TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 5, May 2008, Pages 2265–2284
S 0002-9947(07)04425-X
Article electronically published on December 20, 2007

THE BROWN-PETERSON COHOMOLOGY OF THE CLASSIFYING SPACES OF THE PROJECTIVE UNITARY GROUPS $PU(p)$ AND EXCEPTIONAL LIE GROUPS

MASAKI KAMEKO AND NOBUAKI YAGITA

ABSTRACT. For a fixed prime p, we compute the Brown-Peterson cohomologies of classifying spaces of $PU(p)$ and exceptional Lie groups by using the Adams spectral sequence. In particular, we see that $BP^*(BPU(p))$ and $K(n)^*(BPU(p))$ are even dimensionally generated.

1. Introduction

Let p be a fixed odd prime and denote by $BP^*(X)$ (resp. $P(m)^*(X)$) the Brown-Peterson cohomology of a space X with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ (resp. $P(m)^* = \mathbb{Z}/p[v_m, v_{m+1}, \cdots]$) where $\deg v_k = -2p^k + 2$. We denote by $PU(n)$ the projective unitary group which is the quotient of the unitary group $U(n)$ by its center S^1. Recall that the cohomologies of $PU(p)$ and the exceptional Lie groups F_4, E_6, E_7, E_8 have odd torsion elements. In this paper, we compute the Brown-Peterson cohomologies of classifying spaces BG of these Lie groups G as BP^*-modules using the Adams spectral sequence. Let us write $H^*(X;\mathbb{Z}/p)$ by simply $H^*(X)$ and let $\mathcal{A}$ be the mod p Steenrod algebra.

Our main result is as follows:

Theorem 1.1. *Let (G, p) be one of cases $(G = PU(p), p)$ for an arbitrary odd prime p and $G = F_4, E_7$ for $p = 3$, and $G = E_8$ for $p = 5$. Then the E_2-terms of the Adams spectral sequences abutting to $BP^*(BG)$ and $P(m)^*(BG)$ for $m \geq 1$,*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(BP), H^*(BG)), \qquad \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(P(m)), H^*(BG)),$$

have no odd degree elements.

An immediate consequence is as follows:

Corollary 1.2. *For (G, p) in Theorem 1.1, the Adams spectral sequences abutting to $BP^*(BG)$ and $P(m)^*(BG)$ in the previous theorem collapse at the E_2-level. In particular $BP^{odd}(BG) = P(m)^{odd}(BG) = 0$.*

Recall $K(m)^*(X) \cong K(m)^* \otimes_{P(m)^*} P(m)^*(X)$ is the Morava K-theory. From the above theorem and corollary, we see $K(m)^{odd}(BPU(p)) = 0$. Then we have the following corollary ([Ko-Ya], [Ra-Wi-Ya]).

Received by the editors November 2, 2005.
2000 *Mathematics Subject Classification.* Primary 55P35, 57T25; Secondary 55R35, 57T05.
Key words and phrases. BP-theory, $PU(p)$, exceptional Lie groups.

Corollary 1.3. *For (G, p) in Theorem 1.1, the following holds:*

(1) *$BP^*(BG)$ is BP^*-flat for $BP^*(BP)$-modules, i.e., $BP^*(BG \times X) \cong BP^*(BG) \otimes_{BP^*} BP^*(X)$ for all finite complexes X.*

(2) *$K(n)^*(BG) \cong K(n)^* \otimes_{BP^*} BP^*(BG)$.*

(3) *$P(n)^*(BG) \cong P(n)^* \otimes_{BP^*} BP^*(BG)$.*

We give the BP^*-module structure of $BP^*(BPU(p))$ more explicitly in this introduction. For the exceptional Lie group cases, the explicit formulas are given in §5.

Theorem 1.4. *There exists a BP^*-algebra exact sequence*

$$0 \to BP^*\widehat{\otimes}M \to grBP^*(BPU(p)) \to BP^*\widehat{\otimes}IN/(f_0, f_1) \to 0$$

where

(1) *$M \cong \mathbb{Z}_{(p)}[x_4, x_6, \ldots, x_{2p}]$ as $\mathbb{Z}_{(p)}$-modules (but not $\mathbb{Z}_{(p)}$-algebras).*
(2) *$IN \cong \mathbb{Z}_{(p)}[x_{2p+2}, x_{2p(p-1)}]\{x_{2p+2}\}$, the principal ideal of $\mathbb{Z}_{(p)}[x_{2p+2}, x_{2p(p-1)}]$ generated by x_{2p+2}.*
(3) *Relations f_0, f_1 are given with modulo $(p, v_1, v_2, \cdots)^2$:*

$$f_0 \equiv v_0 - v_2 x_{2p+2}^{p-1} + \cdots, \quad f_1 \equiv v_1 - v_2 x_{2p(p-1)} + \cdots.$$

Remark. In the above theorem, subscript i of x_i means its degree. The algebra $BP^*(BPU(p))$ does not contain the subalgebra $BP^*\widehat{\otimes}\mathbb{Z}_{(p)}[x_4, \ldots, x_{2p}]$, but contains a subalgebra which is isomorphic as BP^*-modules to the above BP^*-subalgebra.

For an algebraic group G over $\mathbb{C}$, Totaro [To] defines its Chow ring and conjectures that $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(BG)_{(p)}$. Recall that $PGL(p, \mathbb{C})$ is the algebraic group over $\mathbb{C}$ corresponding to the Lie group $PU(p)$.

Theorem 1.5. *There exists an isomorphism*

$$BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(BGL(p, \mathbb{C}))_{(p)}.$$

Hence there exists an additive isomorphism

$$CH^*(BGL(p, \mathbb{C}))_{(p)} \cong \mathbb{Z}_{(p)}[x_4, x_6, \cdots, x_{2p}] \oplus \mathbb{F}_p[x_{2p+2}, x_{2p(p-1)}]\{x_{2p+2}\}.$$

Remark. Vistoli [Vi] also determined the additive structure of the Chow ring and integral cohomology of $BPGL(p, \mathbb{C})$ by using stratified methods of Vezzosi [Ve] (see also [Mo-Vi]). Moreover he shows that for $G = PGL(p, \mathbb{C})$,

$$H^*(G; \mathbb{Z}) \to H^*(BT; \mathbb{Z})^{W_G(T)}$$

is epic.

Let $MGL^{*,*}(X)$ be the motivic cobordism ring defined by V. Voevodsky [Vo1] and $MGL^{2*,*}(X) = \bigoplus_i MGL^{2i,i}(X)$.

Corollary 1.6. $MGL^{2*,*}(BPGL(p, \mathbb{C}))_{(p)} \cong MU^*(BPU(p))_{(p)}$.

We prove Theorem 1.1 by using the Adams spectral sequence converging to the Brown-Peterson cohomology. The E_1-term of the spectral sequence could be given by

$$\mathbb{F}_p[v_0, v_1, \cdots]\widehat{\otimes}H^*(X) \quad \text{with} \quad d_1 x = \sum_{k=0}^{\infty} v_k Q_k x$$

where the Q_k's are Milnor's operations. (Here we identify $\mathbb{F}_p[v_0 = p, v_1, \cdots] = P(0)^*$.)

By the change-of-rings isomorphism, the E_2-term is

$$\mathrm{Ext}_{\mathcal{A}}(H^*(BP), H^*(X)) \cong \mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p, H^*(X))$$

where $\mathcal{E} = \Lambda(Q_0, Q_1, \cdots)$. The E_∞-term is given by $grBP^*(X)$.

To state the cohomology $H^*(BPU(p))$, we recall the Dickson algebra. Let A_n be an elementary abelian p-group of rank n, and

$$H^*(BA_n) \cong \mathbb{F}_p[t_1, ..., t_n] \otimes \Lambda(dt_1, ..., dt_n) \quad \text{with } \beta(dt_i) = t_i.$$

The Dickson algebra is

$$D_n = \mathbb{F}_p[t_1, ..., t_n]^{GL(n,\mathbb{F}_p)} \cong \mathbb{F}_p[c_{n,0}, ..., c_{n,n-1}]$$

with $|c_{n,i}| = 2(p^n - p^i)$. The invariant ring under $SL(n, \mathbb{F}_p)$ is also given:

$$SD_n = \mathbb{F}_p[t_1, ..., t_n]^{SL(n,\mathbb{F}_p)} \cong D_n\{1, e_n, ..., e_n^{p-2}\} \quad \text{with } e_n^{p-1} = c_{n,0}.$$

We also recall Mui's ([Mu]) result by using Q_i according to Kameko and Mimura [Ka-Mi]

$$grH^*(BA)^{SL_n(\mathbb{F}_p)} \cong SD_n/(e_n) \oplus SD_n \otimes \Lambda(Q_0, ..., Q_{n-1})\{u_n\}$$

where $u_n = dt_1...dt_n$ and $e_n = Q_0...Q_{n-1}u_n$.

Theorem 1.7. *There exists a short exact sequence*

$$0 \to M/p \to H^*(BPU(p)) \to N \to 0$$

where M/p is the trivial $\mathcal{E}$-module given in Theorem 1.4 and

$$N = SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \cong \mathbb{F}_p[x_{2p+2}, x_{2(p^2-p)}] \otimes \Lambda(Q_0, Q_1)\{u_2\}$$

identifying $x_{2p+2} = e_2$ and $x_{2(p^2-p)} = c_{2,1}$.

