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EXTENSIONS OF THE MULTIPLICITY CONJECTURE

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ABSTRACT. The Multiplicity conjecture of Herzog, Huneke, and Srinivasan states an upper bound for the multiplicity of any graded k-algebra as well as a lower bound for Cohen-Macaulay algebras. In this note we extend this conjecture in several directions. We discuss when these bounds are sharp, find a sharp lower bound in the case of not necessarily arithmetically Cohen-Macaulay one-dimensional schemes of 3-space, and propose an upper bound for finitely generated graded torsion modules. We establish this bound for torsion modules whose codimension is at most two.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over the field k with its standard grading where $\deg x_i = 1$. Let N be a finitely generated graded R-module. We denote by $e(N)$ the multiplicity of N. When I is a saturated ideal defining a closed subscheme $V \subset \mathbb{P}^{n-1}$, $e(R/I)$ is just the degree, $\deg V$, of V. Consider a minimal free resolution of N:

$$0 \to \bigoplus_{j=1}^{b_s} R(-d_{s,j}) \to \cdots \to \bigoplus_{j=1}^{b_0} R(-d_{0,j}) \to N \to 0.$$

We define $m_i(N) := \min\{d_{i,1}, \ldots, d_{i,b_i}\}$ and $M_i(N) := \max\{d_{i,1}, \ldots, d_{i,b_i}\}$. When there is no danger of ambiguity, we simply write m_i and M_i. The module N has a *pure resolution* if $m_i = M_i$ for all i.

Our first extension of the Multiplicity conjecture of Herzog, Huneke and Srinivasan includes also a discussion of sharpness:

Conjecture 1.1. Let $I \subset R$ be a graded ideal, $c = \operatorname{codim} R/I$ and assume that R/I is Cohen-Macaulay. Then

$$\frac{1}{c!}\prod_{i=1}^{c} m_i \leq e(R/I) \leq \frac{1}{c!}\prod_{i=1}^{c} M_i$$

with equality below (resp. above) if and only if R/I has a pure resolution.

Indeed, the original conjecture consisted only of the two inequalities, but the statement about equality is motivated by the results of [16], where this extended conjecture is shown for Cohen-Macaulay algebras of codimension two and Gorenstein algebras of codimension 3.

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There has been a tremendous amount of activity towards establishing important special cases of the Multiplicity conjecture (cf., e.g., [10], [8], [20], [5], [6], [19], [7], [16], [3], [17], [9], and the survey paper [4]). In this paper, our goal is to give several extensions of this conjecture and to prove some initial cases. First, we discuss non-Cohen-Macaulay k-algebras.

It is conjectured in [8] that the upper bound in Conjecture 1.1 is also true when R/I is not Cohen-Macaulay. However, it is known that the inequality $\frac{1}{c!}\prod_{i=1}^{c} m_i \leq e(R/I)$ does not hold in general. A simple example is given by the coordinate ring B of two skew lines in $\mathbb{P}^3$, where $e(B) = 2$ but $m_1(B) = 2, m_2(B) = 3$. Thus, our first goal in this note is to examine what it is that prevents the lower bound from holding, at least in the case of one-dimensional curves in $\mathbb{P}^3$. In the process, we find a lower bound. Furthermore, we get an improved upper bound.

Specifically, let $M(C)$ denote the Hartshorne-Rao module of C (see the beginning of Section 2 for the definition). $M(C)$ has been studied a great deal in the literature, most successfully in work of Rao for liaison of curves in $\mathbb{P}^3$ (see e.g. [18]). However, in our case it is a *submodule* of $M(C)$ that is important. Namely, if A is the ideal generated by two general linear forms, then we denote by K_A the submodule of $M(C)$ annihilated by A. This has finite length regardless of whether or not $M(C)$ does. It has been studied in [14] and in [13], but this is a new application. We show in Theorem 2.1 that

$$\frac{1}{4}m_1(C)m_2(C) - \dim_k K_A \leq \deg C \leq \frac{1}{2}M_1(C)M_2(C) - \dim_k K_A.$$

Then we show in Section 3 how basic double G-links can be used to approach the Multiplicity conjecture, also in the non-Cohen-Macaulay case. As applications, we use this rather general method to provide a unified framework for giving simple proofs that the Multiplicity conjecture is preserved under regular hypersurface sections and that the Multiplicity conjecture is true for standard determinantal ideals. The latter result has been independently obtained, first by Miró-Roig [17] and then by Herzog and Zheng [9]. The hypersurface section result is also contained in [9].

As a further application of the methods in Section 3 we derive a closed formula for the degree of any standard determinantal ideal (Theorem 3.10).

Note that in [9] Herzog and Zheng have shown that virtually in all cases where the multiplicity bounds were previously known, the bounds are sharp if and only if the algebra R/I is Cohen-Macaulay and has a pure resolution.

Finally, we discuss the case of modules. We propose the following extension of the Multiplicity conjecture for cyclic modules in [8]:

Conjecture 1.2. Let N be a finitely generated graded torsion R-module of codimension $c = n - \dim N$. Then we have that

$$e(N) \leq b_0 \cdot \frac{1}{c!}\prod_{i=1}^{c}(M_i - m_0)$$

with equality if and only if N is Cohen-Macaulay and has a pure resolution.

Extending a result by Herzog and Srinivasan in [8], we first show this conjecture for every Cohen-Macaulay module of any codimension $c > 0$ with a quasi-pure resolution, i.e. $m_i \geq M_{i-1}$ for all $i \geq 1$. This is carried out in Section 4. The result (cf. Theorem 4.2) is used as one ingredient in Section 5 where we prove one of the main results of this paper: Conjecture 1.2 is true for every module whose codimension is at most two (cf. Theorem 5.1). This extends the result for such cyclic modules in [19]. It also provides our strongest evidence for Conjecture 1.2.

We hope that our conjectured extensions of the Multiplicity conjecture and the methods of this paper will stimulate further investigations. We would like to thank Jonas Söderberg for comments related to Sections 4 and 5.

2. A lower bound for non-arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$

Let $C \subset \mathbb{P}^3$ be a one-dimensional scheme; we do not necessarily assume that the saturated ideal $I_C \subset R = k[x_1, x_2, x_3, x_4]$ is unmixed. Let

$$M(C) := \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(t))$$

be the *deficiency module* of C (also known as the *Hartshorne-Rao module* of C). C is arithmetically Cohen-Macaulay if and only if $M(C) = 0$, and C is locally Cohen-Macaulay and equidimensional (i.e. I_C is unmixed) if and only if $M(C)$ has finite length. If L is a general linear form, then L induces a multiplication $(\times L) : M(C) \to M(C)(1)$. The kernel, $K_L = [0 :_{M(C)} (L)]$, has finite length regardless of whether or not $M(C)$ does. To simplify notation we write for any ideal $I \subset R$ throughout the paper, $m_i(I)$ and $M_i(I)$ for $m_i(R/I)$ and $M_i(R/I)$ and similarly $m_i(C)$ and $M_i(C)$ for $m_i(I_C) = m_i(R/I_C)$ and $M_i(I_C) = M_i(R/I_C)$, respectively. Since in this section we will compare resolutions of modules and ideals, we will use for an R-module N (in contrast to the notation in the introduction) $m_i(N)$ and $M_i(N)$ to refer to the $(i-1)$-st syzygy module of N.

If A is the ideal generated by two general linear forms, then we set K_A to be the submodule of $M(C)$ annihilated by A. In [14], it was shown that K_A plays a very interesting role in the study of one-dimensional subschemes of $\mathbb{P}^3$. For instance, a famous result of Dubreil for codimension two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$ (in fact for codimension two arithmetically Cohen-Macaulay subschemes of any projective space) says that

$$\nu(C) \leq m_1(C) + 1,$$

where $\nu(C)$ is the number of minimal generators of the homogenous ideal I_C. This result is not true for non-arithmetically Cohen-Macaulay curves. However, using K_A allows one to generalize Dubreil's theorem:

$$\nu(C) \leq m_1(C) + 1 + \nu(K_A).$$

Here C can in fact be *any* subscheme of projective space of dimension ≤ 1.

