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SYMMETRIES OF THE HYPERGEOMETRIC FUNCTION ${}_mF_{m-1}$

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ABSTRACT. In this paper, we show that the generalized hypergeometric function ${}_mF_{m-1}$ has a one parameter group of local symmetries, which is a conjugation of a flow of a rational Calogero-Mozer system. We use the symmetry to construct fermionic fields on a complex torus, which have linear-algebraic properties similar to those of the local solutions of the generalized hypergeometric equation. The fields admit a nontrivial action of the quaternions based on the above symmetry. We use the similarity between the linear-algebraic structures to introduce the quaternionic action on the direct sum of the space of solutions of the generalized hypergeometric equation and its dual. As a side product, we construct a "good" basis for the monodromy operators of the generalized hypergeometric equation inspired by the study of multiple flag varieties with finitely many orbits of the diagonal action of the general linear group by Magyar, Weyman, and Zelevinsky. As an example of computational effectiveness of the basis, we give a proof of the existence of the monodromy invariant hermitian form on the space of solutions of the generalized hypergeometric equation (in the case of real local exponents) different from the proofs of Beukers and Heckman and of Haraoka. As another side product, we prove an elliptic generalization of Cauchy identity.

1. Introduction

This section is a short seminar-style exposition of the paper from history and motivations to main results to open questions. It should suffice the reader who wants to understand the results without going into too many details.

One of the ways to define the *generalized hypergeometric function* ${}_mF_{m-1}$ is by means of power series:

$$(1.1)\qquad {}_mF_{m-1}\left(\begin{array}{c} b_1, \cdots, b_m \\ c_1, \cdots, c_{m-1} \end{array}\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(b_1)_n \cdots (b_m)_n}{(c_1)_n \cdots (c_{m-1})_n\, n!}\, z^n.$$

Here $(x)_n$ is the *Pochhammer symbol* $(x)_n = \Gamma(x+n)/\Gamma(x)$. The right hand side of (1.1) converges inside the unit circle of the complex plane centered at zero.

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The following ODE is known as the *generalized hypergeometric equation* (GHGE):

$$z\left(z\frac{d}{d\,z}-c_1\right)\cdots\left(z\frac{d}{d\,z}-c_m\right)f(z)=\left(z\frac{d}{d\,z}-b_1\right)\cdots\left(z\frac{d}{d\,z}-b_m\right)f(z). \tag{1.2}$$

The functions

$$z^{b_j}\,{}_mF_{m-1}\left(\begin{array}{c}b_j-c_1,\cdots,b_j-c_m\\ b_j-b_1+1,\cdots,\widehat{b_j-b_j+1},\cdots,b_j-b_m+1\end{array}\middle|\, z\right) \tag{1.3}$$

and

$$z^{c_i}\,{}_mF_{m-1}\left(\begin{array}{c}b_1-c_i,\cdots,b_m-c_i\\ c_1-c_i+1,\cdots,\widehat{c_i-c_i+1},\cdots,c_m-c_i+1\end{array}\middle|\, \frac{1}{z}\right) \tag{1.4}$$

form bases of the spaces of solutions of the GHGE in a vicinity of zero and of infinity respectively, if the *local exponents* at zero b_i are distinct *mod* 1 as well as the local exponents at infinity $-c_i$.

The *monodromy* of the GHGE was found by K. Okubo in [14] and independently by F. Beukers and G. Heckman in [1]. The monodromy matrices of the GHGE give probably the most important example of a *rigid local system* (see page 2563 for the definition). An algorithm to construct all rigid local systems on the Riemann sphere was presented by N. Katz in [9] and translated into the language of linear algebra by M. Dettweiler and S. Reiter in [2]. Y. Haraoka and T. Yokoyama in [7] give an algorithm to construct all *semisimple* rigid local systems in the *Okubo normal form* and prove that the corresponding Fuchsian systems have integral solutions (see also [5] and [6]). However, both algorithms are so computationally complicated that one should choose a Fuchsian system to apply them to very carefully. Here is one possible criterion to pick Fuchsian systems for detailed studies.

The local exponents of a Fuchsian system stratify its space of solution into a flag near each singularity. There is no basis in the space of solutions which is simultaneously "good" for all the flags, so the flags should be considered up to a basis change and thus give rise to flag varieties. One way to find the most important Fuchsian systems is to look for the simplest nontrivial multiple flag varieties. P. Magyar, J. Weyman, and A. Zelevinsky classified in [13] all indecomposable multiple flag varieties with finitely many orbits under the diagonal action of the general linear group (of simultaneous base changes). It turned out that there were three infinite series: the hypergeometric, the odd, and the even and two extra cases E_8 and $\hat{E}_8$. The Fuchsian systems corresponding to all the cases were constructed by the author in [3]. It is no coincidence that when the first major breakthrough in understanding rigid local systems had been made earlier by C. Simpson in [18], the local systems he had constructed were the hypergeometric, the odd, the even, and the extra case $\hat{E}_8$. The same results were obtained with different techniques by V. Kostov in [10]. (See also Kostov's survey of the *Deligne-Simpson problem* in [11].) So in a sense, the Fuchsian systems constructed in [3] are the "more equal animals" from the bestiary of Fuchsian systems on the Riemann sphere. Among them, the Fuchsian system corresponding to the hypergeometric case is definitely the "most equal animal". On the one hand, it is arguably the simplest non-trivial Fuchsian system. On the other hand, it is equivalent to the GHGE as a flat connection: it has the same singularities and the same monodromy. We shall call this system the *m-hypergeometric system* (mHGS). It is the main object of study in this paper. Any information regarding the mHGS we obtain can (and will) be translated into the information regarding the GHGE.

Definition 1.1. Linear operators A, B, and C acting on a complex linear space $\mathbb{C}^m$ are called an additive hypergeometric triple, if
- $A + B + C = 0$;
- $rank\, B = rank\, C = m$;
- A is a diagonalizable operator with two different eigenvalues: a_1 of multiplicity 1 and a_2 of multiplicity $m-1$;
- the operators are generic within the above restrictions.

Let A, B, and C be an additive hypergeometric triple. Let the eigenvalues of B and C be $b_1, \dots, b_m$ and $-c_1, \dots, -c_m$ respectively and let $v_1, \dots, v_m$ and $w_1, \dots, w_m$ be the corresponding eigenvectors. A consequence of the fact that the operators are generic is that their eigenvalues are generic complex numbers in the plane given by the *trace condition*:

$$a_1 + (m-1)a_2 + \sum_{i=1}^{m} b_i - c_i = 0. \tag{1.5}$$

In particular, none of the differences $b_i - b_j$ and $c_i - c_j$ is an integer for $i \neq j$, and neither is $b_i - c_j$ for all i and j.

Let u be the eigenvector of A corresponding to the eigenvalue a_1. Another consequence of the fact that the operators are generic is that there exists a unique way to choose v_i and w_i so that

$$\sum_{i=1}^{m} v_i = \sum_{i=1}^{m} w_i = u.$$

Let $f : \mathbb{CP}^1 \setminus \{0, 1, \infty\} \to \mathbb{C}^m$. The mHGS is the following Fuchsian system:

$$\frac{d\,f}{d\,z} = \left[\frac{A}{z-1} + \frac{B}{z}\right] f(z). \tag{1.6}$$

The additive hypergeometric triple is the triple of the *residue operators* of the mHGS. C is the residue at infinity, so it is not explicitly visible in (1.6). We shall call the space where the residues act the *residue space*. The following feature of the residue space was proven in [3]:

Theorem 1.2. *There exists a unique up to a constant multiple complex symmetric scalar product $(*, *)_r$ on the residue space of the mHGS such that the bases v_i and w_i are simultaneously orthogonal with respect to it, given by*

$$(v_i, v_j)_r = \delta_{ij}\, \nu_{i\,r}^2 \text{ and } (w_i, w_j)_r = \delta_{ij}\, \mu_{i\,r}^2, \tag{1.7}$$

where

$$\mu_{i\,r}^2 = \frac{\prod_{k=1}^{m} (a_2 + b_k - c_i)}{\prod_{\substack{k=1\\k\neq i}}^{m} (c_k - c_i)} \text{ and } \nu_{i\,r}^2 = \frac{\prod_{k=1}^{m} (a_2 + b_i - c_k)}{\prod_{\substack{k=1\\k\neq i}}^{m} (b_i - b_k)}. \tag{1.8}$$

When all the local exponents are real, the form $(*,*)_r$ is real symmetric. It is important to know when the real form is sign-definite. The following lemma is proven in [3]:

Lemma 1.3. *Renumbering if necessary, we can think that $b_1 < \cdots < b_m$ and $c_1 < \cdots < c_m$. If the form $\epsilon\,(*,*)_r$ is positive-definite for $\epsilon = \pm 1$, then $\epsilon = sign(a_2 - a_1)$. If the inequalities of the first column hold, then $a_2 > a_1$. If the inequalities of the second column hold, then $a_2 < a_1$.*

(1.9)

$$\begin{array}{|lcccl|lcccl|}
\hline
c_1 - b_1 & < & a_2 & < & c_2 - b_1 & c_{m-1} - b_m & < & a_2 & < & c_m - b_m \\
c_2 - b_2 & < & a_2 & < & c_3 - b_2 & c_{m-2} - b_{m-1} & < & a_2 & < & c_{m-1} - b_{m-1} \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\
c_{m-1} - b_{m-1} & < & a_2 & < & c_m - b_{m-1} & c_1 - b_2 & < & a_2 & < & c_2 - b_2 \\
c_m - b_m & < & a_2 & & & & & a_2 & < & c_1 - b_1 \\
\hline
\end{array}.$$

Note that we can extend the sign-definite real symmetric form $(*,*)_r$ to complex numbers either in the complex symmetric or in the hermitian way.

Let V be the operator of the basis switch from the standard basis to the basis v_i. Let

$$S = VXV^{-1}, \tag{1.10}$$

where X is the following $m \times m$ matrix:

$$X_{ij} = \begin{cases} \dfrac{1}{b_j - b_i}, & \text{if} \quad i \neq j, \\[2ex] -\displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{m} \dfrac{1}{b_j - b_i}, & \text{if} \quad i = j. \end{cases} \tag{1.11}$$

The following two systems with complex times play a major role in our investigation:

$$\begin{array}{ll} \dot{B}(\tau_1) + [B(\tau_1), S] = k_1 Id, & \dot{C}(\tau_2) - [C(\tau_2), S] = k_2 Id, \\[1ex] B(0) = B, & C(0) = C. \end{array} \tag{1.12}$$

Here $[*,*]$ is the usual commutator, k_1 and k_2 are arbitrary complex constants, and Id is the $m \times m$ identity matrix. The following theorem shows the importance of the systems (1.12) for studying the mHGS.

Theorem 1.4. *If $a_2 = 0$, then mHGS has "nice" Fröbenius series solutions in a local parameter z near zero and in a local parameter z^{-1} near infinity respectively:*

$$(T_0)_i = z^{b_i} \left(\sum_{n=0}^{\infty} \alpha_{in}\, e^{Sn}\, z^n \right) v_i \ \textit{and} \tag{1.13}$$

$$(T_\infty)_i = z^{c_i} \left(\sum_{n=0}^{\infty} \beta_{in}\, e^{-Sn}\, z^{-n} \right) w_i, \ \textit{where} \tag{1.14}$$

(1.15)

$$\alpha_{in} = \prod_{k=1}^{m} \frac{\Gamma(b_i - c_k + n)\,\Gamma(b_i - b_k + 1)}{\Gamma(b_i - c_k)\,\Gamma(b_i - b_k + n + 1)}, \quad \beta_{in} = \prod_{k=1}^{m} \frac{\Gamma(b_k - c_i + n)\,\Gamma(c_k - c_i + 1)}{\Gamma(b_k - c_i)\,\Gamma(c_k - c_i + n + 1)}.$$

The series (1.13) *and* (1.14) *converge inside the unit circles centered at zero and at infinity.*

Lemma 1.5. *The eigenvalues of $B(\tau_1)$ and $C(\tau_2)$ evolve linearly: $b_i(\tau_1) = b_i + k_1\tau_1$ and $-c_i(\tau_2) = -c_i + k_2\tau_2$. The corresponding eigenvectors are*

$$v_i(\tau_1) = e^{S\tau_1} v_i, \;\; w_i(\tau_2) = e^{-S\tau_2} w_i, \tag{1.16}$$

where v_i and w_i are the eigenvectors of the original operators B and C.

Thus, up to normalizing factors, the coefficients of the local solutions (1.13) and (1.14) are the eigenvectors $v_i(\tau_1)$ and $w_i(\tau_2)$ of the operators $B(\tau_1)$ and $C(\tau_2)$ for the integral times $\tau_1 = n$ and $\tau_2 = -n$.

The GHGE has the subspace of solutions holomorphic at one of dimension $m-1$, which corresponds to the case $a_2 = 0$ for the mHGS. However, from the point of view of representation theory, the case of the traceless residue operators seems to be more important. Unfortunately, when $a_2 \neq 0$, the nice Fröbenius series for the local solutions similar to the above exist no more. To study the mHGS in this case, one needs an apparatus different from the residue calculus.

In the case $a_2 = 0$, the study of the series (1.13) and (1.14) is completely parallel to the study of the local solutions (1.3) and (1.4) of the GHGE. However, our approach through the dynamical systems (1.12) suggests that taking the times τ_1 and τ_2 along two linearly dependent vectors in the complex plane is a trigonometric degeneration of a more general elliptic situation. So on the one hand, it seems desirable to construct an elliptic analogue of (1.6). On the other hand, we know from [17] that a solution to (1.6) can be realized as a fermionic field. We shall use this idea to treat the case $a_2 \neq 0$ by means of linear algebra rather than the residue calculus.

Definition 1.6. Linear operators M_0, M_1, and M_∞ acting on a complex linear space $\mathbb{C}^m$ are called a multiplicative hypergeometric triple, if

- $M_\infty M_1 M_0 = Id$;
- $rank\, M_0 = rank\, M_1 = m$;
- M_1 is a diagonalizable operator with two different eigenvalues of multiplicities 1 and $m-1$;
- the operators are generic within the above restrictions.

