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APPROXIMATION THEOREMS FOR THE PROPAGATORS OF HIGHER ORDER ABSTRACT CAUCHY PROBLEMS

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Abstract. In this paper, we present two quite general approximation theorems for the propagators of higher order (in time) abstract Cauchy problems, which extend largely the classical Trotter-Kato type approximation theorems for strongly continuous operator semigroups and cosine operator functions. Then, we apply the approximation theorems to deal with the second order dynamical boundary value problems.

1. Introduction and general approximation theorems

In 1958, H. F. Trotter [33] treated the question of convergence of strongly continuous operator semigroups in Banach spaces and gave an approximation theorem. A gap in the proof of the theorem was pointed out and corrected by T. Kato [19]. This theorem is just the well-known Trotter-Kato approximation theorem. Convergence results of a similar nature can be found in T. Kato [20], T. Kurtz [22, 23], A. Pazy [30], T. I. Seidman [32], and K. Yosida [39]. Also, there have been some Trotter-Kato type approximation theorems for various operator families such as for cosine operator functions (cf. [15], [16, Sect.7] and [31]), for integrated semigroups (cf. [29, 35]), and for resolvent families of operators (cf. [27, 28]). Such approximation theorems have proved to be very useful in showing the convergence of solutions of difference equations as well as partial differential equations.

On the other hand, dynamic boundary value problems (DBPs for short) in Banach spaces have been attracting more and more attention (cf., e.g., [1, 3, 4, 5, 6, 8, 10, 11, 12, 14, 17, 21, 25, 36, 38] and references therein) due to their applicability to a lot of practical problems such as those in control theory. There have been a number of developments in the study of many aspects of DBPs, but not yet in the investigation of the approximation problem (among others) for second order (in time) DBPs. Actually, whenever the second order equations involve first order derivatives (damping terms), cosine operator functions will no longer suit the DBPs, and furthermore, without certain strong restrictions on the operators

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in the state equations and on the boundary conditions, it is hard to find appropriate phase spaces on which the operator matrices, corresponding to the second order DBPs, do generate strongly continuous semigroups (cf., e.g., Example 2.5 in the last section and [34, 37, 38]) so that the classical Trotter-Kato approximation theorem can be applied. Thus it is really meaningful to establish corresponding approximation theorems, especially for the solution operators, i.e. the propagators, of second order DBPs. This stimulates us to consider further another and much more general issue of *how to treat the question of convergence of the solution operators* (*the propagators*) *for general higher order* (*in time*) *abstract Cauchy problems*.

In this paper, we devote ourselves to dealing with these two problems. By using general wellposedness concepts from [34, 37], we first obtain two quite general approximation theorems (in Section 1), which extend largely the classical Trotter-Kato approximation theorems for strongly continuous operator semigroups and cosine operator functions. Then, we investigate (in Section 2) approximation issues for second order DBPs as an application of our general results.

For the basic theory on second order and higher order abstract Cauchy problems, we refer the reader to, e.g., [9, 34] (see also [13]).

Consider now the higher order abstract Cauchy problem

$$(ACP_n) \qquad \begin{cases} u^{(n)}(t) + \sum_{i=0}^{n-1} A_i u^{(i)}(t) = 0, \quad t \geq 0, \\ u^{(j)}(0) = u_j, \quad 0 \leq j \leq n-1, \end{cases}$$

and the approximating problems

$$(ACP_n)_m \qquad \begin{cases} u_m^{(n)}(t) + \sum_{i=0}^{n-1} A_{i,m} u_m^{(i)}(t) = 0, \quad t \geq 0, \\ u_m^{(j)}(0) = u_{j,m}, \quad 0 \leq j \leq n-1, \end{cases}$$

$m \in N$, where A_i, $A_{i,m}$ $(m \in N,\ i = 0, \ldots, n-1)$ are closed linear operators with domains $\mathcal{D}(A_i)$, $\mathcal{D}(A_{i,m})$ in a Banach space E. We consider the operator-valued polynomials

$$P(\lambda) := \lambda^n + \sum_{i=0}^{n-1} \lambda^i A_i, \quad P_m(\lambda) := \lambda^n + \sum_{i=0}^{n-1} \lambda^i A_{i,m},$$

and their inverses $R(\lambda) := P(\lambda)^{-1}$, $R_m(\lambda) := P_m(\lambda)^{-1}$, wherever they exist as bounded operators. A core for $[A_0, \ldots, A_{n-1}]$ is a subspace of $\bigcap_{i=0}^{n-1} \mathcal{D}(A_i)$, being dense in $\bigcap_{i=0}^{n-1} \mathcal{D}(A_i)$ for the norm

$$|u| := \|u\| + \sum_{i=0}^{n-1} \|A_i u\|.$$

By $\left[\bigcap_{j=0}^{i} \mathcal{D}(A_j)\right]$, we mean the space $\bigcap_{j=0}^{i} \mathcal{D}(A_j)$ endowed with the norm

$$\|\cdot\|_{\left[\bigcap_{j=0}^{i} \mathcal{D}(A_j)\right]} = \|\cdot\| + \sum_{j=0}^{i} \|A_j u\|.$$

By $C(R^+;\mathcal{L}_s(E))$, we denote the space of all strongly continuous $\mathcal{L}(E)$-valued functions on R^+. The set of complex numbers

$$\rho(A_0,\dots,A_{n-1}) := \{\lambda\in\mathbf{C};\ P(\lambda)^{-1} \text{ exists and } \ R(\lambda)\in\mathcal{L}(E)\}$$

is called the resolvent set of $(A_0,\dots,A_{n-1})$.

By a (strict) solution of (ACP_n), we mean a function $u\in C^n(R^+;E)$ such that for $0\le i\le n-1$, $t\ge 0$, we have $u^{(i)}(t)\in\mathcal{D}(A_i)$, $A_iu^{(i)}(\cdot)\in C(R^+;E)$, and (ACP_n) is satisfied. The (strict) solution of an inhomogeneous higher order abstract Cauchy problem is defined in the same way.

The following definition of strong quasi-wellposedness is a higher order version of [37, Definition 2.6].

