

TWO NEW WEYL-TYPE BOUNDS FOR THE DIRICHLET LAPLACIAN

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ABSTRACT. In this paper, we prove two new Weyl-type upper estimates for the eigenvalues of the Dirichlet Laplacian. As a consequence, we obtain the following *lower* bounds for its counting function. For $\lambda \geq \lambda_1$, one has

$$N(\lambda) > \frac{2}{n+2} \frac{1}{H_n} (\lambda - \lambda_1)^{n/2} \lambda_1^{-n/2}$$

and

$$N(\lambda) > \left(\frac{n+2}{n+4} \right)^{n/2} \frac{1}{H_n} (\lambda - (1 + 4/n) \lambda_1)^{n/2} \lambda_1^{-n/2},$$

where

$$H_n = \frac{2n}{j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1})}$$

is a constant which depends on n , the dimension of the underlying space, and Bessel functions and their zeros.

1. FOUR NEW ESTIMATES

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary. We are interested in bounds for the eigenvalues of the fixed and free membrane whose shape is assumed by Ω . The first problem (also called the Dirichlet problem) is described by the equation

$$(1.1) \quad \begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Its eigenvalues, known to form a discrete countable family with no finite accumulation point (see [19] [25] for example), are denoted (counting multiplicity) by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \rightarrow \infty$. Its associated eigenfunctions, which form an orthonormal basis of *real* functions in $L^2(\Omega)$, are denoted by u_1, u_2, u_3, \dots . The second problem (also called the Neumann problem [20]) is described by

$$(1.2) \quad \begin{aligned} -\Delta v &= \mu v \text{ in } \Omega, \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Its eigenvalues, also discrete and countable, are denoted by $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_k \rightarrow \infty$. In this paper, we show the following.

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Theorem 1.1. For $k \geq 1$, we have

$$(1.3) \quad \lambda_{k+1} - \lambda_1 \leq \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \lambda_1 k^{2/n}$$

and the sharper, average-type, inequality

$$(1.4) \quad \sum_{j=1}^k (\lambda_j - \lambda_1) \leq \frac{n}{n+2} H_n^{2/n} \lambda_1 k^{1+2/n}$$

where

$$(1.5) \quad H_n = \frac{2n}{j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1})}.$$

As corollaries to these two inequalities, we prove (1.7) and (1.8). Here $J_n(x)$ and $j_{n,p}$ denote, respectively, the Bessel function of order n , and the p th zero of this function (see [1]). The proof of this theorem is offered in Section 3. That (1.4) is sharper than (1.3) follows from left Riemann sum considerations (see Figure 1), namely

$$(1.6) \quad \sum_{j=0}^{k-1} j^{2/n} < \frac{k^{1+2/n}}{1+2/n}.$$

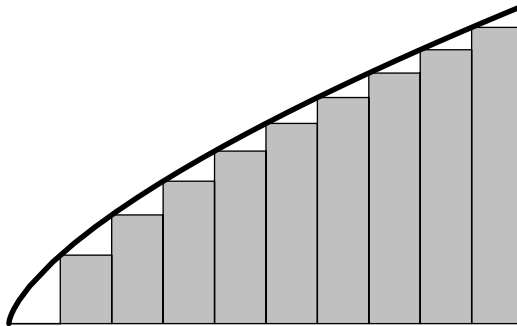


FIGURE 1. Left Riemann sums for $y = x^{2/n}$, for $0 \leq x \leq k$.

To see this, we apply (1.3) to $j = 0, 1, \dots, k - 1$, then sum. We obtain

$$\begin{aligned} \sum_{j=1}^k (\lambda_j - \lambda_1) &\leq \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \lambda_1 \left(\sum_{j=0}^{k-1} j^{2/n}\right) \\ &< \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \lambda_1 \frac{k^{1+2/n}}{1+2/n}. \end{aligned}$$

Inequality (1.4) is tighter since

$$\left(1 + \frac{n}{2}\right)^{2/n} > 1.$$

In fact, we have the following corollary.

Corollary 1.2. *For $k \geq 1$, the eigenvalues of the Dirichlet problem satisfy the estimate*

$$(1.7) \quad \sum_{j=2}^{k+1} \frac{1}{\lambda_j - \lambda_1} \geq \frac{n+2}{n} H_n^{-2/n} \frac{1}{\lambda_1} \frac{k^2}{(k+1)^{1+2/n}}.$$

This of course follows from the Cauchy-Schwarz inequality

$$\begin{aligned} k^2 &= \left(\sum_{j=2}^{k+1} \frac{1}{\sqrt{\lambda_j - \lambda_1}} \sqrt{\lambda_j - \lambda_1} \right)^2 \\ &\leq \sum_{j=2}^{k+1} \frac{1}{\lambda_j - \lambda_1} \sum_{j=2}^{k+1} (\lambda_j - \lambda_1) \end{aligned}$$

and inequality (1.4). Another consequence of (1.4) and H. C. Yang’s [52] (see also [3] [4], [15], [16], [17]) inequality

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n} \right) \frac{1}{k} \sum_{j=1}^k \lambda_j,$$

is the bound

$$(1.8) \quad \lambda_{k+1} \leq \left(1 + \frac{4}{n} \right) \left(1 + \frac{n}{n+2} H_n^{2/n} k^{2/n} \right) \lambda_1.$$

These types of inequalities follow the spirit of Weyl’s asymptotic law, which states that

$$\lambda_k \sim \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}} \text{ as } k \rightarrow \infty,$$

and

$$\mu_{k+1} \sim \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}} \text{ as } k \rightarrow \infty,$$

where $C_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}$ = volume of the unit n -ball, and $|\Omega|$ denotes the volume of Ω . These formulas were proved by Weyl [51] in 1910.

There is a beautiful exposé of the history of this problem in Kac’s paper [30] (see also the equally entertaining paper [46]). Baltés and Hilf [18] trace the history of this type of asymptotic to Pockels (1891) (who proved the discreteness of the spectrum of the Dirichlet Laplacian [19]), Lord Rayleigh (1905), Sommerfeld (1910), and Lorentz (1910). Many asymptotics of this type were developed by Courant and Hilbert [25], Pleijel, and Minakshisundaram (see [18] for further insight and references).

