TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 5, May 2008, Pages 2693–2707
S 0002-9947(07)04400-5
Article electronically published on December 11, 2007

LOCAL AND GLOBAL C-REGULARITY

NİHAT GÖKHAN GÖĞÜŞ

ABSTRACT. A bounded domain D is called c-regular if the plurisubharmonic envelope of every continuous function on $\overline{D}$ extends continuously to $\overline{D}$. We show using Gauthier's Fusion Lemma that a domain is locally c-regular if and only if it is c-regular.

1. INTRODUCTION

The envelopes of functions play an essential role in pluripotential theory. In particular, they are used in the Perron-Bremermann method to solve the Dirichlet problem: find a homogeneous solution of the Monge-Ampère operator with continuous boundary data. So it is important to classify those domains where the solution is also continuous. In [Go] the notion of Jensen measures was used to completely characterize the domains where continuous functions always have continuous envelopes. Such domains are called c-regular. The problem whether c-regularity is a local property has remained unanswered. In this paper we show that a domain is c-regular if and only if it is locally c-regular.

To prove that a locally c-regular domain is c-regular we need a version of Gauthier's Fusion Lemma. Roughly speaking, given a finite number of plurisubharmonic functions v_j on subdomains V_j of a domain D there exists a plurisubharmonic function u on D such that the difference $|u - v_j|$ on V_j is controlled by the sum of the differences of the v_j's. The proof of this form of the Fusion Lemma is given in §3. In §4 we construct an operator $\mathcal{L}$ on $C(\overline{D})$ that arises by taking envelopes of a given $\phi \in C(\overline{D})$ on the V_j's. We prove that the iterates of $\mathcal{L}\phi$ pointwise decrease to the envelope of ϕ on $\overline{D}$. To show that the iterates converge uniformly to a continuous function on $\overline{D}$ we consider the Krasnoselski-Mann iterations of $\mathcal{L}$. A beautiful result of Ishikawa in [I] provides a criteria for such iterations to converge uniformly on $\overline{D}$. It turns out that it is possible to establish Ishikawa's criteria once we prove the Fusion Lemma.

Finally in §5 we show that a c-regular domain is locally c-regular. The proof doesn't follow immediately from the definition. To achieve this we approximate the Jensen measures supported in a neighborhood of an open ball B in $\mathbb{C}^n$ by the Jensen measures supported in $D \cap B$.

The author would like to thank Prof. Evgeny Poletsky, who introduced this problem to him and constantly supported the development and improvement of this paper.

Received by the editors October 28, 2005 and, in revised form, August 2, 2006.
2000 *Mathematics Subject Classification.* Primary 32U15.

2. Basic notation and results

We shall denote an open ball in $\mathbb{C}^n$, $n \geq 2$, of radius r and center z by $B(z, r)$. In the special case where $z = 0$ and $r = 1$ we use B instead of $B(0, 1)$. We will write U for the open unit disc in $\mathbb{C}$. We shall denote by $d\lambda$ the normalized standard Lebesgue measure on the boundary ∂U of U. If $z, w \in \mathbb{C}^n$, then we set

$$(z, w) = \sum_{j=1}^{n} z_j \overline{w}_j.$$

If V is any set in $\mathbb{C}^n$ and f is a function on V, we put

$$||f||_V = \sup_V |f|.$$

Let Ω be a bounded open set in $\mathbb{C}^n$. For any upper (resp. lower) bounded function ψ defined on Ω, we shall denote by ψ^* (resp. by ψ_*) the upper semicontinuous (resp. the lower semicontinuous) regularisation of ψ on $\overline{\Omega}$. That is, for any $z \in \overline{\Omega}$,

$$\psi^*(z) = \limsup_{w \to z,\, w \in \Omega} \psi(w), \quad \psi_*(z) = \liminf_{w \to z,\, w \in \Omega} \psi(w).$$

A regular Borel probability measure μ on $\overline{\Omega}$ is a Jensen measure with barycenter $z \in \overline{\Omega}$ if

$$u^*(z) \leq \int u^* \, d\mu$$

for all u in the set $PSH^b(\Omega)$ of all upper bounded plurisubharmonic functions on Ω. The class of such Jensen measures will be denoted by $\mathcal{J}_z^b$. For any point $z \in \overline{\Omega}$, let

$$\hat{\mathcal{J}}_z = \hat{\mathcal{J}}_z(\Omega) = \{\mu : \mu = \lim \mu_j,\, \mu_j \in \mathcal{J}_{z_j}^b,\, z_j \to z,\, z_j \in \overline{\Omega}\},$$

where the limit above should be understood as the weak-$*$ limit in $C^*(\overline{\Omega})$.

We denote by $PSH^c(\Omega)$ the class of plurisubharmonic functions on Ω which are continuous on $\overline{\Omega}$. Given $z \in \overline{\Omega}$, we write $\mathcal{J}_z^c = \mathcal{J}_z^c(\Omega)$ for the set of all regular Borel measures μ on $\overline{\Omega}$ such that

$$u(z) \leq \int u \, d\mu$$

for every $u \in PSH^c(\Omega)$. The class $\mathcal{J}_z^c$ was introduced in [CCW] and later together with $\mathcal{J}_z^c$, $\mathcal{J}_z^b$ was studied in [W], while the class $\hat{\mathcal{J}}_z$ was introduced in [Go].

Given any function φ defined on $\overline{\Omega}$, we define the envelope

$$E^c\varphi(z) = E^c_\Omega\varphi(z) = \inf\left\{\int \varphi \, d\mu : \mu \in \mathcal{J}_z^c\right\}, \; z \in \overline{\Omega}.$$

For an arbitrary bounded domain $G \subset \mathbb{C}$ and a Borel measurable subset $A \subset \partial G$, the positive harmonic measure of the set A with respect to G is denoted by $\omega(\eta, A, G)$. Recall that the function $\omega(\eta, A, G)$ is harmonic on G.

We denote by $\mathcal{A}(z_0, \Omega)$ the set of all holomorphic mappings f of a neighborhood of the closure $\overline{U}$ of the unit disk $U \subset \mathbb{C}$ into an open set $\Omega \subset \mathbb{C}^n$ such that $f(0) = z_0$. If $f \in \mathcal{A}(z_0, \Omega)$, then the measure

$$\mu_f(E) = \lambda(f^{-1}(E) \cap \partial U) = \omega(0, f^{-1}(E) \cap \partial U, U)$$

is a Jensen measure with barycenter $z_0 = f(0)$. For $z \in \Omega$, let $\mathcal{H}_z = \mathcal{H}_z(\Omega)$ be the collection of all measures μ_f such that $f \in \mathcal{A}(z, \Omega)$. Also let $\overline{\mathcal{H}}_z$ be the closure of $\mathcal{H}_z$ in the weak-$*$ topology induced from $C^*(\overline{\Omega})$. The following is a well-known result of Poletsky (see [P1] and [P2]).

