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THE PARITY OF THE COCHRAN–HARVEY INVARIANTS OF 3–MANIFOLDS

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ABSTRACT. Given a finitely presented group G and an epimorphism $\phi : G \to \mathbb{Z}$ Cochran and Harvey defined a sequence of invariants $\overline{d}_n(G, \phi) \in \mathbb{N}_0, n \in \mathbb{N}_0$, which can be viewed as the degrees of higher–order Alexander polynomials. Cochran and Harvey showed that (up to a minor modification) this is a never decreasing sequence of numbers if G is the fundamental group of a 3–manifold with empty or toroidal boundary. Furthermore they showed that these invariants give lower bounds on the Thurston norm.

Using a certain Cohn localization and the duality of Reidemeister torsion we show that for a fundamental group of a 3–manifold any jump in the sequence is necessarily even. This answers in particular a question of Cochran. Furthermore using results of Turaev we show that under a mild extra hypothesis the parity of the Cochran–Harvey invariant agrees with the parity of the Thurston norm.

1. INTRODUCTION

Let G be a finitely presented group and $\phi : G \to \mathbb{Z}$ an epimorphism. Following [Ha05, Ha06, Co04] we define for $n \geq 0$

$$\delta_n(G, \phi) := \deg(\Delta_n(G, \phi)(t)),$$

where $\Delta_n(G, \phi)(t)$ denotes the n–th higher–order Alexander polynomial of G corresponding to ϕ. These polynomials are elements in $\mathbb{K}_n[t^{\pm1}]$, where $\mathbb{K}_n[t^{\pm1}]$ is a skew Laurent polynomial ring which is associated to the group ring $\mathbb{Z}[G/G_r^{(n+1)}]$ and to the homomorphism ϕ. Here $G_r^{(n+1)}$ denotes the $(n+1)$–st term of Harvey's rational derived series. We adopt the convention that $\deg(0) = -\infty$. We give the precise definitions in Section 3. Note that in [Ha06] these invariants are denoted by $\overline{\delta}_n(G, \phi)$.

These invariants show an interesting behavior for the fundamental groups of 3–manifolds. (Here by a 3–manifold we always mean an oriented, connected and compact 3–manifold.) Let G be the fundamental group of a 3–manifold M with empty or toroidal boundary. Then Harvey [Ha06] showed (cf. also [Co04] and [Fr07]) that if $\phi : G \to \mathbb{Z}$ is an epimorphism such that $\delta_0(G, \phi) \neq -\infty$, then

$$\begin{array}{rcll} \delta_0(G,\phi) - 1 - b_3(M) & \leq & \delta_1(G,\phi) \leq \delta_2(G,\phi) \dots & \text{if } b_1(M) = 1, \text{ and} \\ \delta_0(G,\phi) & \leq & \delta_1(G,\phi) \leq \delta_2(G,\phi) \dots & \text{if } b_1(M) > 1. \end{array}$$

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Here $b_i(M)$ denotes the i–th Betti number of M. Recall that $b_3(M) = 1$ if M is closed and $b_3(M) = 0$ if M has boundary. This result can be viewed as a subtle obstruction to a group being a 3–manifold group. Note that a similar result holds for groups of deficiency one (cf. [Ha06] or [Fr07]).

If $\delta_0(G,\phi) = 0$ and $b_1(M) = 1$, then the proof of Lemma 3.3 shows that $\delta_n(G,\phi) = 0$ for all $n > 0$. We can and will therefore restrict ourselves to the cases $\delta_0(G,\phi) > 0$ or $b_1(M) \geq 2$.

The following theorem strengthens Harvey's result:

Theorem 1.1. *Let G be the fundamental group of a 3–manifold with empty or toroidal boundary. Let $\phi : G \to \mathbb{Z}$ be an epimorphism such that $\delta_0(G,\phi) \neq -\infty$. Then any jump in the sequence of inequalities*

$$\begin{array}{rl} \delta_0(G,\phi) - 1 - b_3(M) \leq \delta_1(G,\phi) \leq \delta_2(G,\phi)\dots & (\textit{if } b_1(M) = 1 \textit{ and } \delta_0(G,\phi) > 0), \\ \delta_0(G,\phi) \leq \delta_1(G,\phi) \leq \delta_2(G,\phi)\dots & (\textit{if } b_1(M) > 1) \end{array}$$

has to be even.

Note that this theorem is not known to hold for groups of deficiency one in general. This theorem in particular answers a question of Cochran (cf. [Co04, Question (5) in Section 14]). Furthermore, together with realization results of Cochran, this completely classifies the sequences of numbers which appear as δ_n–invariants for knots (see Theorem 4.3).

The key difficulty in relating $\delta_{k+1}(G,\phi)$ and $\delta_k(G,\phi)$ is that there are no non–trivial homomorphisms between the skew Laurent polynomial rings $\mathbb{K}_{k+1}[t^{\pm 1}]$ and $\mathbb{K}_k[t^{\pm 1}]$. We circumvent this problem by studying an appropriate Cohn localization which maps non–trivially to $\mathbb{K}_{k+1}[t^{\pm 1}]$ and to $\mathbb{K}_k[t^{\pm 1}]$, and by using the functoriality of Reidemeister torsion. Another major ingredient is a duality theorem for Reidemeister torsion.

The Cochran–Harvey invariants have a close relationship to the geometry of 3–manifolds. Let M be 3–manifold M and $\phi \in H^1(M;\mathbb{Z})$. Then the *Thurston norm* of ϕ is defined as

$$||\phi||_T = \min\{-\chi(\hat{S}) \,|\, S \subset M \text{ properly embedded surface dual to } \phi\}$$

where $\hat{S}$ denotes the result of discarding all connected components of S with positive Euler characteristic. This extends to a seminorm on $H^1(M;\mathbb{R})$. We refer to [Th86] for details. We henceforth identify $H^1(M;\mathbb{Z})$ with $\hom(H_1(M;\mathbb{Z}),\mathbb{Z})$ and $\hom(\pi_1(M),\mathbb{Z})$. Let M be 3–manifold with empty or toroidal boundary and let $G = \pi_1(M)$. In [Ha05, Ha06] Harvey (cf. also [Co04], [Tu02] and [Fr07]) showed that for an epimorphism $\phi : G \to \mathbb{Z}$ with $\delta_0(G,\phi) \neq -\infty$ we have

(1)
$$\begin{array}{rcl} \delta_0(G,\phi) - 1 - b_3(M) & \leq & \delta_1(G,\phi) \leq \delta_2(G,\phi) \dots \leq ||\phi||_T \quad (\text{if } b_1(M) = 1), \\ \delta_0(G,\phi) & \leq & \delta_1(G,\phi) \leq \delta_2(G,\phi) \dots \leq ||\phi||_T \quad (\text{if } b_1(M) > 1). \end{array}$$

Furthermore if (M,ϕ) fibers over S^1 and if $M \neq S^1 \times S^2, S^1 \times D^2$, then all inequalities are in fact equalities. Using classical results of Turaev [Tu86] we can strengthen this geometric interpretation as follows.