In 1954, Pólya conjectured [42] that, for $k = 1, 2, \dots$ (see also the series of papers [43] [44] [45])

$$(1.9) \quad \mu_{k+1} \leq \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}} \leq \lambda_k.$$

He proved his conjecture for the case of tiling domains in a paper dedicated to Weyl in 1961 [43]. The restrictive conditions for the Neumann case in [43] (see also [45]) were relaxed, and the proof was refined and simplified by Kellner [31] “to Pólya’s

pleasure and satisfaction,” reports Hersch (see p. 523 of [29]). In 1984, Urakawa [50] refined the Dirichlet bound to

$$(1.10) \quad \lambda_k \geq \delta_L(\Omega)^{2/n} \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}$$

where $\delta_L(\Omega)$ is the *lattice packing density* of Ω ($\delta_L(\Omega) = 1$ for a tiling domain).

Also in 1954, Payne conjectured [39] that, independently of the Weyl term in (1.9),

$$(1.11) \quad \mu_{k+1} \leq \lambda_k.$$

This question was settled by Friedlander [28] in 1991 for domains with smooth boundaries. More recently, Friedlander’s breakthrough was generalized for domains with non-smooth boundaries by Filonov [27]. On a different track, Li and Yau [37] proved that, for $k \geq 1$,

$$(1.12) \quad \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2 k^{1+2/n}}{(C_n|\Omega|)^{2/n}},$$

from which it obtains that

$$(1.13) \quad \lambda_k \geq \frac{n}{n+2} \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}.$$

Inequalities (1.3), (1.4) and (1.8) can then be thought of as counterparts to these two inequalities of Li-Yau.

In 1992, Kröger [33] produced the Neumann parallels to the Li-Yau inequalities. For $k \geq 1$, he proved

$$(1.14) \quad \sum_{i=1}^k \mu_i \leq \frac{n}{n+2} \frac{4\pi^2 k^{1+2/n}}{(C_n|\Omega|)^{2/n}}$$

and

$$(1.15) \quad \mu_{k+1} \leq \left(1 + \frac{n}{2}\right)^{2/n} \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}.$$

Notice that (1.15) implies that

$$\begin{aligned} \sum_{j=1}^k \mu_j &\leq \left(1 + \frac{n}{2}\right)^{2/n} \frac{4\pi^2}{(C_n|\Omega|)^{2/n}} \left(\sum_{j=0}^{k-1} j^{2/n}\right) \\ &< \left(1 + \frac{n}{2}\right)^{2/n} \frac{4\pi^2}{(C_n|\Omega|)^{2/n}} \frac{k^{1+2/n}}{1 + 2/n}. \end{aligned}$$

Again, we have used the left Riemann sum comparison (1.6) in these two inequalities. Kröger’s inequality (1.14) is tighter, since

$$\frac{(1 + n/2)^{2/n}}{1 + 2/n} > \frac{1}{1 + 2/n} = \frac{n}{n + 2}.$$

Thus, the “averaged” version (namely (1.14)) of Kröger’s two inequalities is sharper.

Our bounds (1.3) and (1.8) are also related to the result of Ashbaugh and Benguria [9] who proved, for $m \geq 1$,

$$(1.16) \quad \frac{\lambda_{2^m}}{\lambda_1} \leq \left(\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \right)^m.$$

Of course, one cannot expect to fare better in the case of $m = 1$ since this is another conjecture by Payne, Pólya, and Weinberger [40], [41] (herein referred to as PPW) which was settled by Ashbaugh and Benguria [5] (see also [6]) in 1991, namely

$$(1.17) \quad \frac{\lambda_2}{\lambda_1} \leq \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2}.$$

The ratio on the RHS of (1.17) is that for the two first eigenvalues of an n -ball. It has the asymptotic expansion [10]

$$(1.18) \quad \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \sim 1 + \frac{4}{n} - \frac{4}{3} c_1 \frac{2^{5/3}}{n^{5/3}} + \frac{12}{n^2} + \frac{4}{3}(c_1^2 - 2c_2) \frac{2^{7/3}}{n^{7/3}} + O(n^{-8/3}),$$

where $c_1 \approx 1.8557571$ and $c_2 \approx 1.033150$ (see [1]).

Payne, Pólya, and Weinberger [40], [41] (see also [9], [49]) proved the weaker form

$$(1.19) \quad \lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{j=1}^k \lambda_j,$$

from which one can infer that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq 1 + 4/n$$

and

$$(1.20) \quad \frac{\lambda_k}{\lambda_1} \leq (1 + 4/n)^{k-1}.$$

Note that (1.16) can be put in the form

$$(1.21) \quad \frac{\lambda_k}{\lambda_1} \leq \left(\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \right)^{\left[\frac{\log k}{\log 2} \right]}$$

where $[x]$ stands for the integer part of x . This bound can be thought of as one of the form

$$(1.22) \quad \left(\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \right)^{\frac{\log k}{\log 2}} = k^{\frac{\log(j_{n/2,1}^2/j_{n/2-1,1}^2)}{\log 2}}.$$

By virtue of the expansion (1.18), the power $\frac{\log(j_{n/2,1}^2/j_{n/2-1,1}^2)}{\log 2}$ has the asymptotic form

$$\frac{1}{\log 2} \left(\frac{4}{n} - \frac{4}{3} c_1 \frac{2^{5/3}}{n^{5/3}} + \frac{4}{n^2} + \frac{4}{3}(c_1^2 - 2c_2) \frac{2^{7/3}}{n^{7/3}} \right) + O(n^{-8/3}),$$

or

$$(1.23) \quad \frac{\log \left(j_{n/2,1}^2 / j_{n/2-1,1}^2 \right)}{\log 2} \sim \frac{5.77078}{n} - 6.10703 c_1 \frac{1}{n^{5/3}} + O \left(\frac{1}{n^2} \right).$$

Thus, while tight at the bottom of the spectrum (viz. (1.17)), (1.16) does not capture the expected Weyl behavior of $k^{2/n}$. Inequalities (1.3) and (1.8) remedy this.