The main purpose of this section is to show how K_A can be used to similarly extend the lower bound of the Multiplicity conjecture for non-arithmetically Cohen-Macaulay one-dimensional schemes. In this geometric context we prefer to write $\deg C$ for $e(R/I_C)$. Recall that the standard formulation is known to be false: for instance, if C is the union of two skew lines, then $\frac{1}{2} m_1(C) m_2(C) = \frac{1}{2}(2)(3) = 3 > 2 = \deg C$. In [16] it was shown that even a bound of the form $\frac{1}{p} m_1(C) m_2(C) \leq \deg C$ is false, for any $p \geq 1$ (see Example 2.8 below), even if we were to restrict to unmixed one-dimensional schemes (which in fact we will not have to do). Here we show that K_A is the right "correction term." The main result of this section is:

Theorem 2.1. *Let $C \subset \mathbb{P}^3$ be a one-dimensional scheme. Then we have*

$$\frac{1}{4}m_1(C)m_2(C) - \dim_k K_A \leq \deg C \leq \frac{1}{2}M_1(C)M_2(C) - \dim_k K_A.$$

The proof is based on the following slightly technical result.

Lemma 2.2. *Let $C \subset \mathbb{P}^3$ be a one-dimensional scheme with saturated ideal I_C. Let $A = (L_1, L_2)$ be the ideal generated by two general linear forms in $R = k[x_0, x_1, x_2, x_3]$. Let $J = \frac{I_C + A}{A} \subset R/A := T \cong k[x, y]$. Let*

$$m_2 = \min\{m_2(C), m_1(K_A) + 2\}.$$

Then

$$\frac{1}{2}m_1(C)m_2 - \dim_k K_A \leq \deg C \leq \frac{1}{2}M_1(C)M_2(C) - \dim_k K_A.$$

Proof. From [13, page 49] we know that

$$K_A(-2) \cong \frac{I_C \cap A}{I_C \cdot A}.$$

Now consider the exact sequence

$$\begin{array}{ccccccccc} 0 & \rightarrow & \dfrac{I_C \cap A}{I_C \cdot A} & \rightarrow & \dfrac{I_C}{I_C \cdot A} & \rightarrow & \dfrac{I_C}{I_C \cap A} & \rightarrow & 0. \\ & & \| & & & & \| & & \\ & & K_A(-2) & & & & \dfrac{I_C + A}{A} & & \end{array} \tag{2.1}$$

The latter module, $\frac{I_C + A}{A} =: J$, is isomorphic to an ideal in $k[x_0, x_1, x_2, x_3]/A \cong k[x, y] =: T$. As such, it satisfies the Multiplicity conjecture. Observe that the graded Betti numbers of the T-module $\frac{I_C}{I_C \cdot A}$, over T, are the same as those of I_C over R.

Considering resolutions over T, we get the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{K}_3(-2) & & \mathbb{F}_3 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{K}_2(-2) & & \mathbb{F}_2 & & \mathbb{G}_2 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{K}_1(-2) & & \mathbb{F}_1 & & \mathbb{G}_1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & K_A(-2) & \rightarrow & \left(\dfrac{I_C}{A \cdot I_C}\right) & \rightarrow & J & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.2}$$

The mapping cone immediately gives that

$$\mathbb{K}_3(-2) \cong \mathbb{F}_3, \text{ and } \mathbb{K}_3 \text{ splits in the mapping cone}$$

and that

$$\mathbb{K}_2(-2) \text{ is a direct summand of } \mathbb{F}_2 \text{ and splits in the mapping cone.} \tag{2.3}$$

Now, using (2.1), we have for any t,

$$-\dim_k J_t = -\dim_k \left(\frac{I_C}{A \cdot I_C} \right)_t + \dim_k (K_A)_{t-2},$$

so

$$\begin{aligned} \dim_k (T/J)_t &= \Delta^2 \dim_k (R/I_C)_t + \dim_k (K_A)_{t-2} \\ &= [\Delta \dim_k (R/I_C)_t] - [\Delta \dim_k (R/I_C)_{t-1}] + \dim_k (K_A)_{t-2}. \end{aligned}$$

Now, we know that for $t \gg 0$ we have $\Delta \dim_k (R/I_C)_t = \deg C$ and $\dim_k (K_A)_t = 0$, while for $t < 0$ we have $\Delta \dim_k (R/I_C)_t = \dim_k (K_A)_t = 0$. Now let $t_0 \gg 0$ and sum over all $t \leq t_0$:

$$\begin{aligned} \deg C &= \Delta \dim_k (R/I_C)_{t_0} \\ &= \sum_{t=0}^{t_0} [\Delta \dim_k (R/I_C)_t - \Delta \dim_k (R/I_C)_{t-1}] \\ &= \sum_{t=0}^{t_0} \dim_k (T/J)_t - \dim_k (K_A)_{t-2} \\ &= e(T/J) - \dim_k K_A. \end{aligned} \tag{2.4}$$

Now, as noted above, $J \subset T \cong k[x, y]$ satisfies the Multiplicity conjecture. Note also that from the definition of J it follows immediately that

$$m_1(J) = m_1 \left(\frac{I_C}{A \cdot I_C} \right) = m_1(C),$$

and from the mapping cone associated to (2.2) we have

$$m_2(J) \geq m_2.$$

Now we get

$$\begin{aligned} \deg C &= e(T/J) - \dim_k K_A \\ &\geq \tfrac{1}{2} m_1(J) m_2(J) - \dim_k K_A \\ &\geq \tfrac{1}{2} m_1(C) m_2 - \dim_k K_A, \end{aligned}$$

proving the lower bound.

For the upper bound, the same argument works as above (reversing the inequalities and replacing m_i by M_i), once we have shown that

$$M_2(J) \leq M_2(C).$$

To see this, the mapping cone construction for (2.2) shows that

$$M_2(J) \leq \max\{M_2(C), M_1(K_A(-2))\}.$$

But using (2.3), we see that

$$M_2(C) \geq M_2(K_A(-2)) > M_1(K_A(-2)),$$

so we are done. $\square$

Corollary 2.3. *With the notation of Lemma 2.2, we furthermore let H be the plane in $\mathbb{P}^3$ defined by L_1 and let $C \cap H$ be the geometric hyperplane section (i.e. it is defined by the saturation of $\frac{I_C + (L_1)}{(L_1)}$ in $R/(L_1)$, which we denote by $I_{C \cap H}$).*

Then:

