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ZJ-THEOREMS FOR FUSION SYSTEMS

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Abstract. For p an odd prime, we generalise the Glauberman-Thompson p-nilpotency theorem (Gorenstein, 1980) to arbitrary fusion systems. We define a notion of $Qd(p)$-free fusion systems and show that if $\mathcal{F}$ is a $Qd(p)$-free fusion system on some finite p-group P, then $\mathcal{F}$ is controlled by $W(P)$ for any Glauberman functor W, generalising Glauberman's ZJ-theorem (Glauberman, 1968) to arbitrary fusion systems.

1. Introduction

Let p be an odd prime, let G be a finite group, let P be a Sylow-p-subgroup of G and let $J(P)$ be the Thompson subgroup of P (generated by the set of abelian subgroups of P of maximal order). The p-nilpotency theorem of Glauberman and Thompson [5, Ch. 8, Theorem 3.1] states that G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent. By a theorem of Frobenius [4, 8.6], G is p-nilpotent if and only if P controls G-fusion in P, or equivalently, if and only if $\mathcal{F}_P(G) = \mathcal{F}_P(P)$, where the notation is as described in §2 below. The p-nilpotency theorem has been generalised to p-blocks of finite groups in [7], and the following theorem proves an analogue for arbitrary fusion systems.

Theorem A. *Let p be an odd prime and let $\mathcal{F}$ be a fusion system on a finite p-group P. We have $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $N_\mathcal{F}(Z(J(P))) = \mathcal{F}_P(P)$.*

The proof of Theorem A is given in §4. A finite group A is said to be *involved* in another finite group G if there are subgroups H, K of G such that $K \trianglelefteq H$ and $A \cong H/K$. Glauberman's ZJ-Theorem in [3] asserts that if p is odd and $Qd(p)$ is not involved in G, then $N_G(Z(J(P)))$ controls strong p-fusion in G, where P is a Sylow-p-subgroup of G and where $Qd(p)$ is the semi-direct product $(C_p \times C_p) \rtimes SL_2(p)$ with $SL_2(p)$ acting naturally on the elementary abelian group of rank 2. By [4, 14.8] the conclusion holds in fact with ZJ replaced by any Glauberman functor (cf. 1.2 below). In order to extend this to arbitrary fusion systems, we introduce the following notation and terminology. By [1, 4.3], if $\mathcal{F}$ is a fusion system on a finite p-group P and Q is an $\mathcal{F}$-centric fully normalised subgroup of P, there is, up to isomorphism, a unique finite group $L = L_Q^\mathcal{F}$ having $N_P(Q)$ as a Sylow-p-subgroup such that $C_L(Q) = Z(Q)$ and $N_\mathcal{F}(Q) = \mathcal{F}_{N_P(Q)}(L)$.

Definition 1.1. A fusion system $\mathcal{F}$ on a finite p-group P is called *$Qd(p)$-free* if $Qd(p)$ is not involved in any of the groups $L_Q^\mathcal{F}$, with Q running over the set of $\mathcal{F}$-centric radical fully normalised subgroups of P.

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Definition 1.2 (cf. [8, 1.3])**.** A *positive characteristic p-functor* is a map sending any finite p-group P to a characteristic subgroup $W(P)$ of P such that $W(P) \neq 1$ if $P \neq 1$ and such that any isomorphism of finite p-groups $P \cong Q$ maps $W(P)$ onto $W(Q)$. A *Glauberman functor* is a positive characteristic p-functor with the following additional property: whenever P is a Sylow-p-subgroup of a finite group L which satisfies $C_L(O_p(L)) = Z(O_p(L))$ and which does not involve $Qd(p)$, then $W(P)$ is normal in L.

Any of the maps sending a finite p-group P to $Z(J(P))$ or $K_\infty(P)$ or $K^\infty(P)$ are Glauberman functors, where $J(P)$ is the Thompson subgroup of P, and where K_∞, K^∞ are as defined in [4, Section 12].

Theorem B. *Let p be an odd prime, let W be a Glauberman functor and let $\mathcal{F}$ be a fusion system on a finite p-group P. If $\mathcal{F}$ is $Qd(p)$-free, then $\mathcal{F} = N_{\mathcal{F}}(W(P))$.*

For fusion systems of finite groups this is Glauberman's ZJ-theorem; for fusion systems of p-blocks of finite groups this has also been noted by G. R. Robinson, generalising [8, 1.4] where it was shown that the conclusion of Theorem B holds under the slightly stronger assumption that $SL_2(p)$ is not involved in any of the automorphism groups $\mathrm{Aut}_{\mathcal{F}}(Q)$, with Q running over the set of $\mathcal{F}$-centric radical subgroups of P. The proof of Theorem B, given in §7, follows the pattern of the proof of [8, 1.4].

Since there exist Glauberman functors mapping P to a subgroup $W(P)$ satisfying $C_P(W(P)) = Z(W(P))$ (for example, K_∞, K^∞ have this property), the above theorem in conjunction with [1, 4.3] implies that a $Qd(p)$-free fusion system on a finite p-group P is in fact equal to the fusion system of a finite group L having P as Sylow-p-subgroup and satisfying $C_L(O_p(L)) \subseteq O_p(L)$. In particular, a $Qd(p)$-free fusion system is the underlying fusion system of a unique p-local finite group in the sense of [2].

2. Background material on fusion systems

Let p be a prime and let P be a finite p-group. Following the terminology of [11], a *category on* P is a category $\mathcal{F}$ with the subgroups of P as objects and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ consisting of injective group homomorphisms, for any two subgroups Q, R of P, such that the following hold. Composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms, and for any morphism $\varphi : Q \to R$ in $\mathcal{F}$, the induced isomorphism $Q \cong \varphi(Q)$, its inverse and the inclusion $\varphi(Q) \subseteq R$ are all morphisms in $\mathcal{F}$ as well. Given a category $\mathcal{F}$ on P and a subgroup Q of P, we say that

- Q is *fully $\mathcal{F}$-normalised* if $|N_P(Q)| \geq |N_P(\varphi(Q))|$ for every morphism $\varphi : Q \to P$ in $\mathcal{F}$;
- Q is *fully $\mathcal{F}$-centralised* if $|C_P(Q)| \geq |C_P(\varphi(Q))|$ for every morphism $\varphi : Q \to P$ in $\mathcal{F}$;
- Q is *$\mathcal{F}$-centric* if $C_P(\varphi(Q)) = Z(\varphi(Q))$ for every morphism $\varphi : Q \to P$ in $\mathcal{F}$; and
- Q is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Aut}_Q(Q)$.

Following [2, 1.2], if $\varphi : Q \to P$ is a morphism in a category $\mathcal{F}$ on P, we denote by N_φ the subgroup of $N_P(Q)$ consisting of all $y \in N_P(Q)$ for which there is $z \in N_P(\varphi(Q))$ such that $\varphi(yuy^{-1}) = z\varphi(u)z^{-1}$ for all $u \in Q$. If Q, R are subgroups of P we denote by $\mathrm{Hom}_P(Q, R)$ the set of all group homomorphisms from Q to R induced by conjugation with elements in P. If $Q = R$ we write $\mathrm{Aut}_P(Q) = \mathrm{Hom}_P(Q,Q)$; note that $\mathrm{Aut}_P(Q) \cong N_P(Q)/C_P(Q)$.

A *fusion system on* P is a category $\mathcal{F}$ on P whose morphism sets contain all morphisms induced by conjugation with elements in P, and which has furthermore the following properties.

(I-S) $\mathrm{Aut}_P(P)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$;

(II-S) for every morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalised, there is a morphism $\psi : N_\varphi \to P$ such that $\varphi = \psi|_Q$.

This concept is due to Puig [12]. The above definition appears in [9] and is equivalent to the definition of what is called a *saturated fusion system* in [2, 1.2]; in particular, it is shown in [9] that the axioms (I-S) and (II-S) imply the axioms used in [2, 1.2]:

(I-BLO) if Q is a fully $\mathcal{F}$-normalised subgroup of P, then Q is fully $\mathcal{F}$-centralised and $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$;

(II-BLO) for every morphism $\varphi : Q \to P$ such that $\varphi(Q)$ is fully $\mathcal{F}$-centralised, there is a morphism $\psi : N_\varphi \to P$ such that $\varphi = \psi|_Q$.

