THE BROWN-PETTERSON COHOMOLOGY
OF THE CLASSIFYING SPACES
OF THE PROJECTIVE UNITARY GROUPS $PU(p)$
AND EXCEPTIONAL LIE GROUPS

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ABSTRACT. For a fixed prime $p$, we compute the Brown-Peterson cohomologies
of classifying spaces of $PU(p)$ and exceptional Lie groups by using
the Adams spectral sequence. In particular, we see that $BP^*(BPU(p))$ and
$K(n)^*(BPU(p))$ are even dimensionally generated.

1. Introduction

Let $p$ be a fixed odd prime and denote by $BP^*(X)$ (resp. $P(m)^*(X)$) the Brown-
Peterson cohomology of a space $X$ with the coefficient ring $BP^* = \mathbb{Z}(p)[v_1, v_2, \cdots]$ (resp. $P(m)^* = \mathbb{Z}/p[v_m, v_{m+1}, \cdots]$) where $\deg v_k = -2p^k + 2$. We denote by
$PU(n)$ the projective unitary group which is the quotient of the unitary group
$U(n)$ by its center $S^1$. Recall that the cohomologies of $PU(p)$ and the exceptional
Lie groups $F_4, E_6, E_7, E_8$ have odd torsion elements. In this paper, we compute
the Brown-Peterson cohomologies of classifying spaces $BG$ of these Lie groups $G$ as $BP^*$-modules using the Adams spectral sequence. Let us write $H^*(X; \mathbb{Z}/p)$ by simply $H^*(X)$ and let $A$ be the mod $p$ Steenrod algebra.

Our main result is as follows:

Theorem 1.1. Let $(G, p)$ be one of cases $(G = PU(p), p)$ for an arbitrary odd
prime $p$ and $G = F_4, E_7$ for $p = 3$, and $G = E_8$ for $p = 5$. Then the $E_2$-terms of
the Adams spectral sequences abutting to $BP^*(BG)$ and $P(m)^*(BG)$ for $m \geq 1$,

$$\text{Ext}^s_A(H^*(BP), H^*(BG)), \quad \text{Ext}^s_A(H^*(P(m)), H^*(BG)),$$

have no odd degree elements.

An immediate consequence is as follows:

Corollary 1.2. For $(G, p)$ in Theorem 1.1, the Adams spectral sequences abutting
to $BP^*(BG)$ and $P(m)^*(BG)$ in the previous theorem collapse at the $E_2$-level. In
particular $BP^{odd}(BG) = P(m)^{odd}(BG) = 0$.

Recall $K(m)^*(X) \cong K(m)^* \otimes_{P(m)^*} P(m)^*(X)$ is the Morava $K$-theory. From
the above theorem and corollary, we see $K(m)^{odd}(BPU(p)) = 0$. Then we have the
following corollary ([Ko-Ya], [Ra-Wi-Ya]).
Corollary 1.3. For $(G, p)$ in Theorem 1.1, the following holds:

1. $BP^*(BG)$ is $BP^*$-flat for $BP^*(BP)$-modules, i.e., $BP^*(BG \times X) \cong BP^*(BG) \otimes_{BP^*} BP^*(X)$ for all finite complexes $X$.
2. $K(n)^*(BG) \cong K(n)^* \otimes_{BP^*} BP^*(BG)$.
3. $P(n)^*(BG) \cong P(n)^* \otimes_{BP^*} BP^*(BG)$.

We give the $BP^*$-module structure of $BP^*(BPU(p))$ more explicitly in this introduction. For the exceptional Lie group cases, the explicit formulas are given in §5.

Theorem 1.4. There exists a $BP^*$-algebra exact sequence

$$0 \to BP^* \otimes M \to grBP^*(BPU(p)) \to BP^* \otimes IN/(f_0, f_1) \to 0$$

where

1. $M \cong \mathbb{Z}(p)[x_4, x_6, \ldots, x_{2p}]$ as $\mathbb{Z}(p)$-modules (but not $\mathbb{Z}(p)$-algebras).
2. $IN \cong \mathbb{Z}(p)[x_{2p+2}, x_{2p(p-1)}] \{x_{2p+2}\}$, the principal ideal of $\mathbb{Z}(p)[x_{2p+2}, x_{2p(p-1)}]$ generated by $x_{2p+2}$.
3. Relations $f_0, f_1$ are given with modulo $(p, v_1, v_2, \cdots)^2$:

$$f_0 \equiv v_0 - v_2 x_{2p}^{p-1} + \cdots, \quad f_1 \equiv v_1 - v_2 x_{2p(p-1)} + \cdots.$$ 

Remark. In the above theorem, subscript $i$ of $x_i$ means its degree. The algebra $BP^*(BPU(p))$ does not contain the subalgebra $BP^* \otimes \mathbb{Z}(p)[x_4, \ldots, x_{2p}]$, but contains a subalgebra which is isomorphic as $BP^*$-modules to the above $BP^*$-subalgebra.

For an algebraic group $G$ over $\mathbb{C}$, Totaro [To] defines its Chow ring and conjectures that $BP^*(BG) \otimes_{BP^*} \mathbb{Z}(p) \cong CH^*(BG)_{(p)}$. Recall that $PGL(p, \mathbb{C})$ is the algebraic group over $\mathbb{C}$ corresponding to the Lie group $PU(p)$.

Theorem 1.5. There exists an isomorphism

$$BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}(p) \cong CH^*(BGL(p, \mathbb{C}))_{(p)}.$$ 

Hence there exists an additive isomorphism

$$CH^*(BGL(p, \mathbb{C}))_{(p)} \cong \mathbb{Z}(p)[x_4, x_6, \cdots, x_{2p}] \oplus \mathbb{F}_p[x_{2p+2}, x_{2p(p-1)}]\{x_{2p+2}\}.$$ 

Remark. Vistoli [Vi] also determined the additive structure of the Chow ring and integral cohomology of $BPG(p, \mathbb{C})$ by using stratified methods of Vezzosi [Ve] (see also [Mo-Vi]). Moreover he shows that for $G = PGL(p, \mathbb{C})$,

$$H^*(G; \mathbb{Z}) \to H^*(BT; \mathbb{Z})^{W_G(T)}$$

is epic.

Let $\text{MGL}^{*,*}(X)$ be the motivic cohomology ring defined by V. Voevodsky [Vo1] and $\text{MGL}^{2i,*}(X) = \bigoplus_i \text{MGL}^{2i,i}(X)$.

Corollary 1.6. $\text{MGL}^{2i,*}(BPU(p, \mathbb{C}))_{(p)} \cong MU^*(BPU(p))_{(p)}$.

We prove Theorem 1.1 by using the Adams spectral sequence converging to the Brown-Peterson cohomology. The $E_1$-term of the spectral sequence could be given by

$$\mathbb{F}_p[v_0, v_1, \cdots] \otimes H^*(X) \quad \text{with} \quad d_1 x = \sum_{k=0}^{\infty} v_k Q_k x$$
where the $Q_k$’s are Milnor’s operations. (Here we identify $\mathbb{F}_p[v_0 = p, v_1, \cdots] = P(0)^*$.)

By the change-of-rings isomorphism, the $E_2$-term is

$$\text{Ext}_A(H^*(BP), H^*(X)) \cong \text{Ext}_E(\mathbb{F}_p, H^*(X))$$

where $E = A(Q_0, Q_1, \cdots)$. The $E_\infty$-term is given by $gr\!BP^*(X)$.

To state the cohomology $H^*(BP(p))$, we recall the Dickson algebra. Let $A_n$ be an elementary abelian $p$-group of rank $n$, and

$$H^*(BA_n) \cong \mathbb{F}_p[t_1, \ldots, t_n] \otimes A(dt_1, \ldots, dt_n) \quad \text{with } \beta(dt_i) = t_i.$$ 

The Dickson algebra is

$$D_n = \mathbb{F}_p[t_1, \ldots, t_n]^{GL(n, \mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,0}, c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL(n, \mathbb{F}_p)$ is also given:

$$SD_n = \mathbb{F}_p[t_1, \ldots, t_n]^{SL(n, \mathbb{F}_p)} \cong D_n\{1, e_n, \ldots, e_n^{p-2}\} \quad \text{with } e_n^{p-1} = c_{n,0}.$$ 

We also recall Mui’s ([Mu]) result by using $Q_i$ according to Kameko and Mimura [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)} \cong SD_n/\{e_n\} \oplus SD_n \otimes A(Q_0, \ldots, Q_{n-1})\{u_n\}$$

where $u_n = dt_1 \cdots dt_n$ and $e_n = Q_0 \cdots Q_{n-1}u_n$.