Theorem 2.1. *Let Ω be an open set in $\mathbb{C}^n$. If φ is an upper semicontinuous function on Ω, then the function*

$$S\varphi(z) = \sup\{u(z) : u \le \varphi \text{ is plurisubharmonic on } \Omega\}$$

is plurisubharmonic on Ω and equal to

$$E\varphi(z) = \inf\left\{\int \varphi\, d\mu : \mu \in \mathcal{H}_z\right\},$$

the plurisubharmonic envelope of φ.

We use the definition of c-regular domains from [Go].

Definition 2.2. A bounded open set Ω in $\mathbb{C}^n$ is called c-regular if the envelope of every continuous function on $\overline{\Omega}$ is continuous on Ω and extends continuously to $\overline{\Omega}$.

Definition 2.3. A bounded domain D in $\mathbb{C}^n$ is locally c-regular if for each $\zeta \in \partial D$ there exists an open neighborhood N of ζ such that the set $V := N \cap D$ is c-regular.

3. Fusion of plurisubharmonic functions

The next lemma is a slightly modified version of Gauthier's Fusion Lemma in [G], and the proof is a copy of Gauthier's proof.

Lemma 3.1. *Let $V \subset \mathbb{C}^n$ and $U_j \subset \mathbb{C}^n$, $j = 1, 2$, be bounded domains with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Then there exists a number $c > 0$ and for any functions $v_1 \in PSH^b(U_1 \cup V)$ and $v_2 \in PSH^b(U_2 \cup V)$ there exists a function $u \in PSH^b(U_1 \cup V \cup U_2)$ so that*

$$||u - v_j||_{U_j \cup V} \le c||v_1 - v_2||_V. \tag{1}$$

If $v_j \in PSH^c(U_j \cup V)$, then u can be chosen to lie in $PSH^c(U_1 \cup V \cup U_2)$.

Proof. If $||v_1 - v_2||_V = 0$, then we set u equal to v_1 on $U_1 \cup V$ and v_2 on U_2. Since $v_1 = v_2$ on V, u is plurisubharmonic on $U_1 \cup V \cup U_2$. We may assume that $||v_1 - v_2||_V \ne 0$. Take $\chi_1 \in C^\infty(\mathbb{C}^n)$ with $-1 \le \chi_1 \le 0$, $\chi_1 = -1$ on $\overline{U}_2$ and $\chi_1 = 0$ on $\overline{U}_1$. Set $\chi_2 = -1 - \chi_1$. Let $\delta(z) = |z|^2$ and choose a number $\lambda > 0$ so small that $\delta + \lambda\chi_j$ are both plurisubharmonic for $j = 1, 2$. We let

$$\eta = \lambda^{-1}||v_1 - v_2||_V. \tag{2}$$

Set

$$u_j = v_j^* + \eta(\delta + \lambda\chi_j) \tag{3}$$

on $U_j \cup \overline{V}$ and $u_j = -\infty$ elsewhere. Finally, we set

$$u = \max\{u_1, u_2\}.$$

Clearly, u is upper bounded and plurisubharmonic on $(U_1 \cup V \cup U_2) \setminus \partial V$. Suppose $z_0 \in \partial V \cap U_1$. Since $\chi_1 = 0$ and $\chi_2 = -1$ on U_1, by (2) and (3) for all points $z \in U_1 \cap \overline{V}$ near z_0,

$$u_2(z) = v_2^*(z) + \eta(\delta - \lambda) = u_1(z) + (v_2^*(z) - v_1(z)) - \lambda\eta \le u_1(z).$$

Since $u_2(z) = -\infty$ for $z \in U_1 \setminus \overline{V}$, we have that $u_2(z) \le u_1(z)$ for all $z \in U_1$ near z_0. As z_0 was an arbitrary point of $\partial V \cap U_1$, u is upper bounded and plurisubharmonic on a neighborhood of $\partial V \cap U_1$. A similar argument shows that u is upper bounded and plurisubharmonic on a neighborhood of $\partial V \cap U_2$. Thus $u \in PSH^b(U_1 \cup V \cup U_2)$. If $v_j \in PSH^c(U_j \cup V)$, then we define the u_j's the same way on $\overline{U}_j \cup \overline{V}$. By the construction, u is in $PSH^c(U_1 \cup V \cup U_2)$.

We now need to verify the required estimates. Let M be the supremum of δ on the set $U_1 \cup V \cup U_2$. On $U_j \setminus V$, $j = 1, 2$,

$$|u(z) - v_j(z)| = |u_j(z) - v_j(z)| = \eta\delta(z) \leq \lambda^{-1}M||v_1 - v_2||_V.$$

Take $z \in V$ and for $j = 1$ or 2 suppose first that $|u(z) - v_j(z)| = u(z) - v_j(z)$. Since $|\delta(z) + \lambda\chi_j(z)| \leq M + \lambda$,

$$\begin{aligned} |u(z) - v_j(z)| &\leq |\max\{v_1(z), v_2(z)\} - v_j(z)| + \eta(M + \lambda) \\ &\leq (\lambda^{-1}M + 2)||v_1 - v_2||_V. \end{aligned}$$

If $|u(z) - v_j(z)| = v_j(z) - u(z)$, then

$$\begin{aligned} |u(z) - v_j(z)| &= v_j(z) - \max\{f_1, f_2\} \\ &\leq v_j(z) - \max\{v_1(z), v_2(z)\} + \eta(M + \lambda) \\ &\leq (\lambda^{-1}M + 2)||v_1 - v_2||_V. \end{aligned}$$

Thus, for $c = \lambda^{-1}M + 2$, (1) holds. This completes the proof. □

The fusion lemma allows us to fuse plurisubharmonic functions over several domains.