If G is a finite group having P as Sylow-p-subgroup, we denote by $\mathcal{F}_P(G)$ the category on P whose morphism sets are the group homomorphisms induced by conjugation with elements in G; that is, $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$ for any two subgroups Q, R of P. It is well known and easy to verify that $\mathcal{F}_P(G)$ is a fusion system; we call $\mathcal{F}_P(G)$ the fusion system of the finite group G on P. In particular, $\mathcal{F}_P(P)$ is the fusion system on P whose morphisms are exactly those induced by inner automorphisms in P. Note that $\mathcal{F}_P(P) \subseteq \mathcal{F}$ for any fusion system on P.

Let $\mathcal{F}$ be a fusion system on P and let Q be a subgroup of P. We denote by $C_{\mathcal{F}}(Q)$ the category on $C_P(Q)$ such that, for any two subgroups R, R' of $C_P(Q)$, the morphism set in $C_{\mathcal{F}}(Q)$ from R to R' consists of all group homomorphisms $\varphi : R \to R'$ such that there exists a morphism $\psi : RQ \to RQ'$ in $\mathcal{F}$ satisfying $\psi|_R = \varphi$ and $\psi|_Q = \mathrm{Id}_Q$. Similarly, we denote by $N_{\mathcal{F}}(Q)$ the category on $N_P(Q)$ such that, for any two subgroups R, R' of $N_P(Q)$, the morphism set in $N_{\mathcal{F}}(Q)$ from R to R' consists of all group homomorphisms $\varphi : R \to R'$ such that there exists a morphism $\psi : RQ \to RQ'$ in $\mathcal{F}$ satisfying $\psi|_R = \varphi$ and $\psi(Q) = Q$. By a result of Puig, if Q is fully $\mathcal{F}$-centralised, then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$, and if Q is fully $\mathcal{F}$-normalised, then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$. Both statements are in fact particular cases of a more general result; see e.g. [2, Appendix, Prop. A6] for a proof. If $\mathcal{F} = N_{\mathcal{F}}(Q) = N_{\mathcal{F}}(R)$ for normal subgroups Q, R of P, then one easily checks that $\mathcal{F} = N_{\mathcal{F}}(QR)$. Thus the following definition makes sense:

Definition 2.1. Let $\mathcal{F}$ be a fusion system on a finite p-group P. We denote by $O_p(\mathcal{F})$ the largest normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(O_p(\mathcal{F}))$.

Lemma 2.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. For any subgroup Q of P there is a morphism $\varphi : N_P(Q) \to P$ in $\mathcal{F}$ such that $\varphi(Q)$ is fully $\mathcal{F}$-normalised.*

Proof. Let $\psi : Q \to P$ be a morphism in $\mathcal{F}$ such that $\psi(Q)$ is fully $\mathcal{F}$-normalised. Thus $\mathrm{Aut}_P(\psi(Q))$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Therefore $\psi \circ \mathrm{Aut}_P(Q) \circ \psi^{-1}$ is conjugate to a subgroup of $\mathrm{Aut}_P(\psi(Q))$, say $\tau \circ \psi \circ \mathrm{Aut}_P(Q) \circ \psi^{-1} \circ \tau^{-1} \subset \mathrm{Aut}_P(\psi(Q))$ for some $\tau \in \mathrm{Aut}_{\mathcal{F}}(\psi(Q))$. Thus $\varphi = \tau \circ \psi$ has the property that $N_\varphi = N_P(Q)$; hence φ extends to a morphism $N_P(Q) \to P$, and $\varphi(Q) = \psi(Q)$ is fully $\mathcal{F}$-normalised. $\square$

Lemma 2.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. For any fully $\mathcal{F}$-normalised subgroup Q of P and any morphism $\varphi : N_P(Q) \to P$ the subgroup $\varphi(Q)$ is fully $\mathcal{F}$-normalised.*

Proof. If $\varphi : N_P(Q) \to P$ is a morphism in $\mathcal{F}$, then $\varphi(N_P(Q)) \subseteq N_P(\varphi(Q))$, hence this inclusion is an equality whenever Q is fully $\mathcal{F}$-normalised. $\square$

Given two subgroups Q, R of P such that $Q \leq R$ we denote by $\mathrm{Aut}_{\mathcal{F}}(Q \leq R)$ the subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$ consisting of all $\varphi \in \mathrm{Aut}_{\mathcal{F}}(R)$ satisfying $\varphi(Q) = Q$. Restriction to Q induces a group homomorphism $\mathrm{Aut}_{\mathcal{F}}(Q \leq R) \to \mathrm{Aut}_{\mathcal{F}}(Q)$. The following lemma is a reformulation of a well-known fact following from the extension axiom for fusion systems.

Lemma 2.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q, R be subgroups of P such that $QC_P(Q) \leq R$. Suppose that Q is fully $\mathcal{F}$-centralised and that $\mathrm{Aut}_R(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$. The group homomorphism $\mathrm{Aut}_{\mathcal{F}}(Q \leq R) \to \mathrm{Aut}_{\mathcal{F}}(Q)$ induced by restriction from R to Q is surjective.*

Proof. For any $y \in N_P(Q)$ denote by c_y the automorphism of Q given by conjugation with y. Let $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Since $\mathrm{Aut}_R(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$, for any $y \in R$ there is $z \in R$ such that $\varphi \circ c_y \circ \varphi^{-1} = c_z$. In particular, we have $R \subseteq N_\varphi$. Thus φ extends to a morphism $\psi : R \to P$. Then, for all $u \in Q$, we have $(\varphi \circ c_y \circ \varphi^{-1})(u) = (\psi \circ c_y \circ \psi^{-1})(u) = \psi({}^y(\psi^{-1}(u))) = {}^{\psi(y)}u$, or equivalently, $\varphi \circ c_y \circ \varphi^{-1} = c_{\psi(y)}$. Since $C_P(Q) \subseteq R$ we get $\psi(R) = R$ and hence $\psi \in \mathrm{Aut}_{\mathcal{F}}(Q \leq R)$. $\square$

Lemma 2.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P. Then Q is contained in any $\mathcal{F}$-centric radical subgroup R of P.*

Proof. Let R be a fully $\mathcal{F}$-normalised centric radical subgroup of P. The hypothesis $\mathcal{F} = N_{\mathcal{F}}(Q)$ implies that $\mathrm{Aut}_{QR}(R)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(R)$. Since R is radical, this forces $\mathrm{Aut}_{QR}(R) = \mathrm{Aut}_R(R)$, hence $N_{QR}(R) \subseteq RC_P(R)$. As R is also centric, we get $N_{QR}(R) = R$, and hence $QR = R$, or equivalently, $Q \subseteq R$. $\square$

Besides $C_{\mathcal{F}}(Q)$ and $N_{\mathcal{F}}(Q)$ we need another particular case of Puig's result in [2, Appendix, Prop. A6].

Proposition 2.6. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a fully $\mathcal{F}$-normalised subgroup of P. Then there is a fusion system $N_P(Q)C_{\mathcal{F}}(Q)$ on $N_P(Q)$ contained in $N_{\mathcal{F}}(Q)$ such that, for any two subgroups R, R' of $N_P(Q)$, the morphism set $\mathrm{Hom}_{N_P(Q)C_{\mathcal{F}}(Q)}(R, R')$ consists of all group homomorphisms $\varphi : R \to R'$ such that there exists a homomorphism $\psi : RQ \to R'Q$ in $N_{\mathcal{F}}(Q)$ and an element $y \in P$ satisfying $\psi|_R = \varphi$ and $\psi(u) = {}^yu$ for all $u \in Q$.*

Proof. In the notation of [2, Appendix, Prop. A 6], this is the case where $K = \mathrm{Aut}_P(Q)$ applied to the fusion system $N_{\mathcal{F}}(Q)$ on $N_P(Q)$. $\square$

We will frequently use Alperin's fusion theorem in the following form (see e.g. [2, Appendix, Theorem A 10] for a proof):

Theorem 2.7 (Alperin's fusion theorem for fusion systems)**.** *Let $\mathcal{F}$ be a fusion system on a finite p-group P. Any isomorphism in $\mathcal{F}$ can be written as a composition of isomorphisms $\varphi : Q \cong Q'$ for which there exists a fully $\mathcal{F}$-normalised centric* *radical subgroup R of P containing Q, Q' and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\alpha|_Q = \varphi$.*

Let $\mathcal{F}$ be a fusion system on a finite p-group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P. We define a category $\mathcal{F}/Q$ on P/Q as follows: for any two subgroups R, S of P containing Q, a group homomorphism $\psi : R/Q \to S/Q$ is a morphism in $\mathcal{F}/Q$ if there is a morphism $\varphi : R \to S$ in $\mathcal{F}$ satisfying $\psi(uQ) = \varphi(u)Q$ for all $u \in R$. The following result is due to Puig [12].

