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THE CENTER OF THE CATEGORY OF $(\mathfrak{g}, K)$–MODULES

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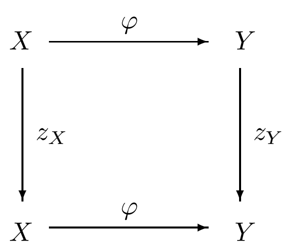
ABSTRACT. Let G be a semi-simple connected Lie group. Let K be a maximal compact subgroup of G and $\mathfrak{g}$ the complexified Lie algebra of G. In this paper we describe the center of the category of $(\mathfrak{g}, K)$-modules.

1. Introduction

Let $\mathcal{A}$ be an Abelian category. We write $Obj(\mathcal{A})$ for the collection of all objects of $\mathcal{A}$. If X, Y are the objects of $\mathcal{A}$, then we let $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ be the abelian group of all morphisms from X to Y. In particular, $\mathrm{End}_{\mathcal{A}}(X) := \mathrm{Hom}_{\mathcal{A}}(X, X)$ is a ring with identity id_X. Objects X and Y in $Obj(\mathcal{A})$ are equivalent (or isomorphic) if there are morphisms $\varphi \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $\psi \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ such that

$$\psi \circ \varphi = id_X \text{ and } \varphi \circ \psi = id_Y.$$

The center of category $\mathcal{A}$ is the ring $Z(\mathcal{A})$ defined as follows. As a set, $Z(\mathcal{A})$ is the set of all families $\{z_X\}_X$, where z_X is in $\mathrm{End}_{\mathcal{A}}(X)$, such that for any two objects X and Y, the following diagram



commutes for every $\varphi \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$. In particular, z_X is contained in $Z(\mathrm{End}_{\mathcal{A}}(X))$, the center of the algebra $\mathrm{End}_{\mathcal{A}}(X)$. The structure of the ring with identity on $Z(\mathcal{A})$ is defined naturally:

$$\begin{cases} \{z_X\}_X + \{z'_X\}_X = \{z_X + z'_X\}_X, \\ \{z_X\}_X \cdot \{z'_X\}_X = \{z_X \circ z'_X\}_X, \\ id_{Z(\mathcal{A})} = \{id_X\}_X \text{ is the identity.} \end{cases}$$

Let G be a semi-simple connected Lie group and fix K, a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexified Lie algebras of G and K. Let $\mathcal{M}(\mathfrak{g})$ be

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the category of all $U(\mathfrak{g})$ modules and $\mathcal{M}(\mathfrak{g}, K)$ be the category of $(\mathfrak{g}, K)$–modules [W1]. Obviously, $\mathcal{M}(\mathfrak{g}, K)$ is an Abelian category. It is a full subcategory of $\mathcal{M}(\mathfrak{g})$. Let $Z(\mathfrak{g}, K)$ be the center of the category $\mathcal{M}(\mathfrak{g}, K)$. Let $\hat{K}$ be the set of equivalence classes of irreducible representations of K. If $\gamma \in \hat{K}$, we fix a representation that represents this class and denote it by the same letter. Differentiating the action of K on γ we obtain the representation of $\mathfrak{k}$ that we denote by the same letter.

If X is an object of $\mathcal{M}(\mathfrak{g}, K)$, then we decompose X as a K–representation as follows:

$$X = \bigoplus_{\gamma \in \hat{K}} X(\gamma),$$

where $X(\gamma)$ is the γ–isotypic component. Let Z_G be the group center of G. Clearly, the category $\mathcal{M}(\mathfrak{g}, K)$ can be decomposed as

$$\mathcal{M}(\mathfrak{g}, K) \simeq \prod_{\chi \in \hat{Z}_G} \mathcal{M}(\mathfrak{g}, K)_\chi$$

where Z_G acts by the character χ on any object in $\mathcal{M}(\mathfrak{g}, K)_\chi$. Thus, we have the following isomorphism of the rings:

$$Z(\mathfrak{g}, K) \simeq \prod_{\chi \in \hat{Z}_G} Z(\mathfrak{g}, K)_\chi$$

where $Z(\mathfrak{g}, K)_\chi$ is the center of the category $\mathcal{M}(\mathfrak{g}, K)_\chi$.

Our description of the center is given in terms of its action on the principal series representations. In order to explain, we need to introduce some additional notation. The choice of the maximal compact subgroup K in G gives the Cartan decomposition of the real Lie algebra $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be the maximal commutative subalgebra in $\mathfrak{p}_0$. Then $\mathfrak{a}_0$ defines a relative root system $\Phi(\mathfrak{g}, \mathfrak{a})$, and we fix a set of simple roots $\Delta(\mathfrak{g}, \mathfrak{a})$ for it. We then define a minimal parabolic subgroup $P = MAN$ as follows. First, A is a subgroup of G with the Lie algebra $\mathfrak{a}_0$. The compact group M is the centralizer of A in K, and N is a unipotent subgroup normalized by A corresponding to the choice of simple roots, in the usual fashion. For every irreducible representation δ of M and ν in $\mathfrak{a}^*$ one can define (see [W1]) a normalized principal series representation

$$H^{(\delta,\nu)}.$$

The corresponding $(\mathfrak{g}, K)$-module will be denoted by $H_K^{(\delta,\nu)}$. The relative Weyl group is defined by

$$W(G : A) = N_K(A)/Z_K(A).$$

This group is equal to the Weyl group of the relative root system $\Phi(\mathfrak{g}, \mathfrak{a})$. Note that $W(G : A)$ acts, by conjugation, on the representations of M. Two principal series $H^{(\delta,\nu)}$ and $H^{(\delta',\nu')}$ are called *associated* if there exists an element w in $W(G : A)$ such that $w(\delta) = \delta'$ and $w(\nu) = \nu'$.

