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# THE LUSIN AREA FUNCTION AND LOCAL ADMISSIBLE CONVERGENCE OF HARMONIC FUNCTIONS ON HOMOGENEOUS TREES

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Abstract. We prove admissible convergence to the boundary of functions that are harmonic on a subset of a homogeneous tree by means of a discrete Green formula and an analogue of the Lusin area function.

## 1. Introduction

A classical result in analysis, the Fatou non-tangential convergence theorem, characterizes the boundary behavior of harmonic functions in the Euclidean half plane $\mathbb{R}^2_+$ as follows. The boundary of $\mathbb{R}^2_+$ is identified with the real line $\mathbb{R}$. For any $z \in \mathbb{R}$ and any $\alpha > 0$, the cone $\Gamma_\alpha(z)$ with vertex $z$ is the set

$$\Gamma_\alpha(z) = \left\{(x,y) \in \mathbb{R}^2_+ : |x-z| < \alpha y\right\}.$$

If $f(x,y)$ is defined at those points in $\mathbb{R}^2_+$ near a boundary point $z$, then $f$ has a *non-tangential (or admissible) limit* at $z$, say equal to $l$, if for every $\alpha > 0$ the conditions $(x,y) \in \Gamma_\alpha(z)$ and $(x,y) \to z$ imply that $f(x,y) \to l$. Moreover, $f$ is called *non-tangentially (or admissibly) bounded* at $z$ if, for some $\alpha$, $f$ is bounded in $\Gamma_\alpha(z)$ (by a constant which depends on $\alpha$ and $z$). The Fatou theorem states that if $f$ is harmonic in $\mathbb{R}^2_+$, then the non-tangential boundedness of $f$ and the existence of its non-tangential limits are almost everywhere equivalent.

This result can be localized as follows. For every point $x$ of any measurable subset $E$ of $\mathbb{R}$, choose a cone (of arbitrary width) in $\mathbb{R}^2_+$ with vertex in $x$, and form the union $\tilde{E}$ of all these cones. Then the local version of the Fatou theorem asserts that, if $f$ is defined in $\tilde{E}$ and harmonic, then the non-tangential boundedness of $f$ and the existence of its non-tangential limits are almost everywhere equivalent in $E$.

There is another condition, known as the Lusin area theorem, which is equivalent to non-tangential boundedness of harmonic functions. This condition is expressed in terms of the area integral introduced by Lusin (see [18]). For every $\alpha > 0$ the area integral of $f$ is a function defined on $\mathbb{R}$ which applies to $z \in \mathbb{R}$ the integral on $\Gamma_\alpha(z)$ of the square of the gradient of $f$.

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The area theorem states the almost everywhere equivalence between non-tangential boundedness and the finiteness of the area integral for harmonic functions on $\mathbb{R}^2_+$.

The local Fatou theorem has been proved by Privalov, and the area theorem is due to Marcinkiewicz and Zygmund and Spencer (see [18]).

The general version of the local Fatou theorem and the area theorem for Euclidean half-spaces is due to A. P. Calderon [2] and E. M. Stein [16] (see also [17, chapter VII]).

The approach of Stein relies on the differential properties of the Laplacian. An important tool is the Green formula which is used to transform the area integral of a harmonic function over smooth compact domains to an integral over their boundary curves.

In analogy with the case of Euclidean spaces, the local Fatou theorem and the area theorem have been naturally generalized to the Poincarè half-plane or hyperbolic disc [3]. Observe that in the hyperbolic metric the cone $\Gamma_\alpha(z)$ becomes a tube around a geodesic whose end point is $z$. For general symmetric spaces of rank one similar results have been proved by A. Korànyi and R. Putz [7], via a suitable extension of Stein's approach based on the Green formula. P. Malliavin and M. P. Malliavin [10] extended the area integral to the product of two hyperbolic discs and used it to prove the local Fatou theorem in this framework; shortly later A. Korànyi and R. Putz [8] generalized this argument to products of rank one symmetric spaces. Except for these degenerate cases, the asymptotic behavior of harmonic functions has never been studied for higher rank symmetric spaces via the geometric approach based on the Lusin area integral and the Green formula, although the area integral has been used in a probabilistic approach to describe local admissible convergence almost everywhere of functions harmonic everywhere on a Riemannian space of negative curvature [11]. Indeed, a higher-dimensional extension of the area integral which could be suitable to investigate non-tangential limits of harmonic functions in a geometric way has never been obtained. In higher rank, the algebra of invariant differential operators is not generated by the Laplace operator only: it has as many independent generators as the rank. Instead of considering harmonic functions, one studies the jointly harmonic functions of these generators. Admissible convergence of harmonic functions in this sense has been studied by other methods (see, for instance, [15]). However, these methods rely upon the Poisson integral of boundary data, and therefore yield global harmonic functions: by their very nature, they cannot provide local admissible convergence results for functions that are harmonic only locally. Instead, to prove a local Fatou theorem we would need an area integral and an approach similar to [7] and [17], based upon surrounding the region of harmonicity with a contour and transporting the area integral over that region to a lower dimensional contour integral.

A natural discrete counterpart of semisimple symmetric spaces of non-compact type of rank one is given by homogeneous spaces of semisimple p-adic groups. It consists of the so-called Bruhat-Tits buildings. In the rank one case, these buildings are nothing but homogeneous and semihomogeneous trees (see [14]). A natural Laplace operator on a homogeneous tree is the nearest neighbor isotropic transition operator. This leads to a fruitful theory of harmonic functions defined on the vertices of homogeneous trees (see [5]). Non-tangential convergence of harmonic functions on homogeneous trees has been studied in [6].

The area integral has another important application in classical harmonic analysis. It was proved in [1] that the non-tangential maximal function of a harmonic function $f$ in $\mathbb{R}^2_+$ is $L^p$-bounded if and only if its area function is $L^p$-bounded (and if and only if its square function, defined in terms of martingales associated with $f$, is also $L^p$-bounded). This theorem is the building block of a wide-reaching extension of the classical theory of $H^p$ spaces. For the purpose of investigating $H^p$ spaces, the area function has been introduced in [9] on a large class of trees, which includes the homogeneous ones. This, in turn, sheds light on the boundary behavior of harmonic functions thereon, although this subject was only briefly mentioned at the end of [9]. The methods used in this reference are probabilistic. An independent approach to $H^p$ theory, inspired by Stein's methods (good lambda inequalities), was developed in [4]: this approach is more geometric, because it makes use of a natural adaptation of the Green formula (hence it carries to the discrete framework of trees methods which are typical of differential analysis). This viewpoint is crucial for the spirit of the present paper.