Proposition 2.8. *Let $\mathcal{F}$ be a fusion system on a finite p-group P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some normal subgroup Q of P. Then the category $\mathcal{F}/Q$ is a fusion system on P/Q.*

3. On central extensions of fusion systems

The following result is well known.

Proposition 3.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Z be a subgroup of $Z(P)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Set $\bar{P} = P/Z$ and $\bar{\mathcal{F}} = \mathcal{F}/Z$. For any subgroup Q of P containing Z the canonical map $Q \to \bar{Q}$ induces a surjective group homomorphism*

$$\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$$

whose kernel is an abelian p-group, where $\bar{Q}$ is the image of Q in $\bar{P}$. In particular, if Q is $\mathcal{F}$-radical, then $\bar{Q}$ is $\bar{\mathcal{F}}$-radical. Moreover, if $\bar{Q}$ is $\bar{\mathcal{F}}$-centric, then Q is $\mathcal{F}$-centric and if Q is $\mathcal{F}$-centric radical, then $\bar{Q}$ is $\mathcal{F}$-centric radical.

Proof. If φ is an automorphism of Q inducing the identity on Z and on $\bar{Q}$, then, for all $u \in Q$, we have $\varphi(u) = u\zeta(u)$ for some group homomorphism $\zeta : Q \to Z$, and hence the group of all such automorphisms is isomorphic to the abelian p-group $\mathrm{Hom}(Q, Z)$ with group structure induced by that of Z. Thus if Q is $\mathcal{F}$-radical, the kernel of the map $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$ is contained in $\mathrm{Aut}_Q(Q)$. Therefore, in that case, $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q) \cong \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})/\mathrm{Aut}_{\bar{Q}}(\bar{Q})$. If $\bar{Q}$ is $\bar{\mathcal{F}}$-centric, for every subgroup R of P isomorphic to Q in $\mathcal{F}$ we have $C_{\bar{P}}(\bar{R}) = Z(\bar{R}) \subseteq \bar{R}$, and therefore $C_P(R) \subseteq R$, which implies that $C_P(R) = Z(R)$ and hence that Q is $\mathcal{F}$-centric. For the last statement, we may assume that $\bar{Q}$ is fully $\bar{\mathcal{F}}$-centralised. The kernel K of the canonical map $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\bar{\mathcal{F}}}(\bar{Q})$ is a p-group. Thus, if Q is $\mathcal{F}$-radical, then $K \subseteq \mathrm{Aut}_Q(Q)$. Let C be the inverse image in P of $C_{\bar{P}}(\bar{Q})$. That is, the image in $\bar{P}$ of any element in C centralises $\bar{Q}$, and hence $\mathrm{Aut}_C(Q) \subseteq K$. This implies $C \subseteq QC_P(Q)$. Thus, if in addition Q is $\mathcal{F}$-centric, we get that $C \subseteq Q$, and hence $C_{\bar{P}}(\bar{Q}) \subseteq \bar{Q}$, which shows that $\bar{Q}$ is $\bar{\mathcal{F}}$-centric. □

Proposition 3.2. *Let $\mathcal{F}$, $\mathcal{F}'$ be fusion systems on a finite p-group P and let Z be a subgroup of $Z(P)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ and $\mathcal{F}' = C_{\mathcal{F}'}(Z)$. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$. Then $\mathcal{F} = \mathcal{F}'$ if and only if $\mathcal{F}/Z = \mathcal{F}'/Z$.*

Proof. Suppose that $\mathcal{F}/Z = \mathcal{F}'/Z$. Let Q be an $\mathcal{F}'$-centric radical subgroup of P. Then, by Proposition 3.1, the kernel K of the canonical map $\mathrm{Aut}_{\mathcal{F}'}(Q) \to \mathrm{Aut}_{\mathcal{F}'/Z}(Q/Z)$ is contained in $\mathrm{Aut}_Q(Q)$. Thus, K is also the kernel of the canonical map $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}/Z}(Q/Z)$. Since $\mathrm{Aut}_{\mathcal{F}/Z}(Q/Z) = \mathrm{Aut}_{\mathcal{F}'/Z}(Q/Z)$ it follows that $\mathrm{Aut}_{\mathcal{F}}(Q)$ and $\mathrm{Aut}_{\mathcal{F}'}(Q)$ have the same order. The assumption $\mathcal{F} \subseteq \mathcal{F}'$ implies $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}'}(Q)$. The equality $\mathcal{F} = \mathcal{F}'$ follows then from Theorem 2.7. The converse is trivial. □

Corollary 3.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Z be a subgroup of $Z(P)$ such that $\mathcal{F} = C_{\mathcal{F}}(Z)$ and let $\mathcal{G}$ be a fusion system on P such that $\mathcal{G} \subseteq \mathcal{F}$. Set $\bar{\mathcal{F}} = \mathcal{F}/Z$ and $\bar{\mathcal{G}} = \mathcal{G}/Z$. Let Q be a normal subgroup of P containing Z and set $\bar{Q} = Q/Z$. We have $\mathcal{G} = N_{\mathcal{F}}(Q)$ if and only if $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{Q})$.*

Proof. Suppose that $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{Q})$. Then $\bar{Q}$ is contained in any $\bar{\mathcal{G}}$-centric radical subgroup of $\bar{P}$. It follows from the last statement of Proposition 3.1 that Q is contained in any $\mathcal{G}$-centric radical subgroup of P, and hence $\mathcal{G} = N_{\mathcal{G}}(Q)$. Since $\mathcal{G} \subseteq \mathcal{F}$ this implies $\mathcal{G} \subseteq N_{\mathcal{F}}(Q)$. Clearly, $N_{\mathcal{F}}(Q)/Z = N_{\bar{\mathcal{F}}}(\bar{Q})$, and thus $\mathcal{G} = N_{\mathcal{F}}(Q)$ by Proposition 3.2. The converse is trivial. □

Proposition 3.4. *Let P be a finite p-group, let Q be a normal subgroup of P and let $\mathcal{F}$, $\mathcal{G}$ be fusion systems on P such that $\mathcal{F} = PC_{\mathcal{F}}(Q)$ and such that $\mathcal{G} \subseteq \mathcal{F}$. Let R be a normal subgroup of P containing Q. We have $\mathcal{G} = N_{\mathcal{F}}(R)$ if and only if $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$.*

Proof. Suppose $\mathcal{G}/Q = N_{\mathcal{F}/Q}(R/Q)$. In order to show the equality $\mathcal{G} = N_{\mathcal{F}}(R)$ we proceed by induction over the order of Q. If $Q = 1$ there is nothing to prove. Suppose $Q \neq 1$. Since Q is normal in P, the group $Z = Q \cap Z(P)$ is nontrivial. The assumption $\mathcal{F} = PC_{\mathcal{F}}(Q)$ implies that $\mathcal{F} = C_{\mathcal{F}}(Z)$. Set $\bar{\mathcal{F}} = \mathcal{F}/Z$ and $\bar{\mathcal{G}} = \mathcal{G}/Z$. Similarly, set $\bar{P} = P/Z$, $\bar{Q} = Q/Z$ and $\bar{R} = R/Z$. Then $\bar{\mathcal{F}}$, $\bar{\mathcal{G}}$ are fusion systems on $\bar{P}$ satisfying $\bar{\mathcal{F}} = \bar{P}C_{\bar{\mathcal{F}}}(\bar{Q})$ and $\bar{\mathcal{G}} \subseteq \bar{\mathcal{F}}$. We have isomorphisms of fusion systems $\bar{\mathcal{G}}/\bar{Q} \cong \mathcal{G}/Q$ and $N_{\bar{\mathcal{F}}/\bar{Q}}(\bar{R}/\bar{Q}) \cong N_{\mathcal{F}/Q}(R/Q)$ induced by the canonical isomorphism $\bar{P}/\bar{Q} \cong P/Q$. Thus $\bar{\mathcal{G}}/\bar{Q} = N_{\bar{\mathcal{F}}/\bar{Q}}(\bar{R}/\bar{Q})$. By induction we get that $\bar{\mathcal{G}} = N_{\bar{\mathcal{F}}}(\bar{R})$, where $\bar{R} = R/Z$. But then Corollary 3.3 implies $\mathcal{G} = N_{\mathcal{F}}(R)$ as required. The converse is trivial. □

4. Proof of Theorem A

Let p be an odd prime, let P be a finite p-group and let $\mathcal{F}$ be a fusion system on P. We have $\mathcal{F}_P(P) \subseteq N_{\mathcal{F}}(Z(J(P))) \subseteq \mathcal{F}$. Thus if $\mathcal{F} = \mathcal{F}_P(P)$, then trivially $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$. In order to prove the converse, let $\mathcal{F}$ be a minimal counterexample to Theorem A; that is, the number of morphisms $|\mathcal{F}|$ of $\mathcal{F}$ is minimal such that $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ but $\mathcal{F} \neq \mathcal{F}_P(P)$. We proceed in a series of steps as in [7].

