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A MODEL CATEGORY STRUCTURE FOR EQUIVARIANT ALGEBRAIC MODELS

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Abstract. In the equivariant category of spaces with an action of a finite group, algebraic 'minimal models' exist which describe the rational homotopy for G-spaces which are 1-connected and of finite type. These models are diagrams of commutative differential graded algebras. In this paper we prove that a model category structure exists on this diagram category in such a way that the equivariant minimal models are cofibrant objects. We show that with this model structure, there is a Quillen equivalence between the equivariant category of rational G-spaces satisfying the above conditions and the algebraic category of the models.

1. Introduction

A model category is a category in which it is possible to 'do homotopy theory' as inspired by the homotopy theory of spaces. Model categories were first developed by Quillen as a way of abstracting the ideas of homotopy theory to more general settings. This has proved to apply to a wide variety of categories and to be useful in many contexts.

A model category has three distinguished classes of maps, called *weak equivalences, fibrations* and *cofibrations*, which satisfy certain axioms; these are listed for example in [DS, Hi, Q1]. These axioms are designed to mimic the structure seen in the category of spaces. They allow one to form a 'homotopy category' $h\mathcal{C}$ by inverting the weak equivalences in the model category $\mathcal{C}$, and perform other homotopy constructions. Quillen also developed conditions for an equivalence between two model categories; such an equivalence induces an isomorphism of homotopy categories, and furthermore preserves much of the homotopy theoretic structure. Quillen used these definitions to show that there is such an equivalence between the homotopy categories of rational 1-connected spaces and differential graded Lie algebras [Q2].

A different algebraic model for rational homotopy was introduced by Sullivan [DGMS, Su]. This is based on the category of commutative differential graded algebras (CDGAs). These 'minimal models' have some advantages over Quillen's original approach in that they encode geometric information more transparently; in fact the algebraic structure of the minimal model closely reflects the rational Postnikov tower of the space. Bousfield and Gugenheim have combined Sullivan's

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minimal models with Quillen's model theory approach, and in [BG] they show that there is a model category structure on CDGAs in which Sullivan's minimal models are cofibrant objects; and that with this model structure, there is a Quillen equivalence of homotopy categories between CDGAs and rational spaces which are nilpotent and of finite type.

For spaces with the action of a finite group G, equivariant minimal models were developed by Triantafillou in [T]. These models live in a category of diagrams of CDGAs. They behave very much like the Sullivan minimal models, encoding the equivariant rational Postnikov tower of suitable spaces.

When the indexing category of the diagrams is sufficiently nice, satisfying finiteness conditions developed by Reedy in [R], it is possible to extend the model category on CDGAs to the diagram category where the equivariant models live. When the group acting is Hamiltonian, this idea was used by Golasinski to define a model structure in [G]. He used this to define an alternate notion of equivariant 'minimal models' which are the cofibrant objects under this model category structure.

This paper develops a different model structure on diagrams of CDGAs. The new definitions apply to actions of any finite group, removing the Hamiltonian condition; since an arbitrary finite group does not lead to a Reedy indexing category, an alternate approach is required. This approach also has the advantage that the existing minimal models are cofibrant objects; thus the geometric and model category notion of minimal models agree. We can also use this model structure to offer an interpretation of the ubiquitous 'injectivity' condition that comes up when dealing with equivariant minimal models.

The main results of this paper are:

- to produce a model category structure for diagrams of CDGAs which applies to actions of any finite group (Section 3);
- to show that the equivariant minimal models of Triantafillou [T], whose definition is geometrically motivated, are cofibrant objects in this model structure (Section 4);
- to prove that under this model structure, there is a Quillen equivalence between the category of equivariant rational spaces which are 1-connected and of finite $\mathbb{Q}$-type and the category of the diagrams of CDGAs with analogous restrictions (Section 5).

The Quillen equivalence induces an isomorphism of homotopy categories and preserves much of the relevant structure present. Thus we get a model category interpretation of the statement that the equivariant minimal models encode all rational homotopy information.

2. Equivariant homotopy and the algebra of diagrams

In this section we recall the connection between equivariant spaces and diagrams, and develop some of the basic algebra of diagrams which we will use for discussing equivariant algebraic models.

The equivariant homotopy type of a G-space X depends not only on the homotopy type of the space itself but also on the homotopy type of the fixed point subspaces $\{X^H\}$ for all closed subgroups $H \subseteq G$: a G-map $f : X \to Y$ is a G-homotopy equivalence if and only if $f^H : X^H \to Y^H$ is a homotopy equivalence for each H. Together with the natural inclusions and maps induced by the action of G, these fixed sets $\{X^H\}$ form a diagram of spaces indexed by the *orbit category* $\mathcal{O}_G$.

When we define algebraic invariants for G-spaces we often define them using diagrams reflecting fixed point data. Therefore much of our work is in understanding the behaviour of diagrams indexed by $\mathcal{O}_G$.

The indexing category $\mathcal{O}_G$ has as its objects the canonical orbits G/H for all subgroups $H \subseteq G$, with morphisms defined by equivariant maps between these G-spaces. For simplicity of notation we will abbreviate objects of $\mathcal{O}_G$ from G/H to H. Elementary group theory tells us what equivariant maps are possible between these canonical orbit spaces: if K is not subconjugate to H, then the set of morphisms $\mathcal{O}_G(K,H)$ is empty; $\mathcal{O}_G(H,H) = Aut(H) \simeq N(H)/H$; and if K is subconjugate to H, then $\mathcal{O}_G(K,H) \simeq (G/H)^K$, which can be broken up into disjoint copies of $Aut(H) = N(H)/H$, one copy for each conjugacy class gHg^{-1} of H which contains K.

In defining rational invariants, the underlying diagram category consists of functors from $\mathcal{O}_G$ to rational vector spaces, with morphisms given by natural transformations; we will denote this diagram category by $\mathbb{Q}^{\mathcal{O}_G}$. If $\underline{V}$ is a diagram in $\mathbb{Q}^{\mathcal{O}_G}$, then for any subgroup $H \subseteq G$, there is a restriction functor from $\mathbb{Q}^{\mathcal{O}_G}$ to $\mathbb{Q}[Aut(H)]$-modules given by $U_H : \underline{V} \to \underline{V}(H)$, which evaluates the functor at H. In the other direction, there are two functors from $\mathbb{Q}[Aut(H)]$-vector spaces to $\mathbb{Q}^{\mathcal{O}_G}$ which we will be considering, the 'projective' P_H and the 'injective' I_H.

If V is a $\mathbb{Q}[Aut(H)]$-vector space, then we define a projective diagram:

Definition 2.1. $P_H(V)$ is the diagram defined by

$$P_H(V)(K) = \mathbb{Q}[Hom_{\mathcal{O}_G}(H,K)] \otimes_{\mathbb{Q}[Aut(H)]} V$$

with structure maps defined by $P_H(V)(\alpha)(\phi \otimes v) = (\phi\alpha) \otimes v$ for a map $\alpha : K \to K'$ in $\mathcal{O}_G$.

Dually we define an injective digram:

Definition 2.2. $I_H(V)$ is the diagram defined by

$$I_H(V)(K) = Hom_{\mathbb{Q}[Aut(H)]}(\mathbb{Q}[Hom_{\mathcal{O}_G}(K,H)], V^*)$$

where V^* is the dual of the module V.

Note that if V is a finitely generated $\mathbb{Q}[Aut(H)]$-module and we evaluate the functor $I_H(V)$ at H, the entry $I_H(V)(H)$ is just the original module V. If K is not subconjugate to H, $I_H(V)(K) = 0$; and if K is subconjugate to H, then $\mathbb{Q}[\mathcal{O}_G(K,H)] = \mathbb{Q}[(G/H)^K]$ is a free $\mathbb{Q}[Aut(H)]$-module indexed by conjugates of H containing K, and so $I_H(V)(K)$ is a direct product of copies of V, one for each conjugate gHg^{-1} of H containing K.

With these definitions, we have the following adjunctions.

Lemma 2.3. *Let H be an object of $\mathcal{O}_G$, W a $\mathbb{Q}[Aut(H)]$-module and $\underline{V}$ a diagram in $\mathbb{Q}^{\mathcal{O}_G}$. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\mathbb{Q}^{\mathcal{O}_G}}(P_H(W), \underline{V}) \simeq \mathrm{Hom}_{\mathbb{Q}[Aut(H)]}(W, \underline{V}(H))$$

and

$$\mathrm{Hom}_{\mathbb{Q}^{\mathcal{O}_G}}(\underline{V}, I_H(W)) \simeq \mathrm{Hom}_{\mathbb{Q}[Aut(H)]}(\underline{V}(H), W).$$

Proof. Let $f : W \to \underline{V}(H)$ be a morphism of $\mathbb{Q}[Aut(H)]$-modules. We define the natural transformation $\hat{f} : P_H(W) \to \underline{V}$ as follows. For any object K of $\mathcal{O}_G$, the map $\hat{f}(K) : \mathbb{Q}[\mathcal{O}_G(H,K)] \otimes_{\mathbb{Q}[Aut(H)]} W \to \underline{V}(K)$ is given by $\hat{f}(K)(\phi \otimes w) =$

$\underline{V}(\phi)(f(w))$. The fact that this correspondance gives an isomorphism is the Yoneda lemma. The other adjunction is defined dually. $\square$

For geometric reasons, in what follows we are more interested in the injective diagrams, so we focus on these from here. Although we don't need them, the duals of the following statements also hold.

