)}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)}\right)^{\frac{1}{2H}} \varepsilon^{-1/H}. \end{aligned}$$

The particular case of this relation for $H = 1/2$ and $\rho \equiv 1$, which corresponds to the ordinary Brownian motion, reads

$$\ln P\{||W \otimes \overline{B}_1| \leq \varepsilon\} \sim -\pi^{-4}\zeta^2(2) \ln P\{||W| \leq \varepsilon\} \sim -\frac{1}{288\varepsilon^2}, \quad \varepsilon \rightarrow 0,$$

and coincides with what follows from Example 2 in [Li].

2. The next example deals with the random field $W^H \otimes B \otimes (\overline{W})_1$, $0 < H \leq 1/2$.

If $H < 1/2$, then we see from Theorem 3.1 in [NaNi2] that our random field satisfies the conditions of Corollary 6.2. Substituting the eigenvalues of fast factors to (6.2), we get finally, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \ln P\{|\sqrt{\rho}W^H \otimes B \otimes (\overline{W})_1| \leq \varepsilon\} \\ \sim -\pi^{-\frac{3}{H}}\zeta^{\frac{2H+1}{2H}} \left(\frac{4}{2H+1}\right)\zeta^{\frac{2H+1}{2H}} \left(\frac{2}{2H+1}\right) \\ \cdot \frac{J_{2H+1}H}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)} \left(\frac{J_{2H+1}\Gamma(2H+1)\sin(\pi H)}{(2H+1)\sin\left(\frac{\pi}{2H+1}\right)}\right)^{\frac{1}{2H}} \varepsilon^{-1/H}. \end{aligned}$$

If $H = 1/2$, then applying successively Proposition 5.1 (with regard to Remark 6) and Proposition 6.3 we obtain, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \ln P\{\|\sqrt{\rho}W \otimes B \otimes (\overline{W})_1\| \leq \varepsilon\} \\ \sim \pi^{-4}\zeta^2(2) \ln P\{\|\sqrt{\rho}W \otimes B\| \leq \varepsilon\} \\ \sim -\frac{(\int_0^1 \rho(t)^{1/2} dt)^2}{72\pi^2} \ln^2(1/\varepsilon)\varepsilon^{-2}. \end{aligned}$$

Even for $\rho \equiv 1$ when the field considered has the exactly known spectrum

$$\lambda_{jkl} = \pi^{-8}(j - 1/2)^{-2}k^{-2}\ell^{-4}, \quad j, k, \ell \geq 1,$$

the result is new and is not covered by [Li].

Now we cite an instance when the eigenvalues $\lambda_n^{(Y)}$ cannot be written down explicitly but nevertheless the coefficient on the right-hand side of (6.1) can be derived.

Proposition 6.4. *Let the eigenvalues λ_n from (1.4) have the asymptotics (4.1) with $p = 2$. Then, as $\varepsilon \rightarrow 0$,*

$$\ln P\{\|\sqrt{\rho}X \otimes (W)_3^{[0,1,1]}\| \leq \varepsilon\} \sim \frac{1}{144} \cdot \ln P\{\|X\|_\rho \leq \varepsilon\}.$$

Proof. Denote by \mathcal{I} the operator of integration from zero:

$$(\mathcal{I}f)(x) = \int_0^x f(t) dt.$$

It is shown in [Na1], Theorem 4.1, that the function $G_{(W)_3^{[0,1,1]}}(t, s)$ is the kernel of the operator $(\mathcal{I}^*)^2\mathcal{I}^2(\mathcal{I}^*)^2\mathcal{I}^2$. The square root of this operator is the integral operator $(\mathcal{I}^*)^2\mathcal{I}^2$ having the kernel

$$K(t, s) = \frac{1}{2}(1 - s)(1 - t)(1 - s \vee t) - \frac{1}{6}(1 - s \vee t)^3$$

(in fact $K(t, s) = G_{(W)_1^{[0]}}(1 - t, 1 - s)$ but it is inessential). Due to the trace formula for integral operators we have

$$\sum_n (\lambda_n^{((W)_3^{[0,1,1]})})^{1/2} = \mathbf{Tr} ((\mathcal{I}^*)^2\mathcal{I}^2) = \int_0^1 K(t, t) dt = \frac{1}{12},$$

and the statement follows. □

7. FIELDS-SUMS

We start with the additive processes of the form (1.7). It is clear that the covariance of the Gaussian field \mathbb{X} reads

$$G_{\mathbb{X}}(x, y) = \sum_{j=1}^d G_{X_j}(x_j, y_j).$$

Assume that $\rho \equiv 1$ and that the processes X_j have the spectral asymptotics (1.6).