**Theorem 1.7.** There exists a short exact sequence

$$0 \to M/p \to H^*(BP(p)) \to N \to 0$$

where $M/p$ is the trivial $E$-module given in Theorem 1.4 and

$$N = SD_2 \otimes A(Q_0, Q_1)\{u_2\} \cong \mathbb{F}_p[x_{2p+2}, x_{2(p^2-p)}] \otimes A(Q_0, Q_1)\{u_2\}$$

identifying $x_{2p+2} = e_2$ and $x_{2(p^2-p)} = c_{2,1}$.

This theorem is proved by using the following facts. The group $G = PU(p)$ has just two conjugacy classes of maximal elementary abelian $p$-subgroups, one of which is toral and the other is nontoral $A$ of rank $p = 2$. The cohomology $H^*(BG)$ is detected by these two subgroups. The restriction image to the nontoral subgroup is $i_A^*(H^*(PU(p)) \cong H^*(BA)^{SL(2, \mathbb{F}_p)}$. Similar (but not the same) facts also hold for the exceptional Lie groups given in Theorem 1.1.

The algebraic main result in this paper is as follows:

**Theorem 1.8.** For $m \geq 0$, define $f_0, \ldots, f_{n-1}$ in $P(m)^* \otimes SD_n$ by

$$d_1u_n = \sum_{k \geq m} v_k Q_k(u_n) = f_0 Q_0 u_n + \cdots + f_{n-1} Q_{n-1} u_n.$$ 

Then the sequence $f_0, \ldots, f_{n-1}$ is a regular sequence in $P(m)^* \otimes SD_n$.

With the notation in this theorem, we prove that the complex

$$C = (P(m)^* \otimes SD_n \otimes A(Q_0, Q_1, \ldots, Q_{n-1})\{u_n\}, d_1)$$

with the differential $d_1u_n = \sum_{i=0}^{n-1} f_i Q_i u_n$ is a Koszul complex. This means that

$$H_i(C, d_1) = \begin{cases} P(m)^* \otimes SD_n\{e_n\}/(f_0, \ldots, f_{n-1}) & \text{for } i = 0 \\ 0 & \text{for } i \geq 1. \end{cases}$$

Thus Theorem 1.1 follows from the above theorem.
Remark about the convergence of the Adams spectral sequence. By Theorem 15.6 in Boardman’s paper [Bo2], since $H^*(BP)$ is of finite type, the above Adams spectral sequence is conditionally convergent. Moreover, since we prove the above Adams spectral sequence collapses at the $E_2$-level, by the remark after Theorem 7.1 in [Bo1], the above Adams spectral sequence is strongly convergent, so that we know the Brown-Peterson cohomology up to group extension.

We now give an outline of the paper. In §2, we note the relation of the Koszul complex and the $E_1$-term of the Adams spectral sequence for $\Lambda(Q_0, \ldots, Q_{n-1})$-free $H^*(X)$. Section 3 is stated about the Dickson invariant rings and the Milnor $Q_i$-operation on the cohomology of classifying spaces of elementary abelian $p$-groups. In §4, we compute $H^*(BG)$, $MU^*(BG)$, $CH^*(BG)$ for $G = PU(p)$. The similar computations are done in §5 for exceptional groups. The last section is devoted to the proof of Theorem 1.8.

2. The Koszul complex and the Adams spectral sequence

For spaces $X, Y$, the Adams spectral sequence converging to the stable homotopy class of $(X, Y)$ has the $E_2$-term

$$E_2^{*, *} \cong \Ext_A(H^*(Y), H^*(X)) \Longrightarrow \{X, Y\}.$$ We want to consider the case $Y = P(m)$.

Recall that $A \cong H^*(BP) \otimes \mathcal{E}$ with $\mathcal{E} = \Lambda(Q_0, Q_1, \ldots)$. Similarly $H^*(P(m)) \cong H^*(BP) \otimes \Lambda(Q_0, \ldots, Q_{m-1})$. Hence the $E_2$-term of the Adams spectral sequence abutting to $P(m)^*(X)$ is given by

$$\Ext_A(H^*(P(m)), H^*(X)) \cong \Ext_{H^*(BP) \otimes \mathcal{E}}(H^*(BP) \otimes \Lambda(Q_0, \ldots, Q_{m-1}), H^*(X)) \cong \Ext_{\mathcal{E}}(\mathbb{F}_p, H^*(X))$$

where $\mathcal{E}_m = \Lambda(Q_m, Q_{m+1}, \ldots)$.

We will study the case

$$0 \rightarrow M \rightarrow H^*(X) \rightarrow N \rightarrow 0$$

where $M$ is $\mathcal{E}$-trivial and $N$ is a free $\Lambda(Q_0, \ldots, Q_{n-1})$-module having the following properties. (For example, $X = BU(p)$; recall Theorem 1.7.)

For a $\mathbb{Z}/p$-algebra $S$, suppose that $\mathcal{E}$ acts on

$$N = S \otimes \Lambda(Q_0, \ldots, Q_{n-1})$$

such that $Q_i$ acts as a derivation and $Q_n(s \otimes t) = s \otimes Q_i(t)$ for $s \in S, t \in N$. Then for each $k \geq 0$, there exist $\alpha_{0,k}, \ldots, \alpha_{n-1,k} \in S$ such that

$$Q_k x = \alpha_{0,k} Q_0 x + \cdots + \alpha_{n-1,k} Q_{n-1} x$$

for all $x \in N$ (compare the proof of Proposition 3.3 below).

Let $R = P(m)^* \otimes S$. For $i = 0, \ldots, n - 1$, let

$$f_i = v_m \alpha_{i,m} + v_{m+1} \alpha_{i,m+1} + \cdots \in R.$$ The derived functor $\Ext_{\mathcal{E}_m}(\mathbb{F}_p, N)$ is the cohomology of a cochain complex

$$E_1 = P(m)^* \otimes N = R \otimes \Lambda(Q_0, \ldots, Q_{n-1})$$

with a differential

$$d_1(x) = \sum_{k \geq m} v_k Q_k(x) = f_0 Q_0(x) + \cdots + f_{n-1} Q_{n-1}(x).$$
Proposition 2.1. With the notation above, if \( f_0, \ldots, f_{n-1} \) is a regular sequence in \( R = P(m)^* \otimes S \), then
\[
\text{Ext}_A(H^*(P(m), N) = \text{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, N) \cong R/(f_0, \ldots, f_{n-1}).
\]

Hence Theorem 1.1 (for \( G = PU(p) \)) follows from Theorem 1.7, 1.8 and the above proposition. (We will show Theorem 1.7 in §4 and Theorem 1.8 in §3,6.)

To see the above proposition, we recall the Koszul complex. Let \( R \) be a commutative algebra and let \( g_0, \ldots, g_{n-1} \) be a sequence of \( R \). Let \( J \) be the ideal of \( R \) generated by \( g_0, \ldots, g_{n-1} \). Let \( \Lambda^r = \Lambda^r(x_0, \ldots, x_{n-1}) \) be \( r \)-homogeneous parts with deg \( (x_i) = 1 \). The Koszul complex is a chain complex
\[
0 \rightarrow R \otimes \Lambda^n \xrightarrow{d} \cdots \xrightarrow{d} R \otimes \Lambda^1 \xrightarrow{d} R \otimes \Lambda^0 \rightarrow 0
\]
with the differential
\[
d(x_{i_1} \cdots x_{i_r}) = \sum_{j=1}^r (-1)^{j+1} g_{i_j} x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_r}.
\]

Let us denote by \( H_k(R \otimes \Lambda^*, d) \) the induced homology. The following proposition is well known.

Proposition 2.2. If \( g_0, \ldots, g_{n-1} \) is a regular sequence in \( R \), then
\[
H_k(R \otimes \Lambda^*, d) \cong \begin{cases} R/J & \text{for } k = 0, \\ 0 & \text{for } k \geq 1. \end{cases}
\]

Proof of Proposition 2.1. Let \( g_i = (-1)^i f_i \). It is clear that \( g_0, \ldots, g_{n-1} \) is a regular sequence if and only if \( f_0, \ldots, f_{n-1} \) is a regular sequence. Let
\[
\mu : R \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \rightarrow R \otimes \Lambda^*
\]
be an \( R \)-module isomorphism defined by
\[
\mu(Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_r} \cdots Q_{n-1}) = x_{i_1} \cdots x_{i_r}
\]
where \( 0 \leq i_1 < \cdots < i_r \leq n - 1 \). We verify that this homomorphism \( \mu \) commutes with the differentials \( d_1, d \). On the one hand,
\[
d \circ \mu(Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_r} \cdots Q_{n-1}) = d(x_{i_1} \cdots x_{i_r}) = \sum_{j=1}^r (-1)^{j+1} g_{i_j} x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_r}.
\]