Lemma 2.4. *$I_H(V)$ is an injective object in the category $\mathbb{Q}^{\mathcal{O}_G}$.*

Proof. Given a commutative diagram

$$\begin{tikzcd} \underline{A} \arrow[r, hook] \arrow[d] & \underline{B} \arrow[dl, dashed] \\ I_H(V) & \end{tikzcd}$$

we can evaluate all functors at the subgroup H to get a similar commutative diagram of $\mathbb{Q}[Aut(H)]$-modules. Since $\mathbb{Q}[Aut(H)]$ is semi-simple, the inclusion $A(H) \hookrightarrow B(H)$ splits and we can fill in the required morphism from $\underline{B}(H)$ to $I_H(V)(H) = V$. Then Lemma 2.3 says this is equivalent to a morphism in $\mathbb{Q}^{\mathcal{O}_G}$ from $\underline{B}$ to $I_H(V)$ as required. $\square$

Not only is the diagram $I_H(V)$ injective, but in fact all injective objects are made up of such diagrams:

Proposition 2.5 (Triantafillou [T]). *A diagram $\underline{A}$ in $\mathbb{Q}^{\mathcal{O}_G}$ is injective if and only if it is of the form $\underline{A} = \bigoplus_H I_H(V_H)$ for some collection of $\mathbb{Q}[Aut(H)]$-modules $\{V_H\}$.*

For an arbitrary system of vector spaces $\underline{V}$, we define an embedding into an injective object.

Definition 2.6. Let $\underline{V}$ be any object of $\mathbb{Q}^{\mathcal{O}_G}$. For each conjugacy class of subroups $[H]$, choose one representative H and define the $\mathbb{Q}[Aut(H)]$-module $V_H = \bigcap_{K \supset H} \ker \underline{V}(\hat{e}_{H,K})$ where $\hat{e}_{H,K} : H \to K$ is the projection $G/H \to G/K$ and $\underline{V}(\hat{e}_{H,K})$ is the induced structure map on the functor $\underline{V}$. We understand this to mean that V_G is all of $\underline{V}(G)$. Let $\underline{I} = \bigoplus_{[H]} I_H(V_H)$. Then there is an an injection $\underline{V} \hookrightarrow \underline{I}$ extending the natural inclusions of V_H. We say that $\underline{I}$ is the *injective envelope* of $\underline{V}$.

3. Model category structure for diagrams

In this section we define the model category structure which is central to this paper.

Before examining the category of diagrams of CDGAs which we are interested in, we first look at the underlying category of chain complexes of objects in $\mathbb{Q}^{\mathcal{O}_G}$. There are two different model categories commonly discussed for chain complexes in an Abelian category; the one which will be relevant for us is the injective model structure.

To be precise, the category we use is the category of chain complexes of diagrams in $\mathbb{Q}^{\mathcal{O}_G}$ which are non-negatively graded, where we take our differential d to raise degree. Then we have the following.

Theorem 3.1. *There is a model category structure on the category of non-negatively graded chain complexes of* $\mathbb{Q}^{\mathcal{O}_G}$ *such that a morphism* $f : \underline{A} \to \underline{B}$ *is*

- *a* weak equivalence *if* $f^* : H^*(\underline{A}(H)) \to H^*(\underline{B}(H))$ *is an isomorphism for all* H*,*
- *a* cofibration *if* $f^n(H)$ *is injective for all* H*, and all* $n \geq 1$*,*
- *a* fibration *if* $f^n(H)$ *is surjective for all* H *and* n*, and the kernel of* f^n *in each degree* $n \geq 1$ *is an injective object.*

This model structure is defined by Quillen [Q1] for chain complexes which are bounded below; a slight adaptation in degree 0 is needed to get non-negatively graded complexes. It is dual to the projective model structure on non-negatively graded chain complexes given as an example in [DS].

To make the connection with rational homotopy theory, we use the category $CDGA^{\mathcal{O}_G}$ defined by functors from $\mathcal{O}_G$ to commutative differential graded algebras. We will assume that all CDGAs A are based in the sense that $A^0 = \mathbb{Q}$, and that the structure maps are also based maps. Note that this means that all diagrams $\mathcal{A}$ in $CDGA^{\mathcal{O}_G}$ have a basing map $\underline{\mathbb{Q}} \to \mathcal{A}$ where $\underline{\mathbb{Q}}$ is the constant diagram with $\mathbb{Q}$ in degree 0 and no other generators. This diagram $\underline{\mathbb{Q}}$ is the initial object of the category $CDGA^{\mathcal{O}_G}$.

We will prove that the model structure defined in Theorem 3.1 can be adapted to this category.

Theorem 3.2. *There is a model category structure on* $CDGA^{\mathcal{O}_G}$ *such that a morphism* $f : \mathcal{A} \to \mathcal{B}$ *is*

- *a* weak equivalence *if* $f^* : H^*(\mathcal{A}(H)) \to H^*(\mathcal{B}(H))$ *is an isomorphism for all* H*,*
- *a* fibration *if it is a fibration in the underlying category of chain complexes in* $\mathbb{Q}^{\mathcal{O}_G}$*.*

To defined the cofibrations, we will use the notion of a 'lifting property'. If we consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

we say that the map i has the *left lifting property* (LLP) with respect to the map p if the dotted arrow can be filled in; similarly p has the *right lifting property* (RLP) with respect to i.

Given a class of weak equivalences, the cofibrations and fibrations determine each other in a model category: the axioms imply that the cofibrations are exactly the maps which have the LLP with respect to acyclic fibrations (maps which are both fibrations and weak equivalences); similarly the fibrations are the maps which have the RLP with respect to acyclic cofibrations. We use this to define cofibrations in $CDGA^{\mathcal{O}_G}$.

We also want to use the lifting property to understand fibrations in the model structure. In Definition 6.3, we define a set $\mathbf{T} = \{\underline{T}_H\}$ of vector space diagrams in $\mathbb{Q}^{\mathcal{O}_G}$, such that the inclusions $\underline{T}_H \hookrightarrow \underline{I}_{T_H}$ into their injective envelopes can be used to detect injectivity. These test objects, together with some projective vector space diagrams $\underline{P}_H$, can be turned into diagrams of CDGAs; then the lifting property with respect to a specific set of maps involving these $CDGA^{\mathcal{O}_G}$ objects obtained can be used to define fibrations and acyclic fibrations. Having such a way of recognizing fibrations helps in checking the axioms of the model structure.

To turn a vector space diagram into an object of $CDGA^{\mathcal{O}_G}$, we use the following constructions.

Definition 3.3. If $\underline{V}$ is a diagram of vector spaces and n is any positive integer, we define $\mathcal{V}^n = \mathbb{Q}(\underline{V})$ to be the diagram of free CDGAs generated by $\underline{V}$ in degree n with $d = 0$. We also define $\widehat{\mathcal{V}}^n = \mathbb{Q}(s^{-1}\underline{V} \oplus \underline{V})$ to be the diagram of free acyclic CDGAs generated by $\underline{V}$ in degree n and an isomorphic copy $s^{-1}\underline{V}$ in degree $n-1$, with $d(s^{-1}\underline{V}) = \underline{V}$.

We use these to define two special sets of maps in $CDGA^{\mathcal{O}_G}$.

Definition 3.4. $\mathcal{I}$ is the the collection of the inclusions i and i':

- For any H, let $\mathbb{Q}[Aut(H)]$ be the free $Aut(H)$-vector space on one generator, and define $\underline{P} = P_H(\mathbb{Q}(Aut(H))$ to be the projective diagram of vector spaces as in Definition 2.1. Then for any $n > 0$, we form the diagrams $\mathcal{P}^n$ and $\widehat{\mathcal{P}}^n$ in $CDGA^{\mathcal{O}_G}$ as described in Definition 3.3; and there is an inclusion $i : \mathcal{P}^n \hookrightarrow \widehat{\mathcal{P}}^n$.
- For any object $\underline{T}_H$ of the test diagrams $\mathbf{T}$, and any $n > 0$, the inclusion $\underline{T}_H \hookrightarrow \underline{I}_{T_H}$ of $\underline{T}_H$ into its injective envelope $\underline{I}_H$ induces an inclusion $i' : \mathcal{T}^n \hookrightarrow \widehat{\mathcal{I}}^n$ in $CDGA^{\mathcal{O}_G}$.

Definition 3.5. $\mathcal{J}$ is the collection of the inclusions j and j':

- For any H, let $\mathbb{Q}[Aut(H)]$ be the free $Aut(H)$-vector space on one generator, and define $\underline{P} = P_H(\mathbb{Q}(Aut(H))$ to be the projective diagram of vector spaces as in Definition 2.1. Then for any $n > 0$, we form the acyclic diagram $\widehat{\mathcal{P}}^n$ in $CDGA^{\mathcal{O}_G}$ as described in Definition 3.3; then there is an inclusion $j : \underline{\mathbb{Q}} \hookrightarrow \widehat{\mathcal{P}}^n$.
- For any object $\underline{T}_H$ of $\mathbf{T}$ and any $n > 0$, the inclusion $\underline{T}_H \hookrightarrow \underline{I}_{T_H}$ of $\underline{T}_H$ into its injective envelope $\underline{I}_{T_H}$ induces an inclusion $j' : \widehat{\mathcal{T}}^n \hookrightarrow \widehat{\mathcal{I}}^n$ in $CDGA^{\mathcal{O}_G}$.

We focus on these particular maps because of the next two propositions.

Proposition 3.6. *A map in $CDGA^{\mathcal{O}_G}$ is a fibration if and only if it has the RLP with respect to the maps in $\mathcal{J}$.*

Proposition 3.7. *A map in $CDGA^{\mathcal{O}_G}$ is a fibration and a weak equivalence if and only if it has the RLP with respect to the maps in $\mathcal{I}$.*

The proofs of these results are somewhat long, so they will be deferred until Section 6.

The last ingredient needed for the proof of Theorem 3.2 is the idea of smallness. We recall the following from [Hi].

Definition 3.8. An object A in a category $\mathcal{C}$ is *small* with respect to a set of morphisms $\mathcal{D}$ of $\mathcal{C}$ if there is a cardinal κ such that for every regular cardinal $\lambda \geq \kappa$ and every λ-sequence

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots (\beta < \lambda)$$

such that the maps $X_\beta \to X_{\beta+1}$ are in $\mathcal{D}$ for each β with $\beta + 1 < \lambda$, then the map of sets

$$colim_{\beta<\lambda} Hom_{\mathcal{C}}(A, X_\beta) \to Hom_{\mathcal{C}}(A, colim_{\beta<\lambda} X_\beta)$$

is an isomorphism.