Definition 1.1. Let $\bigcap_{i=0}^{n-1}\mathcal{D}(A_i)$ be dense in E. (ACP_n) is called strongly quasi-wellposed if

(i) (ACP_n) has a (strict) solution for every $u_j\in\bigcap_{i=0}^{j}\mathcal{D}(A_i)$, $j=0,\dots,n-1$;

(ii) there exist n propagators

$$S_k(\cdot)\in C\left(R^+;\mathcal{L}_s\left(\left[\bigcap_{j=0}^{k}\mathcal{D}(A_j)\right]\right)\right),\quad k=0,\ \dots,\ n-2, \tag{1.1}$$

$$S_{n-1}(\cdot)\in C(R^+;\mathcal{L}_s(E)), \tag{1.2}$$

satisfying, for every $k=1,\dots,n-1$,

$$S_k(\cdot)u\in C^k(R^+;E),\quad u\in\bigcap_{j=0}^{k}\mathcal{D}(A_j), \tag{1.3}$$

$$S_{n-1}(\cdot)u\in C^{k-1}(R^+;[\mathcal{D}(A_k)]),\qquad u\in E,$$

$$\left\|S_{k-1}^{(k-1)}(t)\right\|_{\mathcal{L}\left(\left[\bigcap_{j=0}^{k-1}\mathcal{D}(A_j)\right]\right)}\le Me^{\omega t},\qquad t\ge 0, \tag{1.4}$$

$$\left\|S_{n-1}^{(n-1)}(t)\right\|_{\mathcal{L}(E)},\ \left\|A_kS_{n-1}^{(k-1)}(t)\right\|_{\mathcal{L}(E)}\le Me^{\omega t},\qquad t\ge 0,$$

for some constants M, $\omega\ge 0$, such that any (strict) solution to (ACP_n) can be expressed as

$$u(t)=\sum_{k=0}^{n-1}S_k(t)u^{(k)}(0),\quad t\ge 0.$$

Remark 1.2. The propagator $S_0(\cdot)$ reduces to a strongly continuous semigroup when $n=1$, and to a cosine operator function when $n=2$ and the term A_1u' vanishes.

The vector-valued Laplace transform will be our main tool (see [2, 34]) and we use the following terminology from [34].

Definition 1.3. A function $\mathcal{F}:(a,\ \infty)\to\mathcal{L}(E)$ is in the class $LT-\mathcal{L}(E)$ if there exists a strongly continuous function $\mathcal{H}(\cdot):R^+\to\mathcal{L}(E)$ such that $\{e^{-at}\mathcal{H}(t);t\ge 0\}$ is uniformly bounded for some $a>0$ with

$$\mathcal{F}(\lambda)u=\int_0^\infty e^{-\lambda t}\mathcal{H}(t)u\,dt\quad \text{for all } \ \lambda>a,\ u\in E.$$

Arguing as in the proof of [37, Proposition 2.8] we can characterize the strong quasi-wellposedness using the Laplace transform.

Lemma 1.4. *(ACP_n) is strongly quasi-wellposed if and only if $\bigcap_{i=0}^{n-1}\mathcal{D}(A_i)$ is dense in E, $(\omega,\ \infty)\subset\rho(A_0,\ \dots,\ A_{n-1})$ for some $\omega>0$, and*

$$\lambda\mapsto\lambda^{n-1}R(\lambda),\quad \lambda\mapsto\lambda^{k-1}A_kR(\lambda)\in LT-\mathcal{L}(E),\qquad k=1,\ \dots,\ n-1.$$

In this case, we have for $u\in E$ and λ large enough,

$$\lambda^{n-1}R(\lambda)u=\int_0^\infty e^{-\lambda t}S_{n-1}^{(n-1)}(t)udt,$$
$$\lambda^{k-1}A_kR(\lambda)=\int_0^\infty e^{-\lambda t}A_kS_{n-1}^{(k-1)}(t)udt,\qquad k=1,\ \dots,\ n-1.$$

For our approximation problem the following lemma from [35] will be crucial.

Lemma 1.5. *For each $m\in N$, let $f_m\in C(R^+,E)$ satisfy*

$$\|f_m(t)\|\le Me^{\omega t},\quad \textit{for all } t\ge 0,$$

and let $\mathcal{F}_m$ be defined by

$$\mathcal{F}_m(\lambda)=\int_0^\infty e^{-\lambda t}f_m(t)dt,\quad \lambda>\omega.$$

Then the following assertions are equivalent.

(i) *$\{f_m;\ m\in N\}$ is equicontinuous at each point $t\in[0,\infty)$, and $\lim_{m\to\infty}\mathcal{F}_m(\lambda)$ exists for $\lambda>\omega$.*

(ii) *$\lim_{m\to\infty}f_m(t)$ exists for $t\ge 0$ and the convergence is uniform on bounded t-intervals.*

We are now in a position to give our main result.

Theorem 1.6. *Let each $(ACP_n)_m$ be strongly quasi-wellposed such that*

$$\left\|S_{n-1,m}^{(n-1)}(t)\right\|,\ \left\|A_kS_{n-1,m}^{(k-1)}(t)\right\|\le Me^{\omega t},\quad t\ge 0,\ 1\le k\le n-1,\tag{1.5}$$

where M, ω are constants independent of m. Let D be a core for $[A_0,\dots,A_{n-1}]$. Assume that $\bigcap_{i=0}^{n-1}\mathcal{D}(A_i)$ is dense in E, and $(\omega,\infty)\subset\rho(A_0,\dots,A_{n-1})$. Then the following statements are equivalent.

(i) *For each $u\in D$, there exists $u_m\in\bigcap_{i=0}^{n-1}\mathcal{D}(A_{i,m})$ such that*

$$\lim_{m\to\infty}u_m=u,\quad \lim_{m\to\infty}A_{k,m}u_m=A_ku,\quad 0\le k\le n-1.\tag{1.6}$$

(ii) *For each $u\in E$, $\lambda>\omega$, $1\le k\le n-1$,*

$$\lim_{m\to\infty}R_m(\lambda)u=R(\lambda)u,\quad \lim_{m\to\infty}A_{k,m}R_m(\lambda)u=A_kR(\lambda)u.\tag{1.7}$$

(iii) *(ACP_n) is strongly quasi-wellposed, and for all $u\in E$, $t\ge 0$,*

$$\lim_{m\to\infty}S_{n-1,m}^{(n-1)}(t)u=S_{n-1}^{(n-1)}(t)u,\tag{1.8}$$

$$\lim_{m\to\infty}A_{k,m}S_{n-1,m}^{(k-1)}(t)u=A_kS_{n-1}^{(k-1)}(t)u,\quad 1\le k\le n-1.\tag{1.9}$$

Moreover, the convergence in statement (iii) *is uniform on bounded t-intervals.*

Proof. (i) $\Longrightarrow$(ii).