On the other hand, since
\[
d_1(Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_r} \cdots Q_{n-1})
\]
\[
= (f_0 Q_0 + \cdots + f_{n-1} Q_{n-1}) Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_r} \cdots Q_{n-1}
\]
\[
= (-1)^i f_i Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_2} \cdots \hat{Q}_{i_r} \cdots Q_{n-1}
\]
\[
+ (-1)^{i-2} f_{i_2} Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_2} \cdots \hat{Q}_{i_r} \cdots Q_{n-1}
\]
\[
\vdots
\]
\[
+ (-1)^{i-(r-1)} f_r Q_0 \cdots \hat{Q}_{i_1} \cdots \hat{Q}_{i_2} \cdots \hat{Q}_{i_r} \cdots Q_{n-1},
\]
we get 

\[ \mu \circ d_1(Q_0 \cdots \hat{Q}_i \cdots Q_{n-1}) = \sum_{j=1}^{n} (-1)^{j-1} f_{ij} x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_r} \]

\[ = \sum_{j=1}^{n} (-1)^{j+1} g_{ij} x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_r}. \]

Therefore, \( \mu \) is an isomorphism of chain complexes, so that by Proposition 2.2, Proposition 2.1 holds.

\[ \Box \]

3. The Dickson Invariant and Milnor \( Q_i \) Operation

Let \( A \) be an elementary abelian \( p \)-group of rank \( n \). The mod \( p \) cohomology \( H^*(BA) \) is isomorphic to the polynomial tensor exterior algebra

\[ \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n) \]

as a graded algebra. It is also clear that the finite general linear group \( GL(n, \mathbb{F}_p) \) and the finite special linear group \( SL(n, \mathbb{F}_p) \) act on \( A, BA \) and \( H^*(BA) \). Dickson computed the ring of invariants of \( \mathbb{F}_p[t_1, \ldots, t_n] \) with respect to the action of \( GL(n, \mathbb{F}_p) \). In the case of \( G = GL(n, \mathbb{F}_p) \), the ring of invariants is a polynomial algebra

\[ D_n = \mathbb{F}_p[c_{n,0}, \ldots, c_{n,n-1}] \]

whose generators are given by the equation

\[ O_n(X) = \prod_{t \in \mathbb{F}_p[t_1, \ldots, t_n]} (X + t) = X^{p^n} + \sum_{j=0}^{n-1} c_{n,j} X^{p^j}. \]

Remark. In [Ka], [Ka-Mi], notation \( c_{n,i} \) is used as \( (-1)^{n-i} c_{n,i} \) in this paper. For the invariant ring under \( SL(n, \mathbb{F}_p) \), we have

\[ SD_n = D_n \{ 1, e_n, \ldots, e_n^{p-2} \} \] with \( e_n^{p-1} = c_{n,0}. \)

Mui computed the ring of invariants of \( \mathbb{F}_p[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n) \) with respect to the action of \( SL(n, \mathbb{F}_p) \). Of course \( u_n = dt_1 \cdots dt_n \) is invariant under \( SL(n, \mathbb{F}_p) \). In terms of Milnor’s operation, we may state Mui’s result in the following form.

Theorem 3.1 (Mui [Mu], Kameko-Mimura [Ka-Mi]).

\[ H^*(BA)^{SL(n, \mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,1}, \ldots, c_{n,n-1}] \oplus SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \{ u_n \} \]

\[ \cong SD_n/(e_n) \oplus SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \{ u_n \} \]

where \( Q_0 \ldots Q_{n-1} u_n = e_n. \)

In this section, we compute \( f_0, \ldots, f_{n-1} \) in §2 for the case \( S = SD_n \). We study some important relations between \( Q_i \) and \( c_{n,j}. \)

Lemma 3.2. For \( x \in H^*(BA) \), it follows that

\[ (Q_n + \sum_{i=1}^{n} c_{n,i} Q_i)(x) = 0, \quad Q_0 \cdots Q_i Q_n(u_n) = (-1)^{n-i} c_{n,i} e_n. \]
Proof. Note \( Q_i(dt_j) = t_j^p \). When \( x = dt_j \), we see
\[
(Q_n + \sum c_{n,i}Q_i)(dt_j) = t_j^p + \sum c_{n,i}t_j^p = O_n(t_j) = 0.
\]
Since \( Q_i \) is a derivation, we get the first equation.
Applying \( Q_0...Q_i...Q_{n-1} \) to the first equation, we get the second equation. □

By using the relation \( Q_{k+1} = [Q_k, \varphi^k] \) and the above lemma, we may define an \( \mathcal{E} \)-module structure on \( SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \{u_n\} \). Let us write (recall the definition in §2)
\[
Q_k u_n = \alpha_{0,k}Q_0u_n + \cdots + \alpha_{n-1,k}Q_{n-1}u_n.
\]
With the following proposition, we verify that there is an algebra homomorphism \( \mathcal{E} \to SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \) such that the evaluation at \( u_n \) induces an \( \mathcal{E} \)-module homomorphism \( SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \to H^*(BA) \).

**Proposition 3.3.** For each \( x \) in \( SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \{u_n\} \subset H^*BA \),
\[
Q_k x = (\alpha_{0,k}Q_0 + \cdots + \alpha_{n-1,k}Q_{n-1})x.
\]
**Proof.** It is immediate that \( Q_k Q_i \cdots Q_r = (-1)^r Q_i \cdots Q_r Q_k \) and so
\[
Q_k Q_i \cdots Q_r u_n = (-1)^r Q_i \cdots Q_r Q_k u_n
\]
\[
= (-1)^r Q_i \cdots Q_r (\alpha_{0,k}Q_0u_n + \cdots + \alpha_{n-1,k}Q_{n-1}u_n)
\]
\[
= (\alpha_{k,0}Q_0 + \cdots + \alpha_{n-1,k}Q_{n-1})Q_i \cdots Q_r u_n.
\]
□

The following propositions are only used in the last section to prove Theorem 1.8.

**Proposition 3.4.** For \( i \geq 1 \), \( \alpha_{i,0} = 0 \) and \( \alpha_{0,0} = 1 \). Letting \( \alpha_{-1,k} = 0 \), we have
\[
\alpha_{i,k+1} = \alpha_{i-1,k}^p - \alpha_{n-1,k}^p c_{n,i}.
\]
**Proof.** Since \( Q_{k+1} = [Q_k, \varphi^k] \), we have
\[
Q_{k+1} u_n = \varphi^k Q_k u_n = \varphi^k \sum_{i=0}^{n-1} \alpha_{i,k}Q_i u_n = \sum_{i=0}^{n-1} \alpha_{i,k}^p Q_{i+1} u_n.
\]
The last equation follows from \( \deg \alpha_{i,k} = 2p^k - 2p^i \) and
\[
\varphi^k \alpha_{i,k} Q_i u_n = \varphi^k - p^i \alpha_{i,k} \cdot \varphi^i Q_i u_n = \alpha_{i,k}^p Q_{i+1} u_n.
\]
We also get
\[
\sum_{i=0}^{n-1} \alpha_{i,k}^p Q_{i+1} u_n = \sum_{i=0}^{n-2} \alpha_{i,k}^p Q_{i+1} u_n + \alpha_{n-1,k}^p Q_n u_n
\]
\[
= \sum_{i=1}^{n-1} \alpha_{i-1,k}^p Q_i u_n - \sum_{i=0}^{n-1} \alpha_{n-1,k}^p c_{n,i} Q_i u_n.
\]
Here the last equation follows from Lemma 3.2. Hence the last element is equal to \( Q_{k+1} u = \sum \alpha_{i,k+1} Q_i u_n \). Thus we get the equation in this proposition. □

**Proposition 3.5.** For \( k \leq n-1 \), \( \alpha_{i,k} = \delta_{i,k} \).
Proposition 3.6. We have $\alpha_{i,n} = -c_{n,i}$ and $\alpha_{n-1,k} \neq 0$ for $k \geq n$; in particular,

$$\alpha_{n-1,n+k} = c_{n,n-1}^k + \text{terms lower with respect to } c_{n,n-1}. $$

In particular, $\alpha_{0,n} = -c_{n,0}, \ldots, \alpha_{n-1,n} = -c_{n,n-1}, \alpha_{i,j} = \delta_{i,j}$ for $i, j = 0, \ldots, n-1$. That is, $f_i$ is expressed as

$$\begin{equation}
\begin{aligned}
f_0 &= v_0 - v_n c_{n,0} + \cdots \\
f_1 &= v_1 - v_n c_{n,1} + \cdots \\
&\vdots \\
f_{n-1} &= v_{n-1} - v_n c_{n,n-1} + \cdots
\end{aligned}
\end{equation}$$

4. Projective unitary groups $PU(p)$

The additive structure of the cohomology $H^*(BPU(p))$ was first determined by the first author computing the Poincaré series of the $E_2$-term of the Rothenberg-Steenrod spectral sequence. (A. Vistoli [Vi] also gets the same results by completely different methods.) Indeed, we have

$$PS(H^*(BPU(p), t)) = \sum \text{rank}_p H^i(BPU(p)) t^i = \frac{1}{(1 - t^4)(1 - t^6) \cdots (1 - t^{2p})} + \frac{t^2 + t^3 + t^{2p+1} + t^{2p+2}}{(1 - t^{2p+2})(1 - t^{2p+1})}. $$

However we give it here by a short argument using the results of Vavpetic and Viruel [Va-Vi] and Kono and Yagita [Ko-Ya].