Quite obviously, the monodromy operators of the mHGS form a multiplicative hypergeometric triple. The eigenvalues of M_0 and M_∞ are $e^{2\pi\sqrt{-1}\,b_i}$ and $e^{-2\pi\sqrt{-1}\,c_i}$ respectively. M_1 is diagonalizable and has two eigenvalues: $e^{2\pi\sqrt{-1}\,a_1}$ of multiplicity one and $e^{2\pi\sqrt{-1}\,a_2}$ of multiplicity $m-1$. The triple loop on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ passing around zero, one, and infinity once in the positive (counter clockwise) direction is contractible, so $M_\infty M_1 M_0 = Id$. Note that the product of the operators is taken in the opposite order: that is because they act on the right taking linear combinations of columns of the fundamental matrix.

Let p_i and q_i be the eigenvectors of M_0 and M_∞ corresponding to the eigenvalues $e^{2\pi\sqrt{-1}\,b_i}$ and $e^{-2\pi\sqrt{-1}\,c_i}$ respectively. Let r be the eigenvector of M_1 corresponding to the eigenvalue a_1. The operators are generic, so it is uniquely possible to choose the p_i and q_i so that

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i = r. \tag{1.17}$$

The following theorem is proven in [1] for the GHGE. We reprove this theorem for the mHGS with any a_2 in this paper. The basis we work in is different from the basis used by Beukers and Heckman and is more convenient in some situations.

Theorem 1.7. *If all the local exponents are real, then there exists a unique up to a constant multiple monodromy invariant hermitian product $(*,*)_{trig}$ on the space of solutions such that the bases p_i and q_i are simultaneously orthogonal with respect to it, given by*

$$(p_i, p_j)_{trig} = \delta_{ij}\, \nu^2_{i\,trig} \text{ and } (q_i, q_j)_{trig} = \delta_{ij}\, \mu^2_{i\,trig}, \tag{1.18}$$

where

$$\mu^2_{i\,trig} = \frac{\prod\limits_{k=1}^{m} \sin \pi(a_2 + b_k - c_i)}{\prod\limits_{\substack{k=1\\k\neq i}}^{m} \sin \pi(c_k - c_i)} \text{ and } \nu^2_{i\,trig} = \frac{\prod\limits_{k=1}^{m} \sin \pi(a_2 + b_i - c_k)}{\prod\limits_{\substack{k=1\\k\neq i}}^{m} \sin \pi(b_i - b_k)}. \tag{1.19}$$

The initial motivation for the author to study the subject was to understand why the formulae (1.7) and (1.18) look so similar despite the different nature of the products: the first is complex symmetric, whereas the second is hermitian.

The object which explains this similarity is the following: further in the paper we construct four m-tuples of fermionic fields $(F_0)_i$, $(F_0)^\dagger_i$, $(F_\infty)_j$, and $(F_\infty)^\dagger_j$ such that their *vacuum expectation values* (see page 2560 for the definition) are

$$\left\langle (F_\infty)^\dagger_j\ (F_0)_i \right\rangle = \frac{1}{sn(a_2 + b_i - c_j)} = \left\langle (F_\infty)_j\ (F_0)^\dagger_i \right\rangle. \tag{1.20}$$

Here sn is the elliptic sine of Jacobi and a_2, b_i and $-c_i$ are the local exponents of the mHGS. We use the pairing (1.20) to identify the linear spaces spanned by $(F_0)_i$ and $(F_\infty)^\dagger_i$ as well as by $(F_0)^\dagger_i$ and $(F_\infty)_i$. Let us call the resulting spaces $\mathbb{H}$ and $\mathbb{H}'$ respectively. These spaces resemble the residue space and the space of solutions of the mHGS in many ways. In particular, if we rewrite the formulae (1.7) and (1.18) as

$$\left(\frac{v_i}{\nu^2_{i\,r}}, \frac{v_j}{\nu^2_{j\,r}}\right)_r = \frac{\delta_{ij}}{\nu^2_{i\,r}}, \qquad \left(\frac{w_i}{\mu^2_{i\,r}}, \frac{w_j}{\mu^2_{j\,r}}\right)_r = \frac{\delta_{ij}}{\mu^2_{i\,r}},$$

$$\left(\frac{p_i}{\nu^2_{i\,trig}}, \frac{p_j}{\nu^2_{j\,trig}}\right)_{trig} = \frac{\delta_{ij}}{\nu^2_{i\,trig}}, \qquad \left(\frac{w_i}{\mu^2_{i\,trig}}, \frac{w_j}{\mu^2_{j\,trig}}\right)_{trig} = \frac{\delta_{ij}}{\mu^2_{i\,trig}},$$

then Theorems 1.2 and 1.7 start looking very similar to the following:

Theorem 1.8. *There exists a unique up to a constant multiple complex symmetric scalar product $(*,*)_{ell}$ on $\mathbb{H}$ such that the bases $(F_0)_i$ and $(F_\infty)^\dagger_i$ are simultaneously orthogonal with respect to it, given by*

$$\left((F_0)_i, (F_0)_j\right)_{ell} = \frac{\delta_{ij}}{\nu^2_{i\,ell}} \text{ and } \left((F_\infty)^\dagger_i, (F_\infty)^\dagger_j\right)_{ell} = \frac{\delta_{ij}}{\mu^2_{i\,ell}}, \tag{1.21}$$

where

$$\mu^2_{i\,ell} = \frac{\prod_{k=1}^{m} sn(a_2 + b_k - c_i)}{\prod_{\substack{k=1\\k\neq i}}^{m} sn(c_k - c_i)} \text{ and } \nu^2_{i\,ell} = \frac{\prod_{k=1}^{m} sn(a_2 + b_i - c_k)}{\prod_{\substack{k=1\\k\neq i}}^{m} sn(b_i - b_k)}. \tag{1.22}$$

Similar formulae hold for $\mathbb{H}'$.

The eigenvalues of the monodromy operators of a linear regular system are more important than the local exponents due to the following reason. Consider the linear regular matrix system

$$\frac{dT(z)}{dz} = R(z)T(z). \tag{1.23}$$

The *gauge transformation* $T \mapsto g(z)T$ replaces R by

$$\frac{dg}{dz}g^{-1} + gRg^{-1}.$$

Most often g are taken holomorphic and holomorphically invertible away from the poles of the original system, so that the new system has the same singularities as the old one. The only invariant under such a transformation is the monodromy group of the system; the residue matrices are not preserved. Combining this perspective with the real local exponents, we can think that the local exponents belong to the semi-interval $[0, 1)$ right away. Renumbering if necessary, we can, similar to Lemma 1.3, think that $0 \le b_1 < \cdots < b_m < 1$ and $0 \le c_1 < \cdots < c_m < 1$.

Let the period lattice of the elliptic sine in (1.20) be generated by $2\omega_1$ and ω_2; see page 2561. The fact that, unlike the residue space and the space of solutions, the spaces $\mathbb{H}$ and $\mathbb{H}'$ naturally come in a pair, gives rise to the following two parts theorem:

Theorem 1.9.

1. Let us restrict ourselves to the case $\omega_1 = 1$ *and* $\omega_2 = \sqrt{-1}s$*, where* $s \in \mathbb{R}$. *Let* a_2*,* b_i *and* c_i *be generic real numbers from* $[0, 1)$ *satisfying the positivity conditions* (1.9). *Then the formulae* (1.21) *give hermitian forms on* $\mathbb{H}$ *and* $\mathbb{H}'$. *If the inequalities* (1.9) *are not satisfied, then the forms have nontrivial signatures.*

2. The quaternions act on $\mathbb{H} \oplus \mathbb{H}'$ *by means of the the following formulae:*

(1.24)

$$\begin{aligned}
&\mathbf{i}\,(F_0)_i = -\,(F_0)^\dagger_i\,, && \mathbf{j}\,(F_0)_i = -\sqrt{-1}\,(F_0)_i\,, && \mathbf{k}\,(F_0)_i = \sqrt{-1}\,(F_0)^\dagger_i\,,\\
&\mathbf{i}\,(F_0)^\dagger_i = (F_0)_i\,, && \mathbf{j}\,(F_0)^\dagger_i = \sqrt{-1}\,(F_0)^\dagger_i\,, && \mathbf{k}\,(F_0)^\dagger_i = \sqrt{-1}\,(F_0)_i\,,\\
&\mathbf{i}\,(F_\infty)_i = -\,(F_\infty)^\dagger_i\,, && \mathbf{j}\,(F_\infty)_i = -\sqrt{-1}\,(F_\infty)_i\,, && \mathbf{k}\,(F_\infty)_i = \sqrt{-1}\,(F_\infty)^\dagger_i\,,\\
&\mathbf{i}\,(F_\infty)^\dagger_i = (F_\infty)_i\,, && \mathbf{j}\,(F_\infty)^\dagger_i = \sqrt{-1}\,(F_\infty)^\dagger_i\,, && \mathbf{k}\,(F_\infty)^\dagger_i = \sqrt{-1}\,(F_\infty)_i\,.
\end{aligned}$$

If the hermitian forms are sign-definite, then the quaternionic action is hyperkähler, which explains the simultaneous presence of the complex symmetric and hermitian forms; see [8].

A construction of the quaternionic action (1.24) further in the paper shows that the spaces $\mathbb{H}$ and $\mathbb{H}'$ are naturally dual to each other. Moreover, the sum of the solution space and its dual turns out to be equivalent to the trigonometric limit of $\mathbb{H} \oplus \mathbb{H}'$, whereas the sum of the residue space and its dual is equivalent to the rational limit. The following commuting diagram illustrates how it works:

$$
\begin{array}{ccccc}
? & \simeq & \left\langle (F_\infty)_j^\dagger \ (F_0)_i \right\rangle & = & \frac{1}{sn(a_2+b_i-c_j)} \\
 & & trig \downarrow & \omega_2 \to \infty & \downarrow trig \\
\left(\frac{q_j}{\mu_{j\,trig}^2}, \frac{p_i}{\nu_{i\,trig}^2}\right)_{trig} & \simeq & \left\langle (F_\infty)_{j\,trig}^\dagger \ (F_0)_{i\,trig} \right\rangle & = & \frac{\pi}{\sin \pi(a_2+b_i-c_j)} \\
 & & r \downarrow & \omega_1 \to \infty & \downarrow r \\
\left(\frac{w_j}{\mu_{j\,r}^2}, \frac{v_i}{\nu_{i\,r}^2}\right)_r & \simeq & \left(\frac{1}{\mu_{j\,r}^2} \frac{w_j \oplus -\sqrt{-1}\,w_j}{\sqrt{2}}, \frac{1}{\nu_{i\,r}^2} \frac{v_i \oplus \sqrt{-1}\,v_i}{\sqrt{2}}\right) & = & \frac{1}{a_2+b_i-c_j}
\end{array}
$$

The equivalences in the diagram occur at the level of formal linear algebra. Geometrically, p_i and q_j are multivalued functions on $\mathbb{CP}^1 \setminus \{0,1,\infty\}$, which can be expressed by convergent Fröbenius series or by Euler integrals, undergo monodromy transformations, etc. Their counterparts $(F_0)_{i\,trig}$ and $(F_\infty)_{j\,trig}^\dagger$ are infinite sums of annihilation/creation operators, which obtain geometric meaning when applied to vectors of the corresponding Fock space. The vectors v_i and w_j are the eigenvectors of the residue operators B and C of the mHGS, whereas the $v_i \oplus \sqrt{-1}\,v_i$ and $w_j \oplus -\sqrt{-1}\,w_j$ should be thought of as isotropic generators of some Clifford algebra. Still, some linear-algebraic information can be dragged from one side to the other in a meaningful fashion. In particular, the quaternionic action (1.24) can be introduced on the sums of the solution and the residue spaces of the mHGS with their respective duals, explaining that in fact both the complex symmetric and the hermitian structures are present in both cases. The limiting procedure explains the similarity between the formulae (1.7) and (1.18) for the complex symmetric product $(*,*)_r$ on the residue space and the hermitian product $(*,*)_{trig}$ on the space of solutions of the mHGS.

The question mark in the diagram stands for the unknown Fuchsian system on the torus, which becomes the mHGS as the period $\omega_2 \to \infty$. A construction of such a system would deepen the above equivalences from the linear-algebraic to the geometric level. In particular, it should allow one to drag the action of the monodromy group of the mHGS to the field side of the diagram in a meaningful geometric fashion. The author plans to construct this system in a subsequent publication.

2. More results

In this section, some further results are presented. We also give a couple of proofs, which are essential for understanding the introduced objects and some of the results.

2.1. **Symmetries of the mHGS and the Calogero–Moser flow.** The local solutions (1.13) and (1.14) of the mHGS at zero and at infinity are constructed in terms of the operator $S = VXV^{-1}$. The matrix X (1.11) depends on the eigenvalues b_i of the residue operator B only. V is the operator of the basis switch from the standard basis to the basis v_i of the eigenvectors of B. To restore the symmetry of the construction, let us consider the following $m \times m$ matrix Y:

$$Y_{ij} = \begin{cases} \dfrac{1}{c_i - c_j}, & \text{if } \; i \neq j, \\ -\sum\limits_{\substack{j=1 \\ j\neq i}}^{m} \dfrac{1}{c_i - c_j}, & \text{if } \; i = j. \end{cases} \tag{2.25}$$

Lemma 2.1. • *The Jordan normal form of both X and Y is a single block of full size m with the eigenvalue zero. The corresponding eigenvector of both X and Y is $e = (1, \dots, 1)$.*

• *Let $B_d = diag(b_1, \dots, b_m)$ and $C_d = diag(-c_1, \dots, -c_m)$ be the Jordan normal forms of the operators B and C respectively. Then the matrices X and Y satisfy a special type of the rational Calogero–Moser equation:*

$$[X, B_d] = [Y, C_d] = e \otimes e - Id. \tag{2.26}$$

• *Let W be the operator of the basis switch from the standard basis to the basis w_i of the eigenvectors of the residue operator C. Then*

$$VXV^{-1} = -WYW^{-1}. \tag{2.27}$$

The last formula of the lemma shows that X and $-Y$ are matrices of the operator S from the dynamical equations (1.12) in the bases v_i and w_i respectively. The formula $Xe = 0$ is equivalent to $Su = 0$, which implies that

$$\sum_{i=1}^{m} v_i(\tau_1) = \sum_{i=1}^{m} w_i(\tau_2) = u \tag{2.28}$$

for any complex times τ_1 and τ_2.