Theorem 1.2. *Let M be a 3–manifold which is either closed or is the exterior of a link in S^3. Let $G := \pi_1(M)$ and let $\phi : G \to \mathbb{Z}$ be an epimorphism such that*

$\delta_0(G,\phi) \neq -\infty$. *Then*

$$\begin{aligned} \delta_0(G,\phi) - 1 - b_3(M) &\equiv \delta_1(G,\phi) \equiv \delta_2(G,\phi) \dots \\ &\equiv ||\phi||_T \, mod\, 2 \;\; (\textit{if } b_1(M) = 1,\ \delta_0(G,\phi) \geq 1) \\ \delta_0(G,\phi) \equiv \delta_1(G,\phi) &\equiv \delta_2(G,\phi) \cdots \equiv ||\phi||_T \, mod\, 2 \;\; (\textit{if } b_1(M) > 1). \end{aligned}$$

In a forthcoming paper the first author and Shelly Harvey [FH07] will show that for a 3–manifold group G the map $\phi \mapsto \delta_n(G,\phi)$ defines a seminorm on $H^1(G;\mathbb{R})$. This further strengthens the close relationship between the Cochran–Harvey invariants and the Thurston norm.

The paper is organized as follows. In Section 2 we recall several facts about Reidemeister torsion and fix some notation. We also define the Alexander polynomials and recall the relationship to Reidemeister torsion. In Section 3 we recall several definitions from [Ha06] which will then allow us to give in Section 4 precise formulations of (slight generalizations of) Theorems 1.1 and 1.2. We conclude with the proofs of our results in Section 5.

2. Reidemeister torsion and Alexander polynomials

2.1. **Reidemeister torsion of manifolds.** Throughout this paper we will only consider associative rings R with $1 \neq 0$ with the property that if $r \neq s$, then R^r is not isomorphic to R^s. For such a ring R, let $\mathrm{GL}(R) := \varinjlim \mathrm{GL}(R,n)$ and $K_1(R) := \mathrm{GL}(R)/[\mathrm{GL}(R),\mathrm{GL}(R)]$.

In particular $K_1(R)$ is an abelian group. If $\mathbb{K}$ is a (skew) field, then the Dieudonné determinant gives an isomorphism $\det : K_1(\mathbb{K}) \to \mathbb{K}^\times/[\mathbb{K}^\times,\mathbb{K}^\times]$ where $\mathbb{K}^\times := \mathbb{K}\setminus\{0\}$. For more details we refer to [Ro94] or [Tu01].

Let X be a CW–complex. In this paper by a CW–complex we will always mean a connected CW–complex. Denote the universal cover of X by $\tilde{X}$. We view the chain complex $C_*(\tilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$–module via deck transformations. Let $\varphi : \mathbb{Z}[\pi_1(X)] \to R$ be a homomorphism. This equips R with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the right R–module chain complex $C_*(X;R) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R$. We denote the homology of $C_*(X;R)$ by $H_i(X;R)$.

Now assume that X is in fact a finite CW–complex. If $H_*(X;R) = 0$, then the *Reidemeister torsion* $\tau(X,R) \in K_1(R)/\pm\varphi(\pi_1(X))$ is defined (cf. [Tu01]). We will write $K_1(R)/\pm\pi_1(X)$ for $K_1(R)/\pm\varphi(\pi_1(X))$ for convenience, if φ is clear from the context. If $H_*(X;R) \neq 0$, then we write $\tau(X,R) := 0$. It is known that $\tau(X,R)$ only depends on the homeomorphism type of X. Hence we can define $\tau(M,R)$ for a 3–manifold M by picking any finite CW–structure for M. We refer to the excellent book of Turaev [Tu01] for filling in the details.

2.2. **Alexander polynomials.** Let $\mathbb{K}$ be a (skew) field and $\gamma : \mathbb{K} \to \mathbb{K}$ a ring homomorphism. Denote by $\mathbb{K}[t^{\pm1}]$ the skew Laurent polynomial ring over $\mathbb{K}$ (associated to γ). The elements in $\mathbb{K}[t^{\pm1}]$ are formal sums $\sum_{i=-r}^{s} a_i t^i$ with $a_i \in \mathbb{K}$, and multiplication in $\mathbb{K}[t^{\pm1}]$ is induced from the rule $t^i a = \gamma^i(a) t^i$ for any $a \in \mathbb{K}$. It is known that $\mathbb{K}[t^{\pm1}]$ embeds in its (skew) quotient field (denoted by $\mathbb{K}(t)$) which is flat over $\mathbb{K}[t^{\pm1}]$ (cf. e.g. [DLMSY03]).

Let X be a CW–complex with finitely many cells in dimension 1 and 2. Given a ring homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \to \mathbb{K}\backslash[t^{\pm1}]$ to a skew Laurent polynomial ring we consider the finitely presented $\mathbb{K}[t^{\pm1}]$–module $H_1(X;\mathbb{K}[t^{\pm1}])$. This module was studied in [Co04, Ha05, Tu02, LM06]. Since $\mathbb{K}[t^{\pm 1}]$ is a principal ideal domain (PID) (cf. [Co04, Proposition 4.5]) we can decompose

$$H_1(X;\mathbb{K}[t^{\pm 1}]) \cong \bigoplus_{i=1}^{l} \mathbb{K}[t^{\pm 1}]/(p_i(t))$$

for $p_i(t) \in \mathbb{K}[t^{\pm 1}]$, $1 \le i \le l$. We define $\Delta^{\varphi}(t) := \prod_{i=1}^{l} p_i(t) \in \mathbb{K}[t^{\pm 1}]$. $\Delta^{\varphi}(t)$ is called the *Alexander polynomial* of (X, φ), and it is well-defined up to a certain indeterminacy which is determined in [Fr07, Theorem 3.1] (cf. also [Co04, p. 367] and [Co85]). If $X = K(G,1)$ for a group G, then we also call $\Delta^{\varphi}(t)$ the Alexander polynomial of (G, φ).