(a) *If* $m_2(C) \le m_1(K_A) + 2$, *then*

$$\frac{1}{2} m_1(C) m_2(C) - \dim_k K_A \le \deg C.$$

(b) *If* $m_1(K_A) + 2 < m_2(C)$, *then the following both hold:*

(i) $\frac{1}{2} m_1(C) m_2(C \cap H) - \dim_k K_A \le \deg C.$

(ii) $\frac{1}{4} m_1(C) m_2(C) - \dim_k K_A \le \deg C.$

Proof. Part (a) is immediate from Theorem 2.2. For (b), we first prove (i). Consider the exact sequence

$$0 \to K \to K_{L_1}(-1) \xrightarrow{\times L_2} K_{L_1} \to B \to 0,$$

where K is the kernel and B is the cokernel, respectively, of the multiplication by L_2. Note that $K = K_A(-1)$. The Socle Lemma ([11], Corollary 3.11) then gives that $m_1(K) > m_1(\mathrm{Soc}(B))$. Now, consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & K_A(-2) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I_C(-2) & \xrightarrow{\times L_1} & I_C(-1) & \to & I_{C\cap H}(-1) & \to & K_{L_1}(-2) & \to & 0 \\
 & & \downarrow \times L_2 & & \downarrow \times L_2 & & \downarrow \times L_2 & & \downarrow \times L_2 & & \\
0 & \to & I_C(-1) & \xrightarrow{\times L_1} & I_C & \to & I_{C\cap H} & \to & K_{L_1}(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J(-1) & \xrightarrow{\times L_1} & J & \to & \dfrac{I_{C\cap H}}{L_2 \cdot I_{C\cap H}} & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where $J = I_C/(L_2 \cdot I_C)$ and $\frac{I_{C\cap H}}{L_2 \cdot I_{C\cap H}}$ is isomorphic to an ideal, J', in $T := R/(L_1, L_2)$ with the same graded Betti numbers as $I_{C\cap H}$ in $R/(L_1)$. We denote by $\mathrm{Soc}(C \cap H)$ the socle of T/J'. We get that

$$\begin{array}{rcl}
m_2 & = & m_1(K_A) + 2 \\
 & = & m_1(K) + 1 \\
 & \ge & m_1(\mathrm{Soc}(B)) + 2 \\
 & \ge & m_1(\mathrm{Soc}(C \cap H)) + 2 \\
 & = & m_2(C \cap H).
\end{array}$$

Then Theorem 2.2 gives the desired result.

For (ii), recall that $K_A(-2) \cong \frac{I_C \cap A}{I_C \cdot A}$. Let $d = m_1(K_A(-2)) \ge m_1(C)$. Let $F \in I_C$ be a generator of minimal degree, $m_1(C)$. We claim that there is an element $G \in (I_C)_d$ that is not a multiple of F. Indeed, since A corresponds to a general line, we can assume that F does not vanish identically on this line. If every element of $(I_C)_d$ were a multiple of F, then in particular any element of $(I_C \cap A)_d$ is of the form HF, where $H \in A$, contradicting the fact that $K_A(-2) = \frac{I_C \cap A}{I_C \cdot A}$ begins in degree d. It follows, since F and G have at worst a Koszul syzygy and G may not itself be a minimal generator, that

$$m_2(C) \le m_1(C) + m_1(K_A(-2)) \le 2m_1(K_A(-2)).$$

Now, in Theorem 2.2 we set $m_2 = \min\{m_2(C), m_1(K_A)+2\}$, which in our current hypothesis is equal to $m_1(K_A(-2))$. We showed that

$$\deg C \geq \frac{1}{2}m_1(C)\cdot m_2 - \dim_k K_A.$$

Now we obtain

$$\begin{aligned}\deg C &\geq \frac{1}{2}m_1(C)\left(\frac{1}{2}m_2(C)\right) - \dim_k K_A \\ &= \frac{1}{4}m_1(C)m_2(C) - \dim_k K_A\end{aligned}$$

as claimed. □

Summing up we get:

Proof of Theorem 2.1. This is an immediate consequence of Lemma 2.2 and Corollary 2.3. □

A very strong generalization of the following result can be found in Theorem 5.1, but we include it here as an interesting special case.

Corollary 2.4. *Let $C \subset \mathbb{P}^3$ be a one-dimensional scheme. Then*

$$\deg C \leq \frac{1}{2}M_1(C)M_2(C),$$

with equality if and only if C is arithmetically Cohen-Macaulay with pure resolution.

Proof. It follows immediately from [16, Corollary 1.3], and the observation that $K_A = 0$ if and only if C is arithmetically Cohen-Macaulay. □

Remark 2.5. The analogous statements for the lower bound are not true. First, it is well known that

$$\frac{1}{2}m_1(C)m_2(C) \leq \deg C$$

is false in general (consider for instance two skew lines). Furthermore, if

$$\frac{1}{2}m_1(C)m_2(C) = \deg C,$$

it does not follow that C is arithmetically Cohen-Macaulay. A simple example is a rational quartic curve in $\mathbb{P}^3$.

Corollary 2.6. *Let $C \subset \mathbb{P}^3$ be a curve; i.e., I_C is an unmixed one-dimensional scheme (so $\dim_k M(C) < \infty$). Then*

$$\frac{1}{4}m_1(C)m_2(C) - \dim_k M(C) \leq \deg C.$$

Note, though, that in general $\dim_k M(C)$ can be much greater than $\dim_k K_A$. See the following example.

Example 2.7. In Theorem 2.2, if it were true that

$$m_2(J) \geq m_2\left(\frac{I_C}{A\cdot I_C}\right) = m_2(C),$$

then in fact we would be able to prove that the first assertion of Corollary 2.3 always holds:

$$\frac{1}{2}m_1(C)m_2(C) \leq \deg C + \dim_k K_A.$$

Unfortunately, we now show that these assertions are both false in general.

Let C be the disjoint union of two complete intersections of type $(16, 16)$. Then $\dim_k M(C) = 65,536$, $\dim_k K_L = 2,736$ (where K_L is the submodule of $M(C)$ annihilated by one general linear form) and $\dim_k K_A = 171$. We also have $\deg C = 512$, and one can compute

$$\begin{array}{llll} m_1(C) = 32 & m_1(K_A) + 2 = 42 & m_1(J) = 32 & m_1(C \cap H) = 31 \\ m_2(C) = 48 & m_2(K_A) + 2 = 48 & m_2(J) = 42 & m_2(C \cap H) = 33 \\ m_3(C) = 64 & m_3(K_A) + 2 = 64. & & \end{array}$$

In particular, $m_2(J) < m_2(C)$ and

$$768 = \frac{1}{2} m_1(C) m_2(C) > \deg C + \dim_k K_A = 683.$$

Example 2.8. In [16, Remark 2.4], we began the discussion of what might be the right approach to a lower bound for the multiplicity for non-arithmetically Cohen-Macaulay space curves. We noted that a bound of the form

$$\frac{1}{p} m_1(C) m_2(C) \leq \deg C$$

is not possible, for any fixed value of $p \geq 1$. We gave as an example the curve C with saturated ideal

$$I_C = (x_0, x_1)^t + (F), \tag{2.5}$$

where F is smooth along the line defined by (x_0, x_1) and $d := \deg F \geq t + 1$, and we noted that $\deg C = t$, $m_1(C) = t$ and $m_2(C) = t + 1$. Hence we would need $p \geq t + 1$, and obviously this can be made arbitrarily large.

Since this example was used to illustrate the difficulty of finding a nice lower bound, we would like to remark here that our Corollary 2.3 fits nicely with this example. In Proposition 2.9 we will show that a curve of this form *always* gives equality in Corollary 2.3 (a). First, though, we give a specific numerical example to illustrate not only the sharpness, but the huge difference that is possible between $\dim_k M(C)$ and $\dim_k K_A$ even in the context of Corollary 2.3 (a).

