

UNIVERSAL TODA BRACKETS OF RING SPECTRA

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ABSTRACT. We construct and examine the *universal Toda bracket* of a highly structured ring spectrum R . This invariant of R is a cohomology class in the Mac Lane cohomology of the graded ring of homotopy groups of R which carries information about R and the category of R -module spectra. It determines for example all triple Toda brackets of R and the first obstruction to realizing a module over the homotopy groups of R by an R -module spectrum.

For periodic ring spectra, we study the corresponding theory of *higher* universal Toda brackets. The real and complex K -theory spectra serve as our main examples.

1. INTRODUCTION

In this paper, we study a question about highly structured ring spectra. More specifically, we construct a cohomological invariant γ_R of a ring spectrum R , called its universal Toda bracket, and examine which information about R is encoded in γ_R .

We use the term *ring spectrum* for what is called an S -algebra in [EKMM97], a symmetric ring spectrum in [HSS00], or an orthogonal ring spectrum in [MMSS01]. A ring spectrum R has an associated module category $\text{Mod-}R$, which is a stable model category and has a triangulated homotopy category $\text{Ho}(\text{Mod-}R)$.

For an object X of $\text{Ho}(\text{Mod-}R)$, its stable homotopy groups $\pi_*(X)$ form a graded $\pi_*(R)$ -module. One of our aims is to better understand the resulting functor $\pi_*(-): \text{Ho}(\text{Mod-}R) \rightarrow \text{Mod-}\pi_*(R)$. Particularly, we want to examine under which conditions a $\pi_*(R)$ -module M is *realizable*, that is, arises as the homotopy groups of an R -module spectrum.

There is an obstruction theory associated to this problem, with obstructions $\kappa_i(M) \in \text{Ext}_{\pi_*(R)}^{i, 2-i}(M, M)$ for $i \geq 3$. The first obstruction $\kappa_3(M)$ is always defined and unique. It vanishes if and only if M is a retract of a realizable module. For $i \geq 4$, $\kappa_i(M)$ is only defined if $\kappa_{i-1}(M)$ vanishes and there are choices involved. We examine these obstructions and show how they depend on the structure of R .

The obstruction theory is the special case of an obstruction theory for realizability in a triangulated category \mathcal{T} described in [BKS04, Appendix A]. In this generality, it can be used to find out whether a module over the graded endomorphism ring $\mathcal{T}(N, N)_*$ of a compact object N can be realized as $\mathcal{T}(N, X)_*$ for some object X of \mathcal{T} . An algebraic instance of this problem is to realize a module over the cohomology of a differential graded algebra A as the cohomology of a differential graded A -module.

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Because of this analogy between ring spectra and differential graded algebras, the following result is a motivation for our work: for a differential graded algebra A over a field k , Benson, Krause, and Schwede [BKS04] study a class $\gamma_A \in \mathrm{HH}_k^{3,-1}(H^*(A))$ in the Hochschild cohomology of the cohomology ring of A . It determines by evaluation all triple (matric) Massey products of $H^*(A)$. Moreover, via the map

$$- \otimes^L \mathrm{id}_M : \mathrm{HH}_k^{3,-1}(H^*(A)) \rightarrow \mathrm{Ext}_{H^*(A)}^{3,-1}(M, M),$$

it determines the first realizability obstruction $\kappa_3(M)$ for every $H^*(A)$ -module M .

We develop a similar theory for ring spectra. Though the obstruction theory for the realizability problem takes place completely in triangulated categories, the definition of a cohomology class with that property needs information from an underlying ‘model’. In the case of the differential graded algebra A , the A_∞ -structure of $H^*(A)$ can be used to define γ_A [BKS04, Remark 7.4]. In the case of ring spectra, there is no such A_∞ -structure. The appropriate replacement is to use that choosing representatives in the model category of maps in the homotopy category is in general not associative with respect to the composition. This non-associativity leads to obstructions which assemble to a well defined cohomology class.

The formulation of our main results uses Mac Lane cohomology groups, denoted by HML. We define this cohomology theory for graded rings using the normalized cohomology of categories [BW85]. Its ungraded version is equivalent to Mac Lane’s original definition [JP91]. This theory is, for various reasons, an appropriate replacement of the Hochschild cohomology used in [BKS04]. One reason is that one can, similar to Hochschild cohomology, evaluate a representing cocycle on a sequence of composable maps. If the sequence of maps is a complex, it makes sense to ask the evaluation to be an element of the Toda bracket of the complex.

One main result is the following special case of Theorem 8.1:

Theorem 1.1. *Let R be a ring spectrum. Then there exists a well defined cohomology class $\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R))$ which, by evaluation, determines all triple matric Toda brackets of $\pi_*(R)$. For a $\pi_*(R)$ -module M which admits a resolution by finitely generated free $\pi_*(R)$ -modules, the product $\mathrm{id}_M \cup \gamma_R \in \mathrm{Ext}_{\pi_*(R)}^{3,-1}(M, M)$ is the first realizability obstruction $\kappa_3(M)$.*

The term *universal Toda bracket* for such a cohomology class, as well as the usage of cohomology of categories, are motivated by Baues’ study of universal Toda brackets for subcategories of the homotopy category of topological spaces [Bau97, BD89]. The recent preprint [BM] is concerned with a class similar to the γ_R of the last theorem, but studies different properties, namely a relation to ‘quadratic pair algebras’.

Theorem 1.1 applies for example to the real K -theory spectrum KO . As KO has non-vanishing triple Toda brackets, γ_{KO} is non-trivial. Moreover, the obstructions determined by γ_{KO} detect the non-realizable $\pi_*(KO)$ -module $(\pi_*(KO)) \otimes \mathbb{Z}/2$. We discuss in Remark 8.5 how this contradicts a claim of Wolbert [Wol98, Theorems 20 and 21].

The proof of Theorem 1.1 divides into two parts. In Section 5, we give a general construction of the universal Toda bracket of a small subcategory of the homotopy category of a stable model category. Specializing to the subcategory of finitely generated free modules in $\mathrm{Ho}(\mathrm{Mod}\text{-}R)$, this defines γ_R . Theorem 4.15 shows how

a cohomology class which determines Toda brackets also determines the obstructions κ_3 .

Many examples of ring spectra have the property that their ring of homotopy groups is concentrated in degrees divisible by n for some $n \geq 2$. Then all realizability obstructions κ_3 vanish for degree reasons. The first realizability obstruction not vanishing for degree reasons is determined by a *higher* universal Toda bracket, which we also introduce in Theorem 8.1.

The higher universal Toda bracket of a ring spectrum R becomes particularly nice if $\pi_*(R)$ is a graded Laurent polynomial ring on a central generator of degree n . In this case, the higher universal Toda bracket γ_R^{n+2} can be defined as an element of $\text{HML}^{n+2}(\pi_0(R))$. As in Theorem 1.1, it determines $(n + 2)$ -fold Toda brackets and realizability obstructions κ_{n+2} . But now there is a better chance to actually identify γ_R^{n+2} , since computations of (ungraded) Mac Lane-cohomology groups are known in relevant cases. For example, the universal Toda bracket γ_{KU}^4 of the complex K -theory spectrum KU is an element of $\text{HML}^4(\mathbb{Z}) \cong \mathbb{Z}/2$. We prove in Proposition 8.15 that it is the non-zero element.

The calculation of γ_{KU}^4 is a consequence of a different kind of information detected by universal Toda brackets. The Toda brackets of a ring spectrum R can be considered as higher order information about zero divisors in $\pi_*(R)$ and its matrix rings. It turns out that the universal Toda bracket also knows about the units of R and its matrix rings.

To make the slogan precise, recall that for a ring spectrum R and $q \geq 1$, there is a path connected space $BGL_q R$. It is the classifying space of the topological monoid given by the invertible path components of the mapping space $\text{Map}_R(R^q, R^q)$. The algebraic K -theory of R can be built from the spaces $BGL_q R$ [Wal78, EKMM97].

If $\pi_*(R)$ is concentrated in degrees divisible by n for some $n \geq 1$, we know that $\pi_k(BGL_q R) = 0$ for $1 < k < n + 1$. The following corollary follows from Corollary 8.11 and Theorem 8.7. It is related to [Igu82]; see also [BD89, Example 4.9, Theorem 3.10].

Corollary 1.2. *Let R be a ring spectrum such that $\pi_*(R)$ is a Laurent polynomial ring on a central generator in degree n . The restriction map*

$$\text{HML}^{n+2}(\pi_0(R)) \rightarrow H^{n+2}(\pi_1(BGL_q R), \pi_{n+1}(BGL_q R))$$

sends γ_R^{n+2} to the first k -invariant of $BGL_q R$ not vanishing for dimensional reasons.

Moreover, with an additional assumption on $\text{HML}^{n+1}(\pi_0(R))$, we interpret the vanishing of γ_R^{n+2} in terms of algebraic K -theory in Proposition 8.14.

Organization. The main results can be found in Section 8. There we also discuss the examples mentioned in the Introduction.

In the second section, we briefly review cohomology of categories and Mac Lane cohomology, including a version for graded rings, and define the cup product used in Theorem 1.1. In the third section, we explain the obstruction theory for realizability in triangulated categories. The fourth section is devoted to (higher) Toda brackets in triangulated categories. We explain how Toda brackets determine realizability obstructions.

The fifth section is the technical backbone of this paper. We give a general construction of the universal Toda bracket in the framework of stable topological

model categories. Section 6 features a comparison of different definitions of Toda brackets. In Section 7, we show how universal Toda brackets are related to k -invariants of classifying spaces. The Appendix consists of a brief discussion of topological model categories and provides a technical result needed in Section 5.

2. MAC LANE COHOMOLOGY

We review the definition of Mac Lane cohomology via cohomology of categories and introduce a cup pairing between the Mac Lane cohomology of a graded ring Λ and a group of Λ -module homomorphisms. As a reference for Mac Lane cohomology, we recommend the last chapter of Loday's book [Lod98, Chapter 13].

2.1. Cohomology of categories and HML. Let \mathcal{C} be a small category. A \mathcal{C} -bimodule is a functor $D: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{A}b$. For a map $f: X \rightarrow Y$ in \mathcal{C} , we denote the abelian group $D(X, Y)$ by D_f . For maps $g: X' \rightarrow X$, $h: Y \rightarrow Y'$, and $f: X \rightarrow Y$, the \mathcal{C} -bimodule structure induces actions $g^*: D_f \rightarrow D_{fg}$ and $h_*: D_f \rightarrow D_{hf}$. If A is a ring and \mathcal{C} is the category of A -modules, the bifunctor $\text{Hom}_A(-, -)$ provides an example for a \mathcal{C} -bimodule.

To define the cohomology of a category \mathcal{C} with coefficients in a \mathcal{C} -bimodule D we consider the cochain complex $C^*(\mathcal{C}, D)$ with

$$C^n(\mathcal{C}, D) = \begin{cases} \{c: N_n(\mathcal{C}) \rightarrow \prod_{g \in \text{Mor}(\mathcal{C})} D_g \mid c(g_1, \dots, g_n) \in D_{g_1 \dots g_n}\} & \text{for } n \geq 1, \\ \{c: \text{Ob}(\mathcal{C}) \rightarrow \prod_{X \in \text{Ob}(\mathcal{C})} D(X, X) \mid c(X) \in D(X, X)\} & \text{for } n = 0. \end{cases}$$

Here $N(\mathcal{C})$ is the nerve of \mathcal{C} , so an element of $N_n(\mathcal{C})$ is a sequence (g_1, \dots, g_n) of n composable maps in \mathcal{C} . The abelian group structure on $C^n(\mathcal{C}, D)$ is given by the pointwise addition in D_g . For $n > 1$, the differential $\delta: C^{n-1}(\mathcal{C}, D) \rightarrow C^n(\mathcal{C}, D)$ is

$$\begin{aligned} (\delta c)(g_1, \dots, g_n) &= (g_1)_* c(g_2, \dots, g_n) + \sum_{i=1}^n (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_n) \\ &\quad + (-1)^{n+1} (g_n)_* c(g_1, \dots, g_{n-1}). \end{aligned}$$

For $n = 1$, it is $(\delta c)(g_1: X_1 \rightarrow X_0) = (g_1)_* c(X_1) - (g_1)_* c(X_0)$. It is easy to verify $\delta^2 = 0$.

Definition 2.2 ([BW85, Definition 1.4]). The *cohomology* $H^*(\mathcal{C}, D)$ of the category \mathcal{C} with coefficients in the \mathcal{C} -bimodule D is the cohomology of $(C^*(\mathcal{C}, D), \delta)$.

There is a normalized version of this. A category is *pointed* if it has a preferred zero object $*$, i.e., $*$ is both initial and terminal. A *zero morphism* in a pointed category is a map which factors through the zero object. If \mathcal{C} is a pointed category, a \mathcal{C} -bimodule D is *normalized* if $D(*, X) = 0 = D(X, *)$ holds for all objects X .

For a pointed category \mathcal{C} and a normalized \mathcal{C} -bimodule D , we consider the subgroup $\overline{C}^n(\mathcal{C}, D) = \{c \in C^n(\mathcal{C}, D) \mid c(g_1, \dots, g_n) = 0 \text{ if } g_i \text{ is zero for some } i\}$ of normalized cochains in $C^n(\mathcal{C}, D)$. As D is normalized, $\overline{C}^*(\mathcal{C}, D)$ is a subcomplex of $C^*(\mathcal{C}, D)$. By [BD89, Theorem 1.1], the inclusion $\overline{C}^*(\mathcal{C}, D) \rightarrow C^*(\mathcal{C}, D)$ induces an isomorphism in cohomology. Therefore, we can assume representing cochains to be normalized as soon as we consider the cohomology of a *pointed* category with coefficients in a *normalized* bimodule.

Cohomology of categories has good naturality properties. If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor and D a \mathcal{D} -bimodule, there is an induced \mathcal{C} -bimodule F^*D , and F induces

maps $F^* : C^*(\mathcal{D}, D) \rightarrow C^*(\mathcal{C}, F^*D)$ and $F^* : H^*(\mathcal{D}, D) \rightarrow H^*(\mathcal{C}, F^*D)$. The latter map is an isomorphism if F is an equivalence of categories [BW85, Theorem 1.11].

For a ring A , we denote the category of finitely generated free right A -modules by $F(A)$. To avoid set theoretic problems, we assume $F(A)$ to be small, i.e., we require it to contain only one element from each isomorphism class of objects. The category $F(A)$ is pointed by the trivial module, and for an A -bimodule M , the functor $\text{Hom}_A(-, - \otimes_A M)$ is a normalized $F(A)$ -module.

Definition 2.3. Let A be a ring and let M be an A -bimodule. The *Mac Lane cohomology* of A with coefficients in M is defined by

$$\text{HML}^s(A, M) = H^s(F(A), \text{Hom}_A(-, - \otimes_A M)).$$

If M equals A , we adopt the convention $\text{HML}^s(A) = \text{HML}^s(A, A)$.

Mac Lane cohomology was originally defined by Mac Lane in 1956 [ML57]. Jibladze and Pirashvili [JP91] proved the equivalence of Mac Lane’s definition to the one we use. Mac Lane cohomology is also isomorphic to Ext-groups in the abelian category $\mathcal{F}(A)$ of functors from $F(A)$ to $\text{Mod-}A$ (see [JP91]) and to *topological Hochschild cohomology* (see [Bok85] for the definition and [PW92] or [Sch01, Theorem 6.7] for the equivalence). The computation of Mac Lane cohomology is known for many examples, including the cases $\text{HML}^*(\mathbb{F}_p)$ (see [FLS94]) and $\text{HML}^*(\mathbb{Z})$ (see [FP98]) we encounter in Section 8.

For later use we prove

Lemma 2.4. *Let A be a ring, let M and P be A -modules with P projective, let $I : F(A) \rightarrow \text{Mod-}A$ be the inclusion functor and let $T : F(A) \rightarrow \text{Mod-}A$ be any functor. For $i \geq 1$, there is an isomorphism*

$$H^i(F(A), \text{Hom}_A(I(-) \oplus P, T(-) \oplus M)) \cong H^i(F(A), \text{Hom}_A(-, T(-))).$$

Proof. By [JP91, Corollary 3.11], this translates to a statement about the Ext-group $\text{Ext}_{\mathcal{F}(A)}^i(I(-) \oplus P, T(-) \oplus M)$. Since the constant functor represented by P is projective in $\mathcal{F}(A)$, it cancels out in the first variable. As I is reduced, i.e., $I(0) = 0$, it has a projective resolution by reduced functors. Since there is only the trivial map from a reduced functor to the constant functor M , the M cancels out as well. □

2.5. Mac Lane cohomology of graded rings. If Λ is a graded ring, the morphisms between graded Λ -modules M and N form a graded abelian group by setting $\text{Hom}_\Lambda^i(M, N) = \text{Hom}_\Lambda(M, N[i]) = \text{Hom}_\Lambda(M, N)_{-i}$.

Definition 2.6. A graded ring, a graded abelian group, or a graded module is *n-sparse* if it is concentrated in degrees divisible by n . A full subcategory \mathcal{C} of $\text{Mod-}\Lambda$ is *n-split* if for each pair of objects M and N in \mathcal{C} , the graded abelian group $\text{Hom}_\Lambda(M, N)_*$ is n -sparse.

For a graded ring Λ , let $F(\Lambda)$ be the category of finitely generated free graded right Λ -modules. The objects of $F(\Lambda)$ are finite sums of shifted copies of the free module of rank 1. If Λ is n -sparse for $n \geq 1$, the full subcategory of $F(\Lambda)$ given by the n -sparse Λ -modules is denoted by $F(\Lambda, n)$. For $n = 1$, we have $F(\Lambda) = F(\Lambda, n)$. The category $F(\Lambda, n)$ is an example of an n -split subcategory of $\text{Mod-}\Lambda$.

Definition 2.7. Let Λ be an n -sparse graded ring, and let M be a graded right Λ -module. The *graded n -split Mac Lane cohomology* of Λ with coefficients in M is defined by

$$\text{HML}_{n\text{-sp}}^s(\Lambda, M) = H^s(F(\Lambda, n), \text{Hom}_\Lambda(-, - \otimes_\Lambda M)).$$

If $M = \Lambda[t]$, a t -fold shift of Λ for some $t \in \mathbb{Z}$, we adopt the convention

$$\text{HML}_{n\text{-sp}}^{s,t}(\Lambda) = \text{HML}_{n\text{-sp}}^s(\Lambda, \Lambda[t]).$$

If $n = 1$, we drop ‘1-sp’ from the notation and write $\text{HML}^s(\Lambda, M)$ or $\text{HML}^{s,t}(\Lambda)$.

The graded Mac Lane cohomology is related to the ungraded theory. If Λ is n -sparse, the functor $- \otimes_{\Lambda_0} \Lambda: F(\Lambda_0) \rightarrow F(\Lambda, n)$ satisfies

$$(- \otimes_{\Lambda_0} \Lambda)^* \text{Hom}_\Lambda(-, - \otimes_\Lambda \Lambda[-n]) \cong \text{Hom}_{\Lambda_0}(-, - \otimes_{\Lambda_0} \Lambda_n)$$

and therefore induces a restriction map $\text{HML}_{n\text{-sp}}^{*, -n}(\Lambda) \rightarrow \text{HML}^*(\Lambda_0, \Lambda_n)$.

A central unit u of degree n in Λ is a homogeneous element u of degree n which is a unit and is central in the graded sense. If Λ has a central unit, $- \otimes_{\Lambda_0} \Lambda$ is an equivalence of categories, and Λ_n is isomorphic to Λ_0 as Λ_0 -bimodules. This proves

Lemma 2.8. *Let Λ be an n -sparse graded ring with a central unit u of degree n . Then the restriction induces an isomorphism $\text{HML}_{n\text{-sp}}^{*, -n}(\Lambda) \rightarrow \text{HML}^*(\Lambda_0)$.*

2.9. Relation to group cohomology. We review some well known maps from Mac Lane cohomology to group cohomology.

For an object X in a category \mathcal{C} , we denote its group of automorphisms by $\text{Aut}(X)$. The category with a single object X and $\text{Hom}(X, X) = \text{Aut}(X)$ is denoted by $\mathcal{A}ut(X)$. It comes with a canonical inclusion functor $\mathcal{A}ut(X) \rightarrow \mathcal{C}$. If D is an $\mathcal{A}ut(X)$ -bimodule, the automorphism group $\text{Aut}(X)$ acts via the conjugation action $gx = (g^{-1})^*(g_*(x))$ from the left on the abelian group $D(X, X)$.

Proposition 2.10. *Let \mathcal{C} be a small category, let X be an object of \mathcal{C} , and let D be a \mathcal{C} -bimodule. The inclusion functor $F: \mathcal{A}ut(X) \rightarrow \mathcal{C}$ induces a restriction map*

$$\Theta: H^*(\mathcal{C}, D) \rightarrow H^*(\mathcal{A}ut(X), F^*D) \xrightarrow{\cong} H^*(\text{Aut}(X), D(X, X))$$

from the cohomology of \mathcal{C} with coefficients in D to the cohomology of the group $\text{Aut}(X)$ with coefficients in the $\text{Aut}(X)$ -module $D(X, X)$.