By Lemma 1.4, we have, for $m \in N$, $\lambda > \omega$, $u \in E$,

$$\lambda^{n-1} R_m(\lambda) u = \int_0^\infty e^{-\lambda t} S_{n-1,m}^{(n-1)}(t) u dt, \tag{1.10}$$

$$\lambda^{k-1} A_{k,m} R_m(\lambda) u = \int_0^\infty e^{-\lambda t} A_{k,m} S_{n-1,m}^{(k-1)}(t) u dt, \quad 1 \le k \le n-1. \tag{1.11}$$

Therefore

$$\left\|\lambda^{n-1} R_m(\lambda)\right\|, \ \left\|\lambda^{k-1} A_{k,m} R_m(\lambda)\right\| \le \frac{M}{\lambda - \omega}, \qquad m \in N, \ \lambda > \omega, \ 1 \le k \le n-1. \tag{1.12}$$

Fix $\lambda > \omega$. Let $u \in P(\lambda)D$; then $R(\lambda)u \in D$. By hypothesis there exists $v_m \in \bigcap_{i=0}^{n-1} \mathcal{D}(A_{i,m})$ such that

$$\lim_{m\to\infty} v_m = R(\lambda)u, \quad \lim_{m\to\infty} A_{k,m} v_m = A_k R(\lambda) u, \quad 0 \le k \le n-1.$$

This combined with (1.12) yields that

(1.13)

$$\begin{aligned} R_m(\lambda)u &= R_m(\lambda) P(\lambda) R(\lambda) u \\ &= R_m(\lambda)\left(P(\lambda)R(\lambda)u - P_m(\lambda)v_m\right) + v_m \\ &= v_m - R_m(\lambda)\left\{\lambda^n (v_m - R(\lambda)u) + \sum_{i=0}^{n-1} \lambda^i \left[A_{i,m} v_m - A_i R(\lambda) u\right]\right\} \\ &\longrightarrow R(\lambda)u \qquad \text{as } m \to \infty. \end{aligned}$$

Notice that $P(\lambda)D$ is dense in $P(\lambda)\left(\bigcap_{i=0}^{n-1} \mathcal{D}(A_i)\right)$, and $P(\lambda)\left(\bigcap_{i=0}^{n-1} \mathcal{D}(A_i)\right) = E$. We infer that $P(\lambda)D$ is dense in E. By (1.12), the first equality of (1.7) follows immediately from (1.13).

Similarly, we obtain the other equalities of (1.7) by noting that (1.12) and the identity

$$A_{0,m} R_m(\lambda) = I - \lambda^n R_m(\lambda) - \sum_{i=1}^{n-1} \lambda^i A_{i,m} R_m(\lambda) \tag{1.14}$$

imply that

$$\|A_{0,m} R_m(\lambda)\| \le \text{ const} \qquad \text{for all } m \in N.$$

(ii) $\Longrightarrow$(i).

Take $u \in D$. We choose $\lambda \in (\omega, \infty)$ and set

$$u_m = R_m(\lambda)\left(P(\lambda)u\right).$$

Then, by (1.7) we obtain

$$\begin{aligned} &\lim_{m\to\infty} u_m = R(\lambda)\left(P(\lambda)u\right) = u, \\ &\lim_{m\to\infty} A_{k,m} u_m = A_k R(\lambda)\left(P(\lambda)u\right) = A_k u, \end{aligned}$$

valid for $0 \le k \le n-1$, noting (1.14).

(i) $\Longrightarrow$(iii).

Clearly D is dense in E since $\bigcap_{i=0}^{n-1}\mathcal{D}(A_i)$ is dense in E. Let $u\in D$. By assumption, there exists $u_m\in\bigcap_{i=0}^{n-1}\mathcal{D}(A_{i,m})$ such that (1.6) holds. From (1.10) we see that for $m\in N$, $\lambda>\omega$,

$$\lambda^{n-1}R_m(\lambda)u_m=\int_0^\infty e^{-\lambda t}S_{n-1,m}^{(n-1)}(t)u_m dt,$$

$$\begin{aligned}\lambda^{n-1}R_m(\lambda)u_m &= \lambda^{-1}u_m-\sum_{i=0}^{n-1}\lambda^{i-1}R_m(\lambda)A_{i,m}u_m\\ &= \int_0^\infty e^{-\lambda t}\left[u_m-\sum_{i=0}^{n-1}\int_0^t\frac{(t-\sigma)^{n-i-1}}{(n-i-1)!}S_{n-1,m}^{(n-1)}(\sigma)A_{i,m}u_m d\sigma\right]dt.\end{aligned}$$

Therefore for $t\geq 0$, $m\in N$,

$$S_{n-1,m}^{(n-1)}(t)u_m=u_m-\sum_{i=0}^{n-1}\int_0^t\frac{(t-\sigma)^{n-i-1}}{(n-i-1)!}S_{n-1,m}^{(n-1)}(\sigma)A_{i,m}u_m d\sigma, \tag{1.15}$$

by the uniqueness theorem for Laplace transforms. By (1.11) we deduce that for $m\in N$, $\lambda>\omega$, $1\leq k\leq n-1$,

$$\lambda^{k-1}A_{k,m}R_m(\lambda)u_m=\int_0^\infty e^{-\lambda t}A_{k,m}S_{n-1,m}^{(k-1)}(t)u_m dt,$$

$$\lambda^{k-1}A_{k,m}R_m(\lambda)u=\lambda^{k-n-1}A_{k,m}u_m-\sum_{i=0}^{n-1}\lambda^{k+i-n-1}A_{k,m}R_m(\lambda)A_{i,m}u_m$$

$$\begin{aligned}=\int_0^\infty e^{-\lambda t}\Bigg[&\frac{t^{n-k}}{(n-k)!}A_{k,m}u_m\\ &-\sum_{i=0}^{n-1}\int_0^t\frac{(t-\sigma)^{n-i-1}}{(n-i-1)!}A_{k,m}S_{n-1,m}^{(k-1)}(\sigma)A_{i,m}u_m d\sigma\Bigg]dt.\end{aligned}$$

So for $t\geq 0$, $m\in N$, $1\leq k\leq n-1$,

$$\begin{aligned}A_{k,m}S_{n-1,m}^{(k-1)}(t)u_m &= \frac{t^{n-k}}{(n-k)!}A_{k,m}u_m\\ &-\sum_{i=0}^{n-1}\int_0^t\frac{(t-\sigma)^{n-i-1}}{(n-i-1)!}A_{k,m}S_{n-1,m}^{(k-1)}(\sigma)A_{i,m}u_m d\sigma.\end{aligned} \tag{1.16}$$

Just writing $S_{n-1,m}^{(n-1)}(t)u$ as $S_{n-1,m}^{(n-1)}(t)(u-u_m)+S_{n-1,m}^{(n-1)}(t)u_m$, we see by (1.5), (1.6) and (1.15) that

$$\left\{S_{n-1,m}^{(n-1)}(\cdot)u;\ m\in N\right\}\text{ is equicontinuous at each point }t\in[0,\infty).$$