Specifically, let $t = 12$ and $\deg F = 15$. Using `macaulay` [2], we have computed $\dim_k M(C) = 56,056$, but $\dim_k K_A = 66$. Furthermore, $m_1(K_A) = 13, m_2(K_A) = 24$ and $m_3(K_A) = 25$ (over T). Then the hypothesis of (a) holds, and in fact we have $\deg C = 12$ and

$$78 = \frac{1}{2} m_1(C) m_2(C) = \deg C + \dim_k K_A.$$

Proposition 2.9. *Let C be the non-reduced curve with ideal I_C given in* (2.5). *Then*

$$\frac{1}{2} m_1(C) m_2(C) = \deg C + \dim_k K_A.$$

Proof. We have from (2.4) that

$$\deg C = e(T/J) - \dim_k K_A,$$

where $J = \frac{I_C + A}{A}$. In our case, without loss of generality we may choose $A = (x_2, x_3)$. Note that then

$$\begin{aligned} J &= \frac{(x_0, x_1)^t + (x_2, x_3) + (F)}{(x_2, x_3)} \\ &= (x_0, x_1)^t \end{aligned}$$

viewed in the ring $T = k[x_0, x_1, x_2, x_3]/(x_2, x_3)$. (Note that in [16, Remark 2.4], we could even have taken $\deg F = t$ and we would still have this equality.) Hence we know that $e(T/J) = \binom{t+1}{2}$, so we compute

$$\dim_k K_A = e(T/J) - \deg C = \binom{t+1}{2} - t = \binom{t}{2}.$$

On the other hand, we know that $m_1(C) = t, m_2(C) = t + 1$. Indeed, the entire minimal free resolution of I_C can be computed from a mapping cone using the exact sequence

$$0 \to (x_0, x_1)^{t-1}(-d) \to (x_0, x_1)^t \oplus R(-d) \to I_C \to 0.$$

We then immediately see that

$$\begin{aligned} \frac{1}{2} m_1(C) m_2(C) &= \binom{t+1}{2} \\ &= t + \binom{t}{2} \\ &= \deg C + \dim_k K_A \end{aligned}$$

as desired. □

3. Basic double G-links

In this section we prove that under suitable assumptions in arbitrary codimension, basic double G-links preserve the property of satisfying the upper and lower bounds of the Multiplicity conjecture. We use this to provide a unified framework for some results that also have been independently obtained by Miró-Roig [17] and by Herzog and Zheng [9]. Furthermore, we establish a formula for the degree of any standard determinantal ideal. We believe that this framework will provide further applications.

We first recall the setup.

Proposition 3.1 ([12], Lemma 4.8, Proposition 5.10)**.** *Let $I \subset J$ be homogenous ideals of $R = k[x_0, \dots, x_n]$ such that codim $I + 1 =$ codim $J = c + 1$. Let $L \in R$ be a form of degree d such that $I : L = I$. Let $J_1 = I + L \cdot J$. Then we have*

(i) *$e(R/J_1) = d \cdot e(R/I) + e(R/J)$.*
(ii) *If R/I satisfies property G_0 (Gorenstein in codimension 0) and if J is unmixed, then J and J_1 are Gorenstein linked in two steps.*
(iii) *We have a short exact sequence*

$$0 \to I(-d) \to J(-d) \oplus I \to J_1 \to 0,$$

where the first map is given by $F \mapsto (LF, F)$ and the second map is given by $(F, G) \mapsto F - LG$.

Remark 3.2. Statement (ii) of Proposition 3.1 explains why this process is called Basic Double G-linkage. However, we will not need this fact here.

Recall our convention from Section 2 that we write for an ideal $I \subset R$, $m_i(I)$ for $m_i(R/I)$ and similarly $M_i(I)$ for $M_i(R/I)$.

Theorem 3.3. *Under the assumptions of Proposition* 3.1*, assume that I and J both satisfy the upper bound of the Multiplicity conjecture. Assume further that*

$$M_i(J_1) \geq M_i(J) + d, \quad 1 \leq i \leq c + 1, \tag{3.1}$$

and

$$M_i(J_1) \geq M_{i-1}(I) + d, \quad 2 \leq i \leq c+1. \tag{3.2}$$

Then J_1 also satisfies the upper bound of the Multiplicity conjecture.

Proof. We have the exact sequence

$$0 \to I(-d) \to J(-d) \oplus I \to J_1 \to 0. \tag{3.3}$$

Let $\mathbb{F}_\bullet$ be a minimal free resolution for I and $\mathbb{G}_\bullet$ be a minimal free resolution for J. Consider the exact diagram

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & 0 & & \mathbb{G}_{c+1}(-d) & & & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathbb{F}_c(-d) & & \mathbb{G}_c(-d) & \oplus & \mathbb{F}_c & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathbb{F}_{c-1}(-d) & & \mathbb{G}_{c-1}(-d) & \oplus & \mathbb{F}_{c-1} & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \vdots & & & & \vdots & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathbb{F}_3(-d) & & \mathbb{G}_3(-d) & \oplus & \mathbb{F}_3 & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathbb{F}_2(-d) & & \mathbb{G}_2(-d) & \oplus & \mathbb{F}_2 & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathbb{F}_1(-d) & & \mathbb{G}_1(-d) & \oplus & \mathbb{F}_1 & & & & \\
 & & \downarrow & & & & \downarrow & & & & \\
0 & \to & I(-d) & \to & J(-d) & \oplus & I & \to & J_1 & \to & 0 \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & 0 & & & & 0 & & & &
\end{array} \tag{3.4}$$

From this diagram with the induced maps, the mapping cone gives us the following free resolution for J_1:

$$0 \to \begin{array}{c} \mathbb{F}_c(-d) \\ \oplus \\ \mathbb{G}_{c+1}(-d) \end{array} \to \begin{array}{c} \mathbb{F}_{c-1}(-d) \\ \oplus \\ \mathbb{G}_c(-d) \\ \oplus \\ \mathbb{F}_c \end{array} \to \begin{array}{c} \mathbb{F}_{c-2}(-d) \\ \oplus \\ \mathbb{G}_{c-1}(-d) \\ \oplus \\ \mathbb{F}_{c-1} \end{array} \to \dots \tag{3.5}$$
$$\to \begin{array}{c} \mathbb{F}_1(-d) \\ \oplus \\ \mathbb{G}_2(-d) \\ \oplus \\ \mathbb{F}_2 \end{array} \to \begin{array}{c} \mathbb{F}_1 \\ \oplus \\ \mathbb{G}_1(-d) \end{array} \to J_1 \to 0$$

For our purposes we only need information about the largest shift in any free module.

We first note that our hypotheses (3.1) and (3.2) are implied by the following statement:

(3.1′). *In the i-th free module in the resolution* (3.5)*, there is at least one summand of $\mathbb{G}_i(-d)$ of (total) degree $M_i(J) + d$ that is not split off.*

This is a very natural situation, but it is not *always* true, as seen in Example 3.4 below.

An examination of the diagram (3.4) and the corresponding mapping cone (including the maps) reveals that the only possible splitting in (3.5) comes in canceling a term $R(-t-d)$ of $\mathbb{F}_i(-d)$ in the $(i+1)$-st free module with a corresponding one of $\mathbb{G}_i(-d)$ in the i-th free module. This shows right away that

$$M_i(J_1) \geq M_i(I), \quad 1 \leq i \leq c, \tag{3.6}$$

since no summand of $\mathbb{F}_i$ in the i-th free module of (3.5) is split off.

Now we are ready to establish the bound. Using the inequalities (3.2) and (3.6), we get for $i = 1, \dots, c+1$:

$$\begin{aligned}
&M_1(J_1) \cdots M_i(J_1) \cdot [M_{i+1}(J_1) - d] \cdots [M_{c+1}(J_1) - d] \\
&\quad = \quad M_1(J_1) \cdots M_{i-1}(J_1) \cdot [M_i(J_1) - d + d] \cdot [M_{i+1}(J_1) - d] \cdots [M_{c+1}(J_1) - d] \\
&\quad \geq \quad M_1(J_1) \cdots M_{i-1}(J_1) \cdot [M_i(J_1) - d] \cdots [M_{c+1}(J_1) - d] \\
&\qquad\qquad + d \cdot M_1(J_1) \cdots M_{i-1}(J_1) \cdot [M_{i+1}(J_1) - d] \cdots [M_{c+1}(J_1) - d] \\
&\quad \geq \quad M_1(J_1) \cdots M_{i-1}(J_1) \cdot [M_i(J_1) - d] \cdots [M_{c+1}(J_1) - d] \\
&\qquad\qquad + d \cdot M_1(I) \cdots M_{i-1}(I) \cdot M_i(I) \cdots M_c(I) \\
&\quad \geq \quad M_1(J_1) \cdots M_{i-1}(J_1) \cdot [M_i(J_1) - d] \cdots [M_{c+1}(J_1) - d] + d \cdot c! \cdot e(R/I)
\end{aligned}$$