Theorem 3.2. *Let D be a bounded domain and N_j, $j = 1, \ldots, m$, be open sets in $\mathbb{C}^n$ so that*

$$\overline{D} \subset \bigcup_{j=1}^{m} N_j.$$

For any integer $j = 1, \ldots, m$, let $V_j = D \cap N_j$. Then there exists a constant $c > 0$ (depending only on the N_j's) so that if $v_j \in PSH^b(V_j)$, then there exists a function $u \in PSH^b(D)$ such that for all $j = 1, \ldots, m$,

$$||u - v_j||_{V_j} \leq c \sum_{1 \leq k < l \leq m} ||v_k - v_l||_{V_k \cap V_l}.$$

Moreover, if $v_j \in PSH^c(V_j)$ for all $j = 1, \ldots, m$, then the function u can be taken to lie in $PSH^c(D)$.

Proof. The proof is by induction on the number m. We start with $m = 2$. We fix open subsets $L_j \Subset N_j$ with

$$\overline{D} \subset \bigcup_{j=1}^{2} L_j.$$

Set $A_j = D \cap L_j$. Let $U_1 = D \setminus \overline{L}_2$ and $U_2 = D \setminus \overline{L}_1$. Then $\overline{U}_1 \cap \overline{U}_2 = \overline{D} \setminus (L_1 \cup L_2) = \emptyset$. Also let $V = V_1 \cap V_2$. Note that $V_j = U_j \cup V$ and thus $v_j \in PSH^b(U_j \cup V)$, $j = 1, 2$. Apply Lemma 3.1 to v_j to get a constant $c > 0$ and a function $u \in PSH^b(D)$ so that for all $z \in V_j$,

$$|u(z) - v_j(z)| \leq c||v_1 - v_2||_V.$$

If $v_j \in PSH^c(V_j)$, then by Lemma 3.1 the function u is in $PSH^c(D)$.

We have thus established the beginning of an inductive argument. Suppose that we have shown the theorem when $\overline{D}$ is contained in the union of m open sets for

$m \geq 2$. We may now assume that $\overline{D}$ is contained in the union of open sets N_j, $j = 1, \ldots, m+1$. There exist open subsets $L_j \Subset N_j$, $j = 1, \ldots, m$, with

$$\overline{D} \subset N_{m+1} \cup \left(\bigcup_{j=1}^{m} L_j \right).$$

Let $D_m = D \cap \left(\bigcup_{j=1}^{m} L_j \right)$ and $A_j = D \cap L_j$. Since $\overline{D}_m \subset \bigcup_{j=1}^{m} N_j$, by the induction assumption there exist a function $u_0 \in PSH^b(D_m)$ and a constant $c_m > 0$ such that

$$|u_0(z) - v_j(z)| \leq c_m \sum_{1 \leq k < l \leq m} ||v_k - v_l||_{V_k \cap V_l}$$

for all $z \in A_j$.

Hence we can use the first part of the proof to fuse u_0 and v_{m+1} on D. There exist a function $u \in PSH^b(D)$ and a constant $c_0 > 0$ such that for all $z \in D_m$,

$$|u(z) - u_0(z)| \leq c_0 ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}}$$

and for all $z \in V_{m+1}$,

$$|u(z) - v_{m+1}(z)| \leq c_0 ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}}.$$

We only need to check the estimates. Take a number $\delta > 0$ and a point $z \in D_m \cap V_{m+1}$ so that

$$||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} < |u_0(z) - v_{m+1}(z)| + \delta.$$

Then $z \in A_k$ for some $k \in \{1, \ldots, m\}$ and

$$\begin{aligned} ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} &< |u_0(z) - v_{m+1}(z)| + \delta \\ &\leq |u_0(z) - v_k(z)| + |v_k(z) - v_{m+1}(z)| + \delta \\ &\leq (c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l} + \delta. \end{aligned}$$

Since this is true for all $\delta > 0$,

$$||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} \leq (c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}.$$

If $z \in V_j$ for some $j \in \{1, \ldots, m\}$, then there exists k with $z \in A_k$. Therefore

$$\begin{aligned} |u(z) - v_j(z)| &\leq |u(z) - u_0(z)| + |u_0(z) - v_k(z)| + |v_k(z) - v_j(z)| \\ &\leq (c_0 + 1)(c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}. \end{aligned}$$

If $z \in V_{m+1}$, then

$$|u(z) - v_{m+1}(z)| \leq c_0(c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}.$$

Set $c = (c_0 + 1)(c_m + 1)$. The continuity of u on $\overline{D}$ follows immediately from the construction when all the v_k's are continuous on $\overline{V}_k$. $\square$

4. Localization of envelopes

We give here a general argument to approximate the plurisubharmonic envelope of functions on a domain by envelopes with respect to the subdomains of the domain. Then we use this result and the fusion result of the previous section to show that local c-regularity of the boundary implies global c-regularity. The following lemma is so well known that we omit the proof.

Lemma 4.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $N_1, \dots, N_m$ be open subsets of $\mathbb{C}^n$ so that*

$$\overline{D} \subset \bigcup_{j=1}^{m} N_j.$$

Then there exist open subsets $L_j \Subset N_j$ and functions $\chi_j \in C^\infty(\mathbb{C}^n)$, $j = 1, \dots, m$, so that

(p1) $\overline{D} \subset \bigcup_{j=1}^{m} L_j$ *and* $\sum_{j=1}^{m} \chi_j = 1$ *on* $\overline{D}$,

(p2) $\chi_j = 0$ *on* $\overline{D} \setminus N_j$ *and* $\chi_j \geq 1/m$ *on* $\overline{L}_j \cap \overline{D}$.

Let D, N_j and χ_j be as in Lemma 4.1. Set $V_j = D \cap N_j$. If ϕ is a bounded function on $\overline{D}$, we define

$$\mathcal{L}\phi = \sum_{j=1}^{m} \chi_j (E_{V_j}\phi)^*. \tag{4}$$

Thus if we denote by $\mathcal{B}$ the set of bounded functions on $\overline{D}$, then we have an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$. Note that for any $\phi \in \mathcal{B}$, $\mathcal{L}\phi \leq \phi$ and $\mathcal{L}$ is monotonic in the sense that $\varphi \leq \psi$ implies $\mathcal{L}\varphi \leq \mathcal{L}\psi$. We set $\mathcal{L}^0\phi = \phi$ and for any integer $k \geq 1$,

$$\mathcal{L}^k\phi = \mathcal{L}(\mathcal{L}^{k-1}\phi).$$

We endow the set $\mathcal{B}$ with the supremum norm

$$||f||_{\overline{D}} = \sup_{\overline{D}} |f|.$$

Thus $\mathcal{B}$ together with this norm is a Banach space and $\mathcal{L}$ is a nonlinear operator on $\mathcal{B}$. Let $\mathcal{S}$ be the set of bounded functions on $\overline{D}$ so that

$$\phi(z) = \limsup_{w \to z,\, w \in \overline{D}} \phi(w)$$

for all $z \in \overline{D}$. That is, $\mathcal{S}$ is the set of bounded upper semicontinuous functions on $\overline{D}$. Then $\mathcal{S}$ is a closed subset of $\mathcal{B}$. The operator $\mathcal{L}$ maps $\mathcal{S}$ into $\mathcal{S}$. It's interesting to observe that the fixed points of $\mathcal{L}$ in $\mathcal{S}$ are precisely the bounded plurisubharmonic functions on D.