Likewise, for every $1\leq k\leq n-1$,

$$\left\{A_{k,m}S_{n-1,m}^{(k-1)}(\cdot)u;\ m\in N\right\}\text{ is equicontinuous at each point }t\in[0,\infty),$$

due to (1.16). Thus recalling (i) implies (1.7) and we can apply Lemma 1.5 to (1.10), (1.11). This yields that for each $t \geq 0$, the following limits exist:

$$U(t)u := \lim_{m\to\infty} S^{(n-1)}_{n-1,m}(t)u, \tag{1.17}$$

$$V_k(t)u := \lim_{m\to\infty} A_{k,m}S^{(k-1)}_{n-1,m}(t)u, \quad 1 \leq k \leq n-1, \tag{1.18}$$

and the convergence is uniform on bounded t-intervals. Now, combining (1.5)–(1.7), (1.10), (1.11), (1.17) and (1.18) yields that for $u \in D$, $1 \leq k \leq n-1$,

$$\|U(t)u\|,\ \|V_k(t)u\| \leq 2Me^{\omega t}\|u\|, \quad t \geq 0, \tag{1.19}$$

$$\lambda^{n-1}R(\lambda)u = \int_0^\infty e^{-\lambda t}U(t)udt, \quad \lambda > \omega, \tag{1.20}$$

$$\lambda^{k-1}A_kR(\lambda)u = \int_0^\infty e^{-\lambda t}V_k(t)udt, \quad \lambda > \omega. \tag{1.21}$$

The density of D indicates that $U(t)$ and $V_k(t)$ can be extended to all of E as bounded linear operators, which we denote by the same symbols, and that (1.17)–(1.21) hold for all $u \in E$. Thus, making use of Lemma 1.4 we conclude that (ACP_n) is strongly quasi-wellposed. Comparing (1.20), (1.21) with the corresponding equations in Lemma 1.4, we see that for $t \geq 0$, $1 \leq k \leq n-1$,

$$S^{(n-1)}_{n-1}(t) = U(t), \quad A_kS^{(k-1)}_{n-1}(t) = V_k(t). \tag{1.22}$$

This and (1.17), (1.18) together lead to (1.8) and (1.9).

(iii) $\Longrightarrow$(ii). From (1.1), (1.10) and (1.11), we have

$$\lim_{m\to\infty} \lambda^{n-1}R_m(\lambda)u = \int_0^\infty e^{-\lambda t}S^{(n-1)}_{n-1}(t)udt = \lambda^{n-1}R(\lambda)u,$$

$$\lim_{m\to\infty} \lambda^{k-1}A_{k,m}R_m(\lambda)u = \int_0^\infty e^{-\lambda t}A_{k,m}S^{(k-1)}_{n-1,m}(t)udt = \lambda^{k-1}A_kR(\lambda)u,$$

for $u \in E$, $\lambda > \omega$, $1 \leq k \leq n-1$. So, (1.7) follows immediately. The proof is then complete. □

Next, we consider a slightly different concept of wellposedness.

Definition 1.7. Let $\bigcap_{i=0}^{n-1}\mathcal{D}(A_i)$ be dense in E. (ACP_n) is said to be strongly wellposed if (i) and (ii) of Definition (1.1) are satisfied with (1.1) and (1.2) replaced by

$$S_k(\cdot) \in C\left(R^+; \mathcal{L}_s(E)\right), \quad k = 0,\ \ldots,\ n-1,$$

(1.3) replaced by

$$S_k(\cdot)u \in C^k(R^+; E), \qquad u \in E,$$

and (1.4) replaced by

$$\left\|S^{(k-1)}_{k-1}(t)\right\|_{\mathcal{L}(E)} \leq Me^{\omega t}, \qquad t \geq 0,$$

respectively.

Remark 1.8. By [34, Theorem 1.4 (p.47) and Theorem 1.6 (p.52)], the definition of strong wellposedness is equivalent to that in [34, Definition 1.3, p.46], and strong wellposedness implies strong quasi-wellposedness.

Theorem 1.9. *For $m \in N$, let $(ACP_n)_m$ be strongly wellposed such that*

$$\|S_{0,m}(t)\|,\ \left\|S_{k,m}^{(k)}(t)\right\|,\ \left\|A_k S_{n-1,m}^{(k-1)}(t)\right\| \le Me^{\omega t}, \quad t \ge 0,\ 1 \le k \le n-1, \tag{1.23}$$

where M, ω are constants independent of m. Let D be a core for $[A_0, \dots, A_{n-1}]$. Assume that $\bigcap_{i=0}^{n-1} \mathcal{D}(A_i)$ is dense in E and $(\omega, \infty) \subset \rho(A_0, \dots, A_{n-1})$. Then statement (i) *or* (ii) *in Theorem* 1.6 *is equivalent to*

(iii)$'$ *(ACP_n) is strongly wellposed, and for any $u \in E$, $t \ge 0$,*

$$\begin{aligned} &\lim_{m\to\infty} S_{k,m}^{(k)}(t)u = S_k^{(k)}(t)u, \quad 0 \le k \le n-1, \\ &\lim_{m\to\infty} A_{k,m} S_{n-1,m}^{(k-1)}(t)u = A_k S_{n-1}^{(k-1)}(t)u, \quad 1 \le k \le n-1. \end{aligned} \tag{1.24}$$

Moreover, the convergence in statement (iii)$'$ *is uniform on bounded intervals of $t \ge 0$.*

Proof. (i) $\Longrightarrow$ (iii)$'$.

First, we proceed as in the proof of the implication (i) $\Longrightarrow$(iii) of Theorem 1.6. Then in view of [34, Remark 2.5, p.65] and (1.10), we obtain, for $m \in N$, $\lambda > \omega$, and $1 \le k \le n-1$,

$$\lambda^{k-1} R_m(\lambda) A_{k,m} u_m = \int_0^\infty e^{-\lambda t} \left[S_{k-1,m}^{(k-1)}(t) - S_{k,m}^{(k)}(t)\right] u_m dt,$$

$$\lambda^{k-1} R_m(\lambda) A_{k,m} u_m = \int_0^\infty e^{-\lambda t} \left[\int_0^t \frac{(t-\sigma)^{n-k-1}}{(n-k-1)!} S_{n-1,m}^{(n-1)}(\sigma) A_{k,m} u_m d\sigma\right] dt;$$

hence for $t \ge 0$, $m \in N$, $1 \le k \le n-1$,

$$\left[S_{k-1,m}^{(k-1)}(t) - S_{k,m}^{(k)}(t)\right] u_m = \int_0^t \frac{(t-\sigma)^{n-k-1}}{(n-k-1)!} S_{n-1,m}^{(n-1)}(\sigma) A_{k,m} u_m d\sigma.$$