where we also used the assumption on I. Applying this estimate repeatedly as well as Condition (3.1), we get

$$\begin{aligned}
&\frac{1}{(c+1)!} M_1(J_1) \cdots M_{c+1}(J_1) \\
&\quad \geq \quad \frac{1}{(c+1)!} M_1(J_1) \cdots M_c(J_1) \cdot [M_{c+1}(J_1) - d] + \frac{d}{c+1} \cdot e(R/I) \\
&\quad \geq \quad \frac{1}{(c+1)!} M_1(J_1) \cdots M_{c-1}(J_1) \cdot [M_c(J_1) - d][M_{c+1}(J_1) - d] + \frac{2d}{c+1} \cdot e(R/I) \\
&\quad \geq \quad \dots \\
&\quad \geq \quad \frac{1}{(c+1)!} [M_1(J_1) - d] \cdots [M_{c+1}(J_1) - d] + d \cdot e(R/I) \\
&\quad \geq \quad \frac{1}{(c+1)!} M_1(J) \cdots M_{c+1}(J) + d \cdot e(R/I) \\
&\quad \geq \quad e(R/J) + d \cdot e(R/I) \\
&\quad = \quad e(R/J_1),
\end{aligned}$$

as claimed. □

Example 3.4. Let $R = k[x, y, z]$ and let

$$\begin{aligned} J &= (x, y^9, z^6), \\ I &= (y^9, z^6), \\ J_1 &= (x^2, y^9, z^6). \end{aligned}$$

Then J_1 is a basic double G-link as described above with $d = 1$ and $L = x$, and we see that

$$\begin{array}{lllllllll} M_1(J) & = & 9 & M_1(I) & = & 9 & M_1(J_1) & = & 9 \\ M_2(J) & = & 15 & M_2(I) & = & 15 & M_2(J_1) & = & 15 \\ M_3(J) & = & 16 & & & & M_3(J_1) & = & 17. \end{array}$$

Furthermore, in all six cases there is only one summand of top degree in the corresponding free module. Hence it follows that (3.1) and (3.1′) are false in this example.

Remark 3.5. We have not been able to find an example where Condition (3.2) is not satisfied. In fact, we suspect that this condition is always true.

For later use, we record when we have equality in Theorem 3.3. It is an immediate consequence of its proof.

Corollary 3.6. *Adopt the notation and assumptions of Theorem* 3.3*. Then the upper bound of the Multiplicity conjecture is sharp for J_1 if and only if the upper bound of the Multiplicity conjecture is sharp for J as well as for I and*

$$\begin{array}{lllll} M_i(J_1) & = & M_i(J) + d, & & i = 1, \ldots, c + 1, \\ M_i(J_1) & = & M_i(I), & & i = 1, \ldots, c, \\ M_i(J_1) & = & M_{i-1}(I) + d, & & i = 2, \ldots, c + 1. \end{array}$$

For the lower bound of the Multiplicity Conjecture, there is a similar statement.

Corollary 3.7. *Under the assumptions of Proposition* 3.1*, assume that I and J both satisfy the lower bound of the Multiplicity conjecture* 1.1 *(though we do not assume that R/I nor R/J are Cohen-Macaulay). Assume further that*

$$m_i(J_1) \le m_{i-1}(I) + d \quad \textit{for all } 2 \le i \le c + 1. \tag{3.7}$$

Then J_1 also satisfies the lower bound of the Multiplicity conjecture.

Moreover, the lower bound of the Multiplicity conjecture is sharp for J_1 if and only if the lower bound of the Multiplicity conjecture is sharp for J as well as for I and

$$\begin{array}{lllll} m_i(J_1) & = & m_i(J) + d, & & i = 1, \ldots, c + 1, \\ m_i(J_1) & = & m_i(I), & & i = 1, \ldots, c, \\ m_i(J_1) & = & m_{i-1}(I) + d, & & i = 2, \ldots, c + 1. \end{array}$$

Proof. The proof is almost identical to that of Theorem 3.3. This time we need information about the smallest shift in any free module. An analysis similar to that in Theorem 3.3 shows immediately that

$$m_i(J_1) \le m_i(I), \quad 1 \le i \le c, \tag{3.8}$$

since no summand of $\mathbb{F}_i$ in the i-th free module of (3.5) is split off. But furthermore, if a splitting of a summand $R(-t-d)$ between $\mathbb{F}_i(-d)$ (in the $(i+1)$-st free module) and $\mathbb{G}_i(-d)$ (in the i-th free module) occurs, then $\mathbb{F}_i$ (in the i-th free module) contains the summand $R(-t)$, and $t < t + d$. Hence

$$m_i(J_1) \le m_i(J) + d, \ 1 \le i \le c. \tag{3.9}$$

With the three inequalities (3.7), (3.8) and (3.9), the proof is almost identical to the proof of the upper bound in Theorem 3.3 and is left to the reader. □

The following consequence of Theorem 3.3 and Corollary 3.7 has been proven independently in [9] using different methods.

Corollary 3.8. *Let J_1 be the hypersurface section of I by F, i.e. $J_1 = I + (F)$ where F is a homogeneous polynomial of degree $d > 0$ such that $I : F = I$. Let $c+1$ be the codimension of J_1. Assume that the lower and upper bounds of the Multiplicity conjecture hold for I. Then J_1 satisfies the conjectured bound, that is:*

$$\frac{1}{(c+1)!} m_1(J_1)m_2(J_1)\ldots m_{c+1}(J_1) \le e(R/J_1)$$
$$\le \frac{1}{(c+1)!} M_1(J_1)M_2(J_1)\ldots M_{c+1}(J_1).$$

Proof. The proof is very similar to the proof above, but now there is no need for any extra hypotheses. Indeed, we begin with the exact sequence

$$0 \to I(-d) \to I \oplus R(-d) \to J_1 \to 0.$$

We again consider a minimal free resolution $\mathbb{F}_\bullet$ for I, and now a mapping cone gives the following free resolution for J_1:

$$0 \to \mathbb{F}_c(-d) \to \begin{matrix} \mathbb{F}_{c-1}(-d) \\ \oplus \\ \mathbb{F}_c \end{matrix} \to \begin{matrix} \mathbb{F}_{c-2}(-d) \\ \oplus \\ \mathbb{F}_{c-1} \end{matrix} \to \cdots$$
$$\to \begin{matrix} \mathbb{F}_1(-d) \\ \oplus \\ \mathbb{F}_2 \end{matrix} \to \begin{matrix} \mathbb{F}_1 \\ \oplus \\ R(-d) \end{matrix} \to J_1 \to 0.$$

This time, there is no possible splitting, so this is a minimal free resolution. For the lower bound, we observe the (smaller) set of inequalities

$$\begin{array}{lll} m_i(J_1) & \le & m_i(I) \qquad \text{for } 1 \le i \le c, \\ m_1(J_1) & \le & d, \\ m_i(J_1) & \le & m_{i-1}(I) + d \quad \text{for } 2 \le i \le c+1. \end{array} \tag{3.10}$$

Then almost the same proof as given above (again using the trick of adding and subtracting d several times) yields the desired bound. It is simpler since we only have to "convert" terms involving $m_i(J_1)$ to terms involving $m_i(I)$; we do not have any $m_i(J)$ involved. In the very last step we use the bound $m_1(J_1) \le d$. We omit the details. The upper bound is proven similarly. □

Now we will discuss how Theorem 3.3 and Corollary 3.7 can be applied to show that the Multiplicity conjecture 1.1 is true for standard determinantal ideals.