The key to the new results is an observation by Ashbaugh and Benguria—the extent and limitations of which are discussed on p. 561 of [11]. If one identifies μ_{k+1} with $\lambda_{k+1} - \lambda_1$ and $|\Omega|^{-2/n}$ with λ_1 , then the RHS of the PPW inequality (1.17) can be seen as maximizing the ratio λ_2/λ_1 in the same vein as the quantity $C_n^{2/n} p_{n/2,1}^2$ maximizes $|\Omega|^{-2/n} \mu_2$ for any domain Ω ($p_{\nu,k}$ denotes the k th positive zero of the derivative of $x^{1-\nu} J_\nu(x)$ and C_n is as defined above, i.e. the volume of the unit n -ball). The latter is a result of Szegő in 2 dimensions and Weinberger in n dimensions. The maximum for both is assumed when Ω is an n -ball. The strategy of proof for both is similar, though the first is considerably more involved [5], [6]. This loose analogy can also be seen in comparing the methods of proof and results for

$$(1.24) \quad \sum_{j=1}^n (\lambda_{j+1} - \lambda_1) \leq 4 \lambda_1$$

and

$$(1.25) \quad \sum_{j=1}^n \mu_{j+1} \leq n \left(\frac{C_n}{|\Omega|} \right)^{2/n} p_{n/2,1}^2$$

both of which were proved by Ashbaugh and Benguria in [10] and [11]. Inequality (1.24) is the extension to n dimensions of a result in [41]. (Note that (1.25) was proved with the further restriction that Ω is invariant with respect to 90° rotations in the coordinate planes.) Our new inequality (1.4) can be viewed as an extension for $k \neq n$ of (1.24). (See Section 4 for a comparison with existing results.)

The loose correspondence can also be traced in the analogy between

$$(1.26) \quad \sum_{k=1}^n \frac{1}{\lambda_{k+1} - \lambda_1} \geq \frac{2j_{n/2-1,1}^2 + n(n-4)}{6\lambda_1} > \frac{n^2}{4\lambda_1}$$

and

$$(1.27) \quad \sum_{k=1}^n \frac{1}{\mu_{k+1}} \geq \frac{n \left(\frac{|\Omega|}{C_n} \right)^{2/n}}{p_{n/2,1}^2}.$$

Both of these bounds are also results found in [10] and [11] (with (1.27) also true under rotational symmetry of the base domain Ω). Inequality (1.26) is an extension and improvement of earlier results of Chiti [24]. The $n^2/4\lambda_1$ term in (1.26) is what corresponds to (1.24) via the “usual Cauchy-Schwarz connection” (viz. the proof of Corollary 1.2). On the other hand, there is also a conjectured inequality, from which, if proved, (1.26) would follow via the Cauchy-Schwarz argument. That

inequality would be (1.24), but with its RHS replaced by

$$\frac{6n^2\lambda_1}{2j_{n/2-1,1}^2 + n(n-4)}.$$

Our new inequality (1.7) can be viewed as an extension, for $k \neq n$, of the Ashbaugh-Benguria-Chiti inequality (1.26).

We complete this section by giving the asymptotic expansions for the coefficients appearing in (1.3), and (1.8) (see [10] and [34] for similar estimates):

$$(1.28) \quad \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \sim 1 + \frac{2}{3n} \log \frac{4n^4}{b_0^6} + 2^{8/3}(b_1 - c_1) \left(\frac{1}{n}\right)^{5/3} + O\left(\frac{1}{n}\right)^2.$$

In the case of (1.8), the expansion reads

$$(1.29) \quad \left(1 + \frac{4}{n}\right) \frac{n}{n+2} H_n^{2/n} \sim 1 + \frac{2}{3n} \left(3 + \log \frac{32n}{b_0^6}\right) + 2^{8/3} (b_1 - c_1) \left(\frac{1}{n}\right)^{5/3} + O\left(\frac{1}{n}\right)^2.$$

Here (see [1]) $b_0 \approx 1.1131028$, $b_1 \approx 1.484606$ and $c_1 \approx 1.8557571$.

2. THE COUNTING FUNCTION

One can motivate these inequalities in terms of the *counting function*,

$$N(\lambda) = \sum_{\lambda_k \leq \lambda} 1 = \sup_{\lambda_k \leq \lambda} k.$$

Our Theorem 1.1 can then be restated.

Theorem 2.1. *For $\lambda \geq \lambda_1$, we have the lower bounds*

$$(2.1) \quad N(\lambda) > \frac{2}{n+2} \frac{1}{H_n} (\lambda - \lambda_1)^{n/2} \lambda_1^{-n/2}$$

and

$$(2.2) \quad N(\lambda) > \left(\frac{n+2}{n+4}\right)^{n/2} \frac{1}{H_n} (\lambda - (1 + 4/n) \lambda_1)^{n/2} \lambda_1^{-n/2}.$$

Remark. While (2.1) is a direct corollary to an earlier result of Laptev (see Cor. 4.4 in [35]) and Chiti’s inequality (3.8) below (see [23], [24], [14]), (2.2) is new and in fact sharper. We refer the reader to the discussion in Section 4.

Proof. Inequalities (2.1) and (2.2) follow from (1.3) and (1.8), respectively. The proof for both is similar. We show (2.1) for illustration. Let $N(\lambda) = j$. Then, from the definition of $N(\lambda)$, $\lambda_{j+1} > \lambda$. Inequality (1.3) then implies

$$\lambda - \lambda_1 < \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \lambda_1 j^{2/n}.$$

Reversing, one gets (2.1). □

Remark. Using an entirely different method, Safarov (see ineq. (2.9) of [48]) proved

$$(2.3) \quad N(\lambda) \geq \frac{2}{n+2} e^{-1/4\pi} L_n^{cl} (\lambda - \lambda_1)^{n/2} \lambda_1^{-n/2},$$

where $L_n^{cl} = C_n/(2\pi)^n$. In Table 1 we have listed numerical values of the coefficients appearing in (2.2), (2.1), and (2.3). These inequalities complete bounds of the form

TABLE 1. Comparison of the coefficients appearing in (2.2), (2.1), and (2.3).