Indeed, in this paper, following an idea of A. Korànyi, we return to the subject of admissible convergence of harmonic functions on homogeneous trees, and focus our attention on the local boundary behavior, i.e., the local Fatou theorem. The Green formula of [4] allows us to transport to trees the approach of [17] and [7]. Our proof follows the arguments of [17] and for the last part (Theorem 5.3) an idea of C. Fefferman extended to rank-one symmetric spaces in [7]. However, as usual when transporting to trees methods tailored for continuous environments, we must handle several intriguing complications. A part of the proof (Theorems 5.1 and 5.2) could be derived by estimates for the area functions given in [4], but the proof of [4], based on the combinatorial-probabilistic tools of [9], is considerably more complicated (it is aimed to $L^p$ estimates, not just pointwise estimates almost everywhere).[1]

The motivation for this work is the hope of opening the way for future extensions of these results to higher rank Bruhat-Tits buildings. Buildings are still simplicial complexes and their vertex set is countable. Therefore an appropriate definition of area function, suitable to handle admissible convergence of harmonic functions, is a purely combinatorial task, therefore presumably simpler than for symmetric spaces. We share with Adam Korànyi the feeling that such a definition would shed light on how to introduce an area function on higher rank symmetric spaces. Unfortunately, buildings have a very intriguing and complicated combinatorial structure, and an extension of our results is a very challenging task.

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[1] *Note added in proof:* Pointwise estimates of this type, based on the approach and the results of [9], were also obtained in [12], whose arguments, phrased in a more probabilistic and less combinatorial fashion, do not share our aim to emphasize the geometric analogy between trees and symmetric spaces (the Green formula is not used).

## 2. Notation and preliminaries

2.1. **Trees.** We follow most of the terminology established in [4]. Here is a review. A *tree* $T$ is a connected, simply connected, locally finite graph. With abuse of notation we shall also write $T$ for the set of vertices of the tree. We suppose that $T$ is *homogeneous*, that is, every vertex of $T$ belongs exactly to $q+1$ edges, where $q \geqslant 2$ is a constant. For $x,\ y \in T$ we write $x \sim y$ if $x,\ y$ are neighbours. For any $x,\ y \in T$ there exist a unique $n \in \mathbb{N}$ and a unique minimal finite sequence $(z_0, \dots, z_n)$ of distinct vertices such that $z_0 = x$, $z_n = y$ and $z_k \sim z_{k+1}$ for all $k < n$; this sequence is called the *geodesic* path from $x$ to $y$ and is denoted by $[x, y]$. The integer $n$ is called the *length* of $[x, y]$ and is denoted by $d(x, y)$; $d$ is a metric on $T$. We fix a reference vertex $o \in T$ and call it the *origin*. The choice of $o$ induces a partial ordering in $T$: $x \leqslant y$ if $x$ belongs to the geodesic from $o$ to $y$. For every $x \in T$, $x \neq o$, there exists exactly one *predecessor* $x^-$ of $x$ in this ordering (i.e., $x^- \leqslant x$ and $x^- \sim x$). For $x \in T$, the length $|x|$ of $x$ is defined as $|x| = d(o, x)$. For any vertex $x$ and any integer $k \leqslant |x|$, $x_k$ is the vertex of length $k$ in the geodesic $[o, x]$. For $\alpha \leqslant |x|$, $x(-\alpha)$ denotes the $\alpha$-*retract* of $x$, that is, the vertex $x_{|x|-\alpha}$ in the geodesic $[o, x]$. For $\alpha > |x|$, we let $x(-\alpha) = o$.

For $0 \leqslant s < k < \infty$, let $[s; k]$ be the *corona* $\{x \in T : s \leqslant |x| \leqslant k\}$ and $[s; \infty) = \{x \in T : |x| \geqslant s\}$. For $k \in \mathbb{N}$ let $S_k$ be the *circle* $\{x \in T : |x| = k\}$.

Let $\Omega$ be the set of infinite geodesics starting at $o$. In analogy with the previous notation, for $\omega \in \Omega$ and $n \in \mathbb{N}$, $\omega_n$ is the vertex of length $n$ in the geodesic $\omega$. For $x \in T$ the *interval* $U(x) \subset \Omega$, generated by $x$, is the set $U(x) = \big\{\omega \in \Omega : x = \omega_{|x|}\big\}$. The sets $U(\omega_n)$, $n \in \mathbb{N}$, form an open basis at $\omega \in \Omega$. Equipped with this topology $\Omega$ is compact and totally disconnected.

For every positive integer $n$ the family $\{U(x) : |x| = n\}$ is a partition of $\Omega$ into $(q+1)q^{n-1}$ open and closed sets. We define the $o$-isotropic measure $\nu$ on the algebra of sets generated by the sets $U(x)$, by

$$\nu\left(U(x)\right) = (q+1)^{-1} q^{1-|x|}. \tag{2.1}$$

The measure $\nu$ extends to a regular Borel probability measure on $\Omega$.

2.2. **Harmonic functions and Poisson kernel.** We refer the reader to [5] for more details on the contents of this section.

**Definition 1.** The simple average operator $L$ on functions $f$ on the vertices of $T$ is

$$Lf(x) = (q+1)^{-1} \sum_{y : d(x,y)=1} f(y).$$

The Laplace operator associated with $L$ is $\Delta f = Lf - f$.

**Definition 2.** A function $f : T \to \mathbb{R}$ is harmonic if $\Delta f(x) = 0$ for every $x \in T$.

We shall say that $f$ is harmonic at $x$ if $\Delta f(x) = 0$ for the vertex $x$.

For $x \in T$ and $\omega \in \Omega$, let $N(x, \omega) \in \mathbb{N}$ be the *bifurcation index*, that is, the number of edges in common between the finite geodesic $[o, x]$ and the infinite geodesic $\omega$.

**Definition 3.** For every $x \in T$, $\omega \in \Omega$ the Poisson kernel $P(x, \omega)$ is

$$P(x, \omega) = q^{2N(x,\omega)-|x|}.$$

For every $x \in T$, $P(x,\cdot)$ is a function on $\Omega$ which is constant on each interval $U(z)$ with $|z| = |x|$. Therefore $P(x,\cdot) \in L^2(\Omega)$ for every $x \in T$. For each $\omega \in \Omega$ the function $P(\cdot,\omega)$ is harmonic in $T$.

The *Poisson integral* of a function $h$ in $L^1(\Omega)$ is defined by

$$\mathbf{P}h(x) = \int_\Omega h(\omega)P(x,\omega)d\nu.$$

More generally, by integrating measures on $\Omega$ against the Poisson kernel one obtains harmonic functions on the vertices of $T$. In this sense the measure $\nu$ on $\Omega$ represents the harmonic function with constant value 1. It is called the *Poisson measure* and its support $\Omega$ is the *Poisson boundary* of $T$.

2.3. **The Green formula.** We need results introduced in [4]. We state them here. Denote by $\Lambda$ the set of all the oriented edges (i.e., ordered pairs of neighbours). For $\sigma \in \Lambda$ denote by $b(\sigma)$ the beginning vertex of $\sigma$ and by $e(\sigma)$ the ending vertex: $\sigma = (b(\sigma), e(\sigma))$. The choice of a reference vertex $o \in T$ (see Subsection 2.1) gives rise to a positive orientation on edges: an edge $\sigma$ is *positively oriented* if $b(\sigma)$ is the predecessor of $e(\sigma)$.

The beginning and ending vertices induce two maps $b : \Lambda \to T$ and $e : \Lambda \to T$ defined as above. These maps induce two different liftings, $f \circ b$ and $f \circ e$, for any $f : T \to \mathbb{R}$.