Thus an object A is small if any map from A into a sufficiently long composition will factor through some stage of the composition.

Definition 3.9. A set of morphisms $\mathcal{D}$ in a category C *permits the small object argument* if the domains of the elements of $\mathcal{D}$ are small relative to $\mathcal{D}$.

The importance of this definition is in the following.

Theorem 3.10 (Small object argument)**.** *Suppose $\mathcal{D}$ permits the small object argument. Then for any morphism f, there is a natural factorization of f as $p(f)i(f)$ where $i(f)$ has the RLP with respect to $\mathcal{D}$, and $p(f)$ is a transfinite pushout of maps in $\mathcal{D}$; this also implies that $p(f)$ has the LLP with respect to maps which have the RLP with respect to $\mathcal{D}$.*

In order to apply these results to our situation, we need the following.

Lemma 3.11. *Every object $\mathcal{A}$ of $CDGA^{\mathcal{O}_G}$ is small with respect to all morphisms.*

Proof. We adapt the proof given by Hovey in [Ho] for R-modules. We use the fact that every object is small in the category of R-modules. Let κ be the first infinite cardinal greater than the cardinal $|\mathcal{A}| \times |\mathcal{A}|$, and let X be a λ-sequence of $CDGA^{\mathcal{O}_G}$s where $\lambda \geq \kappa$. We want to show that $colim_{\beta<\lambda} Hom(\mathcal{A}, X_\beta) \to Hom(\mathcal{A}, colim_{\beta<\lambda} X_\beta)$ is an isomorphism.

Given a map $\phi : \mathcal{A} \to colim X_\beta$, then at each H we get a map of $\mathbb{Q}[Aut(H)]$-modules $\phi^H : \mathcal{A}(H) \to colim X_\beta(H)$. Since every object in this category is small, we can factor each one through $\phi^H_\alpha : \mathcal{A} \to X_{\alpha_H}$. These may not combine to give a map in $CDGA^{\mathcal{O}_G}$. However, there is a set of conditions which must be met to make a map in $CDGA^{\mathcal{O}_G}$: for each H and each pair of elements $a, b \in \mathcal{A}(H)$, the maps defined by $\phi^H(a)\phi^H(b)$ and $\phi^H(ab)$ agree in the colimit, and so coincide at some $X_{\alpha_1(H,a,b)}$; similarly $\phi^H(da)$ and $d\phi^H(a)$ agree in the colimit, and so coincide at some $X_{\alpha_2(H,a)}$. There is also a finite collection of structure maps of $\mathcal{O}_G$, $\xi : H \to K$, which must be respected; for each one, we have that $\phi^K\xi(a)$ and $\xi\phi^H(a)$ agree in the colimit, and so agree at some stage $A_{\alpha_3(a,\xi)}$. Let γ be the supremeum of all these α_i. Then $\gamma < \lambda$ and so the map $\phi : \mathcal{A} \to X_\gamma$ defines a factorization of ϕ in the category $CDGA^{\mathcal{O}_G}$. $\square$

Corollary 3.12. *The sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ in $CDGA^{\mathcal{O}_G}$ permit the small object argument.*

With these preliminaries, we can now prove that the model structure exists as described.

Proof of Theorem 3.2. We check the five axioms of a model category (see for example [DS, Hi, Ho, Q1]).

- Ax1 In the diagram category $CDGA^{\mathcal{O}_G}$, the coproduct and product are defined objectwise.
- Ax2 If two out of three of the maps f, g and fg are weak equivalences, then they are quasi-isomorphisms at each H; so the third map must also be a quasi-isomorphism at each H and so a weak equivalence.

Ax3 Quasi-isomorphisms and surjections are closed under retracts. Therefore it is immediate that weak equivalences are also closed. It is also easy to see that the retract of an injective object is injective; so the retract of a fibration is a fibration. Since the cofibrations are defined by a lifting property, they are also closed under retracts.

Ax5 (a) By Corollary 3.12, we can invoke the small object argument with respect to the set of maps $\mathcal{I}$ to obtain a functorial factorization of any morphism f as $p(f)i(f)$, where $p(f)$ has the RLP with respect to morphisms in $\mathcal{I}$ and $i(f)$ has the has the LLP with respect to maps which have the RLP with respect to $\mathcal{I}$. This implies that $p(f)$ is an acyclic fibration by Proposition 3.7, and that $i(f)$ has the LLP with respect to acyclic fibrations and therefore is a cofibration by definition.

(b) Replacing $\mathcal{I}$ with $\mathcal{J}$, the small object argument gives a factorization $p'(f)i'(f)$ where $p'(f)$ has the RLP with respect to morphisms in $\mathcal{J}$ and $i'(f)$ is a transfinite pushout of maps in $\mathcal{J}$, and has the LLP with respect to maps with the RLP with respect to $\mathcal{J}$. Therefore $p'(f)$ is a fibration by Proposition 3.6, and $i'(f)$ has LLP with respect to fibrations, and therefore is a cofibration. Moreover, all maps in $\mathcal{J}$ are weak equivalences, so $i'(f)$ is also a weak equivalence.

Ax4 (b) Suppose p is an acyclic fibration and i a cofibration. Then by definition i has the LLP with respect to acyclic fibrations and so the required lift exists.

(a) If $p : \mathcal{A} \to \mathcal{B}$ is a fibration and i is an acyclic cofibration, then we apply MC5 to factor i as $\mathcal{A} \xrightarrow{\beta} \mathcal{A}' \xrightarrow{\alpha} \mathcal{B}$, where α is an acyclic cofibration and β is a fibration. Since both i and α are weak equivalences, by MC2, β is also a weak equivalence. Now we consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathcal{A}' \arrow[d, "\beta"] \\
\mathcal{B} \arrow[r, "="] \arrow[ur, dashed, "\gamma"] & \mathcal{B}
\end{tikzcd}$$

Since β is an acyclic fibration, by the previous case a lift γ exists. Then the following diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "="] \arrow[d, "i"] & \mathcal{A} \arrow[r, "="] \arrow[d, "\alpha"] & \mathcal{A} \arrow[d, "i"] \\
\mathcal{B} \arrow[r, "\gamma"] & \mathcal{A}' \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

displays i as a retract of α. Since α has the required lifting property, so does i.

□

4. Minimal models

We now relate the model category structure of the previous section to the geometrically motivated theory of minimal models in $CDGA^{\mathcal{O}_G}$.

When developing the geometric theory, the original paper [T] and its sequels restricted their attention to objects in $CDGA^{\mathcal{O}_G}$ which are injective in the underlying category of diagrams of vector spaces $\mathbb{Q}^{\mathcal{O}_G}$. This restriction introduces considerable technical complications which we will see appearing in the definitions to follow. The model structure of Theorem 3.2 allows us to give an interpretation of this seemingly arbitrary condition: the injectivity condition is equivalent to a diagram being a fibrant object. Thus much of the use of injective envelopes and resolutions can be seen as taking fibrant replacements at various stages. In Section 5 we will examine the relationship between G-spaces and $CDGA^{\mathcal{O}_G}$, and see why it makes sense to restrict our attention to fibrant objects.

We recall the definition of a minimal object in $CDGA^{\mathcal{O}_G}$ from [T]. Given a fibrant object $\mathcal{A}$ in $CDGA^{\mathcal{O}_G}$, a diagram of vector spaces $\underline{V}$ in $\mathbb{Q}^{\mathcal{O}_G}$, and a map $\alpha : \underline{V} \to \underline{Z}^{n+1}(\mathcal{A})$, we construct the *elementary extension* $\mathcal{A}(\underline{V})$ of $\mathcal{A}$ with respect to α as follows. First we take an injective (fibrant) replacement for $\underline{V}$ by forming the injective resolution $\underline{V} \hookrightarrow \underline{V}_0 \xrightarrow{w_0} \underline{V}_1 \xrightarrow{w_1} \cdots$. This is constructed using the injective envelope of Definition 2.6; we define $\underline{V}_0$ to be the injective envelope of $\underline{V}$, and then define $\underline{V}_i$ to be the injective envelope of the cokernel of w_{i-1}. Note that we get a resolution of finite length. Using the injectivity of $\mathcal{A}$, we can construct a commutative diagram

$$(A)\qquad \begin{tikzcd} \underline{V} \arrow[r, hook] \arrow[d, "\alpha"] & \underline{V}_0 \arrow[r, "w_0"] \arrow[d, "\alpha_0"] & \underline{V}_1 \arrow[r, "w_1"] \arrow[d, "\alpha_1"] & \underline{V}_2 \arrow[r] \arrow[d, "\alpha_2"] & \cdots \\ \underline{Z}^{n+1}(\mathcal{A}) \arrow[r, hook] & \mathcal{A}^{n+1} \arrow[r, "d"] & \mathcal{A}^{n+2} \arrow[r, "d"] & \mathcal{A}^{n+3} \arrow[r] & \cdots \end{tikzcd}$$

We produce the maps α_i inductively by observing that $d\alpha_i w_{i-1} = dd\alpha_{i-1} = 0$, so $d\alpha_i \mid_{\text{im } w_{i-1}} = 0$ and by the injectivity of $\mathcal{A}$ we can fill in

$$\begin{tikzcd} \underline{V}_i/\text{im } w_{i-1} \arrow[r, hook] \arrow[d, "d\alpha_i"] & \underline{V}_{i+1} \arrow[dl, dashed, "\alpha_{i+1}"] \\ \mathcal{A}^{n+i+1} & \end{tikzcd}$$

Let $\mathcal{W} = \bigotimes_i \mathbb{Q}(\underline{V}_i)$ be the free injective diagram of CDGAs on generators $\underline{V}_i$ in degree $n+i+1$, with differential given by w_i; this is a fibrant approximation of the free object $\mathcal{V}^{n+1}$ generated by $\underline{V}$ in degree $n+1$ of Definition 3.3. The commutative diagram (A) above defines a map $\mathcal{W} \to \mathcal{A}$ in $CDGA^{\mathcal{O}_G}$. We also have the free acyclic object $\widehat{\mathcal{W}}$ in $CDGA^{\mathcal{O}_G}$ defined by $\widehat{\mathcal{W}} = \otimes_i \mathbb{Q}(s^{-1}\underline{V}_i) \otimes_i \mathbb{Q}(\underline{V}_i)$ where $d = w_i$ on generators $\underline{V}_i$, and $d = w_i + (-1)^i\sigma$ on generators of $s^{-1}\underline{V}$, where $\sigma(s^{-1}x) = x$. There is an obvious inclusion $\mathcal{W} \hookrightarrow \widehat{\mathcal{W}}$.