Proof. The first map is the restriction along the inclusion. The second map is analogous to the Mac Lane isomorphism between the Hochschild homology of a group ring and group homology [Lod98, Proposition 7.4.2]. On a cochain c , the isomorphism is given by $(\varphi(c))(g_1, \dots, g_n) = (g_n^{-1} \cdots g_1^{-1})^*c(g_1, \dots, g_n)$. □

When A is a ring and M is an A -bimodule, we write as usual $\text{GL}_q A$ for the group of invertible $(q \times q)$ -matrices, which acts on the abelian group $\text{Mat}_q M$ of all $(q \times q)$ -matrices with entries in M by conjugation. The last proposition specializes to Mac Lane cohomology for graded and ungraded rings:

Corollary 2.11. *Let Λ be an n -sparse graded ring, let A be a ring, and let M be an A -bimodule. For $q \geq 1$, there are restriction maps*

$$\text{HML}_{n\text{-sp}}^{*, -n}(\Lambda) \rightarrow H^*(\text{GL}_q \Lambda_0, \text{Mat}_q \Lambda_n) \text{ and } \text{HML}^*(A, M) \rightarrow H^*(\text{GL}_q A, \text{Mat}_q M).$$

If $A = \Lambda_0$ and $M = \Lambda_n$, the first map factors through the second map and the restriction $\text{HML}_{n\text{-sp}}^{, -n}(\Lambda) \rightarrow \text{HML}^*(\Lambda_0, \Lambda_n)$.*

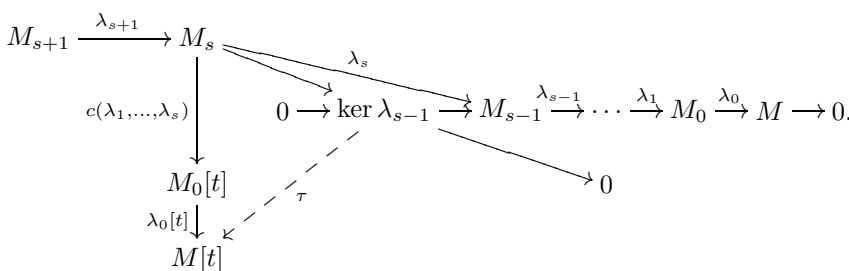


FIGURE 1. Defining the cup product

2.12. **The cup-product.** In the following, Ext-groups are understood in the sense of Yoneda. For a graded ring Λ , shifting of modules gives rise to a bigrading on Ext, that is, $\text{Ext}^{s,t}(M, N) = \text{Ext}^s(M, N[t])$.

Construction 2.13. Let Λ be an n -sparse graded ring. Let M and N be Λ -modules such that M admits a resolution by objects in $F(\Lambda, n)$. Then there is a well defined map

$$\text{Hom}_\Lambda(M, N) \times \text{HML}_{n\text{-sp}}^{s,t}(\Lambda) \rightarrow \text{Ext}_\Lambda^{s,t}(M, N), \quad (f, \gamma) \mapsto f \cup \gamma$$

which we refer to as the *cup product*. It is bilinear and natural in the sense that $(gf) \cup \gamma = g_*(f \cup \gamma)$ holds for composable maps of Λ -modules f and g .

To define the cup product, we choose a resolution $\dots \rightarrow M_1 \xrightarrow{\lambda_1} M_0 \xrightarrow{\lambda_0} M$ of M by objects M_i of $F(\Lambda, n)$ and a normalized cocycle

$$c \in \overline{\mathcal{C}}^s(F(\Lambda, n), \text{Hom}_\Lambda(-, - \otimes_\Lambda \Lambda[t]))$$

representing the cohomology class $\gamma \in \text{HML}_{n\text{-sp}}^{s,t}(\Lambda)$.

Since $\delta(c) = 0$ and the λ_i form a resolution, evaluating $\delta(c)$ on $(\lambda_1, \dots, \lambda_{s+1})$ yields $\lambda_0[t]c(\lambda_1, \dots, \lambda_s)\lambda_{s+1} = (-1)^s \lambda_0[t]\lambda_1[t]c(\lambda_2, \dots, \lambda_{s+1}) = 0$. This implies that there is a dotted arrow τ such that diagram of Figure 1 commutes.

If $\Psi \in \text{Ext}_\Lambda^{s,0}(M, \ker \lambda_{s-1})$ denotes the Yoneda class of the extension

$$0 \rightarrow \ker \lambda_{s-1} \rightarrow M_{s-1} \rightarrow \dots \rightarrow M_0 \rightarrow M \rightarrow 0,$$

we define $f \cup \gamma$ to be $(-1)^{\frac{n(n+3)}{2}}((f[t])\tau)_*(\Psi) \in \text{Ext}_\Lambda^{s,t}(M, N)$. The mysterious sign is built in to cancel out with another sign arising in Lemma 4.12. (This will keep signs out of the statements of the main results.)

The bilinearity and the naturality with respect to composition of maps are obvious. In Lemma 2.16 and Lemma 2.17 we show that the Ext-class of $f \cup \gamma$ doesn't depend on the choice of the cocycle representing γ and the resolution of M .

Remark 2.14. If E is a graded k -algebra over a field k , the tensor product of a right module with a bimodule has a left derived functor

$$- \otimes^L -: \text{Hom}_E(P, Q) \times \text{HH}_k^{s,t}(E) \rightarrow \text{Ext}_E^{s,t}(P, Q).$$

Our cup-product should be thought of as similar to this. The relation becomes clearer when HML^* is defined via Ext-groups in the category $\mathcal{F}(A)$ of functors $F(A) \rightarrow \text{Mod-}A$. We sketch the ungraded case.

The self-extensions $\text{Ext}_{\mathcal{F}(A)}^*(I, I)$ in $\mathcal{F}(A)$ of the inclusion functor I are isomorphic to $\text{HML}^*(A)$ [JP91]. We can enlarge $F(A)$ by a bigger small additive

subcategory \mathcal{C} of $\text{Mod-}A$ that contains M without changing the Ext -group [JP91, §2 and Corollary 3.11]. Evaluating an element of $\text{Ext}_{\mathcal{C}}(I, I)$ on M gives an element of $\text{Ext}_A^*(M, M)$, and inspecting the proof of the isomorphism $\text{Ext}_{\mathcal{F}(A)}^*(I, I) \cong \text{HML}^*(A)$ [JP91, Theorem B], we see that this recovers the cup-product. We do not go into the details, as we only use the description of the product given above.

Lemma 2.15. *Let $0 \rightarrow M' \xrightarrow{g} M_{n-1} \rightarrow \dots \rightarrow M_0 \rightarrow M$ be an exact sequence in $\text{Mod-}\Lambda$ representing a class $\Psi \in \text{Ext}_{\Lambda}^n(M, M')$. Assume that M_0, \dots, M_{n-1} are free. For maps $f: M' \rightarrow N$ and $h: M_{n-1} \rightarrow N$, we have $(f + hg)_*(\Psi) = f_*(\Psi)$.*

Proof. This statement becomes trivial with Ext defined via projective resolutions. □

Lemma 2.16. *The cup product of Construction 2.13 does not depend on the choice of the cocycle representing γ .*

Proof. It is enough to show that the extension associated to a coboundary represents the trivial element in $\text{Ext}_{\Lambda}^{s,t}(M, N)$. Let $b \in \overline{\mathcal{C}}^{s-1}(F(\Lambda, n), \text{Hom}_{\Lambda}(-, - \otimes_{\Lambda} \Lambda[t]))$ be a normalized cochain. Evaluating $\delta(b)$ on $(\lambda_1, \dots, \lambda_s)$ yields $\lambda_0[t]\delta(b)(\lambda_1, \dots, \lambda_s) = (-1)^s \lambda_0[t]b(\lambda_1, \dots, \lambda_{s-1})\lambda_s$. Hence the τ associated to $c = \delta(b)$ extends to M_{s-1} , so $((f[t])\tau)_*(\Psi) = 0$ by the last lemma. □

Lemma 2.17. *The cup product of Construction 2.13 does not depend on the choice of the resolution of M .*

Proof. Suppose we are given another resolution $\dots \xrightarrow{\lambda'_1} M'_0 \xrightarrow{\lambda'_0} M$ of M by objects of $F(\Lambda, n)$. Then there exist maps $\alpha_i: M'_i \rightarrow M_i$ with $\lambda_i \alpha_i = \alpha_{i-1} \lambda'_i$ and $\lambda_0 \alpha_0 = \lambda'_0$. The problem is that in general $(\lambda'_0[t])(c(\lambda'_1, \dots, \lambda'_s)) = (\lambda_0[t])(c(\lambda_1, \dots, \lambda_s))\alpha_s$ does not hold. As we are only interested in the induced maps on Ext -groups, it suffices to show that the two maps give rise to maps $\tau, \tau': \ker \lambda'_{s-1} \rightarrow M[t]$ which induce the same map $\text{Ext}_{\Lambda}^{s,0}(M, \ker \lambda'_{s-1}) \rightarrow \text{Ext}_{\Lambda}^{s,t}(M, N)$.

Using $\lambda_i \alpha_i = \alpha_{i-1} \lambda'_i$ and the definition of δ , we obtain the equation

$$\begin{aligned} 0 &= (\delta c)(\alpha_0, \lambda'_1, \dots, \lambda'_s) + (-1)^s (\delta c)(\lambda_1, \dots, \lambda_s, \alpha_s) \\ &\quad + \sum_{i=1}^{s-1} (-1)^i (\delta c)(\lambda_1, \dots, \lambda_i, \alpha_i, \lambda'_{i+1}, \dots, \lambda'_s) \\ &= \alpha_0[t]c(\lambda'_1, \dots, \lambda'_s) - c(\lambda_1, \dots, \lambda_s)\alpha_s + \lambda_1[t]g + h\lambda'_s \end{aligned}$$

in which $h: M'_{s-1} \rightarrow M_0[t]$ and $g: M'_s \rightarrow M_1[t]$ are maps we don't need to know explicitly. Composing with $\lambda_0[t]$ and applying Lemma 2.15 completes the proof. □

3. REALIZABILITY IN TRIANGULATED CATEGORIES

In this section we give a quick review of the obstruction theory for realizability in triangulated categories described in [BKS04, Appendix A]. The necessary background on triangulated categories can be found in Weibel's book [Wei94].

Let \mathcal{T} be a triangulated category, which we always assume to have infinite coproducts. An object N of \mathcal{T} is *compact* if the functor $\mathcal{T}(N, -)$ preserves arbitrary coproducts. For objects X and Y of \mathcal{T} , we write $\mathcal{T}(X, Y)_*$ for the graded abelian group whose degree k part is $\mathcal{T}(X[k], Y)$.

We fix a compact object N in \mathcal{T} . Under composition, $\Lambda := \mathcal{T}(N, N)_*$ becomes a graded ring, and $\mathcal{T}(N, X)_*$ is a right Λ -module for every object X . The resulting

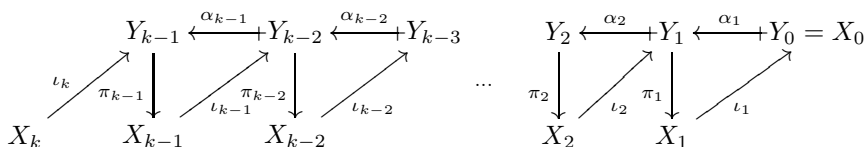
functor $\mathcal{T}(N, -) : \mathcal{T} \rightarrow \text{Mod-}\Lambda$ from \mathcal{T} to graded Λ -modules maps distinguished triangles in \mathcal{T} to long exact sequences. Furthermore, it preserves arbitrary coproducts since N is compact, and it commutes with the shift of \mathcal{T} and $\text{Mod-}\Lambda$.

Definition 3.1. In the above context, a Λ -module M is called *realizable* if there is an object X in \mathcal{T} such that $\mathcal{T}(N, X)_* \cong M$.

The following example for this situation is studied in [BKS04]. Let A be a differential graded algebra over a field k , and let $\mathcal{T} = \mathcal{D}(A)$ be the derived category of dg A -modules. If N is the free module of rank 1, we have $\mathcal{T}(N, N)_{-*} = H^*(A)$. The realizability question amounts to whether a graded module over the cohomology ring $H^*(A)$ is the cohomology of a dg A -module. In Section 8, we will address the corresponding question for a ring spectrum R : when is a module over the homotopy groups of R the homotopy of an R -module spectrum?

3.2. Realizability obstructions. Let \mathcal{T} , N and Λ be as above. An object of \mathcal{T} is called *N -free* if it is a sum of shifted copies of N . We note that $\mathcal{T}(N, -)_*$ restricts to an equivalence between the full subcategory of N -free objects in \mathcal{T} and the category of free Λ -modules.

Definition 3.3 ([BKS04, Definition A.6]). For $k \geq 1$, an *N -exact k -Postnikov system* for a Λ -module M consists of an epimorphism $\mathcal{T}(N, X_0)_* \rightarrow M$ and a diagram



such that all arrows of the form $\longleftarrow \rightarrow$ denote morphisms of degree 1, all triangles are distinguished triangles in \mathcal{T} , and each object X_i is N -free. Moreover, the maps $d_j = \pi_{j-1} \iota_j$ with $j \geq 2$ and $d_1 = \iota_1$ are required to induce an exact sequence

$$\mathcal{T}(N, X_k)_* \xrightarrow{(d_k)_*} \mathcal{T}(N, X_{k-1})_* \xrightarrow{(d_{k-1})_*} \dots \xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_* \rightarrow M \rightarrow 0.$$

An *N -exact Postnikov system* is a collection of distinguished triangles as above which extends infinitely to the left.

Proposition A.19 of [BKS04] shows that a Λ -module M is realizable if there exists an N -exact Postnikov system of M . By realizing the first two steps of a free resolution of M , one can easily see that N -exact 2-Postnikov systems exist for every M . Therefore, the realizability problem can be attacked by extending Postnikov systems stepwise to the left.

By [BKS04, Lemma A.12(iii)], every N -exact k -Postnikov system of M induces an exact sequence

$$\begin{aligned}
 \mathcal{T}(N, X_1)_*[1-k] &\xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_*[1-k] \xrightarrow{\alpha_*} \mathcal{T}(N, Y_{k-1})_* \\
 &\xrightarrow{(\pi_{k-1})_*} \mathcal{T}(N, X_{k-1})_* \xrightarrow{(d_{k-1})_*} \mathcal{T}(N, X_{k-2})_*
 \end{aligned}$$

of Λ -modules, where the map $\alpha: X_0[1 - k] = Y_0[1 - k] \rightarrow Y_{k-1}$ is the composition $\alpha_{k-1} \cdots \alpha_1$. Hence there is an exact sequence

$$(3.4) \quad 0 \rightarrow M[1 - k] \xrightarrow{\eta_{k-1}} \mathcal{T}(N, Y_{k-1})_* \xrightarrow{(\pi_{k-1})_*} \mathcal{T}(N, X_{k-1})_* \\ \xrightarrow{(d_{k-1})_*} \dots \xrightarrow{(d_2)_*} \mathcal{T}(N, X_1)_* \xrightarrow{(d_1)_*} \mathcal{T}(N, X_0)_* \rightarrow M \rightarrow 0$$

of Λ -modules. Its Yoneda class is denoted by $\kappa_{k+1}(M) \in \text{Ext}_\Lambda^{k+1, k-1}(M, M)$ and is called the obstruction class associated to the Postnikov system because of

Lemma 3.5 ([BKS04, Lemma A.18]). *If the class $\kappa_{k+1}(M)$ of an N -exact k -Postnikov system of M is trivial, then there exists an N -exact $(k + 1)$ -Postnikov system for M whose underlying $(k - 1)$ -Postnikov system agrees with that of the given one.*

The class $\kappa_3(M)$ is always defined and unique [BKS04, Proposition 3.4(ii)]. If the higher obstructions $\kappa_i(M)$ for $i \geq 4$ are defined, they may depend on the choice of the Postnikov system.

3.6. A criterion for uniqueness of obstruction classes. To compare the obstruction classes of different Postnikov systems, we need

Definition 3.7. Let $(X_j, Y_j, \alpha_j, \iota_j, \pi_j, M)$ and $(X'_j, Y'_j, \alpha'_j, \iota'_j, \pi'_j, M)$ be two N -exact k -Postnikov systems for M . A morphism between them consists of maps $f_j: X_j \rightarrow X'_j$ and $g_j: Y_j \rightarrow Y'_j$ such that $f_{k-1}d_k = d'_k f_k$ and the following commutativity relations hold for $1 \leq j \leq k - 1$:

$$g_{j-1}\iota_j = \iota'_j f_j \quad (g_j[1])\alpha_j = \alpha'_j g_{j-1} \quad f_j \pi_j = \pi'_j g_j.$$

More generally, for $1 \leq l \leq k$, an l -map of N -exact k -Postnikov systems for M is a map of the underlying N -exact l -Postnikov systems.

A map between two N -exact k -Postnikov systems induces a map of the long exact sequences representing the obstruction classes, and this map is id_M on the outer terms. So the obstruction classes of two Postnikov systems coincide if there is a map between them. Note that this does *not* need the relation $g_{k-1}\iota_k = \iota'_k f_k$, which therefore wasn't required in Definition 3.7. To produce such maps, we use

Lemma 3.8. *Suppose we are given an l -map between two N -exact k -Postnikov systems with $1 \leq l < k$. There is an element in $\text{Ext}_\Lambda^{l, 1-l}(M, M)$ whose vanishing implies the existence of an $(l + 1)$ -map between the Postnikov systems.*

Proof. Since both $(\mathcal{T}(N, X_i)_*, (d_i)_*)$ and $(\mathcal{T}(N, X'_i)_*, (d'_i)_*)$ are exact complexes of free Λ -modules, we can find a $f_{l+1}: X_{l+1} \rightarrow X'_{l+1}$ with $f_l d_{l+1} = d'_{l+1} f_{l+1}$.

Let us assume for a moment our map of Postnikov systems satisfies $g_{l-1}\iota_l = \iota'_l f_l$. Then we could find a $g_l: Y_l \rightarrow Y'_l$ such that $(f_l, g_{l-1}, g_l[1])$ is a map between the distinguished triangles $(-\pi_l[1], \alpha_l, \iota_l)$ and $(-\pi'_l[1], \alpha'_l, \iota'_l)$. The maps f_{l+1} and g_l would complete the required data of an $(l + 1)$ -map.

In general, $\varphi = \iota'_l f_l - g_{l-1}\iota_l \in \mathcal{T}(X_l, Y'_{l-1})$ is non-zero. By applying $\mathcal{T}(X_l, -)$ to the triangle $(\pi'_l, -\alpha'_l[-1], -\iota'_l[-1])$ we see that there is a $\psi \in \mathcal{T}(X_l, Y'_{l-2}[-1])$ with $(\alpha'_{l-1})_*(\psi) = \varphi$.

If we apply $\mathcal{T}(N, -)_*$ to $\mathcal{T}(X_l, Y'_{l-2}[-1])$ and use [BKS04, Lemma A.12(i)-(ii)], ψ defines a class in $\text{Ext}_\Lambda^{l, 1-l}(M, M)$. The vanishing of this Ext-group implies the existence of a $\rho \in \mathcal{T}(X_{l-1}, Y'_{l-2}[-1])$ with $\rho d_l = \psi$.

Now we can change our map of Postnikov systems by replacing g_{l-1} by $\overline{g_{l-1}} = g_{l-1} + (\alpha'_{l-1}[-1])\rho\pi_{l-1}$. The $\overline{g_{l-1}}$ satisfies the required relations. In addition,

$$\overline{g_{l-1}}\iota_l = g_{l-1}\iota_l + (\alpha'_{l-1}[-1])\rho d_l = g_{l-1}\iota_l + \varphi = \iota'_l f_l$$

holds, and the modified l -map extends to an $(l+1)$ -map by the argument above. \square

Recall that a graded abelian group or a graded ring is n -sparse if it is concentrated in degrees divisible by n .

Corollary 3.9. *Suppose that $\Lambda = \mathcal{T}(N, N)_*$ is n -sparse and M is an n -sparse Λ -module. Then there exists an N -exact $(n + 1)$ -Postnikov system of M , and all N -exact $(n + 1)$ -Postnikov systems of M give rise to the same obstruction class $\kappa_{n+2}(M) \in \text{Ext}_\Lambda^{n+2, -n}(M, M)$.*

Proof. The groups $\text{Ext}_\Lambda^{l+1, 1-l}(M, M)$ vanish for $2 \leq l \leq n$ because of the sparseness of Λ and M . Hence there is an N -exact $(n + 1)$ -Postnikov system for M by Lemma 3.5. Similarly, Lemma 3.8 and the vanishing of $\text{Ext}_\Lambda^{l, 1-l}(M, M)$ for $2 \leq l \leq n$ provide the existence of a map between two N -exact $(n + 1)$ -Postnikov systems for M . This implies the uniqueness of the obstruction class $\kappa_{n+2}(M)$. \square

4. TODA BRACKETS AND REALIZABILITY

We recall the definition of Toda brackets in triangulated categories and show how they are related to the realizability obstructions of the last section.

4.1. Definition of higher Toda brackets. Cohen’s definition [Coh68, §2] of higher Toda brackets can be interpreted in the context of triangulated categories. We follow Shipley [Shi02, Appendix A] in doing so.

Definition 4.2 ([Shi02, Definition A.1]). Let \mathcal{T} be a triangulated category and let

$$X_{n-1} \xrightarrow{\lambda_{n-1}} X_{n-2} \xrightarrow{\lambda_{n-2}} \dots \xrightarrow{\lambda_1} X_0$$

be $(n - 1)$ composable maps in \mathcal{T} . An n -filtered object $X \in \{\lambda_1, \dots, \lambda_{n-1}\}$ consists of a sequence of maps $* = F_0X \xrightarrow{i_0} F_1X \xrightarrow{i_1} \dots \xrightarrow{i_n} F_nX = X$ and choices of distinguished triangles $F_jX \xrightarrow{i_j} F_{j+1}X \xrightarrow{p_{j+1}} X_j[j] \xrightarrow{d_j} (F_jX)[1]$ such that $(p_j[1])(d_j) = \lambda_j[j]$.

The maps $X_0 \cong F_1X \rightarrow X$ and $X = F_nX \xrightarrow{p_n} X_{n-1}[n - 1]$ are denoted by σ'_X and σ_X .

Our definition differs from [Shi02, Definition A.1] in the fact that we require the objects $X_j[j]$ to be the cones of the maps i_j , rather than to be isomorphic to the cones. This does not make a difference since triangles isomorphic to distinguished triangles are distinguished again.