Let χ be a character of Z_G and $Cl(\chi)$ be the set of equivalence classes of all irreducible representations of M such that Z_G acts on δ by χ. The action of the component of the center $Z(\mathfrak{g}, K)$ on the principal series representations gives the following:

Theorem 1. *Let $U(\mathfrak{a})$ be the enveloping algebra of $\mathfrak{a}$. For any δ, an irreducible representation of M, let W_δ be the centralizer of δ in $W(G : A)$. Then*

$$Z(\mathfrak{g}, K)_\chi \subseteq \Big(\prod_{\delta \in Cl(\chi)} U(\mathfrak{a})^{W_\delta}\Big)^{W(G:A)}.$$

Up to this point our results are analogous to the description of the center for p-adic groups [Be]. The major difference is that, for real groups, two non-associated principal series representations can have a common subquotient. This does not happen for p-adic groups. For real groups, on the other hand, this already happens for $GL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. In particular, the above result is far from optimal.

At this point we restrict ourselves to the case of a quasi-split group. Then $M = T$ is a compact torus. Let $\mathfrak{t}$ be the complexified Lie algebra of T. Then

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$$

is a maximal Cartan subalgebra of $\mathfrak{g}$. The root system $\Phi(\mathfrak{g}, \mathfrak{h})$ is the absolute root system for the semi-simple Lie algebra $\mathfrak{g}$. Let W be the corresponding (absolute) Weyl group. Let $Z(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$. By the theorem of Harish-Chandra, we have the following isomorphism:

$$Z(\mathfrak{g}) \cong U(\mathfrak{h})^W.$$

Under this identification, the infinitesimal character of the principal series $H^{(\delta,\nu)}$ is $\epsilon+\nu$ where $\epsilon \in \mathfrak{t}^*$ is the differential of the character δ. Let $[\epsilon]$ denote the $W(G : A)$-conjugacy class of ϵ. Define $V_{[\epsilon]} \subseteq \mathfrak{h}^*$ to be the union of all W-conjugates of affine subspaces $\epsilon + \mathfrak{a}^*$. This set depends on the class $[\epsilon]$ of ϵ and represents the set of infinitesimal characters for principal series representations associated to $H^{(\delta,\nu)}$, for all ν in $\mathfrak{a}^*$.

Theorem 2. *Assume that G is quasi-split, linear and without factors of the type BC_n. Let V_χ be the union of all $V_{[\epsilon]}$ such that ϵ is the differential of a character δ with the central character χ. Then*

$$Z(\mathfrak{g}, K)_\chi = \mathbb{C}[V_\chi]^W$$

where $\mathbb{C}[V_\chi]^W$ denotes the algebra of W-invariant regular functions on V_χ. In particular, if G is split, then $V_\chi = \mathfrak{h}^*$ and

$$Z(\mathfrak{g}, K)_\chi = Z(\mathfrak{g}).$$

We finish the introduction with a couple of remarks. We say that a complex valued function on V_χ is regular if it is regular on every $V_{[\epsilon]} \subseteq V_\chi$. The set V_χ is Zariski-dense in $\mathfrak{h}^*$, since it contains the infinitesimal characters of finite-dimensional representations. Thus $Z(\mathfrak{g}, K)_\chi$ contains the center of the enveloping algebra $Z(\mathfrak{g})$. However, if G is not split, then V_χ is a union of infinitely many $V_{[\epsilon]}$ and the set of W-invariant regular functions on V_χ is much larger than the set of W-invariant polynomials on $\mathfrak{h}^*$.

The main tool in proving Theorem 2 is induction in stages and exploiting the fact that non-associated principal series of $GL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ can have a common subquotient.

2. Projective generators

We start the section by reminding the reader that we assume that G is connected. Hence $\mathrm{Hom}_{\mathfrak{g}}(X,Y) \simeq \mathrm{Hom}_{(\mathfrak{g},K)}(X,Y)$, for objects X, Y of the category $\mathcal{M}(\mathfrak{g},K)$. Also, if A is a ring, then we denote by $Z(A)$ its center.

Now, we recall some results from [L] and [LC]. Let I_γ be the annihilator of $\gamma \in \hat{K}$ in $U(\mathfrak{k})$. Let

$$U_\gamma := U(\mathfrak{g})/U(\mathfrak{g})I_\gamma \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathrm{End}(\gamma).$$

The indicated isomorphism is an isomorphism of $\mathfrak{g}$–modules. Note that U_γ is an object of the category $\mathcal{M}(\mathfrak{g},K)$ for the left action of $\mathfrak{g}$ and $\mathfrak{k}$. We have the following proposition:

Proposition 3. *Let X be a $(\mathfrak{g},K)$–module. Let φ be in $\mathrm{Hom}_{\mathfrak{g}}(U_\gamma,\ X)$. Then $\varphi \mapsto \varphi(1+U(\mathfrak{g})I_\gamma)$ defines an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(U_\gamma,\ X) \simeq X(\gamma).$$

In particular, U_γ is a projective object in the category of $(\mathfrak{g},K)$-modules.