n	$\left(\frac{n+2}{n+4}\right)^{n/2} \frac{1}{H_n}$	$\frac{2}{n+2} \frac{1}{H_n}$	$\frac{2}{n+2} e^{-1/4\pi} L_n^{cl}$
2	0.259775	0.194831	0.036745
3	0.201227	0.133333	0.062381
4	0.167459	0.099235	0.000975
5	0.145412	0.077874	0.000142
6	0.129833	0.063395	0.000019
7	0.118201	0.053193	2.5×10^{-6}

$$(2.4) \quad N(\lambda) \leq \tilde{K}_n \lambda^{n/2} |\Omega|$$

found in the works of Lieb [38] and Li-Yau [37] (see Laptev [35]). Of course, Weyl’s asymptotic formula reads

$$(2.5) \quad N(\lambda) \sim \frac{C_n |\Omega| \lambda^{n/2}}{(2\pi)^n} = L_n^{cl} |\Omega| \lambda^{n/2},$$

while the Pólya conjecture states

$$(2.6) \quad N(\lambda) \leq L_n^{cl} \lambda^{n/2} |\Omega|.$$

The Li-Yau bounds can be reformulated as (see [35])

$$(2.7) \quad N(\lambda) \leq \left(\frac{n+2}{n}\right)^{n/2} L_n^{cl} \lambda^{n/2} |\Omega|.$$

In the same spirit, one should note F. Berezin’s inequality [21] (see [47])

$$(2.8) \quad \int_0^\lambda N(\mu) d\mu \leq \frac{2}{n+2} L_n^{cl} \lambda^{n/2+1} |\Omega|.$$

Laptev [35] and Safarov [47] have noted that the Li-Yau bound (2.7) is a corollary of (2.8). Indeed, for $\theta > 0$,

$$(2.9) \quad N(\lambda) \leq \frac{1}{\theta\lambda} \int_0^{\lambda+\theta\lambda} N(\mu) d\mu \leq \frac{2(1+\theta)^{n/2+1}}{(n+2)\theta} L_n^{cl} |\Omega| \lambda^{n/2}.$$

Li-Yau’s bound follows by setting $\theta = 2/n$. As for the Ashbaugh-Benguria inequality (1.16), it can be reworked to appear in the following terms (see [9]): For $\lambda \geq \lambda_1$,

$$(2.10) \quad N(\lambda) \geq 2^{\lceil \log(\lambda/\lambda_1) / \log(j_{n/2,1}^2/j_{n/2-1,1}^2) \rceil}.$$

Notice that the RHS of this inequality assumes the form

$$\left(\frac{\lambda}{\lambda_1}\right)^{\frac{1}{\log_2 j_{n/2,1}^2/j_{n/2-1,1}^2}}.$$

This allows one to restate (2.10) (in view of (1.23)) as

$$N(\lambda) \geq \left(\frac{\lambda}{\lambda_1}\right)^{n/5.77078}.$$

In fact, for $\lambda \geq \lambda_2$, Ashbaugh and Benguria have the sharper bound [9]

$$(2.11) \quad N(\lambda) \geq 2^{1+\lceil \log(\lambda/\lambda_2) / \log(j_{n/2,1}^2/j_{n/2-1,1}^2) \rceil}$$

which, in view of the above considerations, reads as

$$N(\lambda) \geq 2 \left(\frac{\lambda}{\lambda_2} \right)^{n/5.77078}.$$

3. PROOF OF THEOREM 1.1

We begin with the Rayleigh-Ritz estimate for λ_{k+1} ,

$$(3.1) \quad \lambda_{k+1} \leq \inf_{r \geq r_0} \frac{\int_{B_r} \int_{\Omega} |\nabla \phi|^2 d\mathbf{x} d\mathbf{z}}{\int_{B_r} \int_{\Omega} |\phi|^2 d\mathbf{x} d\mathbf{z}},$$

where B_r = a ball of radius $r \geq r_0$, and $r_0 = H_n^{1/n} (1+k)^{1/n} \sqrt{\lambda_1}$. This characterization is suggested by considerations similar to [33]. In fact, the bulk of the arguments follow steps described there. The *test function* ϕ is required to satisfy

$$\phi \perp u_1, u_2, \dots, u_k.$$

It is chosen to be of the form

$$\phi = e^{i\mathbf{x} \cdot \mathbf{z}} u_1(\mathbf{x}) - \sum_{j=1}^k a_j(\mathbf{z}) u_j(\mathbf{x}).$$

The orthogonality conditions lead to $a_j(\mathbf{z}) = \int_{\Omega} u_1 \bar{u}_j e^{i\mathbf{x} \cdot \mathbf{z}} d\mathbf{x}$. We calculate

$$(3.2) \quad \begin{aligned} \int_{\Omega} |\phi|^2 &= \int_{\Omega} |u_1|^2 - 2 \sum_j |a_j|^2 + \sum_{j,\ell} a_j \bar{a}_{\ell} \int_{\Omega} u_j \bar{u}_{\ell} \\ &= 1 - \sum_{j=1}^k |a_j|^2, \end{aligned}$$

since $\int_{\Omega} u_j \bar{u}_{\ell} = \delta_{j\ell}$. One has

$$\int_{\Omega} |\nabla \phi|^2 = \frac{1}{2} \left(\int_{\Omega} \phi(-\Delta \bar{\phi}) + \int_{\Omega} (-\Delta \phi) \bar{\phi} \right),$$

since ϕ and $\bar{\phi}$ satisfy the Dirichlet boundary condition. We let $\phi_0 = e^{i\mathbf{x} \cdot \mathbf{z}}$. Then,

$$\begin{aligned} -\Delta \phi &= (-\Delta \phi_0) u_1 - 2 \nabla \phi_0 \cdot \nabla u_1 + \lambda_1 \phi_0 u_1 - \sum a_j \lambda_j u_j \\ &= (-\Delta \phi_0) u_1 - 2 \nabla \phi_0 \cdot \nabla u_1 + \lambda_1 \left(\phi + \sum a_j u_j \right) - \sum a_j \lambda_j u_j \\ &= (-\Delta \phi_0) u_1 - 2 \nabla \phi_0 \cdot \nabla u_1 + \lambda_1 \phi - \sum a_j (\lambda_j - \lambda_1) u_j. \end{aligned}$$