For a map $\lambda_1 : X_1 \rightarrow X_0$ in \mathcal{T} , the cone C of λ_1 is part of a distinguished triangle $X_1 \rightarrow X_0 \rightarrow C \rightarrow X_1[1]$. With the filtration $* \rightarrow X_0 \rightarrow C$, it is a 2-filtered object in $\{\lambda_1\}$.

If there exists an n -filtered object $X \in \{\lambda_1, \dots, \lambda_{n-1}\}$, each twofold composition $\lambda_i\lambda_{i+1}$ has to be zero since it can be written as a composition of maps which contains two consecutive maps in a distinguished triangle.

Though a filtered object consists of similar data as a Postnikov system, we emphasize the difference: a filtered object starts from a *fixed complex* of maps, while a Postnikov system starts from a *module* and is assumed to have *some* underlying

resolution. Lemma 4.12 shows how in special cases a filtered object gives rise to a Postnikov system.

We will construct filtered objects using

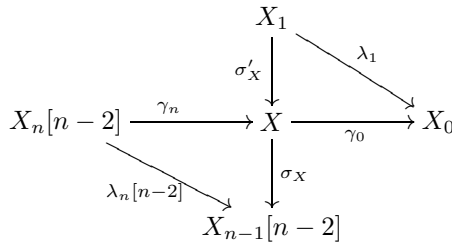
Lemma 4.3 ([Shi02, Lemma A.4]). *Let $\lambda_i: X_i \rightarrow X_{i-1}$ be a sequence of composable maps in a triangulated category \mathcal{T} . An n -filtered object $X \in \{\lambda_2, \dots, \lambda_n\}$ with a map $\alpha: X \rightarrow X_0$ gives rise to an $(n + 1)$ -filtered object $C_\alpha \in \{\alpha\sigma'_X, \lambda_2, \dots, \lambda_n\}$, and an n -filtered object $X \in \{\lambda_1, \dots, \lambda_{n-1}\}$ with a map $\alpha: X_n[n - 1] \rightarrow X$ gives rise to an $(n + 1)$ -filtered object $C_\alpha \in \{\lambda_1, \dots, \lambda_{n-1}, (\sigma_X\alpha)[-n + 1]\}$.*

Proof. The first part uses the octahedral axiom. The second part is immediate. \square

Definition 4.4 ([Shi02, Definition A.2]). Let \mathcal{T} be a triangulated category. A map $\gamma \in \mathcal{T}(X_n[n - 2], X_0)$ lies in the n -fold Toda bracket of

$$X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} X_0$$

if there exist an $(n - 1)$ -filtered object $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ and maps $\gamma_0: X \rightarrow X_0$ and $\gamma_n: X_n[n - 2] \rightarrow X$ such that $\gamma = \gamma_0\gamma_n$ holds and the two triangles in the following diagram commute:



We write $\langle \lambda_1, \dots, \lambda_n \rangle \subseteq \mathcal{T}(X_n[n - 2], X_0)$ for the possibly empty set of all those γ .

We refer to Remark 6.2 for a discussion of other definitions of Toda brackets.

For $n = 3$, this defines the triple Toda bracket $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$. The cone of λ_2 serves as the 2-filtered object. The set $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$ is non-empty iff $\lambda_1\lambda_2 = 0 = \lambda_2\lambda_3$. It is easy to check that two elements of $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$ differ by an element of the set $(\lambda_1)_*(\mathcal{T}(X_3[1], X_1)) + (\lambda_3[1])_*(\mathcal{T}(X_2[1], X_0))$, which we refer to as the *indeterminacy* of the Toda bracket.

Remark 4.5. In the situation of $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1\lambda_2 = 0 = \lambda_2\lambda_3$, there are two more equivalent definitions of $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$ which involve distinguished triangles containing λ_1 or λ_3 instead of λ_2 . By choosing distinguished triangles in the horizontal lines and appropriate extensions, one builds the commutative diagram of Figure 2. Considering the middle line as a filtered object, one sees that $\epsilon_2\tau_3 \in \langle \lambda_1, \lambda_2, \lambda_3 \rangle$ in the sense of Definition 4.4 above. Starting with the upper line, one can first choose ϵ_3 . Since $\epsilon_2\tau_3$ is a choice for extending $\lambda_1\epsilon_3$ to $X_3[1]$, this is an equivalent definition not involving C_2 . A third definition uses the distinguished triangle in the lower line.

4.6. Existence and indeterminacy of higher Toda brackets. A sequence $(\lambda_1, \dots, \lambda_n)$ of composable maps has to satisfy restrictive conditions for its Toda bracket to be non-empty. For example, $0 \in \langle \lambda_2, \dots, \lambda_{n-1} \rangle$ is a necessary condition for the existence of an $(n - 1)$ -filtered object $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ [Shi02, Proposition A.5], and the additional requirement $\lambda_1\lambda_2 = 0 = \lambda_{n-1}\lambda_n$ will in general not be

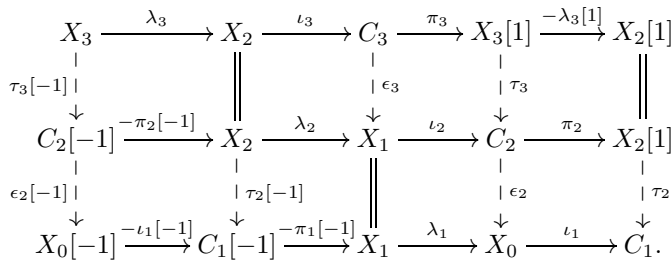


FIGURE 2. Different definitions of triple Toda brackets

sufficient for $\langle \lambda_1, \dots, \lambda_n \rangle$ to be non-empty. We introduce an additional assumption to obtain non-empty Toda brackets with controllable indeterminacy.

Definition 4.7. A full subcategory \mathcal{U} of a triangulated category \mathcal{T} is n -split if $\mathcal{T}(X, Y)_*$ is n -sparse for all objects X and Y of \mathcal{U} .

This is the analog to Definition 2.6 for triangulated categories. If \mathcal{T} has a compact object N for which $\mathcal{T}(N, N)_*$ is n -sparse, the subcategory of sums of copies of N which are shifted by integral multiples of n is n -split.

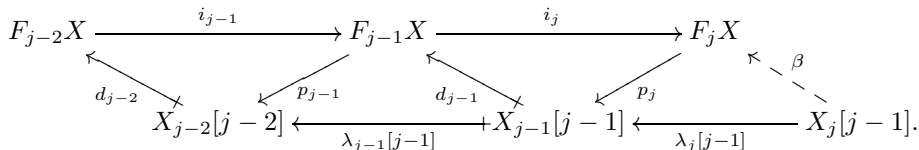
Lemma 4.8. Let \mathcal{U} be an n -split subcategory of a triangulated category \mathcal{T} with $n \geq 2$, let $X_{l-1} \xrightarrow{\lambda_{l-1}} \dots \xrightarrow{\lambda_1} X_0$ be a sequence of maps in \mathcal{U} with $2 \leq l \leq n - 1$, and let $X \in \{\lambda_1, \dots, \lambda_{l-1}\}$ be an l -filtered object. Then for every object Y in \mathcal{U} , we have $\mathcal{T}(Y[l], X) = 0$ and $\mathcal{T}(X, Y[-1]) = 0$.

Proof. To show the first part, we choose a map $\alpha: Y[l] \rightarrow X$. Since the composition $Y[l] \rightarrow X = F_l X \xrightarrow{\sigma_X} X_{l-1}[l-1]$ is zero, α factors through $F_{l-1} X \rightarrow F_l X$. Using inductively that $\mathcal{T}(Y[l], X_j[j]) = 0$ for $j = l - 2, \dots, 0$, we obtain that α factors through $F_0 X \rightarrow F_l X$. Hence $\alpha = 0$ since $F_0 X = *$.

For the second part, we first observe that $\mathcal{T}(F_l X, Y[-1]) \cong \mathcal{T}(X_0, Y[-1]) = 0$. The exact sequence $\mathcal{T}(X_j[j], Y[-1]) \rightarrow \mathcal{T}(F_{j+1} X, Y[-1]) \rightarrow \mathcal{T}(F_j X, Y[-1])$ in which the first term is trivial for $j \leq l - 2$ can be used to show the assertion by induction. \square

Lemma 4.9. Let \mathcal{U} be an n -split subcategory of a triangulated category \mathcal{T} . Then a sequence $X_l \xrightarrow{\lambda_l} X_{l-1} \xrightarrow{\lambda_{l-1}} \dots \xrightarrow{\lambda_1} X_0$ in \mathcal{U} with $\lambda_i \lambda_{i+1} = 0$ admits an $(l + 1)$ -filtered object $X \in \{\lambda_1, \dots, \lambda_l\}$ if $l \leq n + 1$. If $l \leq n$, the $(l + 1)$ -filtered object is unique up to isomorphism.

Proof. The map from X_0 to the cone of $\lambda_1: X_1 \rightarrow X_0$ gives the data of a 2-filtered object in $\{\lambda_1\}$. Inductively, we assume that $X \in \{\lambda_1, \dots, \lambda_{j-1}\}$ is a j -filtered object with $j \leq n$ and consider the solid arrow diagram



The map $(p_{j-1}d_{j-1})(\lambda_j[j-1])$ is trivial as a shift of $\lambda_{j-1}\lambda_j$. Hence $d_{j-1}(\lambda_j[j-1])$ lifts along i_{j-1} and factors through $F_{j-2}X$. We have $\mathcal{T}(X_{j-1}[j-1], F_{j-2}X) = 0$ by

the last lemma, hence $d_{j-1}(\lambda_j[j-1]) = 0$. This provides the existence of the dotted arrow β . By Lemma 4.3, the cone of β is a $(j+1)$ -filtered object in $\{\lambda_1, \dots, \lambda_j\}$.

We prove uniqueness by inductively constructing isomorphisms $f_j: F_j X \rightarrow F'_j X$ compatible with all structure maps. This is trivial for the 1-filtered objects. Assume we are given an isomorphism $f_{j-1}: F_{j-1} X \rightarrow F'_{j-1} X$. The compatibility yields $p'_{j-1}d'_{j-1} = \lambda_{j-1}[j-1] = p_{j-1}f_{j-1}d_{j-1}$. Hence the exact sequence resulting from applying $\mathcal{T}(X_{j-1}[j-2], -)$ to

$$F'_{j-2} X \xrightarrow{\iota'_{j-1}} F'_{j-1} X \xrightarrow{p'_{j-1}} X_{j-2}[j-2]$$

shows that $d'_{j-1} - f_{j-1}d_{j-1}$ is in the image of $(\iota'_{j-1})_*$. Since $\mathcal{T}(X_{j-1}[j-2], F_{j-2} X')$ is trivial for $j \leq n+1$ by Lemma 4.8, this implies $d'_{j-1} = f_{j-1}d_{j-1}$. Completing $(\text{id}_{X_{j-1}[j-1]}, f_{j-1})$ to a map of triangles yields the desired f_j . \square

Proposition 4.10. *Let \mathcal{U} be an n -split subcategory of a triangulated category \mathcal{T} and let*

$$X_{n+2} \xrightarrow{\lambda_{n+2}} X_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} X_0$$

be a sequence of maps in \mathcal{U} with $\lambda_i \lambda_{i+1} = 0$. Then the Toda bracket $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$ is defined, is non-empty, and has the indeterminacy

$$(\lambda_1)_*(\mathcal{T}(X_{n+2}[n], X_1)) + (\lambda_{n+2}[n])^*(\mathcal{T}(X_{n+1}[n], X_0)).$$

Proof. An $(n+1)$ -filtered object $X \in \{\lambda_2, \dots, \lambda_{n+1}\}$ exists and is unique by Lemma 4.9. To construct γ_{n+2} , we consider the exact sequence

$$\mathcal{T}(X_{n+1}[n], X) \xrightarrow{(\sigma_X)_*} \mathcal{T}(X_{n+2}[n], X_{n+1}[n]) \rightarrow \mathcal{T}(X_{n+1}[n], F_n X[1]).$$

The last term is trivial by Lemma 4.8. Hence there is a γ_{n+2} with $\sigma_X \gamma_{n+2} = \lambda_{n+2}[n]$.

To obtain γ_0 , we use $F_1 X \xrightarrow{\cong} X_1$ and λ_1 to get a map $F_1 X \rightarrow X_0$. It can be extended to $F_2 X$ since $\lambda_1 \lambda_2 = 0$. Inductively, we can extend it to a map $\gamma_0: X = F_{n+1} X \rightarrow X_0$: the obstruction for extending a map $F_{j-1} X \rightarrow X_0$ to $F_j X$ lies in $\mathcal{T}(X_{j-1}[j-2], X_0)$, which is trivial for $3 \leq j \leq n+1$.

Next we compute the indeterminacy. Since we have an exact sequence

$$\mathcal{T}(X_{n+2}[n], F_n X) \xrightarrow{(i_n)_*} \mathcal{T}(X_{n+2}[n], F_{n+1} X) \xrightarrow{(\sigma_X)_*} \mathcal{T}(X_{n+2}[n], X_{n+1}[n]),$$

we know that two different choices of γ_{n+2} differ by an element in the image of $(i_n)_*$. Using the same argument as in Lemma 4.8, we see that every map $X_{n+2}[n] \rightarrow F_n X$ factors through $\sigma'_X: X_1 \cong F_1 X \rightarrow F_n X$. Therefore, the possible difference is in the image of $(\sigma'_X)_*$, and after composing with any choice for γ_0 we obtain that this part of the indeterminacy is $(\lambda_1)_* \mathcal{T}(X_{n+2}[n], X_1)$.

To examine the other part of the indeterminacy, we first construct an auxiliary n -filtered object $F'_n X \in \{\lambda_3[1], \dots, \lambda_{n+1}[1]\}$. For $0 \leq j \leq n$, we define $F'_j X$ to be part of a distinguished triangle $X_1 \rightarrow F_{j+1} X \rightarrow F'_j X$.

The exact sequence

$$\mathcal{T}(F'_n X, X_0) \rightarrow \mathcal{T}(X, X_0) \rightarrow \mathcal{T}(X_1, X_0)$$

shows that the difference $\bar{\gamma}_0$ of two choices for γ_0 is in the image of $\mathcal{T}(F'_n X, X_0)$. Since $\mathcal{T}(F'_{n-1} X, X_0)$ vanishes by Lemma 4.8, there is an $\omega: X_{n+1}[n] \rightarrow X_0$ with $\omega \sigma_X = \bar{\gamma}_0$. If we apply $(\gamma_{n+2})^*$ to $\omega \sigma_X$, we see that this part of the indeterminacy is given by $(\lambda_{n+2})^*(\mathcal{T}(X_{n+1}[n], X_0))$. \square

4.11. Relation to realizability obstructions. In this section we exhibit the link between Toda brackets and realizability obstructions. More precisely, we use the cup product of Construction 2.13 to turn the slogan ‘the Toda brackets of the resolution are realizability obstructions’ into a theorem. The first step is the relation between filtered objects in the sense of Definition 4.2 and Postnikov systems in the sense of Definition 3.3.

Lemma 4.12. *Let $X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} X_0$ be a sequence of maps in \mathcal{T} such that each X_i is N -free and $\mathcal{T}(N, -)$ maps it to an exact sequence of Λ -modules. Let M be the cokernel of the map $(\lambda_1)_*: \mathcal{T}(N, X_1)_* \rightarrow \mathcal{T}(N, X_0)$. An $(n + 1)$ -filtered object $X \in \{\lambda_1, \dots, \lambda_n\}$ determines all data of an N -exact $(n + 1)$ -Postnikov system of M except the map $X_{n+1} \rightarrow Y_n$ by setting $Y_l = (F_{l+1})[-l]$ for $0 \leq l \leq n$ and*

$$\pi_l = (-1)^l p_{l+1}[-l], \quad \iota_l = (-1)^l d_l[-l], \quad \text{and} \quad \alpha_l = (-1)^{l+1} i_l[-l + 1]$$

for $1 \leq l \leq n$. Then $\alpha = (-1)^{\frac{n(n+3)}{2}} \sigma'_X[-n]$, and the $(-1)(\lambda_i)_$ form the underlying resolution of the Postnikov system.*

Proof. The triangles $(\alpha_l, \iota_l, \pi_l)$ are distinguished since the (d_l, p_{l+1}, i_l) are. The signs needed for this imply $\pi_{l-1} \iota_l = -\lambda_l$ as well as the sign relating $\alpha = \alpha_n \cdots \alpha_1$ to $\sigma'_X = i_n \cdots i_1$. □

Before stating the main theorem of this section, we explain why the Mac Lane cohomology groups of Definition 2.7 provide an appropriate tool for the systematic study of Toda brackets.

Definition 4.13. Let \mathcal{T} be a triangulated category with a compact object N such that $\Lambda = \mathcal{T}(N, N)_*$ is n -sparse. $F_{\mathcal{T}}(N, n)$ is defined to be the full subcategory of \mathcal{T} given by finite sums of copies of N which are shifted by integral multiples of n .

The functor $\mathcal{T}(N, -)_*$ induces an equivalence between $F_{\mathcal{T}}(N, n)$ and the category $F(\Lambda, n)$. This equivalence induces an isomorphism between the Mac Lane-cohomology group $\text{HML}_{n-\text{sp}}^{*, -n}(\Lambda)$ and the normalized cohomology of $F_{\mathcal{T}}(N, n)$ with coefficients in $\mathcal{T}(-, -)_n$

Suppose we are given a sequence of composable maps $(\lambda'_1, \dots, \lambda'_{n+2})$ in $F(\Lambda, n)$ with $\lambda_{i+1} \lambda_i = 0$ for all i . We define the Toda bracket $\langle \lambda'_1, \dots, \lambda'_{n+2} \rangle$ of this sequence of maps in $\text{Mod-}\Lambda$ to be the Toda bracket of the sequence $(\lambda_1, \dots, \lambda_{n+2})$ in $F_{\mathcal{T}}(N, n)$ associated to it under the equivalence $\mathcal{T}(N, -)_*$. If \mathcal{T} is the derived category of a dga A , this defines Toda brackets in the cohomology ring $H^*(A)$ via the Toda brackets in the derived category $\mathcal{D}(A)$. One can check that this recovers the usual notion of Massey products.

Remark 4.14. In the situation above, the indeterminacy of $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$ is

$$(\lambda_1)_*(\mathcal{T}(X_{n+2}[n], X_1)) + (\lambda_{n+2}[n])^*(\mathcal{T}(X_{n+1}[n], X_0))$$

by Proposition 4.10. Now suppose we are given a normalized cocycle c representing a cohomology class $\gamma \in H^{n+2}(F_{\mathcal{T}}(N, n), \mathcal{T}(-, -)_n)$. Then $c(\lambda_1, \dots, \lambda_{n+2}) \in \mathcal{T}(X_{n+2}[n], X_0)$. If we change c by adding a coboundary $\delta(b)$, the evaluation on $(\lambda_1, \dots, \lambda_{n+2})$ changes by an element of

$$(\lambda_1)_*(\mathcal{T}(X_{n+2}[n], X_1)) + (\lambda_{n+2}[n])^*(\mathcal{T}(X_{n+1}[n], X_0)).$$

Hence the *evaluation of a cohomology class* has the same indeterminacy as the $(n + 2)$ -fold Toda bracket. Consequently, it makes sense to ask the evaluation

of a cohomology class $\gamma \in \text{HML}_{n-\text{sp}}^{n+2,-n}(\Lambda)$ on a complex of n -split Λ -modules $(\lambda'_1, \dots, \lambda'_{n+2})$ to be the Toda bracket $\langle \lambda'_1, \dots, \lambda'_{n+2} \rangle$ without having to mention indeterminacies. In other words, the indeterminacy of Toda brackets is built into the cohomology of categories. For $n = 3$, this observation was used for the study of (triple) universal Toda brackets in [BD89].

Theorem 4.15. *Let \mathcal{T} be a triangulated category, and let N be a compact object such that $\Lambda = \mathcal{T}(N, N)_*$ is n -sparse. Let M be a Λ -module admitting a resolution*

$$\dots \xrightarrow{\lambda'_1} M_0 \xrightarrow{\lambda'_0} M \rightarrow 0$$

by finitely generated free n -sparse Λ -modules. Let $\gamma \in \text{HML}_{n-\text{sp}}^{n+2,-n}(\Lambda)$ be a cohomology class such that the evaluation $\gamma(\lambda'_1, \dots, \lambda'_{n+2})$ is the Toda bracket $\langle \lambda'_1, \dots, \lambda'_{n+2} \rangle$. Then the product $\text{id}_M \cup \gamma \in \text{Ext}_{\Lambda}^{n+2,-n}(M, M)$ coincides with the unique obstruction class $\kappa_{n+2}(M)$ of Corollary 3.9.

Proof. We denote the realization of the resolution of M by N -free objects by

$$X_{n+2} \xrightarrow{\lambda_{n+2}} X_{n+1} \xrightarrow{\lambda_{n+1}} \dots \xrightarrow{\lambda_1} X_0,$$

so $(\lambda_i)_* = \lambda'_i$. By Lemma 4.9, there is a unique n -filtered object $Z \in \{\lambda_2, \dots, \lambda_n\}$. Since the $(n + 1)$ -fold Toda bracket of $(\lambda_1, \dots, \lambda_{n+1})$ contains only zero for degree reasons, we can find maps $\alpha: Z \rightarrow X_0$ and $\beta: X_{n+1}[n - 1] \rightarrow Z$ with $\sigma_Z \beta = \lambda_{n+1}[n - 1]$ and $\alpha \sigma'_Z = \lambda_1$ such that $\alpha \beta = 0 \in \langle \lambda_1, \dots, \lambda_{n-1} \rangle$.