Recall that the vectors $v_i(\tau_1)$ and $w_i(\tau_2)$ are the eigenvectors of the operators $B(\tau_1)$ and $C(\tau_2)$ corresponding to the eigenvalues $b_i(\tau_1)$ and $-c_i(\tau_2)$ respectively; see Lemma 1.5. Let us introduce the following notation:

$$\mu_i^2(\tau) = \frac{\prod\limits_{k=1}^{m} (a_2 + b_k - c_i + \tau)}{\prod\limits_{\substack{k=1 \\ k\neq i}}^{m} (c_k - c_i)}, \;\; \nu_i^2(\tau) = \frac{\prod\limits_{k=1}^{m} (a_2 + b_i - c_k + \tau)}{\prod\limits_{\substack{k=1 \\ k\neq i}}^{m} (b_i - b_k)}. \tag{2.29}$$

The following theorem looks very similar to Theorems 1.2, 1.7, and 1.8. In fact, it underlies all of them:

Theorem 2.2. *For any complex times τ_1 and τ_2, there exists a unique up to a constant multiple complex symmetric scalar product $(*, *)_{\tau_1,\tau_2}$ on $\mathbb{C}^m$ such that the bases $v_i(\tau_1)$ and $w_i(\tau_2)$ are simultaneously orthogonal with respect to it, given by*

$$\begin{aligned} \left(v_i(\tau_1), v_j(\tau_1)\right)_{\tau_1,\tau_2} &= \delta_{ij}\, \nu_i^2\left(\tau_1 + \tau_2\right), \\ \left(w_i(\tau_2), w_j(\tau_2)\right)_{\tau_1,\tau_2} &= \delta_{ij}\, \mu_i^2\left(\tau_1 + \tau_2\right). \end{aligned} \tag{2.30}$$

Also,

$$\left(v_i(\tau_1), w_j(\tau_2)\right)_{\tau_1,\tau_2} = \frac{\nu_i^2\left(\tau_1 + \tau_2\right) \mu_j^2\left(\tau_1 + \tau_2\right)}{a_2 + b_i - c_j + \tau_1 + \tau_2}. \tag{2.31}$$

Let us define the operator $A(\tau_1, \tau_2)$ by the formula $A(\tau_1, \tau_2) + B(\tau_1) + C(\tau_2) = 0$.

Lemma 2.3. *1. $A(\tau_1,\tau_2)$ is a diagonalizable operator with the eigenvalue*

$$a_1(\tau_1,\tau_2) = a_1 + (1-k_1-m)\tau_1 + (1-k_2-m)\tau_2 \tag{2.32}$$

of multiplicity 1 *and the eigenvalue*

$$a_2(\tau_1,\tau_2) = a_2 + (1-k_1)\tau_1 + (1-k_2)\tau_2 \tag{2.33}$$

of multiplicity $m-1$.

2. The vector u is the eigenvector of $A(\tau_1,\tau_2)$ corresponding to the eigenvalue $a_1(\tau_1,\tau_2)$.

3. For any vector $x \in \mathbb{C}^m$,

$$A(\tau_1,\tau_2)\,x = a_2(\tau_1,\tau_2)\,x - (x,u)_{\tau_1,\tau_2}u. \tag{2.34}$$

4. $(u,u)_{\tau_1,\tau_2} = a_2(\tau_1,\tau_2) - a_1(\tau_1,\tau_2)$.

Let H_{τ_1,τ_2} be a copy of $\mathbb{C}^m$ with two distinguished bases: $v_i(\tau_1)$ and $w_i(\tau_2)$, equipped with the complex symmetric scalar product (2.30). The space $H_{\tau_1,\tau_2} \oplus H_{\tau_2,\tau_1}$ admits the natural action of the quaternions:

$$\begin{aligned}
\mathbf{i} &= \begin{bmatrix} 0 & \sqrt{-1}\,e^{S(\tau_1-\tau_2)} \\ \sqrt{-1}\,e^{S(\tau_2-\tau_1)} & 0 \end{bmatrix}, \\
\mathbf{j} &= \begin{bmatrix} 0 & -e^{S(\tau_1-\tau_2)} \\ e^{S(\tau_2-\tau_1)} & 0 \end{bmatrix}, \\
\mathbf{k} &= \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.
\end{aligned} \tag{2.35}$$

The quaternions act on $H_{-\tau_2,-\tau_1} \oplus H_{-\tau_1,-\tau_2}$ by means of (2.35) as well. The action on $H_{\tau_1,\tau_2} \oplus H_{\tau_2,\tau_1}$ preserves the scalar product $(*,*)_{\tau_1,\tau_2} \oplus (*,*)_{\tau_2,\tau_1}$ up to the sign. Over the reals, the space $H_{\tau_1,\tau_2} \oplus H_{\tau_2,\tau_1}$ is spanned by $v_i(\tau_1)$, $v_i(\tau_2)$, $\sqrt{-1}\,v_i(\tau_1)$, and $\sqrt{-1}\,v_i(\tau_2)$. So in the case when the form $(*,*)_{\tau_1,\tau_2}$ (and thus $(*,*)_{\tau_2,\tau_1}$) is sign-definite, the quaternionic action is hyperkähler. The formulae (2.35) also show that the spaces H_{τ_1,τ_2} and H_{τ_2,τ_1} are dual to each other: the exponents acting on them have opposite signs.

2.2. **The GHGE, the mHGS and the elliptic Cauchy identity.** Since we are going to take a closer look at the GHGE, let us temporarily restrict ourselves to the case $a_2 = 0$ when working with the mHGS.

It is not clear from [1] how the vectors p_i and q_i of Theorem 1.7 correspond to the solutions (1.3) and (1.4) of the GHGE at zero and at infinity. The following lemma relates the classical analytic approach to the linear algebra of [1]:

Lemma 2.4.
$$p_j = \sum_{i=1}^{m} \frac{e^{\pi\sqrt{-1}\,(a_1+b_j-c_i)}\,\nu_{j\,trig}^2}{\sin\pi(a_2+b_j-c_i)}\,q_i.$$

The formula

$$\begin{aligned}
{}_mF_{m-1}\left(\begin{matrix} b_1,\dots,b_m \\ c_1,\dots,c_{m-1}\end{matrix}\,\middle|\, z\right) &= \sum_{i=1}^{m} \prod_{\substack{k=1\\k\neq i}}^{m} \frac{\Gamma(b_k-b_i)}{\Gamma(b_k)} \prod_{k=1}^{m-1} \frac{\Gamma(c_k)}{\Gamma(c_k-b_i)} \\
&\times(-z)^{-b_i}\,{}_mF_{m-1}\left(\begin{matrix} b_i-c_1+1,\dots,b_i-c_{m-1}+1,\,b_i \\ b_i-b_1+1,\dots,\widehat{b_i-b_i+1},\dots,b_i-b_m+1\end{matrix}\,\middle|\,\frac{1}{z}\right)
\end{aligned} \tag{2.36}$$

is proven in [19] for $m = 2$. Their argument works for any m without major change. Using the famous reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{2.37}$$

we rewrite (2.36) as

$$\begin{aligned} & z^{b_j}\, {}_mF_{m-1}\left(\begin{matrix} b_j - c_1, \ldots, b_j - c_m \\ b_j - b_1 + 1, \ldots, \widehat{b_j - b_j + 1}, \ldots, b_j - b_m + 1 \end{matrix} \middle| \, z \right) \\ & = \sum_{i=1}^{m} (-1)^{m-1} \prod_{k=1}^{m} \frac{\Gamma(b_k - c_i)\,\Gamma(b_j - b_k + 1)}{\Gamma(c_k - c_i + 1)\,\Gamma(b_j - c_k)} \frac{e^{\pi\sqrt{-1}\,(c_i - b_j)}\,\mu^2_{i\,trig}}{\sin \pi(b_j - c_i)} \\ & \times z^{c_i}\, {}_mF_{m-1}\left(\begin{matrix} b_1 - c_i, \ldots, b_m - c_i \\ c_1 - c_i + 1, \ldots, \widehat{c_i - c_i + 1}, \ldots, c_m - c_i + 1 \end{matrix} \middle| \, \frac{1}{z} \right). \end{aligned} \tag{2.38}$$

Similarly, for the mHGS

$$(T_0)_j = \sum_{i=1}^{m} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k - c_i)}{\prod\limits_{\substack{k=1 \\ k\neq i}}^{m} \Gamma(c_k - c_i)} \frac{\prod\limits_{\substack{k=1 \\ k\neq j}}^{m} \Gamma(b_j - b_k)}{\prod\limits_{k=1}^{m} \Gamma(b_j - c_k)} \frac{e^{\pi\sqrt{-1}\,(c_i - b_j)}\,\mu^2_{i\,trig}}{\sin \pi(b_j - c_i)} \,(T_\infty)_i\,. \tag{2.39}$$

This formula follows from Theorem 3.4 combined with (2.36). Note that all the multiples in (2.38) and in (2.39) either bear the index i or j, except for the term

$$\frac{1}{\sin \pi(b_j - c_i)}.$$

So the rest of them are nothing but normalizing coefficients. Lemma 2.4 is true for any a_2, but for now $a_2 = 0$, so comparing the formula of the lemma to (2.38) and to (2.39) tells us how the p_i and q_i are related to the hypergeometric functions of the GHGE and the mHGS respectively. Also, comparing (2.39) to (2.38) is the way of going back and forth between the mHGS and the GHGE promised in Section 1. Finally, let us mention here that the explicit formulae for the local solutions (1.13) and (1.14) of the mHGS similar to (1.3) and (1.4) for the GHGE are given in Theorem 3.4.

Let D_r (r being the first letter of the word "rational") be the following $m \times m$ matrix:

$$(D_r)_{ij} = \frac{1}{a_2 + b_i - c_j + \tau}. \tag{2.40}$$

The explicit formula for the inverse of this matrix

$$(D_r)^{-1}_{ij} = \frac{\mu_i^2(\tau)\,\nu_j^2(\tau)}{a_2 + b_j - c_i + \tau} \tag{2.41}$$

is helpful in various applications. In particular, the proof of Theorem 2.2 is based on it. The factorization of the determinant

$$det\left(\frac{1}{b_i - c_j}\right) = \frac{\prod\limits_{1\le i<j\le m} (b_i - b_j)(c_j - c_i)}{\prod\limits_{i,j=1}^{m} (b_i - c_j)} \tag{2.42}$$

is called *Cauchy identity* (although known to L'Hospital and probably earlier). It was pointed out to the author by A. Borodin that combining Cauchy identity with Cramer's rule proves (2.41) in the case $a_2 = \tau = 0$. From here, the general case is obtained by renaming the variables. We shall call (2.41) the *rational Cauchy identity* because of its equivalence, in view of Cramer's rule, to (2.42). The trigonometric analogue of (2.41) for the matrix

$$(D_{trig})_{ij} = \frac{1}{\sin \pi(a_2 + b_i - c_j)} \tag{2.43}$$

is given by a similar formula:

$$(D_{trig})^{-1}_{ij} = \frac{\mu^2_{i\,trig}\,\nu^2_{j\,trig}}{\sin \pi(a_2 + b_j - c_i)}. \tag{2.44}$$

We shall call (2.44) the *trigonometric Cauchy identity.* It is easy to obtain from the rational Cauchy identity. Start from the case $a_2 = \tau = 0$. Replace b_i and c_j by $e^{2\pi\sqrt{-1}\,b_i}$ and $e^{2\pi\sqrt{-1}\,c_j}$ respectively. Then use the identity

$$e^{2\pi\sqrt{-1}\,b_i} - e^{2\pi\sqrt{-1}\,c_j} = 2\sqrt{-1}\,e^{\pi\sqrt{-1}\,(b_i + c_j)}\,\sin \pi(b_i - c_j),$$

simplify, and finally rename the variables one last time.

Let us call $\mathcal{M}$ and $\mathcal{N}$ the diagonal $m \times m$ matrices with μ_i and ν_i on the diagonal (we stay at the formal level and not specify the branches of the square roots). Then, in both the rational and the trigonometric case, we can rewrite the Cauchy identities as

$$(\mathcal{N}\,D\,\mathcal{M})^t = (\mathcal{N}\,D\,\mathcal{M})^{-1}\,. \tag{2.45}$$

The rational version of this formula lies at the core of proving the existence of the complex symmetric product (2.30) of Theorem 2.2. The trigonometric Cauchy identity is used to give a proof to Theorem 1.7 different from those of Beukers and Heckman and of Haraoka.

Looking at (2.41) and (2.44), one has an itch to guess the *elliptic Cauchy identity* for the matrix

$$(D_{ell})_{ij} = \frac{1}{sn(a_2 + b_i - c_j)}. \tag{2.46}$$

The problem of inverting (2.46) turns up quite often in the theory of elliptic functions and, according to a specialist in the field (E. Rains), has not been solved to date. We shall also need to invert (2.46) for our own purposes.

Theorem 2.5.

$$(D_{ell})^{-1}_{ij} = \frac{\mu^2_{i\,ell}\,\nu^2_{j\,ell}}{sn(a_2 + b_j - c_i)}. \tag{2.47}$$

Note that the trigonometric and elliptic versions of the determinantal identity (2.42) are easy to obtain from (2.44) and (2.47) by means of Cramer's rule.

2.3. **Fermionic fields.** Let ω_1 and ω_2 be a pair of linearly independent complex numbers and let $\Omega = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$. From this point on, let $\tau_1 = n_1\omega_1$ and $\tau_2 = n_2\omega_2$.