This theorem is proved by using the following facts. The group $G = PU(p)$ has just two conjugacy classes of maximal elementary abelian p-subgroups, one of which is toral and the other is nontoral A of $rank_p = 2$. The cohomology $H^*(BG)$ is detected by these two subgroups. The restriction image to the nontoral subgroup is $i_A^*(H^*PU(p)) \cong H^*(BA)^{SL(2,\mathbb{F}_p)}$. Similar (but not the same) facts also hold for the exceptional Lie groups given in Theorem 1.1.

The algebraic main result in this paper is as follows:

Theorem 1.8. *For $m \geq 0$, define $f_0, ..., f_{n-1}$ in $P(m)^*\widehat{\otimes} SD_n$ by*

$$d_1 u_n = \sum_{k \geq m} v_k Q_k(u_n) = f_0 Q_0 u_n + \cdots + f_{n-1} Q_{n-1} u_n.$$

Then the sequence $f_0, \ldots, f_{n-1}$ is a regular sequence in $P(m)^*\widehat{\otimes} SD_n$.

With the notation in this theorem, we prove that the complex

$$C = (P(m)^*\widehat{\otimes} SD_n \otimes \Lambda(Q_0, Q_1, \ldots, Q_{n-1})\{u_n\}, d_1)$$

with the differential $d_1 u_n = \sum_{i=0}^{n-1} f_i Q_i u_n$ is a Koszul complex. This means that

$$H_i(C, d_1) = \begin{cases} P(m)^*\widehat{\otimes} SD_n\{e_n\}/(f_0, \ldots, f_{n-1}) & \text{for } i = 0 \\ 0 \quad \text{for } i \geq 1. \end{cases}$$

Thus Theorem 1.1 follows from the above theorem.

Remark about the convergence of the Adams spectral sequence. By Theorem 15.6 in Boardman's paper [Bo2], since $H^*(BP)$ is of finite type, the above Adams spectral sequence is conditionally convergent. Moreover, since we prove the above Adams spectral sequence collapses at the E_2-level, by the remark after Theorem 7.1 in [Bo1], the above Adams spectral sequence is strongly convergent, so that we know the Brown-Peterson cohomology up to group extension.

We now give an outline of the paper. In §2, we note the relation of the Koszul complex and the E_1-term of the Adams spectral sequence for $\Lambda(Q_0, ..., Q_{n-1})$-free $H^*(X)$. Section 3 is stated about the Dickson invariant rings and the Milnor Q_i-operation on the cohomology of classifying spaces of elementary abelian p-groups. In §4, we compute $H^*(BG)$, $MU^*(BG)$, $CH^*(BG)$ for $G = PU(p)$. The similar computations are done in §5 for exceptional groups. The last section is devoted to the proof of Theorem 1.8.

2. The Koszul complex and the Adams spectral sequence

For spaces X, Y, the Adams spectral sequence converging to the stable homotopy $\{X, Y\}$ has the E_2-term

$$E_2^{*,*} \cong \mathrm{Ext}_{\mathcal{A}}(H^*(Y), H^*(X)) \Longrightarrow \{X, Y\}.$$

We want to consider the case $Y = P(m)$.

Recall that $\mathcal{A} \cong H^*(BP) \otimes \mathcal{E}$ with $\mathcal{E} = \Lambda(Q_0, Q_1, ...)$. Similarly $H^*(P(m)) \cong H^*(BP) \otimes \Lambda(Q_0, ..., Q_{m-1})$. Hence the E_2-term of the Adams spectral sequence abutting to $P(m)^*(X)$ is given by

$$\begin{aligned}\mathrm{Ext}_{\mathcal{A}}(H^*(P(m)), H^*(X)) &\cong \mathrm{Ext}_{H^*(BP)\otimes\mathcal{E}}(H^*(BP) \otimes \Lambda(Q_0, ..., Q_{m-1}), H^*(X)) \\ &\cong \mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, H^*(X)) \qquad \text{where } \ \mathcal{E}_m = \Lambda(Q_m, Q_{m+1}, \cdots).\end{aligned}$$

We will study the case

$$0 \to M \to H^*(X) \to N \to 0$$

where M is $\mathcal{E}$-trivial and N is a free $\Lambda(Q_0, \ldots, Q_{n-1})$-module having the following properties. (For example, $X = BPU(p)$; recall Theorem 1.7.)

For a $\mathbb{Z}/p$-algebra S, suppose that $\mathcal{E}$ acts on

$$N = S \otimes \Lambda(Q_0, \ldots, Q_{n-1})$$

such that Q_i acts as a derivation and $Q_n(s \otimes t) = s \otimes Q_i(t)$ for $s \in S, t \in N$. Then for each $k \geq 0$, there exist $\alpha_{0,k}, \ldots, \alpha_{n-1,k} \in S$ such that

$$Q_k x = \alpha_{0,k} Q_0 x + \cdots + \alpha_{n-1,k} Q_{n-1} x$$

for all $x \in N$ (compare the proof of Proposition 3.3 below).

Let $R = P(m)^* \widehat{\otimes} S$. For $i = 0, \ldots, n-1$, let

$$f_i = v_m \alpha_{i,m} + v_{m+1} \alpha_{i,m+1} + \cdots \in R.$$

The derived functor $\mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, N)$ is the cohomology of a cochain complex

$$E_1 = P(m)^* \widehat{\otimes} N = R \otimes \Lambda(Q_0, \ldots, Q_{n-1})$$

with a differential

$$d_1(x) = \sum_{k \geq m} v_k Q_k(x) = f_0 Q_0(x) + \cdots + f_{n-1} Q_{n-1}(x).$$

Proposition 2.1. *With the notation above, if $f_0, \ldots, f_{n-1}$ is a regular sequence in $R = P(m)^*\widehat{\otimes} S$, then*

$$\mathrm{Ext}_{\mathcal{A}}(H^*(P(m), N) = \mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, N) \cong R/(f_0, \ldots, f_{n-1}).$$

Hence Theorem 1.1 (for $G = PU(p)$) follows from Theorem 1.7, 1.8 and the above proposition. (We will show Theorem 1.7 in §4 and Theorem 1.8 in $\S 3, 6$.)

To see the above proposition, we recall the Koszul complex. Let R be a commutative algebra and let $g_0, \ldots, g_{n-1}$ be a sequence of R. Let J be the ideal of R generated by $g_0, \ldots, g_{n-1}$. Let $\Lambda^r = \Lambda^r(x_0, \ldots, x_{n-1})$ be r-homogeneous parts with $\deg(x_i) = 1$. The Koszul complex is a chain complex

$$0 \to R \otimes \Lambda^n \xrightarrow{d} \cdots \to R \otimes \Lambda^1 \xrightarrow{d} R \otimes \Lambda^0 \to 0$$

with the differential

$$d(x_{i_1} \cdots x_{i_r}) = \sum_{j=1}^{r} (-1)^{j+1} g_{i_j} x_{i_1} \cdots \widehat{x}_{i_j} \cdots x_{i_r}.$$

Let us denote by $H_k(R \otimes \Lambda^*, d)$ the induced homology. The following proposition is well known.

Proposition 2.2. *If $g_0, \ldots, g_{n-1}$ is a regular sequence in R, then*

$$H_k(R \otimes \Lambda^*, d) \cong \begin{cases} R/J & for\ k = 0, \\ 0 & for\ k \geq 1. \end{cases}$$

Proof of Proposition 2.1. Let $g_i = (-1)^i f_i$. It is clear that $g_0, \ldots, g_{n-1}$ is a regular sequence if and only if $f_0, \ldots, f_{n-1}$ is a regular sequence. Let

$$\mu : R \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \to R \otimes \Lambda^*$$

be an R-module isomorphism defined by

$$\mu(Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1}) = x_{i_1} \cdots x_{i_r}$$

where $0 \leq i_1 < \cdots < i_r \leq n-1$. We verify that this homomorphism μ commutes with the differentials d_1, d. On the one hand,

$$\begin{aligned} d \circ \mu(Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1}) &= d(x_{i_1} \cdots x_{i_r}) \\ &= \sum_{j=1}^{r} (-1)^{j+1} g_{i_j} x_{i_1} \cdots \widehat{x_{i_j}} \cdots x_{i_r}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} & d_1(Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1}) \\ = \ & (f_0 Q_0 + \cdots + f_{n-1} Q_{n-1}) Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1} \\ = \ & (-1)^{i_1} f_{i_1} Q_0 \cdots Q_{i_1} \cdots \widehat{Q}_{i_2} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1} \\ & + (-1)^{i_2 - 1} f_{i_2} Q_0 \cdots \widehat{Q}_{i_1} \cdots Q_{i_2} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1} \\ & \vdots \\ & + (-1)^{i_r - (r-1)} f_{i_r} Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_2} \cdots Q_{i_r} \cdots Q_{n-1}, \end{aligned}$$

we get

$$\begin{aligned}\mu \circ d_1(Q_0 \cdots \widehat{Q}_{i_1} \cdots \widehat{Q}_{i_r} \cdots Q_{n-1}) &= \sum_{j=1}^{r} (-1)^{i_j - j+1} f_{i_j} x_{i_1} \cdots \widehat{x_{i_j}} \cdots x_{i_r} \\ &= \sum_{j=1}^{r} (-1)^{j+1} g_{i_j} x_{i_1} \cdots \widehat{x_{i_j}} \cdots x_{i_r}.\end{aligned}$$

Therefore, μ is an isomorphism of chain complexes, so that by Proposition 2.2, Proposition 2.1 holds. □

3. The Dickson invariant and Milnor Q_i operation

Let A be an elementary abelian p-group of rank n. The mod p cohomology $H^*(BA)$ is isomorphic to the polynomial tensor exterior algebra

$$\mathbb{F}_p[t_1, \dots, t_n] \otimes \Lambda(dt_1, \dots, dt_n)$$

as a graded algebra. It is also clear that the finite general linear group $GL(n, \mathbb{F}_p)$ and the finite special linear group $SL(n, \mathbb{F}_p)$ act on A, BA and $H^*(BA)$. Dickson computed the ring of invariants of $\mathbb{F}_p[t_1, \dots, t_n]$ with respect to the action of $GL(n, \mathbb{F}_p)$. In the case of $G = GL(n, \mathbb{F}_p)$, the ring of invariants is a polynomial algebra

$$D_n = \mathbb{F}_p[c_{n,0}, \dots, c_{n,n-1}]$$

whose generators are given by the equation

$$\mathcal{O}_n(X) = \prod_{t \in \mathbb{F}_p\{t_1, \cdots, t_n\}} (X + t) = X^{p^n} + \sum_{j=0}^{n-1} c_{n,j} X^{p^j}.$$

Remark. In [Ka], [Ka-Mi], notation $c_{n,i}$ is used as $(-1)^{n-i} c_{n,i}$ in this paper.