4.1. *Any fusion system $\mathcal{G}$ on P which is properly contained in $\mathcal{F}$ is equal to $\mathcal{F}_P(P)$.*

Proof. Since $N_{\mathcal{G}}(Z(J(P))) \subseteq N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ we have $N_{\mathcal{G}}(Z(J(P))) = \mathcal{F}_P(P)$ and hence $\mathcal{G} = \mathcal{F}_P(P)$ by the minimality assumption on $\mathcal{F}$. □

4.2. *We have $O_p(\mathcal{F}) \neq 1$.*

Proof. Since $\mathcal{F} \neq \mathcal{F}_P(P)$, Alperin's fusion theorem implies that there is a fully $\mathcal{F}$-normalised subgroup Q of P such that $N_{\mathcal{F}}(Q) \neq \mathcal{F}_R(R)$, where $R = N_P(Q)$. Amongst all such subgroups choose Q such that $R = N_P(Q)$ has maximal possible order. We are going to show that $R = P$. Assume that $R \neq P$. We may choose Q such that $Z(J(R))$ is also fully $\mathcal{F}$-normalised; indeed, by Lemma 2.2 there is a morphism $\varphi : N_P(Z(J(R))) \to P$ such that $\varphi(Z(J(R)))$ is fully $\mathcal{F}$-normalised, and since $N_P(Q) = R \subseteq N_P(R) \subseteq N_P(Z(J(R)))$ it follows from Lemma 2.3 that $\varphi(Q)$ is still fully $\mathcal{F}$-normalised. Having replaced Q by $\varphi(Q)$, consider the fusion system $N_{\mathcal{F}}(Z(J(R)))$ on $N_P(Z(J(R)))$. Note that since R is a proper subgroup of P it is also a proper subgroup of $N_P(R)$, hence of $N_P(Z(J(R)))$. The choice of Q implies that $N_{\mathcal{F}}(Z(J(R))) = \mathcal{F}_{N_P(Z(J(R)))}(N_P(Z(J(R))))$. Then in particular $N_{N_{\mathcal{F}}(Q)}(Z(J(R))) = \mathcal{F}_R(R)$. The minimality assumption on $\mathcal{F}$ implies the contradiction $N_{\mathcal{F}}(Q) = \mathcal{F}_R(R)$. Thus $R = P$, or equivalently, $Q \trianglelefteq P$. Since $N_{\mathcal{F}}(Q) \neq \mathcal{F}_P(P)$ we get that $N_{\mathcal{F}}(Q) = \mathcal{F}$ by 4.1; thus $Q \subseteq O_p(\mathcal{F})$. □

Now set $Q = O_p(\mathcal{F})$. Note that Q is a proper subgroup of P as the equality $Q = P$ would imply the contradiction $\mathcal{F} = N_{\mathcal{F}}(P) = N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$, where the second equality uses the fact that $Z(J(P))$ is characteristic in P. In particular, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{N_{\mathcal{F}}(Z(J(P)))}(P) = \mathrm{Aut}_P(P)$.

4.3. *We have $PC_{\mathcal{F}}(Q) = \mathcal{F}_P(P)$.*

Proof. Assume that $PC_{\mathcal{F}}(Q) \neq \mathcal{F}_P(P)$. Then $\mathcal{F} = PC_{\mathcal{F}}(Q)$ by 4.1. It follows from Proposition 3.4 that $\mathcal{F}/Q \neq \mathcal{F}_P(P)/Q = \mathcal{F}_{P/Q}(P/Q)$. Since $Q \neq 1$, the minimality assumption on $\mathcal{F}$ implies that $N_{\mathcal{F}/Q}(Z(J(P/Q))) \neq \mathcal{F}_{P/Q}(P/Q)$. Let R be the inverse image of $Z(J(P/Q))$ in P. Then $R \trianglelefteq P$ and $N_{\mathcal{F}}(R) \neq \mathcal{F}_P(P)$, by Proposition 3.4. But then $N_{\mathcal{F}}(R) = \mathcal{F}$ by 4.1, contradicting the fact that R contains $Q = O_p(\mathcal{F})$ properly. □

4.4. *The subgroup Q of P is $\mathcal{F}$-centric.*

Proof. Set $R = QC_P(Q)$. In order to show that Q is $\mathcal{F}$-centric it suffices to show that $R = Q$, since $Q = O_p(\mathcal{F})$. Assume that $R \neq Q$. Then $R \trianglelefteq P$ but $N_{\mathcal{F}}(R) \neq \mathcal{F}$. Thus $N_{\mathcal{F}}(R) = \mathcal{F}_P(P)$ by 4.1. Since Q is normal in P it is in particular fully $\mathcal{F}$-normalised, and hence the restriction map $\mathrm{Aut}_{\mathcal{F}}(R) \to \mathrm{Aut}_{\mathcal{F}}(Q)$ is surjective, by the extension axiom (II-S). Since $N_{\mathcal{F}}(R) = \mathcal{F}_P(P)$ we have $\mathrm{Aut}_{\mathcal{F}}(R) = \mathrm{Aut}_P(R)$; hence $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_P(Q)$. Let S be a fully $\mathcal{F}$-normalised centric radical subgroup of P. Then $Q \subseteq S$ by Lemma 2.5. Let $\sigma : \mathrm{Aut}_{\mathcal{F}}(S) \to \mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_P(Q)$ be the restriction homomorphism. Then $\ker(\sigma)$ is a subgroup of $\mathrm{Aut}_{PC_{\mathcal{F}}(Q)}(S) = \mathrm{Aut}_P(S)$. Thus $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_P(S)$ as claimed. Alperin's fusion theorem yields the contradiction $\mathcal{F} = \mathcal{F}_P(P)$. Thus $R = Q$, or equivalently, Q is $\mathcal{F}$-centric. □

We conclude the proof of Theorem A as in [7]. Since $\mathcal{F} = N_{\mathcal{F}}(Q)$ for some $\mathcal{F}$-centric normal subgroup Q of P it follows from [1, 4.3] that there is a finite group L having P as Sylow-p-subgroup such that $Q \trianglelefteq L$, $C_L(Q) = Z(Q)$ and $\mathcal{F} = \mathcal{F}_P(L)$. Then $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(N_L(Z(J(P))))$. Since $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ it follows from Frobenius' theorem [4, 8.6] that $N_L(Z(J(P)))$ is p-nilpotent. But the nilpotency theorem [5, Ch. 8, Theorem 3.1], of Glauberman and Thompson implies then that L itself is p-nilpotent, or equivalently, $\mathcal{F}_P(L) = \mathcal{F}_P(P)$. This however yields the contradiction $\mathcal{F} = \mathcal{F}_P(P)$, and the proof of Theorem A is complete.

5. Local control of characteristic p-functors

Definition 5.1. Let $\mathcal{F}$ be a fusion system on a finite p-group P and let W be a positive characteristic p-functor. Let Q be a subgroup of P. Set $W_1(Q) = Q$ and $P_1(Q) = N_P(Q)$. For any positive integer i define inductively $W_{i+1}(Q) = W(P_i(Q))$ and $P_{i+1}(Q) = N_P(W_{i+1}(Q))$. We will say that Q is $(\mathcal{F}, W)$-*well-placed* if $W_i(Q)$ is fully $\mathcal{F}$-normalised for all positive integers i.