Proof. Since I_γ annihilates $1+U(\mathfrak{g})I_\gamma$, we must have $\varphi(1+U(\mathfrak{g})I_\gamma)$ in $X(\gamma)$. Moreover, since $1+U(\mathfrak{g})I_\gamma$ generates U_γ, the injectivity follows. Next, we need to show the surjectivity of that map. Let $x \in X(\gamma)$. Then x is annihilated by I_γ. Moreover, we have the unique $\mathfrak{g}$–module morphism $U(\mathfrak{g}) \to X$ sending 1 to x. The left ideal $U(\mathfrak{g})I_\gamma$ is mapped to zero since x is annihilated by I_γ. This proves the surjectivity. It follows that the functor $X \mapsto \mathrm{Hom}_{\mathfrak{g}}(U_\gamma, X)$ is exact, and this shows that U_γ is projective. □

Proposition 4. *The center $Z(\mathfrak{g},K)$ is isomorphic to a subalgebra of*

$$\prod_{\gamma\in\hat{K}} Z(\mathrm{End}_{\mathfrak{g}}(U_\gamma))$$

consisting of elements $(z_\gamma)_{\gamma\in\hat{K}}$ such that for every φ in $\mathrm{Hom}_{\mathfrak{g}}(U_\gamma, U_{\gamma'})$,

$$z_{\gamma'} \circ \varphi = \varphi \circ z_\gamma.$$

Proof. Let X be a $(\mathfrak{g},K)$-module and γ any K-type. Define a $(\mathfrak{g},K)$-module P_X by

$$P_X = \bigoplus_{\gamma\in\hat{K}} U_\gamma \otimes \mathrm{Hom}_{\mathfrak{g}}(U_\gamma,\ X),$$

where $\mathfrak{g}$ acts on the first factor. The previous proposition implies that P_X is projective and that the canonical homomorphism

$$p_X : P_X \to X$$

is surjective. Next, notice that we have the following inclusion:

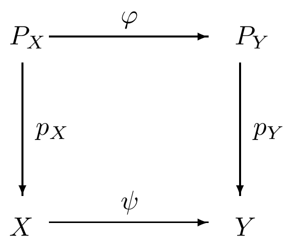
$$\mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{g}}(U_\gamma,\ X)) \subseteq \mathrm{End}_{\mathfrak{g}}(U_\gamma \otimes \mathrm{Hom}_{\mathfrak{g}}(U_\gamma,\ X)).$$

It follows that

$$Z(\mathrm{End}_{\mathfrak{g}}(P_X)) = \prod_{\gamma, X(\gamma)\neq 0} Z(\mathrm{End}_{\mathfrak{g}}(U_\gamma)).$$

This shows that z_{P_X} is completely determined by all z_{U_γ}. The same holds for z_X since $p_X \circ z_{P_X} = z_X \circ p_X$.

Thus, a family (z_γ), with z_γ in $Z(\mathrm{End}_{\mathfrak{g}}(U_\gamma))$, defines uniquely an element z_X for every X. We need to understand when the family of z_X's is an element of the center $Z(\mathfrak{g}, K)$. Clearly, the condition given in the statement of the proposition is necessary. We need to show that it is sufficient. To that end, let (X, Y) be a pair of $(\mathfrak{g}, K)$-modules, and ψ in $\mathrm{Hom}_{\mathfrak{g}}(X, Y)$. Since P_X is projective there exists $\varphi : P_X \to P_Y$ such that the following diagram commutes:



Since $\varphi \circ z_{P_X} = z_{P_Y} \circ \varphi$, it follows that $\psi \circ z_X = z_Y \circ \psi$. The proposition is proved. $\square$

3. The center $Z(\mathfrak{g}, K)$ in terms of Hecke algebras

Our next task is to determine the center of the algebra $\mathrm{End}_{\mathfrak{g}}(U_\gamma)$. Let $U(\mathfrak{g})^K$ be the centralizator in $\mathfrak{k}$ in $U(\mathfrak{g})$. Let

$$\mathcal{H}_\gamma = U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap I_\gamma.$$

Notice that $\mathcal{H}_\gamma$ acts on U_γ from the right. This action, clearly, commutes with the left action of $\mathfrak{g}$ on U_γ; thus

$$\mathcal{H}_\gamma \hookrightarrow \mathrm{End}_{\mathfrak{g}}(U_\gamma).$$

Proposition 5. *The canonical inclusion, by right multiplication, of $\mathcal{H}_\gamma$ into the endomorphism algebra of U_γ gives a bijection*

$$Z(\mathcal{H}_\gamma) \cong Z(\mathrm{End}_{\mathfrak{g}}(U_\gamma)).$$

Proof. We start with the following abstract lemma.

Lemma 6. *Let A be a ring with identity 1. Let $J \subset A$ be a left ideal of A. We regard A/J as a left A–module. Let A^J be the largest subring of A where J is a two–sided ideal, i.e. $A^J = \{a \in A;\ Ja \subset J\}$. Then the ring of endomorphisms $\mathrm{End}_A(A/J)$ of the A–module A/J is isomorphic to A^J/J under the canonical right action of A^J on A/J:*

$$x + J \mapsto a.(x + J) = xa + J \text{ where } a \text{ is in } A^J/J.$$

In particular, the center of $\mathrm{End}_A(A/J)$ is isomorphic to the center $Z(A^J/J)$ of the ring A^J/J.