Let $G$ be a compact Lie group. Quillen showed that the following restriction map $r$ is an F-isomorphism:

$$r : H^*(BG) \to \lim_{\mathcal{A}} H^*(BA) $$

where $\mathcal{A}$ runs over a set of conjugacy classes of elementary abelian $p$-subgroups of $G$. Here an F-isomorphism means that its kernel is generated by nilpotent elements and for each $x \in H^*(BG)$ there is $s \geq 0$ such that $x^{p^s} \in Im(r)$.

Recently A. Vavpetic and A. Viruel gave the following short proof of the fact that the map $r$ is injective for $G = PU(p)$.

There is only one conjugacy class of nontoral abelian groups $A$ of rank $p = 2$. Let $T_G$ be a maximal torus of $G$ and $N_G = N_G(T)$ the normalizer of the torus $T_G$. Let $N_p(G)$ be the $p$-normalizer of $T_G$, namely, the preimage of a $p$-Sylow subgroup in the Weyl group $W_G = N_G/T_G$. For example, it is known that

$$N_p(U(p)) \cong S^1 \times \mathbb{Z}/p \cong \{ \text{diag}(z_1, \ldots, z_p) \in U(p) \mid z_j \in \mathbb{C}^*, 0 \leq i \leq p-1 \} $$

where $P \in U(p)$ is a permutation matrix of order $p$.

Lemma 4.1. $N_p(SU(p)) \cong N_p(PU(p))$.

The cofiber sequence $(\mathbb{C}^* \cong) S^1 \to BSU(p) \to BU(p)$ induces the Gysin exact sequence

$$\to H^{*-2}(BN_p(U(p))) \to H^*(BN_p(U(p))) \to H^*(BN_p(SU(p))) \to . $$
Considering restrictions to $A$ and tori of $U(p)$ and $SU(p)$, Vavpetič and Viruel prove:

**Lemma 4.2** ([Va-Vi]). $H^*(BN_p(SU(p)))$ is detected by $A$ and $T_{SU(p)}$.

**Theorem 4.3** ([Va-Vi]). $H^*(BPU(p))$ is detected by $A$ and $T_{PU(p)}$.

**Proof.** Consider the restrictions

$$H^*(BPU(p)) \xrightarrow{i_{BPU}} H^*(BN_{PU(p)}) \xrightarrow{i_{BU}} H^*(BN_p(PU(p))) \cong H^*(BN_p(SU(p))).$$

Recall the transfer $tr$ and Gottlieb transfer $Gtr$ so that $Gtr.i_{NPU} = id$ and $tr.i_{NP} = id$. Hence the above restrictions are injective. \hfill \Box

Note that $W_G(A) = SL(2,F_p)$ for $G = PU(p)$. Next by using arguments ([Ko-Ya]), we will show for $G = PU(p)$ that the following map $pi_A$ is epic:

$$pi_A : H^*(BG) \xrightarrow{i} H^*(BA)^{W_G(A)} = SD_2/(c_2) \oplus SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \xrightarrow{proj.} SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\}.$$  

Let $E$ be the subgroup of $SU(p)$ generated by 

$$c = \text{diag}(\xi, \ldots, \xi), \quad a = \text{diag}(1, \xi, \ldots, \xi^{p-1}), \quad b = P \quad \text{(permutation matrix)}$$

where $\xi$ is the $p$-th primitive root of 1. Then we see that 

$$a^p = b^p = c^p = 1, \quad [a, b] = c \in \text{center of } E.$$ 

Hence $E$ is isomorphic to the extraspecial $p$-group $p^{1+2}$ of order $p^3$. We consider the commutative diagram for fiberings

$$
\begin{array}{ccc}
\mathbb{Z}/p & \xrightarrow{=} & SU(p) \\
\uparrow & & \uparrow \\
\mathbb{Z}/p = \langle c \rangle & \xrightarrow{=} & E & \xrightarrow{=} & A = \langle a, b \rangle.
\end{array}
$$

The above diagram induces the map $f_r : E(U)^{r,*} \rightarrow E(E)^{r,*}$ from the spectral sequence converging to $H^*(BSU(p))$ to that converging to $H^*(BE)$.

The differential of $E(E)^{r,*}$ is well known [Te-Ya]. Let us write 

$$E(E)^2_{2,*} \cong H^*(BA; H^*(BZ/p)) \cong F_p[t_1, t_2, w] \otimes \Lambda(dt_1, dt_2, dw)$$

with $\beta dw = w$. Then it is known that 

$$d_2(dw) = dt_1 dt_2 = u_2.$$ 

Since $dw \in \text{Im}(f_2)$, we see that $u_2 \in \text{Im}(f_3)$ and so in $\text{Im}(pi_A^*)$.

Here we also see that $c_{2,1}, c_{2,0} \in \text{Im}(i_A^*)$. Let $\rho$ be the canonical representation of $SU(p)$ and 

$$\lambda = \rho \otimes \rho^{-1} : SU(p) \rightarrow SU(p^2).$$

Since $\rho \otimes \rho^{-1}|\mathbb{Z}/p$ is trivial, we can identify $\lambda$ as the representation of $PU(p)$. For $a^ib^j \in A \cong (\mathbb{Z}/p)^2$, we easily compute $\chi_\lambda(a^ib^j) = p^2$ if $i = j = 0$ and $= 0$ otherwise. This means that $\lambda|A$ is the regular representation. So the total Chern class is 

$$c(\lambda|A) = \Pi_{(\lambda_1 \lambda_2) \in (\mathbb{Z}/p)^2} (1 + \lambda_1 t_1 + \lambda_2 t_2) = 1 + c_{2,1} + c_{2,0}.$$ 

Hence we have $c_{p^2-1}(\lambda)|A = c_{2,1}$ and $c_{p^2-1}(\lambda)|A = c_{2,0}$. 

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Lemma 4.4. The map $pi_A$ is epic.

(From Lemma 3.2, we also have $e_2 = Q_0Q_1u_2$, $e_p^{p-1} = c_2,0$ and $Q_2Q_0(u_2) = c_{2,1}e_2$.

Thus we get the short exact sequence

$$0 \to \ker(pi_A) \to H^*(BPU(p)) \to SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \to 0.$$  

By the theorem by Vavpetič and Viruel, we have the injection

$$i_T \times i_A : \ker(pi_A) \subset H^*(BPU(p)) \to H^*(BT_{PU(p)}) \times SD_2/(e_2).$$

In particular $\ker(pi_A)$ is generated by even-dimensional elements.

Now we consider the Bockstein spectral sequence

$$E_1 = H^*(X) \Longrightarrow \mathbb{F}_p \otimes H^*(X; \mathbb{Z})/(\text{torsions}).$$

The first differential is the Bockstein operation $Q_0$. Since $\text{Im}(pi_A)$ is $Q_0$-free, so $H(\text{Im}(pi_A); Q_0) = 0$. Hence we have

$$E_2 = H(H^*(BPU(p); Q_0)) \cong \ker(pi_A).$$

Since $\ker(pi_A)$ is generated by even-dimensional elements, we get

$$E_\infty \cong E_2 \cong \ker(pi_A).$$

It is well known that

$$H^*(BPU(p); \mathbb{Q}) \cong H^*(BSU(p); \mathbb{Q}) \cong \mathbb{Q}[c_2, \ldots, c_p].$$

Since $\text{rank}_p(\mathbb{F}_p \otimes (H^i(X; \mathbb{Z})/(\text{torsions}))) = \text{rank}_Q H^i(X; \mathbb{Q})$, we get

Lemma 4.5. There exist additive isomorphisms

$$\ker(pi_A) \cong \mathbb{F}_p \otimes H^*(BPU(p); \mathbb{Z})/(\text{torsions}) \cong \mathbb{F}_p[x_2, \ldots, x_p] \text{ with } |x_i| = 2i.$$  

Remark. This is not a ring isomorphism; e.g., for $p = 3$ we have the ring isomorphism

$$\ker(pi_A) \cong \mathbb{F}_3[c_2, c_3, c_6]/(c_2^3 - c_2) \not\cong \mathbb{F}_3[c_2, c_3].$$

Thus we have Theorem 1.7 in the introduction.

Theorem 4.6. There exists a short exact sequence of rings

$$0 \to M/p \to H^*(BPU(p)) \to SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \to 0$$

with additively (not rings) $M/p \cong \mathbb{F}_p[x_2, \ldots, x_p]$.

