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LOCAL AND GLOBAL C-REGULARITY

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ABSTRACT. A bounded domain D is called c-regular if the plurisubharmonic envelope of every continuous function on $\overline{D}$ extends continuously to $\overline{D}$. We show using Gauthier's Fusion Lemma that a domain is locally c-regular if and only if it is c-regular.

1. Introduction

The envelopes of functions play an essential role in pluripotential theory. In particular, they are used in the Perron-Bremermann method to solve the Dirichlet problem: find a homogeneous solution of the Monge-Ampère operator with continuous boundary data. So it is important to classify those domains where the solution is also continuous. In [Go] the notion of Jensen measures was used to completely characterize the domains where continuous functions always have continuous envelopes. Such domains are called c-regular. The problem whether c-regularity is a local property has remained unanswered. In this paper we show that a domain is c-regular if and only if it is locally c-regular.

To prove that a locally c-regular domain is c-regular we need a version of Gauthier's Fusion Lemma. Roughly speaking, given a finite number of plurisubharmonic functions v_j on subdomains V_j of a domain D there exists a plurisubharmonic function u on D such that the difference $|u - v_j|$ on V_j is controlled by the sum of the differences of the v_j's. The proof of this form of the Fusion Lemma is given in §3. In §4 we construct an operator $\mathcal{L}$ on $C(\overline{D})$ that arises by taking envelopes of a given $\phi \in C(\overline{D})$ on the V_j's. We prove that the iterates of $\mathcal{L}\phi$ pointwise decrease to the envelope of ϕ on $\overline{D}$. To show that the iterates converge uniformly to a continuous function on $\overline{D}$ we consider the Krasnoselski-Mann iterations of $\mathcal{L}$. A beautiful result of Ishikawa in [I] provides a criteria for such iterations to converge uniformly on $\overline{D}$. It turns out that it is possible to establish Ishikawa's criteria once we prove the Fusion Lemma.

Finally in §5 we show that a c-regular domain is locally c-regular. The proof doesn't follow immediately from the definition. To achieve this we approximate the Jensen measures supported in a neighborhood of an open ball B in $\mathbb{C}^n$ by the Jensen measures supported in $D \cap B$.

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2. Basic notation and results

We shall denote an open ball in $\mathbb{C}^n$, $n \geq 2$, of radius r and center z by $B(z, r)$. In the special case where $z = 0$ and $r = 1$ we use B instead of $B(0, 1)$. We will write U for the open unit disc in $\mathbb{C}$. We shall denote by $d\lambda$ the normalized standard Lebesgue measure on the boundary ∂U of U. If $z, w \in \mathbb{C}^n$, then we set

$$(z, w) = \sum_{j=1}^{n} z_j \overline{w}_j.$$

If V is any set in $\mathbb{C}^n$ and f is a function on V, we put

$$||f||_V = \sup_V |f|.$$

Let Ω be a bounded open set in $\mathbb{C}^n$. For any upper (resp. lower) bounded function ψ defined on Ω, we shall denote by ψ^* (resp. by ψ_*) the upper semicontinuous (resp. the lower semicontinuous) regularisation of ψ on $\overline{\Omega}$. That is, for any $z \in \overline{\Omega}$,

$$\psi^*(z) = \limsup_{w \to z,\, w \in \Omega} \psi(w), \quad \psi_*(z) = \liminf_{w \to z,\, w \in \Omega} \psi(w).$$

A regular Borel probability measure μ on $\overline{\Omega}$ is a Jensen measure with barycenter $z \in \overline{\Omega}$ if

$$u^*(z) \leq \int u^* \, d\mu$$

for all u in the set $PSH^b(\Omega)$ of all upper bounded plurisubharmonic functions on Ω. The class of such Jensen measures will be denoted by $\mathcal{J}_z^b$. For any point $z \in \overline{\Omega}$, let

$$\hat{\mathcal{J}}_z = \hat{\mathcal{J}}_z(\Omega) = \{\mu : \mu = \lim \mu_j,\ \mu_j \in \mathcal{J}_{z_j}^b,\ z_j \to z,\ z_j \in \overline{\Omega}\},$$

where the limit above should be understood as the weak-$*$ limit in $C^*(\overline{\Omega})$.

We denote by $PSH^c(\Omega)$ the class of plurisubharmonic functions on Ω which are continuous on $\overline{\Omega}$. Given $z \in \overline{\Omega}$, we write $\mathcal{J}_z^c = \mathcal{J}_z^c(\Omega)$ for the set of all regular Borel measures μ on $\overline{\Omega}$ such that

$$u(z) \leq \int u \, d\mu$$

for every $u \in PSH^c(\Omega)$. The class $\mathcal{J}_z^c$ was introduced in [CCW] and later together with $\mathcal{J}_z^c$, $\mathcal{J}_z^b$ was studied in [W], while the class $\hat{\mathcal{J}}_z$ was introduced in [Go].

Given any function φ defined on $\overline{\Omega}$, we define the envelope

$$E^c\varphi(z) = E_\Omega^c\varphi(z) = \inf\left\{\int \varphi \, d\mu : \mu \in \mathcal{J}_z^c\right\},\ z \in \overline{\Omega}.$$

For an arbitrary bounded domain $G \subset \mathbb{C}$ and a Borel measurable subset $A \subset \partial G$, the positive harmonic measure of the set A with respect to G is denoted by $\omega(\eta, A, G)$. Recall that the function $\omega(\eta, A, G)$ is harmonic on G.

We denote by $\mathcal{A}(z_0, \Omega)$ the set of all holomorphic mappings f of a neighborhood of the closure $\overline{U}$ of the unit disk $U \subset \mathbb{C}$ into an open set $\Omega \subset \mathbb{C}^n$ such that $f(0) = z_0$. If $f \in \mathcal{A}(z_0, \Omega)$, then the measure

$$\mu_f(E) = \lambda(f^{-1}(E) \cap \partial U) = \omega(0,\, f^{-1}(E) \cap \partial U,\, U)$$

is a Jensen measure with barycenter $z_0 = f(0)$. For $z \in \Omega$, let $\mathcal{H}_z = \mathcal{H}_z(\Omega)$ be the collection of all measures μ_f such that $f \in \mathcal{A}(z, \Omega)$. Also let $\overline{\mathcal{H}}_z$ be the closure of $\mathcal{H}_z$ in the weak-$*$ topology induced from $C^*(\overline{\Omega})$. The following is a well-known result of Poletsky (see [P1] and [P2]).