Proof. It is clear that A^J is a subring of A containing 1. Also, by the definition, $J \subset A^J$ is a two-sided ideal and A^J is the largest subring with this property. Next, if $\varphi \in \mathrm{End}_A(A/J)$, then we compute:

$$\varphi(x + J) = \varphi(x.(1 + J)) = x.\varphi(1 + J) = x.(x_\varphi + J) = xx_\varphi + J,$$

where we let $\varphi(1+J) = x_\varphi + J$. Next, if $x \in J$, then we must have $J = \varphi(x+J) = xx_\varphi + J$. Hence $x_\varphi \in A^J$. Conversely, if $a \in A^J$, then $x + J \mapsto xa + J$ is a well–defined element of $\mathrm{End}_A(A/J)$. The remaining claims of the lemma are clear. □

Now, we apply this lemma to the $U(\mathfrak{g})$–module $U_\gamma = U(\mathfrak{g})/U(\mathfrak{g})I_\gamma$. In this case, we have

$$U(\mathfrak{g})^{U(\mathfrak{g})I_\gamma} = \{x \in U(\mathfrak{g});\ I_\gamma x \subset U(\mathfrak{g})I_\gamma\}.$$

By [LC, Theorem 5.4],

$$U(\mathfrak{g})^{U(\mathfrak{g})I_\gamma} = U(\mathfrak{g})^K U(\mathfrak{k}) + U(\mathfrak{g})I_\gamma,$$

and the multiplication induces an isomorphism of algebras

$$U(\mathfrak{g})^{U(\mathfrak{g})I_\gamma}/U(\mathfrak{g})I_\gamma \simeq U(\mathfrak{k})/I_\gamma \otimes \mathcal{H}_\gamma.$$

Since $U(\mathfrak{k})/I_\gamma$ is isomorphic to the matrix algebra $\mathrm{End}_{\mathbb{C}}(\gamma)$, it follows that

$$Z(U_\gamma^{U(\mathfrak{g})I_\gamma}) \cong Z(\mathcal{H}_\gamma).$$

The proposition follows from the lemma. □

Summarizing, we have identified every element z in $Z(\mathfrak{g}, K)$ with $(z_\gamma)_{\gamma\in\hat{K}}$ where z_γ is in the center of the Hecke algebra $\mathcal{H}_\gamma$. Our next task is to determine which $(z_\gamma)_{\gamma\in\hat{K}}$ arise from elements of $Z(\mathfrak{g}, K)$. To that end, let

$$U(\mathfrak{g})^{\gamma',\gamma} = \{x \in U(\mathfrak{g});\ I_{\gamma'} x \subset U(\mathfrak{g})I_\gamma\},\quad \gamma' \in \hat{K}.$$

(We remark that $U(\mathfrak{g})^{\gamma',\gamma}$ is $A^{\gamma',\gamma}$ in the notation of [LC].) Notice that $U(\mathfrak{g})^{\gamma,\gamma'}$ is defined so that

$$U(\mathfrak{g})^{\gamma,\gamma'} \cdot X(\gamma) \subseteq X(\gamma')$$

for every $(\mathfrak{g}, K)$-module X. The following is the main result of this section:

Theorem 7. *The image of $Z(\mathfrak{g}, K)$ in $\prod_{\gamma\in\hat{K}} Z(\mathcal{H}_\gamma)$ consists of all elements $(z_\gamma)_{\gamma\in\hat{K}}$ such that*

$$yz_\gamma - z_{\gamma'}y \in U(\mathfrak{g})I_\gamma,$$

for every $y \in U(\mathfrak{g})^{\gamma',\gamma}$.

Proof. Recall that the family (z_γ) defines an element of the center if and only if $\varphi \circ z_\gamma = z_{\gamma'} \circ \varphi$ for every element φ in $\mathrm{Hom}_{\mathfrak{g}}(U_\gamma, U_{\gamma'})$.

Lemma 8. *Let φ be in $\mathrm{Hom}_{\mathfrak{g}}(U_{\gamma'},\ U_\gamma)$. Then there exists an element x_φ in $U(\mathfrak{g})^{\gamma',\gamma}$ such that*

$$\varphi(x + U(\mathfrak{g})I_{\gamma'}) = x \cdot x_\varphi + U(\mathfrak{g})I_\gamma,\quad x \in U(\mathfrak{g}).$$

Moreover, x_φ is uniquely determined by that condition modulo $\mathcal{U}(\mathfrak{g})I_\gamma$. Conversely, if $x_\varphi \in U(\mathfrak{g})^{\gamma',\gamma}$, then the formula defines an element of $\mathrm{Hom}_{\mathfrak{g}}(U_{\gamma'},\ U_\gamma)$.

Proof. As in the proof of Proposition 3 any φ in $\mathrm{Hom}_{\mathfrak{g}}(U_{\gamma'},\ U_\gamma)$ is of the form

$$\varphi(x + U(\mathfrak{g})I_{\gamma'}) = x \cdot x_\varphi + U(\mathfrak{g})I_\gamma,\quad x \in U(\mathfrak{g}),$$

where $x_\varphi \in U(\mathfrak{g})$ is uniquely determined by that condition modulo $U(\mathfrak{g})I_\gamma$. Since for $x \in U(\mathfrak{g})I_{\gamma'}$ we must have

$$U(\mathfrak{g})I_\gamma = \varphi(x + U(\mathfrak{g})I_{\gamma'}) = x \cdot x_\varphi + \mathcal{U}(\mathfrak{g})I_\gamma,$$

we see $U(\mathfrak{g})I_{\gamma'}x_\varphi \subset U(\mathfrak{g})I_\gamma$. Hence $I_{\gamma'}x_\varphi \subset U(\mathfrak{g})I_\gamma$. This means $x_\varphi \in U(\mathfrak{g})^{\gamma',\gamma}$. The converse is clear. □

The theorem follows from the lemma. □

4. The action of the Hecke algebra $\mathcal{H}_\gamma$ on principal series representations

In this section we recall the computation of the action of $U(\mathfrak{g})^K$ on the representations in the principal series [W1, 3.5]. (See also [L].) This will be done in several steps. First, the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ implies the following decomposition:

$$U(\mathfrak{g}) = U(\mathfrak{a}) \otimes U(\mathfrak{k}) \oplus \mathfrak{n}U(\mathfrak{g}).$$