Then the elementary extension $\mathcal{A}(\underline{V})$ with respect to α is defined to be the pushout

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\alpha"] \arrow[d, "i"] & \mathcal{A} \arrow[d] \\ \widehat{\mathcal{W}} \arrow[r] & \mathcal{A}(\underline{V}) \end{tikzcd}$$

Observe that in creating $\mathcal{A}(\underline{V})$ from $\mathcal{A}$, the only generators which affect the cohomology of the resulting diagram are those from $s^{-1}\underline{V}$ associated with the original vector space diagram $\underline{V}$; these have degree n, and the differential on $s^{-1}\underline{V}$ is given by the map α into $\underline{Z}^{n+1}(\mathcal{A})$. We say that such an extension has degree n. Although there are choices made in the definition, it is shown in [Sc] that all choices give the same result up to isomorphism. Therefore this construction depends only on $\underline{V}$ and α.

Definition 4.1. A system $\mathcal{M}$ in $CDGA^{\mathcal{O}_G}$ is *minimal* if $\mathcal{M} = \bigcup_n \mathcal{M}(n)$ where $\mathcal{M}(0) = \mathcal{M}(1) = \underline{\mathbb{Q}}$ and $\mathcal{M}(n) = \mathcal{M}(n-1)(\underline{V}_n)$ is an elementary extension of degree n for some system of vector spaces $\underline{V}_n$.

If $\mathcal{A}$ is a fibrant diagram such that all entries are finitely generated and cohomologically 1-connected, then there exists a *minimal model* $\mathcal{M}$ of $\mathcal{A}$, which is a minimal object of $CDGA^{\mathcal{O}_G}$ with a weak equivalence $\mathcal{M} \to \mathcal{A}$. This can be interpreted in terms of our model category.

Theorem 4.2. *If $\mathcal{M}$ is a minimal object of $CDGA^{\mathcal{O}_G}$, then $\mathcal{M}$ is cofibrant in the model category structure on $CDGA^{\mathcal{O}_G}$.*

Corollary 4.3. *A minimal model $\mathcal{M} \to \mathcal{A}$ is a cofibrant approximation of $\mathcal{A}$.*

Theorem 4.2 follows from the next lemmas.

Lemma 4.4. *If $\underline{V}$ is any diagram of $\mathbb{Q}^{\mathcal{O}_G}$ and $\underline{I}$ is its injective envelope, then the inclusion of the induced diagrams of free CDGAs $\mathcal{V}^n \hookrightarrow \widehat{\mathcal{I}}^n$ is a cofibration for any n.*

Proof. We need to show that the inclusion $\mathcal{V}^n \hookrightarrow \widehat{\mathcal{I}}^n$ has the LLP with respect to acyclic fibrations. So suppose that f is an acyclic fibration and we have the outer commutative diagram

(C)
$$\begin{tikzcd} \mathcal{V}^n \arrow[r, "g"] \arrow[d, "i'"'] & \mathcal{A} \arrow[d, "f"] \\ \widehat{\mathcal{I}}^n \arrow[r, "\phi"] \arrow[ur, dashed, "\hat{\beta}"] & \mathcal{B} \end{tikzcd}$$

We can restrict to a commutative diagram of chain complexes in $\mathbb{Q}^{\mathcal{O}_G}$

$$\begin{tikzcd} \underline{V} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, "f"] \\ s^{-1}\underline{I} \oplus \underline{I} \arrow[r] \arrow[ur, dashed, "\beta"] & \mathcal{B} \end{tikzcd}$$

By Theorem 3.1, in the model category of chain complexes of $\mathbb{Q}^{\mathcal{O}_G}$ the map f is a fibration and a weak equivalence, and the map $\underline{V} \hookrightarrow s^{-1}\underline{I} \oplus \underline{I}$ is an injection and thus a cofibration. So by the axioms of this model structure a lift $\hat{\beta} : s^{-1}\underline{I} \oplus \underline{I} \to \mathcal{A}$ exists in $\mathbb{Q}^{\mathcal{O}_G}$. We extend this to a map $\widehat{\mathcal{I}}^n \to \mathcal{A}$ filling in the diagram. □

Lemma 4.5. *For any diagram $\underline{V}$ in $\mathbb{Q}^{\mathcal{O}_G}$, let $\mathcal{W}$ (resp. $\widehat{\mathcal{W}}$) be the free diagram (resp. free acyclic diagram) generated by the injective resolution of $\underline{V}$ used in creating an elementary extension by $\underline{V}$. Then the inclusion $\mathcal{W} \to \widehat{\mathcal{W}}$ is a cofibration.*

Proof. In any model category, the pushout of a cofibration is always a cofibration. We will write the desired inclusion map as a composition of pushouts of cofibrations. The system $\mathcal{W}$ is freely generated by the injective resolution $\underline{V}_0 \xrightarrow{w_0} \underline{V}_1 \xrightarrow{w_1} \cdots \to \underline{V}_m$. For each k, $\underline{V}_k$ is part of the injective resolution of $\underline{V}$, so it is injective and equal to its injective envelope. Therefore the inclusion $i : \mathcal{V}_k^{n+k+1} \to \widehat{\mathcal{V}}_k^{n+k+1}$ is a cofibration by Lemma 4.4. Then to build $\widehat{\mathcal{W}}$ from $\mathcal{W}$ using pushouts of cofibrations, we start at the end of the resolution and work down. In the first step we add $s^{-1}\underline{V}_m$; we form the pushout

$$\begin{tikzcd} \mathcal{V}_m^{n+m+1} \arrow[r, "\theta"] \arrow[d, "i"] & \mathcal{W} \arrow[d] \\ \widehat{\mathcal{V}}_m^{n+m+1} \arrow[r] & \mathcal{W}_n \end{tikzcd}$$

along the map $\theta : \mathcal{V}_m \to \mathcal{V}$ induced by the inclusion $\underline{V}_n \hookrightarrow \underline{V}_n$. Then we add the next lower step in the resolution by a pushout

$$\begin{tikzcd} \mathcal{V}_{m-1}^{n+m} \arrow[r] \arrow[d, "i"] & \mathcal{W}_n \arrow[d] \\ \widehat{\mathcal{V}}_{m-1}^{n+m} \arrow[r] & \mathcal{W}_{n-1} \end{tikzcd}$$

along the map $\underline{V}_{n-1} \to \mathcal{V}(s^{-1}\underline{V}_n)$ defined by $\theta + (-1)^{n-1}w_{n-1}$. Continuing down the resolution we eventually produce the system $\mathcal{W}_0 = \widehat{\mathcal{W}}$; each stage is inductively a pushout by Lemma 4.4. $\square$

Proof of Theorem 4.2. By definition, a minimal object is created by a sequence of pushouts of inclusions $\mathcal{W} \to \widehat{\mathcal{W}}$; these maps are cofibrations by Lemma 4.5. Therefore the inclusion $\underline{\mathbb{Q}} \to \mathcal{M}$ is a cofibration. $\square$

5. Rational equivariant homotopy

In this section we explain the connection between the previous results and rational equivariant homotopy. Recall that our motivation for looking at diagram categories is that we generally study a G-space by looking at the diagram of its fixed sets. We can justify this by an equivalence of homotopy categories.

Let $G\mathcal{S}$ denote the category of based G-spaces, and $\mathcal{S}^{\mathcal{O}_G^{op}}$ denote the diagram category of contravariant functors from $\mathcal{O}_G$ to based spaces. There is a natural functor $G\mathcal{S} \to \mathcal{S}^{\mathcal{O}_G^{op}}$ given by associating to a G-space X the diagram defined by $\underline{X}(H) = X^H$.

We can define a model category structure on $\mathcal{S}^{\mathcal{O}_G^{op}}$ coming from the model category of spaces: a weak equivalence (resp. fibration) between diagrams $f : \underline{X} \to \underline{Y}$ is defined as a map such that $f(H)$ is a weak equivalence (resp. fibration) for each H; a cofibration is a map which has the LLP with respect to all acyclic fibrations. Then a theorem of Elmendorf shows that the functor defined by taking fixed points induces an equivalence of the homotopy categories between $hG\mathcal{S}$ and $h\mathcal{S}^{\mathcal{O}_G^{op}}$ (see [Hi]; [M], Chapter VI).

This motivates our view of equivariant rationalization. A G-space X is said to be rational if each fixed set X^H is rational. Then the rationalization of a G-space X is a G-space $X_{\mathbb{Q}}$ with a map $X \to X_{\mathbb{Q}}$ such that for all H, the restriction of the map to the fixed set $X^H \to (X_{\mathbb{Q}})^H$ is the rationalization of X^H. With this definition, rationalization exists for any nilpotent G-space ([M], Chapter II). Thus in the diagram category $\mathcal{S}^{\mathcal{O}_G^{op}}$, we are simply rationalizing entrywise.