Consider the following vectors in $H_{\tau_1,\tau_2} \oplus H_{\tau_2,\tau_1}$:

$$
\begin{aligned}
(f_0)_i\,(\tau_1,\tau_2) &= \frac{1}{\nu_i^2(\tau_1+\tau_2)}\,\frac{v_i(\tau_1)\oplus\sqrt{-1}\,v_i(\tau_2)}{\sqrt{2}},\\
(f_0)_i^\dagger\,(\tau_1,\tau_2) &= \frac{1}{\nu_i^2(\tau_1+\tau_2)}\,\frac{v_i(\tau_1)\oplus-\sqrt{-1}\,v_i(\tau_2)}{\sqrt{2}},\\
(f_\infty)_i\,(\tau_1,\tau_2) &= \frac{1}{\mu_i^2(\tau_1+\tau_2)}\,\frac{w_i(\tau_2)\oplus\sqrt{-1}\,w_i(\tau_1)}{\sqrt{2}},\\
(f_\infty)_i^\dagger\,(\tau_1,\tau_2) &= \frac{1}{\mu_i^2(\tau_1+\tau_2)}\,\frac{w_i(\tau_2)\oplus-\sqrt{-1}\,w_i(\tau_1)}{\sqrt{2}}.
\end{aligned}
\tag{2.48}
$$

Let us introduce a modified symmetric scalar product on $H_{\tau_1,\tau_2} \oplus H_{\tau_2,\tau_1}$. For $\tau_1+\tau_2\neq 0$,

$$
\begin{aligned}
(g_1\oplus g_2, h_1\oplus h_2) = (-1)^{n_1}\Bigg(&(g_1,h_1)_{\tau_1,\tau_2} + (g_2,h_2)_{\tau_2,\tau_1}\\
&-\frac{1}{\tau_1+\tau_2}\left((u,g_1)_{\tau_1,\tau_2}\,(u,h_1)_{\tau_1,\tau_2} + (u,g_2)_{\tau_2,\tau_1}\,(u,h_2)_{\tau_2,\tau_1}\right)\Bigg).
\end{aligned}
\tag{2.49}
$$

For $\tau_1+\tau_2=0$, let us define

$$
(g_1\oplus g_2, h_1\oplus h_2) = (g_1,h_1)_{\tau_1,\tau_2} + (g_2,h_2)_{\tau_2,\tau_1}\,. \tag{2.50}
$$

Here is the multiplication table for (2.49):

(2.51)

$\Big((f_0)_i\,(\tau_1,\tau_2),(f_0)_j\,(\tau_1,\tau_2)\Big) = \Big((f_0)_i^\dagger\,(\tau_1,\tau_2),(f_0)_j^\dagger\,(\tau_1,\tau_2)\Big) = 0$
$\Big((f_0)_i\,(\tau_1,\tau_2),(f_0)_j^\dagger\,(\tau_1,\tau_2)\Big) = (-1)^{n_1}\left(\frac{\delta_{ij}}{\nu_i^2(\tau_1+\tau_2)} - \frac{1}{\tau_1+\tau_2}\right)$
$\Big((f_\infty)_i\,(\tau_1,\tau_2),(f_\infty)_j\,(\tau_1,\tau_2)\Big) = \Big((f_\infty)_i^\dagger\,(\tau_1,\tau_2)\,(f_\infty)_j^\dagger\,(\tau_1,\tau_2)\Big) = 0$
$\Big((f_\infty)_i\,(\tau_1,\tau_2),(f_\infty)_j^\dagger\,(\tau_1,\tau_2)\Big) = (-1)^{n_1}\left(\frac{\delta_{ij}}{\mu_i^2(\tau_1+\tau_2)} - \frac{1}{\tau_1+\tau_2}\right)$
$\Big((f_0)_i\,(\tau_1,\tau_2),(f_\infty)_j^\dagger\,(\tau_1,\tau_2)\Big) = \Big((f_0)_i^\dagger\,(\tau_1,\tau_2),(f_\infty)_j\,(\tau_1,\tau_2)\Big)$ $= (-1)^{n_1}\left(\frac{1}{a_2+b_i-c_j+\tau_1+\tau_2} - \frac{1}{\tau_1+\tau_2}\right)$

Let H be a complex vector space of even dimension endowed with a non-degenerate symmetric scalar product $(*,*)$. A subspace $I\subset H$ is called *isotropic*, if $(v,v)=0$ for any $v\in I$. Let $H=I\oplus I^\dagger$ be a decomposition of H into a direct sum of maximal isotropic subspaces. Let us choose bases v_j and $v_i^\dagger$ of I and $I^\dagger$ respectively. Then $(v_i,v_j)=(v_i^\dagger,v_j^\dagger)=0$. It is customary to take the dual bases for I and $I^\dagger$ so that $(v_i,v_j^\dagger)=\delta_{ij}$, but we shall not do so in this paper. The vectors v_i and $v_i^\dagger$ are called *annihilation operators* and *creation operators* respectively. Both the annihilation and creation operators also bear the common name of *fermions.*

A *Clifford algebra* CA is the associative algebra generated by the vectors of H with relations

$$fh+hf=(f,h).$$

For a Clifford algebra CA, let us call Ann and Cr the spaces of annihilation and creation operators respectively. Then the left and right CA-modules $Fock=CA/CA\,Ann$ and $Fock^\dagger=Cr\,CA\backslash CA$ are called the *Fock space* and the *dual Fock space* respectively. The generators $1\,mod\,CA\,Ann$ and $1\,mod\,Cr\,CA$ are denoted by $|0\rangle$ and $\langle 0|$ and called the *vacuum vector* and the *dual vacuum vector.* The spaces $Fock$ and $Fock^\dagger$ are dual via the bilinear form $(\langle 0|f,h|0\rangle)\mapsto\langle fh\rangle$ where

$$\begin{aligned}
&\langle 1\rangle=1;\\
&\langle fh\rangle=(f,h),\ \text{ if } f,h\in W;\\
&\langle h_1\cdots h_r\rangle=\begin{cases}0, & \text{if } r \text{ is odd},\\ \sum_\sigma \operatorname{sign}(\sigma)\langle h_{\sigma(1)}h_{\sigma(2)}\rangle\cdots\langle h_{\sigma(r-1)}h_{\sigma(r)}\rangle, & \text{if } r \text{ is even}.\end{cases}
\end{aligned}\tag{2.52}$$

The sum runs over all the permutations σ satisfying $\sigma(1)<\sigma(2),\dots,\sigma(r-1)<\sigma(r)$ and $\sigma(1)<\sigma(3)<\cdots<\sigma(r-1)$, in other words, over all ways of grouping the h_i into pairs. The equation (2.52) is called *Wick's rule.* The number $\langle h_1\cdots h_r\rangle$ is called the *vacuum expectation value.*

Let $H=\bigoplus_{\tau_1+\tau_2\in\Omega}H_{\tau_1,\tau_2}\oplus H_{\tau_2,\tau_1}$ equipped with the scalar product (2.49). Let CA be the Clifford algebra generated by H. Consider the following fermionic fields:

$$\begin{aligned}
(F_0)_i\,(z_1,z_2)&=\sum_{\tau_1+\tau_2\in\Omega}(f_0)_i\,(\tau_1,\tau_2)\,z_1^{n_1}\,z_2^{n_2},\\
(F_0)_i^\dagger\,(z_1,z_2)&=\sum_{\tau_1+\tau_2\in\Omega}(f_0)_i^\dagger\,(\tau_1,\tau_2)\,z_1^{n_1}\,z_2^{n_2},\\
(F_\infty)_i\,(z_1,z_2)&=\sum_{\tau_1+\tau_2\in\Omega}(f_\infty)_i\,(\tau_1,\tau_2)\,z_1^{-n_1}\,z_2^{-n_2},\\
(F_\infty)_i^\dagger\,(z_1,z_2)&=\sum_{\tau_1+\tau_2\in\Omega}(f_\infty)_i^\dagger\,(\tau_1,\tau_2)\,z_1^{-n_1}\,z_2^{-n_2}.
\end{aligned}\tag{2.53}$$

Theorem 2.6.

$$\left\langle (F_\infty)_j^\dagger\ (F_0)_i\right\rangle=\frac{1}{sn(a_2+b_i-c_j)}=\left\langle (F_\infty)_j\ (F_0)_i^\dagger\right\rangle,$$

where $\langle *\rangle$ *is the vacuum expectation value.*

Proof. It immediately follows from the last formula of the multiplication table (2.51) that

(2.54)

$$\left\langle (F_\infty)_j^\dagger\ (F_0)_i \right\rangle = \frac{1}{a_2+b_i-c_j} + \sum_{\tau_1+\tau_2\in\Omega\backslash\{0\}} (-1)^{n_1}\left(\frac{1}{a_2+b_i-c_j+\tau_1+\tau_2} - \frac{1}{\tau_1+\tau_2}\right).$$

The decomposition

$$\sqrt{\mathcal{P}(u)-e_3} = \frac{1}{\omega_1}\sum_{n_2\in\mathbb{Z}} \frac{\pi}{\sin\pi\left(\frac{u}{\omega_1}+\frac{n_2\omega_2}{\omega_1}\right)} \tag{2.55}$$

is proven in different notations (see below) at the end of Chapter 2 of [4]. Here $\mathcal{P}(z)$ is the $\mathcal{P}$-function of Weierstrass and $e_3 = \mathcal{P}(\omega_2/2)$. Expanding (2.55) further by means of the famous identity

$$\frac{\pi}{\sin\pi z} = \frac{1}{z} + \sum_{n_1=1}^{\infty}(-1)^{n_1}\left(\frac{1}{z+n_1}+\frac{1}{z-n_1}\right) \tag{2.56}$$

produces the following formula:

(2.57)

$$\begin{aligned}&\sqrt{\mathcal{P}(a_2+b_i-c_j)-e_3}\\ &= \frac{1}{a_2+b_i-c_j} + \sum_{\tau_1+\tau_2\in\Omega\backslash\{0\}} (-1)^{n_1}\left(\frac{1}{a_2+b_i-c_j+\tau_1+\tau_2} - \frac{1}{\tau_1+\tau_2}\right).\end{aligned}$$

The last term appears in (2.57) to make the sum converge uniformly. Similarly, one can rewrite (2.56) as

$$\frac{\pi}{\sin\pi z} = \frac{1}{z} + \sum_{n_1\in Z\backslash\{0\}}^{\infty}(-1)^{n_1}\left(\frac{1}{z+n_1}-\frac{1}{n_1}\right).$$

The elliptic sine of Jacobi $sn(z)$ is sometimes defined as

$$\frac{1}{sn(z)} = \sqrt{\mathcal{P}(z)-e_3}. \tag{2.58}$$

It is customary in the theory of Jacobi elliptic functions to use notation different from those of Weierstrass:

$$\omega = \frac{\omega_1}{2},\quad \omega' = \frac{\omega_2}{2},\quad \tau = \frac{\omega'}{\omega} = \frac{\omega_2}{\omega_1},\quad h = e^{\pi\sqrt{-1}\,\tau}.$$

The generators 4ω and $2\omega'$ of the period's lattice are chosen so that $\operatorname{Im}\tau > 0$.

It is standard to add the following normalizing condition to the definition (2.58) of $sn(z)$:

$$\omega = \frac{\pi}{2}\left(\sum_{n=-\infty}^{\infty} h^{n^2}\right)^2, \tag{2.59}$$

and to treat $sn(z) = sn(z;\tau)$ rather than $sn(z) = sn(z;\omega,\omega')$; see [4] or [19] for more information on elliptic functions. We shall use the definition (2.58) of $sn(z)$ without the condition (2.59). The only reason we switch notation from $\mathcal{P}$ to sn is that the latter takes less space. This remark concludes the proof. □

Let us call $\mathbb{H}_0$, $\mathbb{H}_0^\dagger$, $\mathbb{H}_\infty$, and $\mathbb{H}_\infty^\dagger$ the m-dimensional spaces spanned by $(F_0)_i$, $(F_0)_i^\dagger$, $(F_\infty)_i$, and $(F_\infty)_i^\dagger$ respectively. We use the pairing of Theorem 2.6 to identify $\mathbb{H}_0$ with $\mathbb{H}_\infty^\dagger$ and $\mathbb{H}_0^\dagger$ with $\mathbb{H}_\infty$. The resulting vector spaces are $\mathbb{H}$ and $\mathbb{H}'$ of Theorem 1.8.

2.4. **Trigonometrization.** Let us restrict ourselves to the case $\omega_1 = 1$ and let us consider what happens when $\omega_2 \to \infty$. The limits of $(f_0)_i(\tau_1,\tau_2)$, $(f_0)_i^\dagger(\tau_1,\tau_2)$, $(f_\infty)_i(\tau_1,\tau_2)$, and $(f_\infty)_i^\dagger(\tau_1,\tau_2)$ are all zero unless $\tau_2 = 0$. So in the limit we get

$$
\begin{aligned}
&\lim_{\omega_2\to\infty} (F_0)_i(z_1,z_2) = (F_0)_{i\,trig}(z) = \sum_{n\in\mathbb{Z}} (f_0)_i(n,0)\, z^n,\\
&\lim_{\omega_2\to\infty} (F_0)_i^\dagger(z_1,z_2) = (F_0)_{i\,trig}^\dagger(z) = \sum_{n\in\mathbb{Z}} (f_0)_i^\dagger(n,0)\, z^n,\\
&\lim_{\omega_2\to\infty} (F_\infty)_i(z_1,z_2) = (F_\infty)_{i\,trig}(z) = \sum_{n\in\mathbb{Z}} (f_\infty)_i(n,0)\, z^{-n},\\
&\lim_{\omega_2\to\infty} (F_\infty)_i^\dagger(z_1,z_2) = (F_\infty)_{i\,trig}^\dagger(z) = \sum_{n\in\mathbb{Z}} (f_\infty)_i^\dagger(n,0)\, z^{-n}.
\end{aligned}
\tag{2.60}
$$

This time the vacuum expectation values are

$$
\begin{aligned}
&\left\langle (F_\infty)_{j\,trig}^\dagger\, (F_0)_{i\,trig}\right\rangle = \left\langle (F_\infty)_{j\,trig}\, (F_0)_{i\,trig}^\dagger\right\rangle\\
&= \frac{1}{a_2+b_i-c_j} + \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n \left(\frac{1}{a_2+b_i-c_j+n} - \frac{1}{n}\right)\\
&= \frac{\pi}{\sin\pi(a_2+b_i-c_j)} = \lim_{\omega_2\to\infty} \frac{1}{sn(a_2+b_i-c_j)}.
\end{aligned}
\tag{2.61}
$$

Let us call $H_{trig} = \bigoplus_{n\in\mathbb{Z}} H_{n0} \oplus H_{0n}$ and CA_{trig} the Clifford algebra generated by H_{trig}. We shall call $\mathbb{H}_{0\,trig}$, $\mathbb{H}_{0\,trig}^\dagger$, $\mathbb{H}_{\infty\,trig}$, and $\mathbb{H}_{\infty\,trig}^\dagger$ the m-dimensional spaces spanned by $(F_0)_{i\,trig}$, $(F_0)_{i\,trig}^\dagger$, $(F_\infty)_{i\,trig}$, and $(F_\infty)_{i\,trig}^\dagger$ respectively. We use the pairing (2.61) to identify $\mathbb{H}_{0\,trig}$ with $\mathbb{H}_{\infty\,trig}^\dagger$. We call the resulting vector space $\mathbb{H}_{trig}$. Similarly, we identify $\mathbb{H}_{0\,trig}^\dagger$ with $\mathbb{H}_{\infty\,trig}$ and call the resulting space $\mathbb{H}'_{trig}$. The following theorem is a trigonometric version of Theorems 1.8 and 1.9 combined:

Theorem 2.7. *Let the local exponents of the mHGS $b_1,\dots,b_m$ and $-c_1,\dots,-c_m$ be generic real numbers. Then there exist unique up to a constant multiple hermitian and complex symmetric scalar products $(*,*)_{trig}$ on $\mathbb{H}_{trig}$ such that the bases $(F_0)_{i\,trig}$ and $(F_\infty)_{i\,trig}^\dagger$ are simultaneously orthogonal with respect to them, given by*

$$
\Big((F_0)_{i\,trig}, (F_0)_{j\,trig}\Big)_{trig} = \frac{\delta_{ij}}{\nu_{i\,trig}^2} \ \textit{and}\ \Big((F_\infty)_{i\,trig}^\dagger, (F_\infty)_{j\,trig}^\dagger\Big)_{trig} = \frac{\delta_{ij}}{\mu_{i\,trig}^2}.
\tag{2.62}
$$

A similar statement holds for $\mathbb{H}'_{trig}$.