**Definition 4.** For any function $f : T \to \mathbb{R}$, the gradient $\nabla f : \Lambda \to \mathbb{R}$ is

$$\nabla f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

For $x \in T$, let $\Lambda(x) = \{\sigma \in \Lambda : b(\sigma) = x\}$ be the *star* of $x$.

**Definition 5.** For $x \in T$, let

$$\|\nabla f(x)\|^2 = (q+1)^{-1} \sum_{\sigma \in \Lambda(x)} |\nabla f(\sigma)|^2.$$

**Definition 6.** The *boundary* $\partial Q$ of a subset $Q \subset T$ is $\partial Q = \{\sigma \in \Lambda : b(\sigma) \in Q, e(\sigma) \notin Q\}$. The *trace* $b(A)$ of a subset $A \subset \Lambda$ is $b(A) = \{b(\sigma) : \sigma \in A\}$. For $Q \subset T$, the set $b(\partial Q) = \{x \in Q : y \notin Q \quad \text{for some} \quad y \sim x\}$ is also called the *frontier* of $Q$.

The Green formulas are well known in the continuous setup. In this discrete context an interesting analogue has been observed in [4].

**Proposition 2.1** (The Green formula)**.** *If $f$ and $h$ are functions on $T$ and $Q$ is a finite subset of $T$, then*

$$\sum_Q (h\Delta f - f\Delta h) = (q+1)^{-1} \sum_{\partial Q} (h \circ b \nabla f - f \circ b \nabla h).$$

**Proposition 2.2** (The Green identity)**.** *For all real valued functions $f$ on $T$ and for every $x \in T$,*

$$\Delta(f^2)(x) = \|\nabla f(x)\|^2 + 2f(x)\Delta f(x).$$

## 3. Non-tangential convergence, local Fatou theorem and the area function

For $x \in T$ and $\omega \in \Omega$ we consider the distance $d(x,\omega) = \min_{j\in\mathbb{N}} d(x,\omega_j)$.

**Definition 7.** Let $\alpha \geqslant 0$ be an integer; the tube $\Gamma_\alpha(\omega)$ around the geodesic $\omega \in \Omega$ is

$$\Gamma_\alpha(\omega) = \{x \in T : d(x,\omega) \leqslant \alpha\}\,.$$

Recalling the definition of $\alpha$-retract given in Subsection 2.1 we observe that $x \in \Gamma_\alpha(\omega) \Leftrightarrow \omega \in U(x(-\alpha))$, in particular $x \in \Gamma_0(\omega) \Leftrightarrow \omega \in U(x)$.

**Definition 8.** The area function of $f$ on $T$ is the function on $\Omega$ defined by

$$A_\alpha f(\omega) = \left( \sum_{x\in\Gamma_\alpha(\omega)} \|\nabla f(x)\|^2 \right)^{\frac{1}{2}}\,.$$

Observe that, if $f \in L^1(T)$, $A_\alpha f(\omega) < \infty$ for every $\alpha$, $\omega$.

**Definition 9.** A function $f$ on $T$ has non-tangential limit at $\omega \in \Omega$ if, for every integer $\alpha \geqslant 0$, $\lim f(x)$ exists as $|x| \to \infty$ and $x \in \Gamma_\alpha(\omega)$.

**Definition 10.** A function $f$ on $T$ is non-tangentially bounded at $\omega \in \Omega$ if, for some $M > 0$, one has $|f(x)| \leqslant M$ for $x \in \Gamma_0(\omega)$.

*Remark.* The terminology adopted in the previous definition is meaningful because it amounts to the more elegant statement that, for some non-negative integer $\alpha = \alpha(\omega)$, $f$ is bounded on a tube of width $\alpha$ around the geodesic $\Gamma_0(\omega)$ (by a different constant depending on $\alpha$).

The main goal of this paper is the following result, which extends to homogeneous trees the celebrated Lusin area theorem [18]. In analogy with the previous Remark, condition (iii) below is equivalent to the classical statement that at almost every boundary point there is a width $\alpha \geqslant 0$ such that the area function of width $\alpha$ is finite.

**Main Theorem.** *Let $E$ be a measurable subset of $\Omega$ and $f$ a harmonic function on $T$. Then the following are equivalent:*

(i) *$f$ is non-tangentially bounded at almost every $\omega \in E$;*
(ii) *$f$ has non-tangential limit at almost every $\omega \in E$;*
(iii) *for almost every $\omega \in E$*

$$A_0 f(\omega) < \infty;$$

(iv) *for every fixed $\alpha \geqslant 0$, $A_\alpha f(\omega) < \infty$ for almost every $\omega \in E$.*

## 4. Uniform estimates

We need some more notation and some lemmas.

**Definition 11.** The tube $W_\alpha(E)$ over a measurable subset $E$ of $\Omega$ is

$$W_\alpha(E) = \bigcup_{\omega\in E} \Gamma_\alpha(\omega).$$

For any integer $s > 0$ let $W^s_\alpha(E) = \bigcup_{\omega\in E} \Gamma_\alpha(\omega) \cap [s;\infty)$.

Now we want to impose some uniform conditions.

**Lemma 4.1.** *For every measurable subset $E$ of $\Omega$,*

$$\lim_n \frac{\nu(E \cap U(\omega_n))}{\nu(U(\omega_n))} = 1$$

*for almost every $\omega \in E$.*

*Proof.* Given a function $F \in L^1(\Omega)$, consider the martingale associated with $F$; that is, the sequence of functions

$$F_n(\omega) = \frac{1}{\nu(U(\omega_n))} \int_{U(\omega_n)} F d\nu.$$

See [5] for more details. It follows from the Martingale Convergence Theorem (see [9, page 225]) that $\lim_n F_n = F$ almost everywhere. To prove the lemma it is sufficient to take $F = \chi_E$, the characteristic function of $E$. □

**Proposition 4.2.** *Let $E$ be a measurable subset of $\Omega$ and let $\varepsilon > 0$. There exists a closed set $D$ with $D \subset E$ and $\nu(E \setminus D) < \varepsilon$ such that for any integers $\alpha$ and $\beta$, there exists an integer $s$ such that $\Gamma_\beta(\omega) \cap [s; \infty) \subset W_\alpha(E)$ for every $\omega \in D$.*

*Proof.* Since $W_0(E) \subset W_\alpha(E)$ for every $\alpha > 0$, it is sufficient to prove the assertion for $\alpha = 0$. Given $\eta < 1$ and $\varepsilon > 0$, by Lemma 4.1 there exists a set $D \subset E$ such that $\nu(E \setminus D) < \varepsilon$ and an integer $m$, independent of $\omega$, such that

$$\frac{\nu(U(\omega_j) \cap E)}{\nu(U(\omega_j))} > \eta \tag{4.1}$$

for $j \geqslant m$ and all $\omega \in D$. Indeed, this is an immediate consequence of Egoroff's theorem (see [13]). We can choose $D$ to be closed because $\nu$ is regular.