Theorem 4.2. *If ϕ is in $\mathcal{S}$, then $\phi = (E\phi)^*$ on $\overline{D}$ if and only if $\mathcal{L}\phi = \phi$ on $\overline{D}$.*

Proof. If $\phi = (E\phi)^*$ on $\overline{D}$, then $\phi = (E\phi)^* \leq (E_{V_j}\phi)^* \leq \phi$ on $\overline{V}_j$. Hence $\phi = (E_{V_j}\phi)^*$ on $\overline{V}_j$ for each j. Therefore

$$\mathcal{L}\phi = \chi_1(E_{V_1}\phi)^* + \ldots + \chi_m(E_{V_m}\phi)^* = \phi.$$

Suppose now that $\mathcal{L}\phi = \phi$. Take $z_0 \in D$. Let $J = \{j : z_0 \in V_j\}$ and $V = \bigcap_{j \in J} V_j$. Suppose ϕ is not plurisubharmonic around z_0. Then there exists a function $f \in \mathcal{A}(z_0, V)$ such that

$$\int \phi \, d\mu_f < \phi(z_0).$$

Since $\chi_k(z_0) = 0$ if $k \notin J$,

$$\mathcal{L}\phi(z_0) = \sum_{j \in J} \chi_j(z_0) E_{V_j}\phi(z_0) \leq \int \phi \, d\mu_f < \phi(z_0),$$

a contradiction to our assumption. Therefore, ϕ is plurisubharmonic on D. Hence the equality $\phi = (E\phi)^*$ holds on $\overline{D}$. □

As a corollary, we show that the iterations of $\mathcal{L}\phi$ converge to the plurisubharmonic envelope of ϕ.

Corollary 4.3. *For any ϕ in $\mathcal{S}$,*

$$\lim_k \mathcal{L}^k \phi(z) = (E_D\phi)^*(z)$$

for all $z \in \overline{D}$.

Proof. The functions $\mathcal{L}^k\phi$ form a decreasing sequence of upper semicontinuous functions on $\overline{D}$. Then the function defined by

$$\psi = \lim_k \mathcal{L}^k \phi$$

is also in $\mathcal{S}$. Moreover,

$$\begin{aligned} \psi = \lim_k \mathcal{L}(\mathcal{L}^{k-1}\phi) &= \sum_{j=1}^{m} \chi_j \lim_k (E_{V_j}\mathcal{L}^{k-1}\phi) \\ &= \sum_{j=1}^{m} \chi_j E_{V_j}\psi = \mathcal{L}\psi. \end{aligned}$$

Hence $\psi = (E_D\psi)^*$ on $\overline{D}$ by Theorem 4.2. Note that $\mathcal{L}^k\phi \leq \phi$ implies that $\psi \leq \phi$. Consequently, $\psi = (E_D\psi)^* \leq (E_D\phi)^*$. Also for each k, $(E_D\phi)^* \leq \mathcal{L}^k\phi$; hence $(E_D\phi)^* \leq \psi$ and the equality follows. □

We need some helpful concepts from functional analysis (see [I]).

If $(X, \| \, . \, \|)$ is a Banach space, $x_0 \in X$, C is a convex subset of X and $T : C \to X$ is any map, we define the Krasnoselski-Mann iterations

$$x_{k+1} = (1 - t_k)x_k + t_k T x_k, \tag{5}$$

where t_k is a sequence in $[0, 1]$. Throughout the following we will impose the conditions that $\sum_k t_k = \infty$ and $0 \leq t_k \leq b < 1$ for all k.

Definition 4.4. A mapping $T : C \to X$ is called nonexpansive if for all $x, y \in C$ we have

$$\| Tx - Ty \| \leq \| x - y \| .$$

Lemma 4.5. *The operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is nonexpansive.*

Proof. Take φ and ψ in $\mathcal{B}$ and let $\delta = \| \varphi - \psi \|_{\overline{D}}$. Then on $\overline{D}$,

$$\varphi - \delta \leq \psi \leq \varphi + \delta;$$

therefore

$$\mathcal{L}\varphi - \delta \leq \mathcal{L}\psi \leq \mathcal{L}\varphi + \delta.$$

We have $|\mathcal{L}\varphi - \mathcal{L}\psi| \leq \delta$ on $\overline{D}$; hence $\| \mathcal{L}\varphi - \mathcal{L}\psi \|_{\overline{D}} \leq \| \varphi - \psi \|_{\overline{D}}$. □

For any subset A of X and a point $x \in X$, the distance of x to A is denoted by $\operatorname{dist}(x, A)$. That is,

$$\operatorname{dist}(x, A) := \inf\{\| x - y \| : y \in A\}.$$

In ([I]), Ishikawa proves the following very useful theorem.

Theorem 4.6. *Let A be a closed subset of a Banach space X and let $T : A \to X$ be a nonexpansive mapping with a nonempty fixed points set F in A. Suppose that $x_k \in A$ for all $k \geq 1$, where $\{x_k\}$ is the sequence of iterations defined in* (5)*. Suppose there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that*

$$\| x - Tx \| \geq f(\operatorname{dist}(x, F))$$

for all $x \in A$. Then the sequence converges to a member of F.