Let A be a homogeneous $t \times (t+c-1)$ matrix with entries in R, i.e. such that multiplication by A defines a graded homomorphism $\varphi : F \to G$ of free R-modules where $\operatorname{rank}(F) = t+c-1$, $\operatorname{rank}(G) = t$, and that all entries of degree zero are zero. Then we denote the ideal generated by the maximal minors of A by $I(\varphi) = I(A)$. Its codimension is at most c. We call $I \subset R$ a *standard determinantal ideal* if $I = I(A)$ for some homogeneous $t \times (t+c-1)$ matrix A and $\operatorname{codim} I = c$. As is well known, the Eagon-Northcott complex gives a resolution of the ideal $I(A)$ generated by the maximal minors if $I(A)$ contains a regular sequence of length at least $\operatorname{rank}(F) - \operatorname{rank}(G) + 1 = c$.

Corollary 3.9. *The Multiplicity conjecture is true for all standard determinantal ideals.*

Furthermore, the following conditions are equivalent for every standard determinantal ideal $I = I(A)$ of codimension c:

(i) $e(R/I) = \frac{1}{c!}\prod_{i=1}^{c} M_i$;
(ii) $e(R/I) = \frac{1}{c!}\prod_{i=1}^{c} m_i$;
(iii) *I has a pure minimal free resolution;*
(iv) *all the entries in A have the same degree.*

Proof. Our method of proof is essentially the same as the one in [16] to show the claim in codimension two. This approach has also recently been carried out by Miró-Roig [17] to establish the bounds, so we just give enough details to discuss sharpness of the bounds. The result has also independently been shown by Herzog and Zheng [9].

First, we discuss the bounds. We will switch to the notation of the basic double link results. Let $J_1 = I(A) \subset R$ be a standard determinantal ideal defined by the homogeneous $t \times (t+c-1)$ matrix A $(t \geq 1)$. We will show the claim by induction on $t \geq 1$. If $t = 1$, then J_1 is a complete intersection and the bounds are shown in [8]. Since the bounds are trivial for principal ideals, we may assume that $c \geq 2$ and that the claims are shown for ideals of codimension $\leq c-1$.

Let $t \geq 2$. By reordering rows and columns we can arrange that the degrees of the entries of A increase from bottom to top and from left to right. Let us write the resulting degree matrix of A as follows:

$$\partial A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} & & * & \\ & a_{21} & a_{22} & \dots & a_{2c} & & \\ & & \ddots & \ddots & & \ddots & \\ * & & & a_{t1} & a_{t2} & \dots & a_{tc} \end{bmatrix}. \tag{3.11}$$

Note that our ordering of degrees means that

$$a_{i-1,j} \leq a_{ij} \leq a_{i,j+1}. \tag{3.12}$$

Observe that the whole degree matrix ∂A is completely determined by the entries specified in (3.11). Moreover, the Eagon-Northcott complex shows that the graded Betti numbers of J_1 are completely determined by ∂A. Hence, it suffices to show the bounds for $J_1 = I(A)$ where

$$A = \begin{bmatrix} x_1^{a_{11}} & x_2^{a_{12}} & \dots & x_c^{a_{1c}} & & 0 & \\ & x_1^{a_{21}} & x_2^{a_{22}} & \dots & x_c^{a_{2c}} & & \\ & & \ddots & \ddots & & \ddots & \\ 0 & & & x_1^{a_{t1}} & x_2^{a_{t2}} & \dots & x_c^{a_{tc}} \end{bmatrix}. \tag{3.13}$$

Denote by B the matrix that is obtained from A by deleting the last column and by A' the matrix that one gets after deleting the last row of B. Then $J := I(A')$ and $I := I(B)$ are standard determinantal ideals satisfying the relation

$$J_1 = x_c^{a_{tc}} \cdot J + I,$$

where $I : x_c^{a_{tc}} = I$, i.e. J_1 is a basic double link of J.

The Eagon-Northcott complex provides that

$$\begin{aligned} m_1(J_1) &= a_{11} + a_{21} + \ldots + a_{t1}, \\ m_i(J_1) &= m_{i-1}(J_1) + a_{ti}, \quad i = 2, \ldots, c, \\ M_1(J_1) &= a_{1c} + a_{2c} + \ldots + a_{tc}, \\ M_i(J_1) &= M_{i-1}(J_1) + a_{t,c+1-i}, \quad i = 2, \ldots, c, \end{aligned}$$

and similar formulas for J and I. Hence, using our ordering (3.12), it is easy to see that the assumptions of Theorem 3.3 and Corollary 3.7 are satisfied; thus it follows that the bounds are true for J_1.

Now we discuss sharpness. It suffices to show that either condition (i) or (ii) implies (iv). Assume (i) is true. Then Corollary 3.6 and the induction hypothesis show that all the entries of B have the same degree. But Corollary 3.6 also provides $M_{i-1}(I) = M_i(J_1)$, which forces the entries in the last column of A to have the same degree as the entries of B.

The proof that (ii) implies (iv) is similar; we omit the details. $\square$

As a further application of basic double links, we conclude this section with a closed formula for the degree of a standard determinantal ideal.

Theorem 3.10. *Let* $\varphi : \bigoplus_{j=1}^{t+c-1} R(-d_j) \to F := \bigoplus_{i=1}^{t} R(-b_i)$ *be a homomorphism of graded* R*-modules such that* $\operatorname{im}\varphi \subset \mathfrak{m} \cdot F$ *and* $\operatorname{coker}\varphi$ *has codimension* $c \geq 1$*. Assume that* $b_1 \leq b_2 \leq \ldots \leq b_t$ *and* $d_1 \leq d_2 \leq \ldots \leq d_{t+c-1}$*. Then the ideal of maximal minors of* φ *has degree*

$$\deg I(\varphi) = \sum_{i_c=1}^{t} a_{i_c,c} \left\{ \sum_{i_{c-1}=1}^{i_c} a_{i_{c-1},c-1} \left\{ \ldots \left\{ \sum_{i_2=1}^{i_3} a_{i_2,2} \cdot \prod_{i_1=1}^{i_2} a_{i_1,1} \right\} \ldots \right\} \right\}$$

where $a_{ij} := d_{i+j-1} - b_i$, $i = 1, \ldots, t$ *and* $j = 1, \ldots, c$ *and the formula reads for* $c = 1$ *as* $\deg I(\varphi) = \prod_{i_1=1}^{t} a_{i_1,1}$.

Proof. We use the approach employed in the proof of Corollary 3.9. We may assume that the map φ is represented by the matrix A as specified in (3.13). Denoting the matrices B and A' obtained from A by deleting the last column and row as in the proof of Corollary 3.9, we get

$$I(A) = x_c^{a_{tc}} \cdot I(A') + I(B);$$

thus

$$\deg I(A) = a_{tc} \cdot \deg I(A') + \deg I(B).$$

The cases $c = 1$ and $t = 1$ being trivial, our claim follows now easily by induction on c and t. $\square$

Remark 3.11. In the classical case $b_1 = \ldots = b_t = 0$ and $d_1 = \ldots = d_{t+c-1} = 1$, i.e. all entries of A have degree one, Theorem 3.10 specializes to

$$\deg I(\varphi) = \binom{t+c-1}{c},$$

which of course also follows by the classical Porteous formula.

4. Modules with quasi-pure resolutions

Herzog and Srinivasan proved in [8, Section 1] the Multiplicity conjecture for k-algebras with quasi-pure resolutions. In this section we generalize this result to the module case. Let N be a finitely generated graded R-module. Let

$$F_{\bullet} : 0 \to \bigoplus_{j=1}^{b_p} R(-d_{pj}) \to \cdots \to \bigoplus_{j=1}^{b_0} R(-d_{0j}) \to 0$$

be the minimal graded free resolution of N and define the invariants M_i and m_i as in the Introduction. We say that N has a *quasi-pure resolution* if $m_i \geq M_{i-1}$ for all $1 \leq i \leq p$. We follow the line of proof in [8] with the necessary changes.