Let $s \geqslant m + \beta$. Suppose that for some $x \in \Gamma_\beta(\omega) \cap [s; \infty)$, with $\omega \in D$, there does not exist any $\omega' \in E$ such that $x \in \Gamma_0(\omega')$. Then $U(x) \cap E = \emptyset$. Let $\omega_j$ be the confluence point of $[o, x]$ and $\omega$. Then $j \geqslant |x| - \beta \geqslant s - \beta$, since $\omega \in U(x(-\beta))$ and $|x| \geqslant s$. Moreover $U(\omega_j) \supset U(x)$ and

$$\frac{\nu(U(\omega_j) \cap E)}{\nu(U(\omega_j))} \leqslant \frac{\nu(U(\omega_j)) - \nu(U(x))}{\nu(U(\omega_j))} \leqslant 1 - \frac{1}{q^\beta}.$$

This yields a contradiction if $\eta \geqslant 1 - \frac{1}{q^\beta}$. □

**Corollary 4.3.** *Let $E$ be a measurable subset of $\Omega$ and $f$ be non-tangentially bounded at every $\omega \in E$. For $\varepsilon > 0$ there exists a closed set $D$ with $D \subset E$, $\nu(E \setminus D) < \varepsilon$ such that for any fixed integer $\alpha \geqslant 0$, there exists a constant $M = M(\alpha, \varepsilon)$ (independent of $\omega$) with $|f| \leqslant M$ on $W_\alpha(D)$.*

*Proof.* Since $f$ is non-tangentially bounded on $E$, $f$ is bounded on $\Gamma_0(\omega)$ for every $\omega \in E$. Let $E_k = \{\omega \in E : |f| \leqslant k \quad \text{in} \quad \Gamma_0(\omega)\}$. Then $E_k \subset E_{k+1}$ and $\bigcup_k E_k = E$ because $f$ is non-tangentially bounded on $E$. For $\varepsilon > 0$ we can choose $k_0$ and $\tilde{D} = E_{k_0}$ such that $\nu(E \setminus \tilde{D}) < \frac{\varepsilon}{2}$ and $|f| \leqslant k_0$ on $W_0(\tilde{D})$. By Proposition 4.2 there exists $s > 0$ and a closed set $D$ with $D \subset \tilde{D}$, $\nu(\tilde{D} \setminus D) < \frac{\varepsilon}{2}$ such that $f$ is bounded in $W_\alpha^s(D)$ and then in $W_\alpha(D)$ since the number of vertices in $[0; s]$ is finite. □

Now we consider the Green function on $T$ (see [5], and in a more general environment [9]).

**Definition 12** (Green function)**.** Let $g : T \to \mathbb{R}$, $g(x) = \nu(U(x))$ for every $x \in T$.

Observe that $g$ is harmonic in $T \setminus \{o\}$ [9].

**Lemma 4.4.** *Let $f$ be a non-negative function on $T$ and suppose that*

$$\sum_{W_0(E)} f(x)g(x) < \infty.$$

*Then $\sum_{x\in\Gamma_\alpha(\omega)} f(x) < \infty$ for all $\alpha \geqslant 0$ and for almost every $\omega \in E$.*

*Proof.* As $f$ is positive, we only need to prove that $\int_E d\nu \sum_{x\in\Gamma_\alpha(\omega)} f(x) < \infty$. By Proposition 4.2, we may assume $\alpha = 0$.

With $\chi$ the characteristic function of $\Gamma_0(\omega)$ one has:

$$\int_E d\nu \sum_{x\in\Gamma_0(\omega)} f(x) = \int_E d\nu \sum_{x\in W_0(E)} \chi(x)f(x) = \sum_{x\in W_0(E)} f(x) \int_E \chi(x)d\nu.$$

Now

$$\begin{aligned}\int_E \chi(x)d\nu = \nu\left\{\omega \in E : x \in \Gamma_0(\omega)\right\} &= \nu\left(E \cap U(x)\right) \\ &\leqslant \nu\left(U(x)\right) = g(x),\end{aligned}$$

and the assertion follows. □

**Lemma 4.5.** *Let $f$ be a non-negative function on $T$. Assume that, for each $\omega \in E$, $\sum_{\Gamma_0(\omega)} f(x) < \infty$. Then for every $\varepsilon > 0$, $\beta \geqslant 0$, there exists a closed set $D \subset E$ such that $\nu(E \setminus D) < \varepsilon$ and $\sum_{W_\beta(D)} f(x)g(x) < \infty$.*

*Proof.* Let $E_k = \left\{\omega \in E : \sum_{\Gamma_0(\omega)} f(x) \leqslant k\right\}$. Observe that $E_k \subset E_{k+1}$. Moreover $\bigcup E_k = E$. Therefore, as in Corollary 4.3, for $\varepsilon > 0$ we can choose $\tilde{k}$ such that $B = E_{\tilde{k}}$ satisfies $\nu(E \setminus B) < \frac{\varepsilon}{2}$ and $\sum_{\Gamma_0(\omega)} f(x) \leqslant \tilde{k}$ for all $\omega \in B$. By Lemma 4.1, there exists a closed set $D \subset B$ with $\nu(B \setminus D) < \frac{\varepsilon}{2}$ and an integer $m$ such that, for $\omega \in D$ and $n \geqslant m$,

$$\nu(B \cap U(\omega_n)) \geqslant (1-\varepsilon)\nu(U(\omega_n)).$$

By Proposition 4.2 we may assume that $\beta = 0$. Let $\omega \in D$ and $\chi$ be the characteristic function of $\Gamma_0(\omega)$. Then

$$\begin{aligned}\tilde{k}\cdot\nu(B) &\geqslant \int_B d\nu \sum_{\Gamma_0(\omega)} f(x) = \int_B d\nu \sum_{W_0(B)} \chi(x)f(x) \\ &= \sum_{W_0(B)} f(x) \int_B \chi(x)\, d\nu = \sum_{W_0(B)} f(x)\nu(B \cap U(x)) \\ &\geqslant \sum_{W_0^s(D)} f(x)(1-\varepsilon)g(x),\end{aligned}$$

where we have chosen $s = m$.

This completes the proof, as the set $[o; s]$ has finite cardinality. □

The next step is to approximate the region $W^s_\alpha(E)$ by a family of regions $\{Q_k(E)\}_{k>0}$ with boundaries to which we can apply the Green formulas.

**Definition 13** (Approximating slabs)**.** For $k > s$ and for every set $E \subset \Omega$ we write

$$Q_k(E) = W_\alpha(E) \cap [s; k].$$

Then $Q_k(E)$ is a finite region with boundary, $Q_k(E) \subset Q_{k+1}(E)$ and $\bigcup_k Q_k(E) = W_\alpha^s(E)$. Let $I_k(E) = \partial Q_k(E)$.

When there is no reason of confusion, we simply write $Q_k$ for $Q_k(E)$ and $I_k$ for $I_k(E)$. One needs estimates on $I_k$ and on $b(I_k)$. The boundary $I_k$ splits into its inward, lateral and outward parts:

$$\begin{aligned} I_k^1 &= I_k \cap \{\sigma : |b(\sigma)| = s\}, \\ I_k^2 &= I_k \cap \{\sigma : s < |b(\sigma)| < k\}, \\ I_k^3 &= I_k \cap \{\sigma : |b(\sigma)| = k\}\,. \end{aligned} \tag{4.2}$$

Similarly, $b(I_k)$ splits as a disjoint union $b(I_k^1) \cup b(I_k^2) \cup b(I_k^3)$. We remark that $b(I_k^1)$ is a portion of the circle of radius $s$ and so its cardinality is independent of $k$.