For the invariant ring under $SL(n, \mathbb{F}_p)$, we have

$$SD_n = D_n\{1, e_n, ..., e_n^{p-2}\} \quad \text{with } e_n^{p-1} = c_{n,0}.$$

Mui computed the ring of invariants of $\mathbb{F}_p[t_1, \dots, t_n] \otimes \Lambda(dt_1, \dots, dt_n)$ with respect to the action of $SL(n, \mathbb{F}_p)$. Of course $u_n = dt_1 ... dt_n$ is invariant under $SL(n, \mathbb{F}_p)$. In terms of Milnor's operation, we may state Mui's result in the following form.

Theorem 3.1 (Mui [Mu], Kameko-Mimura [Ka-Mi])**.**

$$\begin{aligned}H^*(BA)^{SL(n,\mathbb{F}_p)} &\cong \mathbb{F}_p[c_{n,1}, ..., c_{n,n-1}] \oplus SD_n \otimes \Lambda(Q_0, ..., Q_{n-1})\{u_n\} \\ &\cong SD_n/(e_n) \oplus SD_n \otimes \Lambda(Q_0, ..., Q_{n-1})\{u_n\}\end{aligned}$$

where $Q_0 ... Q_{n-1} u_n = e_n$.

In this section, we compute $f_0, \dots, f_{n-1}$ in §2 for the case $S = SD_n$. We study some important relations between Q_i and $c_{n,j}$.

Lemma 3.2. *For* $x \in H^*(BA)$, *it follows that*

$$(Q_n + \sum_{i=1}^{n} c_{n,i} Q_i)(x) = 0, \quad Q_0 ... \hat{Q}_i ... Q_n(u_n) = (-1)^{n-i} c_{n,i} e_n.$$

Proof. Note $Q_i(dt_j) = t_j^{p^i}$. When $x = dt_j$, we see

$$(Q_n + \sum c_{n,i}Q_i)(dt_j) = t_j^{p^n} + \sum c_{n,i}t_j^{p^i} = \mathcal{O}_n(t_j) = 0.$$

Since Q_i is a derivation, we get the first equation.

Applying $Q_0...\hat{Q}_i...Q_{n-1}$ to the first equation, we get the second equation. $\square$

By using the relation $Q_{k+1} = [Q_k, \wp^{p^k}]$ and the above lemma, we may define an $\mathcal{E}$-module structure on $SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1})\{u_n\}$. Let us write (recall the definition in §2)

$$Q_k u_n = \alpha_{0,k}Q_0u_n + \cdots + \alpha_{n-1,k}Q_{n-1}u_n.$$

With the following proposition, we verify that there is an algebra homomorphism $\mathcal{E} \to SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1})$ such that the evaluation at u_n induces an $\mathcal{E}$-module homomorphism $SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1}) \to H^*(BA)$.

Proposition 3.3. *For each x in $SD_n \otimes \Lambda(Q_0, \ldots, Q_{n-1})\{u_n\} \subset H^*BA$,*

$$Q_k x = (\alpha_{0,k}Q_0 + \cdots + \alpha_{n-1,k}Q_{n-1})x.$$

Proof. It is immediate that $Q_kQ_{i_1}\cdots Q_{i_r} = (-1)^r Q_{i_1}\cdots Q_{i_r}Q_k$ and so

$$\begin{aligned} Q_kQ_{i_1}\cdots Q_{i_r}u_n &= (-1)^r Q_{i_1}\cdots Q_{i_r}Q_k u_n \\ &= (-1)^r Q_{i_1}\cdots Q_{i_r}(\alpha_{0,k}Q_0u_n + \cdots + \alpha_{n-1,k}Q_{n-1}u_n) \\ &= (\alpha_{k,0}Q_0 + \cdots + \alpha_{n-1,k}Q_{n-1})Q_{i_1}\cdots Q_{i_r}u_n. \end{aligned}$$

$\square$

The following propositions are only used in the last section to prove Theorem 1.8.

Proposition 3.4. *For $i \geq 1$, $\alpha_{i,0} = 0$ and $\alpha_{0,0} = 1$. Letting $\alpha_{-1,k} = 0$, we have*

$$\alpha_{i,k+1} = \alpha_{i-1,k}^p - \alpha_{n-1,k}^p c_{n,i}.$$

Proof. Since $Q_{k+1} = [Q_k, \wp^{p^k}]$, we have

$$Q_{k+1}u_n = \wp^{p^k}Q_ku_n = \wp^{p^k}\sum_{i=0}^{n-1}\alpha_{i,k}Q_iu_n = \sum_{i=0}^{n-1}\alpha_{i,k}^pQ_{i+1}u_n.$$

The last equation follows from $\deg \alpha_{i,k} = 2p^k - 2p^i$ and

$$\wp^{p^k}\alpha_{i,k}Q_iu_n = \wp^{p^k - p^i}\alpha_{i,k} \cdot \wp^{p^i}Q_iu_n = \alpha_{i,k}^pQ_{i+1}u_n.$$

We also get

$$\begin{aligned} \sum_{i=0}^{n-1}\alpha_{i,k}^pQ_{i+1}u_n &= \sum_{i=0}^{n-2}\alpha_{i,k}^pQ_{i+1}u_n + \alpha_{n-1,k}^pQ_nu_n \\ &= \sum_{i=1}^{n-1}\alpha_{i-1,k}^pQ_iu_n - \sum_{i=0}^{n-1}\alpha_{n-1,k}^pc_{n,i}Q_iu_n. \end{aligned}$$

Here the last equation follows from Lemma 3.2. Hence the last element is equal to $Q_{k+1}u = \sum \alpha_{i,k+1}Q_iu_n$. Thus we get the equation in this proposition. $\square$

Proposition 3.5. *For $k \leq n-1$, $\alpha_{i,k} = \delta_{i,k}$.*

Proposition 3.6. *We have $\alpha_{i,n} = -c_{n,i}$ and $\alpha_{n-1,k} \neq 0$ for $k \geq n$; in particular,*

$$\alpha_{n-1,n+k} = c_{n,n-1}^{p^k+\cdots+p+1} + \textit{terms lower with respect to } c_{n,n-1}.$$

In particular, $\alpha_{0,n} = -c_{n,0}, \ldots, \alpha_{n-1,n} = -c_{n,n-1}$, $\alpha_{i,j} = \delta_{i,j}$ for $i, j = 0, \ldots, n-1$. That is, f_i is expressed as

$$\begin{cases} f_0 = v_0 & - v_n c_{n,0} + \cdots \\ f_1 = \quad v_1 & - v_n c_{n,1} + \cdots \\ \vdots \qquad \ddots & \\ f_{n-1} = \qquad v_{n-1} & - v_n c_{n,n-1} + \cdots \end{cases}$$

4. Projective unitary groups $PU(p)$

The additive structure of the cohomology $H^*(BPU(p))$ was first determined by the first author computing the Poincaré series of the E_2-term of the Rothenberg-Steenrod spectral sequence. (A. Vistoli [Vi] also gets the same results by completely different methods.) Indeed, we have

$$PS(H^*(BPU(p),t)) = \sum rank_p H^i(BPU(p))t^i$$
$$= \frac{1}{(1-t^4)(1-t^6)...(1-t^{2p})} + \frac{t^2+t^3+t^{2p+1}+t^{2p+2}}{(1-t^{2p+2})(1-t^{2p(p-1)})}.$$

However we give it here by a short argument using the results of Vavpetič and Viruel [Va-Vi] and Kono and Yagita [Ko-Ya].

Let G be a compact Lie group. Quillen showed that the following restriction map r is an F-isomorphism:

$$r : H^*(BG) \to \varprojlim_{A} H^*(BA)$$

where A runs over a set of conjugacy classes of elementary abelian p-subgroups of G. Here an F-isomorphism means that its kernel is generated by nilpotent elements and for each $x \in H^*(BG)$ there is $s \geq 0$ such that $x^{p^s} \in Im(r)$.

Recently A.Vavpetič and A.Viruel gave the following short proof of the fact that the map r is injective for $G = PU(p)$.

There is only one conjugacy class of nontoral abelian groups A of $rank_p = 2$. Let T_G be a maximal torus of G and $N_G = N_G(T)$ the normalizer of the torus T_G. Let $N_p(G)$ be the p-normalizer of T_G, namely, the preimage of a p-Sylow subgroup in the Weyl group $W_G = N_G/T_G$. For example, it is known that

$$N_p(U(p)) \cong S^1 \wr \mathbb{Z}/p \cong \{diag(z_1, ..., z_p)P^i \in U(p) | z_j \in \mathbb{C}^*, 0 \leq i \leq p-1\}$$

where $P \in U(p)$ is a permutation matrix of order p.