If $d = 2$ we can apply Theorem 3.6, part 2. The operators $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathcal{T}}_2$ in this case are integral operators with kernels G_{X_1} and G_{X_2} , while $\widetilde{\mathcal{T}}_1$ and \mathcal{T}_2 are integral operators with kernel 1, i.e., orthogonal projectors to one-dimensional space of

constants. Then Theorem 4.2 gives us the small ball asymptotics for \mathbb{X} . In the general case we use Theorem 3.6 successively $d - 1$ times.

Now we consider some examples.

Proposition 7.1. *Let $H_j \in]0, 1[$, $m_j \in \mathbb{Z}_+$, $j = 1, \dots, d$. Then, as $\varepsilon \rightarrow 0$,*

$$\ln P \left\{ \left\| \sum_{j=1}^d (W^{H_j})_{m_j}(x_j) \right\| \leq \varepsilon \right\} \sim k^{\frac{2(m+H)+1}{2(m+H)}} \ln P \left\{ \| (W^H)_m \| \leq \varepsilon \right\},$$

where $m = \min_j m_j$, $H = \min_{j: m_j=m} H_j$ while $k = \#\{j : m_j = m, H_j = H\}$.

Proposition 7.2. *Let $\alpha_j > 0$, $j = 1, \dots, d$. Then, as $\varepsilon \rightarrow 0$, with the same notations as in Proposition 7.1,*

$$\begin{aligned} \ln P \left\{ \left\| \sum_{j=1}^d (U_{(\alpha_j)}^{H_j})_{m_j}(x_j) \right\| \leq \varepsilon \right\} \\ \sim \left(\sum_{\substack{m_j=m \\ H_j=H}} (2\alpha_j)^{\frac{1}{2(m+H)+1}} \right)^{\frac{2(m+H)+1}{2(m+H)}} \ln P \left\{ \| (W^H)_m \| \leq \varepsilon \right\}. \end{aligned}$$

As in §5, the processes X_j can be multiparameter. We consider one example.

Proposition 7.3. *Let $x^{(j)} = (x_1^{(j)}, \dots, x_{\ell_j}^{(j)})$, $\ell_j \in \mathbb{N}$, $H_j \in]0, 1[$, $m_j \in \mathbb{Z}_+$, $j = 1, \dots, d$. Then, as $\varepsilon \rightarrow 0$,*

$$\ln P \left\{ \left\| \sum_{j=1}^d (\mathbb{W}_{\ell_j}^{H_j})_{m_j}(x^{(j)}) \right\| \leq \varepsilon \right\} \sim k^{\frac{2(m+H)+1}{2(m+H)}} \ln P \left\{ \| (\mathbb{W}^H)_m \| \leq \varepsilon \right\},$$

where $m = \min_j m_j$, $H = \min_{j: m_j=m} H_j$, $\ell = \max_{\substack{m_j=m \\ H_j=H}} \ell_j$ while $k = \#\{j : m_j = m, H_j = H, \ell_j = \ell\}$.

All these results can be generalized to the case when the weight function has the form (5.2). We also give only one example in this context.

Proposition 7.4. *Let ρ_j , $j = 1, \dots, d$, be summable nonnegative functions on $[0, 1]$. Then, as $\varepsilon \rightarrow 0$, with the same notation as in Proposition 7.1,*

$$\begin{aligned} \ln P \left\{ \left\| \sum_{j=1}^d (W^{H_j})_{m_j}(x_j) \right\|_{\rho} \leq \varepsilon \right\} \\ \sim J_1^{\frac{1}{2(m+H)}} \left(\sum_{\substack{m_j=m \\ H_j=H}} \frac{J_{2(m+H)+1}^{(j)}}{(J_1^{(j)})^{\frac{1}{2(m+H)+1}}} \right)^{\frac{2(m+H)+1}{2(m+H)}} \cdot \ln P \left\{ \| W^H \| \leq \varepsilon \right\} \end{aligned}$$

(we recall that $J_h^{(j)} = \int_0^1 \rho_j(t)^{1/h} dt$ and $J_h = \int_{[0,1]^d} \rho(x)^{1/h} dx$).