Now we consider the Adams spectral sequence converging to $P(m)^*(BPU(p))$. By the change of rings,

$$\text{Ext}_A^{s,t}(H^*(P(m), H^*(X)) \cong \text{Ext}_A^{s,t}(\mathbb{F}_p, H^*(X)).$$

From the above theorem, Theorem 1.8 implies

Lemma 4.7. There exists a short exact sequence

$$0 \to P(m)^* \otimes M \to \text{Ext}_A(\mathbb{F}_p, H^*(BPU(p)) \to P(m)^* \otimes SD_2/(f_0, f_1)\{e_2\} \to 0.$$
Thus we have Theorem 1.1 in the introduction for $G = PU(p)$.

Next we consider the Chow ring for $BPU(p)$. Recall that $N_p(U(p)) \cong S^1 \times \mathbb{Z}/p$. By Totaro, it is known that $CH^*(S^1 \times \mathbb{Z}/p)\otimes_p$ is epic. Moreover he showed (his conjecture)

$$CH^*(BPU(p)) \cong MGL^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}.$$ 

for this group $G = S^1 \times \mathbb{Z}/p$. By using $CH^*(-)/p$ (and $BP^*(-)$) version of the Gysin sequence given just before Lemma 4.2, we also see (*) for $G = N_p(SU(p))$. Since Chow rings $CH^*(-)(p)$ (and $BP^*(-) \otimes_{BP^*} \mathbb{Z}(p)$) have transfer (and Gottlieb transfer), the isomorphism (*) for $G = PU(p)$ follows from the arguments of the proof of Theorem 4.3. Recall $PGL(p, \mathbb{C})$ is the algebraic group corresponding to $PU(p)$.

**Theorem 4.8.** $CH^*(BPGL(p, \mathbb{C}))(p) \cong BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}.$

Thus we get Theorem 1.5 in the introduction.

**Remark.** Vezzosi first studied $CH^*(BPGL(3, \mathbb{C}))(3) = CH^*(PU(3))(3)$. He showed (Theorem 1.1 in [Ve]) if some element $\chi \in CH^6(PU(3))(3)$ is zero, then we have the isomorphism in the above theorem. Indeed, we have seen $\chi = 0$.

Let $H^{*,*}(-; \mathbb{Z})$ be the motivic cohomology defined by Suslin and Voevodsky so that $H^{2*,*}(X; \mathbb{Z}) \cong CH^*(X)$. Here $H^{2*,*}(X; \mathbb{Z}) = \bigoplus_i H^{2i,*}(X; \mathbb{Z})$. Recall that $MGL^{*,*}(-)$ is the motivic cohomology defined by Voevodsky ([Vo1]) by using spectrum $MGL$ in the $\mathbb{A}^1$ stable homotopy category. Of course there is the natural map $MGL^{2*,*}(X) \to MU^{2*,*}(X)$.

**Corollary 4.9.** $MGL^{2*,*}(BGL(p, \mathbb{C}))(p) \cong MU^{2*,*}(BPU(p))(p)$.

**Proof.** Let $X = BPGL(p, \mathbb{C})$. Consider the motivic Atiyah-Hirzebruch spectral sequence

$$E_2 = H^{*,*}(X; MU^*) \Longrightarrow MGL^{*,*}(X).$$

When $X$ is smooth and $m > 2n$, it is known that $H^{m,n}(X; \mathbb{Z}) = 0$. This implies ((1.2) in [Ya])

$$MGL^{2*,*}(X) \otimes_{MU^*} \mathbb{Z} \cong H^{2*,*}(X; \mathbb{Z}) \cong CH^*(X).$$

From the above theorem, we only need to prove that there is a relation such that

$$f_1 x_{2p+2} = v_1 e_2 + \ldots = 0 \quad \text{also in } MGL^{2*,*}(X)$$

($x_{2p+2} = e_2$ in the notation of Theorem 1.4 and Theorem 4.6). By the solution of the Bloch-Kato conjecture of degree 2 by Merkurjev-Suslin, we get

$$u_2 \in H^{2,2}(X; \mathbb{Z}/p) \quad \text{and} \quad Q_0(u_2) \in H^{3,2}(X; \mathbb{Z}(p)),$$

and hence $e = Q_1 Q_0(u_2) \in H^{2p+2, p+1}(X; \mathbb{Z}(p)) = CH^{p+1}(X)(p)$. The first differential of the Atiyah-Hirzebruch spectral sequence is given by ((4.6) in [Ya])

$$d_{p-1}(Q_0 u_2) = v_1 \otimes Q_1 Q_0(u_2) = v_1 \otimes e_2.$$

Hence $v_1 e_2 = 0$ in $gr MGL^{2*,*}(X)$.
5. The exceptional Lie groups cases

With the result on Poincaré series of cotorsion products, the first author proves that the Rothenberg-Steenrod spectral sequence for the mod p cohomology of $BG$ collapses at the $E_2$-level and the Quillen homomorphism $r$ is a monomorphism in the cases $G = E_8$ and $p = 5$, $G = F_4, E_6, E_7$ and $p = 3$. In these cases, each exceptional Lie group has two conjugacy classes of maximal elementary abelian $p$-groups. One is the subgroup of a maximal torus and the other is a nontoral $A$. Let us write $i_A : BA \to BG$ and $i_T : BT_G \to BG$ for the induced maps from the inclusions. (However note that for $G = E_8$ and $p = 3$, there are two conjugacy classes of nontoral maximal elementary abelian 3-subgroups.)

Theorem 5.1 ([Ka]). In the case $G = E_8$ and $p = 5$, $G = F_4, E_6, E_7$ and $p = 3$, the induced homomorphism $i_A^* \times i_T^* : H^*(BG) \to H^*(BA) \times H^*(BT_G)$ is a monomorphism.

Moreover $\text{Im}(i_A^*)$ is also known for these groups.

Case I. Consider the cases $G = E_8$ and $p = 5$, $G = F_4$ and $p = 3$. In these cases $W_G(A) = \text{SL}(3, \mathbb{F}_p)$. Let

$$pi_A : H^*(BG) \to H^*(BA)^{W_G(A)} \to SD_3 \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\}.$$  

Let $K = \text{Ker}(pi_A)$ and $IM = \text{Im}(pi_A)$. The map $pi_A$ is not epic, indeed $u_3 \notin IM$, and we can show

Lemma 5.2. There exists a short exact sequence

$$0 \to IM \to SD_3 \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\} \to \mathbb{F}_p\{u_3\} \to 0.$$  

Thus we get

$$\text{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, IM) \cong P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_2)\{e_3\} \oplus P(m)^*\{\delta u_3\}, \quad |\delta u_3| = 4.$$  

Since $K = \text{Ker}(pi_A)$ is a trivial $\mathcal{E}_m$-module generated by even-dimensional elements, we get Theorem 1.1 for these groups.

Proposition 5.3. For $G$ in case I, $\text{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, H^*(BG))$ is isomorphic to

$$P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_2)\{e_3\} \oplus P(m)^*\{\delta u_3\} \oplus P(m)^* \hat{\otimes} K.$$  

Next consider the Bockstein spectral sequence

$$E_1 = H^*(BG) \Longrightarrow \mathbb{F}_p \otimes H^*(BG; \mathbb{Z})/(\text{torsions}).$$  

Note $IM$ is not $\Lambda(Q_0)$-free but

$$E_2 = H(H^*(BG); Q_0) \cong K \oplus \mathbb{F}_p\{Q_0 u_3\}.$$  

Hence the Poincaré series $PS_R(\mathbb{F}_p, -t)$ (for graded $R$-vector spaces) is given by

$$PS_{\mathbb{Z}/p}(K \oplus \mathbb{Z}/p[\delta_3], t) = PS_{\mathbb{Z}/p}(K, t) + t^4$$  

$$= PS_{\mathbb{Z}/p}(K \oplus \mathbb{Z}/p\{Q_0 u_3\}, t) = PS_Q(H^*(BG; Q), t).$$  

Thus we get

Theorem 5.4. For the case I, there exists a $P(m)^*$-algebra exact sequence

$$0 \to P(m)^* \hat{\otimes} M \to \text{gr}P(m)^*(BG) \to P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_2)\{e_3\} \to 0.$$  

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(1) $M \cong \mathbb{Z}_3[x_4, x_{12}, x_{16}, x_{24}]$ as $\mathbb{Z}_3$-modules for $G = F_4$, $p = 3$.

(2) $M \cong \mathbb{Z}_3[x_4, x_{16}, x_{24}, x_{28}, x_{36}, x_{40}, x_{48}, x_{60}]$ as $\mathbb{Z}_3$-modules for $G = E_8$, $p = 5$.

The mod 3 cohomology of $BF_4$ is completely determined by Toda.

Theorem 5.5 ([Toda]). The cohomology $H^*(BF_4)$ is isomorphic to

$$\mathbb{Z}/3[x_{36}, x_{48}] \otimes (\mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} + \mathbb{Z}/3[x_{26}] \otimes A(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\})$$

where the above two terms have the intersection $\{1, x_{20}\}$.