Let $f(t) \in \mathbb{K}[t^{\pm 1}]$. If $f(t) = 0$, then we write $\deg(f(t)) = -\infty$. Otherwise, for $f(t) = \sum_{i=m}^{n} a_i t^i \in \mathbb{K}[t^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f(t)) := n - m$. Note that $\deg(\Delta^{\varphi}(t))$ is well–defined (cf. [Co04]). In fact we have the following interpretation of the degree of the Alexander polynomial:

$$\deg(\Delta^{\varphi}(\theta)) = \dim_{\mathbb{K}}(H_1(X;\mathbb{K}[t^{\pm 1}])). \tag{2}$$

Given $\varphi : \mathbb{Z}[\pi_1(X)] \to \mathbb{K}[t^{\pm 1}]$ we also consider the induced map $\mathbb{Z}[\pi_1(X)] \to \mathbb{K}[t^{\pm 1}] \to \mathbb{K}(t)$. In particular we can consider $\tau(X, \mathbb{K}(t)) \in K_1(\mathbb{K}(t))/\pm\pi_1(X) \cup \{0\}$. Note that setting $\deg(f(t)g(t)^{-1}) = \deg(f(t)) - \deg(g(t))$ for $f(t), g(t) \in \mathbb{K}\backslash[t^{\pm 1}]$ we can extend $\deg : \mathbb{K}\backslash[t^{\pm 1}] \setminus \{0\} \to \mathbb{Z}$ to a homomorphism $\deg : \mathbb{K}(t)^{\times} \to \mathbb{Z}$ which in turn induces (using the Dieudonné determinant) a homomorphism $\deg : K_1(\mathbb{K}(t)) \to \mathbb{Z}$. Note that a ring homomorphism $\mathbb{Z}[\pi_1(X)] \xrightarrow{\varphi} \mathbb{K}[t^{\pm 1}]$ sends $g \in \pi_1(X)$ to an invertible element in $\mathbb{K}[t^{\pm 1}]$, i.e. to an element of the form kt^l, whose degree is zero. We therefore see that $\deg : K_1(\mathbb{K}(t)) \to \mathbb{Z}$ passes to

$$\deg : K_1(\mathbb{K}(t))/\pm\pi_1(X) \cup \{0\} \to \mathbb{Z} \cup \{-\infty\}.$$

Let $\phi \in H^1(X;\mathbb{Z})$. We say $\phi \in H^1(X;\mathbb{Z})$ is *primitive* if the induced homomorphism $\phi : \pi_1(X) \to \mathbb{Z}$ is surjective. Following Turaev [Tu02] we call a homomorphism $\varphi : \mathbb{Z}[\pi_1(X)] \to \mathbb{K}[t^{\pm 1}]$ *ϕ–compatible* if for any $g \in \pi_1(X)$ we have $\varphi(\gamma) = kt^{\phi(g)}$ for some $k \in \mathbb{K}$.

Theorem 2.1. *Let M be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(M;\mathbb{Z})$ be primitive and let $\varphi : \mathbb{Z}[\pi_1(M)] \to \mathbb{K}[t^{\pm 1}]$ be a ϕ–compatible homomorphism. If $\Delta^{\varphi}(\theta) \neq 0$, then $\tau(M, \mathbb{K}(t)) \neq 0$. Furthermore, if $\varphi(\pi_1(M)) \subset \mathbb{K}[t^{\pm 1}]$ is cyclic, then*

$$\deg(\tau(M, \mathbb{K}(t))) = \deg(\Delta^{\varphi}(\theta)) - (1 + b_3(M)),$$

otherwise

$$\deg(\tau(M, \mathbb{K}(t))) = \deg(\Delta^{\varphi}(\theta)).$$

Proof. Assume that $\Delta^{\varphi}(\theta) \neq 0$; then $\tau(M, \mathbb{K}(t)) \neq 0$ by [Fr07, Theorem 1.2]. It follows from [Fr07, Theorem 1.1] that $\deg(\det(\tau(M, \mathbb{K}(t))))$ is the alternating sum of the degrees of the Alexander polynomials corresponding to $H_i(M;\mathbb{K}\backslash[t^{\pm 1}]), i = 0, 1, 2$. A direct computation of $H_0(M;\mathbb{K}\backslash[t^{\pm 1}])$ and $H_2(M;\mathbb{K}\backslash[t^{\pm 1}])$ as in [Fr07, Lemmas 4.3 and 4.4] concludes the proof. $\square$

3. Admissible pairs and triples

It is known that for a locally indicable and amenable group G the group ring $\mathbb{Z}[G]$ embeds in its (skew) quotient field which is flat over $\mathbb{Z}[G]$ (cf. e.g. [Hi40], [DLMSY03, Corollary 6.3] and [Ra98, p. 99]). We denote the quotient field by $\mathbb{K}(G)$. For instance, the poly-torsion-free-abelian (henceforth PTFA) groups studied in [COT03, Co04, Ha05] are locally indicable and amenable (cf. e.g. [St74]).

Definition 3.1. Let π be a group, $\phi : \pi \to \mathbb{Z}$ an epimorphism and $\varphi : \pi \to G$ an epimorphism to a locally indicable and amenable group G such that there exists a map $\phi_G : G \to \mathbb{Z}$ (which is necessarily unique) such that

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & G \arrow[d, "\phi_G"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes. Following [Ha06, Definition 1.4] we call (φ, ϕ) an *admissible pair for* π.

Let $(\varphi : \pi \to G, \phi)$ be an admissible pair for π. Let $G' := \ker\{\phi_G : G \to \mathbb{Z}\}$. Then G' is still a locally indicable and amenable group, hence embeds in its (skew) quotient field $\mathbb{K}(G')$. We pick an element $\mu \in G$ such that $\phi_G(\mu) = 1$ and define $\gamma : \mathbb{K}(G') \to \mathbb{K}(G')$ to be the homomorphism induced by $\gamma(g) := \mu g \mu^{-1}$ for $g \in G'$. Then we obtain a Laurent polynomial ring $\mathbb{K}(G')[t^{\pm 1}]$ with multiplication induced via γ as described in the first paragraph in Section 2.2. We point out that different choices of μ give isomorphic rings. Consider the ring homomorphism

$$\begin{array}{rcl} \mathbb{Z}[G] & \to & \mathbb{K}(G')[t^{\pm 1}], \\ \sum n_g g & \mapsto & \sum n_g g \mu^{-\phi_G(g)} t^{\phi_G(g)}. \end{array}$$

Clearly the induced homomorphism $\varphi : \mathbb{Z}[\pi] \to \mathbb{Z}[G] \to \mathbb{K}(G')[t^{\pm 1}]$ is ϕ–compatible. Also the map $f(t)g(t)^{-1} \mapsto f(\mu)g(\mu)^{-1}$ defines isomorphisms $\mathbb{K}(G')(t) \to \mathbb{K}(G)$ and $\mathbb{Z}[G'][t^{\pm 1}] \to \mathbb{Z}[G]$ (cf. also [Co04, p. 364]).