Moreover,

$$\lim_{m\to\infty} \lambda^{k-1} R_m(\lambda) A_{k,m} u_m = \lambda^{k-1} R(\lambda) A_k u, \quad \lambda > \omega,\ 1 \le k \le n-1,$$

by (1.12) and (1.6). Accordingly, an application of Lemma 1.5 yields that for any $1 \le k \le n-1$,

$$W_k(t)u := \lim_{m\to\infty} \left[S_{k-1,m}^{(k-1)}(t) - S_{k,m}^{(k)}(t)\right] u_m \tag{1.25}$$

exists, uniformly on bounded intervals of $t \ge 0$. Therefore, for $u \in D$, $1 \le k \le n-1$,

$$\|W_k(t)u\| \le 2Me^{\omega t}\|u\|, \quad t \ge 0, \tag{1.26}$$

$$\lambda^{k-1} R(\lambda) A_k u = \int_0^\infty e^{-\lambda t} W_k(t) u dt, \quad \lambda > \omega. \tag{1.27}$$

Since D is dense in E, $W_k(t)$ can be extended to all of E as a bounded linear operator, which we denote by the same symbol, (1.26) holds for all $u \in E$, and (1.27) holds for all $u \in \bigcap_{i=0}^{n-1} \mathcal{D}(A_i)$. Thus (1.15)–(1.17), (1.26) and (1.27) enable us to apply [34, Theorem 2.3, p.57], and deduce that (ACP_n) is strongly wellposed. Comparing (1.27) with the corresponding equation in [34, Remark 2.5, p.65], we see that for $t \ge 0$, $1 \le k \le n-1$,

$$S_{k-1}^{(k-1)}(t) - S_k^{(k)}(t) = W_k(t),$$

and so

$$S_k^{(k)}(t) = S_{n-1}^{(n-1)}(t) + \sum_{i=k+1}^{n-1} W_i(t), \quad 0 \le k \le n-2,\ t \ge 0. \tag{1.28}$$

Note from (1.25) and (1.23) that as $m \to \infty$,

$$\left(S_{k-1,m}^{(k-1)}(t) - S_{k,m}^{(k)}(t)\right) u \longrightarrow W_k(t)u, \quad 1 \le k \le n-1,$$

uniformly on bounded intervals of $t \ge 0$, valid for all $u \in E$. Accordingly, (1.24) follows from (1.28). □

2. Approximation of dynamic boundary value problems

Let E and X be Banach spaces. We study the following mixed initial boundary value problem:

$$\begin{cases} u''(t) + Au(t) + Bu'(t) = 0, \quad t \ge 0, \\ x''(t) + F_0 x(t) + F_1 x'(t) = G_0 u(t) + G_1 u'(t), \quad t \ge 0, \\ x(t) = \mathcal{P}u(t), \quad t \ge 0, \\ u(0) = u_0,\ x(0) = x_0,\ u'(0) = u_1,\ x'(0) = x_1. \end{cases} \tag{2.1}$$

Here and in the sequel,

$$A : \mathcal{D}(A) \subset E \to E, \quad B : \mathcal{D}(B) \subset E \to E,$$

$$F_0 : \mathcal{D}(F_0) \subset X \to X, \quad F_1 : \mathcal{D}(F_1) \subset X \to X,$$

$$G_0 : \mathcal{D}(G_0) \subset E \to X, \quad G_1 : \mathcal{D}(G_1) \subset E \to X,$$

$$\mathcal{P} : \mathcal{D}(A) \to X$$

are all linear operators. Note that the boundary condition (i.e., the second equation in (2.1)) is of dynamical type.

As a companion of the boundary operator $\mathcal{P}$, we introduce a linear operator $\mathcal{P}_B$ from $\mathcal{D}(B)$ to the quotient space X/X_0 (X_0 a closed linear subspace of X) satisfying the following relation with $\mathcal{P}$:

$$\mathcal{P}u \in \mathcal{P}_B u, \quad u \in \mathcal{D}(A) \cap \mathcal{D}(B). \tag{2.2}$$

Setting

$$\mathbb{A} := \begin{pmatrix} A & 0 \\ -G_0 & F_0 \end{pmatrix}, \quad \mathcal{D}(\mathbb{A}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(A) \cap \mathcal{D}(G_0)) \times \mathcal{D}(F_0);\ \ x = \mathcal{P}u \right\},$$

$$\mathbb{B} := \begin{pmatrix} B & 0 \\ -G_1 & F_1 \end{pmatrix}, \quad \mathcal{D}(\mathbb{B}) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(B) \cap \mathcal{D}(G_1)) \times \mathcal{D}(F_1);\ \ x \in \mathcal{P}_B u \right\},$$

$$y(t) := \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}, \quad y_0 := \begin{pmatrix} u_0 \\ x_0 \end{pmatrix}, \quad y_1 := \begin{pmatrix} u_1 \\ x_1 \end{pmatrix},$$

we then transform (2.1) (with (2.2)) into an abstract Cauchy problem in $\mathbb{E} := E \times X$:

$$(ACP_2; \mathbb{A}, \mathbb{B}) \qquad \begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = 0, & t \geq 0, \\ y(0) = y_0,\ y'(0) = y_1. \end{cases}$$

We introduce below two special spaces and two operators (corresponding to A and B):

$[\mathcal{D}(A)]_{\mathcal{P}}$: the space $\mathcal{D}(A)$ equipped with the norm

$$\|u\|_{A,\mathcal{P}} := \|u\| + \|Au\| + \|\mathcal{P}u\|;$$

$[\mathcal{D}(B)]_{\mathcal{P}_B}$: the space $\mathcal{D}(B)$ equipped with the norm

$$\|u\|_{B,\mathcal{P}_B} := \|u\| + \|Bu\| + \|\mathcal{P}_B u\|_{X/X_0};$$

and

$$A_0 := A\Big|_{\ker \mathcal{P}}, \quad B_0 := B\Big|_{\ker \mathcal{P}_B}.$$

We will use the following hypotheses:

(H_1) The spaces $[\mathcal{D}(A)]_{\mathcal{P}}$ and $[\mathcal{D}(B)]_{\mathcal{P}_B}$ are complete, and $\mathcal{P}(\mathcal{D}(A) \cap \mathcal{D}(B)) = X$.

(H_2) $(ACP_2; A_0, B_0)$ is strongly quasi-wellposed.

Lemma 2.1 ([37]). *Suppose that* (H_1) *and* (H_2) *hold. Let E_1 be a Banach space such that*

$$[\mathcal{D}(A)]_{\mathcal{P}} \hookrightarrow E_1 \hookrightarrow E \quad \text{and} \quad \lambda \mapsto (\lambda^2 + A_0 + \lambda B_0)^{-1} \in LT - \mathcal{L}(E, E_1).$$

If

$$G_0 \in \mathcal{L}(E_1, X), \quad G_1 \in \mathcal{L}(E, X), \quad F_0, F_1 \in \mathcal{L}(X),$$

then $(ACP_2; \mathbb{A}, \mathbb{B})$ is strongly quasi-wellposed.