We consider the tensor product algebra structure on $U(\mathfrak{a}) \otimes U(\mathfrak{k})$. The projection of $U(\mathfrak{k})^K$ on $U(\mathfrak{g})/\mathfrak{n}U(\mathfrak{g}) \cong U(\mathfrak{a}) \otimes U(\mathfrak{k})^M$ gives an anti-holomorphic embedding

$$p : U(\mathfrak{g})^K \hookrightarrow U(\mathfrak{a}) \otimes U(\mathfrak{k})^M$$

(see [L, Proposition 3.3]). Let $\gamma \in \hat{K}$ and I_γ be the annihilator of γ in $U(\mathfrak{k})$, as before. Note that I_γ is M-invariant. Let p_γ be the map obtained by composing p and the natural projection

$$U(\mathfrak{a}) \otimes U(\mathfrak{k})^M \to U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_\gamma)^M.$$

Then the kernel of p_γ is $U(\mathfrak{g})^K \cap U(\mathfrak{g})I_\gamma$ (as follows easily from [W1, Lemma 3.5.9]). It follows that p_γ is an anti-holomorphic embedding

$$p_\gamma : \mathcal{H}_\gamma \hookrightarrow U(\mathfrak{a}) \otimes (U(\mathfrak{k})/I_\gamma)^M.$$

Let $\gamma(M)$ be the set of irreducible representations of M appearing in the restriction of γ to M. Then

$$\gamma|_M \simeq \bigoplus_{\delta \in \gamma(M)} m_\delta \cdot \delta$$

for some positive integers m_δ. Since $U(\mathfrak{k})/I_\gamma = \mathrm{End}(\gamma)$, it follows that

$$(U(\mathfrak{k})/I_\gamma)^M \cong \prod_{\delta \in \gamma(M)} \mathrm{Mat}(m_\delta, \mathbb{C}).$$

Summarizing, p_γ gives an embedding

$$p_\gamma : \mathcal{H}_\gamma \hookrightarrow \prod_{\delta \in \gamma(M)} U(\mathfrak{a}) \otimes \mathrm{Mat}(m_\delta, \mathbb{C}).$$

We let

$$\tilde{p_\gamma} = (\mu_\rho \otimes id) \circ p_\gamma : \mathcal{H}_\gamma \hookrightarrow \prod_{\delta \in \gamma(M)} U(\mathfrak{a}) \otimes \mathrm{Mat}(m_\delta, \mathbb{C}),$$

where μ_ρ is the automorphism of $U(\mathfrak{a})$ generated by the inclusion $\mathfrak{a} \hookrightarrow U(\mathfrak{a}), H \longmapsto H + \rho(H)$.

We regard $U(\mathfrak{a})$ as a polynomial algebra on $\mathfrak{a}^*$. Thus, every $\nu \in \mathfrak{a}^*$ defines an evaluation homomorphism $U(\mathfrak{a}) \to \mathbb{C}$ denoted by the same letter. Let β_δ be the projection of $(U(\mathfrak{k})/I_\gamma)^M$ on the factor $\mathrm{Mat}(m_\delta, \mathbb{C})$. Clearly, every irreducible module of the algebra $\prod_{\delta \in \gamma(M)} U(\mathfrak{a}) \otimes \mathrm{Mat}(m_\delta, \mathbb{C})$ is isomorphic to

$$R_{\nu,\delta} = \nu \otimes \beta_\delta,$$

for some $\nu \in \mathfrak{a}^*$ and $\delta \in \gamma(M)$.

Finally, let $\delta \in \gamma(M)$. We consider the $(\mathfrak{g}, K)$–module $H_K^{(\delta,\nu)}$ $(\nu \in \mathfrak{a}^*)$. The canonical action of $U(\mathfrak{g})^K$ on $\mathrm{Hom}_K(\gamma,\ H_K^{(\delta,\nu)}(\gamma))$ factors down to the action of $\mathcal{H}_\gamma$. This representation of $\mathcal{H}_\gamma$ is isomorphic to

$$r_{\nu,\delta,\gamma} = R_{\nu,\delta} \circ \tilde{p_\gamma},$$

composed with the transpose map on $\mathrm{Mat}(m_\delta, \mathbb{C})$, but we ignore this since we are interested only in the action of the center $Z(\mathcal{H}_\gamma)$; see [W1, 3.5.8]. Here, of course, ρ is half the sum of the roots in $\mathfrak{n}$ taken with multiplicities. Now, we have the following result (see [W1, Proposition 3.5.4] and ([LC, Theorem 4.9]):

Proposition 9. *Fix $\gamma \in \hat{K}$. For every $\delta \in \gamma(M)$ the representation $r_{\nu,\delta}$ is an irreducible $\mathcal{H}_\gamma$-module for ν in a Zariski dense subset of $\mathfrak{a}^*$ and*

$$r_{\nu,\delta,\gamma} \cong r_{w(\nu),w(\delta),\gamma}$$

for every w in the restricted Weyl group $W(G : A)$.