If $z \in D$ and $B(z, r) \subset D$ is a ball in D, then $B(z, r)$ is c-regular by [Wa]. In the rest of the section we will assume that D is a locally c-regular domain. By compactness of $\overline{D}$ this means there exists a number of open sets N_j, $j = 1, \ldots, m$, so that $\overline{D}$ is contained in the union of the N_j's and the sets $V_j = D \cap N_j$ are c-regular for each j. Throughout, $\mathcal{L}$ will denote the operator on $C(\overline{D})$ defined by (4). We remark here that the set of fixed points of our operator $\mathcal{L} : C(\overline{D}) \to C(\overline{D})$ is precisely $PSH^c(D)$ by Theorem 4.2. For any function $\varphi \in C(\overline{D})$, let $\varphi_1 = \varphi$ and

$$\varphi_{k+1} = (1 - t_k)\varphi_k + t_k \mathcal{L}\varphi_k$$

for $k \geq 1$. Now we can prove the following:

Theorem 4.7. *Let D be a bounded locally c-regular domain in $\mathbb{C}^n$. Then there exists a constant $c > 0$ so that for all $\varphi \in C(\overline{D})$,*

$$\operatorname{dist}(\varphi, PSH^c(D)) \leq c \| \varphi - \mathcal{L}\varphi \|_{\overline{D}} .$$

Proof. There exist open sets $N_1, \ldots, N_m$ in $\mathbb{C}^n$ such that $\overline{D} \subset \bigcup_{j=1}^m N_j$ and for each j, $V_j = N_j \cap D$ is c-regular. Let φ be a function in $C(\overline{D})$. The functions $v_j = E_{V_j}\varphi$ are now in $PSH^c(V_j)$.

As in Lemma 4.1 we find open subsets $L_j \Subset N_j$ and a partition of unity χ_j for $\overline{D}$ satisfying (p1) and (p2). Let $A_j = L_j \cap D$. Then using (p2), it's not difficult to see that for each $j = 1, \ldots, m$,

$$\| \varphi - v_j \|_{\overline{A}_j} < m \| \varphi - \mathcal{L}\varphi \|_{\overline{D}} . \tag{6}$$

Let

$$d_j = \operatorname{dist}(\varphi, PSH^c(A_j))$$

and find functions $u_j \in PSH^c(A_j)$ so that

$$||u_j - \varphi||_{\overline{A}_j} \leq 2d_j.$$

Using (6),

$$\max_{1 \leq j \leq m} d_j \leq \max_{1 \leq j \leq m} ||\varphi - v_j||_{\overline{A}_j} \leq m||\varphi - \mathcal{L}\varphi||_{\overline{D}}.$$

Now we fuse the functions $u_1, \ldots, u_m$ on $\overline{D}$. By Theorem 3.2, there exist a function $u \in PSH^c(D)$ and a constant $c_0 > 0$ so that

$$||u - u_j||_{A_j} \leq c_0 \sum_{1 \leq k < l \leq m} ||u_k - u_l||_{\overline{A_k \cap A_l}}$$

for each $j = 1, \dots, m$. On the other hand, since $||u_j - \varphi||_{\overline{A_j}} \le 2d_j$,

$$||u_k - u_l||_{\overline{A_k \cap A_l}} \le 4 \max_{1 \le j \le m} d_j.$$

Taking the sum over such terms,

$$\sum_{1 \le k < l \le m} ||u_k - u_l||_{\overline{A_k \cap A_l}} \le 2m(m-1) \max_{1 \le j \le m} d_j.$$

Then

$$\begin{aligned} \operatorname{dist}(\varphi, PSH^c(D)) &\le ||u - \varphi||_{\overline{D}} \le \max_{1 \le j \le m} (||u - u_j||_{\overline{A_j}} + ||u_j - \varphi||_{\overline{A_j}}) \\ &\le (2c_0 m(m-1) + 2) \max_{1 \le j \le m} d_j \\ &\le (2c_0 m^2(m-1) + 2m)||\varphi - \mathcal{L}\varphi||_{\overline{D}}. \end{aligned}$$

Now we set $c = 2c_0 m^2(m-1) + 2m$ to finish the proof. □

Using the above theorem and Ishikawa's result, it's now possible to prove that if a domain is locally c-regular, then it is c-regular.

Theorem 4.8. *If a domain D is locally c-regular, then it is c-regular.*

Proof. Let D be locally c-regular and $\varphi \in C(\overline{D})$. Recall that we define $\varphi_1 = \varphi$ and for every integer $k \ge 1$,

$$\varphi_{k+1} = (1 - t_k)\varphi_k + t_k \mathcal{L}\varphi_k.$$

Let us show by induction that for all $k \ge 1$,

$$\mathcal{L}^k \varphi \le \varphi_k \le \varphi.$$

If $k = 1$, then

$$\mathcal{L}\varphi \le (1 - t_1)\varphi + t_1 \mathcal{L}\varphi = \varphi_1 \le \varphi.$$

Suppose for some k, $\mathcal{L}^k \varphi \le \varphi_k \le \varphi$. Then

$$\mathcal{L}^{k+1}\varphi = \mathcal{L}\mathcal{L}^k \varphi \le \mathcal{L}\varphi_k \le \varphi_{k+1} \le \varphi.$$

By Theorem 4.7 and Theorem 4.6 the sequence of iterations φ_k converges uniformly on $\overline{D}$ to a function $u \in PSH^c(D)$. On the other hand by Corollary 4.2 the sequence of functions $\mathcal{L}^k \varphi$ decreases pointwise to $(E\varphi)^*$ on $\overline{D}$. Thus

$$(E\varphi)^* \le u \le \varphi.$$

Since u is in $PSH^c(D)$, it follows that $u = (E\varphi)^* \in PSH^c(D)$. □

5. Locally c-regular domains

The purpose of this section is to prove that global c-regularity is sufficient for local c-regularity. For any open ball $K = B(z, r)$ in $\mathbb{C}^n$ and a number $\delta > 1$ we let $K_\delta = B(z, \delta r)$.

The main idea of the proof is contained in the following observations.