Finally we study more general sums with infinite-dimensional operators $\tilde{\mathcal{T}}_1$ and \mathcal{T}_2 . Let X_j and Y_j , $j = 1, 2$, be independent Gaussian processes on $[0, 1]$. Consider the following Gaussian field on $[0, 1]^2$:

$$\mathbb{X} = X_1 \otimes Y_1 + X_2 \otimes Y_2.$$

If X_1 and Y_2 have the same one-term spectral asymptotics (4.1) while X_2 and Y_1 are the fast factors, we can apply Theorem 3.6, part **3**, and then Theorem 4.2. We restrict ourselves to the following example.

Proposition 7.5. *Let $0 < H < 1/2$. Then, as $\varepsilon \rightarrow 0$,*

$$\begin{aligned} & \ln P \{ \|B^H \otimes W + \overline{B} \otimes W^H\| \leq \varepsilon \} \\ & \sim \left(2^{\frac{2}{2H+1}} + 2^{-\frac{1-2H}{2H+1}} - 1 \right)^{\frac{2H+1}{2H}} \pi^{-\frac{1}{H}} \zeta^{\frac{2H+1}{2H}} \left(\frac{2}{2H+1} \right) \cdot \ln P \{ \|W^H\| \leq \varepsilon \}. \end{aligned}$$

8. SOME GENERALIZATIONS

To consider a more general class of weighted norms we need the following abstract result. Here we do not make distinctions between the function Ψ and the corresponding multiplication operator.

Theorem 8.1. *Let Ω and $\tilde{\Omega}$ be bounded domains in \mathbb{R}^n and $\mathbb{R}^{\tilde{n}}$, correspondingly. Let two compact self-adjoint nonnegative operators \mathcal{T} and $\tilde{\mathcal{T}}$ (in $L_2(\Omega)$ and $L_2(\tilde{\Omega})$, correspondingly) satisfy the following conditions:*

- 1. *For any cubes $\omega \subset \Omega$, $\tilde{\omega} \subset \tilde{\Omega}$, as $t \rightarrow 0+$,*

$$\begin{aligned} \mathcal{N}(t, \chi_\omega \mathcal{T} \chi_\omega) & \sim \frac{\varphi(1/t)}{t^{1/p}} \cdot \int_\omega a(x) \, dx, \\ \mathcal{N}(t, \chi_{\tilde{\omega}} \tilde{\mathcal{T}} \chi_{\tilde{\omega}}) & \sim \frac{\tilde{\varphi}(1/t)}{t^{1/p}} \cdot \int_{\tilde{\omega}} \tilde{a}(\tilde{x}) \, d\tilde{x}, \end{aligned}$$

where χ_A stands for the indicator of the set A , a and \tilde{a} are nonnegative continuous functions in Ω and $\tilde{\Omega}$, correspondingly, $p > 0$ while φ and $\tilde{\varphi}$ are SVFs satisfying (3.11).

- 2. *For any pairs of nonintersecting open cubes $\omega, \omega_1 \subset \Omega$ and $\tilde{\omega}, \tilde{\omega}_1 \subset \tilde{\Omega}$*

$$\Delta_{p,\varphi}(|\chi_\omega \mathcal{T} \chi_{\omega_1}|) = 0, \quad \Delta_{p,\tilde{\varphi}}(|\chi_{\tilde{\omega}} \tilde{\mathcal{T}} \chi_{\tilde{\omega}_1}|) = 0.$$

Then for any nonnegative function $\Psi \in \mathcal{C}(\overline{\Omega \times \tilde{\Omega}})$, as $t \rightarrow 0+$,

$$(8.1) \quad \mathcal{N}(t, \Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi) \sim \frac{\phi(1/t)}{t^{1/p}} \cdot \int_{\tilde{\Omega}} \int_{\Omega} \Psi^{2/p}(x, \tilde{x}) \frac{a(x)\tilde{a}(\tilde{x})}{p} \, dx d\tilde{x},$$

where $\phi(\tau) = (\varphi * \tilde{\varphi})(\tau)$.

Proof. Step 1. Let $\Psi(x, \tilde{x}) = c\chi_\omega(x)\chi_{\tilde{\omega}}(\tilde{x})$. Then

$$\Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi = c^2(\chi_\omega \mathcal{T} \chi_\omega) \otimes (\chi_{\tilde{\omega}} \tilde{\mathcal{T}} \chi_{\tilde{\omega}}).$$

Applying Theorem 3.4, part **1**, and taking into account (3.2), we obtain (8.1).