From now on we denote by $c_q$ any positive constant (not always the same) which depends only on $q$. Instead $c(\alpha)$ is any positive constant which depends only on $\alpha$ and possibly on $q$.

**Lemma 4.6.** *With $g$ as in Definition* 12 *and $Q_k$ as in Definition* 13*,*

i) $|\nabla g(\sigma)| \leqslant c_q g \circ b(\sigma)$ *for all* $\sigma \in \Lambda$*;*
ii) $\nabla g(\sigma) < 0$ *if $\sigma$ is positively oriented;*
iii) $\sum_{I_k} g \circ b \leqslant c_q \sum_{b(I_k^1)} g(x) \leqslant c_q$.

*Proof.* By the definition of $g$, i) and ii) are easy consequences of (2.1). We observe that for a more general positive harmonic function $g$, i) is an immediate consequence of Harnack's inequality (see for example [4, page 260]). Moreover, the first inequality in iii) is proved in Proposition 3 of [4]. The second inequality is obvious since $b(I_k^1) \subset S_s$ and $\sum_{S_s} g = \nu(\Omega) = 1$. □

Recall that the Poisson kernel (see Definition 3) has the following properties, that can be easily verified

($a$) $P(x_{k+k_1}, \omega) = q^{-k_1} P(x_k, \omega)$ for each $\omega \notin U(x_k)$;
($a$) $\int_\Omega P(x, \omega) d\nu = 1$ for every $x \in T$.

**Lemma 4.7.** *Let $\varepsilon > 0$, $M > 0$ and $h \in L^\infty(\Omega)$. Assume that $\|h\|_\infty \leqslant M$ and $h(\omega) > r + \varepsilon$ for all $\omega \in U(x_k)$ with some $r \in \mathbb{R}$, $x \in T$, $k \in \mathbb{N}$. Then there exists $R > 0$ (which depends only on $\varepsilon$ and $M$) such that* $\mathbf{P}h(x) > r$ *when* $|x| = k + R$.

*Proof.* By the hypothesis, $r + \varepsilon < M$. Write

$$\mathbf{P}h(x) = \int_\Omega h(\omega) P(x,\omega) d\nu = \int_{U(x_k)} h(\omega) P(x,\omega) d\nu + \int_{\Omega \setminus U(x_k)} h(\omega) P(x,\omega) d\nu.$$

For each $\omega \notin U(x_k)$, $P(x, \omega) = q^{-R} P(x_k, \omega)$. Therefore if $R$ is sufficiently large

$$\Big| \int_{\Omega \setminus U(x_k)} h(\omega) P(x,\omega) d\nu \Big| \leqslant M q^{-R} \int_{\Omega \setminus U(x_k)} P(x_k, \omega) d\nu \leqslant M q^{-R} < \frac{\varepsilon}{2}\,.$$

Now, by $(a)$ and $(b)$, $\int_{U(x_k)} h(\omega)P(x,\omega)d\nu > (r+\varepsilon)(1-q^{-R})$. Then, recalling that $r+\varepsilon < M$ and $Mq^{-R} < \frac{\varepsilon}{2}$, now we have

$$\mathbf{P}h(x) > (r+\varepsilon)(1-q^{-R}) - Mq^{-R} > r+\varepsilon - 2Mq^{-R} > r+\varepsilon-\varepsilon = r\,.$$

$\square$

## 5. Proof of the Main Theorem

The proof of the Main Theorem is rather long: we prove the chain of its implications as separate theorems. Note that (ii) → (i) and (iv) → (iii) are trivial. Then it is enough to show that (i) → (iv), (iii) → (i) and (iii)+ (i) → (ii).

We also note that the equivalence of (iii) and (iv) is an immediate consequence of Proposition 4.2, Lemma 4.4 and Lemma 4.5.

We prove first that (i) implies (iv).

**Theorem 5.1.** *For every $\alpha \geqslant 0$, for every measurable set $E \subset \Omega$, for every harmonic function $f$ non-tangentially bounded almost everywhere on $E$, the area function of $f$ is finite almost everywhere on $E$.*

*Proof.* Since $f$ is non-tangentially bounded almost everywhere on $E$, by Corollary 4.3, given any non-negative integer $n$ we can find a closed set $D_n \subset E$ with $\nu(E\setminus D_n) < \frac{1}{n}$ and a constant $M = M(\alpha,n)$ such that $|f| \leqslant M$ on $W_{\alpha+1}(D_n)$. Since $\nu(E\setminus \bigcup_{n=1}^{\infty} D_n) = 0$, the finiteness of $A_\alpha f$ at almost every point of each $D_n$ implies the finiteness of $A_\alpha f$ at almost every point of $E$.

Therefore we can reduce the proof of the theorem to the case where $E$ is closed and (after multiplication by a suitable non-zero constant) $|f| \leqslant 1$ on $W_{\alpha+1}(E)$.

By Lemma 4.4, it is sufficient to prove that for every $s > 0$

$$\sum_{x\in W^s_\alpha(E)} \|\nabla f(x)\|^2 g(x) < \infty. \tag{5.1}$$

Let $Q_k = Q_k(E)$ as in Definition 13. We approximate $W^s_\alpha(E)$ with the slabs $Q_k$. It is enough to show that $\sum_{Q_k} \|\nabla f(x)\|^2 g(x)$ is uniformly bounded in $k$.

Recall that $g$ is harmonic in $T\setminus\{0\}$. Then, by Proposition 2.2,

$$\sum_{Q_k} \|\nabla f(x)\|^2 g(x) = \sum_{Q_k} g\Delta f^2 = \sum_{Q_k} (g\Delta f^2 - f^2\Delta g).$$

Now the Green's formula (Proposition 2.1) yields

$$\sum_{Q_k} (g\Delta f^2 - f^2\Delta g) = (q+1)^{-1}\sum_{I_k}(g\circ b\nabla f^2 - f^2\circ b\nabla g) = R_1 - R_2$$

where $R_1 = (q+1)^{-1}\sum_{I_k} g\circ b\nabla f^2$ and $R_2 = (q+1)^{-1}\sum_{I_k} f^2\circ b\nabla g$.

One has

$$|R_1| \leqslant \sum_{I_k} g\circ b|f\circ e - f\circ b||f\circ e + f\circ b| \leqslant \sup_{W_{\alpha+1}(E)} (2|f|)^2 \sum_{I_k} g\circ b$$

and

$$|R_2| \leqslant \sup_{W_{\alpha+1}(E)} |f|^2 \sum_{I_k} |\nabla g|.$$

Since $|f|$ is bounded, i) and iii) of Lemma 4.6 yield

$$\sum_{Q_k} \|\nabla f(x)\|^2 g(x) \leqslant c(\alpha) \sum_{x \in b(I_k^1)} g(x) \leqslant c(\alpha).$$

This proves (5.1). □

We prove next that (iii) implies (i).