Lemma 4.1. $N_p(SU(p)) \cong N_p(PU(p))$.

The cofiber sequence $(\mathbb{C}^* \cong) S^1 \to BSU(p) \to BU(p)$ induces the Gysin exact sequence

$$\to H^{*-2}(BN_p(U(p))) \to H^*(BN_p(U(p))) \to H^*(BN_p(SU(p))) \to .$$

Considering restrictions to A and tori of $U(p)$ and $SU(p)$, Vavpetič and Viruel prove :

Lemma 4.2 ([Va-Vi]). *$H^*(BN_p(SU(p)))$ is detected by A and $T_{SU(p)}$.*

Theorem 4.3 ([Va-Vi]). *$H^*(BPU(p))$ is detected by A and $T_{PU(p)}$.*

Proof. Consider the restrictions

$$H^*(BPU(p)) \overset{i_{N_{PU}}}{\to} H^*(BN_{PU(p)}) \overset{i_{N_p}}{\to} H^*(BN_p(PU(p))) \cong H^*(BN_p(SU(p))).$$

Recall the transfer tr and Gottlieb transfer Gtr so that $Gtr.i_{N_{PU}} = id$ and $tr.i_{N_p} = id$. Hence the above restrictions are injective. ☐

Note that $W_G(A) = SL(2,\mathbb{F}_p)$ for $G = PU(p)$. Next by using arguments ([Ko-Ya]), we will show for $G = PU(p)$ that the following map pi_A is epic:

$$pi_A : H^*(BG) \overset{i_A}{\to} H^*(BA)^{W_G(A)} = SD_2/(e_2) \oplus SD_2 \otimes \Lambda(Q_0,Q_1)\{u_2\}$$
$$\overset{proj.}{\to} SD_2 \otimes \Lambda(Q_0,Q_1)\{u_2\}.$$

Let E be the subgroup of $SU(p)$ generated by

$$c = diag(\xi,...,\xi), \quad a = diag(1,\xi,...,\xi^{p-1}), \quad b = P \ (permutation\ matrix)$$

where ξ is the p-th primitive root of 1. Then we see that

$$a^p = b^p = c^p = 1, \quad [a,b] = c \in \text{ center of } E.$$

Hence E is isomorphic to the extraspecial p-group p_+^{1+2} of order p^3. We consider the commutative diagram for fiberings

$$\begin{array}{ccccc} \mathbb{Z}/p & \longrightarrow & SU(p) & \longrightarrow & PSU(p) = PU(p) \\ {\scriptstyle =}\uparrow & & \uparrow & & \uparrow \\ \mathbb{Z}/p = \langle c\rangle & \longrightarrow & E & \longrightarrow & A = \langle a,b\rangle. \end{array}$$

The above diagram induces the map $f_r : E(U)_r^{*,*} \to E(E)_r^{*,*}$ from the spectral sequence converging to $H^*(BSU(p))$ to that converging to $H^*(BE)$.

The differential of $E(E)_r^{*,*}$ is well known [Te-Ya]. Let us write

$$E(E)_2^{*,*} \cong H^*(BA; H^*(B\mathbb{Z}/p)) \cong \mathbb{F}_p[t_1,t_2,w] \otimes \Lambda(dt_1,dt_2,dw)$$

with $\beta dw = w$. Then it is known that

$$d_2(dw) = dt_1dt_2 = u_2.$$

Since $dw \in Im(f_2)$, we see that $u_2 \in Im(f_3)$ and so in $Im(pi_A^*)$.

Here we also see that $c_{2,1}, c_{2,0} \in Im(i_A^*)$. Let ρ be the canonical representation of $SU(p)$ and

$$\lambda = \rho \otimes \rho^{-1} : SU(p) \to SU(p^2).$$

Since $\rho \otimes \rho^{-1}|\mathbb{Z}/p$ is trivial, we can identify λ as the representation of $PU(p)$. For $a^ib^j \in A \cong (\mathbb{Z}/p)^2$, we easily compute $\chi_\lambda(a^ib^j) = p^2$ if $i = j = 0$ and $= 0$ otherwise. This means that $\lambda|A$ is the regular representation. So the total Chern class is

$$c(\lambda|A) = \Pi_{(\lambda_1,\lambda_2)\in(\mathbb{Z}/p)^2}(1 + \lambda_1t_1 + \lambda_2t_2) = 1 + c_{2,1} + c_{2,0}.$$

Hence we have $c_{p^2-p}(\lambda)|A = c_{2,1}$ and $c_{p^2-1}(\lambda)|A = c_{2,0}$.

Lemma 4.4. *The map pi_A is epic.*

(From Lemma 3.2, we also have $e_2 = Q_0Q_1u_2$, $e_2^{p-1} = c_{2,0}$ and $Q_2Q_0(u_2) = c_{2,1}e_2$.)

Thus we get the short exact sequence

$$0 \to Ker(pi_A) \to H^*(BPU(p)) \to SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \to 0.$$

By the theorem by Vavpetič and Viruel, we have the injection

$$i_T \times i_A : Ker(pi_A) \subset H^*(BPU(p)) \to H^*(BT_{PU(p)}) \times SD_2/(e_2).$$

In particular $Ker(pi_A)$ is generated by even-dimensional elements.

Now we consider the Bockstein spectral sequence

$$E_1 = H^*(X) \Longrightarrow \mathbb{F}_p \otimes H^*(X;\mathbb{Z})/(torsions).$$

The first differential is the Bockstein operation Q_0. Since $Im(pi_A)$ is Q_0-free, so $H(Im(pi_A);Q_0) = 0$. Hence we have

$$E_2 = H(H^*(BPU(p);Q_0)) \cong Ker(pi_A).$$

Since $Ker(pi_A)$ is generated by even-dimensional elements, we get

$$E_\infty \cong E_2 \cong Ker(pi_A).$$

It is well known that

$$H^*(BPU(p);\mathbb{Q}) \cong H^*(BSU(p);\mathbb{Q}) \cong \mathbb{Q}[c_2, ..., c_p].$$

Since $rank_p(\mathbb{F}_p \otimes (H^i(X;\mathbb{Z})/(torsions))) = rank_{\mathbb{Q}} H^i(X;\mathbb{Q})$, we get

Lemma 4.5. *There exist additive isomorphisms*

$$Ker(pi_A) \cong \mathbb{F}_p \otimes H^*(BPU(p);\mathbb{Z})/(torsions) \cong \mathbb{F}_p[x_2, ..., x_p] \quad with\ |x_i| = 2i.$$

Remark. This is *not* a ring isomorphism; e.g., for $p = 3$ we have the ring isomorphism

$$Ker(pi_A) \cong \mathbb{F}_3[c_2, c_3, c_6]/(c_2^3 - c_3^2) \not\cong \mathbb{F}_3[c_2, c_3].$$

Thus we have Theorem 1.7 in the introduction.

Theorem 4.6. *There exists a short exact sequence of rings*

$$0 \to M/p \to H^*(BPU(p)) \to SD_2 \otimes \Lambda(Q_0, Q_1)\{u_2\} \to 0$$

with additively (not rings) $M/p \cong \mathbb{F}_p[x_2, ..., x_p]$.

Now we consider the Adams spectral sequence converging to $P(m)^*(BPU(p))$. By the change of rings,

$$\mathrm{Ext}_A^{s,t}(H^*(P(m), H^*(X)) \cong \mathrm{Ext}_{\mathcal{E}_m}^{s,t}(\mathbb{F}_p, H^*(X)).$$

From the above theorem, Theorem 1.8 implies

Lemma 4.7. *There exists a short exact sequence*

$$0 \to P(m)^*\hat{\otimes}M \to \mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, H^*(BPU(p)) \to P(m)^*\hat{\otimes}SD_2/(f_0, f_1)\{e_2\} \to 0.$$

Thus we have Theorem 1.1 in the introduction for $G = PU(p)$.

Next we consider the Chow ring for $BPU(p)$. Recall that $N_p(U(p)) \cong S^1 \wr \mathbb{Z}/p$. By Totaro, it is known that $CH^*(S^1 \wr \mathbb{Z}/p)/p$ is detected by $CH^*((BS^1)^{\times p})/p$ and $CH^*(B(S^1 \times \mathbb{Z}/p))/p$ (see page 271 in [To]) and so by elementary p-subgroups. We also know that the restriction map

$$CH^*(B(S^1 \wr \mathbb{Z}/p)) \to CH^*((BS^1)^{\times p})^{\mathbb{Z}/p}/p \cong \mathbb{Z}/p[y_1, ..., y_p]^{\mathbb{Z}/p}$$

is epic. Moreover he showed (his conjecture)

$$(*) \qquad CH^*(BG) \cong BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)}$$

for this group $G = S^1 \wr \mathbb{Z}/p$. By using $CH^*(-)/p$ (and $BP^*(-)$) version of the Gysin sequence given just before Lemma 4.2, we also see $(*)$ for $G = N_p(SU(p))$. Since Chow rings $CH^*(-)_{(p)}$ (and $BP^*(-) \otimes_{BP^*} \mathbb{Z}_{(p)}$) have transfer (and Gottlieb transfer), the isomorphism $(*)$ for $G = PU(p)$ follows from the arguments of the proof of Theorem 4.3. Recall $PGL(p, \mathbb{C})$ is the algebraic group corresponding to $PU(p)$.