Theorem 2.1. *Let Ω be an open set in $\mathbb{C}^n$. If φ is an upper semicontinuous function on Ω, then the function*

$$S\varphi(z) = \sup\{u(z) : u \leq \varphi \text{ is plurisubharmonic on } \Omega\}$$

is plurisubharmonic on Ω and equal to

$$E\varphi(z) = \inf\left\{\int \varphi\, d\mu : \mu \in \mathcal{H}_z\right\},$$

the plurisubharmonic envelope of φ.

We use the definition of c-regular domains from [Go].

Definition 2.2. A bounded open set Ω in $\mathbb{C}^n$ is called c-regular if the envelope of every continuous function on $\overline{\Omega}$ is continuous on Ω and extends continuously to $\overline{\Omega}$.

Definition 2.3. A bounded domain D in $\mathbb{C}^n$ is locally c-regular if for each $\zeta \in \partial D$ there exists an open neighborhood N of ζ such that the set $V := N \cap D$ is c-regular.

3. Fusion of plurisubharmonic functions

The next lemma is a slightly modified version of Gauthier's Fusion Lemma in [G], and the proof is a copy of Gauthier's proof.

Lemma 3.1. *Let $V \subset \mathbb{C}^n$ and $U_j \subset \mathbb{C}^n$, $j = 1, 2$, be bounded domains with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Then there exists a number $c > 0$ and for any functions $v_1 \in PSH^b(U_1 \cup V)$ and $v_2 \in PSH^b(U_2 \cup V)$ there exists a function $u \in PSH^b(U_1 \cup V \cup U_2)$ so that*

$$||u - v_j||_{U_j \cup V} \leq c||v_1 - v_2||_V. \tag{1}$$

If $v_j \in PSH^c(U_j \cup V)$, then u can be chosen to lie in $PSH^c(U_1 \cup V \cup U_2)$.

Proof. If $||v_1 - v_2||_V = 0$, then we set u equal to v_1 on $U_1 \cup V$ and v_2 on U_2. Since $v_1 = v_2$ on V, u is plurisubharmonic on $U_1 \cup V \cup U_2$. We may assume that $||v_1 - v_2||_V \neq 0$. Take $\chi_1 \in C^\infty(\mathbb{C}^n)$ with $-1 \leq \chi_1 \leq 0$, $\chi_1 = -1$ on $\overline{U}_2$ and $\chi_1 = 0$ on $\overline{U}_1$. Set $\chi_2 = -1 - \chi_1$. Let $\delta(z) = |z|^2$ and choose a number $\lambda > 0$ so small that $\delta + \lambda\chi_j$ are both plurisubharmonic for $j = 1, 2$. We let

$$\eta = \lambda^{-1}||v_1 - v_2||_V. \tag{2}$$

Set

$$u_j = v_j^* + \eta(\delta + \lambda\chi_j) \tag{3}$$

on $U_j \cup \overline{V}$ and $u_j = -\infty$ elsewhere. Finally, we set

$$u = \max\{u_1, u_2\}.$$

Clearly, u is upper bounded and plurisubharmonic on $(U_1 \cup V \cup U_2) \setminus \partial V$. Suppose $z_0 \in \partial V \cap U_1$. Since $\chi_1 = 0$ and $\chi_2 = -1$ on U_1, by (2) and (3) for all points $z \in U_1 \cap \overline{V}$ near z_0,

$$u_2(z) = v_2^*(z) + \eta(\delta - \lambda) = u_1(z) + (v_2^*(z) - v_1(z)) - \lambda\eta \leq u_1(z).$$

Since $u_2(z) = -\infty$ for $z \in U_1 \setminus \overline{V}$, we have that $u_2(z) \leq u_1(z)$ for all $z \in U_1$ near z_0. As z_0 was an arbitrary point of $\partial V \cap U_1$, u is upper bounded and plurisubharmonic on a neighborhood of $\partial V \cap U_1$. A similar argument shows that u is upper bounded and plurisubharmonic on a neighborhood of $\partial V \cap U_2$. Thus $u \in PSH^b(U_1 \cup V \cup U_2)$. If $v_j \in PSH^c(U_j \cup V)$, then we define the u_j's the same way on $\overline{U}_j \cup \overline{V}$. By the construction, u is in $PSH^c(U_1 \cup V \cup U_2)$.

We now need to verify the required estimates. Let M be the supremum of δ on the set $U_1 \cup V \cup U_2$. On $U_j \setminus V$, $j = 1, 2$,

$$|u(z) - v_j(z)| = |u_j(z) - v_j(z)| = \eta\delta(z) \leq \lambda^{-1}M||v_1 - v_2||_V.$$

Take $z \in V$ and for $j = 1$ or 2 suppose first that $|u(z) - v_j(z)| = u(z) - v_j(z)$. Since $|\delta(z) + \lambda\chi_j(z)| \leq M + \lambda$,

$$\begin{aligned} |u(z) - v_j(z)| &\leq |\max\{v_1(z), v_2(z)\} - v_j(z)| + \eta(M + \lambda) \\ &\leq (\lambda^{-1}M + 2)||v_1 - v_2||_V. \end{aligned}$$

If $|u(z) - v_j(z)| = v_j(z) - u(z)$, then

$$\begin{aligned} |u(z) - v_j(z)| &= v_j(z) - \max\{f_1, f_2\} \\ &\leq v_j(z) - \max\{v_1(z), v_2(z)\} + \eta(M + \lambda) \\ &\leq (\lambda^{-1}M + 2)||v_1 - v_2||_V. \end{aligned}$$

Thus, for $c = \lambda^{-1}M + 2$, (1) holds. This completes the proof. $\square$

The fusion lemma allows us to fuse plurisubharmonic functions over several domains.

Theorem 3.2. *Let D be a bounded domain and N_j, $j = 1, \ldots, m$, be open sets in $\mathbb{C}^n$ so that*

$$\overline{D} \subset \bigcup_{j=1}^{m} N_j.$$

For any integer $j = 1, \ldots, m$, let $V_j = D \cap N_j$. Then there exists a constant $c > 0$ (depending only on the N_j's) so that if $v_j \in PSH^b(V_j)$, then there exists a function $u \in PSH^b(D)$ such that for all $j = 1, \ldots, m$,

$$||u - v_j||_{V_j} \leq c \sum_{1 \leq k < l \leq m} ||v_k - v_l||_{V_k \cap V_l}.$$

Moreover, if $v_j \in PSH^c(V_j)$ for all $j = 1, \ldots, m$, then the function u can be taken to lie in $PSH^c(D)$.