5. The action of $Z(\mathfrak{g}, K)$ on the representations in the principal series

Let $\gamma \in \hat{K}$. In this section we describe the center $Z(\mathcal{H}_\gamma)$ and its action on representations in the principal series. Let $\nu \in \mathfrak{a}^*$ and let $\delta \in \gamma(M)$. If $r_{\delta,\nu,\gamma}$ is irreducible, then by the Schur lemma the center acts by a character denoted by

$$\chi_{\nu,\delta,\gamma} \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(Z(\mathcal{H}_\gamma),\ \mathbb{C}).$$

Recall that the representation $r_{\nu,\delta,\gamma}$ is obtained by composing

$$\mathcal{H}_\gamma \overset{\tilde{p_\gamma}}{\to} \prod_{\delta\in\gamma(M)} U(\mathfrak{a}) \otimes \mathrm{Mat}(m_\delta, \mathbb{C}) \to \mathbb{C}_\nu \otimes \mathrm{Mat}(m_\delta, \mathbb{C})$$

where $\mathbb{C}_\nu \cong \mathbb{C}$ is the one-dimensional representation of $U(\mathfrak{a})$ obtained by evaluating elements of $U(\mathfrak{a})$ at ν. Since $r_{\nu,\delta}$ is irreducible for ν in a Zariski-dense set, the Schur lemma implies the following:

Lemma 10. *For ν in a Zariski dense subset of $\mathfrak{a}^*$, the image of the center Z_γ under $r_{\nu,\delta,\gamma}$ consists of scalar matrices in* $\mathrm{Mat}(m_\delta, \mathbb{C})$.

Now, we can give the first description of the center $\mathcal{H}_\gamma$. We write 1_δ for the identity in $\mathrm{Mat}(m_\delta, \mathbb{C})$. Now,

Proposition 11. *Under the inclusion of $\mathcal{H}_\gamma$ into $\prod_{\delta\in\gamma(M)} U(\mathfrak{a}) \otimes \mathrm{Mat}(m_\delta, \mathbb{C})$ the center $Z(\mathcal{H}_\gamma)$ is contained in $\prod_{\delta\in\gamma(M)} U(\mathfrak{a}) \otimes 1_\delta$.*

Proof. Let $z \in Z_\gamma$. We decompose it according to the direct sum $z = \sum_\delta z_\delta$. Let us write $z_\delta = \sum_i a_{\delta,i} \otimes X_i$, where $\{a_{\delta,i}\}$ is a linearly independent set in $U(\mathfrak{a})$ and $\{X_i\}$ is a linearly independent set in $\mathrm{Mat}(m_\delta, \mathbb{C})$. Since the set $\{a_{\delta,i}\}$ is finite, by the above lemma, we can find $\nu \in \mathfrak{a}^*$, such that $\nu(a_i) \neq 0$, for all i, and the representation $r_{\nu,\delta,\gamma}$ is irreducible. Hence we see that $\sum_i \nu(a_i)X_i \in \mathbb{C}1_\delta$. The proposition follows. □

Next, the group $W(G : A)$ acts on $U(\mathfrak{a})$ as a group of automorphisms. We define the action on the ring $\prod_{\delta\in\gamma(M)} U(\mathfrak{a}) \otimes 1_\delta$ as follows:

$$w(a_\delta \otimes 1_\delta) = w(a_\delta) \otimes 1_{w(\delta)}.$$

Clearly, this induces the action of $W(G:A)$ by algebra automorphisms. Also, this action restricts to an action on the subalgebra

$$\prod_{\delta\in\gamma(M)} U(\mathfrak{a})^{W_\delta}\otimes 1_\delta .$$

Now, we prove the first main result of this section:

Theorem 12. *Under the inclusion of $\mathcal{H}_\gamma$ into $\prod_{\delta\in\gamma(M)} U(\mathfrak{a})\otimes \mathrm{Mat}(m_\delta,\mathbb{C})$ the center $Z(\mathcal{H}_\gamma)$ is contained in*

$$\Big(\prod_{\delta\in\gamma(M)} U(\mathfrak{a})^{W_\delta}\otimes 1_\delta\Big)^{W(G:A)} = \Big(\prod_{\delta\in\gamma(M)} U(\mathfrak{a})\otimes 1_\delta\Big)^{W(G:A)}.$$

Proof. Let $z\in Z_\gamma$. Let us decompose $z=\sum_\delta a_\delta\otimes 1_\delta$. Then $w(z)=\sum_\delta w(a_\delta)\otimes 1_{w(\delta)}$. We have to show $w(a_\delta)=a_{w(\delta)}$. Since this might be regarded as an equality of polynomials on $\mathfrak{a}^*$, it is enough to show $a_\delta(\nu)=a_{w(\delta)}(w(\nu))$, for ν in a Zariski dense subset of $\mathfrak{a}^*$. Since $r_{\nu,\delta,\gamma}$ is isomorphic to $r_{w(\nu),w(\delta),\gamma}$ on a Zariski-dense set, we have the equality of the central characters

$$a_{w(\delta)}(w(\nu)) = \chi_{w(\nu),w(\delta),\gamma}(z) = \chi_{\nu,\delta,\gamma}(z) = a_\delta(\nu)$$

on the Zariski-dense set in $\mathfrak{a}^*$. The theorem is proved. □

From now on, we shall drop 1_δ, so any element z_γ of the center of the Hecke algebra $\mathcal{H}_\gamma$ is identified with

$$(a_\delta)_{\delta\in\gamma(M)} \in \Big(\prod_{\delta\in\gamma(M)} U(\mathfrak{a})^{W_\delta}\Big)^{W(G:A)}.$$

As a direct consequence of the previous theorem, we have the following corollary:

Corollary 13. *Let $z=(z_\gamma)_{\gamma\in\hat{K}}$ in $\prod_{\gamma\in\hat{K}} Z(\mathcal{H}_\gamma)$ be an element of the center $Z(\mathfrak{g},K)$. Let α and α' be any two K-types. Under the correspondence of the previous theorem, the components z_α and $z_{\alpha'}$ of z can be written as $z_\alpha=(a_\delta)_{\delta\in\alpha(M)}$ and $z_{\alpha'}=(a'_\delta)_{\delta\in\alpha'(M)}$. If α and α' share $\delta\in\hat{M}$, then*

$$a'_\delta = a_\delta .$$

In particular, for every central character χ, we have the following embedding of $\mathbb{C}$–algebras:

$$Z(\mathfrak{g},K)_\chi \hookrightarrow \Big(\prod_{\delta\in Cl(\chi)} U(\mathfrak{a})^{W_\delta}\Big)^{W(G:A)}, \quad z\mapsto (a_\delta)_{\delta\in Cl(\chi)}.$$