Similarly,

$$-\Delta \bar{\phi} = (-\Delta \bar{\phi}_0) \bar{u}_1 - 2 \nabla \bar{\phi}_0 \cdot \nabla \bar{u}_1 + \lambda_1 \bar{\phi} - \sum \bar{a}_j (\lambda_j - \lambda_1) \bar{u}_j.$$

Therefore,

$$\int_{\Omega} (-\Delta \phi) \bar{\phi} = \lambda_1 \int_{\Omega} |\phi|^2 + \int_{\Omega} (-\Delta \phi_0 u_1 - 2 \nabla \phi_0 \cdot \nabla u_1) \bar{\phi} - \sum a_j (\lambda_j - \lambda_1) \int_{\Omega} u_j \bar{\phi}.$$

Orthogonality makes $\int_{\Omega} u_j \bar{\phi} = 0$.

$$(3.3) \quad \int_{\Omega} (-\Delta\phi)\bar{\phi} = \lambda_1 \int_{\Omega} |\phi|^2 + \int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{\phi}.$$

We now concentrate on the quantity $\int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{\phi}$. It is equal to

$$\int_{\Omega} (-\Delta\phi_0 \bar{\phi}_0 |u_1|^2 - 2\nabla\phi_0 \cdot \nabla u_1 \bar{\phi}_0 \bar{u}_1) - \sum \bar{a}_j \int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{u}_j.$$

Since

$$\frac{\partial\phi_0}{\partial x_\ell} = i z_\ell \phi_0$$

and

$$-\Delta\phi_0 = |\mathbf{z}|^2 \phi_0,$$

it obtains that

$$(3.4) \quad \begin{aligned} \int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{\phi} &= |\mathbf{z}|^2 \int_{\Omega} |u_1|^2 + 2 i \sum_{\ell=1}^n z_\ell \int_{\Omega} \bar{u}_1 \frac{\partial u_1}{\partial x_\ell} \\ &- \sum_{j=1}^k \bar{a}_j \int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{u}_j. \end{aligned}$$

We now use the identity $-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1 = -\Delta(\phi_0 u_1) + \phi_0 \Delta u_1$ (and the Dirichlet boundary condition) to reduce the second integral to

$$\int_{\Omega} (-\Delta\phi_0 u_1 - 2\nabla\phi_0 \cdot \nabla u_1) \bar{u}_j = (\lambda_j - \lambda_1) a_j.$$

Substituting this into (3.4) and ultimately in (3.3) we conclude (since $\int_{\Omega} |u_1|^2 = 1$)

$$(3.5) \quad \int_{\Omega} (-\Delta\phi)\bar{\phi} = \lambda_1 \int_{\Omega} |\phi|^2 + |\mathbf{z}|^2 + 2 i \sum_{\ell=1}^n z_\ell \int_{\Omega} \bar{u}_1 \frac{\partial u_1}{\partial x_\ell} - \sum_{j=1}^k |a_j|^2 (\lambda_j - \lambda_1).$$

The term $\int_{\Omega} \bar{u}_1 \frac{\partial u_1}{\partial x_\ell}$ is of course real, since u_1 was assumed to be real. Conjugating, we obtain

$$(3.6) \quad \int_{\Omega} \phi(-\Delta\bar{\phi}) = \lambda_1 \int_{\Omega} |\phi|^2 + |\mathbf{z}|^2 - 2 i \sum_{\ell=1}^n z_\ell \int_{\Omega} \bar{u}_1 \frac{\partial u_1}{\partial x_\ell} - \sum_{j=1}^k |a_j|^2 (\lambda_j - \lambda_1).$$

Substituting (3.2), (3.5), and (3.6) into the Rayleigh-Ritz ratio (3.1) yields

$$\lambda_{k+1} - \lambda_1 \leq \frac{\int_{B_r} \left(|\mathbf{z}|^2 - \sum_{j=1}^k |a_j(\mathbf{z})|^2 (\lambda_j - \lambda_1) \right) d\mathbf{z}}{\int_{B_r} \left(1 - \sum_{j=1}^k |a_j(\mathbf{z})|^2 \right) d\mathbf{z}}$$

or

$$(3.7) \quad \lambda_{k+1} - \lambda_1 \leq \frac{\int_{B_r} |\mathbf{z}|^2 d\mathbf{z} - \sum_{j=1}^k (\lambda_j - \lambda_1) \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z}}{|B_r| - \sum_{j=1}^k \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z}}.$$

We are now ready for our second reduction. We rescale the Fourier coefficient $a_j(\mathbf{z})$ by defining

$$\tilde{a}_j(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \int_{\Omega} u_1 \bar{u}_j e^{i\mathbf{x}\cdot\mathbf{z}} d\mathbf{x} = \frac{a_j(\mathbf{z})}{(2\pi)^{n/2}}.$$

By Parseval’s identity

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z} &= \int_{B_r} |\tilde{a}_j(\mathbf{z})|^2 d\mathbf{z} \\ &\leq \int_{\mathbb{R}^n} |\tilde{a}_j(\mathbf{z})|^2 d\mathbf{z} \\ &= \int_{\Omega} |u_1|^2 |u_j|^2 d\mathbf{x} \\ &\leq \operatorname{ess\,sup} |u_1|^2 \int_{\Omega} |u_j(\mathbf{x})|^2 d\mathbf{x}. \end{aligned}$$

We now use the following result of Chiti [23] (see also [24], [14]):

$$(3.8) \quad \operatorname{ess\,sup} |u_1| \leq \left(\frac{\lambda_1}{\pi}\right)^{n/4} \frac{2^{1-n/2}}{\Gamma(n/2)^{1/2} j_{n/2-1,1} J_{n/2}(j_{n/2-1,1})}.$$

Therefore,

$$(3.9) \quad 0 < \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z} \leq \tilde{C}_n \lambda_1^{n/2},$$

where

$$\tilde{C}_n = \frac{2^2 \pi^{n/2}}{\Gamma(n/2) j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1})}.$$

We note that

$$\pi^{n/2} = \frac{n C_n \Gamma(n/2)}{2}$$

(C_n is the volume of the unit n -ball). Moreover, the constant H_n defined in (1.5) is given by

$$H_n = \frac{\tilde{C}_n}{C_n}.$$

Remark. Safarov obtained (2.3) using the following result of E. B. Davies [26]:

$$(3.10) \quad \operatorname{ess\,sup} |u_1| \leq e^{1/8\pi} \lambda_1^{n/4}.$$

Chiti’s statement (3.8) is an isoperimetric inequality. It saturates when Ω is an n -ball. Note that $e^{1/8\pi} \approx 1.04059$, while the constant in (3.8) takes the values listed in Table 2.