Next, we consider the inhomogeneous problem:

$$\begin{cases} y''(t) + \mathbb{A}y(t) + \mathbb{B}y'(t) = h(t), & t \in [0, T], \\ y(0) = y_0,\ y'(0) = y_1. \end{cases} \tag{2.3}$$

Lemma 2.2 ([37]). *Suppose that the hypotheses of Lemma 2.1 hold. Let $h \in C^1([0,T]; \mathbb{E})$, $y_0 \in \mathcal{D}(\mathbb{A})$, and $y_1 \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$. Then*

(1) *problem (2.3) has a unique strict solution $y(\cdot)$ given by*

$$y(t) = \mathbb{C}(t)y_0 + \mathbb{S}(t)y_1 + \int_0^t \mathbb{S}(t-s)h(s)ds, \quad t \in [0, T], \tag{2.4}$$

where $\mathbb{C}(\cdot)$ and $\mathbb{S}(\cdot)$ are the two propagators of $(ACP_2; \mathbb{A}, \mathbb{B})$;

(2) *the solution $y(\cdot)$ satisfies*

$$\begin{aligned} &y'(\cdot) \in C([0,T]; E_1 \times X), \\ &\quad \|y''(t)\| + \|y(t)\|_{[\mathcal{D}(\mathbb{A})]} + \|y'(t)\|_{[\mathcal{D}(\mathbb{B})]} + \|y'(t)\|_{E_1 \times X} \\ &\leq M\Big(\|h\|_{C^1([0,T];\mathbb{E})} + \|y_0\|_{[\mathcal{D}(\mathbb{A})]} + \|y_1\|_{[\mathcal{D}(\mathbb{A})]} + \|y_1\|_{[\mathcal{D}(\mathbb{B})]}\Big), \quad t \in [0, T], \end{aligned}$$

for some constant $M > 0$.

Suppose that

$$A_m : \mathcal{D}(A) \to E, \quad B_m : \mathcal{D}(B_m) \to E, \qquad m \in N,$$

$$F_{0,m} : \mathcal{D}(F_0) \to X, \quad F_{1,m} : \mathcal{D}(F_1) \to X, \qquad m \in N,$$

$$G_{0,m} : \mathcal{D}(G_0) \to X, \quad G_{1,m} : \mathcal{D}(G_1) \to X, \qquad m \in N$$

are six sequences of linear operators, where $\mathcal{D}(B_m)$ is independent of $m \in N$. In the case of $B \notin \mathcal{L}(E)$, we assume for simplicity that $\mathcal{D}(B_1) = \mathcal{D}(B)$ and let $\mathcal{P}_{B_1} = \mathcal{P}_B$. When $B \in \mathcal{L}(E)$, we let $\mathcal{P}_{B_1}$ be a linear operator from $\mathcal{D}(B_1)$ to the quotient space X/X_1 (X_1 a closed linear subspace of X) such that

$$\mathcal{P}u \in \mathcal{P}_{B_1} u \quad (u \in \mathcal{D}(A) \cap \mathcal{D}(B_1)).$$

For $m = 1, 2, 3, \ldots$, we put

$$\mathbb{A}_m := \begin{pmatrix} A_m & 0 \\ -G_{0,m} & F_{0,m} \end{pmatrix}, \quad \mathcal{D}(\mathbb{A}_m) := \mathcal{D}(\mathbb{A}),$$

$$\mathbb{B}_m := \begin{pmatrix} B_m & 0 \\ -G_{1,m} & F_{1,m} \end{pmatrix},$$

$$\mathcal{D}(\mathbb{B}_m) := \left\{ \begin{pmatrix} u \\ x \end{pmatrix} \in (\mathcal{D}(B_1) \cap \mathcal{D}(G_1)) \times \mathcal{D}(F_1); \quad x \in \mathcal{P}_{B_1} u \right\},$$

$$A_{0,m} = A_m\Big|_{\ker \mathcal{P}}, \quad B_{0,m} = B_m\Big|_{\ker \mathcal{P}_{B_1}}.$$

By $\mathbb{S}_m(\cdot)$ (resp. $\mathbb{S}(\cdot)$, $S_{E,m}(\cdot)$, $S_{X,m}(\cdot)$) we denote the second propagator (if it exists) of $(ACP_2; \mathbb{A}_m, \mathbb{B}_m)$ (resp. $(ACP_2; \mathbb{A}, \mathbb{B})$, $(ACP_2; A_{0,m}, B_{0,m})$, $(ACP_2; F_{0,m}, F_{1,m})$).

Theorem 2.3. *Let the conditions of Lemma* 2.1 *hold. Assume that*

(i) *for $m \in N$, $[\mathcal{D}(A_m)]_{\mathcal{P}}$ and $[\mathcal{D}(B_m)]_{\mathcal{P}_{B_1}}$ are complete, and $\mathcal{P}(\mathcal{D}(A) \cap \mathcal{D}(B_1)) = X$;*

(ii) *for $m \in N$, $((ACP_2)_m; A_{0,m}, B_{0,m})$ is strongly quasi-wellposed such that*

$$\left\|S'_{E,m}(t)\right\|_{\mathcal{L}(E)}, \ \left\|B_{0,m} S_{E,m}(t)\right\|_{\mathcal{L}(E)}, \ \left\|S_{E,m}(t)\right\|_{\mathcal{L}(E,E_1)} \le M e^{\omega t}, \quad t \ge 0, \tag{2.5}$$

M, ω being constants independent of m;

(iii) *for $m \in N$, $[\mathcal{D}(A_m)]_{\mathcal{P}} \hookrightarrow E_1$, and*

$$F_{0,m}, F_{1,m} \in \mathcal{L}(X), \quad G_{0,m} \in \mathcal{L}(E_1, X), \quad G_{1,m} \in \mathcal{L}(E, X);$$

(iv) *as $m \to \infty$,*

$$A_m u \to Au \ (u \in \mathcal{D}(A)), \quad B_m u \to Bu \ (u \in \mathcal{D}(B_1)),$$

$$F_{0,m} x \to F_0 x \ (x \in X), \quad F_{1,m} x \to F_1 x \ (x \in X),$$

$$G_{0,m} u \to G_0 u \ (u \in E_1), \quad G_{1,m} u \to G_1 u \ (u \in E).$$

Then, for all $y \in \mathbb{E}$,

$$\mathbb{S}'_m(t)y \to \mathbb{S}'(t)y, \quad \mathbb{B}_m\mathbb{S}_m(t)y \to \mathbb{B}\mathbb{S}(t)y,$$

$$\mathbb{A}_m \int_0^t \mathbb{S}(\sigma)yd\sigma \to \mathbb{A}\int_0^t \mathbb{S}(\sigma)yd\sigma,$$

as $m \to \infty$, uniformly on bounded intervals of $t \geq 0$.