Proof. The proof is by induction on the number m. We start with $m = 2$. We fix open subsets $L_j \Subset N_j$ with

$$\overline{D} \subset \bigcup_{j=1}^{2} L_j.$$

Set $A_j = D \cap L_j$. Let $U_1 = D \setminus \overline{L}_2$ and $U_2 = D \setminus \overline{L}_1$. Then $\overline{U}_1 \cap \overline{U}_2 = \overline{D} \setminus (L_1 \cup L_2) = \emptyset$. Also let $V = V_1 \cap V_2$. Note that $V_j = U_j \cup V$ and thus $v_j \in PSH^b(U_j \cup V)$, $j = 1, 2$. Apply Lemma 3.1 to v_j to get a constant $c > 0$ and a function $u \in PSH^b(D)$ so that for all $z \in V_j$,

$$|u(z) - v_j(z)| \leq c||v_1 - v_2||_V.$$

If $v_j \in PSH^c(V_j)$, then by Lemma 3.1 the function u is in $PSH^c(D)$.

We have thus established the beginning of an inductive argument. Suppose that we have shown the theorem when $\overline{D}$ is contained in the union of m open sets for

$m \geq 2$. We may now assume that $\overline{D}$ is contained in the union of open sets N_j, $j = 1, \ldots, m+1$. There exist open subsets $L_j \Subset N_j$, $j = 1, \ldots, m$, with

$$\overline{D} \subset N_{m+1} \cup \left(\bigcup_{j=1}^{m} L_j \right).$$

Let $D_m = D \cap \left(\bigcup_{j=1}^{m} L_j \right)$ and $A_j = D \cap L_j$. Since $\overline{D}_m \subset \bigcup_{j=1}^{m} N_j$, by the induction assumption there exist a function $u_0 \in PSH^b(D_m)$ and a constant $c_m > 0$ such that

$$|u_0(z) - v_j(z)| \leq c_m \sum_{1 \leq k < l \leq m} ||v_k - v_l||_{V_k \cap V_l}$$

for all $z \in A_j$.

Hence we can use the first part of the proof to fuse u_0 and v_{m+1} on D. There exist a function $u \in PSH^b(D)$ and a constant $c_0 > 0$ such that for all $z \in D_m$,

$$|u(z) - u_0(z)| \leq c_0 ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}}$$

and for all $z \in V_{m+1}$,

$$|u(z) - v_{m+1}(z)| \leq c_0 ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}}.$$

We only need to check the estimates. Take a number $\delta > 0$ and a point $z \in D_m \cap V_{m+1}$ so that

$$||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} < |u_0(z) - v_{m+1}(z)| + \delta.$$

Then $z \in A_k$ for some $k \in \{1, \ldots, m\}$ and

$$\begin{aligned} ||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} &< |u_0(z) - v_{m+1}(z)| + \delta \\ &\leq |u_0(z) - v_k(z)| + |v_k(z) - v_{m+1}(z)| + \delta \\ &\leq (c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l} + \delta. \end{aligned}$$

Since this is true for all $\delta > 0$,

$$||u_0 - v_{m+1}||_{D_m \cap V_{m+1}} \leq (c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}.$$

If $z \in V_j$ for some $j \in \{1, \ldots, m\}$, then there exists k with $z \in A_k$. Therefore

$$\begin{aligned} |u(z) - v_j(z)| &\leq |u(z) - u_0(z)| + |u_0(z) - v_k(z)| + |v_k(z) - v_j(z)| \\ &\leq (c_0 + 1)(c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}. \end{aligned}$$

If $z \in V_{m+1}$, then

$$|u(z) - v_{m+1}(z)| \leq c_0(c_m + 1) \sum_{1 \leq k < l \leq m+1} ||v_k - v_l||_{V_k \cap V_l}.$$

Set $c = (c_0 + 1)(c_m + 1)$. The continuity of u on $\overline{D}$ follows immediately from the construction when all the v_k's are continuous on $\overline{V}_k$. $\square$

4. Localization of envelopes

We give here a general argument to approximate the plurisubharmonic envelope of functions on a domain by envelopes with respect to the subdomains of the domain. Then we use this result and the fusion result of the previous section to show that local c-regularity of the boundary implies global c-regularity. The following lemma is so well known that we omit the proof.

Lemma 4.1. *Let $D \subset \mathbb{C}^n$ be a bounded domain and $N_1, \ldots, N_m$ be open subsets of $\mathbb{C}^n$ so that*

$$\overline{D} \subset \bigcup_{j=1}^{m} N_j.$$

Then there exist open subsets $L_j \Subset N_j$ and functions $\chi_j \in C^\infty(\mathbb{C}^n)$, $j = 1, \ldots, m$, so that

(p1) *$\overline{D} \subset \bigcup_{j=1}^{m} L_j$ and $\sum_{j=1}^{m} \chi_j = 1$ on $\overline{D}$,*
(p2) *$\chi_j = 0$ on $\overline{D} \setminus N_j$ and $\chi_j \geq 1/m$ on $\overline{L}_j \cap \overline{D}$.*

Let D, N_j and χ_j be as in Lemma 4.1. Set $V_j = D \cap N_j$. If ϕ is a bounded function on $\overline{D}$, we define

$$\mathcal{L}\phi = \sum_{j=1}^{m} \chi_j (E_{V_j}\phi)^*. \tag{4}$$

Thus if we denote by $\mathcal{B}$ the set of bounded functions on $\overline{D}$, then we have an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$. Note that for any $\phi \in \mathcal{B}$, $\mathcal{L}\phi \leq \phi$ and $\mathcal{L}$ is monotonic in the sense that $\varphi \leq \psi$ implies $\mathcal{L}\varphi \leq \mathcal{L}\psi$. We set $\mathcal{L}^0\phi = \phi$ and for any integer $k \geq 1$,

$$\mathcal{L}^k\phi = \mathcal{L}(\mathcal{L}^{k-1}\phi).$$

We endow the set $\mathcal{B}$ with the supremum norm

$$||f||_{\overline{D}} = \sup_{\overline{D}} |f|.$$

Thus $\mathcal{B}$ together with this norm is a Banach space and $\mathcal{L}$ is a nonlinear operator on $\mathcal{B}$. Let $\mathcal{S}$ be the set of bounded functions on $\overline{D}$ so that

$$\phi(z) = \limsup_{w \to z,\, w \in \overline{D}} \phi(w)$$

for all $z \in \overline{D}$. That is, $\mathcal{S}$ is the set of bounded upper semicontinuous functions on $\overline{D}$. Then $\mathcal{S}$ is a closed subset of $\mathcal{B}$. The operator $\mathcal{L}$ maps $\mathcal{S}$ into $\mathcal{S}$. It's interesting to observe that the fixed points of $\mathcal{L}$ in $\mathcal{S}$ are precisely the bounded plurisubharmonic functions on D.