TABLE 2. Values of the constant in Chiti’s bound (3.8) as a function of the dimension n .

n	Chiti bound
2	0.451909
3	0.225079
4	0.103129
5	0.044409
6	0.018199
7	0.007157

We now prove by induction the following lemma.

Lemma 3.1. For $r \geq r_0(k) = H_n^{1/n} (1+k)^{1/n} \sqrt{\lambda_1}$, $k \geq 1$,

$$(3.11) \quad \lambda_{k+1} - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \sum_{j=1}^k (\lambda_j - \lambda_1)}{C_n r^n - k \tilde{C}_n \lambda_1^{n/2}}.$$

Proof. We first note that

$$\int_{B_r} |\mathbf{z}|^2 d\mathbf{z} = \frac{n}{n+2} C_n r^{n+2},$$

while

$$|B_r| = \int_{B_r} d\mathbf{z} = C_n r^n.$$

These two facts reduce (3.7) to

$$(3.12) \quad \lambda_{k+1} - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \sum_{j=1}^k (\lambda_j - \lambda_1) \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z}}{C_n r^n - \sum_{j=1}^k \int_{B_r} |a_j(\mathbf{z})|^2 d\mathbf{z}}.$$

For $k = 1$

$$\lambda_2 - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2}}{C_n r^n - \int_{B_r} |a_1(\mathbf{z})|^2 d\mathbf{z}}.$$

It then obtains by virtue of (3.9) that, for $r \geq r_0(1) = H_n^{1/n} 2^{1/n} \sqrt{\lambda_1}$ (this condition guarantees that the denominator is positive)

$$(3.13) \quad \lambda_2 - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2}}{C_n r^n - \tilde{C}_n \lambda_1^{n/2}},$$

as desired. Suppose now that, for $r \geq r_0(k-1) = H_n^{1/n} k^{1/n} \sqrt{\lambda_1}$,

$$\lambda_k - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \sum_{j=1}^{k-1} (\lambda_j - \lambda_1)}{C_n r^n - (k-1) \tilde{C}_n \lambda_1^{n/2}}.$$

Then, this is also true for $r \geq r_0(k)$ as well (since $r_0(k) > r_0(k-1)$). This implies

$$\lambda_k - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \left(\sum_{j=1}^{k-1} (\lambda_j - \lambda_1) + (\lambda_k - \lambda_1) \right)}{C_n r^n - (k-1) \tilde{C}_n \lambda_1^{n/2} - \tilde{C}_n \lambda_1^{n/2}}$$

or

$$(3.14) \quad \lambda_k - \lambda_1 \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \sum_{j=1}^k (\lambda_j - \lambda_1)}{C_n r^n - k \tilde{C}_n \lambda_1^{n/2}}.$$

(We have used the equivalence $\alpha \leq \frac{A}{B} \Leftrightarrow \alpha \leq \frac{A-\alpha\beta}{B-\beta}$, for $B - \beta \geq 0$.)

We also notice that if $A_j \geq 0$, then

$$\frac{\sum_{j=1}^k A_j (\lambda_j - \lambda_1)}{\sum_{j=1}^k A_j} \leq \lambda_k - \lambda_1.$$

Hence, by virtue of (3.14), and for $A_j = \tilde{C}_n \lambda_1^{n/2} - \int_{\Omega} |a_j(\mathbf{z})|^2 d\mathbf{z}$,

$$(3.15) \quad \frac{\sum_{j=1}^k (\lambda_j - \lambda_1) \left(\tilde{C}_n \lambda_1^{n/2} - \int_{\Omega} |a_j(\mathbf{z})|^2 d\mathbf{z} \right)}{\sum_{j=1}^k \left(\tilde{C}_n \lambda_1^{n/2} - \int_{\Omega} |a_j(\mathbf{z})|^2 d\mathbf{z} \right)} \leq \frac{\frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \sum_{j=1}^k (\lambda_j - \lambda_1)}{C_n r^n - k \tilde{C}_n \lambda_1^{n/2}}.$$

For simplicity we let

$$A = \frac{n}{n+2} C_n r^{n+2} - \tilde{C}_n \lambda_1^{n/2} \sum_{j=1}^k (\lambda_j - \lambda_1),$$

$$B = C_n r^n - k \tilde{C}_n \lambda_1^{n/2},$$

$$C = \sum_{j=1}^k (\lambda_j - \lambda_1) \left(\tilde{C}_n \lambda_1^{n/2} - \int_{\Omega} |a_j(\mathbf{z})|^2 d\mathbf{z} \right)$$

and

$$D = \sum_{j=1}^k \left(\tilde{C}_n \lambda_1^{n/2} - \int_{\Omega} |a_j(\mathbf{z})|^2 d\mathbf{z} \right).$$

By (3.12),

$$(3.16) \quad \lambda_{k+1} - \lambda_1 \leq \frac{A + C}{B + D}.$$

Combining (3.15) and (3.16) (since $A/B = \Lambda$ and $C/D \leq \Lambda$ imply $(A + C)/(B + D) \leq \Lambda$), one obtains (3.11), and the proof of the lemma is now complete. \square