Lemma 4.1. *Let N be a graded R-module of codimension $c \geq 1$. Then*

$$\sum_{i=0}^{p}(-1)^i \sum_{j=1}^{b_i} d_{ij}^k = \begin{cases} 0 & \text{if } 1 \leq k < c, \\ (-1)^c c! \cdot e(N) & \text{if } k = c. \end{cases}$$

Proof. Let $H_N(t)$ be the Hilbert series of N. Since N has a quasi-pure resolution, we can compute this series as

$$H_N(t) = \frac{\sum_{i=0}^{p} \sum_{j=1}^{b_i} (-1)^i t^{d_{ij}}}{(1-t)^n}.$$

On the other hand we know that

$$H_N(t) = \frac{Q(t)}{(1-t)^d}$$

where $Q(t)$ is a polynomial such that $Q(1) = e(N)$ equals the multiplicity of N and $d = \dim N$. Thus we get that

$$\sum_{i=0}^{p} \sum_{j=1}^{b_i} (-1)^i t^{d_{ij}} = Q(t)(1-t)^c.$$

Let $P(t)$ be the polynomial on the left hand side of this equation. It follows that

$$P^{(k)}(1) = \begin{cases} 0 & \text{for } 1 \leq k < c, \\ (-1)^c c! \cdot e(N) & \text{for } k = c. \end{cases} \tag{4.1}$$

We prove by induction on $k \in \{1, \dots, c\}$ and another induction on $l \in \{1, \dots, k\}$ that

$$P^{(k)}(1) = \sum_{i=0}^{p} \sum_{j=1}^{b_i} (-1)^i d_{ij}^l (d_{ij} - 1) \dots (d_{ij} - k + 1 + l - 1).$$

The cases for $l = k$ together with (4.1) give the desired formula of the lemma. For $k = l = 1$ and $k > 1$, $l = 1$ we have by the definition of $P(t)$ that

$$P^{(k)}(1) = \sum_{i=0}^{p} \sum_{j=1}^{b_i} (-1)^i d_{ij}(d_{ij} - 1) \dots (d_{ij} - k + 1).$$

Observe that we do not have to distinguish whether the d_{ij} are greater than or equal to k since we can add the remaining terms which are zero. Assume that

$k > 1$ and $l > 1$ and we know by the second induction hypothesis for l that

$$P^{(k)}(1) = \sum_{i=0}^{p}\sum_{j=1}^{b_i}(-1)^i d_{ij}^{l-1}(d_{ij}-1)\dots(d_{ij}-k+1+(l-1)-1).$$

It follows from the induction hypothesis on k that

$$P^{(k-1)}(1) = \sum_{i=0}^{p}\sum_{j=1}^{b_i}(-1)^i d_{ij}^{l-1}(d_{ij}-1)\dots(d_{ij}-k+2+(l-1)-1).$$

On the other hand we know $P^{(k-1)}(1) = 0$ as was already observed in (4.1). Hence we have that

$$P^{(k)}(1) = P^{(k)}(1)+(k-l+1)P^{(k-1)}(1) = \sum_{i=0}^{p}\sum_{j=1}^{b_i}(-1)^i d_{ij}^{l}(d_{ij}-1)\dots(d_{ij}-k+1+l-1)$$

as desired. □

For modules with quasi-pure resolution we get the following result, which hints at a possible generalization of the Multiplicity conjecture to the case of modules.

Notice the degree of a maximal module is simply equal to its rank. Thus, we exclude this case and focus on torsion modules throughout the remainder of this note.

Theorem 4.2. *Let $N = \bigoplus_{i\in\mathbb{Z}} N_i$ be a finitely generated graded R-module with $p = \operatorname{proj}\dim N$. Assume that N is Cohen-Macaulay,* $\operatorname{rank} N = 0$*, and that N has a quasi-pure resolution. Then*

$$b_0 \cdot \prod_{i=1}^{c}(m_i - M_0)/p! \le e(N) \le b_0 \cdot \prod_{i=1}^{c}(M_i - m_0)/p!$$

with equality below (resp. above) if and only if N has a pure resolution.

Proof. We consider the $(p+1)\times(p+1)$-square matrix

$$A = \begin{pmatrix} b_0 & b_1 & \cdots & b_p \\ \sum_{j=1}^{b_0} d_{0j} & \sum_{j=1}^{b_1} d_{1j} & \cdots & \sum_{j=1}^{b_p} d_{pj} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{b_0} d_{0j}^p & \sum_{j=1}^{b_1} d_{1j}^p & \cdots & \sum_{j=1}^{b_p} d_{pj}^p \end{pmatrix}.$$

By replacing the first column of A by the alternating sum of all columns of A, we obtain a matrix A' such that $\det(A) = \det(A')$. Since $\sum_{i=0}^{p} b_i = \operatorname{rank} N = 0$ and, by Lemma 4.1, $\sum_{i=0}^{p}(-1)^i\sum_{j=1}^{b_i} d_{ij}^p = (-1)^p p! e(N)$, it follows that the first column of A' is the transpose of the vector $(0,\dots,0,(-1)^p p! e(N))$. Hence by expanding the determinant of A' with respect to the first column, we get

$$\det(A) = \det(A') = p! e(N)\det(B),$$

where B is the $p\times p$ matrix

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_p \\ \sum_{j=1}^{b_1} d_{1j} & \sum_{j=1}^{b_2} d_{2j} & \cdots & \sum_{j=1}^{b_p} d_{pj} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{b_1} d_{1j}^{p-1} & \sum_{j=1}^{b_2} d_{2j}^{p-1} & \cdots & \sum_{j=1}^{b_p} d_{pj}^{p-1} \end{pmatrix}.$$

Hence

$$\det(A) = p!e(N)\det(B) = p!e(N)\sum_{j_1=1}^{b_1}\sum_{j_2=1}^{b_2}\cdots\sum_{j_p=1}^{b_p} U(d_{1j_1},\ldots,d_{pj_p}) \tag{4.2}$$

with the Vandermonde determinants

$$U(d_{1j_1},\ldots,d_{pj_p}) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_{1j_1} & d_{2j_2} & \cdots & d_{pj_p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{1j_1}^{p-1} & d_{2j_2}^{p-1} & \cdots & d_{pj_p}^{p-1} \end{pmatrix}.$$

We may also directly compute the determinant of A as

$$\det(A) = \sum_{j_0=1}^{b_0}\sum_{j_1=1}^{b_1}\cdots\sum_{j_p=1}^{b_p} V(d_{0j_0},\ldots,d_{pj_p}) \tag{4.3}$$

with the corresponding Vandermonde determinants

$$V(d_{0j_0},\ldots,d_{pj_p}) = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ d_{0j_0} & d_{1j_1} & \cdots & d_{pj_p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{0j_0}^{p} & d_{1j_1}^{p} & \cdots & d_{pj_p}^{p} \end{pmatrix}.$$

Since N has a quasi-pure resolution we have that $d_{ij} \geq d_{i-1k}$ for all i, j, k and thus all the involved Vandermonde determinants are always non-negative. Observe that

$$V(d_{0j_0},\ldots,d_{pj_p}) = \prod_{i=1}^{p}(d_{ij_i} - d_{0j_0})U(d_{1j_1},\ldots,d_{pj_p})$$

and thus

$$\prod_{i=1}^{p}(m_i - M_0)\cdot U(d_{1j_1},\ldots,d_{pj_p}) \leq V(d_{0j_0},\ldots,d_{pj_p}) \leq \prod_{i=1}^{p}(M_i - m_0)\cdot U(d_{1j_1},\ldots,d_{pj_p})$$

with equality below or above for all involved indices if and only if N has a pure resolution. Hence it follows from (4.2) and (4.3) that

$$b_0 \cdot \prod_{i=1}^{p}(m_i - M_0) \leq p!e(N) \leq b_0 \cdot \prod_{i=1}^{p}(M_i - m_0)$$

with equality below or above if and only if N has a pure resolution. This concludes the proof. □

For modules with pure resolution we get the following nice formula for the multiplicity. It generalizes the Huneke-Miller formula for the multiplicity of ideals with a pure resolution [10].