Theorem 4.2. *If ϕ is in $\mathcal{S}$, then $\phi = (E\phi)^*$ on $\overline{D}$ if and only if $\mathcal{L}\phi = \phi$ on $\overline{D}$.*

Proof. If $\phi = (E\phi)^*$ on $\overline{D}$, then $\phi = (E\phi)^* \leq (E_{V_j}\phi)^* \leq \phi$ on $\overline{V}_j$. Hence $\phi = (E_{V_j}\phi)^*$ on $\overline{V}_j$ for each j. Therefore

$$\mathcal{L}\phi = \chi_1(E_{V_1}\phi)^* + \ldots + \chi_m(E_{V_m}\phi)^* = \phi.$$

Suppose now that $\mathcal{L}\phi = \phi$. Take $z_0 \in D$. Let $J = \{j : z_0 \in V_j\}$ and $V = \bigcap_{j \in J} V_j$. Suppose ϕ is not plurisubharmonic around z_0. Then there exists a function $f \in \mathcal{A}(z_0, V)$ such that

$$\int \phi \, d\mu_f < \phi(z_0).$$

Since $\chi_k(z_0) = 0$ if $k \notin J$,

$$\mathcal{L}\phi(z_0) = \sum_{j \in J} \chi_j(z_0) E_{V_j}\phi(z_0) \le \int \phi \, d\mu_f < \phi(z_0),$$

a contradiction to our assumption. Therefore, ϕ is plurisubharmonic on D. Hence the equality $\phi = (E\phi)^*$ holds on $\overline{D}$. □

As a corollary, we show that the iterations of $\mathcal{L}\phi$ converge to the plurisubharmonic envelope of ϕ.

Corollary 4.3. *For any ϕ in $\mathcal{S}$,*

$$\lim_k \mathcal{L}^k \phi(z) = (E_D\phi)^*(z)$$

for all $z \in \overline{D}$.

Proof. The functions $\mathcal{L}^k\phi$ form a decreasing sequence of upper semicontinuous functions on $\overline{D}$. Then the function defined by

$$\psi = \lim_k \mathcal{L}^k \phi$$

is also in $\mathcal{S}$. Moreover,

$$\begin{aligned} \psi = \lim_k \mathcal{L}(\mathcal{L}^{k-1}\phi) &= \sum_{j=1}^{m} \chi_j \lim_k (E_{V_j}\mathcal{L}^{k-1}\phi) \\ &= \sum_{j=1}^{m} \chi_j E_{V_j}\psi = \mathcal{L}\psi. \end{aligned}$$

Hence $\psi = (E_D\psi)^*$ on $\overline{D}$ by Theorem 4.2. Note that $\mathcal{L}^k\phi \le \phi$ implies that $\psi \le \phi$. Consequently, $\psi = (E_D\psi)^* \le (E_D\phi)^*$. Also for each k, $(E_D\phi)^* \le \mathcal{L}^k\phi$; hence $(E_D\phi)^* \le \psi$ and the equality follows. □

We need some helpful concepts from functional analysis (see [I]).

If $(X, \| \, . \, \|)$ is a Banach space, $x_0 \in X$, C is a convex subset of X and $T : C \to X$ is any map, we define the Krasnoselski-Mann iterations

$$x_{k+1} = (1 - t_k)x_k + t_k T x_k, \tag{5}$$

where t_k is a sequence in $[0, 1]$. Throughout the following we will impose the conditions that $\sum_k t_k = \infty$ and $0 \le t_k \le b < 1$ for all k.

Definition 4.4. A mapping $T : C \to X$ is called nonexpansive if for all x, $y \in C$ we have

$$\| \, Tx - Ty \, \| \le \| \, x - y \, \| \, .$$

Lemma 4.5. *The operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is nonexpansive.*

Proof. Take φ and ψ in $\mathcal{B}$ and let $\delta = \| \, \varphi - \psi \, \|_{\overline{D}}$. Then on $\overline{D}$,

$$\varphi - \delta \le \psi \le \varphi + \delta;$$

therefore

$$\mathcal{L}\varphi - \delta \le \mathcal{L}\psi \le \mathcal{L}\varphi + \delta.$$

We have $|\mathcal{L}\varphi - \mathcal{L}\psi| \le \delta$ on $\overline{D}$; hence $\| \, \mathcal{L}\varphi - \mathcal{L}\psi \, \|_{\overline{D}} \le \| \, \varphi - \psi \, \|_{\overline{D}}$. □

For any subset A of X and a point $x \in X$, the distance of x to A is denoted by $\operatorname{dist}(x, A)$. That is,

$$\operatorname{dist}(x, A) := \inf\{\| x - y \| : y \in A\}.$$

In ([I]), Ishikawa proves the following very useful theorem.

Theorem 4.6. *Let A be a closed subset of a Banach space X and let $T : A \to X$ be a nonexpansive mapping with a nonempty fixed points set F in A. Suppose that $x_k \in A$ for all $k \geq 1$, where $\{x_k\}$ is the sequence of iterations defined in* (5)*. Suppose there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that*

$$\| x - Tx \| \geq f(\operatorname{dist}(x, F))$$

for all $x \in A$. Then the sequence converges to a member of F.