It will be convenient to work with the category of pointed simplicial sets SS in place of based spaces. Equivariantly, we can always arrange a simplicial decomposition of a space X such that G acts simplicially and each fixed set X^H is a simplicial complex by repeated triangulation. The standard model structure on SS is defined by: weak equivalences are maps whose geometric realizations induce weak equivalences of spaces; fibrations are Kan fibrations; and cofibrations are injective maps [Ho]. The model structure on pointed simplicial sets is cofibrantly generated by the set of inclusions $\{i : \partial\Delta[n]_+ \hookrightarrow \Delta[n]_+\}$.

This induces a model category structure on the diagrams $SS^{\mathcal{O}_G^{op}}$ by again defining weak equivalences and fibrations objectwise. By ([Hi], Theorem 11.6.1) the induced model structure on the diagram category $SS^{\mathcal{O}_G^{op}}$ is also cofibrantly generated, and we can take the set of generating cofibrations to be maps induced from generating cofibrations i of SS by the following. For any H, we can define the standard projective diagram $\underline{P}_H$ by $\underline{P}_H(K) = Hom_{\mathcal{O}_G^{op}}(H, K)$. Then we can define maps $i_H = id \otimes i : \underline{P}_H \otimes \partial\Delta[n]_+ \hookrightarrow \underline{P}_H \otimes \Delta[n]_+$. The cofibrations of $SS^{\mathcal{O}_G^{op}}$ are exactly the retracts of transfinite compositions of pushouts of the maps i_H. The usual equivalence of homotopy categories between simplicial sets hSS and spaces $h\mathcal{S}$ induces an equivalence of the diagram homotopy categories $h\mathcal{S}^{\mathcal{O}_G^{op}}$ and $hSS^{\mathcal{O}_G^{op}}$ ([Hi], Theorem 11.6.5). We will continue to think of the diagram of simplicial sets as the fixed points of the space, and will treat the simplicial complex and its geometric realization as equivalent.

Given any G-space, we can triangulate it as a G-CW complex [M]. Attaching a G-cell $G/H_+ \wedge D^n$ along $G/H_+ \wedge S^n$ is equivalent to taking a pushout along a map $i_H = id \otimes i : \underline{P}_H \otimes \partial\Delta[n]_+ \hookrightarrow \underline{P}_H \otimes \Delta[n]_+$, since the fixed points of the orbit G/H are exactly given by $\underline{P}_H$; in particular, the simplicial complex of any G-space is projective and cofibrant in the model category. Therefore when discussing equivariant rational homotopy theory we will be using the full subcategory $SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}}$ of cofibrant diagrams in $SS^{\mathcal{O}_G^{op}}$ such that (the geometric realization of) each entry is rational, 1-connected and of finite type. Note that this subcategory is not itself a model category, since it does not have arbitrary colimits and limits; all the other axioms hold, however.

We now develop the relationship between the rational equivariant category $SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}}$ and the algebraic models we have been discussing. The main tool for passing from geometry to algebra is the de Rham functor $\Omega : SS \to CDGA$ of rational polynomial differential forms. On an n-simplex σ^n, we define $\Omega(\sigma^n)$ to the be collection of differential forms $\Sigma_I f_I(t_0, \ldots, t_n) dt_{i_1} \wedge \cdots \wedge dt_{i_p}$ where f_I is a polynomial with rational coefficients; then a global form on a simplicial set is a collection of forms, one for each simplex, which agree on common faces. The functor Ω has an adjoint functor $F : CDGA \to SS$ defined by $F(A)_p = Hom_{CDGA}(X, \nabla(p, *))$, where $\nabla(p, *)$ is a particular CDGA defined for this purpose in [BG], Chapter 5. Note that Ω and F are contravariant functors.

These functors induce adjoint functors $\underline{\Omega} : SS^{\mathcal{O}_G^{op}} \leftrightarrows CDGA^{\mathcal{O}_G} : \underline{F}$ on the diagram categories by the obvious composition of functors ([Hi], Lemma 11.6.4). We begin by showing that these functors respect the model structure on these categories.

Proposition 5.1. *The functors $\underline{\Omega} : SS^{\mathcal{O}_G^{op}} \leftrightarrows CDGA^{\mathcal{O}_G} : \underline{F}$ are Quillen adjoint functors.*

To show this we use the next lemma.

Lemma 5.2. *The de Rham functor of diagrams $\underline{\Omega} : SS^{\mathcal{O}_G^{op}} \to CDGA^{\mathcal{O}_G}$ takes cofibrations in $SS^{\mathcal{O}_G^{op}}$ to fibrations in $CDGA^{\mathcal{O}_G}$.*

Proof. We start by showing that $\underline{\Omega}$ takes the generating cofibrations $i_H = id \otimes i : \underline{P}_H \otimes \partial\Delta[n]_+ \hookrightarrow \underline{P}_H \otimes \Delta[n]_+$ of $SS^{\mathcal{O}_G^{op}}$ to fibrations in $CDGA^{\mathcal{O}_G}$. We need to show that $\underline{\Omega}(i_H)$ is surjective and that the kernel is injective as a diagram of rational vector spaces. The non-equivariant de Rham differential form functor Ω takes an injection of simplicial sets to a surjection of CDGAs. The generating cofibrations i of SS are injections, so the induced diagram maps i_H are also injective at each object; so $\underline{\Omega}(i_H)$ is surjective as required. To show that the kernel is an injective object of $\mathbb{Q}^{\mathcal{O}_G}$, we observe that the contravariant functor $\underline{\Omega}$ turns a projective diagram $\underline{P}_H \otimes X$ into a diagram of the dual form $I_H(\mathbb{Q}[Aut(H) \otimes \Omega(X)])$ where I_H is the standard injective construction from Definition 2.2. Similarly, if we let K be the kernel of $\Omega(i) : \Omega(\Delta[n]_+) \to \Omega(\partial\Delta[n]_+)$, then the kernel of $\underline{\Omega}(i_H) : \underline{\Omega}(\underline{P}_H \otimes \Delta[n]_+) \to \underline{\Omega}(\underline{P}_H \otimes \partial\Delta[n]_+)$ is exactly $I_H(\mathbb{Q}[Aut(H) \otimes K])$. Therefore $\underline{\Omega}(i_H)$ is surjective with an injective kernel, and so is a fibration.

For a general cofibration, observe that Ω takes pushouts to pullbacks. So if f is a pushout of a map i_H, then $\underline{\Omega}(f)$ is a pullback of the fibration $\underline{\Omega}(i_H)$. Surjections and kernels are preserved under pullbacks, so $\underline{\Omega}(f)$ is also a fibration. Since fibrations are exactly those maps which have the RLP with respect to the maps $\mathcal{J}$ by Proposition 3.6, they are closed under transfinite composition; and fibrations are closed under retracts by the axioms of model categories. Therefore $\underline{\Omega}$ takes any cofibration to a fibration. □

Proof of Proposition 5.1. Since weak equivalences are defined objectwise, the functor $\underline{\Omega}$ preserves weak equivalences of diagrams. The functor $\underline{\Omega}$ also takes cofibrations to fibrations by Lemma 5.2, so it will also take any acyclic cofibration to an acyclic fibration. So by ([Hi], Proposition 8.5.3) the adjoint functors $\underline{\Omega}$ and $\underline{F}$ form a Quillen pair. □

In order to show that these functors give an equivalence between G-spaces and models, we use the correspondence between the equivariant Postnikov deomposition of a space and the structure of its minimal model. The minimal model of X is defined to be the minimal model of $\underline{\Omega}(X)$; so there is a quasi-isomorphism $\mathcal{M}_X \to \underline{\Omega}(X)$, and $\mathcal{M}_X$ is created by a sequence of elementary extensions. A 1-connected G-space X has an equivariant Postnikov decomposition which describes X as an inverse limit of a tower of principal G-fibrations, where each fibre is an equivariant Eilenberg-Mac Lane space $K(\underline{V}, n)$. The next two lemmas show that there is an exact correspondance between these G-fibrations and elementary extensions.

Lemma 5.3 (Triantafillou [T], Lemma 6.3)**.** *If $K(\underline{V}, n) \to E \to X$ is a principal G-fibration, and $\mathcal{M}_X$ is the minimal model of X, then there is an elementary extension $\mathcal{M}_X(\underline{V})$ of degree n such that $\mathcal{M}_X(\underline{V})$ is the minimal model of E.*

Lemma 5.4. *If $\mathcal{A} \to \mathcal{A}(\underline{V})$ is an elementary extension of degree n, then $\underline{F}(\mathcal{A}(\underline{V})) \to \underline{F}(\mathcal{A})$ is a principal G-fibration whose fibre is an equivariant Eilenberg-Mac Lane space $K(\underline{V}, n)$.*

Proof. The elementary extension $\mathcal{A}(\underline{V})$ can be described by a pushout diagram

$$\begin{tikzcd} \mathcal{W} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ \widehat{\mathcal{W}} \arrow[r] & \mathcal{A}(\underline{V}) \end{tikzcd}$$

where $\mathcal{W}$ (resp. $\widehat{\mathcal{W}}$) is the free diagram (resp. free acyclic diagram) generated by the injective resolution of $\underline{V}$ in degree $n+1$. If we apply $\underline{F}$ to this diagram we get the diagram

$$\begin{tikzcd} \underline{F}(\mathcal{A}(\underline{V})) \arrow[r] \arrow[d] & \underline{F}(\widehat{\mathcal{W}}) \arrow[d] \\ \underline{F}(\mathcal{A}) \arrow[r] & \underline{F}(\mathcal{W}) \end{tikzcd}$$

This is a pullback diagram, since F takes pushouts to pullbacks and both can be defined objectwise in the diagram category. If we examine these diagrams at a single object H, we observe that the inclusion of the free CDGA $\mathbb{Q}(\underline{V}(H))$ in $\mathcal{W}(H)$ is a quasi-isomorphism, so $\underline{F}(\mathcal{W})(H)$ is homotopic to an Eilenberg-Mac Lane space $K(\underline{V}(H), n+1)$ for each H; therefore $\underline{F}(\mathcal{W})$ is an equivariant $K(\underline{V}, n+1)$. Similarly, $\widehat{\mathcal{W}}(H)$ is acyclic, and so $\underline{F}(\widehat{\mathcal{W}})(H)$ is contractible for each H, and $\underline{F}(\widehat{\mathcal{W}})$ is equivariantly contractible. Lastly, the inclusion $\mathcal{W}(H) \to \widehat{\mathcal{W}}(H)$ is a composition of pushouts of maps of free CDGAs $\mathbb{Q}(V) \to \mathbb{Q}(s^{-1}V \oplus V)$, and so is a cofibration in the model category of CDGAs ([BG], Chapter 4); therefore $F(\widehat{\mathcal{V}}(H)) \to F(\mathcal{V}(H))$ is a Kan fibration ([BG], Lemma 8.2). Fibrations are defined objectwise in $SS^{\mathcal{O}_G}$, and so this is a G-fibration.