An important family of examples of admissible pairs is provided by Harvey's rational derived series of a group π (cf. [Ha05, Section 3]) which is defined as follows: Let $\pi_r^{(0)} := \pi$ and define inductively for $n \geq 1$

$$\pi_r^{(n)} := \left\{ g \in \pi_r^{(n-1)} \,|\, g^d \in \left[\pi_r^{(n-1)}, \pi_r^{(n-1)}\right] \text{ for some } d \in \mathbb{Z} \setminus \{0\} \right\}.$$

Note that $\pi_r^{(n-1)}/\pi_r^{(n)}$ is isomorphic to $\left(\pi_r^{(n-1)}/\left[\pi_r^{(n-1)}, \pi_r^{(n-1)}\right]\right)/\mathbb{Z}$–torsion. Hence it is torsion–free abelian. By [Ha05, Corollary 3.6] the quotients $\pi/\pi_r^{(n)}$ are PTFA groups for any π and any n. If $\phi : \pi \to \mathbb{Z}$ is an epimorphism, then $(\pi \to \pi/\pi_r^{(n)}, \phi)$ is an admissible pair for π for any $n > 0$. Let $\pi' := \ker\{\phi : \pi \to \mathbb{Z}\}$. Note that $\ker\{\pi/\pi_r^{(n+1)} \to \mathbb{Z}\} = \pi'/\pi_r^{(n+1)}$. We write $\mathbb{K}_n := \mathbb{K}\left(\pi'/\pi_r^{(n+1)}\right)$.

Given a finitely presented group π and a homomorphism $\phi : \pi \to \mathbb{Z}$ we now define

$$\Delta_n(\pi, \phi)(t) \in \mathbb{K}_n[t^{\pm 1}]$$

to be the Alexander polynomial of $(K(\pi, 1), \mathbb{Z}[\pi] \to \mathbb{K}(\pi'/\pi_r^{(n+1)})[t^{\pm 1}])$. This is called the *$n$–th higher–order Alexander polynomial of π corresponding to ϕ*. Note that this definition only works since we can assume that $K(\pi, 1)$ has finitely many cells in dimensions 1 and 2 since π is finitely presented.

Remark. Harvey [Ha06] and Cochran [Co04, Definition 5.3] show the following equality:

$$(3)\quad \delta_n(\pi,\phi) = \dim_{\mathbb{K}(\pi'/\pi_r^{(n+1)})}(\pi_r^{(n+1)}/[\pi_r^{(n+1)},\pi_r^{(n+1)}] \otimes_{\mathbb{Z}[\pi'/\pi_r^{(n+1)}]} \mathbb{K}(\pi'/\pi_r^{(n+1)})).$$

Here $\pi'/\pi_r^{(n+2)}$ acts on the abelian group $\pi^{(n+1)}/\pi_r^{(n+2)}$ via conjugation. This equality can easily be deduced from Lemma 3.2, equality (2) and the fact that Ore localizations are flat.

The next lemma follows immediately from the observation that we can view $K(\pi,1)$ as the result of adding k–cells with $k>2$ to X. It provides the link between group theory and topology.

Lemma 3.2. *Let X be a finite CW–complex with $\pi := \pi_1(X)$ and $\phi : \pi \to \mathbb{Z}$ a non–trivial homomorphism. Then the Alexander polynomials of $(X, \mathbb{Z}[\pi] \to \mathbb{K}(\pi'/\pi_r^{(n+1)})[t^{\pm1}])$ and $(K(\pi,1), \pi \to \mathbb{K}(\pi'/\pi_r^{(n+1)})[t^{\pm1}])$ agree.*

In our applications of Theorem 2.1 we will need the following basic lemma.

Lemma 3.3. *Let π be a group and $n \geq 1$. Then $\pi/\pi_r^{(n)}$ is cyclic if and only if*

(1) *$n=1$ and rank$(H_1(\pi))=1$, or*
(2) *$n \geq 2$, rank$(H_1(\pi))=1$ and $\delta_0(\pi,\phi)=0$ where $\phi:\pi\to\mathbb{Z}$ is either of the two epimorphisms.*

Proof. It is clear that if $\text{rank}(H_1(\pi))>1$, then $\pi/\pi_r^{(n)}$ is not cyclic. Now assume that $\text{rank}(H_1(\pi))=1$. The first statement is immediate. Now let $\phi:\pi\to\mathbb{Z}$ be either of the two epimorphisms. Clearly $\ker(\phi)=\pi_r^{(1)}$. For the second statement, note that by equation (3) we have

$$\delta_0(\pi,\phi) = \dim_{\mathbb{Q}}(\pi_r^{(1)}/[\pi_r^{(1)},\pi_r^{(1)}] \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Recall that $\pi_r^{(1)}/\pi_r^{(2)}$ is torsion free and that $\pi_r^{(1)}/[\pi_r^{(1)},\pi_r^{(1)}] \otimes_{\mathbb{Z}} \mathbb{Q} = \pi_r^{(1)}/\pi_r^{(2)} \otimes_{\mathbb{Z}} \mathbb{Q}$. This means that $\delta_0(\pi,\phi)=0$ if and only if $\pi_r^{(1)}=\pi_r^{(2)}$. But in that case $\pi_r^{(1)}=\pi_r^{(n)}$ for all n. The second statement is now clear. □

Before we state our main theorems we need to introduce admissible triples:

Definition 3.4. Let π be a group and $\phi:\pi\to\mathbb{Z}$ be an epimorphism. Let $\varphi_1:\pi\to G_1$ and $\varphi_2:\pi\to G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2. Following [Ha06, Definition 2.1] we call $(\varphi_1,\varphi_2,\phi)$ an *admissible triple for* π if there exist epimorphisms $\psi:G_1\to G_2$ and $\phi_2:G_2\to\mathbb{Z}$ such that $\varphi_2=\psi\circ\varphi_1$ and $\phi=\phi_2\circ\varphi_2$.

We note that for an admissible triple $(\varphi_1,\varphi_2,\phi)$ for π we have the following commutative diagram and that (φ_i,ϕ) are admissible pairs for π for $i=1,2$:

$$\begin{tikzcd}
 & G_1 \arrow[d, "\psi"] \\
\pi \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[dr, "\phi"'] & G_2 \arrow[d, "\phi_2"] \\
 & \mathbb{Z}.
\end{tikzcd}$$

Clearly for any epimorphism $\phi:\pi\to\mathbb{Z}$ and for any $n\geq m\geq 1$ we get admissible triples $(\pi\to\pi/\pi_r^{(n)}, \pi\to\pi/\pi_r^{(m)}, \phi)$ for π.

4. The statement of the main results

It follows from Theorem 2.1 and Lemmas 3.2 and 3.3 that the following theorem implies Theorem 1.1.