Proof. By the third estimate in (2.5), one knows that for $m \in N$,

$$\lambda \mapsto R_{E,m} := (\lambda^2 + A_{0,m} + \lambda B_{0,m})^{-1} \in LT - \mathcal{L}(E, E_1).$$

It follows from Lemma 2.1 that $(ACP_2; \mathbb{A}, \mathbb{B})$ and $(ACP_2; \mathbb{A}_m, \mathbb{B}_m)$ $(m \in N)$ are strongly quasi-wellposed.

Take $\mu = \omega + 1$. For each $x \in X$ and $m = 2, 3, 4, \ldots$, we have

$$D_{\mu,m}x - D_{\mu,1}x \in \ker(\mathcal{P}),$$

where $D_{\mu,m} := \left(\mathcal{P}\Big|_{\ker(\mu^2+A+\mu B)}\right)^{-1}$. This implies that

$$\begin{aligned} & (\mu^2 + A_m + \mu B_m)D_{\mu,1}x \\ = \; & (\mu^2 + A_m + \mu B_m)(D_{\mu,m}x - D_{\mu,1}x) \\ = \; & (\mu^2 + A_{0,m} + \mu B_{0,m})(D_{\mu,m}x - D_{\mu,1}x). \end{aligned}$$

Therefore, for $x \in X$, $m = 2, 3, 4, \ldots$,

$$D_{\mu,m}x = D_{\mu,1}x + R_{E,m}(\mu)(\mu^2 + A_m + \mu B_m)D_{\mu,1}x. \tag{2.6}$$

It is clear from hypothesis (iv) and (2.5) that

$$\|F_{0,m}\|_{\mathcal{L}(X)}, \|F_{1,m}\|_{\mathcal{L}(X)}, \|G_{0,m}\|_{\mathcal{L}(E_1,X)}, \|G_{1,m}\|_{\mathcal{L}(E,X)} \leq \text{const}, \tag{2.7}$$

$$\|A_m D_{\mu,1}\|_{\mathcal{L}(X,E)}, \|B_m D_{\mu,1}\|_{\mathcal{L}(X,E)}, \|G_{0,m} D_{\mu,1}\|_{\mathcal{L}(X,E)} \leq \text{const}, \tag{2.8}$$

$$\|R_{E,m}(\mu)\|_{\mathcal{L}(E)}, \|B_{0,m}R_{E,m}(\mu)\|_{\mathcal{L}(E)}, \|G_{0,m}R_{E,m}(\mu)\|_{\mathcal{L}(E)} \leq \text{const}, \tag{2.9}$$

for all $m \in N$. Combining (2.6), (2.8) and (2.9) yields that for $m \in N$,

$$\|D_{\mu,m}\|_{\mathcal{L}(X,E)}, \|B_m D_{\mu,m}\|_{\mathcal{L}(X,E)}, \|G_{0,m}D_{\mu,m}\|_{\mathcal{L}(X,E)} \leq \text{const}. \tag{2.10}$$

Moreover, it is not hard to verify by (2.7) that

$$\|S''_{X,m}(t)\| \leq M_1 e^{\omega_1 t}, \quad t \geq 0, \; m \in N, \tag{2.11}$$

where M_1, ω_1 are constants.

We know from the proof of [37, Theorem 3.5] that for λ large enough,

$$\int_0^\infty \mathbb{S}'_m(t)ydt = \begin{pmatrix} \lambda R_{E,m}(\lambda) & \lambda D_{\lambda,m}R_{X,m}(\lambda) \\ 0 & \lambda R_{X,m}(\lambda) \end{pmatrix}$$

$$\left[I - \begin{pmatrix} 0 & 0 \\ (G_{0,m}+\lambda G_{1,m})R_{E,m}(\lambda) & (G_{0,m}+\lambda G_{1,m})D_{\lambda,m}R_{X,m}(\lambda) \end{pmatrix}\right]^{-1} y$$

$$= \int_0^\infty e^{-\lambda t}\left[H_{0,m}(t) + H_{1,m}(t) * \left(\sum_{j=1}^\infty [H_{0,m}(t)]^{*j}\right)\right] ydt,$$

$$y \in \mathbb{E},\ m \in N, \tag{2.12}$$

where $*j$ indicates the j-th convolution power, $R_{X,m}(\lambda) := (\lambda^2 + F_0 + \lambda F_1)^{-1}$, and for $t \geq 0$,

$$H_{0,m}(t) := \begin{pmatrix} S'_{E,m}(t) & J'_m(t) \\ 0 & S'_{X,m}(t) \end{pmatrix},$$

$$H_{1,m}(t) := \begin{pmatrix} 0 & 0 \\ G_{0,m}S_{E,m}(t) + G_{1,m}S'_{E,m}(t) & G_{0,m}J_m(t) + G_{1,m}J'_m(t) \end{pmatrix},$$

$$\begin{aligned} J_m(t) \ &:= \ D_{\mu,m}S_{X,m}(t) - S_{E,m}(t)D_{\mu,m} \\ &+ \mu\int_0^t S_{E,m}(t-s)(B_m+\mu)D_{\mu,m}S_{X,m}(s)ds \\ &- \int_0^t S_{E,m}(t-s)B_mD_{\mu,m}S_{X,m}(s)ds \\ &- \int_0^t S_{E,m}(t-s)D_{\mu,m}S''_{X,m}(s)ds. \end{aligned}$$

According to (2.5), (2.7), (2.10) and (2.11), there exist constants $M_2 > M + M_1$, $\omega_2 > \omega + \omega_1$ such that

$$\|H_{i,m}(t)\| \leq M_2 e^{\omega_2 t}, \quad t \geq 0,\ m \in N.$$

This implies the existence of constants $M_3 > M_2$, $\omega_3 > \omega_2$ such that for all $m \in N$,

$$\|\mathbb{S}'_m(t)\| \leq M_3 e^{\omega_3 t}, \quad t \geq 0 \tag{2.13}$$

due to (2.12). Similarly, we obtain

$$\|\mathbb{B}_m\mathbb{S}_m(t)\| \leq M_4 e^{\omega_4 t}, \quad t \geq 0,\ m \in N, \tag{2.14}$$

with some constants M_4, $\omega_4 > 0$. Finally, hypothesis (iv) ensures that

$$\lim_{m\to\infty} \mathbb{A}_m y = \mathbb{A}y, \quad \lim_{m\to\infty} \mathbb{B}_m y = \mathbb{B}y \tag{2.15}$$

for $y \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B}_1)$. Note that for $m \in N$,

$$\mathcal{D}(\mathbb{A}_m) \cap \mathcal{D}(\mathbb{B}_m) = \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B}_1) = \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B}), \tag{2.16}$$

since

$$\mathcal{D}(\mathbb{B}_1)\begin{cases} = \mathcal{D}(\mathbb{B}) & \text{if } B \notin \mathcal{L}(E), \\ \supset \mathcal{D}(\mathbb{A}) & \text{if } B \in \mathcal{L}(E) \ \ (\text{which implies } \mathcal{D}(\mathbb{B}) = E). \end{cases}$$