The quaternions act on $\mathbb{H}_{trig} \oplus \mathbb{H}'_{trig}$ *according to* (1.24). *If the forms are sign-definite, then the quaternionic action is hyperkähler.*

The map

$$p_i \mapsto \nu^2_{i\,trig}\,(F_0)_{i\,trig}\,,\ q_i \mapsto \mu^2_{i\,trig}\,(F_\infty)^\dagger_{i\,trig}$$

establishes hermitian isometries between $\mathbb{H}_{trig}$ and the spaces of solution of the GHGE and the mHGS; see Theorem 1.7 and Lemma 2.4. Let us repeat here that the isometry is a formal linear-algebraic construction. It allows us to introduce the quaternionic action on the direct sum of the spaces of solutions and their duals, but the action of the monodromy group on the fermionic fields doesn't make geometric sense. The situation should be improved by constructing a Fuchsian system on the torus $T(\omega_1, \omega_2)$ such that as $\omega_2 \to \infty$, the system becomes the mHGS.

3. Proofs

In this section, we prove the above results. The section also contains some more new results, which we use as tools, but which we believe are important for their own sake. These are Theorem 3.4 and an explicit construction of the monodromy operators of the mHGS as well as their eigenvectors in a "good" basis.

3.1. **Rigidity and irreducibility.** Let $R_1, \dots, R_k$ ($M_1, \dots, M_k$) be a k-tuple of linear operators on $\mathbb{C}^m$ such that $R_1 + \cdots + R_k = 0$ ($M_1 \cdot \cdots \cdot M_k = Id$, in which case the M_i are called a *local system*). The k-tuple is called *irreducible*, if the operators do not simultaneously preserve a proper subspace of $\mathbb{C}^m$. Let $R'_1, \dots, R'_k$ ($M'_1, \dots, M'_k$) be any other k-tuple of linear operators on $\mathbb{C}^m$ such that $R'_1 + \cdots + R'_k = 0$ ($M'_1 \cdot \cdots \cdot M'_k = Id$) and that the operators R_i and R'_i (M_i and M'_i) are conjugate to each other for $i = 1, \dots, k$. The tuple is called *rigid* , if all the operators R_i and R'_i (M_i and M'_i) are simultaneously conjugate to each other. The additive and multiplicative hypergeometric triples are rigid and irreducible (see [1], [10], and [18] for proofs).

Consider the following matrices:

$$A_{ij} = \begin{cases} c_i - b_i, & \text{if} \quad i = j, \\ c_i - b_i - a_2, & \text{if} \quad i \neq j, \end{cases}$$

$$B_{ij} = \begin{cases} a_2 + b_i - c_i, & \text{if} \quad i < j, \\ b_i, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i > j, \end{cases} \tag{3.63}$$

$$C_{ij} = \begin{cases} 0, & \text{if} \quad i < j, \\ -c_i, & \text{if} \quad i = j, \\ a_2 + b_i - c_j, & \text{if} \quad i > j. \end{cases}$$

Quite obviously, they add up to zero and have the same eigenvalues (with multiplicities) as the operators A, B, and C we have started with. It is proven in [3] that the following vectors are the eigenvectors of B and C corresponding to the eigenvalues b_i and $-c_i$ respectively:

$$v_i^j = \begin{cases} (b_j - c_j + a_2) \dfrac{\prod\limits_{k=j+1}^{m} (b_i - c_k + a_2)}{\prod\limits_{\substack{k=j \\ k \neq i}}^{m} (b_i - b_k)}, & \text{if } i \geq j, \\ 0, & \text{if } i < j; \end{cases} \tag{3.64}$$

$$w_i^j = \begin{cases} 0, & \text{if } i > j; \\ (b_j - c_j + a_2) \dfrac{\prod\limits_{k=1}^{j-1} (b_k - c_i + a_2)}{\prod\limits_{\substack{k=1 \\ k \neq i}}^{j} (c_k - c_i)}, & \text{if } i \leq j. \end{cases} \tag{3.65}$$

Here and in the sequel, all empty products are understood to be equal to 1.

Let $\mathcal{F}_0$ and $\mathcal{F}_\infty$ be the flags $Span(v_1) \subset Span(v_1, v_2) \subset \cdots \subset \mathbb{C}^m$ and $Span(w_m) \subset Span(w_m, w_{m-1}) \subset \cdots \subset \mathbb{C}^m$. The flags are opposite to each other as $\mathcal{F}_0 = Span(e_1) \subset Span(e_1, e_2) \subset \cdots \subset \mathbb{C}^m$, whereas $\mathcal{F}_\infty = Span(e_m) \subset Span(e_m, e_{m-1}) \subset \cdots \subset \mathbb{C}^m$. Thus, the flags $\mathcal{F}_0$ and $\mathcal{F}_\infty$ are in a general position with respect to each other: if we take a subspace from $\mathcal{F}_0$ and from $\mathcal{F}_\infty$, the dimension of their intersection will be the lowest possible.

Let $u = (b_1 - c_1 + a_2, \ldots, b_m - c_m + a_2)$. It is not hard to see that $Au = a_1 u$. It is proven in [3] that

$$\sum_{i=1}^{m} v_i = \sum_{i=1}^{m} w_i = u.$$

Thus the flag $\mathcal{F}_1 = Span(u) \subset \mathbb{C}^m$ is in a general position with respect to the flags $\mathcal{F}_0$ and $\mathcal{F}_\infty$. The three *spectral flags* being in a general position with respect to each other is the combinatorial part of the condition that the operators A, B, and C are generic (see also [12]). Due to rigidity, any additive hypergeometric triple of matrices with the same eigenvalues is conjugate to the triple above. This fact allows us to make most of the proofs computational.

3.2. **Main technical tools.** For $\tau \in \mathbb{C}^*$, let

$$\xi_i^2(\tau) = \frac{\prod\limits_{k=1}^{m} (b_i - b_k + \tau)}{\prod\limits_{\substack{k=1 \\ k \neq i}}^{m} (b_i - b_k)}, \qquad \theta_i^2(\tau) = \frac{\prod\limits_{k=1}^{m} (c_k - c_i + \tau)}{\prod\limits_{\substack{k=1 \\ k \neq i}}^{m} (c_k - c_j)}, \tag{3.66}$$

$$\Xi^2(\tau) = diag\left(\xi_i^2(\tau)\right)_{i=1,\ldots,m}, \qquad \Theta^2(\tau) = diag\left(\theta_i^2(\tau)\right)_{i=1,\ldots,m}.$$

For $\tau \in \mathbb{C}^*$, let $EX(\tau)$ and $EY(\tau)$ be the following $m \times m$ matrices:

$$EX_{ij}(\tau) = \frac{\xi_j^2(\tau)}{b_j - b_i + \tau}, \; EY_{ij}(\tau) = \frac{\theta_j^2(\tau)}{c_i - c_j + \tau}. \tag{3.67}$$

As $\lim\limits_{\tau\to 0} EX(\tau) = \lim\limits_{\tau\to 0} EY(\tau) = Id$, we naturally set $EX(0) = EY(0) = Id$. As we shall show in Lemma 3.3, in fact

$$EX(\tau) = e^{X\tau} \text{ and } EY(\tau) = e^{Y\tau},$$

where X and Y are the matrices (1.11) and (2.25) respectively.

Lemma 3.1. • *For any $\tau_1, \tau_2 \in \mathbb{C}$, $EX(\tau_1)\, EX(\tau_2) = EX(\tau_1 + \tau_2)$ and $EY(\tau_1)\, EY(\tau_2) = EY(\tau_1 + \tau_2)$.*

• *For $\tau \neq 0$, the Jordan normal form of $EX(\tau)$ and $EY(\tau)$ is a single block with the eigenvalue 1.*

• *Recall that $e = (1, \dots, 1)$. $EX(\tau)\, e = EY(\tau)\, e = e$.*

Proof. All the proofs for EX and EY are the same, so we shall only prove the statements of the lemma for EX.

• To prove the first statement of the lemma, we need to show that

$$(EX(\tau_1)\, EX(\tau_2))_{ij} = EX(\tau_1 + \tau_2)_{ij},$$

i.e.

$$\sum_{k=1}^{m} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_k - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_k - b_l)} \frac{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_j - b_l + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq j}}^{m} (b_j - b_l)} = \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_j - b_l + \tau_1 + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq j}}^{m} (b_j - b_l)}. \tag{3.68}$$

Canceling out common multiples, we rewrite (3.68) as

$$\sum_{k=1}^{m} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_k - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_k - b_l)} \frac{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_j - b_l + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_j - b_l + \tau_1 + \tau_2)} = 1. \tag{3.69}$$

Rational identities of this kind will often appear in the rest of the paper. Let us outline the strategy of proving them here. The left hand side L of an identity to prove will be a homogeneous rational function of b_i, c_i, τ_1, and τ_2. The right hand side R of the identity will be a homogeneous polynomial in the same variables such that $deg(L) = deg(R)$. The degree will not exceed 1. All the denominators of L will be products of linear forms f. The power of every such form in every denominator will be 1. The first step to prove such an identity is to show that L is in fact a polynomial. For that, it is enough to show that $f\,L|_{f=0} = 0$ for every form f from any denominator of the identity. The second step is to check enough points to make sure that $L \equiv R$. We shall write down the proof of (3.69) in full detail. We shall be sketchy with the rest of the proofs, if they follow the strategy outlined here.

Let $1 \leq l_1 < l_2 \leq m$. The only two summands in $(b_{l_1} - b_{l_2})L$ which do not nullify when restricted to $b_{l_1} = b_{l_2}$ are

$$\frac{b_{l_1} - b_{l_2}}{b_{l_1} - b_{l_2}} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_{l_1} - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq l_1\\ l\neq l_2}}^{m} (b_{l_1} - b_l)} \frac{\prod\limits_{\substack{l=1\\ l\neq l_1}}^{m} (b_j - b_l + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_j - b_l + \tau_1 + \tau_2)}$$

and

$$\frac{b_{l_1} - b_{l_2}}{b_{l_2} - b_{l_1}} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_{l_2} - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq l_2\\ l\neq l_1}}^{m} (b_{l_2} - b_l)} \frac{\prod\limits_{\substack{l=1\\ l\neq l_2}}^{m} (b_j - b_l + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_j - b_l + \tau_1 + \tau_2)}.$$

Obviously, these two add up to zero.

For $l_1 = 1, \ldots, \hat{i}, \ldots, m$, let $f = b_j - b_{l_1} + \tau_1 + \tau_2$. This time

$$f\,L|_{f=0} = \frac{\prod\limits_{l=1}^{m} (b_j - b_l + \tau_2)}{\prod\limits_{\substack{l=1\\ l\neq i\\ l\neq l_1}}^{m} (b_j - b_l + \tau_1 + \tau_2)} \sum_{k=1}^{m} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_k - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_k - b_l)} \frac{1}{b_j - b_k + \tau_2}\Bigg|_{b_j - b_{l_1} + \tau_1 + \tau_2 = 0}$$

$$= \frac{\prod\limits_{l=1}^{m} (b_l - b_{l_1} + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq i\\ l\neq l_1}}^{m} (b_l - b_{l_1})} \sum_{k=1}^{m} \frac{\prod\limits_{\substack{l=1\\ l\neq i}}^{m} (b_k - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_k - b_l)} \frac{-1}{b_k - b_{l_1} + \tau_1}.$$

The identity

$$\sum_{k=1}^{m} \frac{\prod\limits_{\substack{l=1\\ l\neq i\\ l\neq l_1}}^{m} (b_k - b_l + \tau_1)}{\prod\limits_{\substack{l=1\\ l\neq k}}^{m} (b_k - b_l)} = 0$$

for $l_1 \neq i$ is proven in [3]; see (7.59) of the Appendix there.

We now know that L is a homogeneous polynomial in b_i and τ_i. The degree of this polynomial is zero, so it must be a constant. Setting $b_k = k$ for $k = 1, \ldots, m$ and $\tau_1 = 0$, we see that all the summands on the left hand side of (3.69) nullify except for when $k = i$. The one remaining equals the one on the opposite side of the identity.