If $z \in D$ and $B(z, r) \subset D$ is a ball in D, then $B(z, r)$ is c-regular by [Wa]. In the rest of the section we will assume that D is a locally c-regular domain. By compactness of $\overline{D}$ this means there exists a number of open sets N_j, $j = 1, \ldots, m$, so that $\overline{D}$ is contained in the union of the N_j's and the sets $V_j = D \cap N_j$ are c-regular for each j. Throughout, $\mathcal{L}$ will denote the operator on $C(\overline{D})$ defined by (4). We remark here that the set of fixed points of our operator $\mathcal{L} : C(\overline{D}) \to C(\overline{D})$ is precisely $PSH^c(D)$ by Theorem 4.2. For any function $\varphi \in C(\overline{D})$, let $\varphi_1 = \varphi$ and

$$\varphi_{k+1} = (1 - t_k)\varphi_k + t_k \mathcal{L}\varphi_k$$

for $k \geq 1$. Now we can prove the following:

Theorem 4.7. *Let D be a bounded locally c-regular domain in $\mathbb{C}^n$. Then there exists a constant $c > 0$ so that for all $\varphi \in C(\overline{D})$,*

$$\operatorname{dist}(\varphi, PSH^c(D)) \leq c \| \varphi - \mathcal{L}\varphi \|_{\overline{D}} .$$

Proof. There exist open sets $N_1, \ldots, N_m$ in $\mathbb{C}^n$ such that $\overline{D} \subset \bigcup_{j=1}^m N_j$ and for each j, $V_j = N_j \cap D$ is c-regular. Let φ be a function in $C(\overline{D})$. The functions $v_j = E_{V_j}\varphi$ are now in $PSH^c(V_j)$.

As in Lemma 4.1 we find open subsets $L_j \Subset N_j$ and a partition of unity χ_j for $\overline{D}$ satisfying (p1) and (p2). Let $A_j = L_j \cap D$. Then using (p2), it's not difficult to see that for each $j = 1, \ldots, m$,

$$\| \varphi - v_j \|_{\overline{A}_j} < m \| \varphi - \mathcal{L}\varphi \|_{\overline{D}} . \tag{6}$$

Let

$$d_j = \operatorname{dist}(\varphi, PSH^c(A_j))$$

and find functions $u_j \in PSH^c(A_j)$ so that

$$||u_j - \varphi||_{\overline{A}_j} \leq 2d_j.$$

Using (6),

$$\max_{1 \leq j \leq m} d_j \leq \max_{1 \leq j \leq m} ||\varphi - v_j||_{\overline{A}_j} \leq m||\varphi - \mathcal{L}\varphi||_{\overline{D}}.$$

Now we fuse the functions $u_1, \ldots, u_m$ on $\overline{D}$. By Theorem 3.2, there exist a function $u \in PSH^c(D)$ and a constant $c_0 > 0$ so that

$$||u - u_j||_{A_j} \leq c_0 \sum_{1 \leq k < l \leq m} ||u_k - u_l||_{\overline{A_k \cap A_l}}$$

for each $j = 1, \ldots, m$. On the other hand, since $||u_j - \varphi||_{\overline{A_j}} \leq 2d_j$,

$$||u_k - u_l||_{\overline{A_k \cap A_l}} \leq 4 \max_{1 \leq j \leq m} d_j.$$

Taking the sum over such terms,

$$\sum_{1 \leq k < l \leq m} ||u_k - u_l||_{\overline{A_k \cap A_l}} \leq 2m(m-1) \max_{1 \leq j \leq m} d_j.$$

Then

$$\begin{aligned}
\operatorname{dist}(\varphi,\, PSH^c(D)) &\leq ||u - \varphi||_{\overline{D}} \leq \max_{1 \leq j \leq m} (||u - u_j||_{\overline{A_j}} + ||u_j - \varphi||_{\overline{A_j}}) \\
&\leq (2c_0 m(m-1) + 2) \max_{1 \leq j \leq m} d_j \\
&\leq (2c_0 m^2(m-1) + 2m)||\varphi - \mathcal{L}\varphi||_{\overline{D}}.
\end{aligned}$$

Now we set $c = 2c_0 m^2(m-1) + 2m$ to finish the proof. ☐

Using the above theorem and Ishikawa's result, it's now possible to prove that if a domain is locally c-regular, then it is c-regular.

Theorem 4.8. *If a domain D is locally c-regular, then it is c-regular.*

Proof. Let D be locally c-regular and $\varphi \in C(\overline{D})$. Recall that we define $\varphi_1 = \varphi$ and for every integer $k \geq 1$,

$$\varphi_{k+1} = (1 - t_k)\varphi_k + t_k \mathcal{L}\varphi_k.$$

Let us show by induction that for all $k \geq 1$,

$$\mathcal{L}^k \varphi \leq \varphi_k \leq \varphi.$$

If $k = 1$, then

$$\mathcal{L}\varphi \leq (1 - t_1)\varphi + t_1 \mathcal{L}\varphi = \varphi_1 \leq \varphi.$$

Suppose for some k, $\mathcal{L}^k \varphi \leq \varphi_k \leq \varphi$. Then

$$\mathcal{L}^{k+1}\varphi = \mathcal{L}\mathcal{L}^k \varphi \leq \mathcal{L}\varphi_k \leq \varphi_{k+1} \leq \varphi.$$

By Theorem 4.7 and Theorem 4.6 the sequence of iterations φ_k converges uniformly on $\overline{D}$ to a function $u \in PSH^c(D)$. On the other hand by Corollary 4.2 the sequence of functions $\mathcal{L}^k\varphi$ decreases pointwise to $(E\varphi)^*$ on $\overline{D}$. Thus

$$(E\varphi)^* \leq u \leq \varphi.$$

Since u is in $PSH^c(D)$, it follows that $u = (E\varphi)^* \in PSH^c(D)$. ☐

5. Locally c-regular domains

The purpose of this section is to prove that global c-regularity is sufficient for local c-regularity. For any open ball $K = B(z, r)$ in $\mathbb{C}^n$ and a number $\delta > 1$ we let $K_\delta = B(z, \delta r)$.

The main idea of the proof is contained in the following observations.