Theorem 4.8. $CH^*(BPGL(p, \mathbb{C}))_{(p)} \cong BP^*(BPU(p)) \otimes_{BP^*} \mathbb{Z}_{(p)}$.

Thus we get Theorem 1.5 in the introduction.

Remark. Vezzosi first studied $CH^*(BPGL(3, \mathbb{C}))_{(3)} = CH^*(PU(3))_{(3)}$. He showed (Theorem 1.1 in [Ve]) if some element $\chi \in CH^6(PU(3))_{(3)}$ is zero, then we have the isomorphism in the above theorem. Indeed, we have seen $\chi = 0$.

Let $H^{*,*}(-; \mathbb{Z})$ be the motivic cohomology defined by Suslin and Voevodsky so that $H^{2*,*}(X; \mathbb{Z}) \cong CH^*(X)$. Here $H^{2*,*}(X; \mathbb{Z}) = \bigoplus_i H^{2i,i}(X; \mathbb{Z})$. Recall that $MGL^{*,*}(-)$ is the motivic cohomology defined by Voevodsky ([Vo1]) by using spectrum MGL in the $\mathbb{A}^1$ stable homotopy category. Of course there is the natural map $MGL^{2*,*}(X) \to MU^{2*}(X)$.

Corollary 4.9. $MGL^{2*,*}(BGL(p, \mathbb{C}))_{(p)} \cong MU^{2*}(BPU(p))_{(p)}$.

Proof. Let $X = BPGL(p, \mathbb{C})$. Consider the motivic Atiyah-Hirzebruch spectral sequence

$$E_2 = H^{*,*}(X; MU^*) \Longrightarrow MGL^{*,*}(X).$$

When X is smooth and $m > 2n$, it is known that $H^{m,n}(X; \mathbb{Z}) = 0$. This implies ((1.2) in [Ya])

$$MGL^{2*,*}(X) \otimes_{MU^*} \mathbb{Z} \cong H^{2*,*}(X; \mathbb{Z}) \cong CH^*(X).$$

From the above theorem, we only need to prove that there is a relation such that

$$f_1 x_{2p+2} = v_1 e_2 + ... = 0 \quad \text{also in } MGL^{2*,*}(X)$$

($x_{2p+2} = e_2$ in the notation of Theorem 1.4 and Theorem 4.6). By the solution of the Bloch-Kato conjecture of degree 2 by Merkurjev-Suslin, we get

$$u_2 \in H^{2,2}(X; \mathbb{Z}/p) \quad \text{and} \quad Q_0(u_2) \in H^{3,2}(X; \mathbb{Z}_{(p)}),$$

and hence $e = Q_1 Q_0(u_2) \in H^{2p+2,p+1}(X; \mathbb{Z}_{(p)}) = CH^{p+1}(X)_{(p)}$. The first differential of the Atiyah-Hirzebruch spectral sequence is given by ((4.6) in [Ya])

$$d_{p-1}(Q_0 u_2) = v_1 \otimes Q_1 Q_0(u_2) = v_1 \otimes e_2.$$

Hence $v_1 e_2 = 0$ in $grMGL^{2*,*}(X)$. □

5. The exceptional Lie groups cases

With the result on Poincaré series of cotorsion products, the first author proves that the Rothenberg-Steenrod spectral sequence for the mod p cohomology of BG collapses at the E_2-level and the Quillen homomorphism r is a monomorphism in the cases $G = E_8$ and $p = 5$, $G = F_4, E_6, E_7$ and $p = 3$. In these cases, each exceptional Lie group has two conjugacy classes of maximal elementary abelian p-groups. One is the subgroup of a maximal torus and the other is a nontoral A. Let us write $i_A : BA \to BG$ and $i_T : BT_G \to BG$ for the induced maps from the inclusions. (However note that for $G = E_8$ and $p = 3$, there are two conjugacy classes of nontoral maximal elementary abelian 3-subgroups.)

Theorem 5.1 ([Ka]). *In the case $G = E_8$ and $p = 5$, $G = F_4, E_6, E_7$ and $p = 3$, the induced homomorphism $i_A^* \times i_T^* : H^*(BG) \to H^*(BA) \times H^*(BT_G)$ is a monomorphism.*

Moreover $Im(i_A^*)$ is also known for these groups.

Case I. *Consider the cases $G = E_8$ and $p = 5$, $G = F_4$ and $p = 3$.* In these cases $W_G(A) = SL(3, \mathbb{F}_p)$. Let

$$pi_A : H^*(BG) \to H^*(BA)^{W_G(A)} \to SD_3 \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\}.$$

Let $K = Ker(pi_A)$ and $IM = Im(pi_A)$. The map pi_A is not epic, indeed $u_3 \notin IM$, and we can show

Lemma 5.2. *There exists a short exact sequence*

$$0 \to IM \to SD_3 \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\} \to \mathbb{F}_p\{u_3\} \to 0.$$

Thus we get

$$\mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, IM) \cong P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_2)\{e_3\} \oplus P(m)^*\{\delta u_3\}, \quad |\delta u_3| = 4.$$

Since $K = Ker(pi_A)$ is a trivial $\mathcal{E}_m$-module generated by even-dimensional elements, we get Theorem 1.1 for these groups.

Proposition 5.3. *For G in case I, $\mathrm{Ext}_{\mathcal{E}_m}(\mathbb{F}_p, H^*(BG))$ is isomorphic to*

$$P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_2)\{e_3\} \oplus P(m)^*\{\delta u_3\} \oplus P(m)^* \hat{\otimes} K.$$

Next consider the Bockstein spectral sequence

$$E_1 = H^*(BG) \Longrightarrow \mathbb{F}_p \otimes H^*(BG; \mathbb{Z})/(torsions).$$

Note IM is not $\Lambda(Q_0)$-free but

$$E_2 = H(H^*(BG); Q_0) \cong K \oplus \mathbb{F}_p\{Q_0 u_3\}.$$

Hence the Poincaré series $PS_R(-, t)$ (for graded R-vector spaces) is given by

$$\begin{aligned} PS_{\mathbb{Z}/p}(K \oplus \mathbb{Z}/p\{\delta_3\}, t) &= PS_{\mathbb{Z}/p}(K, t) + t^4 \\ &= PS_{\mathbb{Z}/p}(K \oplus \mathbb{Z}/p\{Q_0 u_3\}, t) = PS_{\mathbb{Q}}(H^*(BG; \mathbb{Q}), t). \end{aligned}$$

Thus we get

Theorem 5.4. *For the case I, there exists a $P(m)^*$-algebra exact sequence*

$$0 \to P(m)^* \hat{\otimes} M \to grP(m)^*(BG) \to P(m)^* \hat{\otimes} SD_3/(f_0, f_1, f_3)\{e_3\} \to 0.$$

(1) $M \cong \mathbb{Z}_{(3)}[x_4, x_{12}, x_{16}, x_{24}]$ *as* $\mathbb{Z}_{(3)}$*-modules for* $G = F_4$, $p = 3$.
(2) $M \cong \mathbb{Z}_{(5)}[x_4, x_{16}, x_{24}, x_{28}, x_{36}, x_{40}, x_{48}, x_{60}]$ *as* $\mathbb{Z}_{(5)}$*-modules for* $G = E_8$, $p = 5$.

The mod 3 cohomology of BF_4 is completely determined by Toda.

Theorem 5.5 ([Toda]). *The cohomology* $H^*(BF_4)$ *is isomorphic to*

$$\mathbb{Z}/3[x_{36}, x_{48}] \otimes (\mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} + \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\})$$

where the above two terms have the intersection $\{1, x_{20}\}$.

Indeed, we see that $x_{26}|A = e_3,\ x_{36}|A = c_{3,1},\ x_{48}|A = c_{3,2},\ x_4|A = Q_0(u_3),\ x_8|A = Q_1(u_3),\ x_{20}|A = Q_2(u_3),\ x_9|A = Q_0Q_1(u_3),\ x_{21}|A = Q_0Q_2(u_3),\ x_{25}|A = Q_1Q_2(u_3)$. The Brown-Peterson theory $BP^*(BG)$ is also computed in [Ko-Ya] by using the Atiyah-Hirzebruch spectral sequence. The ring structure of M in Theorem 5.4 is quite complicated

Proposition 5.6 ([Ko-Ya]). $M \cong D_3/(c_{3,0}) \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E)$ *with*

$$E = \mathbb{Z}_{(3)}[x_4, x_8]\{ab | a, b \in \{x_4, x_8, x_{20}\}\} \subset \mathbb{Z}_{(3)}[x_4, x_8]\{1, x_{20}, x_{20}^2\}.$$

The Poincaré series of $(\mathbb{Z}/3\{1, 3x_4\} \oplus E/3)$ is $a/(1-t^4)(1-t^8)$ with

$$a = 1 + t^{20} + t^{40} - (t^8 + t^{20})(1-t^4)(1-t^8)$$
$$= 1 - t^8 + t^{12} + t^{16} - t^{20} + t^{24} + t^{28} - t^{32} + t^{40} = (1 - t^8 + t^{16})(1 + t^{12} + t^{24}).$$

Then we have

$$P_{\mathbb{Z}/3}(D_3/(c_{3,0}) \otimes (\mathbb{Z}/3\{1, 3x_4\} \oplus E), t) = \frac{1}{(1-t^{36})(1-t^{48})} \times \frac{a}{(1-t^4)(1-t^8)}$$
$$= \frac{a(1+t^8)}{(1-t^{12})(1+t^{12}+t^{24})(1-t^{24})(1+t^{24})(1-t^4)(1-t^{16})}.$$

Here we see that

$$(1+t^8)a = (1+t^8)(1-t^8+t^{16})(1+t^{12}+t^{24}) = (1+t^{24})(1+t^{12}+t^{24}).$$

Hence the above Poincaré series is indeed equal to

$$P_{\mathbb{Z}/3}(M/3, t) = 1/(1-t^{12})(1-t^{24})(1-t^4)(1-t^{16}).$$

Proposition 5.7. *Let* $cl : CH^*(X)/p \to H^*(X)$ *be the mod* p *cycle map. For* G *of Case I, we have* $pi_A \cdot cl(CH^*(BG)) = SD_3\{e_3\}$.