Note that for all positive integers i we have $W_i(Q) \subseteq P_i(Q)$, and if $P_i(Q)$ is a proper subgroup of P, then in fact $P_i(Q)$ is a proper sugroup of $P_{i+1}(Q)$. In particular, $P_i(Q) = P$ for i large enough. The following result generalises [4, 5.2], [8, 3.1].

Proposition 5.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let W be a positive characteristic p-functor and let Q be a subgroup of P. There is a morphism $\varphi : N_P(Q) \to P$ such that $\varphi(Q)$ is $(\mathcal{F}, W)$-well-placed.*

Proof. Set $W_i = W_i(Q)$ and $P_i = P_i(Q)$ for any positive integer i. Note that $P_i = N_P(W_i)$ for any positive integer i. Let $\varphi_1 : P_1 = N_P(Q) \to P$ be a morphism in $\mathcal{F}$ such that $\varphi_1(Q)$ is fully $\mathcal{F}$-normalised. Thus, after replacing Q by $\varphi_1(Q)$ we may assume that $W_1 = Q$ is fully normalised. Assume now that for some positive integer n the subgroups W_i are fully $\mathcal{F}$-normalised for $1 \le i \le n$. Let $\varphi_{n+1} : P_{n+1} = N_P(W_{n+1}) \to P$ be a morphism in $\mathcal{F}$ such that $\varphi_{n+1}(W_{n+1})$ is fully $\mathcal{F}$-normalised. Since $P_i = N_P(W_i) \subseteq P_{n+1}$ the subgroups $\varphi_{n+1}(W_i)$ are still all fully $\mathcal{F}$-normalised. Note that in particular $P_1 = N_P(Q) \subseteq P_{n+1}$. Thus we may in fact assume that W_i is fully $\mathcal{F}$-normalised for $1 \le i \le n+1$. The result follows by induction. $\square$

The next result generalises [4, 5.5], [8, 3.2] to arbitrary fusion systems, saying that if a positive characteristic p-functor controls fusion locally, it does so globally.

Proposition 5.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let W be a positive characteristic p-functor. Assume that for any nontrivial fully $\mathcal{F}$-normalised subgroup Q of P we have $N_\mathcal{F}(Q) = N_{N_\mathcal{F}(Q)}(W(N_P(Q)))$. Then $\mathcal{F} = N_\mathcal{F}(W(P))$.*

Proof. Suppose the conclusion does not hold. Then there is a fully $\mathcal{F}$-normalised nontrivial subgroup Q of P such that $\mathrm{Aut}_{N_\mathcal{F}(W(P))}(Q)$ is a proper subgroup of $\mathrm{Aut}_\mathcal{F}(Q)$. By the previous proposition we may assume that Q is $(\mathcal{F}, W)$-well-placed. For any positive integer i, set $W_i = W_i(Q)$ and $P_i = P_i(Q)$. Set $\mathcal{F}_i = N_\mathcal{F}(W_i)$ and $\mathcal{G}_i = N_{\mathcal{F}_i}(W_{i+1})$. Since W_i is fully $\mathcal{F}$-normalised, the category $\mathcal{F}_i$ is a fusion system on $P_i = N_P(W_i)$. Clearly $W_{i+1} \subseteq P_i$, and since W_{i+1} is fully $\mathcal{F}$-normalised, W_{i+1} is also fully $\mathcal{F}_i$-normalised, and so $\mathcal{G}_i$ is a fusion system as well. Note that $\mathcal{G}_i \subseteq \mathcal{F}_{i+1}$. Clearly $\mathcal{F}_1 \subseteq \mathcal{F}_i$. In fact, by the assumptions, we have $\mathcal{G}_i = \mathcal{F}_i$. Thus $\mathcal{F}_1 \subseteq \mathcal{F}_i \subseteq \mathcal{F}_{i+1}$. If i is large enough, then $P_i = P$ and hence $\mathcal{F}_i = N_\mathcal{F}(W(P))$. But then also $\mathcal{F}_1 \subseteq N_\mathcal{F}(W(P))$. In particular, $\mathrm{Aut}_{\mathcal{F}_1}(Q) = \mathrm{Aut}_\mathcal{F}(Q)$ is contained in $\mathrm{Aut}_{N_\mathcal{F}(W(P))}(Q)$, contradicting our choice of Q. $\square$

6. On $Qd(p)$-free fusion systems

In this section we prove that if a fusion system $\mathcal{F}$ on a finite p-group P is $Qd(p)$-free, then so are $N_\mathcal{F}(Q)$, $N_P(Q)C_\mathcal{F}(Q)$ and $N_\mathcal{F}(Q)/Q$ for any fully $\mathcal{F}$-normalised subgroup Q of P. In fact, the statements in this section remain true with $Qd(p)$ replaced by any finite group, but we state them as needed in the proof of Theorem B. Given a fusion system $\mathcal{F}$ on a finite p-group P and a fully $\mathcal{F}$-normalised centric subgroup Q of P we denote as before by $L_Q^\mathcal{F}$ the p'-reduced p-constrained group from [1, 4.3] for which there is a short exact sequence

$$1 \longrightarrow Z(Q) \longrightarrow L_Q^\mathcal{F} \longrightarrow \mathrm{Aut}_\mathcal{F}(Q) \longrightarrow 1$$

such that $N_P(Q)$ is a Sylow-p-subgroup of $L_Q^\mathcal{F}$ and such that $N_\mathcal{F}(Q) = \mathcal{F}_{N_P(Q)}(L_Q^\mathcal{F})$. This short exact sequence is represented by an element in $H^2(\mathrm{Aut}_\mathcal{F}(Q); Z(Q))$. Since $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_\mathcal{F}(Q)$ and since $Z(Q)$ is an abelian p-group, the restriction map $H^2(\mathrm{Aut}_\mathcal{F}(Q); Z(Q)) \to H^2(\mathrm{Aut}_P(Q); Z(Q))$ is injective. In other words, the group $L_Q^\mathcal{F}$ is, up to isomorphism, determined by the group

$\mathrm{Aut}_{\mathcal{F}}(Q)$ and the p-group extension

$$1 \longrightarrow Z(Q) \longrightarrow N_P(Q) \longrightarrow \mathrm{Aut}_P(Q) \longrightarrow 1.$$

Proposition 6.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P. If $Qd(p)$ is involved in the group $L_Q^{\mathcal{F}}$ for some fully $\mathcal{F}$-normalised centric subgroup Q of P, then $Qd(p)$ is involved in the group $L_R^{\mathcal{F}}$ for some fully $\mathcal{F}$-normalised centric radical subgroup R of P.*

Proof. Let Q be a fully $\mathcal{F}$-normalised centric subgroup of P. We proceed by induction over the order of Q. If Q is $\mathcal{F}$-radical, there is nothing to prove. Otherwise, let R be the unique subgroup of $N_P(Q)$ containing Q such that $\mathrm{Aut}_R(Q) = O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$. Then R is an $\mathcal{F}$-centric subgroup of P which properly contains Q. In particular, $\mathrm{Aut}_R(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$; hence $N_P(Q) \subseteq N_P(R)$. Let $\psi : N_P(R) \to P$ be a morphism in $\mathcal{F}$ such that $\psi(R)$ is fully $\mathcal{F}$-normalised. Then $\psi(Q)$ is still fully $\mathcal{F}$-normalised. Thus we may assume that both Q and R are fully $\mathcal{F}$-normalised. Let $L_{Q\leq R}$ be the inverse image of $\mathrm{Aut}_{\mathcal{F}}(Q \leq R)$ in $L_R^{\mathcal{F}}$. That is, we have a short exact sequence of groups

$$1 \longrightarrow Z(R) \longrightarrow L_{Q\leq R} \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q \leq R) \longrightarrow 1.$$

By Lemma 2.4 restriction from R to Q induces a surjective group homomorphism $\mathrm{Aut}_{\mathcal{F}}(Q \leq R) \to \mathrm{Aut}_{\mathcal{F}}(Q)$. Since Q is $\mathcal{F}$-centric, the kernel of this group homomorphism is $\mathrm{Aut}_{Z(Q)}(R) \cong Z(Q)/Z(R)$ (cf. [11, 1.12]). Thus we have an exact commutative diagram

$$\begin{tikzcd}
1 \arrow[dr] & & & 1 \arrow[d] & \\
 & Z(Q) \arrow[dr] & & Z(Q)/Z(R) \arrow[d] & \\
1 \arrow[r] & Z(R) \arrow[r] & L_{Q\leq R} \arrow[r] \arrow[dr] & \mathrm{Aut}_{\mathcal{F}}(Q \leq R) \arrow[r] \arrow[d] & 1 \\
 & & & \mathrm{Aut}_{\mathcal{F}}(Q) \arrow[d] \arrow[dr] & \\
 & & & 1 & 1
\end{tikzcd}$$