• To prove the second statement of the lemma, let us introduce the following matrix:

$$G(\tau)_{ij} = \frac{1}{\prod\limits_{\substack{k=1\\k\neq j}}^{m}(b_j - b_k)} \left[1 + \frac{b_j - b_1}{\tau} + \frac{(b_j - b_1)(b_j - b_1 - \tau)}{2!\,\tau^2} + \cdots + \frac{(b_j - b_1)\cdot\cdots\cdot(b_j - b_1 - (m-i-1)\tau)}{(m-i)!\,\tau^{m-i}}\right]. \tag{3.70}$$

To prove that $G(\tau)$ in not degenerate for $\tau \neq 0$, let us prove that

$$\det(G(\tau)) = \frac{1}{1!\,2!\,\cdots(m-1)!\,\tau^{m(m-1)/2}\prod\limits_{1\leq i<j\leq m}(b_j - b_i)}. \tag{3.71}$$

Subtracting the $i+1$ row from the i-th for all the rows of G except for the last one nullifies all the elements of the first column except for the last element. Decomposing the determinant with respect to this column and factoring out common multiples, we see that

$$\begin{aligned} &det\,(G(\tau)) \\ &= \left(1!\,2!\,\cdots(m-1)!\,\tau^{m(m-1)/2}\prod_{k=2}^{m}(b_k - b_1)\prod_{2\leq i<j\leq m}(b_j - b_i)^2\right)^{-1} det\left(\tilde{G}\right), \end{aligned}$$

where $\tilde{G}$ is the following $(m-1)\times(m-1)$ matrix:

$$\tilde{G}_{ij} = \prod_{k=1}^{m-1-i}(b_{j+1} - b_1 - k\tau). \tag{3.72}$$

For example, for $m = 4$,

$$\tilde{G} = \begin{bmatrix} (b_2 - b_1 - \tau)(b_2 - b_1 - 2\tau) & (b_3 - b_1 - \tau)(b_3 - b_1 - 2\tau) & (b_4 - b_1 - \tau)(b_4 - b_1 - 2\tau) \\ b_2 - b_1 - \tau & b_3 - b_1 - \tau & b_4 - b_1 - \tau \\ 1 & 1 & 1 \end{bmatrix}.$$

To prove (3.71), we have to prove that

$$det\left(\tilde{G}\right) = \prod_{2\leq i<j\leq m}(b_j - b_i).$$

Let us show that $det\left(\tilde{G}\right)$ can be turned into the standard Vandermonde without changing the determinant by means of row operations. First, let us take a look at our example. Multiplying the last row by $b_1 + \tau$ and adding the result to the previous row gives us the following matrix:

$$\begin{bmatrix} (b_2 - b_1 - \tau)(b_2 - b_1 - 2\tau) & (b_3 - b_1 - \tau)(b_3 - b_1 - 2\tau) & (b_4 - b_1 - \tau)(b_4 - b_1 - 2\tau) \\ b_2 & b_3 & b_4 \\ 1 & 1 & 1 \end{bmatrix}.$$

Now let us take the last row times $-(b_1+\tau)(b_1+2\tau)$ plus the second row times $2b_1 + 3\tau$ and add them to the first row. The result is the desired Vandermonde matrix:

$$\begin{bmatrix} b_2^2 & b_3^2 & b_4^2 \\ b_2 & b_3 & b_4 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is easy to complete the argument using induction.

Let J be the matrix with ones on the main diagonal and right above it, and zeros elsewhere:

$$J_{ij} = \begin{cases} 1, & \text{if } i = j-1; \\ 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

We want to prove that

$$G(\tau)\, EX(\tau)\, G(\tau)^{-1} = J. \tag{3.73}$$

Consider the following $m \times m$ matrix:

$$J'_{ij} = \begin{cases} -1, & \text{if } i = j-1; \\ 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

J' is invertible, so $G(\tau)\, X(\tau) = J\, G(\tau)$ if and only if $J'\, G(\tau)\, X(\tau) = J'\, J\, G(\tau)$. It is not hard to see that

$$(J' G(\tau))_{ij} = \begin{cases} 0, & \text{if } j = 1 \text{ and } i \neq m, \\ \dfrac{1}{\prod\limits_{k=2}^{m} (b_1 - b_k)}, & \text{if } j = 1 \text{ and } i = m, \\ \dfrac{\prod\limits_{k=0}^{m-i-1} (b_j - b_1 - k\tau)}{(m-i)!\, \tau^{m-i} \prod\limits_{\substack{k=1 \\ k \neq j}}^{m} (b_j - b_k)}, & \text{if } j > 1 \end{cases}$$

and that

$$(J' J\, G(\tau))_{ij} = \begin{cases} 0, & \text{if } j = 1 \text{ and } i < m-1, \\ \dfrac{1}{\prod\limits_{k=2}^{m} (b_1 - b_k)}, & \text{if } j = 1 \text{ and } i = m-1, \\ \dfrac{1}{\prod\limits_{\substack{k=1 \\ k \neq j}}^{m} (b_j - b_k)}, & \text{if } i = m, \\ \dfrac{(b_j - b_1 + \tau) \prod\limits_{k=0}^{m-i-2} (b_j - b_1 - k\tau)}{(m-i)!\, \tau^{m-i} \prod\limits_{\substack{k=1 \\ k \neq j}}^{m} (b_j - b_k)}, & \text{if } j > 1 \text{ and } i < m. \end{cases}$$

Let us prove that

$$\sum_{l=1}^{m} (J'G(\tau))_{il}\, EX(\tau)_{lj} = (J'J\,G(\tau))_{ij}\,. \tag{3.74}$$

In the first case $j = 1$, $i = 1, \ldots, m-2$; after factoring out and cancellation, (3.74) becomes the following identity:

$$\sum_{l=2}^{m} \frac{\prod\limits_{k=2}^{m-i-1} (b_l - b_1 - k\tau)}{\prod\limits_{\substack{k=2\\ k\neq l}}^{m} (b_l - b_k)} = 0.$$

Up to a change of variables, this is identity (7.59) from [3].

In the second case $j = 1$, $i = m-1$; after factoring out and cancellation, (3.74) becomes the following identity:

$$\sum_{l=2}^{m} \frac{\prod\limits_{\substack{k=2\\ k\neq l}}^{m} (b_1 - b_k + \tau)}{\prod\limits_{\substack{k=2\\ k\neq l}}^{m} (b_l - b_k)} = 1. \tag{3.75}$$

Let us employ the strategy introduced at the beginning of the proof to show that the left hand side of (3.75) is a polynomial in b_i and τ of degree zero and thus a constant. Then set $b_1 + \tau = b_2$ to see that the constant is in fact 1.

In the third case $i = m$, $j = 1, \ldots, m$; after some simplification, (3.74) becomes the identity

$$\sum_{l=1}^{m} \frac{\prod\limits_{\substack{k=1\\ k\neq l}}^{m} (b_j - b_k + \tau)}{\prod\limits_{\substack{k=1\\ k\neq l}}^{m} (b_l - b_k)} = 1,$$

which is proven similarly to (3.75).

Finally, for $i = 1, \ldots, m-1$ and $j = 2, \ldots, m$, (3.74) boils down to the following identity:

$$\sum_{l=2}^{m} \frac{\prod\limits_{k=1}^{m-i-1} (b_l - b_1 - k\tau)}{\prod\limits_{k=0}^{m-i-2} (b_j - b_1 - k\tau)} \frac{\prod\limits_{\substack{k=2\\ k\neq l}}^{m} (b_j - b_k + \tau)}{\prod\limits_{\substack{k=2\\ k\neq l}}^{m} (b_l - b_k)} = 1.$$

Our strategy works here again aided at some point by the identity (7.59) from [3].

• The last statement of the lemma follows from the last two statements of Lemma 3.2. Another way to see it is to observe that the first column of $G^{-1}(\tau)$ is $(m-1)!\,\tau^{m-1}e$. □

Let V and W be the matrices composed of the eigenvectors v_i (3.64) of the residue matrix B and of the eigenvectors w_i (3.65) of the residue matrix C as columns respectively. For $\tau \in \mathbb{C}$, let $Z(\tau)$ be the following $m \times m$ matrix:

$$Z_{ij}(\tau) = \frac{\nu_j^2(\tau)}{b_j - c_i + a_2 + \tau}. \tag{3.76}$$

Lemma 3.2. *1.* $Z(0) = W^{-1}V$.

2. $Z_{ij}^{-1}(\tau) = \dfrac{\mu_j^2(\tau)}{b_i - c_j + a_2 + \tau}$.

3. $Z^{-1}(\tau_1)\, Z(\tau_2) = EX(\tau_2 - \tau_1),\ Z(\tau_1)\, Z^{-1}(\tau_2) = EY(\tau_2 - \tau_1)$.

4. $Z(\tau)\, e = e$.

Proof. Let us first prove that

$$W_{ij}^{-1} = \begin{cases} \dfrac{1}{b_j - c_i + a_2} \displaystyle\prod_{k=1}^{j-1} \frac{c_k - c_i}{b_k - c_i + a_2}, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases} \tag{3.77}$$

To prove (3.77), we have to show that

$$\sum_{l=1}^{m} W_{il}\, W_{lj}^{-1} = \delta_{ij}. \tag{3.78}$$

Since both W and W^{-1} are lower triangular, it is clear that the right hand side of (3.78) equals 0 for $j > i$. For $i = j$, the the right hand side of (3.78) has only one summand. It is easy to see that the summand equals 1. Finally, for $i > j$ (3.78) boils down to the identity

$$\sum_{l=j}^{i} \frac{\prod_{k=j+1}^{i-1} (b_k - c_l + a_2)}{\prod_{\substack{k=j \\ k \neq l}}^{i} (c_k - c_l)} = 0,$$

which is up to a change of variables equivalent to (7.59) from [3].

1. We need to show that

$$\sum_{l=1}^{m} W_{il}^{-1} V_{lj} = Z(0)_{ij}$$

or, equivalently,

$$\sum_{l=1}^{\min(i,j)} \frac{\prod_{\substack{k=1 \\ k \neq j}}^{l-1} (b_j - b_k)}{\prod_{k=1}^{l} (b_j - c_k + a_2)} (b_j - c_i + a_2) \frac{\prod_{k=1}^{l-1} (c_k - c_i)}{\prod_{k=1}^{l} (b_k - c_i + a_2)} (b_l - c_l + a_2) = 1. \tag{3.79}$$

Let us add up the summands of (3.79) starting from the end. It is not hard to prove by induction that the sum of (the last) n terms equals

$$S_n = \prod_{k=1}^{\min(i,j)-n} \frac{(b_j - b_k)(c_k - c_i)}{(b_j - c_k + a_2)(b_k - c_i + a_2)}.$$

In particular, $S_{\min(i,j)} = 1$.

2. The second statement of the lemma is equivalent to the rational Cauchy identity (2.41).

3. Let us prove that $\sum_{l=1}^m Z^{-1}(\tau_1)_{il}\, Z(\tau_2)_{lj} = EX(\tau_2-\tau_1)_{ij}$. This is equivalent to proving the following identity:

$$\sum_{l=1}^m \prod_{\substack{k=1\\k\neq i}}^m \frac{b_k-c_l+a_2+\tau_1}{b_j-b_k+\tau_2-\tau_1} \prod_{\substack{k=1\\k\neq l}}^m \frac{b_j-c_k+a_2+\tau_2}{c_k-c_l} = 1.$$

The strategy of Lemma 3.1 works here again aided by identity (7.59) from [3]. The corresponding identity for EY is proven similarly.

4. We need to show that for any $i=1,\ldots,m$,

$$\sum_{j=1}^m \frac{\prod\limits_{\substack{k=1\\k\neq i}}^m (b_j-c_k+\tau)}{\prod\limits_{\substack{k=1\\k\neq j}}^m (b_j-b_k)} = 1.$$

But this, up to a change of variables, is the identity (7.60) from [3]. □

Lemma 3.3.

$$EX(\tau)=e^{X\tau} \text{ and } EY(\tau)=e^{Y\tau}.$$

Proof. We shall only give a proof for $EX(\tau)$. According to the first statement of Lemma 3.1, $EX(\tau)$ is a one-parameter group. It is easy to see for an off-diagonal term that

$$\frac{d}{d\tau}\, EX(\tau)_{ij}\big|_{\tau=0} = \frac{1}{b_j-b_i}.$$

According to the second statement of Lemma 3.1, the Jordan normal form of the derivative at zero is a single block of full size m with the eigenvalue 0. According to the third statement of Lemma 3.1, the corresponding eigenvector is e. Thus, the sum of the elements of every row must add up to zero. □

Proof of Lemma 2.1. The first statement of the lemma is already proven in the course of proving Lemma 3.3. The second is proven by an easy computation. To prove the third statement of the lemma, let us prove that

$$V\, e^{X\tau}\, V^{-1} = W\, e^{-Y\tau}\, W^{-1}. \tag{3.80}$$

Rewriting this formula as

$$Z(0)\, EX(\tau) = EY(-\tau)\, Z(0)$$

and using the third property of Lemma 3.2 proves (3.80). Differentiating (3.80) at zero finishes the proof for the bases v_i (3.64) and w_i (3.65). Recalling that the additive hypergeometric triple is rigid completes the argument. □

Proof of Lemma 1.5. Let $B_d(0)$ and $C_d(0)$ be the Jordan normal forms

$$diag(b_1,\ldots,b_m) \quad \text{and} \quad diag(-c_1,\ldots,-c_m)$$

of the residue operators B and C respectively. Let $B_d(\tau_1) = B_d(0)+k_1\tau_1 Id$ and $C_d(\tau_2)=C_d(0)+k_2\tau_2 Id$. Recalling that $S=VXV^{-1}=-WYW^{-1}$ and using the second statement of Lemma 2.1, it is not hard to show that the formulae

$$B(\tau_1)=Ve^{X\tau_1}B_d(\tau_1)e^{-X\tau_1}V^{-1} \text{ and } C(\tau_2)=We^{Y\tau_2}C_d(\tau_2)e^{-Y\tau_2}W^{-1} \tag{3.81}$$

solve the dynamical equations (1.12), which are equivalent to the statements of Lemma 1.5. □

Proof of Theorem 1.4. Let us first prove the formula for $(T_0)_i$. The fact $S = VXV^{-1}$ implies that

$$e^{nS}v_i = Ve^{nX}e_i = \sum_{j=0}^{m} EX(n)_{ji}\, v_j = \sum_{j=0}^{m} \xi_i^2(n)\, \frac{v_j}{b_i - b_j + n}.$$

Thus, we want to prove that

$$(T_0)_i = z^{b_i}\left(v_i + \sum_{n=1}^{\infty} \alpha_{in}\, \xi_i^2(n)\, z^n \sum_{j=0}^{m} \frac{v_j}{b_i - b_j + n}\right). \tag{3.82}$$

Let us rewrite the mHGS (1.6) as

$$z\frac{df}{dz} = \left[B - A(z + z^2 + z^3 + \cdots)\right] f(z). \tag{3.83}$$

Let us seek solutions to (3.83) in the form

$$(T_0)_i = z^{b_i}\left(\sum_{n=0}^{\infty} (T_0)_{in}\, z^n\right). \tag{3.84}$$

Plugging (3.84) into (3.83), we obtain the following recursive relation on $(T_0)_{in}$:

$$(T_0)_{in} = (B - (b_i + n)Id)^{-1}\, A\, ((T_0)_{i0} + (T_0)_{i1} + \cdots + (T_0)_{i,n-1}), \tag{3.85}$$

where $(T_0)_{i0} = v_i$. Let us prove that

$$(T_0)_{in} = \alpha_{in}\, \xi_i^2(n) \sum_{j=0}^{m} \frac{v_j}{b_i - b_j + n} \tag{3.86}$$

satisfies the recursion (3.85) by induction. First, let us establish the base. According to (2.34), $Av_i = a_2 v_i - (v_i, u)_r\, u$. For the case in consideration, $a_2 = 0$. Recalling that $u = v_1 + \cdots + v_m$ and that the v_i form an orthogonal basis with respect to the product (1.7), we get

$$Av_i = -\nu_{i\,r}^2 \sum_{j=1}^{m} v_j.$$

Applying $(B - (b_i + 1)Id)^{-1}$ to the right hand side of this formula, we obtain that

$$(T_0)_{i1} = \nu_{i\,r}^2 \sum_{j=1}^{m} \frac{v_j}{b_i - b_j + 1}.$$

Observing that $\alpha_{i1}\, \xi_i^2(1) = \nu_{i\,r}^2$ finishes establishing the base.