Proof. From Lemma 9.6 in [Ya] and the affirmative answer to the Bloch-Kato conjecture, there is an element $x \in H^{4,3}(BG; \mathbb{Z}/p)$ with $cl(x) = Q_0 e_3$. Here note

$$Q_iQ_j(x) \in H^{2p^i+2p^j+2, p^i+p^j+1}(BG; \mathbb{Z}/p) = CH^{p^i+p^j+1}(BG)/p.$$

Thus $e_3, c_{3,1}e_3, c_{3,2}e_3$ are in the image of the cycle map. □

Case II. *The groups* G *are* E_6 *and* E_7 *for* $p = 3$.

For each of these cases, G contains a maximal nontoral elementary abelian 3-subgroup $A \cong (\mathbb{Z}/3)^4$. It is known (e.g., [A-G-M-V]) that $W_G(A)$ is the subgroup of $SL(4,\mathbb{F}_3)$ generated by matrices of the form

$$\begin{pmatrix} & & & 0 \\ & SL(3,\mathbb{F}_3) & & 0 \\ & & & 0 \\ * & * & * & \epsilon \end{pmatrix}$$

where $*\in\mathbb{F}_3$, $\epsilon=1$ (resp. $\epsilon=\pm1$) for $G=E_6$ (resp. $G=E_7$).

Recall that

$$\mathcal{O}_3(X)=\Pi_{(\lambda_1,\lambda_2,\lambda_3)\in(\mathbb{Z}/3)^3}(X+\lambda_1t_1+\lambda_2t_2+\lambda_3t_3).$$

Let us denote by $\mathcal{O}$ simply $\mathcal{O}_3(t_4)$ so that $e_4=Q_0Q_1Q_2Q_3(u_4)=e_3\mathcal{O}$. Then the invariant ring is computed by Kameko and Mimura [Ka-Mi]:

$$H^*(BA)^{W_G(A)}\cong\begin{cases} SD_3/(e_3)[\mathcal{O}]\oplus SD_3[\mathcal{O}]\otimes\Lambda(Q_0,Q_1,Q_2)\{u_3,u_4\} \text{ for } G=E_6\\ SD_3/(e_3)[\mathcal{O}^2]\oplus SD_3[\mathcal{O}^2]\otimes\Lambda(Q_0,Q_1,Q_2)\{u_3,\mathcal{O}u_4\} \text{ for } G=E_7.\end{cases}$$

At first consider the case $G=E_6$. Let us write $N=SD_3[\mathcal{O}]\otimes\Lambda(Q_0,Q_1,Q_2)\{u_3,u_4\}$ and let the projection be

$$pi_A:H^*(BG)\to H^*(BA)^{W_G(A)}\to N.$$

Lemma 5.8. $\mathcal{O}u_3=(Q_3+c_{3,2}Q_2+c_{3,1}Q_1+c_{3,0}Q_0)(u_4)$.

Proof. From Lemma 3.2, we have

$$(Q_3+\sum c_{3,i}Q_i)(u_4)=(Q_3+\sum c_{3,i}Q_i)(dt_4)\times dt_1dt_2dt_3=\mathcal{O}(t_4)u_3. \qquad \square$$

Hence we have the decomposition

$$N=SD_3\otimes\Lambda(Q_0,Q_1,Q_2)\{u_3\}\oplus SD_3[\mathcal{O}]\otimes\Lambda(Q_0,Q_1,Q_2,Q_3)\{u_4\}.$$

We can see that $f_0,...,f_3$ is not regular in $SD_3[\mathcal{O}]$ (while it is regular in SD_4) for general $m\geq1$. Hence we assume here $m=0$. Take $\mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p,-)$.

Lemma 5.9. *The module* $\mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p,N)$ *is isomorphic to*

$$BP^*\hat{\otimes}SD_3\{e_3\}/(v_1,v_2)\oplus BP^*\hat{\otimes}SD_3[\mathcal{O}]\{e_3\mathcal{O}\}/(v_1,v_2,v_3).$$

It is known that the image $IM=Im(pi_A)$ is an algebra over $\mathcal{A}$ generated by $Q_0Q_1u_4,Q_0u_3$ and $SD_3,\mathcal{O}$; namely,

$$N/IM\cong SD_3[\mathcal{O}]\{u_3,u_4,Q_0u_4,Q_1u_4,Q_2u_4,Q_3u_4\}.$$

The above fact is also known in Kono-Mimura [Ko-Mi]; indeed, $H^*(BE_6)$ is still computed in [Ko-Mi].

Lemma 5.10. $\mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p,N/IM)$ *is isomorphic to*

$$P(0)^*\hat{\otimes}SD_3[\mathcal{O}]\{u_3,Q_0u_4,Q_1u_4,Q_2u_4,Q_3u_4\}/(3Q_0u_4),$$

which is generated by odd-dimensional elements where $P(0)^*=\mathbb{Z}/3[3,v_1,...]$.

From the above lemmas, we know that $\mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p,Im(pi_A))$ is generated by even-dimensional elements. Of course, so is $\mathrm{Ext}_{\mathcal{E}}(\mathbb{F}_p,Ker(pi_A))$, and hence we have

Theorem 5.11. *For* $(E_6,p=3)$, *there exists a short exact sequence*

$$0\to BP^*\hat{\otimes}M\to grBP^*(BE_6)\to IN\to0.$$

(1) $IN\cong BP^*\hat{\otimes}SD_3[\mathcal{O}]\{\delta Q_0u_4\}\oplus BP^*\hat{\otimes}SD_3\{e_3\}/(v_1,v_2)\oplus BP^*\hat{\otimes} SD_3[\mathcal{O}]\{e_3\mathcal{O}\}/(v_1,v_2,v_3)$.

(2) $M\cong\mathbb{Z}_{(3)}[x_4,x_{10},x_{12},x_{16},x_{18},x_{24}]$ *as additively.*

Next consider the case $(G = E_7, p = 3)$. It is known that

$$i_A(H^*(BG)) \subset SD_3/(e_3)[\mathcal{O}^2] \oplus SD_3[\mathcal{O}^2] \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\}$$

(but not in the submodule $SD_3[\mathcal{O}] \otimes \Lambda(Q_0, Q_1, Q_2)\{\mathcal{O}u_4\}$). Let us write $N = SD_3[\mathcal{O}^2] \otimes \Lambda(Q_0, Q_1, Q_2)\{u_3\}$ and let the projection

$$pi_A : H^*(BG) \to H^*(BA)^{W_G(A)} \to N.$$

We also know that

$$N/Im(pi_A) \cong SD_3[\mathcal{O}^2]\{u_3\}.$$

Hence we get Theorem 1.1 by an argument similar to (but more easily than) the case E_6.

Theorem 5.12. *For* $(G = E_7, p = 3)$, *there exists a short exact sequence*

$$0 \to P(m)^*\hat{\otimes}M \to grP(m)^*(BE_7) \to P(m)^*\hat{\otimes}SD_3[\mathcal{O}^2]\{e_3\}/(f_0, f_1, f_2) \to 0$$

where $M \cong \mathbb{Z}_{(3)}[x_4, x_{12}, x_{16}, x_{20}, x_{24}, x_{28}, x_{36}]$ *as additively.*

6. Proof of Theorem 1.8

For the sake of notational ease, we write $R_{m,n}$ for $P(m)^*\widehat{\otimes}SD_n$ and J_ℓ for the ideal $(f_\ell, \dots, f_{n-1}) \subset P(m)^*\widehat{\otimes}D_n$.

Proof of Theorem 1.8 *in the case* $m = 0$. It is clear that $f_i = v_i - v_n c_{n,i} + \cdots$ for $i = 0, \dots, n-1$. The above $f_0, \dots, f_{n-1}$ is a regular sequence in $R_{0,n}$. Let $\rho : R_{0,n} \to R_{0,n}$ be a ring homomorphism defined by

$$\begin{array}{rcll} \rho(v_k) & = & f_k & \text{for } 0 \le k \le n-1, \\ \rho(v_k) & = & v_k & \text{for } k \ge n, \\ \rho(c_{n,i}) & = & c_{n,i} & \text{for } 0 \le i \le n-1. \end{array}$$

Then, it is clear that for $k = 0, \dots, n-1$, $\rho^p(v_k) = v_k + pf'_k = v_k$ where $f'_k = v_n\beta_{k,n} + v_{n+1}\beta_{k,n+1} + \cdots$. So the ring homomorphism ρ is a ring automorphism. It is also clear that $v_0, \dots, v_{n-1}$ is a regular sequence. Hence, $\rho(v_0), \dots, \rho(v_{n-1})$ is also a regular sequence. Thus, $f_0, \dots, f_{n-1}$ is a regular sequence. □

Proof of Theorem 1.8 *in the case* $n = 1$. In the case $n = 1$, the statement of Theorem 1.8 is that $f_0 = v_m c_{n,0}^{p^{m-1}+\cdots+1} + v_{m+1} c_{n,0}^{p^m+p^{m-1}+\cdots+1} + \cdots$ is not a zerodivisor. It is clear that $R(m,1)$ is an integral domain, so that Theorem 1.8 holds in the case $n = 1$. □

Lemma 6.1. *Let* Δ_m *be the determinant of the matrix*

$$A_m = \begin{pmatrix} \alpha_{0,m} & \cdots & \alpha_{0,m+n-1} \\ \vdots & \ddots & \vdots \\ \alpha_{n-1,m} & \cdots & \alpha_{n-1,m+n-1} \end{pmatrix}.$$

Then $\Delta_0 = 1$ *and*

$$\Delta_{m+1} = \Delta_m^p c_{n,0}$$

for $m \ge 0$. *In particular, the above matrix has an inverse as a matrix of coefficients in* $R_{m,n}[c_{n,0}^{-1}]$.