Indeed, we see that $x_{26} | A = e_3$, $x_{36} | A = c_{3,1}$, $x_{48} | A = c_{3,2}$, $x_4 | A = Q_8(u_1)$, $x_8 | A = Q_1(u_3)$, $x_{20} | A = Q_2(u_3)$, $x_{29} | A = Q_0Q_1(u_3)$, $x_{25} | A = Q_1Q_2(u_3)$. The Brown-Peterson theory $BP^*(BG)$ is also computed in [Ko-Ya] by using the Atiyah-Hirzebruch spectral sequence. The ring structure of $M$ in Theorem 5.4 is quite complicated.

Proposition 5.6 ([Ko-Ya]). $M \cong D_3/(c_{3,0}) \otimes (\mathbb{Z}_3 \cdot \{1, 3x_4\} \oplus E)$ with

$$E = \mathbb{Z}_3[x_4, x_8] \{ab | a, b \in \{x_4, x_8, x_{20}\}) \subset \mathbb{Z}_3[x_4, x_8][x_{20}, x_{20}^2].$$

The Poincaré series of $(\mathbb{Z}/3 \cdot \{1, 3x_4\} \oplus E/3)$ is $a/(1-t^4)(1-t^8)$ with

$$a = 1 + t^{20} + t^{40} - (t^8 + t^{20})(1-t^4)(1-t^8)$$

$= 1 - t^8 + t^{12} + t^{16} - t^{20} + t^{24} + t^{28} - t^{32} + t^{40} = (1 - t^8 + t^{16})(1 + t^{12} + t^{24}).$

Then we have

$$P_{\mathbb{Z}/3}(D_3/(c_{3,0}) \otimes (\mathbb{Z}/3 \cdot \{1, 3x_4\} \oplus E), t) = \frac{1}{(1-t^{36})(1-t^{48})} \times \frac{a}{1-t^8}$$

$$= \frac{a(1+t^8)}{(1-t^{12})(1+t^{12}+t^{24})(1-t^{24})(1+t^{24})(1-t^4)(1-t^{16})}. $$

Here we see that

$$(1 + t^8)a = (1 + t^8)(1-t^8 + t^{16})(1 + t^{12} + t^{24}) = (1 + t^{24})(1 + t^{12} + t^{24}).$$

Hence the above Poincaré series is indeed equal to

$$P_{\mathbb{Z}/3}(M/3, t) = 1/(1-t^{12})(1-t^{24})(1-t^4)(1-t^{16}).$$

Proposition 5.7. Let $cl : CH^*(X)/p \to H^*(X)$ be the mod p cycle map. For $G$ of Case I, we have $pi_A \cdot cl(CH^*(BG)) = SD_3\{e_3\}$.

Proof. From Lemma 9.6 in [Ya] and the affirmative answer to the Bloch-Kato conjecture, there is an element $x \in H^{4,3}(BG; \mathbb{Z}/p)$ with $cl(x) = Q_0e_3$. Here note

$$Q_iQ_j(x) \in H^{2p^i+2p^j+2p^i+p^j+1}(BG; \mathbb{Z}/p) = CH^{p^j+p^j+1}(BG)/p.$$ 

Thus $e_3, c_{3,1}e_3, c_{3,2}e_3$ are in the image of the cycle map.

Case II. The groups $G$ are $E_6$ and $E_7$ for $p = 3$.

For each of these cases, $G$ contains a maximal nontoral elementary abelian 3-subgroup $A \cong (\mathbb{Z}/3)^4$. It is known (e.g., [A-G-M-V]) that $W_G(A)$ is the subgroup
of $SL(4, \mathbb{F}_3)$ generated by matrices of the form
\[
\begin{pmatrix}
0 \\
0 \\
0 \\
* * * \epsilon
\end{pmatrix}
\]
where $* \in \mathbb{F}_3$, $\epsilon = 1$ (resp. $\epsilon = \pm 1$) for $G = E_6$ (resp. $G = E_7$).

Recall that
\[O_3(X) = \Pi_{(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{Z}/3)^3} (X + \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3).\]
Let us denote by $O$ simply $O_3(t_4)$ so that $e_4 = Q_0 Q_1 Q_2 Q_3(u_4) = e_3 O$. Then the invariant ring is computed by Kameko and Mimura [Ka-Mi]:
\[H^*(BA)^{W_G(A)} \cong \begin{cases} 
SD_3/(e_3)[O] \oplus SD_3[O] \oplus \Lambda(Q_0, Q_1, Q_2)[u_3, u_4]) & \text{for } G = E_6 \\
SD_3/(e_3)[O^2] \oplus SD_3[O^2] \oplus \Lambda(Q_0, Q_1, Q_2)[u_3, O u_4)) & \text{for } G = E_7.
\end{cases}\]
At first consider the case $G = E_6$. Let us write $N = SD_3[O] \otimes \Lambda(Q_0, Q_1, Q_2)[u_3, u_4)$ and let the projection be
\[p_i_A : H^*(BG) \to H^*(BA)^{W_G(A)} \to N.\]

**Lemma 5.8.** $O u_3 = (Q_3 + c_{3,2} Q_2 + c_{3,1} Q_1 + c_{3,0} Q_0)(u_4)$.

**Proof.** From Lemma 3.2, we have
\[(Q_3 + \sum c_{3,i} Q_i)(u_4) = (Q_3 + \sum c_{3,i} Q_i)(dt_4) \times dt_1 dt_2 dt_3 = O(t_4) u_3. \]
Hence we have the decomposition
\[N = SD_3 \otimes \Lambda(Q_0, Q_1, Q_2)[u_3) \oplus SD_3[O] \otimes \Lambda(Q_0, Q_1, Q_2, Q_3)[u_4).\]
We can see that $f_0, ..., f_3$ is not regular in $SD_3[O]$ (while it is regular in $SD_4$) for general $m \geq 1$. Hence we assume here $m = 0$. Take $\text{Ext}_\mathcal{E}(\mathbb{F}_p, -)$.

**Lemma 5.9.** The module $\text{Ext}_\mathcal{E}(\mathbb{F}_p, N)$ is isomorphic to
\[BP^* \otimes SD_3[e_3]/(v_1, v_2) \oplus BP^* \otimes SD_3[O](e_3 O)/\langle v_1, v_2, v_3 \rangle.\]

It is known that the image $IM = \text{Im}(p_i_A)$ is an algebra over $A$ generated by $Q_0 Q_1 u_4, Q_0 u_3$ and $SD_3, O$; namely,
\[N/IN \cong SD_3[O][u_3, u_4, Q_0 u_4, Q_1 u_4, Q_2 u_4, Q_3 u_4].\]
The above fact is also known in Kono-Mimura [Ko-Mi]; indeed, $H^*(BE_6)$ is still computed in [Ko-Mi].

**Lemma 5.10.** $\text{Ext}_\mathcal{E}(\mathbb{F}_p, N/IM)$ is isomorphic to
\[P(0)^* \otimes SD_3[O][u_3, Q_0 u_4, Q_1 u_4, Q_2 u_4, Q_3 u_4]/(3Q_0 u_4),\]
which is generated by odd-dimensional elements where $P(0)^* = \mathbb{Z}/3[3, v_1, ...]$. From the above lemmas, we know that $\text{Ext}_\mathcal{E}(\mathbb{F}_p, \text{Im}(p_i_A))$ is generated by even-dimensional elements. Of course, so is $\text{Ext}_\mathcal{E}(\mathbb{F}_p, \text{Ker}(p_i_A))$, and hence we have

**Theorem 5.11.** For $(E_6, p = 3)$, there exists a short exact sequence
\[0 \to BP^* \otimes M \to \text{gr}BP^*(BE_6) \to \text{IN} \to 0.\]

(1) $\text{IN} \cong BP^* \otimes SD_3[O][Q_0 u_4] \oplus BP^* \otimes SD_3[e_3]/(v_1, v_2) \oplus BP^* \otimes SD_3[O][e_3 O]/\langle v_1, v_2, v_3 \rangle.$

(2) $M \cong \mathbb{Z}(3)[x_4, x_{10}, x_{12}, x_{16}, x_{18}, x_{24}]$ as additively.
Next consider the case \((G = E_7, p = 3)\). It is known that
\[ i_A(H^*(BG)) \subset SD_3/(e_3)[O^2] \oplus SD_3(O^2) \otimes \Lambda(Q_0, Q_1, Q_2)(u_3) \]
(but not in the submodule \(SD_3(O) \otimes \Lambda(Q_0, Q_1, Q_2)\{Ou_4\}\). Let us write \(N = SD_3(O^2) \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\}\) and let the projection
\[ p_i : H^*(BG) \to H^*(BA)^{W_G(A)} \to N. \]
We also know that
\[ N/\text{Im}(p_i) \cong SD_3(O^2)\{u_3\}. \]
Hence we get Theorem 1.1 by an argument similar to (but more easily than) the case \(E_6\).