Lemma 5.1. *Let $1 < \delta < 2$, $0 < r \le 1$ and $g : \overline{U} \to B_\delta$ be a holomorphic map from a neighborhood of $\overline{U}$ into B_δ such that $g(0) = z_0$ for some $z_0 \in \partial B_r$. Given a number $\varepsilon > 0$, we define the set*

$$A = \{\eta \in \partial U : |g(\eta) - g(0)| > \varepsilon\}.$$

Then

$$\omega(0,\, A,\, U) = \lambda(A) \le \frac{8}{(\delta\varepsilon)^2}(\delta^2 - r^2).$$

Proof. The function

$$u(z) = \frac{1}{\delta^2}\mathbf{Re}\,(z,\, z_0)$$

is plurisubharmonic and $u(z) \le 1$ on B_δ. Take any point $z \in B_\delta$ with $\varepsilon < |z - z_0|$. Then

$$\varepsilon^2 < |z - z_0|^2 \le |z|^2 - 2\mathbf{Re}\,(z,\, z_0) + 1.$$

Since $\delta < 2$,

$$u(z) = \frac{1}{\delta^2}\mathbf{Re}\,(z,\, z_0) < 1 - \frac{\varepsilon^2}{8}. \tag{7}$$

Let $v(\eta) = u(g(\eta))$ for $\eta \in \overline{U}$. The function v is subharmonic on U, $v \le 1$ on $\overline{U}$, $v|_A \le 1 - \varepsilon^2/8$ and

$$\begin{aligned} \frac{r^2}{\delta^2} &= u(z_0) = v(0) \le \int_{\eta \in \partial U} v(\eta)\, d\lambda(\eta) \\ &= \int_A v(\eta)\, d\lambda(\eta) + \int_{\partial U \setminus A} v(\eta)\, d\lambda(\eta) \\ &\le (1 - \varepsilon^2/8)\lambda(A) + 1 - \lambda(A) = 1 - (\varepsilon^2/8)\lambda(A). \end{aligned}$$

Hence

$$\lambda(A) \le \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2).$$

□

If A is any subset of ∂G, the characteristic function of A, which is equal to 1 if $\eta \in A$ and 0 if $\eta \notin A$, is denoted by $\chi_A(\eta)$. The following remarks on harmonic measure will also be necessary.

Remark 5.2.
(1) Let G be a domain in $\mathbb{C}$ with non-polar boundary. Then for every Borel subset A of ∂G,

$$\omega(\eta,\, A,\, G) = \sup\{v(\eta) : v^* \le \chi_A \text{ is subharmonic on } G\}$$

for every $\eta \in G$ (see [R, Theorem 4.3.3]).

If $A \subset \partial G$ is relatively open in ∂G and ζ is a regular boundary point of G so that ζ is either in A or in $\partial G \setminus \overline{A}$, then by [R, Theorem 4.3.4],

$$\lim_{\eta \to \zeta} \omega(\eta,\, A,\, G) = \chi_A(\zeta).$$

If G is a bounded simply connected domain in $\mathbb{C}$, then all the points of the boundary of G are regular (see [R, Theorem 4.2.1]).
(2) If G is a simply connected domain with locally connected boundary and $p : U \to G$ is a conformal mapping onto G, then by [Po, Theorem 2.1], p extends continuously to $\overline{U}$. If $A \subset \partial G$ is a Borel measurable subset, then

$$\omega(\eta,\, A,\, G) = \omega(p^{-1}(\eta),\, p^{-1}(A),\, U)$$

for all $\eta \in G$ (see [Po, p. 86]).

Theorem 5.3. *Let $0 < r \leq 1 < \delta < 2$, $z_0 \in B_r$ and $f \in \mathcal{A}(z_0, B_\delta)$. Then there exists a conformal mapping $p : \overline{U} \to U$ so that the function $g = f \circ p$ maps $\overline{U}$ into $\overline{B}_r$, $g(0) = z_0$ and for every open ball K in $\mathbb{C}^n$,*

$$\mu_f(K) \leq \mu_g(K) + \frac{8}{\delta^2}(\delta^2 - r^2).$$

Proof. Let H be the connected component of $f^{-1}(B_r)$ in $\overline{U}$ that contains the origin. Note that ∂H is a semi-analytic subset of $\overline{U}$. As Theorem 6.5.12 in [KrP] implies, ∂H is locally connected. By the maximum principle, the set H is simply connected. Take a conformal mapping $p : U \to H$ such that $p(0) = 0$. It follows that the mapping p extends continuously to a mapping of $\overline{U}$ onto $\overline{H}$ (see [Po, Theorem 2.1]). Thus, the function defined by $g(\eta) = f(p(\eta))$ maps $\overline{U}$ into $\overline{B}_r$ and $g(0) = z_0$.

We consider the sets

$$F_\varepsilon = f^{-1}(K_\varepsilon) \cap \partial U, \quad G_\varepsilon = g^{-1}(K_\varepsilon) \cap \partial U, \quad J_\varepsilon = f^{-1}(K_\varepsilon) \cap \partial H$$

for $\varepsilon \geq 1$. To prove the theorem we will first compare the functions $\omega^*(\eta, J_\varepsilon, H)$ and $\omega^*(\eta, F_1, U)$ for $\eta \in \partial H$. One can write ∂H as the union of two disjoint sets Γ_1 and Γ_2, where

$$\Gamma_1 = \partial U \cap \partial H \quad \text{and} \quad \Gamma_2 = U \cap \partial H.$$

Let $\varepsilon > 1$. If $\eta \in J_\varepsilon$, then by (1) in Remark 5.2,

$$\omega^*(\eta, F_1, U) \leq \lim_{\zeta \to \eta} \omega(\zeta, J_\varepsilon, H) = 1.$$

There remain two other cases to investigate. First let us suppose that $\eta \in \Gamma_1 \setminus J_\varepsilon$. Then $\eta \in \partial U \setminus \overline{F}_1$; hence using Remark 5.2 once more,

$$\lim_{\zeta \to \eta} \omega(\zeta, F_1, U) = 0 \leq \omega^*(\eta, J_\varepsilon, H).$$

Finally suppose that $\eta \in \Gamma_2 \setminus J_\varepsilon$. Define the set A as in Lemma 5.1 by

$$A = \{\zeta \in \partial U : |f(\zeta) - f(\eta)| > \varepsilon\}.$$

Note that $F_1 \subset A$ since $f(\eta) \notin K_\varepsilon$ and that $\eta \in \Gamma_2$ implies $f(\eta) \in \partial B_r$. Let c be a conformal mapping of U onto U such that $c(0) = \eta$. Then

$$c^{-1}(A) = \{\zeta \in \partial U : |f \circ c(\zeta) - f \circ c(0)| > \varepsilon\}.$$

We use the fact that the harmonic measure is invariant under conformal mappings of U onto U and apply Lemma 5.1 to the function $f \circ c$ to get

$$\omega(\eta, F_1, U) \leq \omega(\eta, A, U) = \omega(0, c^{-1}(A), U) \leq \frac{8}{\delta^2 \varepsilon^2}(\delta^2 - r^2).$$