Theorem 4.1. *Let M be a 3–manifold with empty or toroidal boundary. Let $(\varphi_1 : \pi_1(M) \to G_1, \varphi_2 : \pi_1(M) \to G_2, \phi)$ be an admissible triple for $\pi_1(M)$. If $\tau(M, \mathbb{K}(G_2')(t)) \neq 0$, then $\tau(M, \mathbb{K}(G_1')(t)) \neq 0$ and*

$$\deg(\tau(M, \mathbb{K}(G_1')(t)) = \deg(\tau(M, \mathbb{K}(G_2')(t)) + 2k$$

for some $k \geq 0$.

The proof of this theorem is postponed to Section 5.2.

Now let M be a 3-manifold. We pick a primitive element $\phi \in H^1(M; \mathbb{Z})$. Note that if $b_1(M) = 1$, then there is a unique ϕ up to sign. Clearly (ϕ, ϕ) is an admissible pair and $\mathbb{K}(\mathbb{Z}')(t) = \mathbb{Q}(t)$, and we obtain the corresponding Reidemeister torsion $\tau(M, \mathbb{Q}(t))$.

Given a link $L \subset S^3$ we write $X(L) = S^3 \setminus N(L)$ for the link exterior, where $N(L)$ denotes an open tubular neighborhood of L. Now we can formulate the following corollary to Theorem 4.1.

Corollary 4.2. *Let M be a 3–manifold and $(\varphi : \pi_1(M) \to G, \phi)$ an admissible pair for $\pi_1(M)$.*

1. *If $M = X(K)$ is a knot exterior, then $\deg(\tau(X(K), \mathbb{K}(G')(t)))$ is odd.*
2. *If M is closed and $\tau(M, \mathbb{Q}(t)) \neq 0$, then $\deg(\tau(M, \mathbb{K}(G')(t)))$ is even.*
3. *If $M = X(L)$ is the exterior of a link $L = L_1 \cup \cdots \cup L_m \subset S^3$ and if $\tau(X(L), \mathbb{Q}(t)) \neq 0$, then*

$$\deg(\tau(X(L), \mathbb{K}(G')(t))) \equiv \sum_{i=1}^{m} \phi(\mu_i)\big(1 + \sum_{j \neq i} lk(L_i, L_j)\big) \, mod \, 2,$$

where μ_i denotes the meridian of L_i, $1 \leq i \leq m$.

Proof of Corollary 4.2. First recall the well–known fact that the Alexander polynomial $\Delta_K(t)$ is of even degree for any knot K. This implies that $\tau(X(K), \mathbb{Q}(t)) = \Delta_K(t)(t-1)^{-1}$ is of odd degree. The first statement now follows immediately from Theorem 4.1 applied to the admissible triple (φ, ϕ, ϕ).

The parity of the multivariable Alexander polynomials for links (cf. [Tu86, Theorem 1.7.1]) and for closed 3–manifolds (cf. [Tu86, p. 141]) are well–known. Using standard results (cf. [Tu86, Theorem 1.1.2] and also [FK05, Theorem 3.4], [FV06]) these can be translated into parity results for the ordinary one–variable Reidemeister torsion $\tau(M, \mathbb{Q}(t))$. These results in turn imply the last two statements by Theorem 4.1. □

We now discuss a special case of Corollary 4.2(1). Let K be a knot and let $\phi : H_1(X(K); \mathbb{Z}) \to \mathbb{Z}$ be an isomorphism. We write $\delta_n(K) := \delta_n(\pi_1(X(K)), \phi)$. By Lemma 3.2 this agrees with Cochran's [Co04] definition. In Question (5) in [Co04, Section 14] Cochran asks if there is a knot K and some $n > 0$ for which $\delta_n(K)$ is a non-zero *even* integer. Theorem 2.1 together with Corollary 4.2 now gives a negative answer to this question. More precisely we have the following result.

Theorem 4.3. *Let $(n_i)_{i\in\mathbb{N}_0}$ be a sequence of non–negative integers. Then there exists a knot K with $\delta_0(K)-1=n_0$ and $\delta_i(K)=n_i$ for $i\geq 1$ if and only if (n_i) is a never decreasing sequence of odd numbers which is bounded.*

Proof. Note that $\delta_0(K)$ is the degree of the (classical) Alexander polynomial of K, hence it is always even. One direction of the theorem now follows from Theorem 2.1 together with Corollary 4.2 and from the sequence of inequalities (1) in Section 1. The realization results in [Co04, Theorem 7.3] can be strengthened to prove the converse (cf. also Question (5) in [Co04]). □

Remarks.

(1) Note that more realization results are given in [Ha05, Section 11].

(2) The statement corresponding to Corollary 4.2 for the Reidemeister torsion of a 3–manifold associated to a general linear representation $\pi_1(M)\to \mathrm{GL}(\mathbb{F},k)$ (where $\mathbb{F}$ is a field) is not known. A careful analysis of Heegard decompositions of closed 3–manifolds as in [He83] could perhaps be used to show that for a closed 3–manifold the degrees of twisted Reidemeister torsions are always even.

Using Corollary 4.2 we will prove in Section 5.3 the following theorem which by Lemmas 3.2 and 3.3 is clearly a slight generalization of Theorem 1.2.

Theorem 4.4. *Let M be a closed 3–manifold or the exterior of a link in S^3. Let $(\varphi:\pi_1(M)\to G,\phi)$ be an admissible pair. If $\tau(M,\mathbb{Q}(t))\neq 0$, then*

$$\max\{0,\deg(\tau(M,\mathbb{K}(G')(t)))\}\equiv ||\phi||_T \ mod\, 2.$$

5. Proof of the theorems

5.1. **Cohn localization.** Let $\psi: G_1\to G_2$ be an epimorphism between locally indicable amenable groups G_1 and G_2. Let Σ be the set of all matrices over $\mathbb{Z}[G_1]$ which become invertible under the map $\mathbb{Z}[G_1]\xrightarrow{\psi}\mathbb{Z}[G_2]\to\mathbb{K}(G_2)$. Denote by $c:\mathbb{Z}[G_1]\to C(\psi)$ the Cohn localization of $\mathbb{Z}[G_1]$ corresponding to Σ. Recall that the Cohn localization is characterized by the following two conditions:

(1) The homomorphism c is Σ–inverting, i.e. any matrix in Σ becomes invertible under c.
(2) The homomorphism c has the universal Σ–inverting property, i.e. if $d:\mathbb{Z}[G_1]\to D$ is another Σ–inverting ring homomorphism, then there exists a unique homomorphism $C(\psi)\to D$ to make the following diagram commute:

$$\begin{tikzcd} \mathbb{Z}[G_1] \arrow[r, "c"] \arrow[dr, "d"'] & C(\psi) \arrow[d] \\ & D. \end{tikzcd}$$

We refer to [Co85, Chapter 7] for more details. Note that for $r\neq s$ we have $C(\psi)^r\not\cong C(\psi)^s$ since $C(\psi)^i\otimes_{C(\psi)}\mathbb{K}(G_2)\cong\mathbb{K}(G_2)^i$.