This lemma will be proved in the last parts of this section.

Lemma 6.2. *The ring* $(R_{m,n}/J_\ell)[c_{n,0}^{-1}]$ *is an integral domain for* $\ell = 0, \dots, n-1$.

Proof. Let
$$f'_i = v_{m+n}\alpha_{i,m+n} + v_{m+n+1}\alpha_{i,m+n+1} + \cdots$$
for $i = 0, \ldots, n-1$. Let
$$\rho_m : R_{m,n}[c_{n,0}^{-1}] \to R_{m,n}[c_{n,0}^{-1}]$$
be a ring homomorphism defined by
$$\begin{array}{rcll} \rho_m(v_{m+k}) & = & v_{m+k} + f'_k & \text{for } 0 \le k \le n-1, \\ \rho_m(v_{m+k}) & = & v_{m+k} & \text{for } k \ge m+n, \\ \rho_m(c_{n,i}) & = & c_{n,i} & \text{for } 0 \le i \le n-1. \end{array}$$
As in the above proof of Theorem 1.8 in the case $m = 0$, $\rho_m^p = 1$, so that ρ_m is a ring automorphism.

We define a ring homomorphism
$$\psi_m : R_{m,n}[c_{n,0}^{-1}] \to R_{m,n}[c_{n,0}^{-1}]$$
by
$$\begin{array}{rcll} \psi_m(v_{m+k}) & = & v_m\alpha_{k,m} + \cdots + v_{m+n-1}\alpha_{k,m+n-1} & \text{for } 0 \le k \le n-1, \\ \psi_m(v_{m+k}) & = & v_{m+k} & \text{for } n \le k, \\ \psi_m(c_{n,i}) & = & c_{n,i} & \text{for } 0 \le i \le n-1. \end{array}$$
By Lemma 6.1, in $R_{m,n}[c_{n,0}^{-1}]$, the inverse of the matrix A_m exists. Hence, ψ_m also has an inverse, that is, ψ_m is also a ring automorphism. Therefore, $\psi_m \circ \rho_m$ is also a ring automorphism and $\psi_m \circ \rho_m$ maps $v_m, \ldots, v_{m+n-1}$ to $f_0, \ldots, f_{n-1}$. It is clear that $\psi_m \circ \rho_m$ induces an isomorphism
$$R_{m,n}[c_{n,0}^{-1}]/(v_{m+\ell}, \ldots, v_{m+n-1}) \to (R_{m,n}/J_\ell)[c_{n,0}^{-1}].$$
It is clear that $R_{m,n}[c_{n,0}^{-1}]/(v_{m+\ell}, \ldots, v_{m+n-1})$ is an integral domain and so is $(R_{m,n}/J_\ell)[c_{n,0}^{-1}]$. ☐

Lemma 6.3. *Let $\phi : R_{m,n} \to R_{m-1,n-1}$ be a ring homomorphism defined by*
$$\begin{array}{rcl} \phi(v_k) & = & v_{k-1}, \\ \phi(c_{n,i}) & = & c_{n-1,i-1}, \\ \phi(c_{n,0}) & = & 0. \end{array}$$
Then, ϕ maps f_k to f_{k-1} for $k = 1, \ldots, n-1$ and $\phi(f_0) = 0$.

Proof. It suffices to show that

(1) $\phi(\alpha_{0,n}) = 0$,

(2) $\phi(\alpha_{i,k}) = \alpha_{i-1,k-1}$ for $i \ge 1$.

It is clear from the fact that $\alpha_{0,n} = -c_{n,0}$ for $k \ge 1$ that (1) holds. We prove this lemma by induction on k. For $i \ge 1$,
$$\begin{array}{rcl} \phi(\alpha_{i,k}) & = & \phi(\alpha_{i-1,k-1}^p - \alpha_{n-1,k-1}^p c_{n,k}) \\ & = & a_{i-2,k-2}^p - \alpha_{n-2,k-2}^p c_{n-1,k-1} \\ & = & \alpha_{i-1,k-1} \end{array}$$
in $R_{m-1,n-1}$. ☐

Proposition 6.4. *If Theorem 1.8 holds for $R_{m-1,n-1}$, then $c_{n,0}$ is not a zerodivisor in $R_{m,n}/J_\ell$ for $1 \le \ell \le n-1$.*

Proof. Recall $J_\ell = (f_\ell, ..., f_{n-1})$. We prove this proposition by induction on $n-1-\ell$. In the case $n-1-\ell = 0$, the proposition holds because $R_{m,n}$ is an integral domain.

Suppose that

$$c_{n,0}a = a_\ell f_\ell + \cdots + a_{n-1}f_{n-1} \quad (1)$$

in $R_{m,n}$. Then,

$$\phi(a_\ell)\phi(f_\ell) + \cdots + \phi(a_{n-1})\phi(f_{n-1}) = 0$$

in $R_{m-1,n-1}$. By the induction hypothesis, $\phi(f_\ell)$, …, $\phi(f_{n-1})$ is a regular sequence in $R_{m-1,n-1}$. Therefore,

$$\phi(a_\ell) = b_{\ell+1}\phi(f_{\ell+1}) + \cdots + b_{n-1}\phi(f_{n-1})$$

for some $b_{\ell+1}$, …. b_{n-1}. Since the kernel of ϕ is a principal ideal $(c_{n,0})$,

$$a_\ell = b'_{\ell+1}f_{\ell+1} + \cdots + b'_{n-1}f_{n-1} + c_{n,0}b \quad (2)$$

for some b'_k, $b \in R_{m,n}$ such that $\phi(b'_k) = b_k$ for $k = \ell+1$, …, $n-1$. From (1), (2),

$$c_{n,0}a = c_{n,0}bf_\ell + (a_{\ell+1} + b'_{\ell+1}f_\ell)f_{\ell+1} + \cdots + (a_{n-1} + b'_{n-1}f_\ell)f_{n-1}$$

in $R_{m,n}$. Hence,

$$c_{n,0}(a - bf_\ell) \equiv 0$$

in $R_{m,n}/J_{\ell+1}$. Since $c_{n,0}$ is not a zerodivisor, $a - bf_\ell = 0$ in $R_{m,n}/J_{\ell+1}$. Hence, $a = 0$ in $R_{m,n}/J_\ell$. □

Remark. Since f_0 is divisible by $c_{n,0}$, $c_{n,0}$ is a zerodivisor in $R_{m,n}/J_0$.

Proposition 6.5. *If Theorem* 1.8 *holds for $R_{m-1,n-1}$, then $R_{m,n}/J_\ell$ is an integral domain for* $1 \le \ell \le n-1$.

Proof. For $1 \le \ell \le n-1$, since $c_{n,0}$ is not a zerodivisor in $R_{m,n}/J_\ell$ the induced homomorphism

$$R_{m,n}/J_\ell \to (R_{m,n}/J_\ell)[c_{n,0}^{-1}]$$

is a monomorphism. Since $(R_{m,n}/J_\ell)[c_{n,0}^{-1}]$ is an integral domain by Lemma 6.2, $R_{m,n}/J_\ell$ is also an integral domain. □

This proposition completes the proof of Theorem 1.8.

Now we prove Lemma 6.1. Firstly, we prove that

$$\det A_{m+1} = (-1)^{n-1}c_{n,0}\det A_m^p$$

for $m \ge n$. For the sake of notational ease, let

$$\alpha = \alpha_{n-1,m}\cdots\alpha_{n-1,m+n-1}.$$

Note that $\alpha_{n-1,m} \ne 0$ for $m \ge n$ from Proposition 3.6. We write $\beta_{i,j}$ for the quotient

$$\alpha_{i,m+j}/\alpha_{n-1,m+j}$$

and $\gamma_{i,j}$ for

$$\beta_{i,j} - \beta_{i,0}$$

where $j = 1$, …, $n-1$.