We use α and β to find distinguished triangles

$$Z \xrightarrow{\alpha} X_0 \rightarrow Y \xrightarrow{\omega} Z[1] \quad \text{and} \quad X_{n+1}[n - 1] \xrightarrow{\beta} Z \xrightarrow{\iota} X \rightarrow X_{n+1}[n].$$

Lemma 4.3 tells us that X is an $(n + 1)$ -filtered object in $\{\lambda_2, \dots, \lambda_{n+1}\}$ and that Y is an $(n + 1)$ -filtered object in $\{\lambda_1, \dots, \lambda_n\}$.

The Toda bracket of $(\lambda_1, \dots, \lambda_{n+2})$ is non-empty by Proposition 4.10. It can be defined using the n -filtered object X . Hence there are maps $\gamma_0: X \rightarrow X_0$ and $\gamma_{n+2}: X_{n+2}[n] \rightarrow X$ with $\gamma_0 \sigma'_X = \lambda_1$ and $\sigma_X \gamma_{n+2} = \lambda_{n+2}[n]$ such that $\gamma' = \gamma_0 \gamma_{n+2}$ is an element of $\langle \lambda_1, \dots, \lambda_{n+2} \rangle$. Looking at the triangle defining X , we see that γ_0 can be constructed by extending $\alpha: Z \rightarrow X_0$ to a map $X \rightarrow X_0$. The relation $\gamma_0 \iota = \alpha$ implies the existence of the map ρ in the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & X_{n+2}[n] & & \\
 & & & & \swarrow \gamma_{n+2} & \downarrow \lambda_{n+2}[n] & \\
 & & & & X & \xrightarrow{\beta[1]} & X_{n+1}[n] & \xrightarrow{\beta[1]} & Z[1] \\
 Z & \xrightarrow{\iota} & X & \xrightarrow{\quad} & X_{n+1}[n] & \xrightarrow{\beta[1]} & Z[1] \\
 \downarrow = & & \downarrow \gamma_0 & & \downarrow \rho & & \downarrow = \\
 Z & \xrightarrow{\alpha} & X_0 & \xrightarrow{\sigma'_Y} & Y & \xrightarrow{\omega} & Z[1].
 \end{array}$$

Here we use the fact that the map $X_0 \rightarrow Y$ from the distinguished triangle defining Y coincides with the map σ'_Y which is part of the data of the n -filtered object Y .

Applying $\mathcal{T}(N, -)_*$ to the last diagram, we obtain the following commutative diagram of Λ -modules:

$$\begin{array}{ccccccc}
 \mathcal{T}(N, X_{n+2}[n])_* & \xrightarrow{(\lambda_{n+2})_*} & \mathcal{T}(N, X_{n+1}[n])_* & \xrightarrow{(\lambda_{n+1})_*} & \mathcal{T}(N, X_n[n])_* & \longrightarrow & \cdots \\
 \gamma'_* \downarrow & & \rho_* \downarrow & & \parallel & & \\
 \mathcal{T}(N, X_0)_* & \xrightarrow{(\sigma'_Y)_*} & \mathcal{T}(N, Y)_* & \xrightarrow{((\sigma_Z[1])\omega)_*} & \mathcal{T}(N, X_n[n])_* & \longrightarrow & \cdots \\
 & \searrow \lambda'_0 & \nearrow & & & &
 \end{array}$$

The lower sequence starting with M in this diagram represents $\text{id}_M \cup \gamma$ up to sign. Inspecting (3.4) and Lemma 4.12, we observe that it, up to signs, represents as well the exact sequence associated to the $(n + 1)$ -Postnikov system obtained from Y . This uses that the map $(\sigma_Z[1])\omega$ equals the map p_{n+1} of the $(n + 1)$ -filtered object Y , and therefore the map $(-1)^n \pi_n[n]$ of the associated Postnikov system. The sign of the latter map cancels with the n factors (-1) by which the maps $(\lambda_i)_*$ differ from the differentials of the resolution induced by the Postnikov system. The remaining sign $(-1)^{\frac{n(n+3)}{2}}$ of the map σ'_Y cancels with the sign built into the cup product. \square

Applications of this theorem will be given in Section 8. We point out that for $n = 1$, the last theorem also leads to an interpretation of the product of a Λ -module homomorphism $f: M \rightarrow M'$ with γ , provided that M satisfies the hypothesis of the theorem: by [BKS04, Proposition 3.4(iv) and Theorem 3.7] and the naturality of the cup product, $f \cup \gamma$ vanishes if and only if f factors through a realizable Λ -module.

5. CONSTRUCTION OF UNIVERSAL TODA BRACKETS

As outlined in the Introduction, the characteristic Hochschild cohomology class γ_A of a dga A considered in [BKS04] is a motivation for the study of the universal Toda bracket γ_R of a ring spectrum R . The class γ_A cannot be recovered from the derived category $\mathcal{D}(A)$ [BKS04, Example 5.15]. This suggests that the construction of γ_R from R will need more input than $\text{Ho}(\text{Mod-}R)$. It turns out that the stable model structure on the category $\text{Mod-}R$ together with the topological enrichment provides the necessary information.

Having the example $\text{Mod-}R$ in mind, we construct the universal Toda bracket of an n -split subcategory of a general stable topological model category in this section. The applications to ring spectra and the link to the realizability obstructions discussed above are given in Section 8.

Besides [BKS04], Baues' work on universal triple Toda brackets [Bau97, BD89] is another motivation for our construction (and its name). He is working mainly in an unstable context, considering subcategories of H -group or H -cogroup objects in the homotopy category of topological spaces, though he points out that these constructions generalize to 'cofibration categories' [Bau97, Remark on p. 271]. We will only work in a stable context, in order to provide the link to triangulated categories. This also avoids certain difficulties in the unstable case arising from maps which are not suspensions (see the correction of [BD89] in [Bau97, Remark on p. 270]). We also do not use Baues' language of 'linear track extensions', as these seem to be only appropriate for the study of triple universal Toda brackets.

Nevertheless, the $n = 1$ case of Proposition A.1 is basically what Baues encodes in a linear track extension.

A motivation for the actual construction of the representing cocycle is the approach of Blanc and Markl to higher homotopy operations [BM03]. For a directed category Γ , the authors use the bar resolution $W\Gamma$ in the sense of Boardman and Vogt [BV73, III, §1] to define general higher homotopy operations. If Γ is the category generated by $n + 2$ composable morphisms, this specializes to the higher Toda brackets we would like to construct. In this case, $W\Gamma$ is just an $(n + 1)$ -dimensional cube. As we are not interested in other indexing categories, we will just use the cubes and do not make use of the bar resolution in our construction.

In what follows, we assume familiarity with model categories. Hovey’s book [Hov99] provides a good reference. Other than in Quillen’s original treatment of model categories [Qui67], we will follow Hovey in assuming our model categories to have all small limits and colimits as well as functorial factorizations. $\mathcal{T}op$ will be the category of compactly generated weak Hausdorff spaces, and $\mathcal{T}op_*$ will be the pointed version. The reason for working with these categories of spaces is that $\mathcal{T}op_*$ is a closed symmetric monoidal model category [Hov99, Corollary 4.2.12]. We will often use stable topological model categories that are built on $\mathcal{T}op_*$. See Appendix A for a brief review.

5.1. Cube systems. Some notation is needed to state the next definition. Let $N(\mathcal{U})$ be the nerve of a small category \mathcal{U} . We write $d_i : N_n(\mathcal{U}) \rightarrow N_{n-1}(\mathcal{U})$ for the i th simplicial face map, and $d_i^{\text{fr}} : N_n(\mathcal{U}) \rightarrow N_i(\mathcal{U})$ and $d_i^{\text{ba}} : N_n(\mathcal{U}) \rightarrow N_i(\mathcal{U})$ for the simplicial ‘front face’ and the ‘back face’ maps. In our notation for sequences of composable maps from the preceding sections, this means for example $d_{n-i}(f_1, \dots, f_n) = (f_1, \dots, f_i f_{i+1}, \dots, f_n)$ and $d_i^{\text{ba}}(f_1, \dots, f_n) = (f_1, \dots, f_i)$. Here we resist from reversing the notation for (f_1, \dots, f_n) to make these formulas more intuitive, since this would be inconsistent with our previous convention, which was chosen since (f_1, \dots, f_n) frequently arose from a projective resolution.

For $n \geq 1$, we denote the n -fold cartesian product of the unit interval by I^n and define I^0 to be the one point space. For $\epsilon \in \{0, 1\}$ and $1 \leq i \leq n$, we have a structure map

$$\epsilon^i : I^{n-1} \rightarrow I^n, \quad (t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{i-1}, \epsilon, t_i, \dots, t_{n-1}).$$

With $\epsilon, \omega \in \{0, 1\}$, these maps satisfy the relation $\epsilon^i \omega^{j-1} = \omega^j \epsilon^i$ if $1 \leq i < j \leq n$. We write $\text{sk}_i I^n$ for the i -skeleton of I^n in the obvious CW-structure. When we consider I^n with $n \geq 1$ as a pointed space, we take $(1, \dots, 1)$ as the basepoint.

For a stable topological model category \mathcal{C} , we will work with the set of maps $\mathcal{T}op(I^n, \text{Map}_{\mathcal{C}}(X, Y))$. The enriched composition μ of \mathcal{C} induces a composition

$$\begin{aligned} \mu_{p,q} : \mathcal{T}op(I^p, \text{Map}_{\mathcal{C}}(Y, Z)) \times \mathcal{T}op(I^q, \text{Map}_{\mathcal{C}}(X, Y)) &\rightarrow \mathcal{T}op(I^{p+q}, \text{Map}_{\mathcal{C}}(X, Z)), \\ (b, b') &\mapsto ((t_1, \dots, t_{p+q}) \mapsto (x \mapsto b_{(t_1, \dots, t_p)}(b'_{(t_{p+1}, \dots, t_{p+q})}(x))))). \end{aligned}$$

The associativity of the enriched composition implies that $\mu_{p,q+r}(\text{id} \times \mu_{q,r})$ and $\mu_{p+q,r}(\mu_{p,q} \times \text{id})$ correspond under the coherence isomorphism for associativity of the 3-fold cartesian product in $\mathcal{T}op$.

The zero map is a canonical basepoint for $\text{Map}_{\mathcal{C}}(X, Y)$. When a possibly different map $g : X \rightarrow Y$ is used as the basepoint, we write $(\text{Map}_{\mathcal{C}}(X, Y), g)$ for the resulting pointed space.

For $\epsilon \in \{0, 1\}$ and $1 \leq i \leq p$, we have $(\epsilon^i)^* : \mathcal{T}op(I^p, T) \rightarrow \mathcal{T}op(I^{p-1}, T)$. If $1 \leq i \leq p + q$, these restrictions satisfy

$$(\epsilon^i)^* \mu_{p,q}(b, b') = \begin{cases} \mu_{p-1,q}((\epsilon^i)^* b, b') & \text{if } i \leq p, \\ \mu_{p,q-1}(b, (\epsilon^{i-p})^* b') & \text{if } i > p. \end{cases}$$

Definition 5.2. Let \mathcal{U} be a small full subcategory of the homotopy category of a stable topological model category \mathcal{C} . A cube system for \mathcal{U} consists of the following data: for every object X of \mathcal{U} , there is a cofibrant and fibrant object $\Phi(X)$ of \mathcal{C} and an isomorphism $\varphi_X : X \rightarrow \Phi(X)$ in $\text{Ho}(\mathcal{C})$. We write $\Phi(\mathcal{U})$ for the set of all those objects. Furthermore, for $0 \leq j \leq n$ there are maps

$$b^j : N_{j+1}(\mathcal{U}) \rightarrow \coprod_{X, Y \in \Phi(\mathcal{U})} \mathcal{T}op(I^j, \text{Map}_{\mathcal{C}}(X, Y))$$

such that

- (i) $b^0(f : X \rightarrow Y) \in \text{Map}_{\mathcal{C}}(\Phi(X), \Phi(Y))$ and $b^0(f)$ represents f on the model category level, i.e., $\varphi_Y^{-1} b^0(f) \varphi_X = f$ in $\text{Ho}(\mathcal{C})$.
- (ii) If one of the maps f_i in (f_1, \dots, f_{j+1}) is a zero map, then $b^j(f_1, \dots, f_{j+1})$ has the zero map in \mathcal{C} as constant value.
- (iii) For $j \geq 1$ and $1 \leq i \leq j$, $b^{j-1} d_{j+1-i} = (1^i)^* b^j$.
- (iv) For $j \geq 1$ and $1 \leq i \leq j$,

$$\mu_{i-1, j-i}(b^{i-1} \times b^{j-i})(d_i^{\text{ba}} \times d_{j+1-i}^{\text{fr}}) \Delta = (0^i)^* b^j,$$

where Δ is the diagonal and $\mu_{i-1, j-i}$ is explained above.

By (iii), $b^j(f_1, \dots, f_{j+1})$ maps the basepoint of I^j to $b^0(f_1 \cdots f_{j+1})$.

A 0-cube system chooses maps in the model category representing maps in the homotopy category. In general, it is not possible to arrange these choices such that $b^0(f_1) b^0(f_2) = b^0(f_1 f_2)$ holds. Nevertheless, these maps are homotopic, and unraveling (ii) and (iii) shows that a 1-cube system specifies a homotopy $b^1(f_1, f_2)$ between them. For $j \geq 2$, the $b^j(f_1, \dots, f_{j+1})$ encode coherence homotopies between different choices of representatives and coherence homotopies of lower degree. Figure 3 (compare [BM03, Figure 2.12]) illustrates the case $n = 3$. In the picture, we write $(f_j \cdots f_k)$ for $b^0(f_j \cdots f_k)$ and $(f_j \cdots f_{k-1}) \circ (f_k \cdots f_l)$ for $b^1(f_j \cdots f_{k-1}, f_k \cdots f_l)$.

Definition 5.3. In the situation of Definition 5.2, a *pre-n-cube system* for \mathcal{U} consists of an $(n - 1)$ -cube system for \mathcal{U} and a map

$$\widehat{b}^n : N_{n+1}(\mathcal{U}) \rightarrow \coprod_{X, Y \in \Phi(\mathcal{U})} \mathcal{T}op(\text{sk}_{n-1} I^n, \text{Map}_{\mathcal{C}}(X, Y))$$

such that \widehat{b}^n and the b^j for $j < n$ satisfy conditions (ii)-(iv) of Definition 5.2. This makes sense since (iii) and (iv) only involve the behavior on $\text{sk}_{n-1} I^n$.

Similar as above, $\widehat{b}^n(f_1, \dots, f_{n+1})$ maps the basepoint to $b^0(f_1 \cdots f_{n+1})$.

Lemma 5.4. *An $(n - 1)$ -cube system for \mathcal{U} can be extended to a pre-n-cube system. The restriction of $\widehat{b}^n(f_1, \dots, f_{n+1})$ to the subcubes $(0^i)(I^{n-1})$ for $1 < i < n$ is determined by the underlying $(n - 2)$ -cube system.*

Proof. Since $\text{sk}_{n-1} I^n$ is the union of the $(n-1)$ -dimensional subcubes $(0^i)(I^{n-1})$ and $(1^i)(I^{n-1})$, we define the restriction of \widehat{b}^n to these subcubes by

$$(1^i)^*\widehat{b}^n := b^{n-1}d_{n+1-i} \quad \text{and} \quad (0^i)^*\widehat{b}^n := \mu_{i-1,n-i}(b^{i-1} \times b^{n-i})(d_i^{\text{ba}} \times d_{n+1-i}^{\text{fr}})\Delta.$$

It remains to check that this is well defined on the intersections.

Let $1 \leq j < k \leq n$. On $1^k(1^j(I^{n-2})) = 1^j(1^{k-1}(I^{n-2}))$ we have

$$(1^{k-1})^*(1^j)^*\widehat{b}^n = b^{n-1}d_{n-(k-1)}d_{n+1-j} = b^{n-1}d_{n-j}d_{n+1-k} = (1^j)^*(1^k)^*\widehat{b}^n.$$

Next we check the compatibility on $1^k(0^j(I^{n-2})) = 0^j(1^{k-1}(I^{n-2}))$. A somewhat lengthy calculation involving the interchange formula for $\mu_{p,q}$ and $(1^i)^*$ mentioned above shows that both $(1^{k-1})^*(0^j)^*\widehat{b}^n$ and $(0^j)^*(1^k)^*\widehat{b}^n$ equal

$$\mu_{j-1,n-j-1}(b^{j-1} \times b^{n-j-1})(d_j^{\text{ba}} \times d_{n-k+1}d_{n+1-j}^{\text{fr}})\Delta.$$

The case of $(0^{k-1})^*(1^j)^*\widehat{b}^n$ and $(1^j)^*(0^k)^*\widehat{b}^n$ is similar. The remaining case of $(0^{k-1})^*(0^j)^*\widehat{b}^n$ and $(0^j)^*(0^k)^*\widehat{b}^n$ follows from the associativity of the maps $\mu_{p,q}$. \square

Proposition 5.5. *Let \mathcal{U} be a small n -split subcategory of the homotopy category of a stable topological model category \mathcal{C} . Then there exists an n -cube system for \mathcal{U} .*

Proof. We start with choosing the representing objects $\Phi(X)$, the isomorphisms φ_X and the representing maps $b^0(f)$, which is always possible. Here we consider $b^0(f)$ as an element of $\mathcal{T}op(I^0, \text{Map}_{\mathcal{C}}(\Phi(X), \Phi(Y)))$. For each pair of composable maps (f_1, f_2) , the maps $b^0(f_1f_2)$ and $b^0(f_1)b^0(f_2)$ are homotopic. After adjoining and forgetting the basepoint, a homotopy $(I^1)_+ \wedge \Phi(X_2) \rightarrow \Phi(X_0)$ gives rise to $b^1(f_1, f_2)$. This completes the 1-cube system.

Inductively, suppose we have constructed a j -cube system for $j < n$. By Lemma 5.4, it induces a pre- $(j+1)$ -cube system. In order to extend $\widehat{b}^{j+1}(f_1, \dots, f_{j+2})$ from $\text{sk}_j I^{j+1}$ to I^{j+1} , it suffices to know that it represents the trivial homotopy

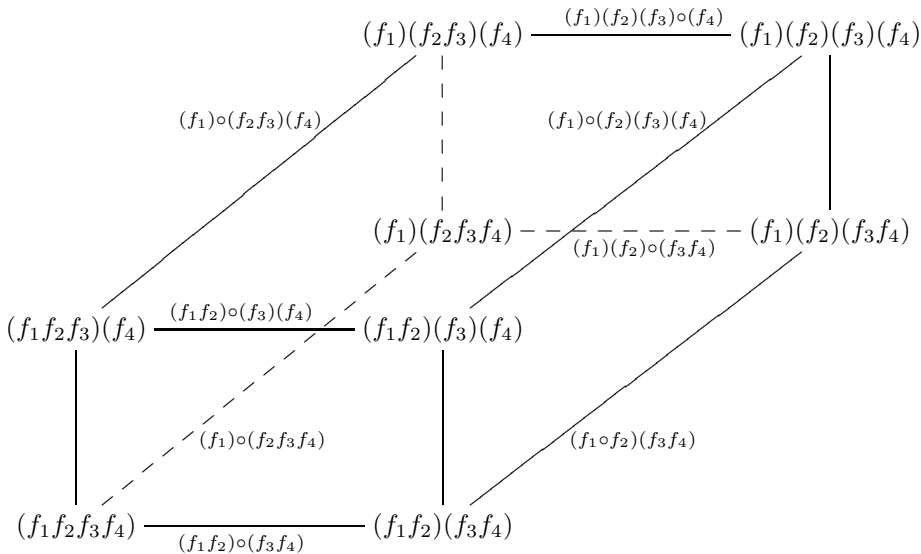


FIGURE 3. A 3-cube.

class in $\pi_j(\text{Map}_{\mathcal{C}}(\Phi(X_{j+2}), \Phi(X_0)), b^0(f_1 \cdots f_{j+2}))$. This group is isomorphic to $[S^j \wedge X_{j+2}, X_0]^{\text{Ho}(\mathcal{C})} \cong [X_{j+2}, X_0]_j^{\text{Ho}(\mathcal{C})}$ by means of the isomorphism $\sigma_{(f_1 \cdots f_{j+2})}$ of Proposition A.1, and hence trivial since \mathcal{U} is n -split. \square

5.6. The universal Toda bracket. We now put the data of a cube system together to get the desired cohomology class.

Construction 5.7. Let \mathcal{C} be a stable topological model category and let \mathcal{U} be a small n -split subcategory of $\text{Ho}(\mathcal{C})$. Then there is a well defined cohomology class $\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$ which determines by evaluation all $(n + 2)$ -fold Toda brackets of complexes of $n + 2$ composable maps in \mathcal{U} .

We choose an n -cube system for \mathcal{U} which is possible by Proposition 5.5, and extend it to a pre- $(n + 1)$ -cube system by Lemma 5.4. Then we define a normalized cochain $c \in \overline{\mathcal{C}}^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$ as follows. Its evaluation on a sequence of $(n + 2)$ composable maps $X_{n+2} \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} \cdots \xrightarrow{f_1} X_0$ in \mathcal{U} is the image of the homotopy class of the (pointed) map $\widehat{b}^{n+1}(f_1, \dots, f_{n+2})$ under the chain of isomorphisms

$$\begin{aligned} & [\text{sk}_n I^{n+2}, (\text{Map}_{\mathcal{C}}(\Phi(X_{n+2}), \Phi(X_0)), b^0(f_1 \cdots f_{n+2}))]^{\text{Ho}(\mathcal{T}op_*)} \\ & \quad \downarrow \sigma_{(f_1 \cdots f_{n+2})} \\ & [\text{sk}_n I^{n+1} \wedge X_{n+2}, X_0]^{\text{Ho}(\mathcal{C})} \xrightarrow{\cong} [X_{n+2}[n], X_0]^{\text{Ho}(\mathcal{C})} \end{aligned}$$

We show in Lemma 5.8 that c is a cocycle. Lemma 5.9 verifies that its cohomology class does not depend on the choice of the cube system. In Proposition 6.1, we show that the evaluation of c on a complex of maps in \mathcal{U} is an element of the Toda bracket of that complex. With the comparison of the indeterminacies in Remark 4.14, it follows that the evaluation of the *cohomology class* $\gamma_{\mathcal{U}}$ of c on a complex in \mathcal{U} yields its Toda bracket. This is why we call the well defined cohomology class $\gamma_{\mathcal{U}}$ the *universal Toda bracket* of \mathcal{U} .