Corollary 4.3. *Let $N = \bigoplus_{i\in\mathbb{N}} N_i$ be a finitely generated graded R-module of codimension c. Assume that N is Cohen-Macaulay,* $\operatorname{rank} N = 0$*, and that N has a pure resolution* $0 \to R^{b_c}(-d_c) \to \cdots \to R^{b_0}(-d_0) \to 0$*. Then*

$$e(N) = b_0 \cdot \prod_{i=1}^{c}(d_i - d_0)/c!$$

5. Modules of codimension 2

In [19] the third author proved Conjecture 1.2 for k-algebras of codimension 2. The goal of this section will be to generalize this result to the module case. As in [19], one of our tools is the use of general hyperplane sections. Let $N = \bigoplus_{i\in\mathbb{N}}$ be a finitely generated graded R-module. Following [1] we call an element $x \in R_1$ *almost regular* for N if

$$(0 :_N x)_a = 0 \text{ for } a \gg 0.$$

A sequence $x_1, \ldots, x_t \in R_1$ is an *almost regular sequence* for N if for all $i \in \{1, \ldots, t\}$ the element x_i is almost regular for $N/(x_1, \ldots, x_{i-1})N$. Since neither the Betti numbers nor the multiplicity of N changes by enlarging the field, we may always assume that k is an infinite field. It is well known that then after a generic choice of coordinates we can achieve that a k-basis of R_1 is almost regular for N.

Let $x \in R_1$ be generic and almost regular for N. We observe that the following holds:

(1) If $\dim N > 0$, then $\dim N/xN = \dim N - 1$.
(2) If $\dim N > 1$, then $e(N) = e(N/xN)$.
(3) If $\dim N = 1$, then $e(N) \le e(N/xN)$ with equality if and only if N is Cohen-Macaulay.

We are ready to prove the main theorem of this section.

Theorem 5.1. *Let $N = \bigoplus_{i\in\mathbb{Z}}$ be a finitely generated graded R-module with* $\operatorname{rank} N = 0$.

(i) *If* $\operatorname{codim} N = 1$*, then*
$$e(N) \le b_0 \cdot (M_1 - m_0)$$
with equality if and only if N is Cohen-Macaulay and N has a pure resolution.

(ii) *If* $\operatorname{codim} N = 2$*, then*
$$e(N) \le b_0 \cdot (M_1 - m_0)(M_2 - m_0)/2$$
with equality if and only if N is Cohen-Macaulay and N has a pure resolution.

Proof. We only prove (ii) since the proof of (i) is simpler and is shown analogously. As noticed above we may assume that $x_1, \ldots, x_n \in R_1$ is a generic almost regular sequence for N.

Let $\mathbf{x} = x_1, \ldots, x_{n-2}$ and consider $\tilde{N} = N/\mathbf{x}N$. Observe that $2 = \operatorname{codim} N = \operatorname{codim} \tilde{N}$ and $e(N) \le e(\tilde{N})$ with equality if and only if N is Cohen-Macaulay. Notice that $\tilde{N}$ is a finitely generated graded $\tilde{R}$-module, where $\tilde{R}$ is the 2-dimensional polynomial ring $R/\mathbf{x}R$. Let

$$M_i = M_i(N) \text{ and } \tilde{M_i} = M_i(\tilde{N}) \text{ for } i = 1, 2.$$

Define similarly $b_0, \tilde{b_0}$ and $m_0, \tilde{m_0}$. We claim that

$$\tilde{b_0} = b_0,\ \tilde{m_0} = m_0,\ \tilde{M_1} \le M_1 \text{ and } \tilde{M_2} \le M_2. \tag{5.1}$$

Note that for a Cohen-Macaulay module N we have equalities everywhere.

Since $\dim \tilde{N} = 0$, the module $\tilde{N}$ is Cohen-Macaulay. Hence $\tilde{M_2} > \tilde{M_1} > \tilde{M_0}$ and $\tilde{m_2} > \tilde{m_1} > \tilde{m_0}$. Let $\tilde{N} \cong \tilde{F}/\tilde{G}$ where $\tilde{F}$ is the first finitely generated graded free $\tilde{R}$-module of a minimal free resolution of $\tilde{N}$ and $\tilde{G}$ the kernel of the map $\tilde{F} \to \tilde{N}$.

Consider the module $\tilde{N}' = \tilde{F}/\tilde{G}_{\geq M_1}$. Observe that $\tilde{N}'$ is still Cohen-Macaulay of codimension 2 because $\dim \tilde{N}' = 0$. Let $\tilde{b}'_i, \tilde{M}'_i, \tilde{m}'_i$ denote the corresponding invariants of $\tilde{N}'$. We see that

$$\tilde{b}'_0 = \tilde{b}_0,\ \tilde{M}'_0 = \tilde{M}_0,\ \tilde{m}'_0 = \tilde{m}_0,\ \tilde{M}'_1 = \tilde{m}'_1 = \tilde{M}_1, \text{ and } \tilde{M}'_2 = \tilde{M}_2.$$

Furthermore we have that

$$\tilde{m}'_2 > \tilde{m}'_1 = \tilde{M}'_1.$$

Thus $\tilde{N}'$ is Cohen-Macaulay of rank 0 and has a quasi-pure resolution. Hence we may apply Theorem 4.2 to the module $\tilde{N}'$. Note that

$$e(\tilde{N}) \leq e(\tilde{N}')$$

with equality if and only if $\tilde{M}_1 = \tilde{m}_1$. All in all we get that

$$\begin{array}{rcl} e(N) & \leq & e(\tilde{N}) \\ & \leq & e(\tilde{N}') \\ & \leq & \tilde{b}'_0(\tilde{M}'_1 - \tilde{m}'_0)(\tilde{M}'_2 - \tilde{m}'_0) \\ & = & \tilde{b}_0 \cdot (\tilde{M}_1 - \tilde{m}_0)(\tilde{M}_2 - \tilde{m}_0) \\ & \leq & b_0 \cdot (M_1 - m_0)(M_2 - m_0) \end{array}$$

with equalities everywhere if and only if N is Cohen-Macaulay and has a pure resolution.

It remains to prove claim (5.1). The first three inequalities can easily be seen: $\tilde{b}_0$ is the number of elements in a minimal system of generators of $\tilde{N}$ and b_0 is the corresponding number for N. $\tilde{m}_0$ is the minimal degree of a minimal generator of $\tilde{N}$ and m_0 is the minimal degree of a minimal generator of N. By Nakayama's lemma we know that the number of elements in a minimal system of generators and their degrees do not change by passing from N to $\tilde{N}$. If $N = F/G$, where F is the first finitely generated graded free R-module of a minimal free resolution of N and G the kernel of the map $F \to N$, then $\tilde{N} \cong \big(F/\mathbf{x}F\big)/\big(G + \mathbf{x}F/\mathbf{x}F\big)$. Thus we see that $\tilde{M}_1 \leq M_1$.

To prove the remaining inequality $\tilde{M}_2 \leq M_2$, we can use the proof of Theorem 2.4 in [19] word by word since that proof holds also in the module case. This completes the argument. □

Remark 5.2. Theorem 5.1 extends Theorem 2.4 in [19] (cf. also [9], Theorem 3.1) from cyclic modules to arbitrary modules (of codimension two). It proves Conjecture 1.2 for modules whose codimension is at most two.

Theorem 4.2 suggests for a Cohen-Macaulay torsion module N of codimension c that $\frac{b_0}{c!}\prod_{i=1}^{c}(m_i - M_0) \leq e(N)$. While this would be an interesting bound in some cases, for example, if all generators of N have the same degree, it does not always give useful information because this lower bound can be a negative number.

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