Moreover, we have

$$\mathbb{A}_m \int_0^t \mathbb{S}_m(\sigma) y d\sigma = y - \mathbb{S}_m'(t)y - \mathbb{B}_m \mathbb{S}_m(t)y, \quad t \geq 0,\ y \in \mathbb{E}, \tag{2.17}$$

for $m \in N$. Thus, according to (2.13)–(2.17) we obtain the conclusions by an application of Theorem 1.6. □

As a consequence of Theorem 2.3 and Lemma 2.2 we have the following result.

Theorem 2.4. *Assume that the conditions of Theorem* 2.3 *are satisfied. For* $m \in N$, *let* h, $h_m \in C^1([0,t_0];\mathbb{E})$, y_0, $y_{0,m} \in \mathcal{D}(\mathbb{A})$, y_1, $y_{1,m} \in \mathcal{D}(\mathbb{A}) \cap \mathcal{D}(\mathbb{B})$ *such that*

$$\|h_m - h\|_{L^1((0,t_0);\mathbb{E})} \to 0, \quad y_{0,m} \to y_0, \quad y_{1,m} \to y_1, \tag{2.18}$$

as $m \to \infty$. *Then, the solution sequence* $y_m(t)$ *of*

$$\begin{cases} y_m''(t) + \mathbb{A}_m y_m(t) + \mathbb{B}_m y_m'(t) = h_m(t), \quad t > 0, \\ y_m(0) = y_{0,m},\ y_m'(0) = y_{1,m} \end{cases}$$

converges to the solution $y(t)$ *of* (2.3) *uniformly for* $t \in [0,t_0]$.

To illustrate Theorem 2.4, we present the following example.

Example 2.5. Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. We consider a sequence of mixed initial-boundary value problems for structurally damped plate-like equations whose boundary conditions are of dynamic natures:

$$\begin{cases} \partial_t^2 u_m + \Delta^2 u_m - \rho_m \Delta \partial_t u_m = f_m, & \text{in } [0,t_0] \times \Omega, \\ \partial_t^2 u_m = \left\langle \dfrac{\partial u_m}{\partial \nu},\ v_m \right\rangle_{L^2(\partial\Omega)} w_m, & \text{in } [0,t_0] \times \partial\Omega, \\ \Delta u_m \Big|_{\partial\Omega} = 0, & \text{in } [0,t_0] \times \partial\Omega, \\ u_m(0,\cdot) = \varphi_0, \quad \partial_t u_m(0,\cdot) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{2.19}$$

where $m \in N \cup \{0\}$, $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$,

$$\begin{aligned} &\varphi_i \in H^2(\Omega) \quad \text{with } \Delta\varphi_i \in H^2(\Omega) \text{ and } \ \Delta\varphi_i\Big|_{\partial\Omega} = 0 \ \ (i = 0,1), \\ &\{v_m\}_{m\in N_0} \subset L^2(\partial\Omega), \quad \{w_m\}_{m\in N_0} \subset H^2(\partial\Omega), \\ &\{f_m\}_{m\in N_0} \subset C^1([0,t_0];L^2(\Omega)), \quad \{\rho_m\}_{m\in N} \subset (0,\infty) \end{aligned}$$

such that

$$\begin{aligned} &\lim_{m\to\infty} \|v_m - v_0\|_{L^2(\partial\Omega)} = 0, \quad \lim_{m\to\infty} \|w_m - w_0\|_{H^2(\partial\Omega)} = 0, \\ &\lim_{m\to\infty} \|f_m - f_0\|_{L^2((0,t_0);H^2(\Omega))} = 0, \quad \lim_{m\to\infty} \rho_m = \rho_0 := 0. \end{aligned}$$

Take

$$E = L^2(\Omega), \quad E_1 = H^2(\Omega), \quad X = X_0 = H^{\frac{3}{2}}(\partial\Omega), \quad B = 0,$$

and for $m \in N$,

$$B_m = -\rho_m \Delta \quad \text{with} \quad \mathcal{D}(B_m) = H^2(\Omega),$$

$$A = A_m = \Delta^2 \quad \text{with} \quad \mathcal{D}(A) = \mathcal{D}(A_m) = \{\varphi \in H^2(\Omega);\ \ \Delta\varphi \in H^2(\Omega),\ \Delta\varphi\Big|_{\partial\Omega} = 0\},$$

$$G_0\varphi = \left\langle \frac{\partial\varphi}{\partial\nu},\ v_0 \right\rangle_{L^2(\partial\Omega)} w_0,$$

$$G_{0,m}\varphi = \left\langle \frac{\partial\varphi}{\partial\nu},\ v_m \right\rangle_{L^2(\partial\Omega)} w_m \quad \text{for } \varphi \in \mathcal{D}(G_0) = \mathcal{D}(G_{0,m}) := \mathcal{D}(A),$$

$$\mathcal{P}\varphi = \varphi\Big|_{\partial\Omega} \quad \text{for } \varphi \in \mathcal{D}(\mathcal{P}) := \mathcal{D}(A),$$

$$\mathcal{P}_B\varphi = X \quad \text{for } \varphi \in \mathcal{D}(\mathcal{P}_B) := E,$$

$$\mathcal{P}_{B_1} = \mathcal{P}, \quad F_0 = 0, \quad F_1 = 0, \quad G_1 = 0, \quad F_{0,m} = 0, \quad F_{1,m} = 0, \quad G_{1,m} = 0.$$

From [34, p.232] one can see that for $t \geq 1$, $m \in N$,

$$\|S'_{E,m}(t)\|_{\mathcal{L}(E)}, \quad \|B_{0,m}S_{E,m}(t)\|_{\mathcal{L}(E)}, \quad \|S_{E,m}(t)\|_{\mathcal{L}(E)} \leq 1.$$

Also the other conditions of Theorem 2.4 are satisfied (cf. [37, Example 5.5]). Therefore, if $u_m(\cdot)$ ($m \in N \cup \{0\}$) is the solution of (2.19), then

$$\lim_{m\to\infty} \sup_{t\in[0,t_0]} \|u_m(t) - u_0(t)\|_{L^2(\Omega)} = 0.$$

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