Therefore $\underline{F}(\mathcal{A}(\underline{V})) \to \underline{F}(\mathcal{A})$ is the pullback of a fibration from a contractible space to an equivariant Eilenberg-Mac Lane space $K(\underline{V}, n+1)$, and so is a principal G-fibration with fibre $K(\underline{V}, n)$. ☐

We can now prove the main result of this section which shows the precise correspondence between the rational homotopy of a G-space and the minimal model. Recall that $SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}}$ consists of cofibrant diagrams of simplicial sets whose geometric realizations are rational, simply connected and of finite type for all entries H. We let $CDGA_0^{\mathcal{O}_G}$ denote the corresponding full subcategory of fibrant diagrams whose entries are all cohomologically finitely generated and 1-connected. The restriction to fibrant diagrams shows up in [T] as a restriction to diagrams of CDGAs which are injective as diagrams of vector spaces, and in the ensuing injective resolutions needed in the creation of elementary extensions and minimal models. Here we have an explanation for this condition: since any G space X gives a cofibrant object of $SS^{\mathcal{O}_G^{op}}$, Proposition 5.1 implies that $\underline{\Omega}(X)$ is always fibrant in $CDGA^{\mathcal{O}_G}$ and so it makes sense to restrict ourselves to such objects.

Proposition 5.5. *The functors $\underline{\Omega}$ and $\underline{F}$ restrict to functors of the subcategories $\underline{\Omega} : SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}} \leftrightarrows CDGA_0^{\mathcal{O}_G} : \underline{F}$.*

Proof. If X is an object of $SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}}$, then X has a G-Postnikov deomposition with $X_0 = X_1 = *$ [T]. By Lemma 5.3, the minimal model of $\underline{\Omega}(X)$ is composed of elementary extensions corresponding to the fibrations of the Postnikov tower.

Therefore $\mathcal{M}$ is finitely generated and 1-connected; since it is quasi-isomorphic to $\underline{\Omega}(X)$, then $\underline{\Omega}(X)$ is in $CDGA_0^{\mathcal{O}_G}$.

Conversely, if $\mathcal{A}$ is in $CDGA_0^{\mathcal{O}_G}$, its minimal model is finitely generated and 1-connected; therefore Lemma 5.4 shows that $\underline{F}(\mathcal{A})$ is an inverse limit of G-fibrations with fibres $K(\underline{V}, n)$ for $n \geq 2$, where $\underline{V}$ is finitely generated. Therefore $\underline{F}(\mathcal{A})$ is in $SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}}$. □

Theorem 5.6. *The functors $\underline{\Omega} : SS_{\mathbb{Q}}^{\mathcal{O}_G^{op}} \leftrightarrows CDGA_0^{\mathcal{O}_G} : \underline{F}$ give a Quillen equivalence.*

Proof. We need to show that if X is in $SS^{\mathcal{O}_G^{op}}$ and $\mathcal{A}$ is in $CDGA^{\mathcal{O}_G}$, then $X \to \underline{F}(\mathcal{A})$ is a weak equivalence in $SS^{\mathcal{O}_G^{op}}$ if and only if $\mathcal{A} \to \Omega(X)$ is a weak equivalence in $CDGA^{\mathcal{O}_G}$.

Since $\mathcal{A}$ is in $CDGA_0^{\mathcal{O}_G}$, it is injective as a diagram of vector spaces and so has a minimal model $\mathcal{M}_{\mathcal{A}}$ with a weak equivalence $\mathcal{M}_{\mathcal{A}} \to \mathcal{A}$ [T]. Then the map $\underline{F}(\mathcal{A}) \to \underline{F}(\mathcal{M}_{\mathcal{A}})$ is a weak equivalence, and so the map $X \to \underline{F}(\mathcal{A})$ is a weak equivalence if and only if the composite map $X \to \underline{F}(\mathcal{A}) \to \underline{F}(\mathcal{M}_{\mathcal{A}})$ is a weak equivalence. By Lemma 5.4, $\underline{F}(\mathcal{M}_{\mathcal{A}})$ has a G-Postnikov decomposition whose fibrations match the elementary extensions of $\mathcal{M}_{\mathcal{A}}$. Now $X \to \underline{F}(\mathcal{M}_{\mathcal{A}})$ is a weak equivalence if and only if the induced map on the G-Postnikov towers of these spaces is an equivalence at each stage, which will happen precisely when the G-Postnikov tower structure of X corresponds exactly to the elementary extensions of $\mathcal{M}_{\mathcal{A}}$.

Now let $\mathcal{M}_X$ be the minimal model of X; by Lemma 5.3 the elementary extensions of $\mathcal{M}_X$ correspond to the fibrations in the G-Postnikov tower of X. Given the map $\mathcal{A} \to \underline{\Omega}(X)$ we can extend to the diagram

$$\begin{tikzcd}
& & \mathcal{M}_X \arrow[d] \\
\mathcal{M}_{\mathcal{A}} \arrow[r] \arrow[urr] & \mathcal{A} \arrow[r] & \underline{\Omega}(X)
\end{tikzcd}$$

The lift $\mathcal{M}_{\mathcal{A}} \to \mathcal{M}_X$ exists since $\mathcal{M}_X \to \underline{\Omega}(X)$ is a weak equivalence and $\mathcal{M}_{\mathcal{A}}$ is minimal ([T], Propostion 5.5), and it is a quasi-isomorphism if and only if it is an isomorphism since both source and target are minimal ([T], Theorem 5.2). But $\mathcal{M}_{\mathcal{A}} \simeq \mathcal{M}_X$ if and only if these two systems are constructed out of isomorphic elementary extensions. Therefore the map $\mathcal{A} \to \Omega(X)$ is a weak equivalence if and only if the G-Postnikov tower structure of X corresponds exactly to the elementary extensions of $\mathcal{M}$.

Thus we see that the condition for the map $X \to \underline{F}(\mathcal{A})$ to be a weak equivalence in $SS^{\mathcal{O}_G^{op}}$ is precisely the same as the condition for $\mathcal{A} \to \underline{\Omega}(X)$ to be a weak equivalence in $CDGA^{\mathcal{O}_G}$. □

Thus we have a preciese version of the statement that all of the rational equivariant homotopy information of G-simply connected spaces of finite type is captured by the algebraic minimal models.

6. Proofs from Section 3

This section contains the proofs of Propositions 3.6 and 3.7 which identify fibrations and acyclic fibrations in terms of lifting properties, and are central to proving the existence of the model structure.

Recall from Definition 3.3 that if $\underline{V}$ is any diagram of vector spaces in $\mathbb{Q}^{\mathcal{O}_G}$, then we form the free $CDGA^{\mathcal{O}_G}$ object $\mathcal{V}^n$ generated by $\underline{V}$ in degree n, and the free acyclic $CDGA^{\mathcal{O}_G}$ object $\widehat{\mathcal{V}}^n$ generated by $s^{-1}\underline{V} \oplus \underline{V}$ in degrees $n-1$ and n. The maps from the relevant propositions used these constructions on two specific types of vector space diagrams, the projective diagrams $\{\underline{P}_H\}$ and the promised injectivity test diagrams $\mathbf{T} = \{\underline{T}_H\}$. We will begin by examining properties of each of these closely, beginning with the projectives.

Recall that for any H we define free projective objects by letting $\underline{P}$ be the projective diagram $\underline{P}_H = P_H(\mathbb{Q}[Aut(H)])$ of Definition 2.1; then we have associated diagrams $\mathcal{P}^n_H$ and $\widehat{\mathcal{P}}^n_H$. We will use this notation throughout this section. These objects have the following useful properties.

Lemma 6.1. (1) *A morphism $\alpha : \mathcal{P}^n_H \to \mathcal{A}$ in $CDGA^{\mathcal{O}_G}$ is equivalent to an element $a \in Z^n\mathcal{A}(H)$, under the correspondence $\alpha \leftrightarrow \alpha(eH)$.*

(2) *A morphism defined by $\alpha : \widehat{\mathcal{P}}^n_H \to \mathcal{A}$ in $CDGA^{\mathcal{O}_G}$ is equivalent to an element $a \in \mathcal{A}^{n-1}(H)$, under the correspondence $\alpha \leftrightarrow \alpha(s^{-1}eH)$.*

(3) *Let $i : \mathcal{P}^n_H \to \widehat{\mathcal{P}}^n_H$ be the inclusion. Then given maps as in the outer commuting diagram*

$$\begin{tikzcd} \mathcal{P}^n_H \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathcal{A} \arrow[d, "p"] \\ \widehat{\mathcal{P}}^n_H \arrow[r, "\beta"] \arrow[ur, dashed, "\gamma"] & \mathcal{B} \end{tikzcd}$$

a lift γ of this diagram is equivalent to an element $a \in \mathcal{A}^{n-1}(H)$ which satisfies $da = \alpha(eH)$ and $p(a) = \beta(s^{-1}eH)$, under the correspondance $\gamma \leftrightarrow \gamma(s^{-1}eH)$.