The Homotopy Addition Theorem [Hu53] will be our tool in the proof of the next two lemmas. We use $T = \{(\epsilon, i) | \epsilon \in \{0, 1\}, 1 \leq i \leq n + 2\}$ as an indexing set for the $(n + 1)$ -dimensional subcubes of I^{n+2} . It is a disjoint union of $T_+ = \{(\epsilon, i) | (-1)^{n+\epsilon+1} = 1\}$ and $T_- = \{(\epsilon, i) | (-1)^{n+\epsilon+1} = -1\}$.

Given a pointed topological space K with abelian fundamental group and a pointed map $f: \text{sk}_n I^{n+2} \rightarrow K$, the Homotopy Addition Theorem states that

$$\sum_{(\epsilon, i) \in T_+} [(\epsilon^i)^* f] - \sum_{(\epsilon, i) \in T_-} [(\epsilon^i)^* f] = 0.$$

Here $[(\epsilon^i)^* f]$ is the homotopy class of the restriction of f to $(\epsilon^i)^*(\text{sk}_n I^{n+2})$, or the image of the operation of a path to the basepoint of $\text{sk}_n I^{n+2}$ on $[(\epsilon)^* f]$ if $(\epsilon^i)^*(\text{sk}_n I^{n+2})$ doesn't contain the basepoint of $\text{sk}_n I^{n+2}$. This is well defined since two such paths are homotopic if $n > 1$ and the π_1 -action is trivial for $n = 1$ as π_1 is assumed to be abelian. Our source for this formulation of the theorem is [Bre97, VII.9.6]. The signs result from specifying an orientation through choosing 'even' and 'odd' faces T_+ and T_- ; compare [Whi66].

Lemma 5.8. *The cochain c of Construction 5.7 is a cocycle.*

Proof. We fix a sequence of $(n + 3)$ composable maps (f_1, \dots, f_{n+3}) in \mathcal{U} . As in Lemma 5.4, the pre- $(n + 1)$ -cube system induces

$$e(f) = e(f_1, \dots, f_{n+3}) : \text{sk}_n I^{n+2} \rightarrow \text{Map}_{\mathcal{C}}(\Phi(X_3), \Phi(X_0))$$

with $(1^i)^* e(f) := (\widehat{b}^{n+1} d_{n+3-i})(f_1, \dots, f_{n+3})$ and

$$(0^i)^* e(f) := (\mu_{i-1, n+2-i}(\widehat{b}^{i-1} \times \widehat{b}^{n+2-i})(d_i^{\text{ba}} \times d_{n+3-i}^{\text{fr}})\Delta)(f_1, \dots, f_{n+3}),$$

where \widehat{b}^i is the restriction of b^i if $i < n + 1$.

We apply to this map the Homotopy Addition Theorem mentioned above to get

$$0 = \sum_{(\epsilon, i) \in T_+} [(\epsilon^i)^* e(f)] - \sum_{(\epsilon, i) \in T_-} [(\epsilon^i)^* e(f)]$$

in $\pi_n(\text{Map}_{\mathcal{C}}(\Phi(X_{n+3}), \Phi(X_0)), b^0(f_1 \cdots f_{n+3}))$. For $n = 1$, this uses that \mathcal{C} being stable implies π_1 is abelian. For $1 < i < n + 2$, the restrictions $(0^i)^* e(f)$ extend to maps $(0^i)^*(\text{sk}_{n+1} I^{n+1})$ as $\widehat{b}^{i-1} = b^{i-1}$ and $\widehat{b}^{n+2-i} = b^{n+2-i}$. Hence their homotopy classes vanish. Now we apply the isomorphism $\sigma_{(f_1 \cdots f_{n+3})}$ to the sum and use the fact that σ is additive and invariant under the action of basepoint changing paths. Hence

$$\begin{aligned} 0 &= (f_1)_* c(f_2, \dots, f_{n+3}) + \sum_{i=1}^{n+2} (-1)^i c(f_1, \dots, f_i f_{i+1}, \dots, f_{n+3}) \\ &\quad + (-1)^{n+3} (f_{n+3})^* c(f_1, \dots, f_{n+2}). \end{aligned}$$

□

Lemma 5.9. *The cohomology class $\gamma_{\mathcal{U}}$ of Construction 5.7 does not depend on the choice of a cube system.*

Proof. In the first step we assume we are given a second $(n - 1)$ -cube system $(\underline{\Phi}, \underline{\varphi}, \underline{b}^j)$ for \mathcal{U} . We show that it extends to an n -cube system giving the same cohomology class.

For every object X in \mathcal{U} , our data specifies an isomorphism $\Phi(X) \rightarrow \underline{\Phi}(X)$ in \mathcal{U} . We realize it by g_X in \mathcal{C} and its inverse by g'_X . For $f_1 : X_1 \rightarrow X_0$ in \mathcal{U} , we know $g_{X_0} b^0(f_1) g'_{X_1} = \underline{b}^0(f_1)$ in \mathcal{U} . Let $h^0(f) : I^1 \rightarrow \text{Map}_{\mathcal{C}}(\underline{\Phi}(X_1), \underline{\Phi}(X_0))$ be a homotopy between them. With similar arguments as in Proposition 5.5, one can iterate the construction to find maps

$$h^j : N_{j+1}(\mathcal{U}) \rightarrow \coprod_{X, Y \in \mathcal{U}} \text{Top}(I^{j+1}, \text{Map}_{\mathcal{C}}(\underline{\Phi}(X), \underline{\Phi}(Y)))$$

for $j < n$ with $(0^1)^* h^j = (g_{X_0})_*(g'_{X_{j+1}})^* b^j$ and $(1^1)^* h^j = \underline{b}^j$. For $j = n$, we use the homotopy extension property of $(\text{sk}_n I^{n+1}) \setminus (1^1)(I^n) \rightarrow I^{n+1}$ to find an h^n whose restriction to $(1^1)(I^n)$ defines \underline{b}^n . When we form the pre- $(n + 1)$ -cube systems associated to the two cube systems, the h^j assemble to a homotopy between $(g_{X_0})_*(g'_{X_{j+2}})^* \widehat{b}^{n+1}$ and \underline{b}^{n+1} . Hence σ associates the same cocycle to them.

Now suppose we are given two n -cube systems for \mathcal{U} . The first part shows that we can assume their underlying $(n - 1)$ -cube systems to coincide. Let $T_{\text{dev}} = \{(1, i) \mid 1 \leq i \leq n + 1\} \cup \{(0, 1), (0, n + 1)\}$. Lemma 5.4 shows that the associated pre- $(n + 1)$ -cube systems can only deviate on the faces specified by T_{dev} .

Let A be the space obtained from $\text{sk}_n I^{n+1}$ by gluing for each $(\epsilon, i) \in T_{\text{dev}}$ one copy of $I^n \cup_{\text{sk}_{n-1} I^n} I^n$ along the right hand side copy of I^n to $(\epsilon^i)(I^n)$. Let

$i: \text{sk}_n I^{n+1} \rightarrow A$ be the canonical inclusion and let $\underline{i}: \text{sk}_n I^{n+1} \rightarrow A$ be the injection which maps $(\epsilon^i)I^n$ to the left copy of I^n in the pushout. Then the two pre-cube systems induce a map

$$\widehat{a}: N_{j+2}(\mathcal{U}) \rightarrow \coprod_{X,Y \in \mathcal{U}} \text{Top}(A, \text{Map}_{\mathcal{C}}(\Phi(X), \Phi(Y)))$$

with $i^*\widehat{a} = \widehat{b}^{n+1}$ and $\underline{i}^*\widehat{a} = \widehat{\underline{b}}^{n+1}$. By the slightly different incarnation of the Homotopy Addition Theorem [Bre97, VII.9.5], the evaluations on (f_1, \dots, f_{n+2}) satisfy

$$\widehat{\underline{b}}^{n+1} = \widehat{b}^{n+1} + \sum_{(\epsilon,i) \in T_{\text{dev}} \cap T_+} [(\iota_{(\epsilon,i)})^*\widehat{a}] - \sum_{(\epsilon,i) \in T_{\text{dev}} \cap T_-} [(\iota_{(\epsilon,i)})^*\widehat{a}],$$

where $\iota_{(\epsilon,i)}: I^n \cup_{\text{sk}_{n-1} I^n} I^n \rightarrow A$ is the inclusion which belongs to $(\epsilon, i) \in T_{\text{dev}}$. The signs arise in the same way as in the last lemma.

Let $a: N_{n+1}(\mathcal{U}) \rightarrow \coprod_{X,Y \in \mathcal{U}} \text{Top}(I^n \cup_{\text{sk}_{n-1} I^n} I^n, \text{Map}_{\mathcal{C}}(\Phi(X), \Phi(Y)))$ be the map which is b^n on the right and \underline{b}^n on the left copy of I^n . Applying $\sigma_{b^0(f_1 \dots f_{n+2})}$ to $[a(f_1, \dots, f_{n+1})]$ defines an $(n+1)$ -cochain $\bar{a} \in \overline{\mathcal{C}}^{n+1}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$. Next we apply $\sigma_{b^0(f_1 \dots f_{n+2})}$ to the sum formula above to get the desired equation

$$c(f_1, \dots, f_{n+2}) = \underline{c}(f_1, \dots, f_{n+2}) + (\delta \bar{a})(f_1, \dots, f_{n+2}).$$

As in Lemma 5.8, the orientations of the subcubes imply the signs needed for the coboundary formula. □

The last lemma completes the proof of $\gamma_{\mathcal{U}}$ being well defined. For later use, we prove two more lemmas closely related to this construction.

Lemma 5.10. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a left Quillen functor between stable topological model categories \mathcal{C} and \mathcal{D} which is compatible with the topological structure. If \mathcal{U} and \mathcal{W} are small n -split subcategories of $\text{Ho}(\mathcal{C})$ and $\text{Ho}(\mathcal{D})$ such that G induces an equivalence $\mathcal{U} \rightarrow \mathcal{W}$ and an isomorphism $G^*([-, -]_n^{\text{Ho}(\mathcal{D})}) \cong [-, -]_n^{\text{Ho}(\mathcal{C})}$, then the induced isomorphism*

$$G^*: H^{n+2}(\mathcal{W}, [-, -]_n^{\text{Ho}(\mathcal{D})}) \rightarrow H^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$$

sends $\gamma_{\mathcal{W}}$ to $\gamma_{\mathcal{U}}$.

Proof. We apply G to the data of an n -cube system for \mathcal{U} . This almost gives an n -cube system for \mathcal{W} . The only missing part is that the objects $G(\Phi(X))$ are not necessarily fibrant. Similarly as in Lemma 5.9, one can construct a cube system for \mathcal{W} such that the resulting cocycle representing $\gamma_{\mathcal{W}}$ becomes, after applying G^* , equivalent to that of the cube system for \mathcal{U} . □

Lemma 5.11. *Let \mathcal{C} be a stable topological model category and let \mathcal{U} be a small n -split subcategory of $\text{Ho}(\mathcal{C})$. If $\gamma_{\mathcal{U}}$ is trivial, then the map b^n of any n -cube system for \mathcal{U} can be changed such that the resulting new n -cube system has the zero cochain as a representing cocycle. In particular, the modified n -cube system can be extended to an $(n + 1)$ -cube system.*

Proof. Since $\gamma_{\mathcal{U}} = 0$, there is an $e \in \overline{\mathcal{C}}^{n+1}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$ with $\delta(e) = c$. We model the n -sphere by gluing two copies of the n -cube I^n together along their boundaries. For every sequence of composable maps (f_1, \dots, f_{n+1}) in \mathcal{U} , there is a map $\widehat{e}(f_1, \dots, f_{n+1}): (I^n \cup_{\partial I^n} I^n) \rightarrow \text{Map}_{\mathcal{C}}(X_{n+1}, X_0)$ such that its restriction

to the left copy of I^n is $b^n(f_1, \dots, f_{n+1})$ and that $[\sigma_{b^0(f_1 \dots f_{n+1})}(\widehat{e}(f_1, \dots, f_{n+1}))] = e(f_1, \dots, f_{n+1})$. We define $\underline{b}^n(f_1, \dots, f_{n+1})$ to be the restriction of $\widehat{e}(f_1, \dots, f_{n+1})$ to the right copy of I^n . Similarly as in Lemma 5.9, the Homotopy Addition Theorem shows the assertion. □

6. COMPARING DEFINITIONS OF TODA BRACKETS

This section is devoted to the proof of

Proposition 6.1. *Let \mathcal{C} be a stable topological model category, let \mathcal{U} be an n -split subcategory of $\text{Ho}(\mathcal{C})$, and let $X_{n+2} \xrightarrow{f_{n+2}} X_{n+1} \xrightarrow{f_{n+1}} \dots \xrightarrow{f_1} X_0$ be a sequence of maps in \mathcal{U} with $f_i f_{i+1} = 0$ for $1 \leq i \leq n + 1$. Let c be the cocycle of Construction 5.7. Its evaluation on (f_1, \dots, f_{n+2}) lies in the Toda bracket $\langle f_1, \dots, f_{n+2} \rangle$ in the sense of Definition 4.4.*

Remark 6.2. Triple Toda brackets were introduced by Toda [Tod52, Tod62] to study the stable homotopy groups of spheres. Higher Toda brackets were introduced in the 1960's, and there are different approaches in the literature. One of them is Cohen's definition using *filtered objects* [Coh68, §2]. We used a variant of this for triangulated categories in Definition 4.4.

Another approach is Spanier's definition of higher Toda brackets [Spa63] using the concept of a *carrier*. A related concept is Klaus' definition of a pyramid [Kla01, 3.4], which is linked to Spanier's definition by [Kla01, Proposition 3.6]. Perhaps the most general approach to Toda brackets and other higher homotopy operations is that of Blanc and Markl [BM03], who define them as obstructions to realizing homotopy commutative diagrams by strictly commutative ones. Their definition of Toda brackets is related to Spanier's [BM03, Example 3.12].

In Lemma 6.3 below we will see that the evaluation of the universal Toda bracket can be interpreted as something similar to a pyramid in the sense of Klaus. Proposition 6.1 shows that this is equivalent to the Toda bracket defined via filtered objects in Definition 4.4. We work out the comparison as far as needed for our purposes in some detail since we were not able to find an appropriate reference in the literature which relates the different approaches.

For the rest of the section, we fix a \mathcal{U} and (f_1, \dots, f_{n+2}) with $f_i f_{i+1} = 0$ for $1 \leq i \leq n + 1$ as in the proposition. We also fix an n -cube system b^j of \mathcal{U} and write \widehat{b}^{n+1} for the associated pre- $(n + 1)$ -cube system and c for the cocycle defined in Construction 5.7. For simplicity, we denote the objects $\Phi(X)$ of \mathcal{C} chosen by the cube system also by X . The map $\widetilde{b}^j(f_1, \dots, f_{j+1}): (I^j)_+ \wedge X_{j+1} \rightarrow X_0$ will always be the adjoint of the map $b^j(f_1, \dots, f_{j+1})$.

We denote by $\widetilde{\partial I^{n+1}}$ the pointed space obtained from $\text{sk}_n I^{n+1}$ by collapsing all n -dimensional subcubes $(1^i)(I^n)$ with $1 \leq i \leq n + 1$ to the basepoint $(1, \dots, 1)$. The space $\widetilde{\partial I^{n+1}}$ is homeomorphic to an n -sphere. Since $f_i f_{i+1} = 0$, the map $\widehat{b}^{n+1}(f_1, \dots, f_{n+2})$ factors through the quotient map $\text{sk}_n I^{n+1} \rightarrow \widetilde{\partial I^{n+1}}$ and induces $\widetilde{c}^{n+1}(f_1, \dots, f_{n+2}): \widetilde{\partial I^{n+1}} \rightarrow (\text{Map}_{\mathcal{C}}(X_{n+2}, X_0), 0)$.

Lemma 6.3. *The map $\widetilde{c}^{n+1}(f_1, \dots, f_{n+2})$ represents the evaluation of c on the complex (f_1, \dots, f_{n+2}) in \mathcal{U} .*

Proof. This follows from Proposition A.1. □

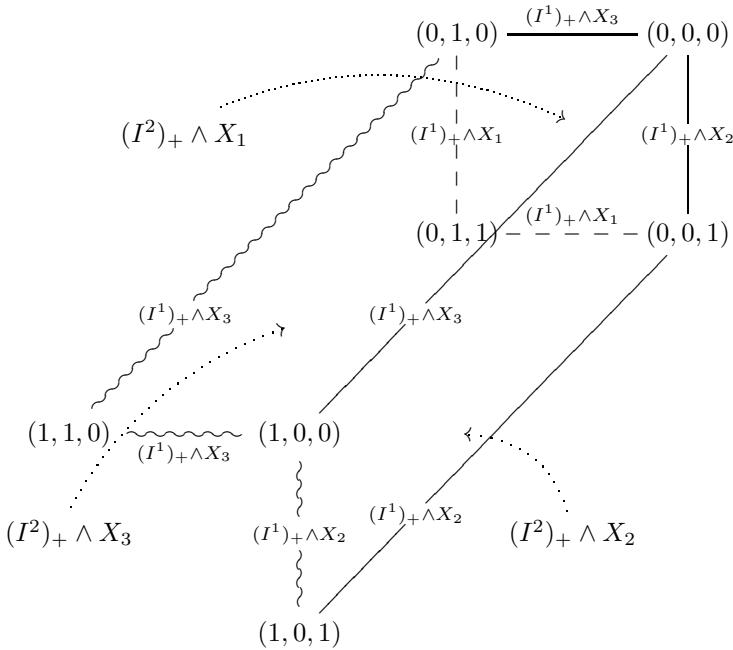


FIGURE 4. The object $F_2(f_2, f_3)$.

Depending on our chosen n -cube system and (f_1, \dots, f_{n+2}) , we now construct objects $F_j = F_j(f_2, \dots, f_{j+1})$ in \mathcal{C} for $j \leq n + 1$. Set

$$A_j = A'_j = \coprod_{1 \leq r < s \leq j+1} (I^{j-1})_+ \wedge X_s \quad \text{and} \quad B_j = \coprod_{1 \leq i \leq j+1} (I^j)_+ \wedge X_i.$$

The object F_j is the coequalizer of two maps $h, k: A_j \amalg A'_j \rightarrow B_j$ to be described next. For this, we think of the copies of I^j in B_j as the $j + 1$ subcubes $(0^i)(I^j)$ of I^{j+1} . The copies of I^{j-1} in A_j are thought of as the $(j - 1)$ -dimensional subcubes $(0^s)(0^r)(I^{j-1})$ of I^{j+1} , and the copies of I^{j-1} in A'_j are thought of as the $(j - 1)$ -dimensional subcubes $(0^s)(1^r)I^{j-1}$.

The map h is given by $(0^r)_+ \wedge X_s$ on the copy of $(I^{j-1})_+ \wedge X_s$ in A_j indexed by (r, s) , and by $(1^r)_+ \wedge X_s$ in the case of A'_j . The map k is the trivial map to the basepoint on A'_j . On the copy of $(I^{j-1})_+ \wedge X_s$ in A_j indexed by (r, s) , it is given by the product of 0^{s-1} and the map $\tilde{b}^{s-r-1}(f_{r+1}, \dots, f_s)$ using the last $(s - r - 1)$ -coordinates of I^{j-1} .

Example 6.4. The case $j = 2$, which becomes relevant for 4-fold Toda-brackets, is displayed in Figure 4. In the diagram, the lines of the shape $\sim\sim\sim$ mark the part which is collapsed to the basepoint. Thinking of all cubes as subcubes of I^3 , we glue the 3 objects $(I^2)_+ \wedge X_1$, $(I^2)_+ \wedge X_2$, and $(I^2)_+ \wedge X_3$ (indexed by $(0^i)(I^2)$ for $1 \leq i \leq 3$) together along two copies of $(I^1)_+ \wedge X_3$ (indexed by $(0^3)(0^2)(I^1)$ and $(0^3)(0^1)(I^1)$) and one copy of $(I^1)_+ \wedge X_2$ (indexed by $(0^2)(0^1)(I^1)$). Furthermore, we collapse two copies of $(I^1)_+ \wedge X_3$ (indexed by $(0^3)(1^2)(I^1)$ and $(0^3)(1^1)(I^1)$) and one copy of $(I^1)_+ \wedge X_2$ (indexed by $(0^2)(1^1)(I^1)$) to the basepoint.

Lemma 6.5. *The data of the cube system induces maps*

$$\xi_j : \widetilde{\partial I^{j+1}} \wedge X_{j+2} \rightarrow F_j(f_2, \dots, f_{j+1}) \quad \text{and} \quad \zeta_j : F_j(f_2, \dots, f_{j+1}) \rightarrow X_0.$$

If $j = n$, the composition $\zeta_n \xi_n$ coincides with $\widetilde{b}^{n+1}(f_1, \dots, f_{n+2})$ of Lemma 6.3.