Therefore, we have shown that for all $\eta \in \partial H$,

$$\omega^*(\eta, F_1, U) - \frac{8}{\delta^2 \varepsilon^2}(\delta^2 - r^2) \leq \omega^*(\eta, J_\varepsilon, H).$$

Now both functions in the last inequality are harmonic in H; thus the inequality holds for all $\eta \in \overline{H}$, in particular for $\eta = 0$. Hence

$$\mu_f(K) = \omega(0, F_1, U) \leq \omega(0, J_\varepsilon, H) + \frac{8}{\delta^2 \varepsilon^2}(\delta^2 - r^2). \tag{8}$$

Since $G_\varepsilon = p^{-1}(J_\varepsilon)$ and $p^{-1}(0) = 0$, by (2) in Remark 5.2,

$$\omega(0, J_\varepsilon, H) = \omega(0, G_\varepsilon, U) = \mu_g(K_\varepsilon).$$

Therefore

$$\mu_f(K) \le \mu_g(K_\varepsilon) + \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2).$$

Letting $\varepsilon \to 1$,

$$\mu_f(K) \le \mu_g(K) + \frac{8}{\delta^2}(\delta^2 - r^2).$$

□

Following the terminology in [P3] we first extend the definition of harmonic measure to subsets of $\overline{U}$. Let $A \subset \overline{U}$ be a closed set. We consider the function

$$\omega(\zeta,\, A,\, U) = \liminf_{\xi \to \zeta,\, \xi \in U} \inf\{v(\xi) : v_* \ge \chi_A \text{ is superharmonic on } U\}.$$

If E is a relatively open subset of $\overline{U}$, then

$$\omega(\zeta,\, E,\, U) = \sup \omega(\zeta, A, U),$$

where the supremum is taken over all closed subsets A of E.

The next definitions and Theorem 5.4 are due to Poletsky ([P3, Theorem 2.1]).

Let $f_j \in \mathcal{A}(z_j,\, D)$ for some points $z_j \in D$. *The cluster* $\operatorname{cl} L$ of the sequence $L = \{f_j\}$ is the set of all points $z \in D$ such that for every $r > 0$ and infinitely many j the sets $f_j(\overline{U}) \cap B(z,\, r) \ne \emptyset$. A point $z \in \operatorname{cl} L$ is called *essential* if

$$\limsup_{j \to \infty} \omega(0,\, f_j^{-1}(V),\, U) > 0$$

for every open set V containing z. The set of essential points of L is denoted by $\operatorname{ess} L$. Other points in $\operatorname{cl} L$ are called *nonessential*.

Theorem 5.4. *Let $z_j \in D$ be points converging to $z \in \overline{D}$, and let $f_j \in \mathcal{A}(z_j,\, D)$ be holomorphic mappings such that μ_{f_j} converges to a measure $\mu \in \hat{J}_z$. Let $L = \{f_j\}$. Then there exist conformal mappings $q_j : U \to U$ so that the functions $g_j = f_j \circ q_j$ belong to $\mathcal{A}(z_j,\, D)$ and if $M = \{g_j\}$, then $\operatorname{cl} M = \operatorname{ess} L = \operatorname{ess} M$ and the μ_{g_j} converge weak-$*$ to μ.*

Theorem 5.4 and Lemma 5.1 allow us to prove the following result.

Theorem 5.5. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose the set $V = D \cap B$ is non-empty. Let $z_0 \in \overline{V}$, $z_j \in V$ be points in V converging to z_0 and $f_j \in \mathcal{A}(z_j,\, D)$ be holomorphic mappings so that μ_{f_j} converges weak-$*$ to a measure μ with* $\operatorname{supp} \mu \subset \overline{V}$. *Then there exist conformal mappings $p_j : U \to U$ so that the functions $g_j = f_j \circ p_j$ belong to $\mathcal{A}(z_j,\, V)$ and μ_{g_j} converges weak-$*$ to μ.*

Proof. Using Theorem 5.4 we can find conformal mappings $q_j : U \to U$ and replace f_j by $f_j \circ q_j$. So without loss of generality we may assume that for $L = \{f_j\}$, $\operatorname{cl} L = \operatorname{ess} L$. First let us show that the set $\operatorname{cl} L$ is contained in $\overline{B}$.

Take numbers $c > 0$, $\varepsilon > 0$ and a point z_1 such that $K = B(z_1,\, \varepsilon) \subset E_c = \{z : |z| \ge e^c\}$. Let

$$u_j(\zeta) = \log|f_j(\zeta)|$$

for any $\zeta \in \overline{U}$. Since D is bounded there exists a number $M > 0$ so that $|f_j(\zeta)| \le e^M$ for every $\zeta \in \overline{U}$ and $j \ge 1$. Let $\varepsilon_j > 0$ be numbers decreasing to 0. Take a harmonic function h_j on U so that if $\zeta \in \partial U$, then $h_j(\zeta) = \varepsilon_j$ when $\log|f_j(\zeta)| < \varepsilon_j$ and $h_j(\zeta) = M$ otherwise. Let

$$A_j = \{\zeta \in \partial U : |f_j(\zeta)| < e^{\varepsilon_j}\}.$$

Since the sequence $\{\mu_{f_j}\}$ converges weak-$*$ to a measure supported in $\overline{B}$, the arclengths $l(A_j)$ of the sets A_j converge to 1. This implies that $h_j(0)$ decreases to 0. Hence if $F_c = \{\zeta \in \overline{U} : h_j(\zeta) \geq c\}$, then $h_j/c \geq 1$ on F_c and

$$\lim_{j\to\infty} \omega(0,\, F_c,\, U) = 0.$$

Note that $u_j \leq h_j$. It follows that $f_j^{-1}(K) \subset f_j^{-1}(E_c) \subset F_c$. Then

$$\lim_{j\to\infty} \omega(0,\, f_j^{-1}(K),\, U) = 0.$$

That is, z_1 is not an essential point of L and $z_1 \notin \operatorname{cl} L$. Therefore $\operatorname{cl} L \subset \overline{B}$.

Now it is possible to see that for every number $\delta > 1$ there exists an integer $j_0 \geq 1$ such that $f_j(\overline{U}) \subset B_\delta$ for every $j \geq j_0$. If not, then there exist $\delta > 1$ and $\zeta_j \in \overline{U}$ so that $z_j = f_j(\zeta_j) \notin B_\delta$ for all j. Since $\overline{D} \setminus B_\delta$ is compact, a subsequence $\{z_{j_k}\}$ converges to a point $z \in \overline{D} \setminus B_\delta$. Then $z \in \operatorname{cl} L$, but not in $\overline{B}$, which is a contradiction.