Proof. Consider the principal series $H_K^{(\delta,\nu)}$. Then z_γ acts on

$${}_K^{(\delta,\nu)}(\gamma) = \mathrm{Hom}_K(\gamma, H_K^{(\delta,\nu)})$$

by the scalar $a_\delta(\nu)$. The action of the central element $(z_\gamma)_{\gamma\in\hat{K}}$ on $H_K^{(\delta,\nu)}$ is obtained by "patching together" the actions of all z_γ. Now, if α and α' both contain δ, then

$$H_K^{(\delta,\nu)}(\alpha)\neq 0 \text{ and } H_K^{(\delta,\nu)}(\alpha')\neq 0.$$

Thus, if $H_K^{(\delta,\nu)}$ is irreducible, then $a_\delta(\nu)=a'_\delta(\nu)$. Since $H_K^{(\delta,\nu)}$ is irreducible on a Zariski-dense set, $a_\delta=a'_\delta$ holds everywhere. The corollary is proved. □

6. Example: $Sp_4(\mathbb{R})$

In this section we shall compute completely the center $Z(\mathfrak{g}, K)$ for $Sp_4(\mathbb{R})$. This example will help the reader to understand a feature which appears for real groups, but not for p-adic groups.

If $G = Sp_4(\mathbb{R})$, then $M = \{\pm 1\} \times \{\pm 1\}$ and $A = \mathbb{R}^+ \times \mathbb{R}^+$. The characters of M can be identified with pairs (ϵ_1, ϵ_2) in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The center $Z_G = \{\pm 1\}$ of G is embedded diagonally in M. Let $+$ and $-$ denote the trivial and non-trivial characters of Z_G, respectively. The corresponding classes of characters of M are

$$Cl(+) = \{\delta_1 = (0,0), \delta_2 = (1,1)\} \text{ and } Cl(-) = \{\delta_3 = (1,0), \delta_4 = (0,1)\}.$$

The centralizer of the two characters in $Cl(+)$ is the full Weyl group W. The Weyl group permutes the two characters in $Cl(-)$. The centralizer of each is a Klein 4-subgroup W'. The center $Z(\mathfrak{g}, K)$ of the category is a direct sum of two components $Z(\mathfrak{g}, K)_+$ and $Z(\mathfrak{g}, K)_-$. Moreover, Corollary 13 implies that

$$\begin{cases} Z(\mathfrak{g}, K)_+ \subseteq U(\mathfrak{a})^W \oplus U(\mathfrak{a})^W, \\ Z(\mathfrak{g}, K)_- \subseteq (U(\mathfrak{a})^{W'} \oplus U(\mathfrak{a})^{W'})^W. \end{cases}$$

This result can be sharpened as follows. We need some representation theory of $GL_2(\mathbb{R})$. Let us write $|\ |^{s_1} sgn^{\epsilon_1} \times |\ |^{s_2} sgn^{\epsilon_2}$ for the principal series representation of $GL_2(\mathbb{R})$, where $s_i \in \mathbb{C}$, $\epsilon_i \in \{0,1\}$. It reduces if and only if $s_1 - s_2 \in \mathbb{Z}_{\neq 0}$ and the *parity* condition

$$s_1 - s_2 \equiv \epsilon_1 - \epsilon_2 + 1 \pmod 2$$

is satisfied. In this case it contains an essentially square integrable subquotient denoted by $\delta(s_1, s_2, \epsilon_1, \epsilon_2)$. We have $\delta(s_1, s_2, \epsilon_1, \epsilon_2) \simeq \delta(s_1, s_2, \epsilon_1 + 1, \epsilon_2 + 1)$. This and induction in stages show that the principal series in $Sp_4(\mathbb{R})$ $(p, t \in \mathbb{Z}, p \neq t)$:

$$\begin{cases} |\ |^p sgn^{p+1} \times |\ |^t sgn^t \rtimes 1 \\ |\ |^p sgn^p \times |\ |^t sgn^{t+1} \rtimes 1 \end{cases}$$

share a common irreducible subquotient that is a subquotient of the representation induced from a Siegel parabolic subgroup of $Sp_4(\mathbb{R})$ of $\delta(p, t, p+1, t) \simeq \delta(p, t, p, t+1)$. Applying this to $t \equiv p + 1 \equiv 0 \pmod 2$ we see that for $\nu = (p, t)$,

$$a_{\delta_1}(\nu) = a_{\delta_2}(\nu).$$

Since the set of such points (p, t) is dense in $\mathfrak{a}^*$, it follows that $a_{\delta_1} = a_{\delta_2}$. This shows that $Z(\mathfrak{g}, K)_+$ is contained in $U(\mathfrak{a})^W$.