Since Q is fully $\mathcal{F}$-normalised, $\mathrm{Aut}_P(Q)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_P(Q)/Z(Q)$; hence its inverse image $\mathrm{Aut}_P(Q \leq R) \cong N_P(Q \leq R)/Z(R)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q \leq R)$. Note that since $N_P(Q) \subseteq N_P(R)$ we have $N_P(Q) = N_P(Q \leq R)$, and this is a Sylow-p-subgroup of $L_{Q\leq R}$. Since a group extension by an abelian p-group is determined by its restriction to Sylow-p-subgroups it follows that $L_Q^{\mathcal{F}} \cong L_{Q\leq R}$ is isomorphic to a subgroup of $L_R^{\mathcal{F}}$. Thus, if $Qd(p)$ is involved in $L_Q^{\mathcal{F}}$, it is involved in $L_R^{\mathcal{F}}$, too. The result follows by induction. $\square$

Proposition 6.2. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a fully $\mathcal{F}$-normalised subgroup of P and let $\mathcal{G}$ be a fusion system on $N_P(Q)$ such that $N_P(Q)C_{\mathcal{F}}(Q) \subseteq \mathcal{G} \subseteq N_{\mathcal{F}}(Q)$. Let R be a subgroup of $N_P(Q)$ containing Q. Then*

R is fully $\mathcal{G}$-centralised if and only if R is fully $\mathcal{F}$-centralised. In particular, R is $\mathcal{G}$-centric if and only if R is $\mathcal{F}$-centric.

Proof. If R is fully $\mathcal{F}$-centralised, then clearly R is fully $\mathcal{G}$-centralised. Suppose conversely that R is fully $\mathcal{G}$-centralised. Let $\varphi : R \to P$ be a morphism in $\mathcal{F}$ such that $\varphi(R)$ is fully $\mathcal{F}$-centralised. Denote by $\psi : \varphi(Q) \to Q$ the isomorphism which is inverse to $\varphi|_Q$. We have $C_P(\varphi(Q)) \subseteq N_\psi$, and we also have $\varphi(R) \subseteq N_\psi$. Indeed, for all $r \in R$ and all $u \in Q$ we have $\psi(\varphi(r)\psi^{-1}(u)\varphi(r)^{-1}) = \psi(\varphi(rur^{-1})) = rur^{-1}$. Since Q is fully $\mathcal{F}$-normalised, ψ extends to a morphism $\tau : \varphi(R)C_P(\varphi(Q)) \to P$. Note that $\tau \circ \varphi$ restricts to the identity on Q; in particular, $\tau \circ \varphi : R \to P$ is a morphism in $\mathcal{G}$. Thus $S = \tau(\varphi(R))$ is a subgroup of $N_P(Q)$ containing Q and isomorphic to R in $\mathcal{G}$. Since $C_P(\varphi(R)) \subseteq C_P(\varphi(Q))$ we get $\tau(C_P(\varphi(R))) \subseteq C_P(S)$; in particular $|C_P(\varphi(R))| \leq |C_P(S)|$. As R was chosen fully $\mathcal{G}$-centralised, it follows that $|C_P(S)| \leq |C_P(R)|$; hence $|C_P(\varphi(R))| \leq |C_P(R)|$. However, $\varphi(R)$ is fully $\mathcal{F}$-centralised, and thus so is R. $\square$

Proposition 6.3. *Let $\mathcal{F}$ be a fusion system on a finite p-group P, let Q be a fully $\mathcal{F}$-normalised subgroup of P and let $\mathcal{G}$ be a fusion system on $N_P(Q)$ such that $N_P(Q)C_\mathcal{F}(Q) \subseteq \mathcal{G} \subseteq N_\mathcal{F}(Q)$. If $\mathcal{F}$ is $Qd(p)$-free, so is $\mathcal{G}$. In particular, if $\mathcal{F}$ is $Qd(p)$-free, so are $N_\mathcal{F}(Q)$ and $N_P(Q)C_\mathcal{F}(Q)$.*

Proof. Suppose that $\mathcal{F}$ is $Qd(p)$-free. Let R be a fully $\mathcal{G}$-normalised radical centric subgroup of $N_P(Q)$. By Lemma 2.5 we have $Q \subseteq R$. By Proposition 6.1 it suffices to show that $L_R^\mathcal{G}$ is $Qd(p)$-free. By Proposition 6.2, R is $\mathcal{F}$-centric. By Lemma 2.3 there is a morphism $\varphi : N_P(R) \to P$ such that $\varphi(R)$ is fully $\mathcal{F}$-normalised. The plan is to show that $L_R^\mathcal{G}$ is isomorphic to a subgroup of $L_{\varphi(R)}^\mathcal{F}$. Conjugation by φ induces a group isomorphism $\mathrm{Aut}_\mathcal{F}(R) \cong \mathrm{Aut}_\mathcal{F}(\varphi(R))$ sending $\rho \in \mathrm{Aut}_\mathcal{F}(R)$ to $\varphi \circ \rho \circ \varphi^{-1}|_{\varphi(R)}$. Restricting this to the subgroup $\mathrm{Aut}_\mathcal{G}(R)$ of $\mathrm{Aut}_\mathcal{F}(R)$ yields an injective group homomorphism

$$\Phi : \mathrm{Aut}_\mathcal{G}(R) \to \mathrm{Aut}_\mathcal{F}(\varphi(R)) .$$

Consider the canonical group extension

$$1 \longrightarrow Z(\varphi(R)) \longrightarrow L_{\varphi(R)}^\mathcal{F} \xrightarrow{\lambda} \mathrm{Aut}_\mathcal{F}(\varphi(R)) \longrightarrow 1.$$

Let L be the pullback of λ and Φ; that is,

$$L = \{(y, \alpha) \in L_{\varphi(R)}^\mathcal{F} \times \mathrm{Aut}_\mathcal{G}(R) \mid \lambda(y) = \Phi(\alpha)\}.$$

The canonical projections yield a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Z(\varphi(R)) \arrow[r] \arrow[d, equal] & L \arrow[r, "\mu"] \arrow[d] & \mathrm{Aut}_\mathcal{G}(R) \arrow[r] \arrow[d, "\Phi"] & 1 \\
1 \arrow[r] & Z(\varphi(R)) \arrow[r] & L_{\varphi(R)}^\mathcal{F} \arrow[r, "\lambda"'] & \mathrm{Aut}_\mathcal{F}(\varphi(R)) \arrow[r] & 1
\end{tikzcd}$$

Since $L_{\varphi(R)}^\mathcal{F}$ is $Qd(p)$-free, so is L. It suffices thus to show that $L \cong L_R^\mathcal{G}$. Note that $\mathrm{Aut}_P(\varphi(R))$ is a Sylow-p-subgroup of $\mathrm{Aut}_\mathcal{F}(\varphi(R))$ because $\varphi(R)$ is fully $\mathcal{F}$-normalised, and similarly, $\mathrm{Aut}_P(Q \leq R) = \mathrm{Aut}_{N_P(Q)}(R)$ is a Sylow-p-subgroup of $\mathrm{Aut}_\mathcal{G}(R)$ because R is fully $\mathcal{G}$-normalised. Let S be the inverse image of $\mathrm{Aut}_P(Q \leq R)$ in L. Then S is a Sylow-p-subgroup of L. The group L is determined, up to isomorphism, by $\mathrm{Aut}_\mathcal{G}(R)$ and the short exact sequence of p-groups

$$1 \longrightarrow Z(\varphi(R)) \longrightarrow S \longrightarrow \mathrm{Aut}_P(Q \leq R) \longrightarrow 1$$

obtained from restricting the first row in the above diagram to Sylow-p-subgroups. In order to show that $L \cong L_R^{\mathcal{G}}$ it suffices to show that this short exact sequence is equivalent to

$$1 \longrightarrow Z(R) \longrightarrow N_P(Q \leq R) \longrightarrow \mathrm{Aut}_P(Q \leq R) \longrightarrow 1.$$