Step 2. Let $\Psi(x, \tilde{x}) \equiv \Psi_N(x, \tilde{x}) = \sum_{m, \tilde{m}=1}^N c_{m\tilde{m}} \chi_{\omega_m}(x) \chi_{\tilde{\omega}_{\tilde{m}}}(\tilde{x})$, where $\{\omega_m\}_1^N$ and $\{\tilde{\omega}_{\tilde{m}}\}_1^N$ are disjoint open cubes in Ω and $\tilde{\Omega}$, correspondingly. Then

$$\Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi = \sum_{j, m, \tilde{j}, \tilde{m}=1}^N c_{j\tilde{j}} c_{m\tilde{m}} \mathcal{S}_{jm} \otimes \tilde{\mathcal{S}}_{\tilde{j}\tilde{m}},$$

where

$$\mathcal{S}_{jm} = \chi_{\omega_j} \mathcal{T} \chi_{\omega_m}, \quad \tilde{\mathcal{S}}_{\tilde{j}\tilde{m}} = \chi_{\tilde{\omega}_{\tilde{j}}} \tilde{\mathcal{T}} \chi_{\tilde{\omega}_{\tilde{m}}}.$$

For $j \neq m$ we have $\Delta_{p,\varphi}(|\mathcal{S}_{jm}|) = 0$. Similarly to the proof of Theorem 3.3 one can show that for all \tilde{j} and \tilde{m}

$$(8.2) \quad \Delta_{p,\phi}(|\mathcal{S}_{jm} \otimes \tilde{\mathcal{S}}_{\tilde{j}\tilde{m}}|) = 0.$$

In the same way, (8.2) holds if $\tilde{j} \neq \tilde{m}$. Therefore, by (3.17a), (3.17b) we obtain

$$(8.3) \quad \lim_{t \rightarrow 0+} \frac{t^{1/p}}{\phi(1/t)} \cdot \mathcal{N}(t, \Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi) = \lim_{t \rightarrow 0+} \frac{t^{1/p}}{\phi(1/t)} \cdot \mathcal{N}(t, \sum_{m, \tilde{m}=1}^N c_{m\tilde{m}}^2 \mathcal{S}_{mm} \otimes \tilde{\mathcal{S}}_{\tilde{m}\tilde{m}}).$$

Next, for $(j, \tilde{j}) \neq (m, \tilde{m})$ the operators $\mathcal{S}_{jj} \otimes \tilde{\mathcal{S}}_{\tilde{j}\tilde{j}}$ and $\mathcal{S}_{mm} \otimes \tilde{\mathcal{S}}_{\tilde{m}\tilde{m}}$ are orthogonal. Hence the variational principle yields

$$(8.4) \quad \mathcal{N}(t, \sum_{m, \tilde{m}=1}^N c_{m\tilde{m}}^2 \mathcal{S}_{mm} \otimes \tilde{\mathcal{S}}_{\tilde{m}\tilde{m}}) = \sum_{m, \tilde{m}=1}^N \mathcal{N}(t, c_{m\tilde{m}}^2 \mathcal{S}_{mm} \otimes \tilde{\mathcal{S}}_{\tilde{m}\tilde{m}}).$$

The asymptotics of summands as $t \rightarrow 0+$ was obtained in Step 1. Substituting (8.4) to (8.3) we get

$$\mathcal{N}(t, \Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi) \sim \frac{\phi(1/t)}{t^{1/p}} \cdot \sum_{m, \tilde{m}=1}^N c_{m\tilde{m}}^{2/p} \int_{\tilde{\omega}_{\tilde{m}}} \int_{\omega_m} \frac{a(x)\tilde{a}(\tilde{x})}{p} dx d\tilde{x},$$

that gives (8.1).

Step 3. Let the function Ψ be uniformly approximated by functions Ψ_N considered in Step 2. In view of (3.17a), (3.17b) for passing to the limit it is sufficient to prove that

$$\Delta_{p,\phi}(|\Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi - \Psi_N(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi_N|) \rightarrow 0, \quad N \rightarrow \infty.$$

By (3.17a) and (3.18) we obtain

$$\begin{aligned} & (\Delta_{p,\phi}(|\Psi(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi - \Psi_N(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi_N|))^{p+1} \\ & \leq (\Delta_{p,\phi}(|(\Psi - \Psi_N)(\mathcal{T} \otimes \tilde{\mathcal{T}})\Psi|))^{p+1} + (\Delta_{p,\phi}(|\Psi_N(\mathcal{T} \otimes \tilde{\mathcal{T}})(\Psi - \Psi_N)|))^{p+1} \\ & \leq \|\Psi - \Psi_N\|^{1/p+1} \cdot (\Delta_{p,\phi}(\mathcal{T} \otimes \tilde{\mathcal{T}}))^{p+1} \cdot \left(\|\Psi\|^{1/p+1} + \|\Psi_N\|^{1/p+1} \right). \end{aligned}$$

Here $\|\cdot\|$ stands for the norms of multiplication operators in $L_2(\Omega \times \tilde{\Omega})$ which coincide with L_∞ -norms of corresponding functions. So, the last product tends to zero as $N \rightarrow \infty$, and (8.1) follows.