To make the step of induction from n to $n+1$, we go along the same lines. In the end, the proof boils down to the identity

$$\prod_{k=1}^{m} \frac{b_j - c_k + \tau}{b_j - b_k + \tau} - \sum_{i=1}^{m} \frac{1}{b_j - b_i + \tau}\, \frac{\prod\limits_{k=1}^{m} (b_i - c_k)}{\prod\limits_{\substack{k=1 \\ k \neq i}}^{m} (b_i - b_k)} = 1$$

(for $\tau = n$), which is proven using the method of Lemma 3.3. The proof for $(T_\infty)_i$ is similar. Finally, convergence follows from the following theorem. □

Theorem 3.4.

(3.87)
$$(T_0)_i^j(z)$$
$$=\begin{cases} z^{b_i} v_i^j \, {}_mF_{m-1}\left(\left.\begin{matrix} b_i-c_1,\ldots,b_i-c_j,b_i-c_{j+1}+1,\ldots,b_i-c_m+1 \\ b_i-b_1,\ldots,b_i-b_{j-1},b_i-b_j+1,\ldots,b_i\widehat{-b_i+1},\ldots,b_i-b_m+1 \end{matrix}\right| z\right), \\ \text{if } i\geq j, \\ z^{b_i+1}\,(b_j-c_j)\,\nu_i^2(0)\,\dfrac{\prod\limits_{k=j+1}^{m}(b_i-c_k+1)}{\prod\limits_{k=j}^{m}(b_i-b_k+1)} \\ \text{if } i<j; \\ \times {}_mF_{m-1}\left(\left.\begin{matrix} b_i-c_1+1,\ldots,b_i-c_j+1,b_i-c_{j+1}+2,\ldots,b_i-c_m+2 \\ b_i-b_1+1,\ldots,b_i\widehat{-b_i}+1,\ldots,b_i-b_{j-1}+1,b_i-b_j+2,\ldots,b_i-b_m+2 \end{matrix}\right| z\right), \end{cases}$$

(3.88)
$$(T_\infty)_i^j\left(\frac{1}{z}\right)$$
$$=\begin{cases} z^{c_i-1}\,(b_j-c_j)\,\mu_i^2(0)\,\dfrac{\prod\limits_{k=1}^{j-1}(b_k-c_i+1)}{\prod\limits_{k=1}^{j}(c_k-c_i+1)} \\ \times_m F_{m-1}\left(\left.\begin{matrix} b_1-c_i+2,\ldots,b_{j-1}-c_i+2,b_j-c_i+1,\ldots,b_m-c_i+1 \\ c_1-c_i+2,\ldots,c_j-c_i+2,c_{j+1}-c_i+1,\ldots,c_i\widehat{-c_i}+1,\ldots,c_m-c_i+1 \end{matrix}\right| \frac{1}{z}\right), \\ \text{if } i<j; \\ z^{c_i}\, w_i^j \, {}_mF_{m-1}\left(\left.\begin{matrix} b_1-c_i+1,\ldots,b_{j-1}-c_i+1,b_j-c_i,\ldots,b_m-c_i \\ c_1-c_i+1,\ldots,c_i\widehat{-c_i+1},\ldots,c_j-c_i+1,c_{j+1}-c_i,\ldots,c_m-c_i \end{matrix}\right| \frac{1}{z}\right), \\ \text{if } i\leq j. \end{cases}$$

Proof. According to (3.86),

$$(T_0)_{in}=\alpha_{in}\,\xi_i^2(n)\sum_{j=0}^{m}\frac{v_j}{b_i-b_j+n}.$$

Let us call $(T_0)_n$ (not to be confused with $(T_0)_i$ from Theorem 1.4) the matrix composed of columns $(T_0)_{in}$ and let us call $\mathcal{A}_n$ the diagonal $m\times m$ matrix with α_{in} on the diagonal. Then

(3.89)
$$(T_0)_n=Ve^{nX}\mathcal{A}_n=WW^{-1}Ve^{nX}\mathcal{A}_n.$$

According to the first statement of Lemma 3.2, $W^{-1}V=Z(0)$. According to the third statement of Lemma 3.2 combined with Lemma 3.3, $e^{nX}=Z^{-1}(0)Z(n)$. Thus, we can rewrite (3.89) as

(3.90)
$$(T_0)_n=WZ(n)\,\mathcal{A}_n=WD_r^t(n)\,\mathcal{N}^2(n)\,\mathcal{A}_n.$$

Let us prove that

(3.91)
$$\left(WD_r^t(n)\right)_{ij}=(b_i-c_i)\,\frac{\prod\limits_{k=1}^{i-1}(b_j-b_k+n)}{\prod\limits_{k=1}^{i}(b_j-c_k+n)}.$$

This boils down to the identity

$$\sum_{l=1}^{i}\frac{\prod\limits_{\substack{k=1\\k\neq l}}^{i}(b_j-c_k+n)}{\prod\limits_{k=1}^{i-1}(b_j-b_k+n)}\,\frac{\prod\limits_{k=1}^{i-1}(b_k-c_l)}{\prod\limits_{\substack{k=1\\k\neq l}}^{i}(c_k-c_l)}=1,$$

proven using the strategy of Lemma 3.3. A look at the definition (1.1) of the generalized hypergeometric function finishes the proof of (3.87).

Similarly,

$$(T_\infty)_n = We^{nY}\mathcal{B}_n = VZ^{-1}(n)\,\mathcal{B}_n = VD_r(n)\,\mathcal{M}^2(n)\,\mathcal{B}_n.$$

Similarly to (3.91),

$$(VD_r(n))_{ij} = (b_i - c_i)\,\frac{\prod_{k=i+1}^{m}(c_k - c_j + n)}{\prod_{k=i}^{m}(b_k - c_j + n)},$$

which proves (3.88). □

The formula (2.39) follows from Theorem 3.4 combined with (2.36).

Proof of Theorems 1.8 *and* 2.5. Recall that the space $\mathbb{H}$ has two special bases $(F_0)_i$ and $(F_\infty)_j^\dagger$, such that $\Big((F_0)_i\,, (F_\infty)_j^\dagger\Big) = 1/sn(a_2+b_i-c_j)$. Let us set $\Big((F_0)_i\,, (F_0)_j\Big)$ $= \delta_{ij}/\nu_{i\,ell}^2$. Let $(F_\infty)_j^\dagger = \sum_{k=1}^m x_{kj}\,(F_0)_k$. Taking the scalar product of both sides of this equality with $(F_0)_i$, we see that

$$(F_\infty)_j^\dagger = \sum_{i=1}^m \frac{\nu_{i\,ell}^2}{sn(a_2+b_i-c_j)}\,(F_0)_i\,.$$

Our construction has the following symmetry: if we switch the points 0 and ∞ on the Riemann sphere and simultaneously switch the b_i and $-c_i$ for all $i = 1,\ldots,m$, all the formulae remain valid. Applying the symmetry to the above formula, we get

$$(F_0)_j = \sum_{i=1}^m \frac{\mu_{i\,ell}^2}{sn(a_2+b_j-c_i)}\,(F_\infty)_i^\dagger\,.$$

Rewriting the last two formulae in the matrix form as

$$F_\infty^\dagger = F_0 D_{ell}^t\,\mathcal{N}_{ell}^2 \text{ and } F_0 = F_\infty^\dagger D_{ell}\,\mathcal{M}_{ell}^2$$

and comparing them to each other proves Theorem 2.5.

The formula $(\mathcal{N}\,D\,\mathcal{M})^t = (\mathcal{N}\,D\,\mathcal{M})^{-1}$ implies that $\Big((F_\infty)_i^\dagger/\mu_{i\,ell}, (F_\infty)_j^\dagger/\mu_{j\,ell}\Big)$ $= \delta_{ij}$. Finally, the dimension count proves the uniqueness of the product. □

Proof of Theorem 1.9. Recall that in Theorem 1.9 $\omega_1 = 1$, ω_2 is an imaginary number, all the local exponents are real, and $0 \le b_1 < \cdots < b_m < 1$ and $0 \le c_1 < \cdots < c_m < 1$. In this case, all the values of $sn(a_2 + b_i - c_j)$ are real for $i, j = 1,\ldots,m$. If in addition the positivity conditions (1.9) are satisfied, then the real symmetric form (1.21) on $\mathbb{H}$ is sign-definite as well as its counterpart on $\mathbb{H}'$. They can be extended to complex numbers either in the hermitian or in the complex symmetric fashion. The quaternionic action (2.35) applied to the fields (2.53) produces the action (1.24). It is not hard to see that in the sign-definite case the action (1.24) is hyperkähler. □

Theorem 2.2 – extended version. • *For any complex times* τ_1 *and* τ_2,

$$v_i(\tau_1) = \sum_{j=1}^{m} \frac{\nu_i^2(\tau_1+\tau_2)}{a_2+b_i-c_j+\tau_1+\tau_2}\, w_j(\tau_2). \tag{3.92}$$

There exists a unique up to a constant multiple complex symmetric scalar product $(*,*)_{\tau_1,\tau_2}$ *on* H_{τ_1,τ_2} *such that the bases* $v_i(\tau_1)$ *and* $w_i(\tau_2)$ *are simultaneously orthogonal with respect to it:*

$$(v_i(\tau_1), v_j(\tau_1))_{\tau_1,\tau_2} = \delta_{ij}\, \nu_i^2\,(\tau_1+\tau_2)\,,\;\; (w_i(\tau_2), w_j(\tau_2))_{\tau_1,\tau_2} = \delta_{ij}\, \mu_i^2\,(\tau_1+\tau_2)\,.$$

(Note that formula (2.31) *of the original Theorem* 2.2 *follows from* (3.92) *and* (2.30) *combined.)*

• *For the bases* $v_i(\tau_1)$ *and* $v_i(\tau_2)$ *such that* $\tau_1 \neq \tau_2$,

$$v_j(\tau_1) = \sum_{i=1}^{m} \frac{\xi_j^2(\tau_1-\tau_2)}{b_j-b_i+\tau_1-\tau_2}\, v_i(\tau_2). \tag{3.93}$$

There exists a unique up to a constant multiple complex symmetric scalar product $(*,*)^+_{\tau_1,\tau_2}$ *such that the bases* $v_i(\tau_1)$ *and* $v_i(\tau_2)$ *are simultaneously orthogonal with respect to it:*

$$(v_i(\tau_1), v_j(\tau_1))^+_{\tau_1,\tau_2} = \delta_{ij}\, \xi_i^2(\tau_1-\tau_2),\;\; (v_i(\tau_2), v_j(\tau_2))^+_{\tau_1,\tau_2} = -\delta_{ij}\, \xi_i^2(\tau_2-\tau_1). \tag{3.94}$$

• *For the bases* $w_i(\tau_1)$ *and* $w_i(\tau_2)$ *such that* $\tau_1 \neq \tau_2$,

$$w_j(\tau_1) = \sum_{i=1}^{m} \frac{\theta_j^2(\tau_1-\tau_2)}{c_i-c_j+\tau_1-\tau_2}\, w_i(\tau_2). \tag{3.95}$$

There exists a unique up to a constant multiple complex symmetric scalar product $(*,*)^-_{\tau_1,\tau_2}$ *such that the bases* $w_i(\tau_1)$ *and* $w_i(\tau_2)$ *are simultaneously orthogonal with respect to it:*

$$(w_i(\tau_1), w_j(\tau_1))^-_{\tau_1,\tau_2} = \delta_{ij}\, \theta_i^2(\tau_1-\tau_2),\;\; (w_i(\tau_2), w_j(\tau_2))^-_{\tau_1,\tau_2} = -\delta_{ij}\, \theta_i^2(\tau_2-\tau_1). \tag{3.96}$$

Proof. • The formula (3.92) in the matrix form reads as $Ve^{X\tau_1} = We^{Y\tau_2}Z(\tau_1+\tau_2)$. By Lemma 3.2, $e^{Y\tau_2} = Z(\tau_1)Z^{-1}(\tau_1+\tau_2)$, so we get $Ve^{X\tau_1} = WZ(\tau_1)$. By the same lemma, $V^{-1}W = Z^{-1}(0)$ and $Z^{-1}(0)Z(\tau_1) = e^{X\tau_1}$. The existence of the product (2.30) follows from the rational Cauchy identity (2.41). Its uniqueness follows from the dimension count.