Lemma 5.1. *Let $1 < \delta < 2$, $0 < r \leq 1$ and $g : \overline{U} \to B_\delta$ be a holomorphic map from a neighborhood of $\overline{U}$ into B_δ such that $g(0) = z_0$ for some $z_0 \in \partial B_r$. Given a number $\varepsilon > 0$, we define the set*

$$A = \{\eta \in \partial U : |g(\eta) - g(0)| > \varepsilon\}.$$

Then

$$\omega(0,\, A,\, U) = \lambda(A) \leq \frac{8}{(\delta\varepsilon)^2}(\delta^2 - r^2).$$

Proof. The function

$$u(z) = \frac{1}{\delta^2}\mathbf{Re}\,(z,\, z_0)$$

is plurisubharmonic and $u(z) \leq 1$ on B_δ. Take any point $z \in B_\delta$ with $\varepsilon < |z - z_0|$. Then

$$\varepsilon^2 < |z - z_0|^2 \leq |z|^2 - 2\mathbf{Re}\,(z,\, z_0) + 1.$$

Since $\delta < 2$,

$$u(z) = \frac{1}{\delta^2}\mathbf{Re}\,(z,\, z_0) < 1 - \frac{\varepsilon^2}{8}. \tag{7}$$

Let $v(\eta) = u(g(\eta))$ for $\eta \in \overline{U}$. The function v is subharmonic on U, $v \leq 1$ on $\overline{U}$, $v|_A \leq 1 - \varepsilon^2/8$ and

$$\begin{aligned}\frac{r^2}{\delta^2} &= u(z_0) = v(0) \leq \int_{\eta\in\partial U} v(\eta)\, d\lambda(\eta) \\ &= \int_A v(\eta)\, d\lambda(\eta) + \int_{\partial U\setminus A} v(\eta)\, d\lambda(\eta) \\ &\leq (1 - \varepsilon^2/8)\lambda(A) + 1 - \lambda(A) = 1 - (\varepsilon^2/8)\lambda(A).\end{aligned}$$

Hence

$$\lambda(A) \leq \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2).$$

□

If A is any subset of ∂G, the characteristic function of A, which is equal to 1 if $\eta \in A$ and 0 if $\eta \notin A$, is denoted by $\chi_A(\eta)$. The following remarks on harmonic measure will also be necessary.

Remark 5.2.
(1) Let G be a domain in $\mathbb{C}$ with non-polar boundary. Then for every Borel subset A of ∂G,

$$\omega(\eta,\, A,\, G) = \sup\{v(\eta) : v^* \leq \chi_A \text{ is subharmonic on } G\}$$

for every $\eta \in G$ (see [R, Theorem 4.3.3]).

If $A \subset \partial G$ is relatively open in ∂G and ζ is a regular boundary point of G so that ζ is either in A or in $\partial G \setminus \overline{A}$, then by [R, Theorem 4.3.4],

$$\lim_{\eta\to\zeta} \omega(\eta,\, A,\, G) = \chi_A(\zeta).$$

If G is a bounded simply connected domain in $\mathbb{C}$, then all the points of the boundary of G are regular (see [R, Theorem 4.2.1]).
(2) If G is a simply connected domain with locally connected boundary and $p : U \to G$ is a conformal mapping onto G, then by [Po, Theorem 2.1], p extends continuously to $\overline{U}$. If $A \subset \partial G$ is a Borel measurable subset, then

$$\omega(\eta,\, A,\, G) = \omega(p^{-1}(\eta),\, p^{-1}(A),\, U)$$

for all $\eta \in G$ (see [Po, p. 86]).

Theorem 5.3. *Let $0 < r \le 1 < \delta < 2$, $z_0 \in B_r$ and $f \in \mathcal{A}(z_0, B_\delta)$. Then there exists a conformal mapping $p : \overline{U} \to U$ so that the function $g = f \circ p$ maps $\overline{U}$ into $\overline{B}_r$, $g(0) = z_0$ and for every open ball K in $\mathbb{C}^n$,*

$$\mu_f(K) \le \mu_g(K) + \frac{8}{\delta^2}(\delta^2 - r^2).$$

Proof. Let H be the connected component of $f^{-1}(B_r)$ in $\overline{U}$ that contains the origin. Note that ∂H is a semi-analytic subset of $\overline{U}$. As Theorem 6.5.12 in [KrP] implies, ∂H is locally connected. By the maximum principle, the set H is simply connected. Take a conformal mapping $p : U \to H$ such that $p(0) = 0$. It follows that the mapping p extends continuously to a mapping of $\overline{U}$ onto $\overline{H}$ (see [Po, Theorem 2.1]). Thus, the function defined by $g(\eta) = f(p(\eta))$ maps $\overline{U}$ into $\overline{B}_r$ and $g(0) = z_0$.

We consider the sets

$$F_\varepsilon = f^{-1}(K_\varepsilon) \cap \partial U, \quad G_\varepsilon = g^{-1}(K_\varepsilon) \cap \partial U, \quad J_\varepsilon = f^{-1}(K_\varepsilon) \cap \partial H$$

for $\varepsilon \ge 1$. To prove the theorem we will first compare the functions $\omega^*(\eta, J_\varepsilon, H)$ and $\omega^*(\eta, F_1, U)$ for $\eta \in \partial H$. One can write ∂H as the union of two disjoint sets Γ_1 and Γ_2, where

$$\Gamma_1 = \partial U \cap \partial H \quad \text{and} \quad \Gamma_2 = U \cap \partial H.$$

Let $\varepsilon > 1$. If $\eta \in J_\varepsilon$, then by (1) in Remark 5.2,

$$\omega^*(\eta, F_1, U) \le \lim_{\zeta \to \eta} \omega(\zeta, J_\varepsilon, H) = 1.$$

There remain two other cases to investigate. First let us suppose that $\eta \in \Gamma_1 \setminus J_\varepsilon$. Then $\eta \in \partial U \setminus \overline{F}_1$; hence using Remark 5.2 once more,

$$\lim_{\zeta \to \eta} \omega(\zeta, F_1, U) = 0 \le \omega^*(\eta, J_\varepsilon, H).$$

Finally suppose that $\eta \in \Gamma_2 \setminus J_\varepsilon$. Define the set A as in Lemma 5.1 by

$$A = \{\zeta \in \partial U : |f(\zeta) - f(\eta)| > \varepsilon\}.$$

Note that $F_1 \subset A$ since $f(\eta) \notin K_\varepsilon$ and that $\eta \in \Gamma_2$ implies $f(\eta) \in \partial B_r$. Let c be a conformal mapping of U onto U such that $c(0) = \eta$. Then

$$c^{-1}(A) = \{\zeta \in \partial U : |f \circ c(\zeta) - f \circ c(0)| > \varepsilon\}.$$