Proof. We define $(0^i)^* \xi_j$ to be the composition of $(I^j)_+ \wedge X_{j+2} \rightarrow (I^j)_+ \wedge X_i$ given by the identity smashed with $\widetilde{b}^{j+1-i}(f_{i+1}, \dots, f_{j+2})$ using the last $(j + 1 - i)$ coordinates of the cube and the canonical map $(I^j)_+ \wedge X_i \rightarrow B_j \rightarrow F_j(f_2, \dots, f_{j+2})$. Its restriction along 1^{k-1} is trivial for $k > i$ since $\widetilde{b}^{j+1-i}(f_{i+1}, \dots, f_{j+2})$ can be replaced by the trivial map $b^{j-i}(f_{i+1}, \dots, f_k f_{k+1}, \dots, f_{j+2})$ there. Its restriction along 1^k is trivial for $k < i$ as well, since these subcubes are mapped to the part of $F_j(f_2, \dots, f_{j+1})$ which gets collapsed. The same arguments as in Lemma 5.4 show that the maps for different i coincide on the intersections. Therefore, we get an induced map $\xi_j : \widetilde{\partial I^{j+1}} \wedge X_{j+2} \rightarrow F_j(f_2, \dots, f_{j+1})$.

Next we define ζ_j . On the copy $(I^j)_+ \wedge X_i$ of B_j indexed by i with $1 \leq i \leq j + 1$, we take $\widetilde{b}^{i-1}(f_1, \dots, f_i)$ using the first $(i - 1)$ coordinates of I^j . This is compatible with the identifications of the coequalizer.

To see $\zeta_n \xi_n = \widetilde{c}^{n+1}(f_1, \dots, f_{n+2})$, we look at its restriction to the subcube $(0^i)(I^n)$. Here $\zeta_n \xi_n$ is $\widetilde{b}^{n+1-i}(f_{i+1}, \dots, f_{n+2})$ using the last $(n + 1 - i)$ coordinates of the cube, composed with $\widetilde{b}^{i-1}(f_1, \dots, f_i)$ using the first $(i - 1)$ -coordinates. This is the adjoint of the map which defines \widetilde{b}^{n+1} on $(0^i)(I^n)$. □

Lemma 6.6. *The object $F_{j+1} := F_{j+1}(f_2, \dots, f_{j+2})$ can be constructed from $F_j := F_j(f_2, \dots, f_{j+1})$ as the mapping cylinder of the map from F_j to the cone C of the map $\xi_j : \widetilde{\partial I^{j+1}} \wedge X_{j+2} \rightarrow F_j$. The inclusion of F_j into the mapping cylinder therefore gives a map $\iota_j : F_j \rightarrow F_{j+1}$.*

Proof. Let \widetilde{I}^{j+1} denote the quotient of I^{j+1} obtained by collapsing the $(1^i)(I^j)$ with $1 \leq i \leq j + 1$ to a point. Then there is a canonical map $\widetilde{\partial I^{j+1}} \rightarrow \widetilde{I}^{j+1}$, and we can interpret \widetilde{I}^{j+1} as a cone on $\widetilde{\partial I^{j+1}}$. Hence we can model the mapping cone of ξ_j by the pushout of $\widetilde{I}^{j+1} \wedge X_{j+2} \leftarrow \widetilde{\partial I^{j+1}} \wedge X_{j+2} \xrightarrow{\xi_j} F_j$.

To replace the map from F_j to the cone by a cofibration, we need a cylinder object for F_j . One choice for this is $(I^1)_+ \wedge F_j$, which amounts to adding one additional coordinate to each $(I^i)_+ \wedge X_k$ that occurred in the construction of F_j . We choose it to be the last coordinate. Hence the mapping cylinder of $F_j \rightarrow C$ is weakly equivalent to the pushout of

$$\widetilde{I}^{j+1} \wedge X_{j+2} \leftarrow \widetilde{\partial I^{j+1}} \wedge X_{j+2} \xrightarrow{((0^1)_+ \wedge F_j)(\xi_j)} (I^1)_+ \wedge F_j.$$

The pushout of this diagram is isomorphic to F_{j+1} as defined above. The case $j = 1$ can easily be deduced from Figure 4. □

Corollary 6.7. *For $j \leq n$, there is a distinguished triangle*

$$X_{j+2}[j] \xrightarrow{\xi_j} F_j(f_2, \dots, f_{j+1}) \xrightarrow{\iota_j} F_{j+1}(f_2, \dots, f_{j+2}) \xrightarrow{\pi_{j+1}} X_{j+2}[j + 1]$$

in $\text{Ho}(\mathcal{C})$.

Proof. This follows from the last lemma and the definition of the distinguished triangles in the homotopy category of a stable model category [Hov99, Chapter 7]. □

Lemma 6.8. *For $0 \leq j \leq n$, the map $X_{j+2}[j] \xrightarrow{\xi_j} F_j(f_2, \dots, f_{j+1}) \xrightarrow{\pi_j} X_{j+1}[j]$ equals $f_{j+2}[j]$ in $\text{Ho}(\mathcal{C})$.*

Proof. The last lemma says that we have a cofibration sequence

$$F_{j-1}(f_2, \dots, f_j) \xrightarrow{\iota_{j-1}} F_j(f_2, \dots, f_{j+1}) \xrightarrow{\pi_j} X_{j+1}[j].$$

Hence π_j is up to homotopy the map from F_j to its quotient obtained by collapsing every subcube $(I^j)_+ \wedge X_i$ of B_j indexed by $2 \leq i \leq j$ to the $(j - 1)$ -dimensional subcube along which it is glued to $(I^j)_+ \wedge X_{j+1}$. To examine the homotopy class of $\pi_j \xi_j$, we hence only need to know what ξ_j does on the subcube $(I^j) \wedge X_{j+2}$ indexed by $j + 1$. As it is defined to be the map $b^0(f_{j+2})$ on that, we are done. \square

Lemma 6.9. *If we consider the $F_j(f_2, \dots, f_{j+1})$ as objects of $\text{Ho}(\mathcal{C})$, the sequence*

$$* \rightarrow X_1 \xrightarrow{\iota_0} F_1(f_2) \xrightarrow{\iota_1} \dots \xrightarrow{\iota_{n-1}} F_n(f_2, \dots, f_{n+1})$$

gives $F_n(f_2, \dots, f_{n+1})$ the structure of an $(n + 1)$ -filtered object in $\{f_2, \dots, f_{n+1}\}$.

Proof. We prove that $F_j(f_2, \dots, f_{j+1})$ is a $(j + 1)$ -filtered object in $\{f_2, \dots, f_{j+1}\}$ by induction. This is clear for $j = 1$. Using that $\pi_j: F_j(f_2, \dots, f_{j+1}) \rightarrow X_{j+1}[j]$ plays the role of the map σ_X for X being the $(j + 1)$ -filtered object $F_j(f_2, \dots, f_{j+1})$, we apply Lemma 4.3 and Corollary 6.7 to see that $F_{j+1}(f_2, \dots, f_{j+2})$ is a $(j + 2)$ -filtered object in $\{f_2, \dots, f_{j+1}, \pi_j \xi_j[-j]\}$. The last lemma provides the remaining fact $(\pi_j \xi_j)[-j] = f_{j+2}$. \square

Proof of Proposition 6.1. As we have seen in Lemma 6.5, the composition $\zeta_n \xi_n$ is the map $\tilde{c}^m(f_1, \dots, f_{n+2})$. Hence it represents the evaluation of c by Lemma 6.3.

Let σ_X and σ'_X denote the structure maps of the filtered object $F_n(f_2, \dots, f_{n+1})$. Lemma 6.9 implies $\sigma_X \xi_n = f_n[n - 2]$. The definition of ζ_n and the fact that σ'_X is the composition $X_1 \xrightarrow{\iota_{(1,1,\dots,1)}} (I^{n+1})_+ \wedge X_1 \rightarrow F_n(f_2, \dots, f_{n+1})$ show $\zeta_n \sigma'_X = f_1$. Hence $\xi_n \zeta_n$ is an element of $\langle f_1, \dots, f_{n+2} \rangle$. \square

7. RELATION TO k -INVARIANTS OF CLASSIFYING SPACES

We saw that the evaluation of the universal Toda bracket on a complex is the Toda bracket of the complex. Since it may as well be evaluated on arbitrary sequences of maps, it will carry more information than just that about the Toda bracket in general. We will now exhibit how its evaluation on a sequence of automorphisms can be expressed. When we apply our theory to ring spectra in Theorem 8.7, this will give us information about the units of ring spectra (and the units of their matrix rings), rather than only the information about zero divisors encoded in the Toda brackets.

A motivation for this comes from Igusa’s results [Igu82] about the first k -invariant of the space $BGL_\infty(Q\Omega X_+)$, which is related to Waldhausen’s algebraic K -theory of spaces [Wal78]: Igusa shows that the first k -invariant of a connected space X is determined by a cohomology class $k_1^H(\Omega X)$ in the cohomology of the monoid $\pi_0(\Omega X)$ with coefficients in $H_1(X)$, where the class $k_1^H(\Omega X)$ is constructed from the A_4 -part of the A_∞ -structure of ΩX [Igu82, B, Property 1.1]. This observation is also used in [BD89, Example 4.9, Theorem 3.10].

We fix a stable topological model category \mathcal{C} , an n -split subcategory \mathcal{U} of $\text{Ho}(\mathcal{C})$ for some $n \geq 1$, and an n -cube system defining $\gamma_{\mathcal{U}}$. We also fix an object X of \mathcal{U} and denote the representing cofibrant and fibrant object of \mathcal{C} which the cube

systems chooses as well by X . Consider the topological space $\text{Map}_{\mathcal{C}}(X, X)$ which is pointed by the zero map in \mathcal{C} . Its homotopy groups are

$$\pi_i(\text{Map}_{\mathcal{C}}(X, X), 0) \cong [S^i, \text{Map}_{\mathcal{C}}(X, X)]^{\text{Ho}(\mathcal{T}op_*)} \cong [S^i \wedge X, X]^{\text{Ho}(\mathcal{C})} \cong [X, X]_i^{\text{Ho}(\mathcal{C})}.$$

As \mathcal{U} is n -split, $\pi_i(\text{Map}_{\mathcal{C}}(X, X), 0)$ is concentrated in degrees divisible by n .

The enriched composition in the category \mathcal{C} equips $\text{Map}_{\mathcal{C}}(X, X)$ with the structure of a topological monoid, and we refer to the composition as the multiplication. Under the identification above, the composition of maps in $\text{Ho}(\mathcal{C})$ corresponds to the multiplication of $\text{Map}_{\mathcal{C}}(X, X)$.

The set $\pi_0(\text{Map}_{\mathcal{C}}(X, X))$ inherits a monoid structure from $\text{Map}_{\mathcal{C}}(X, X)$, and $\text{Map}_{\mathcal{C}}(X, X)^\times$ denotes the union of all path components of $\text{Map}_{\mathcal{C}}(X, X)$ which are invertible with respect to the multiplication on $\pi_0(\text{Map}_{\mathcal{C}}(X, X))$. Therefore, $\text{Map}_{\mathcal{C}}(X, X)^\times$ is a group-like topological monoid.

As the basepoint of $\text{Map}_{\mathcal{C}}(X, X)^\times$ we take id_X , the unit of the multiplication, since the basepoint 0 of $\text{Map}_{\mathcal{C}}(X, X)$ is not in $\text{Map}_{\mathcal{C}}(X, X)^\times$. There are isomorphisms

$$\pi_i(\text{Map}_{\mathcal{C}}(X, X), 0) \cong \pi_i(\text{Map}_{\mathcal{C}}(X, X), \text{id}_X) \cong \pi_i(\text{Map}_{\mathcal{C}}(X, X)^\times, \text{id}_X)$$

for $i \geq 1$. The second isomorphism is the restriction to the path component. For the first one, we take the isomorphism σ_{id_X} of Proposition A.1 combined with an adjunction.

A topological monoid G has a classifying space BG , defined via the bar construction. It comes with a map $\omega: G \rightarrow \Omega BG$. If the topological monoid G is group-like, that is, the monoid $\pi_0(G)$ is a group, then ω is a weak equivalence. In our example we get a space $B\text{Map}_{\mathcal{C}}(X, X)^\times$ with

$$\pi_i(B\text{Map}_{\mathcal{C}}(X, X)^\times) \cong \begin{cases} ([X, X]^{\text{Ho}(\mathcal{C})})^\times & i = 1, \\ 0 & 1 < i \leq n, \\ [X, X]_n^{\text{Ho}(\mathcal{C})} & i = n + 1. \end{cases}$$

The left action of $\pi_1(B\text{Map}_{\mathcal{C}}(X, X)^\times)$ on $\pi_{n+1}(B\text{Map}_{\mathcal{C}}(X, X)^\times)$ corresponds under this isomorphism to the conjugation action $g \cdot \lambda = (g^{-1})_*(g)_*\lambda$ of $[X, X]_0^\times$ on $[X[n], X]_0$.

Theorem 7.1. *Let \mathcal{C} be a stable topological model category, let \mathcal{U} be a small n -split subcategory of $\text{Ho}(\mathcal{C})$, and let X be a cofibrant and fibrant object of \mathcal{C} representing an object in \mathcal{U} . Then the restriction map*

$$\Theta: H^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})}) \rightarrow H^{n+2}(\pi_1(B\text{Map}_{\mathcal{C}}(X, X)^\times), \pi_{n+1}(B\text{Map}_{\mathcal{C}}(X, X)^\times))$$

of Proposition 2.10 sends the universal Toda bracket $\gamma_{\mathcal{U}}$ to the first k -invariant of $B\text{Map}_{\mathcal{C}}(X, X)^\times$ not vanishing for dimensional reasons.

We need an auxiliary lemma for the proof. Let G be a group-like topological monoid and let $\omega: G \rightarrow \Omega BG$ be the map to the group completion. Let $\varphi: (S^n, \text{pt}) \rightarrow (G, g)$ be any map. The adjoint of $\omega\varphi$ is a map from the unreduced suspension $S(S^n)$ to BG . It represents an element in $\pi_{n+1}(BG)$. On the other hand, we can choose an $h \in G$ such that gh is in the component of 1_G . If v is a path from gh to 1_G , we get $[\varphi \cdot h]^v \in \pi_n(G, 1_G)$. This does not depend on v and h , as G being a topological monoid implies the π_1 -action on $\pi_n(G)$ to be trivial. Composing with ω , we get $\omega_*([\varphi \cdot h]^v) \in \pi_{n+1}(BG)$.

Lemma 7.2. *These two ways to associate an element of $\pi_{n+1}(BG)$ to $\varphi: S^n \rightarrow G$ are equivalent.*

Proof. One can use the homotopy extension property to see that $S(S^n) \rightarrow BG$ is homotopic to a map which sends $[0, 1] \times \{\text{pt}\}$ to the basepoint and represents $\omega_*([\varphi \cdot h]^v)$. □

Proof of Theorem 7.1. We fix a sequence of $(n + 2)$ automorphisms (f_1, \dots, f_{n+2}) of X in \mathcal{U} . Let $f = f_1 \cdots f_{n+2}$ be their composition. We write \widehat{b}_f^{n+1} for the map $\widehat{b}^{n+1}(f_1, \dots, f_{n+2}): \text{sk}_n I^{n+1} \rightarrow (\text{Map}_C(X, X), b^0(f))$.

By the definition of c in Construction 5.7 and the restriction map Θ in Proposition 2.10, the evaluation of $\Theta(\gamma_{\mathcal{U}})$ on (f_1, \dots, f_{n+2}) is

$$(f^{-1})^* \sigma_{b^0(f)}(\widehat{b}_f^{n+1}) = \sigma_{b^0(f)b^0(f^{-1})}((\widehat{b}_f^{n+1}) \cdot b^0(f^{-1})) \in [X, X]_n^{\text{Ho}(C)}.$$

We need to examine the image of the homotopy class of this map under

$$\begin{aligned} [X, X]_n^{\text{Ho}(C)} &\cong [S^n, (\text{Map}_C(X, X), 0)] \cong [S^n, (\text{Map}_C(X, X)^\times, \text{id}_X)] \\ &\cong [S^{n+1}, B \text{Map}_C(X, X)^\times]. \end{aligned}$$

Choose a path v from $b^0(f)b^0(f^{-1})$ to id_X in $\text{Map}_C(X, X)$. Then

$$\sigma_{\text{id}_X}^{-1} \sigma_{b^0(f)b^0(f^{-1})}[\widehat{b}_f^{n+1} \cdot b^0(f^{-1})] = \sigma_{\text{id}_X}^{-1} \sigma_{\text{id}_X}[\widehat{b}_f^{n+1} \cdot b^0(f^{-1})]^v = [\widehat{b}_f^{n+1} \cdot b^0(f^{-1})]^v$$

holds by Proposition A.1, and the last term represents $\Theta(\gamma_{\mathcal{U}})(f_1, \dots, f_{n+2})$.

For the next step we use Lemma 7.2. It says that the image of $[\widehat{b}_f^{n+1} \cdot b^0(f^{-1})]^v$ in $\pi_{n+1}(B \text{Map}_C(X, X)^\times)$ is represented by the adjoint of $\omega \widehat{b}_f^{n+1}$ considered as a map $S(\text{sk}_n I^{n+1}) \rightarrow B \text{Map}_C(X, X)^\times$. The unreduced suspension of $S(\text{sk}_n I^{n+1})$ is homotopy equivalent to $\partial \Delta^{n+2}$, the boundary of an $(n + 2)$ -simplex, and we will now explain the resulting map $a_f = a(f_1, \dots, f_{n+2}): \partial \Delta^{n+2} \rightarrow B \text{Map}_C(X, X)^\times$.

We denote the set of vertices of $\partial \Delta^{n+2}$ by $\{1, \dots, n + 3\}$. Then a_f maps every vertex i of $\partial \Delta^{n+2}$ to the basepoint. The 1-simplex of $\partial \Delta^{n+2}$ containing the two vertices $i < j$ is mapped to $B \text{Map}_C(X, X)^\times$ using the path associated to $b^0(f_i \cdots f_{j-1})$ via the map $\omega: \text{Map}_C(X, X)^\times \rightarrow \Omega B \text{Map}_C(X, X)^\times$. Hence every 0-dimensional subcube of $\text{sk}_n I^{n+1}$ specifies a path from the initial to the terminal vertex of $\partial \Delta^{n+2}$. This path runs through the vertex containing $i < j$ if the term $b^0(f_i \cdots f_{j-1})$ occurs in the restriction of the cube system to that 0-dimensional subcube.

The 2-simplices of $\partial \Delta^{n+2}$ containing $i < j < k$ are mapped to $B \text{Map}_C(X, X)^\times$ by the homotopy between the paths associated to $b^0(f_i \cdots f_{j-1})$, $b^0(f_j \cdots f_{k-1})$ and $b^0(f_i \cdots f_{k-1})$ which we get from $b^1(f_i \cdots f_{j-1}, f_j \cdots f_{k-1})$. This time, the 1-dimensional subcubes of $\text{sk}_n I^{n+1}$ correspond to the 2-simplices of $\partial \Delta^{n+2}$.

The case $n = 1$ is displayed in Figure 5, whose right part also appears in [Igu82, B.2.2]. The situation gets a little bit more involved if $n > 1$, since an $(n + 1)$ -cube has $2(n + 1)$ subcubes of dimension n , but the $(n + 2)$ -simplex has only $(n + 3)$ sub- $(n + 1)$ -simplices. In this case, the $2(n + 1) - (n + 3) = n - 1$ codimension 1 subcubes $(0^k)(I^n)$ of $\text{sk}_n I^{n+1}$ with $1 < k < n + 1$ do not contribute new information to the map defined on the boundary of the $(n + 2)$ -simplex. The reason is that the restriction of the pre- $(n + 1)$ -cube system to these subcubes is already determined by the underlying $(n - 1)$ -cube system. We recall that the restriction to these

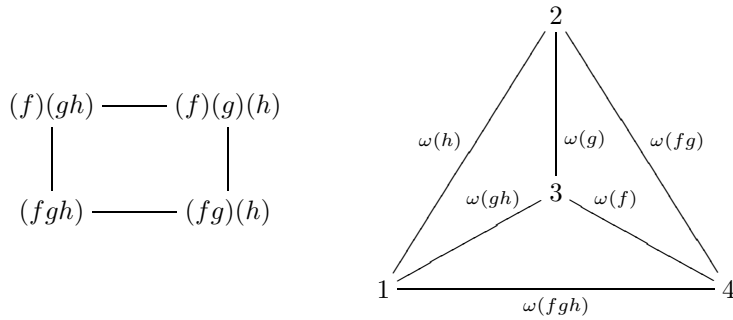


FIGURE 5. The square ... and the associated simplex.

subcube is built from $b^{k-1}(f_1, \dots, f_k)$ and $b^{n+1-k}(f_{k+1}, \dots, f_{n+2})$. Accordingly, it corresponds to the restriction of the map $a_f : \partial\Delta^{n+2} \rightarrow B\text{Map}_{\mathcal{C}}(X, X)^\times$ to the two simplices with the vertices $\{1, \dots, k\}$ and $\{k+1, \dots, n+2\}$. The maps on all other n -dimensional subcubes induce maps on one of the $(n+1)$ -simplices of $\partial\Delta^{n+2}$.

The cochain $(f_1, \dots, f_{n+2}) \mapsto [a(f_1, \dots, f_{n+2})]$ is a representing cocycle for the first k -invariant as described by Eilenberg and Mac Lane in [EML49, §19]. In that reference, the authors also give an equivalence of this definition of a k -invariant to a more commonly used one. \square

7.3. Coherent vanishing of k -invariants. The last theorem says that the vanishing of $\gamma_{\mathcal{U}}$ implies the vanishing of the first k -invariant of the space $B\text{Map}_{\mathcal{C}}(X, X)^\times$ for every cofibrant and fibrant object X of \mathcal{C} representing an object of \mathcal{U} . For our applications, we need a stronger statement in a special case.

For the rest of this section, we assume that \mathcal{C} is a stable topological model category in which all objects are fibrant. Furthermore, we assume the n -split subcategory \mathcal{U} of $\text{Ho}(\mathcal{C})$ to have a fixed object X^1 such that all other objects of \mathcal{U} are finite sums of copies of X^1 . Such a q -fold sum will be denoted by X^q .