**Theorem 5.2.** *Let $f$ be a harmonic function on $T$ and $E$ a measurable subset of $\Omega$. Assume that $A_0 f(\omega) < \infty$ for almost all $\omega \in E$. Then $f$ is non-tangentially bounded almost everywhere on $E$.*

*Proof.* By Lemma 4.4 and Lemma 4.5, $E$ has subsets $D_n$ such that $\nu(E \setminus D_n) < \frac{1}{n}$ and, given any $\alpha \geqslant 0$, $A_\alpha f(\omega) < M(\alpha, n)$ in $D_n$. Arguing as at the beginning of the proof of Theorem 5.1, we reduce the proof to the case where $\alpha \geqslant 0$ is given, $E$ is closed and (up to normalization) $A_{\alpha+1} f(\omega) \leqslant 1$ for every $\omega \in E$. With this assumption, the proof is easily adapted from [4, Proposition 8]. We give the details for the sake of completeness.

Let $0 < \varepsilon < \frac{1}{2}$. By Lemma 4.1 and compactness there exists a closed set $D$, $D \subset E$ with $\nu(E \setminus D) < \varepsilon$ such that, for some $m > 0$, for all $k \geqslant m$ and $\omega \in D$, one has

$$\nu(U(\omega_k) \cap E) \geqslant \frac{1}{2} \nu(U(\omega_k)). \tag{5.2}$$

Let $s \geqslant m + \alpha$. We claim that

$$\sum_{W_\alpha^s(D)} \|\nabla f(x)\|^2 g(x) \leqslant 2. \tag{5.3}$$

With this goal, for $\omega \in E$, let $\chi$ be the characteristic function of $\Gamma_{\alpha+1}(\omega)$. We have:

$$1 \geqslant \nu(E) \geqslant \int_E (A_{\alpha+1} f(\omega))^2 \, d\nu = \int_E \sum_T \|\nabla f(x)\|^2 \chi(x, \omega) d\nu$$

$$\geqslant \int_E \sum_{W_\alpha^s(D)} \|\nabla f(x)\|^2 \chi(x, \omega) d\nu = \sum_{W_\alpha^s(D)} \|\nabla f(x)\|^2 \int_E \chi(x, \omega) d\nu.$$

Observe that $x \in \Gamma_{\alpha+1}(\omega)$ if and only if $U(x(-\alpha-1))$ intersects $E$. Therefore

$$\int_E \chi(x, \omega) = \nu \left\{ E \cap U(x(-\alpha-1)) \right\} \geqslant \nu \left\{ E \cap U(x(-\alpha)) \right\}.$$

If $x \in W_\alpha^s(D)$, there exists $\eta \in D \subset E$ such that $x \in \Gamma_\alpha(\eta)$ and $|x| \geqslant s$. Then $x(-\alpha) = \eta_{|x|-\alpha}$, with $|x| - \alpha \geqslant m$. Therefore, by (5.2)

$$\nu \left\{ E \cap U(\eta_{|x|-\alpha}) \right\} \geqslant \frac{1}{2} \nu(U(\eta_{|x|-\alpha})) \geqslant \frac{1}{2} \nu(U(x)).$$

Thus

$$1 \geqslant \nu(E) \geqslant \int_E (A_{\alpha+1} f(\omega))^2 \, d\nu \geqslant \frac{1}{2} \sum_{W_\alpha^s(D)} \|\nabla f(x)\|^2 g(x).$$

This proves (5.3).

We now replace $W^s_\alpha(D)$ with the approximating slabs $Q_k = Q_k(D)$ (see Definition 13). By (5.3), for every $k$,

$$\frac{1}{2}\sum_{Q_k} \|\nabla f(x)\|^2 g(x) < 1. \tag{5.4}$$

By harmonicity and Proposition 2.2

$$\frac{1}{2}\sum_{Q_k} \|\nabla f(x)\|^2 g(x) = \frac{1}{2}\sum_{Q_k} \left(g\Delta f^2 - f^2\Delta g\right)$$

and by the Green's formula (Proposition 2.1)

$$\frac{1}{2}\sum_{Q_k} \left(g\Delta f^2 - f^2\Delta g\right) = \frac{1}{2}(q+1)^{-1}\sum_{I_k} \left(g\circ b\nabla f^2 - f^2\circ b\nabla g\right).$$

It is easily seen that:

$$\nabla f^2 = 2f\circ b\nabla f + (\nabla f)^2.$$

Hence

$$\sum_{I_k} \left(g\circ b\nabla f^2 - f^2\circ b\nabla g\right) = \sum_{I_k}\left(2(g\circ b)(f\circ b)\nabla f + g\circ b(\nabla f)^2\right) - \sum_{I_k} f^2\circ b\nabla g.$$

We split the boundary $I_k$ in two parts, $I_+(k) = I_k^1$ and $I_-(k) = I_k^2 \cup I_k^3$, where $I_k^j$, $j = 1,2,3$, are as in (4.2). For the sake of simplicity let us write $C = \sum_{I_k}\left(g\circ b\nabla f^2 - f^2\circ b\nabla g\right)$.

Then $C = C_1 + C_2 + C_3$ where $C_1 = \sum_{I_k} -(f^2\circ b)\nabla g$, $C_2 = \sum_{I_k}(g\circ b)(\nabla f)^2$ and $C_3 = 2\sum_{I_k}(g\circ b)(f\circ b)\nabla f$. Moreover we let

$$C_1^+ = \sum_{I_+(k)} -(f^2\circ b)\nabla g \quad\text{and}\quad C_1^- = \sum_{I_-(k)} -(f^2\circ b)\nabla g,$$
$$C_2^+ = \sum_{I_+(k)} (g\circ b)(\nabla f)^2 \quad\text{and}\quad C_2^- = \sum_{I_-(k)} (g\circ b)(\nabla f)^2,$$
$$C_3^+ = 2\sum_{I_+(k)} (g\circ b)(f\circ b)\nabla f \quad\text{and}\quad C_3^- = 2\sum_{I_-(k)} (g\circ b)(f\circ b)\nabla f.$$

Then (5.4) implies that

$$0 \leqslant C = \left(C_1^+ + C_1^- + C_2^+ + C_2^- + C_3^+ + C_3^-\right) \leqslant 2(q+1).$$

From now on, $M$ will denote a generic constant, not always the same but always independent of $k$.

We observe that $C_1^+$, $C_2^+$ and $C_3^+$ are uniformly bounded, since $I_+(k)$ is contained in a bounded set, as we remarked before. Thus the sum on $I_+(k)$ is uniformly bounded. Hence $|C_1^- + C_2^- + C_3^-| \leqslant M$.