As S is the inverse image in L of $\mathrm{Aut}_P(Q \leq R)$, we have

$$S = \{(u, c_v) \in N_P(\varphi(R)) \times \mathrm{Aut}_P(Q \leq R) \mid c_u = \Phi(c_v)\},$$

where c_u, c_v are the automorphisms of $\varphi(R)$, R, induced by conjugation with u, v, respectively. The equality $c_u = \Phi(c_v)$ is equivalent to $c_u = \varphi \circ c_v \circ \varphi^{-1}|_{\varphi(R)} = c_{\varphi(v)}$. Thus $S = \{(\varphi(v), c_v) \mid v \in N_P(Q \leq R)\}$. Hence the map $\psi : N_P(Q \leq R) \to S$ sending $v \in N_P(Q \leq R)$ to $(\varphi(v), c_v)$ is an isomorphism making the diagram

$$\begin{tikzcd}
1 \arrow[r] & Z(R) \arrow[r] \arrow[d, "\varphi|_{Z(R)}"'] & N_P(Q \leq R) \arrow[r] \arrow[d, "\psi"] & \mathrm{Aut}_P(Q \leq R) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Z(\varphi(R)) \arrow[r] & S \arrow[r] & \mathrm{Aut}_P(Q \leq R) \arrow[r] & 1
\end{tikzcd}$$

commutative. The isomorphism $L \cong L_R^{\mathcal{G}}$ follows. $\square$

Proposition 6.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and let Q be a normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$. If $\mathcal{F}$ is $Qd(p)$-free, then $\mathcal{F}/Q$ is $Qd(p)$-free.*

Proof. Let R be a subgroup of P containing Q such that R/Q is $\mathcal{F}/Q$-centric fully normalised. We show first that then R is $\mathcal{F}$-centric fully normalised. Let $\varphi : R \to P$ be a morphism in $\mathcal{F}$ and let $y \in C_P(\varphi(R))$. Then $yQ \in C_{P/Q}(\varphi(R)/Q) = Z(\varphi(R)/Q)$; hence $y \in \varphi(R)$, which implies that R is $\mathcal{F}$-centric. Since $N_{P/Q}(R/Q) = N_P(R)/Q$ it follows also that R is fully $\mathcal{F}$-normalised. Consider the group extension

$$1 \longrightarrow Z(R) \longrightarrow L_R^{\mathcal{F}} \longrightarrow \mathrm{Aut}_{\mathcal{F}}(R) \longrightarrow 1.$$

The fusion system of $L = L_R^{\mathcal{F}}$ on $N_P(R)$ is equal to $N_{\mathcal{F}}(R)$, and hence the fusion system of $\bar{L} = L_R^{\mathcal{F}}/Q$ is equal to $N_{\mathcal{F}}(R)/Q = N_{\mathcal{F}/Q}(R/Q)$. Now R/Q is centric in $\mathcal{F}/Q$; hence R/Q is a p-centric subgroup of the finite group $\bar{L}$. Thus

$$C_{\bar{L}}(R/Q) = Z(R/Q) \times C$$

where $C = O_{p'}(C_{\bar{L}}(R/Q))$. Let $S = C_R(R/Q)$ be the inverse image of $Z(R/Q)$ in R, let K be the inverse image of C in L and let $C_L(R/Q)$ be the inverse image of $C_{\bar{L}}(R/Q)$ in L. Then $C_{\bar{L}}(R/Q)$ is the kernel of the composition of canonical maps

$$L \longrightarrow \mathrm{Aut}_{\mathcal{F}}(R) \longrightarrow \mathrm{Aut}_{\mathcal{F}/Q}(R/Q),$$

and we have $C_{\bar{L}}(R/Q) = SK$ and $S \cap K = Q$. It follows that the canonical maps induce an exact commutative diagram of the form

$$\begin{tikzcd}
1 \arrow[r] & Z(R) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\pi"] & \mathrm{Aut}_{\mathcal{F}}(R) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(R/Q) \arrow[r] & L/K \arrow[r] & \mathrm{Aut}_{\mathcal{F}/Q}(R/Q) \arrow[r] & 1
\end{tikzcd}$$

The second row, when restricted to Sylow-p-subgroups, yields the exact sequence

$$1 \longrightarrow Z(R/Q) \longrightarrow N_{P/Q}(R/Q) \longrightarrow \mathrm{Aut}_{P/Q}(R/Q) \longrightarrow 1,$$

and hence $L/K \cong L^{\mathcal{F}/Q}_{R/Q}$. This group is obviously $Qd(p)$-free as it is a quotient of the $Qd(p)$-free group $L = L^{\mathcal{F}}_R$. Thus the fusion system $\mathcal{F}/Q$ on P/Q is $Qd(p)$-free. □

Remark 6.5. For future reference we point out that the proofs of Propositions 6.1, 6.3 and 6.4 yield the following slightly more precise statements about the groups $L^{\mathcal{F}}_Q$, where $\mathcal{F}$ is a fusion system on a finite p-group P.

(1) If Q is a fully $\mathcal{F}$-normalised centric subgroup of P and $Q \subseteq R \subseteq N_P(R)$ such that $\mathrm{Aut}_R(Q) = O_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, then there is a morphism $\varphi : R \to P$ such that both $\varphi(Q)$, $\varphi(R)$ are fully $\mathcal{F}$-normalised, and for any such φ the group $L^{\mathcal{F}}_{\varphi(Q)}$ is isomorphic to a subgroup of $L^{\mathcal{F}}_{\varphi(R)}$.

(2) If Q is a fully $\mathcal{F}$-normalised subgroup of P and $\mathcal{G}$ a fusion system on $N_P(Q)$ such that $N_P(Q)C_{\mathcal{F}}(Q) \subseteq \mathcal{G} \subseteq N_{\mathcal{F}}(Q)$, then for any fully $\mathcal{G}$-normalised centric radical subgroup R of $N_P(Q)$ there is a morphism $\varphi : R \to P$ in $\mathcal{F}$ such that $\varphi(R)$ is fully $\mathcal{F}$-normalised centric, and for any such φ the group $L^{\mathcal{G}}_R$ is isomorphic to a subgroup of $L^{\mathcal{F}}_{\varphi(R)}$.

(3) If Q is a normal subgroup of P such that $\mathcal{F} = N_{\mathcal{F}}(Q)$, then for any fully $\mathcal{F}/Q$-normalised centric subgroup R/Q of P/Q, its inverse image R in P is fully $\mathcal{F}$-normalised centric, and the group $L^{\mathcal{F}}_R$ has a quotient isomorphic to $L^{\mathcal{F}/Q}_{R/Q}$.

7. Proof of Theorem B

Given a fusion system $\mathcal{F}$ on a finite p-group P we denote as before by $|\mathcal{F}|$ the number of morphisms in $\mathcal{F}$. We argue by induction over $|\mathcal{F}|$. Let $\mathcal{F}$ be a counterexample to Theorem B with $|\mathcal{F}|$ minimal. That is, $\mathcal{F}$ is $Qd(p)$-free, $N_{\mathcal{F}}(W(P)) \neq \mathcal{F}$, where W is a Glauberman functor, but $N_{\mathcal{F}}(W(P')) = \mathcal{F}'$ for any $Qd(p)$-free fusion system $\mathcal{F}'$ on some finite p-group P' such that $|\mathcal{F}'| < |\mathcal{F}|$. We show first that

7.1. $O_p(\mathcal{F}) \neq 1$.

Proof. If $O_p(\mathcal{F}) = 1$, then for any nontrivial fully $\mathcal{F}$-normalised subgroup Q of P we have $N_{\mathcal{F}}(Q) < \mathcal{F}$. Using Proposition 6.3 we get that $N_{\mathcal{F}}(Q)$ is $Qd(p)$-free. Hence $N_{\mathcal{F}}(Q) = N_{N_{\mathcal{F}}(Q)}(W(N_P(Q)))$ by the induction hypothesis. Then Proposition 5.3 implies the contradiction $\mathcal{F} = N_{\mathcal{F}}(W(P))$. This proves 7.1. □

We now set $Q = O_p(\mathcal{F})$ and $R = QC_P(Q)$. We observe next that

7.2. $Q < R$.