We choose a cofibrant (and automatically fibrant) object of \mathcal{C} representing X^1 and also denote it by X^1 . Let the object X^q in \mathcal{U} be represented by the q -fold coproduct $X^1 \vee \dots \vee X^1$ of copies of X^1 in \mathcal{C} , which we also denote by X^q . The difference between objects in $\text{Ho}(\mathcal{C})$ and \mathcal{C} will be emphasized by writing \vee for the coproduct in \mathcal{C} and \oplus for the coproduct in $\text{Ho}(\mathcal{C})$.

We get maps $\text{Map}_{\mathcal{C}}(X^q, X^q) \rightarrow \text{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})$ by adding id_{X^1} on the last summand. The restriction of these maps to the set of invertible path components is multiplicative with respect to the monoid structure. Hence we get a map $t_q : B\text{Map}_{\mathcal{C}}(X^q, X^q)^\times \rightarrow B\text{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})^\times$ for every q .

Here it is convenient to work in a setup with all objects fibrant, since the otherwise necessary fibrant replacement of the sum $X^q \vee X^1$ would mean that we only get a homotopy class of maps $\text{Map}_{\mathcal{C}}(X^q, X^q) \rightarrow \text{Map}_{\mathcal{C}}(X^{q+1}, X^{q+1})$, rather than an actual map.

Denote the mapping telescope of $B\text{Map}_{\mathcal{C}}(X^1, X^1)^\times \rightarrow B\text{Map}_{\mathcal{C}}(X^2, X^2)^\times \rightarrow \dots$ by $B\text{Map}_{\mathcal{C}}^\infty(X, X)^\times$. The vanishing of the first k -invariant of this space does not follow from the vanishing of the first k -invariant of all spaces $B\text{Map}_{\mathcal{C}}(X^q, X^q)^\times$ in general, since this vanishing does not have to be compatible with the maps t_q . The next lemma provides a sufficient condition for this stronger statement.

Lemma 7.4. *Let \mathcal{C} be a stable topological model category in which all objects are fibrant. Let \mathcal{U} be a small n -split subcategory of $\text{Ho}(\mathcal{C})$ such that*

- (i) *there is an object X^1 in \mathcal{U} such that all objects of \mathcal{U} are finite sums of copies of X^1 ,*
- (ii) *$\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$ vanishes,*
- (iii) *$[X, X]_i^{\text{Ho}(\mathcal{C})} = 0$ for all objects X of \mathcal{U} if $i > n$, and*
- (iv) *$H^{n+1}(\mathcal{U}, [(-) \oplus X^q, (-) \oplus X^q]_n^{\text{Ho}(\mathcal{C})}) = 0$ for all $q \geq 1$.*

Then the space $B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times}$ has a vanishing k -invariant k^{n+2} , i.e., it has the Eilenberg-Mac Lane space $|B\pi_1(B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times})|$ as a retract up to homotopy.

Proof. We will construct a section up to homotopy of the π_1 -isomorphism from $B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times}$ to the Eilenberg-Mac Lane space $|B\pi_1(B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times})|$. Condition (iii) implies that it is enough to specify it on $|\text{sk}_{n+2} B\pi_1(B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times})|$. Since $B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times}$ is constructed as a mapping telescope, it is enough to define π_1 -isomorphisms $s_q: |\text{sk}_{n+2} B\pi_1(B\text{Map}_{\mathcal{C}}(X^q, X^q)^{\times})| \rightarrow B\text{Map}_{\mathcal{C}}(X^q, X^q)^{\times}$ with $t_q s_q \simeq s_{q+1}(t_q)_*$.

As we have seen in the proof of the Theorem 7.1, the b^j specify maps from all $(j + 1)$ -simplices of $|B\pi_1(B\text{Map}_{\mathcal{C}}^{\infty}(X, X)^{\times})|$ to $B\text{Map}_{\mathcal{C}}(X^q, X^q)^{\times}$. By the compatibility of the cube system, they assemble to a π_1 -isomorphism s_q^{n+1} defined on the $(n + 1)$ -skeleton of $|B\pi_1(B\text{Map}_{\mathcal{C}}(X^q, X^q)^{\times})|$

In general, $t_q s_q^{n+1} \simeq s_{q+1}^{n+1}(t_q)_*$ will not hold. But without loss of generality, we can build this condition into the cube system: for all $j \leq n$, we require in the inductive construction of the cube system the map $b^j(f_1 \oplus X^1, \dots, f_{j+1} \oplus X^1)$ to be $(-\vee X^1)b^j(f_1, \dots, f_{j+1})$.

We could extend the s_q^{n+1} to the desired maps s^q if we knew that our cube system extends to an $(n + 1)$ -cube system. By Lemma 5.11, we know that $\gamma_{\mathcal{U}} = 0$ implies that we can change the maps $b^n(f_1, \dots, f_{n+1})$ to achieve this. Unfortunately, the modified b^n does not have to be compatible with the t_q anymore.

To fix this, we inductively construct a sequence $e_k \in \overline{\mathcal{C}}^{n+1}(\mathcal{U}, [-, -]_n^{\text{Ho}(\mathcal{C})})$ for $k \geq 0$ with $\delta(e_k) = c$. Since $\gamma_{\mathcal{U}} = 0$, we can find e_0 with $\delta(e_0) = c$. Assume we have built e_k with $\delta(e_k) = c$ and consider $e_k^0, e_k^1 \in \overline{\mathcal{C}}^{n+1}(\mathcal{U}, [(-) \oplus X^{k+1}, (-) \oplus X^{k+1}]_n^{\text{Ho}(\mathcal{C})})$ given by

$$e_k^{\epsilon}(f_1, \dots, f_{n+1}) = \begin{cases} e(f_1 \oplus X^{k+1}, \dots, f_{n+1} \oplus X^{k+1}) & \text{if } \epsilon = 0, \\ (-\vee X^{k+1})e(f_1, \dots, f_{n+1}) & \text{if } \epsilon = 1. \end{cases}$$

We claim $\delta(e_k^0 - e_k^1) = 0$. This follows from b^n being compatible and

$$0 = \delta(e_0^{\epsilon})(f_1, \dots, f_{n+2}) + c(f_1 \oplus X^1, \dots, f_{n+1} \oplus X^1) \text{ for } \epsilon \in \{1, 2\},$$

which holds since the cochain e_k with $\delta(e_k) = c$ can be used to change b^n as in Lemma 5.11. So by (iv), there is an a_k with $\delta(a_k) = e_k^1 - e_k^0$. Now we define e_{k+1} by adding $\delta(a_k)(f_1, \dots, f_{n+1})$ to $e_k(f_1 \oplus X^{k+1}, \dots, f_{n+1} \oplus X^{k+1})$, and leaving e_k unchanged on sequences (f_1, \dots, f_{n+1}) not of this form. Since $\delta^2(a_k) = 0$, we have $\delta(e_{k+1}) = c$. This finishes the construction of the e_k .

We say that a sequence of $(n + 1)$ composable maps (f_1, \dots, f_{n+1}) in \mathcal{U} has filtration k if k is the maximal integer such that there exist maps (f'_1, \dots, f'_{n+1}) with $f_i = f'_i \oplus X^k$. If we change our compatible n -cube system b^j by the cochain e_{k+1} with the procedure of Lemma 5.11, we get an n -cube system b_{k+1}^j which is

compatible on all sequences of filtration up to k , and which extends to an $(n + 1)$ -cube system. Since b_{k+1}^j and b_{k+2}^j coincide on the sequences of filtration up to k , this is enough to get the desired map on the telescope. \square

The hypotheses of the lemma may appear unrealistic at first glance. Nevertheless, condition (iv), probably the strongest condition, will reduce to the vanishing of a single Mac Lane cohomology group when we apply it in Proposition 8.14. This is much easier to verify as to ensure a coherent vanishing of the k -invariants by dealing with the associated obstructions on the level of group cohomology.

8. THE UNIVERSAL TODA BRACKET OF A RING SPECTRUM

We now apply the results of the preceding sections to ring spectra based on topological spaces. These can be the S -algebras of [EKMM97], the symmetric ring spectra of [HSS00] (see [MMSS01] for a version based on topological spaces), or the orthogonal ring spectra introduced in [MMSS01]. For a ring spectrum R , the module category $\text{Mod-}R$ is a topological model category. If \mathcal{C} is the underlying category of spectra, $-\wedge R: \mathcal{C} \rightarrow \text{Mod-}R$ is a left Quillen functor. Hence $\pi_*(R) = [R, R]_*^{\text{Ho}(\text{Mod-}R)} \cong [\mathbb{S}, R]_*^{\text{Ho}(\mathcal{C})}$, and R is compact in $\text{Ho}(\text{Mod-}R)$.

Recall that $\pi_*(R)$ is n -sparse if it is concentrated in degrees divisible by n .

Theorem 8.1. *Let R be a ring spectrum such that $\pi_*(R)$ is n -sparse for some $n \geq 1$. There exists a well defined cohomology class $\gamma_R^{n+2} \in \text{HML}_{n-\text{sp}}^{n+2, -n}(\pi_*(R))$. By evaluation, it determines the $(n + 2)$ -fold Toda bracket of every complex of $(n + 2)$ composable maps between finitely generated free n -sparse $\pi_*(R)$ -modules. For a $\pi_*(R)$ -module M which admits a resolution by such modules, the product $\text{id}_M \cup \gamma_R^{n+2}$ is the unique realizability obstruction $\kappa_{n+2}(M) \in \text{Ext}_{\pi_*(R)}^{n+2, -n}(M, M)$.*

Proof. Let \mathcal{U} be the full subcategory of $\text{Ho}(\text{Mod-}R)$ given by finite sums of copies of the free module of rank 1 which are shifted by integral multiples of n . Construction 5.7 provides a cohomology class $\gamma_{\mathcal{U}} \in H^{n+2}(\mathcal{U}, [-, -]_n)^{\text{Ho}(\text{Mod-}R)}$. The equivalence $\mathcal{U} \rightarrow F(\pi_*(R), n)$ induces an isomorphism $H^{n+2}(\mathcal{U}, [-, -]_n)^{\text{Ho}(\text{Mod-}R)} \cong \text{HML}_{n-\text{sp}}^{n+2, -n}(\pi_*(R))$. The image of $\gamma_{\mathcal{U}}$ in the latter group defines γ_R . By Construction 5.7 and Theorem 4.15, it has the desired properties. \square

We call γ_R^{n+2} the universal Toda bracket of R . Theorem 1.1 is the $n = 1$ case of the last theorem.

Remark 8.2. The restriction to modules with a resolution by finitely generated free $\pi_*(R)$ -modules can be avoided. By [JP91, §2 and Corollary 3.11], replacing $F(\pi_*(R), n)$ by a larger full small additive subcategory doesn't change the cohomology. We choose such a \mathcal{D} so that it consists only of free modules, and that it contains all modules from a given free resolution of M . Then there is a subcategory \mathcal{U} in $\text{Ho}(\text{Mod-}R)$ equivalent to \mathcal{D} that gives rise to a γ_R for which $\text{id}_M \cup \gamma_R$ is defined and equals the obstruction. However, there is no small \mathcal{D} which works for all M simultaneously. Hence we keep the restriction to the M as stated in the theorem, as this seems to be the most natural choice.

The complex K -theory spectrum KU is a ring spectrum [EKMM97, VIII, Theorem 4.2] such that $\pi_*(KU) \cong \mathbb{Z}[u^{\pm 1}]$ with u of degree 2. Its 4-fold universal Toda bracket is an element of $\text{HML}_{2-\text{sp}}^{4, -2}(\pi_*(KU))$, which is isomorphic to $\text{HML}^4(\mathbb{Z})$ by

Lemma 2.8 and therefore isomorphic to $\mathbb{Z}/2$ by [FP98]. We compute $\gamma_{KU}^4 \neq 0$ in Proposition 8.15

By [EKMM97, VIII, Theorem 4.2] or [Joa01], the real K -theory spectrum KO is a ring spectrum. Its graded ring of homotopy groups is given by

$$\pi_*(KO) = \mathbb{Z}[\eta, \omega, \beta^{\pm 1}] / (2\eta, \eta^3, \eta\omega, \omega^2 - 4\beta) \quad \text{with } |\eta| = 1, |\omega| = 4, \text{ and } |\beta| = 8.$$

The universal Toda bracket $\gamma_{KO} \in \text{HML}^{3,-1}(\pi_*(KO))$ of KO is non-trivial, as KO has non-trivial Toda brackets. For the reader's convenience, we recall the well known computation of the easiest example:

Lemma 8.3. *The Toda bracket $\langle 2, \eta, 2 \rangle$ in $\pi_*(KO)$ is defined, has trivial indeterminacy, and contains η^2 .*

Proof. As $2\eta = 0 = \eta 2$ and $\pi_2(KO)$ is 2-torsion, the first two statements hold. The ring spectra map $\mathbb{S} \rightarrow KO$ is a π_i -isomorphism for $0 \leq i \leq 2$, so it suffices to calculate the corresponding Toda bracket for the sphere spectrum. This can be either taken from [Tod62] or computed directly, following [Tod71, Theorem 6.1]: Suppose $0 \in \langle 2, \eta, 2 \rangle$. This would imply the existence of a 4-cell complex X with 2, η and 2 as attaching maps. We consider $H^*(X, \mathbb{Z}/2)$. Since Sq^1 detects 2 and Sq^2 detects η , the existence of X implies that $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1$ acts non-trivially on the bottom dimensional class in $H^*(X, \mathbb{Z}/2)$. But $\text{Sq}^1 \text{Sq}^2 \text{Sq}^1 = \text{Sq}^2 \text{Sq}^2$, and $\text{Sq}^2 \text{Sq}^2$ applied to the bottom class of $H^*(X, \mathbb{Z}/2)$ is trivial for dimensional reasons. \square

The class γ_{KO} detects non-trivial realizability obstructions:

Lemma 8.4. *The first realizability obstruction κ_3 of the $\pi_*(KO)$ -module $\pi_*(KO) \otimes \mathbb{Z}/2$ does not vanish. Hence $\pi_*(KO) \otimes \mathbb{Z}/2$ cannot be the homotopy of a KO -module spectrum.*

Proof. Write M for $\pi_*(KO) \otimes \mathbb{Z}/2$. There is a distinguished triangle $KO \xrightarrow{\cdot 2} KO \rightarrow C(2) \rightarrow KO[1]$ in $\text{Ho}(\text{Mod-}KO)$ which induces a long exact sequence in homotopy. Since M is the cokernel of multiplication with 2 on $\pi_*(KO)$, there is an injection $\iota: M \rightarrow \pi_*(C(2))$. As the two copies of $\pi_*(KO)$ in the long exact sequence are free modules of rank one, $\kappa_3(M)$ vanishes if and only if ι splits. Hence it is enough to show $\pi_2(C(2)) \cong \mathbb{Z}/4$.

From the long exact sequence, we see that $\pi_2(C(2))$ is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let $\rho \in \pi_2(C(2))$ be a lift of $\eta \in \pi_1(KO)$ along the epimorphism $\pi_2(C(2)) \rightarrow \pi_1(KO)$. Consider

$$\begin{array}{ccccccc} KO[-1] & \longrightarrow & C(2)[-1] & & & & \\ \tau \uparrow & & \rho \uparrow & \searrow & & & \\ KO[1] & \xrightarrow{\cdot 2} & KO[1] & \xrightarrow{\eta} & KO & \xrightarrow{\cdot 2} & KO. \end{array}$$

As we have observed in Remark 4.5, a map τ such that the left square commutes is an element of $\langle 2, \eta, 2 \rangle$. Hence $2\rho = 0$ would imply the contradiction $0 = \tau \in \langle 2, \eta, 2 \rangle$. Therefore, ρ cannot be 2-torsion, and $\pi_2(C(2)) \cong \mathbb{Z}/4$. \square

Remark 8.5. The same argument as in the last lemma shows the corresponding statement about the connective real K -theory spectrum. This contradicts [Wol98, Theorem 20]. The reason is an error in [Wol98, 14.1]. In this construction, the author assumes ku_* to be flat as a ko_* -module, which does not hold. Accordingly, the generalization [Wol98, Theorem 21] is false as well.

8.6. Universal Toda brackets and k -invariants. Let R^q denote a cofibrant and fibrant object of $\text{Mod-}R$ representing the free R -module spectrum of rank q . We write $\text{GL}_q R$ for the space $\text{Map}_{\text{Mod-}R}(R^q, R^q)^\times$ considered in Section 7.3. This definition of the ‘general linear group’ of a ring spectrum R is an important ingredient for the construction of the algebraic K -theory of R in the sense of Waldhausen [Wal78], if his definition is interpreted in the modern language of ring spectra [EKMM97, VI.7]. We will encounter the algebraic K -theory of ring spectra in Proposition 8.14.

Theorem 8.7. *Let R be a ring spectrum such that $\pi_*(R)$ is n -sparse for some $n \geq 1$. For $q \geq 1$, the restriction map*

$$\text{HML}_{n\text{-sp}}^{n+2,-n}(\pi_*(R)) \rightarrow H^{n+2}(\pi_1(B\text{GL}_q R), \pi_{n+1}(B\text{GL}_q R))$$

sends the universal Toda bracket γ_R^{n+2} of R to the first k -invariant of the space $B\text{GL}_q R$ not vanishing for dimensional reasons.

Proof. Since $B\text{GL}_q R = B\text{Map}_{\text{Mod-}R}(R^q, R^q)^\times$, this follows from Theorem 7.1 and the description of the restriction map in Corollary 2.11. \square

Before applying this theorem to examples, we describe the image γ_R^{n+2} in the ungraded cohomology group $\text{HML}^{n+2}(\pi_0(R), \pi_n(R))$.

Theorem 8.8. *Let R be a ring spectrum with $\pi_*(R)$ concentrated in degrees 0 and n for some $n \geq 1$. There is a universal Toda bracket $\gamma_R^{n+2} \in \text{HML}^{n+2}(\pi_0(R), \pi_n(R))$. It determines all $(n+2)$ -fold Toda brackets in $\pi_*(R)$ and the realizability obstruction $\kappa_{n+2}(M)$ of a $\pi_*(R)$ -module M which admits a resolution by finitely generated free n -sparse $\pi_*(R)$ -modules. The restriction map*

$$\text{HML}^{n+2}(\pi_0(R), \pi_n(R)) \rightarrow H^{n+2}(\pi_1(B\text{GL}_q R), \pi_{n+1}(B\text{GL}_q R))$$

sends γ_R to the first k -invariant of $B\text{GL}_q R$ not vanishing for dimensional reasons.

Proof. The proof uses the same arguments as that of Theorem 8.1. This time, \mathcal{U} has the finite sums of (unshifted) copies of R as objects. It is equivalent to $F(\pi_0(R))$. The isomorphism $H^{n+2}(\mathcal{U}, [-, -]_n^{\text{Ho}(C)}) \cong \text{HML}^{n+2}(\pi_0(R), \pi_n(R))$ induced by the equivalence enables us to define γ_R^{n+2} in the latter group. \square

Proposition 8.9. *Let R be a ring spectrum such that $\pi_*(R)$ is n -sparse. Let $R_{\geq 0}$ be its connective cover and let $P_n(R_{\geq 0})$ be the first non-trivial Postnikov section of $R_{\geq 0}$. The restriction $\text{HML}_{n\text{-sp}}^{n+2,-n}(\pi_*(R)) \rightarrow \text{HML}^{n+2}(\pi_0(R), \pi_n(R))$ sends the universal Toda bracket of R to the one of $P_n(R_{\geq 0})$.*

Proof. Let \mathcal{U} be the subcategory of $\text{Ho}(\text{Mod-}R)$ given by the finite sums of copies of R which are shifted by integral multiples of n . The class γ_R^{n+2} was defined by applying Construction 5.7 to \mathcal{U} . If \mathcal{U}_0 is the subcategory of \mathcal{U} of finite unshifted copies of R , the map from the graded to the ungraded Mac Lane cohomology is induced by the restriction along the inclusion $\mathcal{U}_0 \rightarrow \mathcal{U}$.

Let $\mathcal{U}_{\geq 0}$ be the subcategory of $\text{Ho}(\text{Mod-}R_{\geq 0})$ which is given by the finite sums of unshifted copies of $R_{\geq 0}$. The left Quillen functor $-\wedge_{R_{\geq 0}} R: \text{Mod-}R_{\geq 0} \rightarrow \text{Mod-}R$ induces an equivalence between $\mathcal{U}_{\geq 0}$ and \mathcal{U}_0 , since the induced map on homotopy groups

$$\text{Mod-}\pi_*(R_{\geq 0}) \rightarrow \text{Mod-}\pi_*(R), \quad M \mapsto M \otimes_{\pi_*(R_{\geq 0})} \pi_*(R)$$

restricts to an equivalence between the subcategories of unshifted copies of the free module of rank 1. Lemma 5.10 shows that this equivalence maps the universal Toda bracket of \mathcal{U}_0 to the one of $\mathcal{U}_{\geq 0}$.

A similar argument applied to $-\wedge_{R_{\geq 0}} P_n(R_{\geq 0})$ shows that $\gamma_{\mathcal{U}_0}$ equals the universal Toda bracket of the subcategory of $\text{Ho}(P_n R_{\geq 0})$ given by the finite sums of unshifted copies of $P_n R_{\geq 0}$. By Theorem 8.8, this is $\gamma_{P_n R_{\geq 0}}^{n+2}$. \square

We consider the example KO again. The restriction of the universal Toda bracket γ_{KO} to $\text{HML}^3(\pi_0(KO), \pi_1(KO)) \cong \text{HML}^3(\mathbb{Z}, \mathbb{Z}/2)$ is $\gamma_{P_1 KO_{\geq 0}}$. The latter group is $\mathbb{Z}/2$ [Lod98, Proposition 13.4.23]. We show that the image of γ_{KO} is the non-zero element, thereby proving once more $\gamma_{KO} \neq 0$. Since $P_1 KO_{\geq 0} \cong P_1 ko \cong P_1 \mathbb{S}$, this is a statement about the sphere spectrum, and computations of Igusa [Igu82] imply

Proposition 8.10. *The universal Toda bracket $\gamma_{P_1 \mathbb{S}}$ of the first Postnikov section of the sphere spectrum is the non-zero element in $\text{HML}^3(\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. Let \mathcal{H}_q^m be the topological monoid of self homotopy equivalences of q copies of the m -sphere. Suspension induces a map $\mathcal{H}_q^m \rightarrow \mathcal{H}_q^{m+1}$, which is $(m - 1)$ -connected by the Freudenthal suspension theorem.