Lemma 5.1. *Let $\psi: G_1\to G_2$ be an epimorphism between locally indicable amenable groups. Let B be an $r\times r$–matrix over $\mathbb{Z}[G_1]$. If $\psi(B):\mathbb{Z}[G_2]^r\to\mathbb{Z}[G_2]^r$ is invertible over $\mathbb{K}(G_2)$, then B is invertible over $\mathbb{K}(G_1)$.*

Proof. First note that $\psi(B):\mathbb{Z}[G_2]^r\to\mathbb{Z}[G_2]^r$ is injective since $\mathbb{Z}[G_2]\subset\mathbb{K}(G_2)$. Now let $H=\ker\{\psi: G_1\to G_2\}$. Clearly H is again locally indicable. Note that

$B : \mathbb{Z}[G_1]^r \to \mathbb{Z}[G_1]^r$ can also be viewed as a map between free $\mathbb{Z}[H]$–modules. Pick any right inverse $\lambda : G_2 \to G_1$ of ψ. It is easy to see that $g \otimes h \mapsto g\lambda(h) \otimes 1, g \in G_1, h \in G_2$ induces an isomorphism

$$\mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[H]} \mathbb{Z}.$$

By assumption $B \otimes \mathrm{id} : \mathbb{Z}[G_1]^r \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to \mathbb{Z}[G_1]^r \otimes_{\mathbb{Z}[H]} \mathbb{Z}$ is injective. Since H is locally indicable it follows immediately from [Ge83] or [HS83] (cf. also [St74] for the case of PTFA groups) that $\mathbb{Z}[G_1]^r \to \mathbb{Z}[G_1]^r$ is injective.

Since $\mathbb{K}(G_1)$ is flat over $\mathbb{Z}[G_1]$ it follows that $B : \mathbb{K}(G_1)^r \to \mathbb{K}(G_1)^r$ is injective. But an injective homomorphism between vector spaces over a skew field of the same dimension is in fact an isomorphism. This shows that B is invertible over $\mathbb{K}(G_1)$. ☐

It follows from the universal property of the Cohn localization and Lemma 5.1 that there exist unique maps $C(\psi) \to \mathbb{K}(G_1)$ and $C(\psi) \to \mathbb{K}(G_2)$ which make the following diagram commute:

$$\begin{tikzcd}
 & \mathbb{K}(G_1) \\
\mathbb{Z}[G_1] \arrow[ur] \arrow[r] \arrow[dr, "\psi"'] & C(\psi) \arrow[u] \arrow[d] \\
 & \mathbb{K}(G_2).
\end{tikzcd}$$

Note that since the homomorphism $\mathbb{Z}[G_1] \to \mathbb{K}(G_1)$ is an injection it follows that the homomorphism $\mathbb{Z}[G_1] \to C(\psi)$ is an injection as well. We will henceforth identify $\mathbb{Z}[G_1]$ with its image in $C(\psi)$.

Lemma 5.2. *Let M be a 3–manifold and $(\varphi_1 : \pi_1(M) \to G_1, \varphi_2 : \pi_1(M) \to G_2, \phi)$ an admissible triple. If $\tau(M, \mathbb{K}(G_2)) \neq 0$, then $\tau(M, C(\psi)) \neq 0$.*

Proof. We have to show that if $H_*(M; \mathbb{K}(G_2)) = 0$, then $H_*(M; C(\psi)) = 0$. Denote the chain complex of the universal cover $\tilde{M}$ by (C_*, d_*). Write $\pi := \pi_1(M)$. By assumption we have $H_*(M; \mathbb{K}(G_2)) = H_*(C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(G_2)) = 0$. Therefore there exist chain homotopies $s_k : C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(G_2) \to C_{k+1} \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(G_2)$ such that for any k

$$d_{k+1} \circ s_k + s_{k-1} \circ d_k = \mathrm{id}.$$

Note that $C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_2]$ is a free $\mathbb{Z}[G_2]$–module and $C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{K}(G_2) = (C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_2]) \otimes_{\mathbb{Z}[G_2]} \mathbb{K}(G_2)$. Since $\mathbb{K}(G_2)$ is the Ore localization of $\mathbb{Z}[G_2]$ the chain homotopies s_k induce $\mathbb{Z}[G_2]$–homomorphisms $s'_k : C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_2] \to C_{k+1} \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_2]$ such that $d_{k+1} \circ s'_k + s'_{k-1} \circ d_k$ is invertible over $\mathbb{K}(G_2)$.

Furthermore note that $C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_1]$ is a free $\mathbb{Z}[G_1]$–module and $C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_2] = (C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_1]) \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2]$. Hence we can find lifts of s'_k to $\mathbb{Z}[G_1]$–homomorphisms $t'_k : C_k \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_1] \to C_{k+1} \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[G_1]$ such that $d_{k+1} \circ t'_k + t'_{k-1} \circ d_k$ becomes invertible under the map $\mathbb{Z}[G_1] \to \mathbb{Z}[G_2] \to \mathbb{K}(G_2)$. From the definition of the Cohn localization it follows that

$$d_{k+1} \circ t'_k + t'_{k-1} \circ d_k : C_k \otimes_{\mathbb{Z}[G_1]} C(\psi) \to C_k \otimes_{\mathbb{Z}[G_1]} C(\psi)$$

is an isomorphism for any k, hence $H_*(M; C(\psi)) = 0$. ☐

5.2. **Proof of Theorem 4.1.** In the following assume that R is a ring equipped with a (possibly trivial) involution $r \mapsto \overline{r}$ such that $\overline{r \cdot s} = \overline{s} \cdot \overline{r}$. This extends to an involution on $K_1(R)$ via $\overline{(a_{ij})} := (\overline{a_{ji}})$ for $(a_{ij}) \in K_1(R)$. Note that if R is a skew field, then $\det(\overline{A}) = \overline{\det(A)}$ for any $A \in K_1(R)$.

We need the following lemma.