If we set $r = r_0(k) = \left(\frac{\tilde{C}_n}{C_n} \right)^{1/n} (1+k)^{1/n} \sqrt{\lambda_1}$ in the statement of Lemma 3.1, we obtain (1.4) (the ‘‘averaged’’ version) in the form

$$(3.17) \quad \sum_{j=1}^{k+1} (\lambda_j - \lambda_1) \leq \frac{n}{n+2} \left(\frac{\tilde{C}_n}{C_n} \right)^{2/n} \lambda_1 (1+k)^{1+2/n}.$$

This choice amounts to setting

$$C_n r^n - k \tilde{C}_n \lambda_1^{n/2} = \tilde{C}_n \lambda_1^{n/2}.$$

If we drop the sum in (3.11) and let $r = \tilde{r}(k) = \left(\frac{\tilde{C}_n}{C_n} \right)^{1/n} k^{1/n} \sqrt{\lambda_1} \left(1 + \frac{n}{2} \right)^{1/n}$ we are led to (1.3), namely,

$$(3.18) \quad \lambda_{k+1} - \lambda_1 \leq \left(1 + \frac{n}{2} \right)^{2/n} \left(\frac{\tilde{C}_n}{C_n} \right)^{2/n} \lambda_1 k^{2/n}.$$

This choice amounts to making

$$C_n r^n - k \tilde{C}_n \lambda_1^{n/2} = \frac{k n}{2} \tilde{C}_n \lambda_1^{n/2}.$$

(Note that $\tilde{r}(k) \geq r_0(k)$ since $n \geq 2$.)

TABLE 3. Bound for $\frac{\lambda_2}{\lambda_1}$ as a function of the dimension n .

n	(1.20)	(1.4)	(1.3)	(1.17)
2	3	6.133	6.133	2.539
3	2.333	4.962	4.832	2.046
4	2	4.556	4.174	1.796
5	1.8	4.171	3.777	1.645
6	1.667	3.986	3.508	1.543
7	1.571	3.856	3.314	1.470

Remark. The case $k = 1$ in Lemma 3.1 provides a class of bounds for $\lambda_2 - \lambda_1$ for $r \geq r_0(1)$. The function

$$\frac{\frac{n}{n+2} C_n r^{n+2}}{C_n r^n - \tilde{C}_n \lambda_1^{n/2}}$$

is non-increasing for $2^{1/n} H_n^{1/n} \sqrt{\lambda_1} \leq r \leq (1 + n/2)^{1/n} H_n^{1/n} \sqrt{\lambda_1}$ and non-decreasing beyond. At $r_0(1)$, it assumes the value of

$$2^{1+2/n} \frac{n}{n+2} H_n^{2/n} \lambda_1.$$

Hence

$$(3.19) \quad \frac{\lambda_2}{\lambda_1} \leq 1 + 2^{1+2/n} \frac{n}{n+2} H_n^{2/n}.$$

At its minimum (viz. $r = (1 + n/2)^{1/n} H_n^{1/n} \sqrt{\lambda_1}$), it assumes the form

$$(3.20) \quad \lambda_2 - \lambda_1 \leq \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \lambda_1.$$

Note that (3.20) is just (1.3) for $k = 1$. Both bounds (3.19) and (3.20) are not expected to fare better than the Ashbaugh-Benguria inequality (1.17)—the best constant of its type (see Table 3). In fact, the first has the asymptotic expansion

$$1 + 2^{1+2/n} \frac{n}{n+2} H_n^{2/n} \sim 3 + \frac{2.53636}{n} - \frac{4}{3} \left(\frac{1}{n} \ln \frac{1}{n}\right) - \frac{4.71333}{n^{5/3}} + O\left(\frac{1}{n^2}\right).$$

Expanding the second, it obtains (see (1.28) above)

$$1 + \left(1 + \frac{n}{2}\right)^{2/n} H_n^{2/n} \sim 2 + \frac{0.495591}{n} - \frac{8}{3n} \ln \frac{1}{n} - \frac{2.35666}{n^{5/3}} + O\left(\frac{1}{n^2}\right).$$

4. COMPARISON WITH EXISTING RESULTS

Consider the convex function

$$\phi_\lambda(t) = (\lambda - t)_+ = \begin{cases} \lambda - t, & \text{if } t \leq \lambda, \\ 0, & \text{if } t \geq \lambda. \end{cases}$$

In [35], Laptev proved (see Theo. 4.1) that

$$(4.1) \quad \sum_j (\lambda - \lambda_j)_+ \geq (\lambda - \lambda_1)^{1+n/2} L_n^{cl} \frac{2}{n+2} \tilde{u}_1^{-2}$$

where $\tilde{u}_1 = \text{ess sup}|u_1|$. When $\lambda = \lambda_2$, (4.1) reduces to (see Cor. 4.2 of [35])

$$(4.2) \quad \lambda_2 - \lambda_1 \leq \left(L_n^{cl} \frac{2}{n+2} \right)^{-2/n} \tilde{u}_1^{4/n}.$$

Combining this with the isoperimetric inequality of Chiti (3.8) gives (3.20). Note that Chiti's inequality (3.8) can be put in the form

$$(4.3) \quad \tilde{u}_1^2 \leq H_n L_n^{cl} \lambda_1^{n/2}.$$

For $\lambda \geq \lambda_1$, Laptev's result (4.1) can also be interpreted as (see Cor. 4.4 of [35])

$$(4.4) \quad N(\lambda) \geq (\lambda - \lambda_1)^{1+n/2} L_n^{cl} \frac{2}{n+2} \tilde{u}_1^{-2}$$

which, when combined with Chiti's ineq. (4.3), results in the statement of (2.1) with the same universal constant $\frac{2}{n+2} \frac{1}{H_n}$.