We use the fact that the harmonic measure is invariant under conformal mappings of U onto U and apply Lemma 5.1 to the function $f \circ c$ to get

$$\omega(\eta, F_1, U) \le \omega(\eta, A, U) = \omega(0, c^{-1}(A), U) \le \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2).$$

Therefore, we have shown that for all $\eta \in \partial H$,

$$\omega^*(\eta, F_1, U) - \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2) \le \omega^*(\eta, J_\varepsilon, H).$$

Now both functions in the last inequality are harmonic in H; thus the inequality holds for all $\eta \in \overline{H}$, in particular for $\eta = 0$. Hence

$$\mu_f(K) = \omega(0, F_1, U) \le \omega(0, J_\varepsilon, H) + \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2). \tag{8}$$

Since $G_\varepsilon = p^{-1}(J_\varepsilon)$ and $p^{-1}(0) = 0$, by (2) in Remark 5.2,

$$\omega(0, J_\varepsilon, H) = \omega(0, G_\varepsilon, U) = \mu_g(K_\varepsilon).$$

Therefore

$$\mu_f(K) \le \mu_g(K_\varepsilon) + \frac{8}{\delta^2\varepsilon^2}(\delta^2 - r^2).$$

Letting $\varepsilon \to 1$,

$$\mu_f(K) \le \mu_g(K) + \frac{8}{\delta^2}(\delta^2 - r^2).$$

□

Following the terminology in [P3] we first extend the definition of harmonic measure to subsets of $\overline{U}$. Let $A \subset \overline{U}$ be a closed set. We consider the function

$$\omega(\zeta,\, A,\, U) = \liminf_{\xi\to\zeta,\, \xi\in U} \inf\{v(\xi) : v_* \ge \chi_A \text{ is superharmonic on } U\}.$$

If E is a relatively open subset of $\overline{U}$, then

$$\omega(\zeta,\, E,\, U) = \sup \omega(\zeta, A, U),$$

where the supremum is taken over all closed subsets A of E.

The next definitions and Theorem 5.4 are due to Poletsky ([P3, Theorem 2.1]).

Let $f_j \in \mathcal{A}(z_j,\, D)$ for some points $z_j \in D$. *The cluster* $\operatorname{cl} L$ of the sequence $L = \{f_j\}$ is the set of all points $z \in D$ such that for every $r > 0$ and infinitely many j the sets $f_j(\overline{U}) \cap B(z,\, r) \neq \emptyset$. A point $z \in \operatorname{cl} L$ is called *essential* if

$$\limsup_{j\to\infty} \omega(0,\, f_j^{-1}(V),\, U) > 0$$

for every open set V containing z. The set of essential points of L is denoted by $\operatorname{ess} L$. Other points in $\operatorname{cl} L$ are called *nonessential*.

Theorem 5.4. *Let $z_j \in D$ be points converging to $z \in \overline{D}$, and let $f_j \in \mathcal{A}(z_j,\, D)$ be holomorphic mappings such that μ_{f_j} converges to a measure $\mu \in \hat{J}_z$. Let $L = \{f_j\}$. Then there exist conformal mappings $q_j : U \to U$ so that the functions $g_j = f_j \circ q_j$ belong to $\mathcal{A}(z_j,\, D)$ and if $M = \{g_j\}$, then $\operatorname{cl} M = \operatorname{ess} L = \operatorname{ess} M$ and the μ_{g_j} converge weak-$*$ to μ.*

Theorem 5.4 and Lemma 5.1 allow us to prove the following result.

Theorem 5.5. *Let D be a bounded domain in $\mathbb{C}^n$. Suppose the set $V = D \cap B$ is non-empty. Let $z_0 \in \overline{V}$, $z_j \in V$ be points in V converging to z_0 and $f_j \in \mathcal{A}(z_j,\, D)$ be holomorphic mappings so that μ_{f_j} converges weak-$*$ to a measure μ with* $\operatorname{supp}\mu \subset \overline{V}$. *Then there exist conformal mappings $p_j : U \to U$ so that the functions $g_j = f_j \circ p_j$ belong to $\mathcal{A}(z_j,\, V)$ and μ_{g_j} converges weak-$*$ to μ.*

Proof. Using Theorem 5.4 we can find conformal mappings $q_j : U \to U$ and replace f_j by $f_j \circ q_j$. So without loss of generality we may assume that for $L = \{f_j\}$, $\operatorname{cl} L = \operatorname{ess} L$. First let us show that the set $\operatorname{cl} L$ is contained in $\overline{B}$.

Take numbers $c > 0$, $\varepsilon > 0$ and a point z_1 such that $K = B(z_1, \varepsilon) \subset E_c = \{z : |z| \ge e^c\}$. Let

$$u_j(\zeta) = \log|f_j(\zeta)|$$

for any $\zeta \in \overline{U}$. Since D is bounded there exists a number $M > 0$ so that $|f_j(\zeta)| \le e^M$ for every $\zeta \in \overline{U}$ and $j \ge 1$. Let $\varepsilon_j > 0$ be numbers decreasing to 0. Take a harmonic function h_j on U so that if $\zeta \in \partial U$, then $h_j(\zeta) = \varepsilon_j$ when $\log|f_j(\zeta)| < \varepsilon_j$ and $h_j(\zeta) = M$ otherwise. Let

$$A_j = \{\zeta \in \partial U : |f_j(\zeta)| < e^{\varepsilon_j}\}.$$

Since the sequence $\{\mu_{f_j}\}$ converges weak-$*$ to a measure supported in $\overline{B}$, the arclengths $l(A_j)$ of the sets A_j converge to 1. This implies that $h_j(0)$ decreases to 0. Hence if $F_c = \{\zeta \in \overline{U} : h_j(\zeta) \geq c\}$, then $h_j/c \geq 1$ on F_c and

$$\lim_{j\to\infty} \omega(0,\, F_c,\, U) = 0.$$

Note that $u_j \leq h_j$. It follows that $f_j^{-1}(K) \subset f_j^{-1}(E_c) \subset F_c$. Then

$$\lim_{j\to\infty} \omega(0,\, f_j^{-1}(K),\, U) = 0.$$

That is, z_1 is not an essential point of L and $z_1 \notin \operatorname{cl} L$. Therefore $\operatorname{cl} L \subset \overline{B}$.