**Theorem 5.12.** For \((G = E_7, p = 3)\), there exists a short exact sequence
\[ 0 \to P(m)^{\hat{\ominus}} M \to \text{gr}P(m)^{\hat{\ominus}}(BE_7) \to P(m)^{\hat{\ominus}}SD_3(O^2)\{e_3\}/(f_0, f_1, f_2) \to 0 \]
where \(M \cong \mathbb{Z}_3[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}]\) as additively.

6. **Proof of Theorem 1.8**

For the sake of notational ease, we write \(R_{m,n}\) for \(P(m)^{\hat{\ominus}}SD_n\) and \(J_\ell\) for the ideal \((f_\ell, \ldots, f_{n-1}) \subset P(m)^{\hat{\ominus}}D_n\).

**Proof of Theorem 1.8 in the case \(m = 0\).** It is clear that \(f_i = v_i - v_nc_{n,i} + \cdots\) for \(i = 0, \ldots, n - 1\). The above \(f_0, \ldots, f_{n-1}\) is a regular sequence in \(R_{0,n}\). Let \(\rho : R_{0,n} \to R_{0,n}\) be a ring homomorphism defined by
\[
\rho(v_k) = f_k \quad \text{for } 0 \leq k \leq n - 1,
\[
\rho(v_k) = v_k \quad \text{for } k \geq n,
\]
\[
\rho(c_{n,i}) = c_{n,i} \quad \text{for } 0 \leq i \leq n - 1.
\]
Then, it is clear that for \(k = 0, \ldots, n - 1\), \(\rho^p(v_k) = v_k + pf_k' = v_k \) where \(f_k' = v_n^{p^k}k + v_{n+1}^{p^k}k_{n+1} + \cdots\). So the ring homomorphism \(\rho\) is a ring automorphism. It is also clear that \(v_0, \ldots, v_{n-1}\) is a regular sequence. Hence, \(\rho(v_0), \ldots, \rho(v_{n-1})\) is also a regular sequence. Thus, \(f_0, \ldots, f_{n-1}\) is a regular sequence. □

**Proof of Theorem 1.8 in the case \(n = 1\).** In the case \(n = 1\), the statement of Theorem 1.8 is that \(f_0 = v_m c_{n,0}^{p^{m-1}+\cdots+1} = v_{m+1} c_{n,0}^{p^m+p^{m-1}+\cdots+1} + \cdots\) is not a zerodivisor. It is clear that \(R(m, 1)\) is an integral domain, so that Theorem 1.8 holds in the case \(n = 1\). □

**Lemma 6.1.** Let \(\Delta_m\) be the determinant of the matrix
\[
A_m = \begin{pmatrix}
\alpha_{0,m} & \cdots & \alpha_{0,m+n-1} \\
\vdots & \ddots & \vdots \\
\alpha_{n-1,m} & \cdots & \alpha_{n-1,m+n-1}
\end{pmatrix}.
\]
Then \(\Delta_0 = 1\) and
\[
\Delta_{m+1} = \Delta_m c_{n,0}.
\]
for \(m \geq 0\). In particular, the above matrix has an inverse as a matrix of coefficients in \(R_{m,n}[c_{n,0}^{-1}]\).

This lemma will be proved in the last parts of this section.

**Lemma 6.2.** The ring \((R_{m,n}/J_\ell)[c_{n,0}^{-1}]\) is an integral domain for \(\ell = 0, \ldots, n - 1\).
Proof. Let

\[ f'_i = v_{m+n\alpha_i,m+n} + v_{m+n+1\alpha_i,m+n+1} + \cdots \]

for \( i = 0, \ldots, n - 1 \). Let

\[ \rho_m : R_{m,n}[c_{n,0}^{-1}] \to R_{m,n}[c_{n,0}^{-1}] \]

be a ring homomorphism defined by

\[
\begin{align*}
\rho_m(v_{m+k}) &= v_{m+k} + f'_k \quad \text{for } 0 \leq k \leq n - 1, \\
\rho_m(v_{m+k}) &= v_{m+k} \quad \text{for } k \geq m + n, \\
\rho_m(c_{n,i}) &= c_{n,i} \quad \text{for } 0 \leq i \leq n - 1.
\end{align*}
\]

As in the above proof of Theorem 1.8 in the case \( m = 0, \rho_m^p = 1 \), so that \( \rho_m \) is a ring automorphism.

We define a ring homomorphism

\[ \psi_m : R_{m,n}[c_{n,0}^{-1}] \to R_{m,n}[c_{n,0}^{-1}] \]

by

\[
\begin{align*}
\psi_m(v_{m+k}) &= v_m \alpha_{k,m} + \cdots + v_{m+n-1} \alpha_{k,m+n-1} \quad \text{for } 0 \leq k \leq n - 1, \\
\psi_m(v_{m+k}) &= v_{m+k} \quad \text{for } n \leq k, \\
\psi_m(c_{n,i}) &= c_{n,i} \quad \text{for } 0 \leq i \leq n - 1.
\end{align*}
\]

By Lemma 6.1, in \( R_{m,n}[c_{n,0}^{-1}] \), the inverse of the matrix \( A_m \) exists. Hence, \( \psi_m \) also has an inverse, that is, \( \psi_m \) is also a ring automorphism. Therefore, \( \psi_m \circ \rho_m \) is also a ring automorphism and \( \psi_m \circ \rho_m \) maps \( v_m, \ldots, v_{m+n-1} \) to \( f_0, \ldots, f_{n-1} \). It is clear that \( \psi_m \circ \rho_m \) induces an isomorphism

\[ R_{m,n}[c_{n,0}^{-1}]/(v_{m+\ell}, \ldots, v_{m+n-1}) \to (R_{m,n}/J_\ell)[c_{n,0}^{-1}]. \]

It is clear that \( R_{m,n}[c_{n,0}^{-1}]/(v_{m+\ell}, \ldots, v_{m+n-1}) \) is an integral domain and so is \( (R_{m,n}/J_\ell)[c_{n,0}^{-1}] \).

\[ \square \]

Lemma 6.3. Let \( \phi : R_{m,n} \to R_{m-1,n-1} \) be a ring homomorphism defined by

\[
\begin{align*}
\phi(v_k) &= v_{k-1}, \\
\phi(c_{n,i}) &= c_{n-1, i-1}, \\
\phi(c_{n,0}) &= 0.
\end{align*}
\]

Then, \( \phi \) maps \( f_k \) to \( f_{k-1} \) for \( k = 1, \ldots, n-1 \) and \( \phi(f_0) = 0 \).

Proof. It suffices to show that

1. \( \phi(\alpha_{0,n}) = 0 \),
2. \( \phi(\alpha_{i,k}) = \alpha_{i-1,k-1} \) for \( i \geq 1 \).

It is clear from the fact that \( \alpha_{0,n} = -c_{n,0} \) for \( k \geq 1 \) that (1) holds. We prove this lemma by induction on \( k \). For \( i \geq 1 \),

\[
\begin{align*}
\phi(\alpha_{i,k}) &= \phi(\alpha_{i-1,k-1} - \alpha_{n-1,k-1}^p c_{n,k}) \\
&= a_{i-2,k-2}^p - \alpha_{n-2,k-2}^p c_{n-1,k-1} \\
&= \alpha_{i-1,k-1}
\end{align*}
\]

in \( R_{m-1,n-1} \).

\[ \square \]

Proposition 6.4. If Theorem 1.8 holds for \( R_{m-1,n-1} \), then \( c_{n,0} \) is not a zerodivisor in \( R_{m,n}/J_\ell \) for \( 1 \leq \ell \leq n-1 \).
Proof. Recall $J_\ell = (f_\ell, \ldots, f_{n-1})$. We prove this proposition by induction on $n-1-\ell$.
In the case $n-1-\ell = 0$, the proposition holds because $R_{m,n}$ is an integral domain.
Suppose that
\[ c_{n,0}a = a_\ell f_\ell + \cdots + a_{n-1}f_{n-1} \quad (1) \]
in $R_{m,n}$. Then,
\[ \phi(a_\ell)\phi(f_\ell) + \cdots + \phi(a_{n-1})\phi(f_{n-1}) = 0 \]
in $R_{m-1,n-1}$. By the induction hypothesis, $\phi(f_\ell), \ldots, \phi(f_{n-1})$ is a regular sequence in $R_{m-1,n-1}$. Therefore,
\[ \phi(a_\ell) = b_{\ell+1}\phi(f_{\ell+1}) + \cdots + b_{n-1}\phi(f_{n-1}) \]
for some $b_{\ell+1}, \ldots, b_{n-1}$. Since the kernel of $\phi$ is a principal ideal $(c_{n,0})$,
\[ a_\ell = b_{\ell+1}f_{\ell+1} + \cdots + b_{n-1}f_{n-1} + c_{n,0}b \quad (2) \]
for some $b'_k, b \in R_{m,n}$ such that $\phi(b'_k) = b_k$ for $k = \ell + 1, \ldots, n-1$. From (1), (2),
\[ c_{n,0}a = c_{n,0}bf_\ell + (a_\ell + b'_{\ell+1}f_\ell)f_{\ell+1} + \cdots + (a_{n-1} + b'_{n-1}f_\ell)f_{n-1} \]
in $R_{m,n}$. Hence,
\[ c_{n,0}(a - bf_\ell) \equiv 0 \]
in $R_{m,n}/J_{\ell+1}$. Since $c_{n,0}$ is not a zerodivisor, $a - bf_\ell = 0$ in $R_{m,n}/J_{\ell+1}$. Hence, $a = 0$ in $R_{m,n}/J_\ell$. \qed

Remark. Since $f_0$ is divisible by $c_{n,0}$, $c_{n,0}$ is a zerodivisor in $R_{m,n}/J_0$.