To obtain (1.3) from Laptev's vantage point, set $\lambda = \lambda_{k+1}$ in (4.1) and observe that

$$k(\lambda_{k+1} - \lambda_1) \geq \sum_{j=1}^k (\lambda - \lambda_j).$$

Hence

$$(4.5) \quad \lambda_{k+1} - \lambda_1 \leq k^{2/n} \left(L_n^{cl} \frac{2}{n+2} \right)^{-2/n} \tilde{u}_1^{4/n}.$$

Again, bounding \tilde{u}_1 using Chiti's isoperimetric inequality (4.3) yields the statement (1.3). In fact, one can write (4.1) in the form

$$k(\lambda_{k+1} - \bar{\lambda}) \geq (\lambda_{k+1} - \lambda_1)^{1+n/2} L_n^{cl} \frac{2}{n+2} \tilde{u}_1^{-2}$$

where $\bar{\lambda} = \sum_{j=1}^k \lambda_j/k$. Therefore, using (3.8),

$$(4.6) \quad \left(1 + \frac{n}{2} \right) H_n \lambda_1^{n/2} k(\lambda_{k+1} - \bar{\lambda}) \geq (\lambda_{k+1} - \lambda_1)^{1+n/2}.$$

This is a class of less accessible Weyl-type universal upper bounds for λ_{k+1} different from both (1.3) and (1.4), but in the same spirit. Indeed, Cor. 4.4 from [35] (and eventually the weaker inequality (1.3)) follows from Theo. 4.1 of [35] by applying the rather rough estimate $(\lambda - \lambda_j)_+ \leq (\lambda - \lambda_1)_+$. Refining this coarse estimate, one can recover (1.4) from Laptev's bound (4.1). Starting with (4.1), one first introduces the Legendre transform $\mathcal{L}\{f\}(p) = \sup_{\lambda \geq 0} (p\lambda - f(\lambda))$. It is then clear (see, e.g., [36]) that

$$(4.7) \quad \mathcal{L} \left\{ \sum_j (\lambda - \lambda_j)_+ \right\} (p) = (p - [p]) \lambda_{[p]+1} + \sum_{j=1}^{[p]} \lambda_j,$$

where $[p]$ designates the integer part of p . The Legendre transform of the right hand side of (4.1) is given by

$$(4.8) \quad \mathcal{L} \left\{ (\lambda - \lambda_1)^{1+n/2} L_n^{cl} \frac{2}{n+2} \tilde{u}_1^{-2} \right\} (p) = \lambda_1 p + \frac{n}{n+2} p^{1+2/n} \tilde{u}_1^{-4/n} (L_n^{cl})^{-2/n}.$$

Since $f(\lambda) \geq g(\lambda)$ for all $\lambda \geq 0$ implies $\mathcal{L}\{f\}(p) \leq \mathcal{L}\{g\}(p)$ for all $p \geq 0$, we have (setting $p = k$)

$$(4.9) \quad \sum_{j=1}^k (\lambda_j - \lambda_1) \leq \frac{n}{n+2} \tilde{u}_1^{4/n} (L_n^{cl})^{-2/n} k^{1+2/n}.$$

Combining the latter inequality with Chiti’s inequality (3.8) we obtain (1.4).

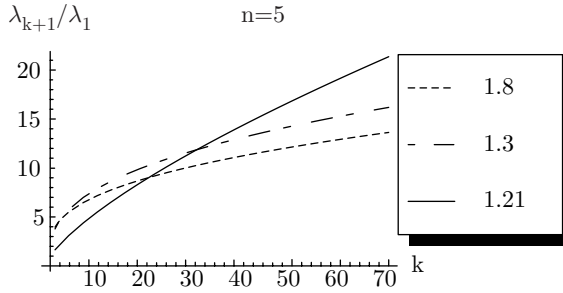


FIGURE 2. Comparison of new and old bounds.

Now, we turn to comparing these bounds. We claim that (1.8) is sharper than (1.3). To see this, we take the limit of the ratio of bounds as $k \rightarrow \infty$. This limit is equal to

$$\frac{1 + \frac{4}{n}}{\left(1 + \frac{n}{2}\right)^{1+2/n}}.$$

It is strictly less than 1 since $1 + 4/n < 1 + n/2 < (1 + n/2)^{1+2/n}$, for $n \geq 3$. This limit is equal to $3/4$ at $n = 2$.

That both (1.3) and (1.8) are sharper than (1.21) (in the form (1.22)) follows from Krahn’s second inequality [32] (see in particular ineq. (22) of [2])

$$|\Omega|^{2/n} \lambda_2 > 2^{2/n} C_n^{2/n} j_{n/2-1,1}^2.$$

For the unit ball in \mathbb{R}^n , $|\Omega| = C_n$ and $\lambda_2 = j_{n/2,1}^2$. Therefore,

$$\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} > 2^{2/n}$$

or

$$(4.10) \quad \frac{\log\left(j_{n/2,1}^2/j_{n/2-1,1}^2\right)}{\log 2} > \frac{2}{n}$$

(see also (1.23) above). This is clearly displayed in Figure 2 where the Ashbaugh-Benguria bound fares better to about $k = 20$. The “averaged” bound (1.8) takes over and—at a latter stage—so does (1.3). Three tables are included in this paper which display this fact as well (see Tables 3-5). The new inequalities—both of which are disguised in earlier work of Laptev—cannot be expected to improve on existing bounds in the case of λ_2/λ_1 . There is a competition (see Table 4) in the case of

TABLE 4. Bound for $\frac{\lambda_{32}}{\lambda_1}$ as a function of the dimension n .

n	(1.20)	(1.8)	(1.3)	(1.16)
2	6.177×10^{14}	122.334	160.112	105.46
3	2.554×10^{11}	31.071	38.811	35.831
4	2.147×10^9	15.606	18.675	18.707
5	8.193×10^7	10.341	11.965	12.052
6	7.539×10^6	7.870	8.878	8.758
7	1.217×10^6	6.491	7.174	6.865

TABLE 5. Bound for $\frac{\lambda_{128}}{\lambda_1}$ as a function of the dimension n .

n	(1.20)	(1.8)	(1.3)	(1.16)
2	3.930×10^{60}	491.885	652.846	679.705
3	5.408×10^{46}	75.911	97.808	149.957
4	1.701×10^{38}	29.539	36.774	60.369
5	2.628×10^{32}	16.814	20.2736	32.621
6	1.496×10^{28}	11.593	11.5934	20.861
7	8.500×10^{24}	8.917	8.917	14.836

λ_{32}/λ_1 between (1.3) and (1.16) (already (1.8) is better than both for $n \geq 3$). In the case of λ_{128}/λ_1 , both new bounds show considerable improvement (see Table 5).

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