Let $B\mathcal{H}_q^m$ be the classifying space of \mathcal{H}_q^m . The map $\text{colim}_m B\mathcal{H}_q^m \rightarrow BGL_q \mathbb{S}$ is a homotopy equivalence by [EKMM97, Proposition VI.8.3]. From [Igu82] (also compare [BD89, (7.6)]), we know that the first k -invariant of $B\mathcal{H}_q^m$ is non-trivial for $q \geq 4$ and $m \geq 3$. The increasing connectivity of the maps in the colimit system therefore implies that the first k -invariant of $BGL_q \mathbb{S}$ does not vanish for $q \geq 4$. Hence the first k -invariant of $BGL_q P_1 \mathbb{S}$ is non-trivial as well. By Theorem 8.8, $\gamma_{P_1 \mathbb{S}}$ has to be non-trivial since the $\text{HML}^3(\mathbb{Z}, \mathbb{Z}/2) \rightarrow H^3(\pi_1(BGL_q P_1(\mathbb{S})), \pi_2(BGL_q P_1(\mathbb{S})))$ sends it to this k -invariant. \square

Focusing on a ring spectrum with polynomial homotopy again, Proposition 8.9 implies

Corollary 8.11. *Let R be a ring spectrum with $\pi_*(R) \cong (\pi_0(R))[u^{\pm 1}]$ for a central unit u in degree n . The isomorphism $\text{HML}_{n-\text{sp}}^{n+2, -n}(\pi_*(R)) \rightarrow \text{HML}^{n+2}(\pi_0(R))$ of Lemma 2.8 sends the universal Toda bracket γ_R^{n+2} to the one of the first non-trivial Postnikov section of its connective cover.*

This reduces the computation of γ_{KU}^4 to that of $\gamma_{P_2 ku}^4$.

Remark 8.12. A ring spectrum R with only two homotopy groups $\pi_0(R)$ and $\pi_n(R)$ has a first k -invariant in the group $\text{Der}^{n+1}(\pi_0 R, \pi_n R) \cong \text{THH}^{n+2}(\pi_0 R, \pi_n R)$ [Laz01, DS06]. Since $\text{THH}^{n+2}(\pi_0 R, \pi_n R) \cong \text{HML}^{n+2}(\pi_0 R, \pi_n R)$ and the universal Toda brackets coincide with the k -invariant in the examples $P_1 \mathbb{S}$ and $P_2 ku$, we expect the universal Toda brackets of first non-trivial Postnikov sections to coincide with these k -invariants in general. We don't have a proof for this. The difficult point is that these two groups are only related by a chain of isomorphisms, and we do not know how to identify the k -invariant or the universal Toda bracket in the intermediate steps.

A proof of this statement would not only be interesting for the computation of universal Toda brackets, it would also relate the first k -invariant of a ring spectrum R with the Toda brackets of R and the first k -invariants of the spaces $BGL_q R$ in a very explicit way.

8.13. A relation to K -theory of ring spectra. For a connective ring spectrum R , there is a map $R \rightarrow H(\pi_0(R))$ from R to the Eilenberg-Mac Lane spectrum of $\pi_0(R)$ which is the identity on π_0 . In view of the last remark, we expect the map $R \rightarrow H(\pi_0(R))$ to split in the homotopy category of ring spectra if R has only two non-trivial homotopy groups and a vanishing universal Toda bracket. Though we are not able to prove this statement, the following proposition will provide a weaker result.

We briefly recall the definition of the algebraic K -theory of a ring spectrum R , following [EKMM97, VI]. To avoid technical difficulties, we assume our ring spectrum R to be an S -algebra in the sense of [EKMM97]. Since all objects in the category of R -modules are fibrant in this case, we obtain maps $BGL_q R \rightarrow BGL_{q+1} R$ as described in Section 7.3.

Let $BGL R$ be the (homotopy) colimit of the spaces $BGL_q R$ with respect to these maps. We apply Quillen's plus construction to the space $BGL R$ to obtain $(BGL R)^+$. For $i \geq 1$, algebraic K -groups of R can be defined as $K_i(R) = \pi_i((BGL R)^+)$. We will not need $K_0(R)$, which has to be defined separately. If R is an Eilenberg-Mac Lane spectrum of a discrete ring A , this definition recovers the algebraic K -groups $K_*(A)$ of A in the sense of Quillen [EKMM97, VI, Theorem 4.3].

We will later need the fact that the algebraic K -theory construction increases connectivity by 1. Recall that map $R \rightarrow R'$ of ring spectra is n -connected if the induced map $\pi_i(R) \rightarrow \pi_i(R')$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. If $R \rightarrow R'$ is n -connected, the induced map $K_i(R) \rightarrow K_i(R')$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$. This fact is due to the appearance of the bar construction in the definition of $K(R)$ and can be proved in a similar way as the corresponding statement about simplicial rings in [Wal78, Proposition 1.1].

Proposition 8.14. *Let R be a ring spectrum with homotopy groups concentrated in degrees 0 and n . Suppose that the universal Toda bracket γ_R^{n+2} of R is trivial and that $\text{HML}^{n+1}(\pi_0(R), \pi_n(R))$ vanishes. Then the map $K_i(R) \rightarrow K_i(\pi_0(R))$ induced by $R \rightarrow H(\pi_0(R))$ splits for all i .*

Proof. It is enough to show that $BGL R \rightarrow BGL(H(\pi_0(R)))$ splits up to homotopy, as this property is preserved by the plus construction. This is equivalent to the splitting of the map $BGL R \rightarrow |B\pi_1(BGL R)|$, since both maps are isomorphisms on the fundamental group and map into an Eilenberg-Mac Lane space.

We show this by applying Lemma 7.4 to the category \mathcal{U} used in the proof of Theorem 8.8. The first three conditions are obviously satisfied. It remains to show that $H^{n+1}(\mathcal{U}, [(-) \oplus R^q, (-) \oplus R^q]_n) = 0$.

The category \mathcal{U} is equivalent to $F(\pi_0(R))$. If we set $A = \pi_0(R)$ and $M = \pi_n(R)$, this equivalence induces an isomorphism between the last cohomology group and

$$H^{n+1}(F(A), \text{Hom}_A((-) \oplus A^q, (- \otimes_A M) \oplus M^q)).$$

By Lemma 2.4, this is isomorphic to $\text{HML}^{n+1}(\pi_0(R), \pi_n(R)) \cong 0$. □

Proposition 8.15. *The universal Toda bracket of KU is the non-zero element of $\text{HML}^4(\mathbb{Z}) \cong \mathbb{Z}/2$.*

Proof. By Corollary 8.11 it is enough to prove $\gamma_{P_2ku}^4 \neq 0$. We assume $\gamma_{P_2ku}^4 = 0$ and show that this leads to a contradiction.

As $\text{HML}^3(\mathbb{Z}) = 0$, Proposition 8.14 would imply that $K_3(P_2ku) \rightarrow K_3(\mathbb{Z})$ is split. This map is onto since $P_2ku \rightarrow H\mathbb{Z}$ is 2-connected. Since $ku \rightarrow P_2ku$ is 4-connected, $K_3(ku) \cong K_3(P_2ku)$, and our assumption implies that $K_3(ku) \rightarrow K_3(\mathbb{Z})$ is split. As the author learned from Ch. Ausoni and J. Rognes, there is a commutative diagram

$$\begin{CD} K_3(ku) @>>> \text{THH}_3(ku) \cong \mathbb{Z} \\ @VVV @VVV \\ \mathbb{Z}/48 \cong K_3(\mathbb{Z}) @>>> \text{THH}_3(\mathbb{Z}) \cong \mathbb{Z}/2 \end{CD}$$

in which the upper and the right arrow are epimorphisms [Aus]. Here the horizontal maps are the Bökstedt trace maps from algebraic K -theory to topological Hochschild homology, and the vertical maps are induced by $ku \rightarrow H(\pi_0(ku)) = H(\mathbb{Z})$. It follows that the lower map is an epimorphism as well. If the left map was split, this would mean that $\mathbb{Z}/48 \twoheadrightarrow \mathbb{Z}/2$ factors through \mathbb{Z} . This is the contradiction implying $\gamma_{P_2ku}^4 \neq 0$. \square

One may argue that it would be easier to derive $\gamma_{kU}^4 \neq 0$ by calculating $k^4(BGL_q P_2ku) \neq 0$ for some $q \geq 1$. Unfortunately, we don't know how to compute this k -invariant for $q > 1$. For $q = 1$, it is trivial. This follows for example from the restriction $\text{HML}^4(\mathbb{Z}) \rightarrow H^4(\mathbb{Z}/2, \mathbb{Z})$ being trivial.

Remark 8.16. The n th Morava K -theory spectrum at a prime p can be represented by a ring spectrum $K(n)$ [Laz01, §11]. Here ‘can be’ refers to the fact that there are non-equivalent choices for this structure.

Since $\pi_*(K(n)) \cong \mathbb{F}_p[v_n^{\pm 1}]$ with $|v_n| = 2(p^n - 1)$, we obtain a universal Toda bracket $\gamma_{K(n)}^{2p^n} \in \text{HML}^{2p^n}(\mathbb{F}_p) \cong \mathbb{Z}/p$. Hence for fixed p and varying n , the universal Toda brackets of the $K(n)$ are elements of $\text{HML}^*(\mathbb{F}_p)$ lying in the same degrees as the multiplicative generators of the graded ring $\text{HML}^*(\mathbb{F}_p)$ [FLS94].

The ring $\pi_*(K(n))$ is a graded field, that is, all $\pi_*(K(n))$ -modules are free. Hence it follows that all $\pi_*(K(n))$ -modules are realizable. Therefore, we cannot detect $\gamma_{K(n)}^{2p^n}$ by finding a non-vanishing realizability obstruction.

We do not know if the universal Toda bracket $\gamma_{K(n)}^{2p^n}$ depends on the choice of a model for $K(n)$, and we also do not know whether it is non-trivial or not. However, in view of Corollary 8.11 and Remark 8.12, we expect $\gamma_{K(n)}^{2p^n}$ to be non-trivial, as the connective Morava K -theory spectrum $k(n)$ has a non-vanishing first k -invariant.

Remark 8.17. As mentioned before, Benson, Krause, and Schwede studied a characteristic cohomology class $\gamma_A \in \text{HH}_k^{3,-1}(H^*(A))$ for a differential graded algebra A over a field k . Though they are only concerned with a ‘triple’ characteristic Hochschild class, their theory easily generalizes to higher classes when we assume $H^*(A)$ to be n -sparse. In this case, the Hochschild cochain m_{n+2} , which is part of the A_∞ -structure of $H^*(A)$ [Kad80], happens to be a Hochschild cocycle. This is easily deduced from the A_∞ -relations. Similar to the triple class, the cohomology class $[m_{n+2}] \in \text{HH}_k^{n+2,-n}(H^*(A))$ is well defined and determines all $(n + 2)$ -fold Massey products in $H^*(A)$.

One may ask whether it is possible to define the higher classes under weaker assumptions, that is, without $H^*(A)$ or $\pi_*(R)$ being n -sparse. To some extent, this is possible if the lower universal classes vanish. We begin by sketching the first step

in the case of a dga A . Suppose that $\gamma_A = [m_3] \in \mathrm{HH}_k^{3,-1}(H^*(A))$ vanishes. Then it is possible to find an equivalent A_∞ -structure (m'_i) on $H^*(A)$ such that the cocycle m_3 is zero. This employs the same kind of argument as used to show that every A_∞ -structure on $H^*(A)$ is trivial if $\mathrm{HH}_k^{n+2,-n}(H^*(A)) = 0$ for $n \geq 1$ [Kad88]. It follows that m'_4 is a Hochschild cocycle, which can be used to define a cohomology class in $\mathrm{HH}_k^{4,-2}(H^*(A))$. This is the candidate for the higher Hochschild class. However, it is not unique in general.

In the case of ring spectra, Lemma 5.11 is the tool for a similar kind of argument. If for example $\gamma_R \in \mathrm{HML}^{3,-1}(\pi_*(R))$ vanishes, the lemma says that we can find a 1-cube system which extends to a 2-cube system. This cube system can be used to define $\gamma_R^4 \in \mathrm{HML}^{4,-2}(\pi_*(R))$ without requiring $\pi_*(R)$ to be 2-sparse. As in the algebraic case, there may be different choices for this class γ_R^4 . Moreover, the relation to the Toda brackets becomes more involved since the indeterminacy is not as easy to control as in the 2-sparse case. This also affects the obstruction theory, as there is no unique obstruction class.

APPENDIX A. DISCARDING BASEPOINTS OF MAPPING SPACES

Let $\mathcal{T}op_*$ be the category of pointed compactly generated weak Hausdorff spaces, equipped with the usual model structure in which the weak equivalences are the weak homotopy equivalences [Hov99, Theorem 2.4.25].

A pointed topological model category \mathcal{C} is a pointed model category which is enriched, tensored, and cotensored over $\mathcal{T}op_*$. This means that there are bifunctors $-\wedge -: \mathcal{T}op_* \times \mathcal{C} \rightarrow \mathcal{T}$, $\mathrm{Map}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \rightarrow \mathcal{T}op_*$, $(-)^{(-)} : \mathcal{C} \times \mathcal{T}op_*^{\mathrm{op}} \rightarrow \mathcal{C}$, adjunction isomorphisms $\mathcal{C}(X, Y^K) \cong \mathcal{C}(K \wedge X, Y) \cong \mathcal{T}op_*(K, \mathrm{Map}_{\mathcal{C}}(X, Y))$, and an enriched composition $\mathrm{Map}_{\mathcal{C}}(Y, Z) \wedge \mathrm{Map}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Map}_{\mathcal{C}}(X, Z)$. The data is asked to satisfy the usual associativity and unit conditions. Moreover, the pushout product axiom is required to ensure compatibility of the model structures. Details can be found in [Hov99, 4.2].

A stable topological model category \mathcal{C} is a pointed topological model category in which the suspension functor $S^1 \wedge -: \mathcal{C} \rightarrow \mathcal{C}$ and the loop functor $(-)^{S^1} : \mathcal{C} \rightarrow \mathcal{C}$ form a Quillen equivalence.

For an object X in a category \mathcal{C} , we write $(X \downarrow \mathcal{C})$ for the category of objects under X . If \mathcal{C} is a model category, $(X \downarrow \mathcal{C})$ inherits a model structure in which a map is a cofibration, fibration, or a weak equivalence if the underlying map in \mathcal{C} is one [Hir03, Theorem 7.6.5.(1)].

Proposition A.1. *Let \mathcal{C} be a stable topological model category. For every map $g: X \rightarrow Y$ between cofibrant and fibrant objects in \mathcal{C} , there is an isomorphism*

$$\sigma_g : [S^n, (\mathrm{Map}_{\mathcal{C}}(X, Y), g)]^{\mathrm{Ho}(\mathcal{T}op_*)} \xrightarrow{\cong} [S^n \wedge X, Y]^{\mathrm{Ho}(\mathcal{C})}.$$

If $h: W \rightarrow X$ and $f: Y \rightarrow Z$ are maps between cofibrant and fibrant objects, the isomorphisms satisfy $(f_*)(\sigma_g) = (\sigma_{fg})(f_*)$ and $(h^*)(\sigma_g) = (\sigma_{hg})(h^*)$. For a path w from g to g' in $\mathrm{Map}_{\mathcal{C}}(X, Y)$, the isomorphisms σ_g and $\sigma_{g'}$ are compatible with the isomorphism of homotopy groups induced by w , i.e., $\sigma_{g'}(-)^w = \sigma_g$. If g is the zero map, σ_g is the adjunction isomorphism.

The proof needs some notation and an auxiliary lemma. If S^n is the n -sphere in $\mathcal{T}op_*$, we consider S^n_+ as an object of $(S^0 \downarrow \mathcal{T}op_*)$. The structure map $S^0 \rightarrow S^n_+$

sends the basepoint of S^0 to the ‘added’ basepoint of S_+^n , and the other point to the ‘original’ basepoint of S^n . If X is an object in a pointed topological model category \mathcal{C} , we consider $S_+^n \wedge X$ as an object in $(X \downarrow \mathcal{C})$ via $X \cong S^0 \wedge X \rightarrow S_+^n \wedge X$. By the pushout product axiom, $S_+^n \wedge X$ is cofibrant if X is. If $f: X \rightarrow Y$ is another object of $(X \downarrow \mathcal{C})$, we write $[S_+^n \wedge X, f]^{\text{Ho}(X \downarrow \mathcal{C})}$ for the set of maps from $X \rightarrow S_+^n \wedge X$ to f in the homotopy category of $(X \downarrow \mathcal{C})$.

The following lemma is a reformulation of the well known fact that S_+^n splits as $S^n \vee S^0$ after suspension.

Lemma A.2. *Let $S_+^n \wedge S^1$ be a space under S^1 via $(S^0 \rightarrow S_+^n) \wedge S^1$, and let $S^{n+1} \vee S^1$ be a space under S^1 via the inclusion of the second summand. Then there is a map $\mu: (S^n \wedge S^1) \vee S^1 \rightarrow S_+^n \wedge S^1$ in $(S^1 \downarrow \mathcal{Top})$ which is a homotopy equivalence. If $p: S_+^n \rightarrow S^n$ is the map which identifies the two basepoints of S_+^n ,*

$$S^n \wedge S^1 \xrightarrow{\text{incl}} (S^n \wedge S^1) \vee S^1 \xrightarrow{\mu} S_+^n \wedge S^1 \xrightarrow{p \wedge S^1} S^n \wedge S^1$$

is the identity.

Proof. S^n is a CW-complex with one 0-cell and one n -cell. The complex $S_+^n \wedge S^1 \cong S^n \times S^1 / (S^n \times \{s_0\})$ has a 0-cell, an 1-cell, and an $(n + 1)$ -cell. The attaching map of the $(n + 1)$ -cell of $S_+^n \wedge S^1$ is null-homotopic for $n \geq 1$. Hence the desired homotopy equivalence exists. If we compose with $p \wedge S^1$, we collapse the 1-cell and do not see the effect of the null-homotopy. This verifies the last assertion. \square

Proof of Proposition A.1. We define a functor $G: \mathcal{C} \rightarrow \mathcal{C}$ by $G(X) = (X^{S^1})^{\text{cof}}$, where $(-)^{\text{cof}}$ is the functorial cofibrant replacement. The adjunction of suspension and loop gives a natural transformation $\tau: S^1 \wedge G(X) \rightarrow \text{id}_{\mathcal{C}}$. Since \mathcal{C} is stable, τ_X is a weak equivalence if X is fibrant [Hov99, Proposition 1.3.13(b)].

Let σ_g be the following chain of isomorphisms:

$$\begin{aligned} & [S^n, (\text{Map}_{\mathcal{C}}(X, Y), g)]^{\text{Ho}(\mathcal{Top}_*)} \stackrel{(i)}{\cong} [S^n, (\text{Map}_{\mathcal{C}}(S^1 \wedge G(X), Y), g\tau_X)]^{\text{Ho}(\mathcal{Top}_*)} \\ & \stackrel{(ii)}{\cong} [S_+^n, (\text{Map}_{\mathcal{C}}(S^1 \wedge G(X), Y), g\tau_X, 0)]^{\text{Ho}(S^0 \downarrow \mathcal{Top}_*)} \\ & \stackrel{(iii)}{\cong} [S_+^n \wedge S^1 \wedge G(X), g\tau_X]^{\text{Ho}(S^1 \wedge G(X) \downarrow \mathcal{C})} \\ & \stackrel{(iv)}{\cong} [((S^n \wedge S^1) \vee S^1) \wedge G(X), g\tau_X]^{\text{Ho}(S^1 \wedge G(X) \downarrow \mathcal{C})} \\ & \stackrel{(v)}{\cong} [S^n \wedge S^1 \wedge G(X), Y]^{\text{Ho}(\mathcal{C})} \stackrel{(vi)}{\cong} [S^n \wedge X, Y]^{\text{Ho}(\mathcal{C})}. \end{aligned}$$

Here (i) is induced by the weak equivalence τ_X , (ii) is adding a basepoint, (iii) results from the Quillen adjunction between $(S^0 \downarrow \mathcal{Top}_*)$ and $(X' \downarrow \mathcal{C})$ induced by $- \wedge X'$ and $\text{Map}_{\mathcal{C}}(X', -)$ with $X' = S^1 \wedge G(X)$, (iv) uses the weak equivalence under S^1 provided by Lemma A.2, (v) results from the Quillen adjunction between $(X' \downarrow \mathcal{C})$ and \mathcal{C} given by $Z \mapsto Z \vee X'$ and the forgetful functor (with $X' = S^1 \wedge G(X)$), and (vi) is induced by τ_X again.

It is easy to see that the construction is natural in X and Y . It is additive since the addition can be defined in terms of the H -cogroup structure of S^n both in the source and the target. Let w be a path from g to g' in $\text{Map}_{\mathcal{C}}(X, Y)$. Following its action through (i)-(iii), it induces a homotopy of maps $S_+^n \wedge S^1 \wedge G(X) \rightarrow Y$ which is itself not a map under S^1 . But after applying (iv) and (v), the representing maps

become homotopic in $\mathrm{Ho}(\mathcal{C})$. If g is the zero map, σ_g reduces to the adjunction isomorphism: Composing with the p of Lemma A.2 is inverse to (ii), hence Lemma A.2 shows the assertion. \square

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