• The formula (3.93) in matrix notations is simply $Ve^{X\tau_1} = Ve^{X\tau_2}e^{X(\tau_1-\tau_2)}$. For $\tau \neq 0$, let us set in the rational Cauchy identity $a_2 = 0$ and $c_i = b_i$ for $i = 1, \ldots, m$. The special case

$$\left[\frac{1}{b_i-b_j+\tau}\right]^{-1}_{ij} = -\frac{\xi_i^2(-\tau)\,\xi_j^2(\tau)}{b_j-b_i+\tau}$$

gives rise to (3.94). Uniqueness follows from the dimension count. The last case is proven similarly. □

Proof of Lemma 2.3. Let us begin with proving the third statement of the lemma first. To prove that $B(\tau_1)x + C(\tau_2)x + a_2(\tau_1,\tau_2)x = (x,u)_{\tau_1,\tau_2}u$ for any $x \in H_{\tau_1,\tau_2}$, it is enough to prove that this formula holds for any basis vector $v_i(\tau_1)$. The formula (3.92) gives us

$$(B(\tau_1)+a_2(\tau_1,\tau_2))\,v_i(\tau_1)=(b_i(\tau_1)+a_2(\tau_1,\tau_2))\sum_{j=1}^{m}\frac{\nu_i^2(\tau_1+\tau_2)}{a_2+b_i-c_j+\tau_1+\tau_2}\,w_j(\tau_2).$$

On the other hand,

$$C(\tau_2)v_i(\tau_1)=\sum_{j=1}^{m}\frac{\nu_i^2(\tau_1+\tau_2)\,c_j(\tau_2)}{a_2+b_i-c_j+\tau_1+\tau_2}\,w_j(\tau_2).$$

But $b_i(\tau_1)+c_j(\tau_2)+a_2(\tau_1,\tau_2)=a_2+b_i-c_j+\tau_1+\tau_2$. Thus,

$$(B(\tau_1)+C(\tau_2)+a_2(\tau_1,\tau_2))\,v_i(\tau_1)=\nu_i^2(\tau_1+\tau_2)\sum_{j=1}^{m}w_j(\tau_2)=(v_i(\tau_1),u)_{\tau_1,\tau_2}u.$$

The first statement easily follows from the third. To prove the fourth statement, we need to show that $\sum_{i=1}^m\nu_i^2(\tau_1+\tau_2)=ma_2(\tau_1,\tau_2)+\sum_{i=1}^m(b(\tau_1)+c(\tau_2))=\sum_{i=1}^m(a_2+b_i-c_j+\tau_1+\tau_2)$. This is formula (7.61) from [3] in different notations. Finally, the second statement follows from the third and the fourth combined. ☐

3.3. **Beukers and Heckman revisited.** The main purpose of this subsection is to prove Lemma 2.4. As a side product, we give a proof to Theorem 1.7 different from those of Beukers and Heckman and of Haraoka.

Recall that p_i are the eigenvectors of the monodromy operator M_0 corresponding to the eigenvalues $e^{2\pi\sqrt{-1}\,b_i}$, q_i are the eigenvectors of the monodromy operator M_∞ corresponding to the eigenvalues $e^{-2\pi\sqrt{-1}\,c_i}$, and r is the eigenvector of the monodromy operator M_1 corresponding to the eigenvalue $e^{2\pi\sqrt{-1}\,a_1}$. Consider the following three flags in the space of solutions: $\mathcal{F}_0=Span(p_1)\subset Span(p_1,p_2)\subset\cdots\subset\mathbb{C}^m$, $\mathcal{F}_\infty=Span(q_m)\subset Span(q_m,q_{m-1})\subset\cdots\subset\mathbb{C}^m$, and $\mathcal{F}_1=Span(r)\subset\mathbb{C}^m$. The first two flags are complete. For two complete flags, one can always choose a basis $e_1,\ldots,e_m$ making the flags opposite: $\mathcal{F}_0=Span(e_1)\subset Span(e_1,e_2)\subset\cdots\subset\mathbb{C}^m$ and $\mathcal{F}_\infty=Span(e_m)\subset Span(e_m,e_{m-1})\subset\cdots\subset\mathbb{C}^m$. In such a basis, the matrix of M_0 will look upper- and the matrix of M_∞ lower-triangular.

Theorem 3.5. *Let $b_1,\ldots,b_m$ and $c_1,\ldots,c_m$ be generic complex numbers and let*

(3.97)
$$(M_0)_{ij}=\begin{cases}0, & \text{if}\quad i>j\\ e^{2\pi\sqrt{-1}\,b_i}, & \text{if}\quad i=j\\ e^{2\pi\sqrt{-1}\left((j-i-1)a_2+b_j+\sum_{k=i+1}^{j-1}(b_k-c_k)\right)}\left(e^{2\pi\sqrt{-1}\,(b_i-c_i+a_2)}-1\right), & \text{if}\quad i<j\end{cases},$$

$$(M_1-e^{2\pi\sqrt{-1}\,a_2}Id)_{ij}=e^{-2\pi\sqrt{-1}\left((i-1)a_2+\sum_{k=1}^{i}(b_k-c_k)\right)}\left(1-e^{2\pi\sqrt{-1}\,(b_i-c_i+a_2)}\right), \tag{3.98}$$

$$(M_\infty)_{ij}=\begin{cases}e^{-2\pi\sqrt{-1}\,(b_i+a_2)}\left(e^{2\pi\sqrt{-1}\,(b_i-c_i+a_2)}-1\right), & \text{if}\quad i>j\\ e^{-2\pi\sqrt{-1}\,c_i}, & \text{if}\quad i=j\\ 0, & \text{if}\quad i<j\end{cases}. \tag{3.99}$$

Then $M_\infty M_1 M_0=Id$.

Proof. Let us first prove that

$$(3.100)\qquad \left(M_0^{-1}\right)_{ij} = \begin{cases} 0, & \textit{if} \quad i > j; \\ e^{-2\pi\sqrt{-1}\,b_i}, & \textit{if} \quad i = j; \\ e^{2\pi\sqrt{-1}\,(a_2-c_i)}\left(e^{2\pi\sqrt{-1}\,(c_i-b_i-a_2)}-1\right), & \textit{if} \quad i < j \end{cases}$$

For $i > j$, $\sum_{k=1}^m \left(M_0^{-1}\right)_{ik}(M_0)_{kj} = 0$, since both M_0 and M_0^{-1} are upper-triangular. For $i = j$, $\sum_{k=1}^m \left(M_0^{-1}\right)_{ik}(M_0)_{kj} = e^{-2\pi\sqrt{-1}\,b_i}\,e^{2\pi\sqrt{-1}\,b_i} = 1$. For $i < j$,

$$\sum_{k=1}^{m}\left(M_0^{-1}\right)_{ik}(M_0)_{kj} = \sum_{k=i}^{j}\left(M_0^{-1}\right)_{ik}(M_0)_{kj}.$$

After factoring out the common multiple $e^{2\pi\sqrt{-1}\,(c_i-b_i-a_2)}-1$, it is not hard to see that the sum telescopes to zero. The proof of the fact that $M_\infty M_1 = M_0^{-1}$ is also a straightforward computation with telescoping sums for three different cases $i > j$, $i = j$, and $i < j$. □

We shall suppress detailed proofs in the remaining part of this subsection. All the formulae below are not hard to prove by direct computation. After factoring out common multiples, simplification, and sometimes telescoping, all of them either boil down to the identities from [3] or become trivial just as above.

For $i = 1, \ldots, m$, the vectors

$$(3.101)\qquad p_i^j = \begin{cases} \left(e^{2\pi\sqrt{-1}\,(a_1-a_2+b_i-b_j)}\left(e^{2\pi\sqrt{-1}\,(b_j-c_j+a_2)}-1\right)\right) \\ \qquad\times\dfrac{\prod\limits_{k=j+1}^{m}\left(e^{2\pi\sqrt{-1}(b_i-c_k+a_2)}-1\right)}{\prod\limits_{\substack{k=j\\k\neq i}}^{m}\left(e^{2\pi\sqrt{-1}\,(b_i-b_k)}-1\right)}, & \text{if } i \geq j, \\ \qquad 0, & \text{if } i < j \end{cases}$$

are the eigenvectors of the matrix (3.97) corresponding to the eigenvalues $e^{2\pi\sqrt{-1}\,b_i}$ and the vectors

$$(3.102)\qquad q_i^j = \begin{cases} \qquad 0, & \text{if } i > j; \\ e^{-2\pi\sqrt{-1}\left(ja_2+\sum\limits_{k=1}^{j}(b_k-c_k)\right)}\left(e^{2\pi\sqrt{-1}\,(b_j-c_j+a_2)}-1\right) \\ \qquad\times\dfrac{\prod\limits_{k=1}^{j-1}\left(e^{2\pi\sqrt{-1}\,(b_k-c_i+a_2)}-1\right)}{\prod\limits_{\substack{k=1\\k\neq i}}^{j}\left(e^{2\pi\sqrt{-1}\,(c_k-c_i)}-1\right)}, & \text{if } i \leq j. \end{cases}$$

are the eigenvectors of the matrix (3.99) corresponding to the eigenvalues $e^{-2\pi\sqrt{-1}\,c_i}$. The vector

$$(3.103)\qquad r^i = e^{-2\pi\sqrt{-1}\left(ia_2+\sum\limits_{k=1}^{i}(b_k-c_k)\right)}\left(e^{2\pi\sqrt{-1}\,(b_i-c_i+a_2)}-1\right)$$

is the eigenvector of the matrix (3.98) corresponding to the eigenvalue $e^{2\pi\sqrt{-1}\,a_1}$ such that

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i = r.$$

Let P and Q be the $m\times m$ matrices composed of the vectors p_i and q_i as columns. Then

(3.104)

$$\left(P^{-1}Q\right)_{ij} = \frac{\prod_{\substack{k=1\\k\neq i}}^{m}\left(e^{2\pi\sqrt{-1}\,(b_k-c_j+a_2)}-1\right)}{\prod_{\substack{k=1\\k\neq j}}^{m}\left(e^{2\pi\sqrt{-1}\,(c_k-c_j)}-1\right)} = e^{-\pi\sqrt{-1}\,(a_1+b_i-c_j)}\,\frac{\prod_{\substack{k=1\\k\neq i}}^{m}\sin\pi(b_k-c_j+a_2)}{\prod_{\substack{k=1\\k\neq j}}^{m}\sin\pi(c_k-c_j)}$$

or, in matrix notations,

$$P^{-1}Q = e^{-\pi\sqrt{-1}\,a_1}\,diag(e^{-\pi\sqrt{-1}\,b_i})\,D_{trig}\,\mathcal{M}^2_{trig}\,diag(e^{\pi\sqrt{-1}\,c_j}), \tag{3.105}$$

where D_{trig} is the matrix (2.43). Inverting (3.105) with the help of the trigonometric Cauchy identity (2.44) proves Lemma 2.4. Theorem 1.7 follows from Lemma 2.4 combined with the trigonometric Cauchy identity.

4. Remarks and open questions

A different and very interesting approach to the generalized hypergeometric function through fermionic fields was made recently in [15] and [16]. The following observation may serve as a starting point in investigating the relations between their view of the generalized hypergeometric function and ours.

Let $d(b)$ and $d(c)$ be $m\times m$ diagonal matrices with the i-th diagonal elements equal to

$$\prod_{\substack{k=1\\k\neq i}}^{m}(b_i-b_k) \quad\text{and}\quad \prod_{\substack{k=1\\k\neq i}}^{m}(c_k-c_i) \tag{4.106}$$

respectively. Let $\tilde{X} = d^{-1}(b)\,X d(b)$ and $\tilde{Y} = d^{-1}(c)\,Y d(c)$. Let us call $Vnd(x,\tau)$ the following $m\times m$ Vandermonde matrix:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1+\tau & x_2+\tau & \cdots & x_m+\tau \\ (x_1+\tau)^2 & (x_2+\tau)^2 & \cdots & (x_m+\tau)^2 \\ \vdots & \vdots & & \vdots \\ (x_1+\tau)^{m-1} & (x_2+\tau)^{m-1} & \cdots & (x_m+\tau)^{m-1} \end{bmatrix}. \tag{4.107}$$

Then $e^{\tau\tilde{X}} = Vnd^{-1}(b,0)\,Vnd(b,\tau)$ and $e^{\tau\tilde{Y}} = Vnd^{-1}(c,0)\,Vnd(c,\tau)$.

Also, it would be interesting to understand what unknown feature of our construction manifests itself through the existence of the "extra" products (3.94) and (3.96) in the extended version of Theorem 2.2.

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References

[1] F. Beukers, G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Inventiones Mathematicae 95 (1989), 325-354 MR974906 (90f:11034)

[2] M. Dettweiler, S. Reiter, *An Algorithm of Katz and its applications to the inverse Galois problem*, J. Symbolic Computations (2000) 30, 761-798 MR1800678 (2001k:12010)

[3] O. Gleizer, *Explicit solutions of the additive Deligne-Simpson problem and their applications*, Advances in Mathematics 178 (2003) 311-374 MR1994222 (2005b:14022)

[4] A. Hurwitz and R. Courant, *Vorlesungen über allgemeine funktionentheorie und elliptische funktionen*, Springer-Verlag, 1964

[5] Y. Haraoka, *Canonical forms of differential equations free from accessory parameters*, SIAM J. of Mathematical Analysis, vol.25, n.4 (1994), 1203-1226 MR1278901 (95g:33004)

[6] Y. Haraoka, *Monodromy representations of systems of differential equations free from accessory parameters*, SIAM J. Math. Anal. 25 (6) (1994) 1595-1621 MR1302165 (95k:34008)

[7] Y. Haraoka and T. Yokoyama, *Construction of rigid local systems and integral representation of their sections*, Math. Nachr. 279 (2006), no. 3, 255-271 MR2200665

[8] N. Hitchin, *Hyperkähler manifolds*, séminaire Bourbaki, 44éme année, 1991-92, n. 748

[9] N. Katz, *Rigid local systems*, Annals of Mathematics Studies, n.139, Princeton University Press (1996). MR1366651 (97e:14027)

[10] V.P. Kostov, *Monodromy groups of regular systems on Riemann's sphere*, prepublication N 401 de l'Universite de Nice, 1994; to appear in Encyclopedia of Mathematical Sciences, Springer

[11] V.P. Kostov, *The Deligne-Simpson problem – a survey*, J. of Algebra, 281 (2004), no. 1, 83-108 MR2091962 (2005g:20074)

[12] P. Magyar, *Bruhat order for two flags and a line*, Journal of Algebraic Combinatorics 21 (2005) MR2130795 (2006d:14055)

[13] P. Magyar, J. Weyman, A. Zelevinsky, *Multiple flag varieties of finite type*, Advances in Mathematics 141, 97-118 (1999) MR1667147 (99m:14095)

[14] K. Okubo, *On the group of Fuchsian equations*, Seminar Reports of Tokyo Metropolitan University, 1987

[15] A. Orlov, D. Scherbin, *Multivariate hypergeometric functions as τ functions of Toda lattice and Kadomtsev-Petviashvili equation*, Phys. D 152/153 (2001), 51-65 MR1837897 (2002h:37147)

[16] A. Orlov, D. Scherbin, *Fermionic representations for basic hypergeometric functions related to Schur polynomials*, preprint arXiv:nlin.SI/0001011 v4, Dec. 1, 2000

[17] M. Sato, M. Jimbo, T. Miwa, *Holonomic quantum fields I, III, IV, V* (in Russian), Mir publishing house, Moscow, Russia, 1983

[18] C. Simpson, *Products of matrices*, AMS Proceedings 1 (1992), 157-185 MR1158474 (93c:15015)

[19] E. Whittaker, G. Watson, *A Course of modern analysis*, Cambridge University Press, 1927 MR1424469 (97k:01072)

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