Lemma 5.3. *Let $\mathbb{K}$ be a skew field with (possibly trivial) involution. Let $\mathbb{K}(t)$ be the (skew) quotient field of a Laurent polynomial ring $\mathbb{K}\backslash[t^{\pm 1}]$ equipped with the involution given by $\overline{kt^l} := t^{-l}\overline{k}$. Assume that $\tau \in K_1(\mathbb{K}(t))$ has the property that $\tau = kt^j\, \overline{\tau}$ for some $k \in \mathbb{K}, j \in \mathbb{Z}$. Then*

$$\deg(\tau) \equiv j \mod 2.$$

Proof. Recall that the Dieudonné determinant defines an isomorphism $K_1(\mathbb{K}(t)) \to \mathbb{K}(t)^\times/[\mathbb{K}(t)^\times, \mathbb{K}(t)^\times]$. We can therefore rewrite the assumption as

$$\det(\tau) = kt^j\, \overline{\det(\tau)} \prod_{i=1}^{n} [f_i(t), g_i(t)] \in \mathbb{K}(t),$$

for some $f_i(t), g_i(t) \in \mathbb{K}(t)^\times$. For $p(t) = a_r t^r + a_{r+1}t^{r+1} + \cdots + a_s t^s \in \mathbb{K}\backslash[t^{\pm 1}]$ with $a_i \in \mathbb{K}, a_r \neq 0, a_s \neq 0, r \leq s$ we define $l(p(t)) := r$ (the lowest exponent), $h(p(t)) := s$ (the highest exponent). Clearly l and h define homomorphisms $\mathbb{K}\backslash[t^{\pm 1}]\backslash\{0\} \to \mathbb{Z}$. Furthermore since $\mathbb{K}(t)$ is the Ore localization of $\mathbb{K}\backslash[t^{\pm 1}]$ we can extend l and h to homomorphisms $l, h : \mathbb{K}(t)^\times \to \mathbb{Z}$ via $l(f(t)g(t)^{-1}) = l(f(t)) - l(g(t))$ and $h(f(t)g(t)^{-1}) = h(f(t)) - h(g(t))$ for $f(t), g(t) \in \mathbb{K}\backslash[t^{\pm 1}] \setminus \{0\}$. Note that l, h vanish on commutators. Then we get the following equalities:

$$\begin{aligned} h(\det(\tau)) &= h(kt^j\, \overline{\det(\tau)} \textstyle\prod_{i=1}^{n} [f_i(t), g_i(t)]) \\ &= j + h(\overline{\det(\tau)}) \\ &= j - l(\det(\tau)). \end{aligned}$$

It follows that

$$\deg(\tau) \equiv \deg(\det(\tau)) \equiv h(\det(\tau)) - l(\det(\tau)) \equiv h(\det(\tau)) + l(\det(\tau)) \equiv j \mod 2.$$

□

We need the following duality theorem.

Theorem 5.4. *Let M be a 3–manifold with empty or toroidal boundary. Assume that R is equipped with a (possibly trivial) involution and that $\varphi : \mathbb{Z}[\pi_1(M)] \to R$ is a homomorphism such that $\varphi(\gamma^{-1}) = \overline{\varphi(\gamma)}$ for all $g \in \pi_1(M)$. If $\tau(M, R) \neq 0$, then*

$$\tau(M, R) = \overline{\tau(M, R)} \in K_1(R)/ \pm \pi_1(M).$$

Proof. Similarly to $\tau(M, R)$ we can define $\tau(M, \partial M, R)$. By [Tu01, Theorem 14.1] it follows that $H_*(M, \partial M; R) = 0$ and

$$\tau(M, R) = \overline{\tau(M, \partial M, R)} \in K_1(R)/ \pm \pi_1(M).$$

In particular we are done if M is closed. If M has boundary, then it follows from the long exact sequence of the pair $(M, \partial M)$ that $H_*(\partial M; R) = 0$. Furthermore by [Tu01, Theorem 3.4] we have

$$\tau(M, R) = \tau(M, \partial M, R)\tau(\partial M, R) \in K_1(R)/ \pm \pi_1(M).$$

It therefore suffices to show that $\tau(\partial M, R) = 1 \in K_1(R)/\pm\pi_1(M)$. But this follows from an easy argument using the standard CW–structure of a torus. □

For a locally indicable amenable group G, we will always consider the group ring $\mathbb{Z}[G]$ together with the involution given by $\overline{g} := g^{-1}$ for $g \in G$. Note that this involution extends to involutions on $\mathbb{K}(G)$ and $K_1(\mathbb{K}(G))$. Recall that for an admissible pair $(\varphi : \pi \to G, \phi)$ for π we have the isomorphism $\mathbb{Z}[G'][t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}[G]$. If we equip $\mathbb{Z}[G'][t^{\pm 1}]$ with the involution given by $\overline{gt^l} := t^{-l}g^{-1}$ for $g \in G'$, then this isomorphism preserves involutions. The extended involutions on $\mathbb{K}(G')(t)$ and $\mathbb{K}(G)$ are also preserved under the corresponding isomorphism, and we have a similar result for $K_1(\mathbb{K}(G')(t))$ and $K_1(\mathbb{K}(G))$.

Proof of Theorem 4.1. Let M be a 3–manifold with empty or toroidal boundary. Let $(\varphi_1 : \pi_1(M) \to G_1, \varphi_2 : \pi_1(M) \to G_2, \phi)$ be an admissible triple for $\pi_1(M)$. Assume that $\tau(M, \mathbb{K}(G_2')(t)) \neq 0$. Harvey [Ha06] showed that this implies that $\tau(M, \mathbb{K}(G_1')(t)) \neq 0$ and furthermore that

$$\deg(\tau(M, \mathbb{K}(G_1')(t))) \geq \deg(\tau(M, \mathbb{K}(G_2')(t))).$$

(See also [Fr07, Theorem 1.3]; we also refer to [Co04] in the case of a knot exterior.) Therefore in order to prove Theorem 4.1 it is enough to show that

$$\deg(\tau(M, \mathbb{K}(G_1')(t))) \equiv \deg(\tau(M, \mathbb{K}(G_2')(t))) \mod 2.$$

Write $\pi := \pi_1(M)$. Note that $\tau(M, C(\psi))$ and $\overline{\tau(M, C(\psi))}$ are non–zero in $K_1(C(\psi))/\pm\pi \cup \{0\}$ by Lemma 5.2. Now pick a representative $\tau^{rep}(M, C(\psi))$ of $\tau(M,C(\psi))$ in $K_1(C(\psi))$. Note that $\overline{\tau^{rep}(M, C(\psi))}$ is a representative of $\overline{\tau(M, C(\psi))}$ in $K_1(C(\psi))$. It follows from Theorem 5.4 that

$$\tau^{rep}(M, C(\psi)) = \epsilon g_1 \overline{\tau^{rep}(M, C(\psi))} \in K_1(C(\psi))$$

for some $\epsilon \in \{1, -1\}, g_1 \in G_1$. (Recall that we identified $\mathbb{Z}[G_1]$ with the image of $\mathbb{Z}[G_1]$ in $C(\psi)$.) Recall that we have homomorphisms $\alpha_i : C(\psi) \to \mathbb{K}(G_i')(t)$ for $i = 1, 2$ induced from the universal property of the Cohn localization $C(\psi)$. These homomorphisms induce homomorphisms $\alpha_i : K_1(C(\psi)) \to K_1(\mathbb{K}(G_i')(t))$ for $i = 1, 2$.