Since $\big||C_1^-| - |C_2^-| - |C_3^-|\big| \leqslant |C_1^- + C_2^- + C_3^-|$, it follows that

$$|C_1^-| \leqslant |C_2^-| + |C_3^-| + M. \tag{5.5}$$

*Claim:*

$$\sum_{b(I_-(k))} f^2 g \leqslant M. \tag{5.6}$$

*Proof of the Claim.* First observe that as $I_-(k)$ consists of positively oriented edges (see page 3331) we have that $\nabla g(\sigma) < 0$ if $\sigma \in I_-(k)$. Then (5.5) yields:

$$0 \leqslant C_1^- = \sum_{I_-(k)} -f^2 \circ b \nabla g \leqslant M + \sum_{I_-(k)} g \circ b (\nabla f)^2 + 2 \sum_{I_-(k)} g \circ b |f \circ b||\nabla f|.$$

Moreover, if $\sigma \in I_-(k)$, $-\nabla g(\sigma) = c_q g(b(\sigma))$ (here $c_q = q(q-1)$ as followed by (2.1)). Then

$$c_q \sum_{I_-(k)} (f^2 \circ b)(g \circ b) \leqslant M + \sum_{I_-(k)} g \circ b (\nabla f)^2 + 2 \sum_{I_-(k)} g \circ b |f \circ b||\nabla f|.$$

For any $x \in W_\alpha(D)$ there is $\omega \in D$ such that $x \in \Gamma_\alpha(\omega)$. As $D \subset E$, we have that $A_{\alpha+1} f(\omega) \leqslant 1$ in $D$, hence $\|\nabla f(x)\| \leqslant 1$ in $W_\alpha(D)$. Recall that, by Definition 5,

$$|\nabla f(\sigma)| \leqslant \sqrt{(q+1)} \|\nabla f(x)\| \tag{5.7}$$

for all $x \in T$ and all $\sigma \in \Lambda(x)$. Therefore $|\nabla f(\sigma)| \leqslant \sqrt{(q+1)}$ for all $x \in W_\alpha(D)$ and all edges $\sigma$ starting at $x$. In particular $|\nabla f| \leqslant \sqrt{(q+1)}$ in $I_-(k)$. Then (choosing $M \geqslant 2$)

$$\sum_{I_-(k)} (f^2 \circ b)(g \circ b) \leqslant M + M \sum_{I_-(k)} g \circ b + M \sum_{I_-(k)} g \circ b |f \circ b|.$$

By iii) of Lemma 4.6, $\sum_{I_-(k)} g \circ b \leqslant c_q$. Moreover, since $\sum_{b(\partial Q)} h \circ b \leqslant \sum_{\partial Q} h \leqslant c_q \sum_{b(\partial Q)} h \circ b$, for every $Q \subset \Lambda$ and every positive function $h$ on $T$, we have:

$$\sum_{b(I_-(k))} f^2 g \leqslant M + M \sum_{b(I_-(k))} g |f|.$$

By Schwarz's inequality

$$\sum_{b(I_-(k))} g|f| \leqslant \left( \sum_{b(I_-(k))} g \right)^{\frac{1}{2}} \left( \sum_{b(I_-(k))} g f^2 \right)^{\frac{1}{2}};$$

hence

$$\sum_{b(I_-(k))} f^2 g \leqslant M + M \left( \sum_{b(I_-(k))} f^2 g \right)^{\frac{1}{2}}.$$

This shows that the left hand side is bounded uniformly with respect to $k$, thereby proving the claim.

*End of the proof of Theorem* 5.2. We will now use the "$l^2$-boundedness" in (5.6) to prove non-tangential boundedness almost everywhere in $D$. We proceed as follows.

We bound the function $f$ by another $F$ whose non-tangential behavior is known. For this goal, let

$$f_k(\omega) = \begin{cases} 0 & \text{if } \omega_j \notin b(I_-(k)) \quad \forall j; \\ |f(\omega_m)| & \text{if } m = \max\{j : \omega_j \in b(I_-(k))\}. \end{cases} \tag{5.8}$$

Indeed we show that

$$\|f_k\|^2_{L^2(\omega)} \leqslant \sum_{x \in b(I_-(k))} f^2(x) g(x) \leqslant M.$$

In fact, let

$$\Omega(x) = \{\omega \in U(x) : \text{ for all } j > |x|,\ \omega_j \notin b(I_-(k))\}.$$

Then $\Omega(x) \subseteq U(x)$ and

$$\|f_k\|^2_{L^2(\omega)} = \sum_{x \in b(I_-(k))} f^2(x)\nu(\Omega(x)) \leqslant \sum_{x \in b(I_-(k))} f^2(x)g(x).$$

Hence, by (5.6)

$$\|f_k\|^2_{L^2(\omega)} \leqslant M. \tag{5.9}$$

Now let $F_k$ be the Poisson integral of $|f_k|$. We shall show that, for an appropriate constant $M$ independent of $k$, we have on $Q_k$

$$|f| \leqslant M + MF_k. \tag{5.10}$$

By the maximum principle it is sufficient to prove (5.10) on $b(\partial Q_k) = b(I_+(k)) \cup b(I_-(k))$ (see also the remark in [4, page 269]). Since the cardinality of $b(I_+(k))$ is finite, we can choose $M$ so large that $|f| \leqslant M$ on $b(I_+(k))$. Now

$$F_k(x) = \int_\Omega f_k(\omega)P(x,\omega)d\nu = \sum_{v \in b(I_-(k))} |f(v)| \int_{\Omega(v)} P(x,\omega)d\nu.$$

By the definition of the Poisson kernel (Definition 3), if $x \in b(I_-(k))$ one has

$$F_k(x) \geqslant |f(x)|q^{|x|}\nu(\Omega(x)).$$

Observe that $\Omega(x) = U(x)$ if $\alpha > 0$. Moreover if $\alpha = 0$ we distinguish two cases: if $x \in b(I_-(k))$ is such that $|x| = k$, then $\Omega(x) = U(x)$; if $|x| < k$, then $\Omega(x) \subset U(x)$ and there exists a vertex $y$ such that $x = y^-$ and $U(y) \subset \Omega(x)$.

So $\nu(\Omega(x)) \approx q^{-|x|}$ by (2.1). Therefore in both cases we have that $F_k(x) \geqslant c_q|f(x)|$. We have proved (5.10).

By (5.9), $f_k$ is a uniformly bounded sequence on $L^2(\Omega)$. Thus there exists a subsequence of $\{f_k\}$ that converges weakly in $L^2(\Omega)$ to, say, $\tilde{f}$. Denote again by $\{f_k\}$ this subsequence. Let $\tilde{F}$ be the Poisson integral of $\tilde{f}$. Then $F_k$ converges pointwise to $\tilde{F}$ and thus (5.10) implies that in $W^s_\alpha(D)$, and consequently in $W_\alpha(D)$,

$$|f| \leqslant M + M\tilde{F}$$

for some $M$.

Since $\tilde{f} \in L^2(\Omega)$ and Poisson integrals of $L^2$-functions are non-tangentially bounded almost everywhere (see [6, theorem 1]), by Proposition 4.2 we know that $f$ is non-tangentially bounded almost everywhere in $D$. As noted at the beginning this completes the proof. □

Finally we prove that (iii) (and (i)) imply (ii).

**Theorem 5.3.** *Let $f$ be a harmonic function on $T$ and $E$ a measurable subset of $\Omega$. Assume that $A_0f(\omega) < \infty$ for almost all $\omega \in E$. Then $f$ has a non-tangential limit almost everywhere on $E$.*

*Proof.* Let $\eta > 0$. By the implication ((iii) $\Rightarrow$ (i)) in Theorem 5.2 and by Corollary 4.3, there exists a closed set $D \subset E$, $\nu(E \setminus D) < \frac{\eta}{2}$ such that $f$ is bounded in $W_\alpha(D)$, say by a constant $M$. We can suppose $\alpha = 0$.