By definition,

$$\det A_m = \begin{vmatrix} \alpha_{0,m} & \alpha_{0,m+1} & \cdots & \alpha_{0,m+n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-2,m} & \alpha_{n-2,m+1} & \cdots & \alpha_{n-2,m+n-1} \\ \alpha_{n-1,m} & \alpha_{n-1,m+1} & \cdots & \alpha_{n-1,m+n-1} \end{vmatrix}.$$

Dividing the j-th column by $\alpha_{n-1,m+j-1}$, we have

$$\det A_m/\alpha = \begin{vmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-2,0} & \beta_{n-2,1} & \cdots & \beta_{n-2,n-1} \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

Subtracting the first column from the j-th column for $j = 2, \ldots, n$, we have

$$\det A_m/\alpha = \begin{vmatrix} \beta_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\ \vdots & \vdots & & \vdots \\ \beta_{n-2,0} & \gamma_{n-2,1} & \cdots & \gamma_{n-2,n-1} \\ 1 & 0 & \cdots & 0 \end{vmatrix}.$$

Hence,

$$\det A_m = (-1)^{n-1}\alpha \begin{vmatrix} \gamma_{0,1} & \gamma_{0,2} & \cdots & \gamma_{0,n-1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n-2,1} & \gamma_{n-2,2} & \cdots & \gamma_{n-2,n-1} \end{vmatrix}.$$

On the other hand, by Proposition 3.4, $\det A_{m+1}$ is equal to

$$\begin{vmatrix} -\alpha^p_{n-1,m}c_{n,0} & \cdots & -\alpha^p_{n-1,m+n-1}c_{n,0} \\ \alpha^p_{0,m} - \alpha^p_{n-1,m}c_{n,1} & \cdots & \alpha^p_{0,m+n-1} - \alpha^p_{n-1,m+n-1}c_{n,1} \\ \vdots & \ddots & \vdots \\ \alpha^p_{n-2,m} - \alpha^p_{n-1,m}c_{n,n-1} & \cdots & \alpha^p_{n-2,m+n-1} - \alpha^p_{n-1,m+n-1}c_{n,n-1} \end{vmatrix}.$$

Dividing the j-th column by $\alpha^p_{n-1,m+j-1}$ for each $j = 1, \ldots, n$, we have

$$\det A_{m+1}/\alpha^p = \begin{vmatrix} -c_{n,0} & -c_{n,0} & \cdots & -c_{n,0} \\ \beta^p_{0,0} - c_{n,1} & \beta^p_{0,1} - c_{n.1} & \cdots & \beta^p_{0,n-1} - c_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^p_{n-2,0} - c_{n,n-1} & \beta^p_{n-2,1} - c_{n,n-1} & \cdots & \beta^p_{n-2,n-1} - c_{n,n-1} \end{vmatrix}.$$

Subtracting the first column from the j-th column for $j = 2, \ldots, n$, we have

$$\det A_{m+1}/\alpha^p = \begin{vmatrix} -c_{n,0} & 0 & \cdots & 0 \\ \beta^p_{0,0} - c_{n,1} & \gamma^p_{0,1} & \cdots & \gamma^p_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^p_{n-2,0} - c_{n,n-1} & \gamma^p_{n-2,1} & \cdots & \gamma^p_{n-2,n-1} \end{vmatrix}.$$

Hence,

$$\det A_{m+1} = -c_{n,0}\alpha^p \begin{vmatrix} \gamma_{0,1} & \gamma_{0,2} & \cdots & \gamma_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-2,1} & \gamma_{n-2,2} & \cdots & \gamma_{n-2,n-1} \end{vmatrix}^p.$$

Therefore, we have

$$\det A_{m+1} = (-1)^n c_{n,0}(\det {A_m}^p)$$

for $m \geq n$.

Secondly, we deal with the case $0 \leq m \leq n-1$. It is clear from Proposition 3.5 that $\det A_0 = 1$. Suppose that $1 \leq m \leq n-1$. From Proposition 3.5, we have $\alpha_{i,k} = \delta_{i,k}$ for $k \leq n-1$.

$$\det A_m \quad = \quad \left|\begin{array}{c|ccc} & \alpha_{0,n} & \cdots & \alpha_{0,n+m-1} \\ 0_{m,n-m} & \vdots & & \vdots \\ & \alpha_{m-1,n} & \cdots & \alpha_{m-1,n+m-1} \\ \hline & \alpha_{m,n} & \cdots & \alpha_{m,n+m-1} \\ I_{n-m} & \vdots & & \vdots \\ & \alpha_{n-1,n} & \cdots & \alpha_{n-1,n+m-1} \end{array}\right|$$

where $0_{m,n-m}$ is the $m \times (n-m)$ zero matrix and I_{n-m} is the $(n-m) \times (n-m)$ identity matrix. So we get

$$\det A_m \quad = \quad (-1)^{(n-1)(n-m)} \begin{vmatrix} \alpha_{0,n} & \cdots & \alpha_{0,n+m-1} \\ \vdots & & \vdots \\ \alpha_{m-1,n} & \cdots & \alpha_{m-1,n+m-1} \end{vmatrix}.$$

Using Proposition 3.5 and Proposition 3.6, subtracting the first column multiplied by $\alpha^p_{n-1,n+j-1}$ from the j-th column, we have

$$\begin{aligned} \det A_{m+1} \quad &= \quad (-1)^{(n-1)(n-m-1)} \begin{vmatrix} -c_{n,0} & 0 & \cdots & 0 \\ -c_{n,1} & \alpha^p_{0,n} & \cdots & \alpha^p_{0,n+m-2} \\ \vdots & \vdots & & \vdots \\ -c_{n,m-1} & \alpha^p_{m-2,n} & \cdots & \alpha^p_{m-2,n+m-2} \end{vmatrix} \\ &= \quad (-1)^{(n-1)(n-m-1)+1} c_{n,0} \begin{vmatrix} \alpha_{0,n} & \cdots & \alpha_{0,n+m-2} \\ \vdots & & \vdots \\ \alpha_{m-2,n} & \cdots & \alpha_{m-2,n+m-2} \end{vmatrix}^p \\ &= \quad (-1)^n c_{n,0} (\det {A_m}^p). \end{aligned}$$

This completes the proof of Lemma 6.1.

References

[A-G-M-V] K. Andersen, J. Grodal, M. Møller and A. Viruel. The classification of p-compact groups for p odd. *preprint arXiv:math. AT/0302346 vl 27 Feb.* (2003).

[Bo1] J.M. Boardman. Operations on the Adams spectral sequence for Brown-Peterson homology and cohomology. *preprint* (1988).

[Bo2] J.M. Boardman. Conditionally Convergent Spectral Sequences. *Contemp. Math.,* **239** (1999) 49-84. MR1718076 (2000m:55024)

[Bo-Jo-Wi] J.M. Boardman, D.C. Johnson and W.S. Wilson. Unstable operations in generalized cohomology. *Handbook of algebraic topology* (1995) 687-828. MR1361900 (97b:55022)

[Ka-Mi] M. Kameko and M. Mimura. Mui invariant and Milnor operations. *preprint* (2003).

[Ka] M. Kameko. Poincaré series of cotorsion products. *Preprint* (2005).

[Ko-Mi] A. Kono and M. Mimura. Cohomology mod 3 of the classifying space of the Lie group E_6.*Math. Scand.* **46** (1978), 223-235. MR591603 (81m:55015)

[Ko-Ya] A. Kono and N. Yagita. Brown-Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups. *Trans. of AMS.* **339** (1993), 781-798. MR1139493 (93m:55006)

[Mo-Vi] L. Molina and A. Vistoli. On the Chow rings of classifying spaces for classical groups. Rend. Sem. Mat. Univ. Padova **116** (2006), 271–298. MR2287351

[Mu] H. Mui. Modular invariant theory and the cohomology algebras of symmetric groups. *J. Fac. Sci. U. of Tokyo* **22** (1975), 319-369. MR0422451 (54:10440)

[Ra-Wi-Ya] D. C. Ravenel, W. S. Wilson and N. Yagita. Brown-Peterson cohomology from Morava K-theory. *K-theory* **15** (1998), 147-199. MR1648284 (2000d:55012)

[Te-Ya] M. Tezuka and N. Yagita. The varieties of the mod p cohomology rings of extraspecial p-groups for an odd prime p. *Math. Proc. Cambridge Phil. Soc.* **94** (1983) 449-459. MR720796 (85g:20069)

[Toda] H. Toda. Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4. *J. Math. Kyoto Univ.* **13** (1973) 97-115. MR0321086 (47:9619)

[To] B. Totaro. The Chow ring of classifying spaces. *Proc.of Symposia in Pure Math. "Algebraic K-theory" (1997:University of Washington, Seattle)* **67** (1999), 248-281. MR1743244 (2001f:14011)

[Va-Vi] A. Vavpetič and A. Viruel. On the mod p cohomology of $BPU(p)$. *Trans. AMS* **357** (2005), 4517-4532. MR2156719 (2006d:55013)

[Ve] G. Vezzosi. On the Chow ring of the classifying stack of $PGL_{3,C}$. *J. Reine Angew. Math.* **523** (2000), 1-54. MR1762954 (2001f:14012)

[Vi] A. Vistoli. On the cohomology and the Chow ring of the classifying space of PGL_p. arXiv:math.AG/0505052 (2005).

[Vo1] V. Voevodsky. The Milnor conjecture. *www.math.unic.edu/K-theory/0170* (1996)

[Vo2] V. Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. IHES* **98** (2003) 1-57. MR2031198 (2005b:14038a)

[Vo3] V. Voevodsky. Motivic cohomology with $\mathbb{Z}/2$ coefficients. *Publ. Math. IHES* **98** (2003) 59-104. MR2031199 (2005b:14038b)

[Ya] N. Yagita. Applications of Atiyah-Hirzebruch spectral sequences for motivic cobordism. *Proc. London Math. Soc.* **90** (2005) 783-816. MR2137831 (2006d:57049)

Faculty of Regional Science, Toyama University of International Studies, Toyama, Japan

E-mail address: `kameko@tuins.ac.jp`

Faculty of Education, Ibaraki University, Mito, Ibaraki, Japan

E-mail address: `yagita@mx.ibaraki.ac.jp`