Then for $i = 1, 2$,

$$\begin{aligned}\alpha_i(\tau^{rep}(M, C(\psi))) &= \alpha_i(\epsilon g_1 \overline{\tau^{rep}(M, C(\psi))}) \\ &= \epsilon\alpha_i(g_1)\alpha_i(\overline{\tau^{rep}(M, C(\psi))}) \in K_1(\mathbb{K}(G_i')(t)).\end{aligned}$$

Note that because Reidemeister torsion is functorial (cf. e.g. [Tu01, Proposition 3.6]), $\alpha_i(\tau^{rep}(M, C(\psi))) \in K_1(\mathbb{K}(G_i')(t))$ is a representative of $\tau(M, \mathbb{K}(G_i')(t)) \in K_1(\mathbb{K}(G_i')(t))/\pm\pi$. Now write $\alpha_i(g_1) = k_i t^{l_i}$ for some $k_i \in \mathbb{K}(G_i')$ and $l_i \in \mathbb{Z}$. Note that $l_i = \phi(g_1)$ since $\mathbb{Z}[\pi] \to \mathbb{K}(G_i')[t^{\pm 1}]$ is ϕ–compatible. In particular $l_1 = l_2$. The theorem now follows from Lemma 5.3 which states that

$$l_i \equiv \deg(\tau(M, \mathbb{K}(G_i')(t))) \mod 2$$

for $i = 1, 2$. □

5.3. **Proof of Theorem 4.4.** We will need the following lemma.

Lemma 5.5. *Let $\phi \in H^1(M)$ be primitive such that $\tau(M, \mathbb{Q}(t)) \neq 0$. If M is closed, then $||\phi||_T$ is even. Assume that ∂M consists of a non–empty collection of tori $N_1 \cup \cdots \cup N_m$. If $\phi|_{H_1(N_i)} = 0$, then let $n_i := 0$, otherwise define*

$$n_i = \max\{n \in \mathbb{N} | \phi|_{H_1(N_i)} = nh \textit{ for some } h \in \hom(H_1(N_i), \mathbb{Z})\},$$

i.e. n_i *is the divisibility of* $\phi|_{H_1(N_i)}$. *Then either* $||\phi||_T = 0$ *or*

$$||\phi||_T \equiv \left(\sum_{i=1}^m n_i\right) \, mod\ 2.$$

Proof. If M is closed, then S is closed, hence $\chi(S)$ is even. Now assume that ∂M is a non–empty collection of tori. Since $\tau(M,\mathbb{Q}(t)) \neq 0$ it follows from [Mc02] that there exists a connected Thurston norm minimizing surface S dual to ϕ. Assume that $||\phi||_T > 0$, then $||\phi||_T = -\chi(S)$. Note that

$$-\chi(S) \equiv b_0(\partial S) \;\; mod\, 2;$$

this follows from the observation that adding a 2–disk to each component of ∂S gives a closed surface, which has even Euler characteristic. Now consider N_i. Clearly $S \cap N_i$ is Poincaré dual to $\phi|_{H_1(N_i)}$. It follows from a standard argument that, modulo 2, $\partial S \cap N_i$ has n_i components. □

Now we can give the proof of Theorem 4.4.

Proof of Theorem 4.4. First assume that $||\phi||_T = 0$. Note that it follows from equation (1) and from Theorem 2.1 that $\deg(\tau(M,\mathbb{K}(G')(t))) \leq ||\phi||_T$. The required equality is now immediate. Now assume that $||\phi||_T > 0$.

Since both sides are $\mathbb{N}$–linear we can restrict ourselves to the case that ϕ is primitive. If M is closed, then $||\phi||_T$ is even and the theorem follows from Corollary 4.2(2). Now assume that $M = X(L)$ where $L = L_1 \cup \dots \cup L_m$ is a link in S^3.

Denote by $\mu_1,\dots,\mu_m$ (respectively $\lambda_1,\dots,\lambda_m$) the meridians (respectively longitudes) of $L_1 \cup \dots \cup L_m$. Denote by $N_1 \cup \dots \cup N_m$ the boundary components of $X(L)$. Let $n_1,\dots,n_m$ as in Lemma 5.5. Then by Lemma 5.5

$$||\phi||_T \equiv \left(\sum_{i=1}^m n_i\right) \, mod\ 2.$$

Clearly μ_i, λ_i is a basis for $H_1(N_i)$ and $\phi|_{H_1(N_i)} \in H^1(N_i)$ is Poincaré dual to $\phi(\mu_i)\mu_i + \big(\sum_{j\neq i}\phi(\mu_j)\mathrm{lk}(L_i,L_j)\big)\lambda_i$. In particular

$$n_i = \gcd\Big(\phi(\mu_i), \sum_{j\neq i}\phi(\mu_j)\mathrm{lk}(L_i,L_j)\Big)$$

where we set $\gcd(a,0) := \gcd(0,a) := 0$.

Without loss of generality we can assume that there exists an r such that $\phi(\mu_i) \equiv 1\, mod\, 2$ for $i \leq r$ and $\phi(\mu_i) \equiv 0\, mod\, 2$ for $i > r$. It follows immediately that

$$\sum_{i=1}^m n_i \equiv r + \sum_{i=r+1}^m \sum_{j\neq i} \phi(\mu_j)\mathrm{lk}(L_i,L_j) \;\; mod\, 2.$$

On the other hand by Corollary 4.2 and by the symmetry of the linking numbers we have

$$\begin{array}{rcll}
\deg(\tau(M,\mathbb{K}(G')(t))) & \equiv & \sum_{i=1}^m \phi(\mu_i) + \sum_{i=1}^m \sum_{j\neq i} \phi(\mu_i)\mathrm{lk}(L_i,L_j) & mod\, 2 \\
& \equiv & r + \sum_{i=1}^m \sum_{j<i} (\phi(\mu_i)+\phi(\mu_j))\mathrm{lk}(L_i,L_j) & mod\, 2 \\
& \equiv & r + \sum_{i=r+1}^m \sum_{j<i} (\phi(\mu_i)+\phi(\mu_j))\mathrm{lk}(L_i,L_j) & mod\, 2 \\
& \equiv & r + \sum_{i=r+1}^m \sum_{j<i} \phi(\mu_j)\mathrm{lk}(L_i,L_j) & mod\, 2 \\
& \equiv & r + \sum_{i=r+1}^m \sum_{j\neq i} \phi(\mu_j)\mathrm{lk}(L_i,L_j) & mod\, 2.
\end{array}$$

Note that we used that $\phi(\mu_i) + \phi(\mu_j) \equiv 0 \, mod \, 2$ for $i, j \leq r$ and $\phi(\mu_j) \equiv 0 \, mod \, 2$ for $j > r$. □

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