Proof. (1) Since $\mathcal{P}^n_H$ is a diagram of free CDGAs generated by $\underline{P}_H$ with $d = 0$, a map $\mathcal{P}^n_H \to \mathcal{A}$ is equivalent to a map $\underline{P}_H \to Z^n\mathcal{A}$ in $\mathbb{Q}^{\mathcal{O}_G}$. Then by Lemma 2.3 this is equivalent to a map $\mathbb{Q}[Aut(H)] \to Z^n\mathcal{A}(H)$, and this is specified uniquely by the element $a \in Z^n\mathcal{A}(H)$ which is the image of eH.

(2) Since $\widehat{\mathcal{P}}^n_H$ is a diagram of free acyclic CDGAs generated by $s^{-1}\underline{P}_H \oplus \underline{P}_H$, a map $\widehat{\mathcal{P}}^n_H \to \mathcal{A}$ is equivalent to a map in $\mathbb{Q}^{\mathcal{O}_G}$ from $s^{-1}\underline{P}_H \to \mathcal{A}^{n-1}$; the map on $\underline{P}_H$ is determined by the differential d. Then as in the previous case, this is specified uniquely by an element $a \in \mathcal{A}^{n-1}(H)$ which is the image of $s^{-1}eH$.

(3) By part (1), the map α of the commutative diagram is determined by $\alpha(eH) \in Z^n\mathcal{A}(H)$, and by part (2), the map β is determined by $\beta(s^{-1}eH) \in \mathcal{A}^{n-1}(H)$. The lift $\gamma : \widehat{\mathcal{P}}^n_H \to \mathcal{A}$ is determined by $a \in \mathcal{A}^{n-1}(H)$ such that $\gamma(s^{-1}eH) = a$. The top triangle commutes if $da = \alpha(eH)$, and the bottom triangle commutes if $p(a) = \beta(s^{-1}eH)$. ☐

Next, we want to define a set of test diagrams $\mathbf{T}$ which will help us characterize injectivity in $\mathbb{Q}^{\mathcal{O}_G}$. For convenience, we use the notation $K > H$ to denote a subgroup K which is contains some conjugate gHg^{-1} of H; similarly $[K] > H$ denotes a conjugacy class of subgroups $[K]$ such that H is contained in some conjugate K of the class.

We begin with the following construction.

Definition 6.2. For each $H \subseteq G$, we define the truncation $Tr_H(\underline{V})$ of a diagram $\underline{V}$ to be

$$Tr_H(\underline{V})(K) = \begin{cases} \underline{V}(K) & \text{if } K > H, \\ 0 & \text{otherwise.} \end{cases}$$

We use this to define our test diagrams.

Definition 6.3. For each conjugacy class $[K]$ of subgroups of G, we define $\underline{I}_{[K]}$ using the injective construction of Definition 2.2 by $\underline{I}_{[K]} = I_K(\mathbb{Q}[Aut(K)])$, where K is some chosen representative of the class $[K]$; note that up to isomorphism, this is independent of the choice of representative K. Then for each $H \subseteq G$ we have an injective diagram $\bigoplus_{[K]>H} \underline{I}_{[K]}$. Our set of test diagrams $\mathbf{T}$ is defined to be the truncations $\{\underline{T}_H = Tr_H(\bigoplus_{[K]>H} \underline{I}_K)\}$.

The key property of these diagrams is the following.

Lemma 6.4. *A diagram $\underline{A}$ is an injective object of $\mathbb{Q}^{\mathcal{O}_G}$ if and only if it satisfies the following condition: given any map $f : \underline{T}_H \to \underline{A}$ from a diagram of $\mathbf{T}$, we can extend this to a map from the injective envelope of $\underline{T}_H$.*

Proof. If $\underline{A}$ is an injective object, then the extension $\underline{T}_H \hookrightarrow \underline{I} \to \underline{A}$ exists because $\underline{T}_H \hookrightarrow \underline{I}$ is an embedding.

Conversely, suppose $\underline{A}$ is **not** injective. Then we have an embedding $\underline{A} \to \underline{I}_A$ where $\underline{I}_A$ is the injective envelope of Definition 2.6. Recall that this is defined by $\underline{I}_A = \bigoplus_{[H]} I_H(\ker_H)$, the injective construction applied to the vector spaces $\ker_H = \bigcap_{K>H} \ker \underline{A}(\hat{e}_{H,K})$, which are the intersections of the kernels of all the structure maps $\underline{A}(H) \to \underline{A}(K)$.

The fact that $\underline{A}$ is not injective means that the embedding $\underline{A} \to \underline{I}_A$ is not an isomorphism. So for some H, this map must fail to be surjective. Choose a maximal such subgroup M, for which the map $\underline{A}(K) \to \underline{I}_A(K)$ is an isomorphism for all $K > M$. Now $I_H(\ker_H)(M)$ is 0 unless $gHg^{-1} \subseteq M$ for some conjugate of H, in which case we get a direct product of copies of $\ker_H$, one for each conjugate of H in M; therefore we can write $\underline{I}_A(M) = \bigoplus_{[H]<M}(\ker_H)$. Let $\alpha = (\alpha_{K_1}, \alpha_{K_2}, \ldots, \alpha_{K_m})$ be an element of $\underline{I}_A(M)$ which is not in the image of $\underline{A}(M)$.

We will use α to define a map $\underline{T}_M \to \underline{A}$. For each $K < M$, we define a map $\mathbb{Q}[Aut(K)] \to \ker_K$ by $1 \to \alpha_K$. This induces a map $\underline{I}_K(\mathbb{Q}[Aut(K)]) \to I_K(\ker_K)$ by Lemma 2.3. Then we use the zero map $\underline{I}_K \to I_H(\ker_H)$ for $[K] \neq [H]$ and put these together to get a map $\underline{\phi} : \bigoplus_{[K]>M} \underline{I}_K \to \underline{I}_A$. We can restrict $\underline{\phi}$ to subgroups $[K] > M$ to get a map from the truncation $T_M = Tr_M(\bigoplus_{[K]>M} \underline{I}_K)$ to $\underline{I}_A$. But $\underline{A} \to \underline{I}_A$ is an isomorphism for all $[K] > M$, and so we have defined a map $\underline{T}_H \to \underline{A}$.

The injective envelope I_{T_H} of $\underline{T}_H$ is just $\bigoplus_{[K]>H} \underline{I}_K$ without truncation. It is clear that this map **cannot** be extended to $I_{T_H} \to \underline{A}$; by design, such a map would need to hit the element α at M to be a coherent diagram map, but α was selected to not be in the image of $\underline{A}(M)$.

Thus we can detect the lack of injectivity of $\underline{A}$ by using a map from one of our test diagrams $\underline{T}_H$. If, on the other hand, all of maps from objects in $\mathbf{T}$ do extend, then $\underline{A}$ must be injective. $\square$

We can now prove the two propositions.

Proof of Proposition 3.6. ($\Leftarrow$) Suppose $p : \mathcal{A} \to \mathcal{B}$ is a map which has the RLP with respect to the maps of $\mathcal{J}$. We need to show that p is a fibration, which means that p is a surjection and that $ker(p)$ is an injective object of $\mathbb{Q}^{\mathcal{O}_G}$ in each degree $n \geq 1$.

Surjectivity: Let $b \in \mathcal{B}(H)$ be an element of degree $n-1$. By Lemma 6.1, there is a map $\alpha : \widehat{\mathcal{P}}^n_H \to \mathcal{B}$ defined by $\alpha(s^{-1}eH) = b$. We can fill in the outer commutative diagram

$$(A) \qquad \begin{tikzcd} \underline{\mathbb{Q}} \arrow[r] \arrow[d, "j"'] & \mathcal{A} \arrow[d, "p"] \\ \widehat{\mathcal{P}}^n_H \arrow[r, "\alpha"] \arrow[ur, dashed, "\beta"] & \mathcal{B} \end{tikzcd}$$

By assumption the lift β exists, and so $\beta(s^{-1}eH) = a$ in $\mathcal{A}$ satisfies $p(a) = b$. Thus any $b \in \mathcal{B}$ is in the image of p.

Injectivity of the kernel: Let $\underline{K}$ be the diagram of vector spaces in $\mathbb{Q}^{\mathcal{O}_G}$ defined by $ker(p)$ in degree $n-1$ for $n \geq 2$, and let $\alpha : \underline{T}_H \to \underline{K}$ be any map from a test diagram of vector spaces in $\mathbf{T}$. Then $\underline{T}_H$ has an injective envelope $\underline{I}_{T_H}$, and we consider the induced inclusion $j' : \widehat{\mathcal{T}}^n_H \to \widehat{\mathcal{I}}^n$. The map $f : \underline{T}_H \to \underline{K}$ and the inclusion of the kernel $\underline{K} \hookrightarrow \mathcal{A}$ induce a map $f' : \widehat{\mathcal{T}}^n_H \to \mathcal{A}$ by f on $s^{-1}\underline{T}_H$, and df on $\underline{T}_H$, extending to a map in $CDGA^{\mathcal{O}_G}$ by freeness. We can extend this to the outer commutative diagram:

$$(B) \qquad \begin{tikzcd} \widehat{\mathcal{T}}^n_H \arrow[r, "f'"] \arrow[d, "j'"'] & \mathcal{A} \arrow[d, "p"] \\ \widehat{\mathcal{I}}^n \arrow[r, "\alpha"] \arrow[ur, dashed, "\beta"] & \mathcal{B} \end{tikzcd}$$

by defining $\widehat{\mathcal{I}}^n \to \mathcal{B}$ to be zero on all positive degree generators. The lift exists by assumption, and must land in $\underline{K} = ker(p)$ in order to commute. If we restrict to $s^{-1}\underline{I} \to \underline{K}$ we get a map in $\mathbb{Q}^{\mathcal{O}_G}$ from the injective envelope $\underline{I}$ of $\underline{T}_H$. Since $\underline{T}_H$ was a general element of our set of test diagrams, Lemma 6.4 implies that $\underline{K}$ is injective.

($\Rightarrow$) Suppose $p : \mathcal{A} \to \mathcal{B}$ is a fibration. We want to show that p has the RLP with respect to any map in $\mathcal{J}$.