It remains to note that the class of weights considered contains $\mathcal{C}(\overline{\Omega \times \tilde{\Omega}})$. \square

Corollary 8.2. *Propositions 5.1 – 5.4 hold true for any $\rho \in \mathcal{C}([0, 1]^d)$.*

Proof. First of all, applying Theorem 8.1 repeatedly we obtain analogous result for the tensor product $\bigotimes_{j=1}^d \mathcal{T}_j$. Next, due to [BS1], Appendix 7, the marginal integral operators under consideration satisfy the conditions of Theorem 8.1. Then we can put $\Psi = \sqrt{\rho}$, and the statement follows. \square

Remark 8. In fact, Theorem 8.1 is true for $\Psi \in L_2(\Omega \times \tilde{\Omega})$ (and therefore Corollary 8.2 is true for $\rho \in L_1([0, 1]^d)$). The proof requires more delicate interpolation techniques. We are going to return to this problem in a separate work.

The results of previous sections can be also generalized to the vector-valued zero mean Gaussian fields $X(x) \in \mathbb{R}^\ell$, $x \in [0, 1]^d$. Let, for example, the components

$X^{(\mu)}$, $\mu = 1, \dots, \ell$, be d -dimensional fractional Brownian sheets with the Hurst parameters H_μ . Let $A(x)$ be real $\ell \times k$ -matrix, $A^{(\mu\tau)}(x) = \prod_{j=1}^d A_j^{(\mu\tau)}(x_j)$ for all $\mu = 1, \dots, \ell$, $\tau = 1, \dots, k$, and $A_j \in L_2([0, 1])$. Then

$$A^T(x)X(x) = \bigotimes_{j=1}^d A_j^T X_j,$$

where the marginal processes X_j are the vector-valued fractional Brownian motions on $[0, 1]$.

Nazarov and Nikitin ([NaNi2], §5) derived the spectral asymptotics for A_j -weighted vector-valued fractional Brownian motions. It was proved that

$$\lambda_n^{(j)} = \widehat{C}^{(j)} n^{-(2H+1)}(1 + o(1)),$$

where $H = \min_\mu H_\mu$ while $\widehat{C}^{(j)}$ can be determined in terms of A_j and marginal cross-covariance matrices $G_j^{(\mu\nu)}(x_j, y_j) = EX_j^{(\mu)}(x_j)X_j^{(\nu)}(y_j)$. The same arguments as in Proposition 5.1 now yield

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{1/H} \ln^{-(d-1)\frac{2H+1}{2H}}(1/\varepsilon) \cdot \ln P \left\{ \int_{[0,1]^d} \|A^T(x)X(x)\|^2 dx \leq \varepsilon^2 \right\} \\ &= - \left(\frac{\prod_{j=1}^d \widehat{C}^{(j)}}{(d-1)!^{2H+1}} \right)^{\frac{1}{2H}} H^{1-\frac{(d-1)(2H+1)}{2H}} \left(\frac{\pi}{(2H+1) \sin\left(\frac{\pi}{2H+1}\right)} \right)^{\frac{2H+1}{2H}}. \end{aligned}$$

In the case when only one (for example, the first) component $X^{(\mu)}$ has the least Hurst parameter, i.e. $H_1 = H$, $H_\mu > H$ for $\mu \geq 2$, the explicit formula for $\widehat{C}^{(j)}$ gives, similarly to (5.6),

$$\prod_{j=1}^d \widehat{C}^{(j)} = \left(\mathcal{J}_{2H+1}^{(1)} \right)^{2H+1} \left(\frac{\Gamma(2H+1) \sin(\pi H)}{\pi^{2H+1}} \right)^d$$

(here $\mathcal{J}_h^{(\mu)} = \int_{[0,1]^d} |A^{(\mu)}(x)|^{2/h} dx$ while $A^{(\mu)}$ is the μ -th row of the matrix A).

The formula obtained holds true for $A \in C([0, 1]^d \rightarrow \mathbb{R}^{\ell k})$. It can be proved by applying the vector-valued analogue of Theorem 8.1.

Analogous statements are true for the vector-valued isotropically integrated fractional Brownian sheet, for the vector-valued fractional Ornstein – Uhlenbeck sheet, and for similar fields.

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