There exist non-increasing numbers $\delta_j > 1$ converging to 1 such that each f_j maps $\overline{U}$ into B_{δ_j}. Let $r_j > 0$ be an increasing sequence of numbers converging to 1 such that $z_j \in B_{r_j}$. By Theorem 5.3 there exist conformal maps p_j from $\overline{U}$ into $\overline{U}$ so that for each j the function $g_j = f_j \circ p_j$ maps $\overline{U}$ into $\overline{B}_{r_j} \cap D \subset V$ and for every open ball K in $\mathbb{C}^n$,

$$\mu_{f_j}(K) \leq S_j = \mu_{g_j}(K) + \frac{8}{{\delta_j}^2}({\delta_j}^2 - r_j^2).$$

Thus $g_j \in \mathcal{A}(z_j,\, V)$.

Suppose there exist a subsequence $\{j_k\}$, an open ball K_0 and a number $a > 0$ such that $\mu(K_0) \,+\, a < S_{j_k}$ for all k. Then a subsequence of $\mu_{g_{j_k}}$, which we call $\{\mu_{g_{j_k}}\}$ again, converges weak-$*$ to a measure μ_0. As $k \to \infty$, $\mu(K) \,+\, a \leq \mu_0(K)$ and for any other open ball K, $\mu(K) \leq \mu_0(K)$. This is not possible since $\mu(\overline{V}) = \mu_0(\overline{V}) = 1$. Thus S_j has a limit as $j \to \infty$ for every open ball K in $\mathbb{C}^n$ and the limit is equal to $\mu(K)$. This implies μ_{g_j} converges weak-$*$ to μ and the proof is finished. $\square$

To prove Lemma 5.1 we used the fact that on the open unit ball in $\mathbb{C}^n$ one can find a plurisubharmonic function that has a uniform estimate as in (7). Our proof does not work if we replace the open ball by an arbitrary open set in $\mathbb{C}^n$. As the following example shows we cannot even take the open polydisk instead of the open ball.

Example 5.6. Consider the open polydisk $U \times U$ in $\mathbb{C}^2$. Let

$$s_j = 1 + \frac{1}{\sqrt{j}}.$$

Then s_j is decreasing to 1 and

$$\lim_{j\to\infty} s_j^j = \infty.$$

Let $z_0 = (0,\, 0)$ and

$$f_j(\zeta) = (s_j \zeta,\, \zeta^j)$$

for all $\zeta \in \overline{U}$. Then $\mu_{f_j} \in \mathcal{H}_{z_0}(U_{s_j^2} \times U_{s_j})$. Let p_j be a conformal mapping from $\overline{U}$ into $\overline{U}$ so that $g_j = f_j \circ p_j$ maps $\overline{U}$ into $U \times U$. Note that μ_{f_j} weak-$*$ converges to the measure

$$\mu = \frac{1}{4\pi^2} d\alpha\, d\beta$$

on $\partial U \times \partial U$, where $d\alpha$, $d\beta$ denote the arclength measures on ∂U. On the other hand since $|s_j p_j(\zeta)| \leq 1$ for every $\zeta \in \overline{U}$,

$$(|p_j(\zeta)|)^j \leq \frac{1}{s_j^j}.$$

Hence $|p_j(\zeta)|^j$ converges uniformly to 0 on $\overline{U}$. It follows that if $0 < \varepsilon < 1$ and $K = B(1, \varepsilon)$, then $K \cap g_j(\overline{U}) = \emptyset$ for j large enough. Thus if μ_0 is the weak-$*$ limit of a subsequence of μ_{g_j}, then $\mu_0(K) = 0 < \mu(K)$.

The following characterization of c-regular domains was proved in [Go, Corollary 4.4].

Theorem 5.7. *A bounded domain D in $\mathbb{C}^n$ is c-regular if and only if for all $z \in \partial D$, $\mathcal{J}_z^b = \mathcal{J}_z^c$ if and only if for all $z \in \partial D$, measures $\mu \in \hat{\mathcal{J}}_z$ and for any sequence of points $\{z_j\} \subset D$ converging to z there exist measures $\mu_j \in \overline{\mathcal{H}}_{z_j}$ that converge weak-$*$ to μ.*

Now let us show that if a domain is c-regular, then it is locally c-regular.

Theorem 5.8. *A c-regular domain is locally c-regular.*

Proof. Let D be a c-regular domain. Take any point $\zeta \in \partial D$ and a ball B_0 so that $V = D \cap B_0 \neq \emptyset$. Without loss of generality we will assume that B_0 is the unit ball B in $\mathbb{C}^n$. We will show that the set V is c-regular.

If $z \in \partial V$, then either $z \in \partial B$ or $z \in B$. For all $z \in \partial B$, the classes $\mathcal{J}_z^b$ and $\mathcal{J}_z^c$ coincide and they are equal to $\{\delta_z\}$, the Dirac measure at z (see [S] and also [W, Corollary 3.8]). Therefore we may assume that $z \in B \cap \partial V$. Suppose there exist $\mu \in \hat{\mathcal{J}}_z(V)$ and a sequence of points $\{z_j\} \subset V$ converging to z. Since D is c-regular and $\hat{\mathcal{J}}_z(V) \subset \hat{\mathcal{J}}_z(D)$ there exist measures $\mu_j \in \overline{\mathcal{H}}_{z_j}(D)$ that converge weak-$*$ to μ in $C^*(\overline{D})$. We point out that the space $C^*(\overline{D})$ is metrizable (see [C, Theorem 5.1]), therefore we can find functions $f_j \in \mathcal{A}(z_j, D)$ so that the measures μ_{f_j} converge weak-$*$ to μ. By Theorem 5.5 there exist functions $g_j \in \mathcal{A}(z_j, V)$ so that the measures $\mu_{g_j} \in \mathcal{H}_{z_j}(V)$ converge weak-$*$ to μ. Thus by Theorem 5.7, V is c-regular and the assertion follows. □

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Department of Mathematics, 215 Carnegie Hall, Syracuse University, Syracuse, New York 13244

E-mail address: `nggogus@syr.edu`

Current address: Faculty of Engineering and Natural Sciences, Sabanci University, Orhanli, Tuzla, 34956, Istanbul, Turkey

E-mail address: `nggogus@sabanciuniv.edu`