Given an outer commutative diagram as in (A) above for some H and n, we define $b \in \mathcal{B}^{n-1}(H)$ by $b = \alpha(s^{-1}eH)$. Since p is a fibration and thus surjective, b has a pre-image $a \in \mathcal{A}^{n-1}(H)$. Then by Lemma 6.1, we can define a map $\beta : \widehat{\mathcal{P}}^n_H \to \mathcal{A}$ by $\beta(s^{-1}eH) = a$. Since $p(a) = b$, the maps $p\beta$ and α are defined by the same element of $\mathcal{B}^{n-1}(H)$, and so by Lemma 6.1 they are the same map. Thus the lift β makes the diagram commute.

Now consider an outer commutative diagram as in (B). By assumption, p is a fibration, and so is surjective with its kernel an injective object of $\mathbb{Q}^{\mathcal{O}_G}$ in each degree $n \geq 1$. Therefore we can decompose the diagrams of vector spaces $\mathcal{A}^{n-1} \simeq \mathcal{B}^{n-1} \oplus \underline{K}$ in $\mathbb{Q}^{\mathcal{O}_G}$ for $n \geq 2$. Since $\underline{K}$ is an injective object, Lemma 6.4 says that we can extend the restriction of $f : s^{-1}\underline{T}_H \to \mathcal{A}^{n-1}$ to the inclusion $s^{-1}\underline{T}_H \hookrightarrow s^{-1}\underline{I} \to \mathcal{A}^{n-1}$ and get a map $\xi : s^{-1}\underline{I} \to \underline{K}$ (note that $s^{-1}\underline{T}_H$ is a test diagram of

T since $\underline{T}_H$ is one). Then we can define the desired lift β on $s^{-1}\underline{I}$ by $\alpha \oplus \xi$ making the diagram commute, and extend to all of $\widehat{\mathcal{I}}^n$ by freeness. $\square$

Proof of Proposition 3.7. ($\Rightarrow$) Suppose that f is an acyclic fibration. We want to show that f has the RLP with respect to any map in $\mathcal{I}$.

Suppose we have a commutative diagram

$$(C) \qquad \begin{tikzcd} \mathcal{T}^n_H \arrow[r, "g"] \arrow[d, "i'"'] & \mathcal{A} \arrow[d, "f"] \\ \widehat{\mathcal{I}}^n \arrow[r, "\phi"] \arrow[ur, dashed, "\hat{\beta}"] & \mathcal{B} \end{tikzcd}$$

The map $\mathcal{T}^n \to \widehat{\mathcal{I}}^n$ is a cofibration by Lemma 4.4, and therefore has the LLP with respect to acyclic fibrations; therefore the lift exists.

Now suppose we have the outer diagram

$$(D) \qquad \begin{tikzcd} \mathcal{P}^n_H \arrow[r, "g"] \arrow[d, "i"'] & \mathcal{A} \arrow[d, "f"] \\ \widehat{\mathcal{P}}^n_H \arrow[r, "\phi"] \arrow[ur, dashed, "\beta"] & \mathcal{B} \end{tikzcd}$$

Then by Lemma 6.1, the map g is specified by an element $a \in Z^n\mathcal{A}$ such that $g(eH) = a$, and ϕ is specified by an element $b \in \mathcal{B}$ such that $\phi(s^{-1}eH) = b$; since the diagram commutes, these elements must satisfy $fg(eH) = f(a) = db$. Then the induced map on cohomology $f^* : H^n(\mathcal{A}(H)) \to H^n(\mathcal{B}(H))$ takes $[a]$ to $[db]$, and since db is a boundary, $[db] = 0$. Since f^* is a cohomology isomorphism, we must have $[a] = 0$ and so a is also a boundary; there is an element $a' \in \mathcal{A}^{n-1}(H)$ such that $da' = a$. Then $\beta(s^{-1}eH) = a'$ defines a map $\beta : \widehat{\mathcal{P}}^n_H \to \mathcal{A}$ by Lemma 6.1; this determines the desired lift.

($\Leftarrow$) Suppose that f has the RLP with respect to all maps of $\mathcal{I}$. We need to show that f is an acyclic fibration. This means that f must be a surjective cohomology isomorphism whose kernel in degree $n \geq 1$ is injective.

We start by showing that f is surjective. To do this, we first show that the induced map f^* is surjective on cohomology. To show this, let $b \in Z^{n-1}\mathcal{B}(H)$ be any cocycle, and define a map $\phi : \widehat{\mathcal{P}}^n_H \to \mathcal{B}$ by $\phi(s^{-1}eH) = b$ using Lemma 6.1. Then $db = 0$, so if we define $g : \underline{P}^n \to \mathcal{A}$ by 0 and extend over the free CDGAs, we get a commutative diagram as in (D). By assumption there exists a lift β, so $\beta(s^{-1}eH) = a \in \mathcal{A}$ is an element such that $f(a) = b$ and $da = 0$. Thus we have shown that for any $b \in Z^{n-1}\mathcal{B}(H)$, there is a pre-image $a \in Z^{n-1}\mathcal{A}(H)$. So when we restrict f to cocycles $Z^{n-1}\mathcal{A}(H) \to Z^{n-1}\mathcal{B}(H)$, it is surjective. This implies that $f^* : H^*(\mathcal{A}(H)) \to H^*(\mathcal{B}(H))$ is also surjective.

Next, consider an element $b \in \mathcal{B}^{n-1}(H)$. Then $db \in Z^n(\mathcal{B})(H)$ is a cocycle, and so by the previous paragraph there is a cocycle $a \in Z^n\mathcal{A}(H)$ such that $f(a) = db$. Now define $g : \mathcal{P}^n_H \to \mathcal{A}$ by $g(eH) = a$; similary define $\phi : \widehat{\mathcal{P}}^n_H \to \mathcal{B}$ by $\phi(s^{-1}eH) = b$. This again defines a commutative diagram as in (D), and by assumption there is a lift $\beta : \widehat{\mathcal{P}}^n_H \to \mathcal{A}$. Then $a' = \beta(s^{-1}eH)$ satisfies $f(a') = b$ and we see that b has a pre-image. Thus f is surjective.

Now we show that $f^* : H^*(\mathcal{A}(H)) \to H^*(\mathcal{B})$ is also injective. Consider a class $[a] \in H^n(\mathcal{A}(H))$ such that $[a]$ is in $ker(f^*)$; then $[a]$ is defined by a cocycle $a \in$

$Z^n\mathcal{A}(H)$ such that $[f(a)] = 0$, so $f(a) = db$ for some $b \in \mathcal{B}^{n-1}(H)$. We define another commutative diagram as in (D) by the maps $g : \mathcal{P}^n_H \to \mathcal{A}$ such that $g(eH) = a$ and $\phi : \widehat{\mathcal{P}}^n_H \to \mathcal{B}$ such that $\phi(s^{-1}eH) = b$. Then the lift β specifies an element $a' = \beta(s^{-1}eH)$ such that $d(a') = g(eH) = a$ and so we also see that $[a] = 0$. Therefore f^* is injective for all H.

We have shown that $f^n(H)$ is surjective, and $f^*(H)$ is injective on cohomology for all H; therefore $f^n(H)$ must be a surjective cohomology isomorphism.

Lastly, we show that as a diagram of vector spaces, the kernel of f is injective in each degree $n \geq 1$. We begin by showing that $Z^n(\mathcal{A}) \cap ker(p)$ is injective for each $n \geq 1$. We will use Lemma 6.4. Let $\underline{Z}^n = Z^n(\mathcal{A}) \cap ker(p)$, and let $\phi : \underline{T}_H \to \underline{Z}^n$ be a map in $\mathbb{Q}^{\mathcal{O}_G}$. Then we can compose with the inclusion $\underline{Z}^n \hookrightarrow Z^n(\mathcal{A})$ and extend to get a map $\alpha : \mathcal{T}^n_H \to \mathcal{A}$ such that the composition $f\alpha$ is zero. So we can define a commutative diagram

$$\begin{tikzcd}
\mathcal{T}^n_H \arrow[r, "\alpha"] \arrow[d, "j"'] & \mathcal{A} \arrow[d, "f"] \\
\widehat{\mathcal{I}}^n \arrow[r] \arrow[ur, dashed, "\beta"] & \mathcal{B}
\end{tikzcd}$$

by using the map $\widehat{\mathcal{I}}^n \to \mathcal{B}$ which is zero on all positive degree generators.

By assumption, a lift β in this diagram exists. Now restricting β to the generating vector space diagram $\underline{I}$ in $\widehat{\mathcal{I}}^n$ gives a map in $\mathbb{Q}^{\mathcal{O}_G}$ from the injective envelope $\underline{I}$ of $\underline{T}_H$ which extends the original map ϕ. Since ϕ was an arbitrary map from a test diagram $\underline{T}_H$ of $\mathbf{T}$, this shows that $\underline{Z}^n$ satisfies the condition of Lemma 6.4 and so is injective.

Now we want to show that the kernel of f is also an injective object. The arguments above show that $\ker(f^n)$ is acyclic, since f is a cohomology isomorphism. Let $\underline{K}^n$ be the kernel of f in degree n; then $\underline{Z}^n \subseteq \underline{K}^n$. Since $\underline{Z}^n$ is injective, we can write $\underline{K}^n \simeq \underline{Z}^n \oplus \underline{L}$; then $\underline{L} \simeq im(d) \subseteq \underline{K}^{n+1}$. But now $\underline{L}$ is exactly the boundaries of $\underline{K}^{n+1}$, and $\ker(f^n)$ is acyclic, so $\underline{L} \simeq \underline{Z}^{n+1}$ which is also an injective diagram of $\mathbb{Q}^{\mathcal{O}_G}$. So $\underline{K}^n$ is the direct sum of two injective diagrams in $\mathbb{Q}^{\mathcal{O}_G}$; by examining the structure of injective objects from Proposition 2.5, we see that this implies that $\underline{K}^n$ is also injective. □

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