But (i) implies (iv) and we also have $\sum_{\Gamma_0(\omega)} \|\nabla f(x)\|^2 < \infty$ almost everywhere in $D$. Let $\varepsilon > 0$ be given and let $R$ be the offset given in Lemma 4.7 corresponding to $\varepsilon$ and to the bound $M$ of $f$. Let $\delta = \frac{\varepsilon^2}{4R^2(q+1)}$. For almost every $\omega \in D$ there exists an integer $k = k(\omega)$ such that

$$\sum_{\Gamma_0(\omega)\cap[k;\infty)} \|\nabla f(x)\|^2 < \delta.$$

Let $D_j = \{\omega \in D : \sum_{\Gamma_0(\omega)\cap[j;\infty)} \|\nabla f(x)\|^2 < \delta\}$. Then $D_j \subset D_{j+1}$ for all $j$, and $\nu(D \setminus \bigcup_j D_j) = 0$. So there is an integer $k_0$ so that $\nu(D \setminus D_{k_0}) < \frac{\eta}{2}$ and $\sum_{\Gamma_0(\omega)\cap[k_0;\infty)} \|\nabla f(x)\|^2 < \delta$ for every $\omega \in D_{k_0}$.

Let $k > k_0 + R$. Let us write $G$ instead of $D_{k_0}$. If $x, y \in W_0^{k_0}(G)$, then $x \in \Gamma_0(\omega)$ and $y \in \Gamma_0(\tilde{\omega})$ for some $\omega, \tilde{\omega} \in G$. Suppose that $|x| = |y| = k$. If $\tilde{\omega} \in U(\omega_{k-R})$, then the geodesic $\gamma$ that joins $x$ and $y$ lies inside $\Gamma_0(\omega) \cup \Gamma_0(\tilde{\omega})$, and its length $l(\gamma)$ is less than or equal to $2R$. By the triangular inequality and by (5.7),

$$|f(y) - f(x)| \leqslant l(\gamma) \max_{\sigma\subset\gamma} |\nabla f(\sigma)| < 2R\sqrt{(q+1)} \max_{\gamma} \|\nabla f(x)\| < \varepsilon. \tag{5.11}$$

For integers $k > k_0 + R$ we consider the new approximating regions

$$\bigcup_{\omega\in G} \Gamma_0(\omega) \cap [k_0; k].$$

By abuse of notation we continue to denote by $Q_k = Q_k(G)$ these regions and by $I_k$ their boundary, which decomposes in three disjoint parts as in (4.2).

Define $f_k$ on $\Omega$ by

$$f_k(\omega) = \begin{cases} f(\omega_k) & \text{if } \omega_k \in b(I_k^3); \\ \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $F_k$ the Poisson integral of $f_k$, and by $\Phi$ the Poisson integral of the characteristic function $\chi_{D'}$ of the complement $D'$ of $G$.
*Claim*: for some positive constants $A$, $B$ independent of $k$, we have, on $Q_k$,

$$F_k + A\Phi + Bg > f - 2\varepsilon. \tag{5.12}$$

By harmonicity it is sufficient to prove (5.12) on $b(I_k)$. Note that $g$ is harmonic in $Q_k$ because $o \notin Q_k$.

*Proof of the Claim.* We first handle the outward part $b(I_k^3)$ of the boundary. Let $x \in b(I_k^3)$ and $\omega \in G$ such that $\omega_k = x$. For $\tilde{\omega} \in U(\omega_{k-R})$ the following inequality holds:

$$(f_k + A\chi_{D'})(\tilde{\omega}) > f_k(\omega) - \varepsilon.$$

Indeed, if $\tilde{\omega} \in D'$ it is enough to choose $A \geqslant 2|f|$ on $W_0(G)$. This is possible because $f$ is assumed to be bounded in $W_0(\tilde{D}) \supset W_0(G)$. On the other hand, if $\tilde{\omega} \in G$, let us denote by $y$ the vertex of length $k$ on $\tilde{\omega}$. Then $y \in b(I_k^3)$ and the inequality holds by (5.11). Now (5.12) follows by Lemma 4.7 because $g$ is positive.

Let us now consider the lateral part $b(I_k^2)$ of the boundary. If $x \in b(I_k^2)$, then there exists $z$ such that $x = z^-$ (see Subsection 2.1) and $z \notin \Gamma_0(\omega)$ for every $\omega \in G$. Then $U(z) \in D'$ because otherwise there would exist $\omega \in G$ such that $\omega \in U(z)$, hence $z \in \Gamma_0(\omega)$, a contradiction. So

$$\Phi(x) = \int_\Omega P(x,\omega)\chi_{D'}(\omega)\, d\nu \geqslant \int_{U(z)} P(x,\omega) d\nu.$$

As $x = z^-$, by Definition 3 we have that $P(x,\omega) = q^{|x|}$ for all $\omega \in U(z)$. On the other hand, by (2.1) we have $\nu(U(z)) = (q+1)^{-1}q^{-|x|}$. Hence $\int_{U(z)} P(x,\omega)d\nu = (q+1)^{-1}$. So the right hand side is a positive constant independent of $k$. Therefore to prove the claim on the lateral part it is sufficient to choose $A$ sufficiently large, because $g$ is positive and $f$ is bounded in $W_0(G)$, so the same is true for $F_k$.

Finally let us consider the inward $b(I_k^1)$. It is obvious that $0 \leqslant \Phi \leqslant 1$ everywhere. We again use the fact that $f$ and $F_k$ are bounded independently of $k$. Moreover each element $u \in b(I_k^1)$ verifies $|u| = k_0$, hence $g(u) = (q+1)^{-1}q^{1-k_0}$. Therefore to have (5.12) on $b(I_k^1)$ it is enough to choose $B$ sufficiently large. Now the claim is proved on all of $b(I_k)$, hence on $Q_k$.

*End of the proof of Theorem* 5.3. By applying the same argument to $-f$ in place of $f$, we have on $Q_k$

$$-F_k + A\Phi + Bg > -f - 2\varepsilon.$$

Now we put these inequalities together and let $k \to +\infty$. Observe that the sequence $\{f_k\}$ is uniformly bounded by $M$. By the Banach-Alaoglu theorem, there exists a subsequence, that we again denote by $\{f_k\}$, such that $\int_\Omega hf_k \to \int_\Omega hf_\infty$, for some bounded function $f_\infty$ and every $L^1$-function $h$ on $\Omega$. Denote by $F$ the Poisson integral of $f_\infty$. Since $P(x,\omega) \in L^1(\Omega)$ for every $x$ in $T$ it follows that $F_k$ converges pointwise to $F$. Thus we have

$$F - A\Phi - Bg - 2\varepsilon < f < F + A\Phi + Bg + 2\varepsilon$$

everywhere in $W_0^{k_0}(G)$. The function $F$ has a non-tangential limit since it is the Poisson integral of a bounded function [6, Theorem 1]. The Green function $g$ has a non-tangential limit 0. By [6, Theorem 1] it follows that $\Phi$ has a non-tangential limit 0 almost everywhere in $G$.

Since $\varepsilon$ and $\eta$ were arbitrary, by Proposition 4.2 it follows that $f$ is admissibly convergent at almost every $\omega \in E$. $\square$

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