Proposition 6.5. If Theorem 1.8 holds for $R_{m-1,n-1}$, then $R_{m,n}/J_\ell$ is an integral domain for $1 \leq \ell \leq n-1$.

Proof. For $1 \leq \ell \leq n-1$, since $c_{n,0}$ is not a zerodivisor in $R_{m,n}/J_\ell$ the induced homomorphism
\[ R_{m,n}/J_\ell \to (R_{m,n}/J_\ell)[c_{n,0}^{-1}] \]
is a monomorphism. Since $(R_{m,n}/J_\ell)[c_{n,0}^{-1}]$ is an integral domain by Lemma 6.2, $R_{m,n}/J_\ell$ is also an integral domain. \qed

This proposition completes the proof of Theorem 1.8.

Now we prove Lemma 6.1. Firstly, we prove that
\[ \det A_{m+1} = (-1)^{n-1}c_{n,0}\det A_m^p \]
for $m \geq n$. For the sake of notational ease, let
\[ \alpha = \alpha_{n-1,m} \cdots \alpha_{n-1,m+n-1}. \]

Note that $\alpha_{n-1,m} \neq 0$ for $m \geq n$ from Proposition 3.6. We write $\beta_{i,j}$ for the quotient
\[ \frac{\alpha_{i,m+j}}{\alpha_{n-1,m+j}} \]
and $\gamma_{i,j}$ for
\[ \beta_{i,j} - \beta_{i,0} \]
where $j = 1, \ldots, n-1$.

By definition,
\[
\det A_m = \begin{vmatrix}
\alpha_{0,m} & \alpha_{0,m+1} & \cdots & \alpha_{0,m+n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n-2,m} & \alpha_{n-2,m+1} & \cdots & \alpha_{n-2,m+n-1} \\
\alpha_{n-1,m} & \alpha_{n-1,m+1} & \cdots & \alpha_{n-1,m+n-1}
\end{vmatrix}.
\]
Dividing the \(j\)-th column by \(\alpha_{n-1,m+j-1}\), we have

\[
\det A_m/\alpha = \begin{vmatrix}
\beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n-2,0} & \beta_{n-2,1} & \cdots & \beta_{n-2,n-1} \\
1 & 1 & \cdots & 1
\end{vmatrix}.
\]

Subtracting the first column from the \(j\)-th column for \(j = 2, \ldots, n\), we have

\[
\det A_m/\alpha = \begin{vmatrix}
\beta_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n-2,0} & \gamma_{n-2,1} & \cdots & \gamma_{n-2,n-1} \\
1 & 0 & \cdots & 0
\end{vmatrix}.
\]

Hence,

\[
\det A_m = (-1)^{n-1} \alpha \begin{vmatrix}
\gamma_{0,1} & \gamma_{0,2} & \cdots & \gamma_{0,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n-2,1} & \gamma_{n-2,2} & \cdots & \gamma_{n-2,n-1}
\end{vmatrix}.
\]

On the other hand, by Proposition 3.4, \(\det A_{m+1}\) is equal to

\[
\begin{vmatrix}
-\alpha^{p}_{n-1,m} c_{n,0} & \cdots & -\alpha^{p}_{n-1,m+n-1} c_{n,0} \\
\alpha^{p}_{0,m} - \alpha^{p}_{n-1,m} c_{n,1} & \cdots & \alpha^{p}_{0,m+n-1} - \alpha^{p}_{n-1,m+n-1} c_{n,1} \\
\vdots & \ddots & \vdots \\
\alpha^{p}_{n-2,m} - \alpha^{p}_{n-1,m} c_{n,n-1} & \cdots & \alpha^{p}_{n-2,m+n-1} - \alpha^{p}_{n-1,m+n-1} c_{n,n-1}
\end{vmatrix}.
\]

Dividing the \(j\)-th column by \(\alpha^{p}_{n-1,m+j-1}\) for each \(j = 1, \ldots, n\), we have

\[
\det A_{m+1}/\alpha^{p} = \begin{vmatrix}
-c_{n,0} & \cdots & -c_{n,0} \\
\beta^{p}_{0,0} - c_{n,1} & \cdots & \beta^{p}_{0,0} - c_{n,1} \\
\vdots & \ddots & \vdots \\
\beta^{p}_{n-2,0} - c_{n,n-1} & \cdots & \beta^{p}_{n-2,0} - c_{n,n-1}
\end{vmatrix}.
\]

Subtracting the first column from the \(j\)-th column for \(j = 2, \ldots, n\), we have

\[
\det A_{m+1}/\alpha^{p} = \begin{vmatrix}
-c_{n,0} & 0 & \cdots & 0 \\
\beta^{p}_{0,0} - c_{n,1} & \gamma^{p}_{0,1} & \cdots & \gamma^{p}_{0,n-1} \\
\vdots & \ddots & \ddots & \vdots \\
\beta^{p}_{n-2,0} - c_{n,n-1} & \gamma^{p}_{n-2,1} & \cdots & \gamma^{p}_{n-2,n-1}
\end{vmatrix}.
\]

Hence,

\[
\det A_{m+1} = -c_{n,0} \alpha^{p} \begin{vmatrix}
\gamma_{0,1} & \gamma_{0,2} & \cdots & \gamma_{0,n-1} \\
\vdots & \ddots & \ddots & \vdots \\
\gamma_{n-2,1} & \gamma_{n-2,2} & \cdots & \gamma_{n-2,n-1}
\end{vmatrix}^{p}.
\]

Therefore, we have

\[
\det A_{m+1} = (-1)^{n} c_{n,0} (\det A_{m}^{p})
\]

for \(m \geq n\).
Secondly, we deal with the case \(0 \leq m \leq n - 1\). It is clear from Proposition 3.5 that \(\det A_0 = 1\). Suppose that \(1 \leq m \leq n - 1\). From Proposition 3.5, we have \(\alpha_{i,k} = \delta_{i,k}\) for \(k \leq n - 1\).

\[
\begin{align*}
\det A_m &= \\
&= \begin{vmatrix}
\alpha_{0,n} & \cdots & \alpha_{0,n+m-1} \\
\vdots & & \vdots \\
\alpha_{m-1,n} & \cdots & \alpha_{m-1,n+m-1} \\
\alpha_{m,n} & \cdots & \alpha_{m,n+m-1} \\
\vdots & & \vdots \\
\alpha_{n-1,n} & \cdots & \alpha_{n-1,n+m-1}
\end{vmatrix}
\end{align*}
\]

where \(0_{m,n-m}\) is the \(m \times (n-m)\) zero matrix and \(I_{n-m}\) is the \((n-m) \times (n-m)\) identity matrix. So we get

\[
\det A_m = (-1)^{(n-1)(n-m)} \begin{vmatrix}
\alpha_{0,n} & \cdots & \alpha_{0,n+m-1} \\
\vdots & & \vdots \\
\alpha_{m-1,n} & \cdots & \alpha_{m-1,n+m-1}
\end{vmatrix}
\]

Using Proposition 3.5 and Proposition 3.6, subtracting the first column multiplied by \(\alpha_{n-1,n+j-1}\) from the \(j\)-th column, we have

\[
\det A_{m+1} = (-1)^{(n-1)(n-m-1)} c_n,0 \\
\begin{vmatrix}
-c_n,0 & 0 & \cdots & 0 \\
-c_n,1 & \alpha_{0,n} & \cdots & \alpha_{0,n+m-2} \\
\vdots & \vdots & \ddots & \vdots \\
-c_n,m-1 & \alpha_{m-2,n} & \cdots & \alpha_{m-2,n+m-2}
\end{vmatrix}
\]

\[
= (-1)^{(n-1)(n-m-1)+1} c_n,0 (\det A_m^p)
\]

This completes the proof of Lemma 6.1.

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