Proof. If $Q = R = QC_P(Q)$, then Q is $\mathcal{F}$-centric, and we have a short exact sequence of finite groups

$$1 \longrightarrow Z(Q) \longrightarrow L_Q \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q) \longrightarrow 1$$

where L_Q is a finite group having P as Sylow-p-subgroup such that $C_{L_Q}(O_p(L_Q)) \subseteq O_p(L_Q)$ and such that $\mathcal{F}_P(L_Q) = \mathcal{F}$. But L_Q is $Qd(p)$-free by the assumptions. Glauberman's Theorem implies that $\mathcal{F}_P(L_Q) = \mathcal{F}_P(N_{L_Q}(W(P)))$ and hence $\mathcal{F} = N_{\mathcal{F}}(W(P))$, contradicting our choice of $\mathcal{F}$. □

The next step is to prove that

7.3. $\mathcal{F} = PC_{\mathcal{F}}(Q)$.

Proof. Assume that $PC_{\mathcal{F}}(Q) < \mathcal{F}$. Note that then $PC_{\mathcal{F}}(Q) = N_{PC_{\mathcal{F}}(Q)}(W(P))$ by induction. We will show that this implies that $\mathcal{F} = N_{\mathcal{F}}(W(P))$, contradicting our choice of $\mathcal{F}$. To show this, we will prove by induction over $[P : S]$ that for any $\mathcal{F}$-centric radical subgroup S of P we have $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$. The equality $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(P)$ is clear because $W(P)$ is a characteristic subgroup of P. Let S be an $\mathcal{F}$-centric radical subgroup of P. Note that then S contains Q. By Alperin's fusion theorem, any automorphism of S can be written as a product of automorphisms of fully $\mathcal{F}$-normalised centric radical subgroups of P of order at least $|S|$, and hence we may assume that S is fully $\mathcal{F}$-normalised. Restriction from S to Q induces a group homomorphism

$$\rho : \mathrm{Aut}_{\mathcal{F}}(S) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q) \ .$$

Set $A = \rho^{-1}(\mathrm{Aut}_S(Q))$. Since $\mathrm{Aut}_S(Q)$ is normal in $\mathrm{Im}(\rho)$, it follows that A is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, and clearly $A \subseteq \mathrm{Aut}_{PC_{\mathcal{F}}(Q)}(S)$. Also, since $\mathrm{Aut}_P(S)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, the intersection $\mathrm{Aut}_P(S) \cap A$ is a Sylow-p-subgroup of A. Setting $T = N_P(S) \cap SC_P(Q) = N_{SR}(S)$ yields

$$\mathrm{Aut}_P(S) \cap A = \mathrm{Aut}_T(S) \ .$$

The Frattini argument implies that

$$\mathrm{Aut}_{\mathcal{F}}(S) = A \cdot N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_T(S)) \ .$$

By our initial induction on fusion systems, we get

$$A \subseteq \mathrm{Aut}_{PC_{\mathcal{F}}(Q)}(S) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S).$$

Thus, in order to prove 7.3 we have to prove that

$$N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_T(S)) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S).$$

We have to consider two cases, depending on whether S contains R or not.

Suppose first that S does not contain R. Then T has greater order than S. By induction, we get $\mathrm{Aut}_{\mathcal{F}}(T) = \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(T)$. Now every automorphism of S which normalises $\mathrm{Aut}_T(S)$ extends to T, by the extension axiom, and so $N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_T(S))$ is contained in the image of the restriction map $\mathrm{Aut}_{\mathcal{F}}(S < T) \to \mathrm{Aut}_{\mathcal{F}}(S)$. Thus indeed $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ in this case.

Consider second the case where $R \subseteq S$. Set $B = \rho^{-1}(\mathrm{Aut}_Q(Q))$. Then B is normal in $\mathrm{Aut}_{\mathcal{F}}(S)$ and $B \subseteq \mathrm{Aut}_{PC_{\mathcal{F}}(Q)}(S) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$. As before, since $\mathrm{Aut}_P(S)$ is a Sylow-p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$, the intersection $\mathrm{Aut}_P(S) \cap B$ is a Sylow-p-subgroup of B. We claim that $\mathrm{Aut}_P(S) \cap B = \mathrm{Aut}_R(S)$. Indeed, if we denote by c_y the automorphism of S given by conjugation with an element $y \in N_P(S)$, we have $\mathrm{Aut}_P(S) \cap B = \{c_y \mid y \in N_P(S),\ \text{there is } x \in Q \text{ such that } c_y|_Q = c_x|_Q\} = \{c_y \mid y \in R\} = \mathrm{Aut}_R(S)$. The Frattini argument implies that

$$\mathrm{Aut}_{\mathcal{F}}(S) = B \cdot N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_R(S)) \ .$$

Since we know already that B is contained in $\mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$, we need to show that $N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_R(S)) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$. To see this, we prove first that $N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_R(S)) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(R))}(S)$. Indeed, let $\varphi \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_R(S))$ and let $x \in R$. As before, denote by c_x the automorphism of S given by conjugation with x. Since φ normalises $\mathrm{Aut}_R(S)$ we have $\varphi \circ c_x \circ \varphi = c_z$ for some $z \in R$. But since $R \subseteq S$ we also have $\varphi \circ c_x \circ \varphi^{-1} = c_{\varphi(x)}$. Thus $z^{-1}\varphi(x) \in C_P(S) = Z(S) \subseteq R$, and hence $\varphi(R) = R$. This shows the inclusion $N_{\mathrm{Aut}_{\mathcal{F}}(S)}(\mathrm{Aut}_R(S)) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(R))}(S)$.

From here we observe that, by induction applied to the fusion system $N_{\mathcal{F}}(R)$, we get $\mathrm{Aut}_{N_{\mathcal{F}}(R)}(S) \subseteq \mathrm{Aut}_{N_{\mathcal{F}}(W(P))}(S)$ as required.

Thus the assumption $PC_{\mathcal{F}}(Q) < \mathcal{F}$ yields the contradiction $\mathcal{F} = N_{\mathcal{F}}(W(P))$, which concludes the proof of 7.3. $\square$

We use Proposition 3.4 to finish the proof of the theorem. Since $\mathcal{F} = PC_{\mathcal{F}}(Q)$ is $Qd(p)$-free, so is the fusion system $\mathcal{F}/Q$ on P/Q by Proposition 6.4. Thus, by induction, we have $\mathcal{F}/Q = N_{\mathcal{F}/Q}(W(P/Q))$. Denoting by V the inverse image of $W(P/Q)$ in P we get from Proposition 3.4 that $\mathcal{F} = N_{\mathcal{F}}(V)$. Hence $V \subseteq Q = O_p(\mathcal{F})$. But $W(P/Q) \neq 1$; hence V contains Q properly. This contradiction concludes the proof of Theorem B. $\square$

References

1. C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver, *Subgroup families controlling p-local finite groups*, Proc. London Math. Soc. (3) **91** (2005), 325-354. MR2167090
2. C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856. MR1992826 (2004k:55016)
3. G. Glauberman, *A characteristic subgroup of a p-stable group*, Canadian J. Math. **20** (1968), 1101–1135. MR0230807 (37:6365)
4. G. Glauberman, *Global and local properties of finite groups*, Finite simple groups (eds. Powell-Higman), Academic Press, London, 1971, pp. 1–64. MR0352241 (50:4728)
5. D. Gorenstein, *Finite Groups*, Chelsea Publishing Company, New York, 1980. MR569209 (81b:20002)
6. B. Huppert, N. Blackburn, *Finite Groups III*, Springer-Verlag, Berlin, Heidelberg, New York, 1982. MR662826 (84i:20001b)
7. R. Kessar, M. Linckelmann, *A block theoretic analogue of a theorem of Glauberman and Thompson*, Proc. Amer. Math. Soc. **131** (2003), 35–40. MR1929020 (2003g:20014)
8. R. Kessar, M. Linckelmann, G. R. Robinson, *Local control in fusion systems of p-blocks of finite groups*, J. Algebra **257** (2002), 393–413. MR1947328 (2003j:20011)
9. R. Kessar, R. Stancu, *A reduction theorem for fusion systems of blocks*, J. Algebra, doi:10-1016
10. B. Külshammer, L. Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), 17–71. MR1069239 (91i:20009)
11. M. Linckelmann, *Simple fusion systems and the Solomon* 2*-local groups*, J. Algebra **296** (2006), 385–401. MR2201048 (2006i:20024)
12. L. Puig, *Unpublished notes.*

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