Now it is possible to see that for every number $\delta > 1$ there exists an integer $j_0 \geq 1$ such that $f_j(\overline{U}) \subset B_\delta$ for every $j \geq j_0$. If not, then there exist $\delta > 1$ and $\zeta_j \in \overline{U}$ so that $z_j = f_j(\zeta_j) \notin B_\delta$ for all j. Since $\overline{D} \setminus B_\delta$ is compact, a subsequence $\{z_{j_k}\}$ converges to a point $z \in \overline{D} \setminus B_\delta$. Then $z \in \operatorname{cl} L$, but not in $\overline{B}$, which is a contradiction.

There exist non-increasing numbers $\delta_j > 1$ converging to 1 such that each f_j maps $\overline{U}$ into B_{δ_j}. Let $r_j > 0$ be an increasing sequence of numbers converging to 1 such that $z_j \in B_{r_j}$. By Theorem 5.3 there exist conformal maps p_j from $\overline{U}$ into $\overline{U}$ so that for each j the function $g_j = f_j \circ p_j$ maps $\overline{U}$ into $\overline{B}_{r_j} \cap D \subset V$ and for every open ball K in $\mathbb{C}^n$,

$$\mu_{f_j}(K) \leq S_j = \mu_{g_j}(K) + \frac{8}{{\delta_j}^2}({\delta_j}^2 - r_j^2).$$

Thus $g_j \in \mathcal{A}(z_j,\, V)$.

Suppose there exist a subsequence $\{j_k\}$, an open ball K_0 and a number $a > 0$ such that $\mu(K_0) \;+\; a < S_{j_k}$ for all k. Then a subsequence of $\mu_{g_{j_k}}$, which we call $\{\mu_{g_{j_k}}\}$ again, converges weak-$*$ to a measure μ_0. As $k \to \infty$, $\mu(K) \;+\; a \leq \mu_0(K)$ and for any other open ball K, $\mu(K) \leq \mu_0(K)$. This is not possible since $\mu(\overline{V}) = \mu_0(\overline{V}) = 1$. Thus S_j has a limit as $j \to \infty$ for every open ball K in $\mathbb{C}^n$ and the limit is equal to $\mu(K)$. This implies μ_{g_j} converges weak-$*$ to μ and the proof is finished. □

To prove Lemma 5.1 we used the fact that on the open unit ball in $\mathbb{C}^n$ one can find a plurisubharmonic function that has a uniform estimate as in (7). Our proof does not work if we replace the open ball by an arbitrary open set in $\mathbb{C}^n$. As the following example shows we cannot even take the open polydisk instead of the open ball.

Example 5.6. Consider the open polydisk $U \times U$ in $\mathbb{C}^2$. Let

$$s_j = 1 + \frac{1}{\sqrt{j}}.$$

Then s_j is decreasing to 1 and

$$\lim_{j\to\infty} s_j^j = \infty.$$

Let $z_0 = (0,\, 0)$ and

$$f_j(\zeta) = (s_j\zeta,\, \zeta^j)$$

for all $\zeta \in \overline{U}$. Then $\mu_{f_j} \in \mathcal{H}_{z_0}(U_{s_j^2} \times U_{s_j})$. Let p_j be a conformal mapping from $\overline{U}$ into $\overline{U}$ so that $g_j = f_j \circ p_j$ maps $\overline{U}$ into $U \times U$. Note that μ_{f_j} weak-$*$ converges to the measure

$$\mu = \frac{1}{4\pi^2} d\alpha\, d\beta$$

on $\partial U \times \partial U$, where $d\alpha$, $d\beta$ denote the arclength measures on ∂U. On the other hand since $|s_j p_j(\zeta)| \leq 1$ for every $\zeta \in \overline{U}$,

$$(|p_j(\zeta)|)^j \leq \frac{1}{s_j^j}.$$

Hence $|p_j(\zeta)|^j$ converges uniformly to 0 on $\overline{U}$. It follows that if $0 < \varepsilon < 1$ and $K = B(1, \varepsilon)$, then $K \cap g_j(\overline{U}) = \emptyset$ for j large enough. Thus if μ_0 is the weak-$*$ limit of a subsequence of μ_{g_j}, then $\mu_0(K) = 0 < \mu(K)$.

The following characterization of c-regular domains was proved in [Go, Corollary 4.4].

Theorem 5.7. *A bounded domain D in $\mathbb{C}^n$ is c-regular if and only if for all $z \in \partial D$, $\mathcal{J}_z^b = \mathcal{J}_z^c$ if and only if for all $z \in \partial D$, measures $\mu \in \hat{\mathcal{J}}_z$ and for any sequence of points $\{z_j\} \subset D$ converging to z there exist measures $\mu_j \in \overline{\mathcal{H}}_{z_j}$ that converge weak-$*$ to μ.*

Now let us show that if a domain is c-regular, then it is locally c-regular.

Theorem 5.8. *A c-regular domain is locally c-regular.*

Proof. Let D be a c-regular domain. Take any point $\zeta \in \partial D$ and a ball B_0 so that $V = D \cap B_0 \neq \emptyset$. Without loss of generality we will assume that B_0 is the unit ball B in $\mathbb{C}^n$. We will show that the set V is c-regular.

If $z \in \partial V$, then either $z \in \partial B$ or $z \in B$. For all $z \in \partial B$, the classes $\mathcal{J}_z^b$ and $\mathcal{J}_z^c$ coincide and they are equal to $\{\delta_z\}$, the Dirac measure at z (see [S] and also [W, Corollary 3.8]). Therefore we may assume that $z \in B \cap \partial V$. Suppose there exist $\mu \in \hat{\mathcal{J}}_z(V)$ and a sequence of points $\{z_j\} \subset V$ converging to z. Since D is c-regular and $\hat{\mathcal{J}}_z(V) \subset \hat{\mathcal{J}}_z(D)$ there exist measures $\mu_j \in \overline{\mathcal{H}}_{z_j}(D)$ that converge weak-$*$ to μ in $C^*(\overline{D})$. We point out that the space $C^*(\overline{D})$ is metrizable (see [C, Theorem 5.1]), therefore we can find functions $f_j \in \mathcal{A}(z_j, D)$ so that the measures μ_{f_j} converge weak-$*$ to μ. By Theorem 5.5 there exist functions $g_j \in \mathcal{A}(z_j, V)$ so that the measures $\mu_{g_j} \in \mathcal{H}_{z_j}(V)$ converge weak-$*$ to μ. Thus by Theorem 5.7, V is c-regular and the assertion follows. $\square$

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