Similarly, using $t \equiv p \equiv 1 \pmod 2$ we can show that $a_{\delta_3} = a_{\delta_4}$. Since, in addition, $w(a_{\delta_3}) = w_{\delta_4}$ for any element w in $W \setminus W'$, it follows that $a_{\delta_3} = a_{\delta_4}$ is W-invariant. This shows that $Z(\mathfrak{g}, K)_-$ is also contained in $U(\mathfrak{a})^W$. Thus we have shown the following:

Theorem 14. *Let $G = Sp_4(\mathbb{R})$. Then*

$$Z(\mathfrak{g}, K)_+ = Z(\mathfrak{g}) \text{ and } Z(\mathfrak{g}, K)_- = Z(\mathfrak{g}).$$

7. Quasi-split groups

Let Φ be a root system, not necessarily irreducible, and let $\underline{G}$ be the corresponding simply-connected Chevalley group. It is a split, semi-simple algebraic group defined over $\mathbb{Z}$. The maximal split torus $\underline{H}$ is described as follows. Let C be the co-root lattice. Then, for any field F,

$$\underline{H}(F) = C \otimes F^{\times}$$

is a maximal (split) torus in $\underline{G}(F)$. Pick Δ to be a set of simple roots, and let

$$\sigma : \Delta \to \Delta$$

be an involution of the set of simple roots, preserving the invariant form $\langle \cdot, \cdot \rangle$ of Φ. Then sigma defines a quasi-split form G of $\underline{G}(\mathbb{R})$. For example, if $\Delta_1 \cup \Delta_2$ such that the roots in Δ_1 are perpendicular to the roots in Δ_2 and $\sigma(\Delta_1) = \Delta_2$, then G is a complex group. The maximally split torus of G is

$$H = \underline{H}(\mathbb{C}^{\times})^{\sigma},$$

where σ is defined by $\sigma(c \otimes z) = \sigma(c) \otimes \bar{z}$.

We shall now give a detailed description of H. Let Δ_+ be the set of simple roots fixed by σ. These are simple real roots. Simple roots α such that $\sigma(\alpha) \neq \alpha$ are called complex. For every real simple root α there exists a homomorphism

$$\varphi_\alpha : SL_2(\mathbb{R}) \to G.$$

Define

$$h_\beta = \varphi_\alpha \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $C_+ \subseteq C$ be the co-root lattice spanned by roots in Δ_+. Then the subgroup of H generated by h_α for all simple real roots is

$$C_+ \otimes \{\pm 1\} \subseteq H.$$

Let C^+ and C^- be the eigen-lattices for σ in the co-root lattice C. Note that $C_+ \subseteq C^+$, but the containment is strict except for split groups when $\Delta_+ = \Delta$. Indeed, C^+ is spanned by real roots and $\alpha + \sigma(\alpha)$ for all complex roots. Let $H = TA$ be the usual decomposition of the torus H where T is a maximal compact subgroup. Let T^0 be the connected component of T. Then

$$A = C^+ \otimes \mathbb{R}^+ \text{ and } T^0 = C^- \otimes \mathbb{T}.$$

Here, of course, $\mathbb{T}$ is the group of complex numbers of norm one. Moreover,

$$T = T^0 \times (C_+ \otimes \{\pm 1\}).$$

A quasi-split group is complex if $\Delta_+ = \emptyset$ and split if $\Delta_+ = \Delta$. Other cases are tabulated below. The meaning of the first three columns is clear. The meaning of the last column will be explained shortly.

Δ	Δ_+	$\lvert T/T^0 \rvert$	$\lvert (L^+/R^+)_2 \rvert$
A_{2n}		1	1
A_{2n+1}	A_1	2	2
D_n	A_{n-2}	2^{n-2}	1 or 2
E_6	A_2	4	1

Proposition 15. *Real roots generate all characters of T trivial on $Z_G \cdot T^0$.*

Proof. Recall that
$$T/T^0 = C_+ \otimes \{\pm 1\} = C_+/2C_+.$$
Let R_+ be the root lattice for Δ_+ and $L_+ \supseteq R_+$ the weight lattice. Since L_+ is a dual lattice of C_+, the characters of $C_+/2C_+$ are parameterized by $L_+/2L_+$. If instead of L_+ we use the root lattice R_+, the obstruction, in getting all characters of $C_+/2C_+$ is measured by the 2-torsion of L^+/R^+. If the 2-torsion $(L_+/R_+)_2$ is trivial, then the real roots give rise to all characters of T/T^0. The data in the table show that the condition is satisfied except when $\Delta_+ = A_n$ with n odd. This happens for Δ of type A_{2n+1} and D_{2n+1} with n odd. Moreover, in each of these cases, real characters vanish on a subgroup of T containing T^0 with index 2. Thus, the proposition follows since in each case one can construct an element of Z_G which is not contained in the connected component T^0. For example, in the case D_n: Let $\alpha_1, \dots, \alpha_n$ be the usual marking of the simple roots in the D_n root system such that α_{n-2} is the branch point. The central element not in the connected component is
$$\alpha_1^\vee(-1) + \alpha_3^\vee(-1) + \dots$$
with the tail ending with $\alpha_{n-1}^\vee(-1)$ if n is even and $\alpha_{n-1}^\vee(i) - \alpha_n^\vee(i)$ if n is odd ($i^2 = -1$). □

8. Cross action of real roots

In this section we reinterpret some results of Vogan [V] necessary for our applications. Let $P = MAN$ be the minimal parabolic subgroup in G corresponding to the choice of simple roots Δ. Recall that $M = T$ since G is quasi-split. For every simple root α there exists a standard parabolic subgroup $P_\alpha = M_\alpha A_\alpha N_\alpha \supseteq P$ such that A_α is the kernel of α on A and
$$M_\alpha = T \cdot \varphi_\alpha(SL_2(\mathbb{R})).$$

Proposition 16. *Let α be a simple real root. Let $Z(M_\alpha)$ be the center of M_α. If the character α is non-trivial on $M = T$, then*
$$M_\alpha = Z(M_\alpha) \times_{\{\pm 1\}} G_1,$$
where G_1 is a subgroup of M_α isomorphic to the subgroup of $GL_2(\mathbb{R})$ of 2×2 matrices of determinant 1 or -1.

Proof. Notice that $Z(M_\alpha)$ is equal to the kernel of α on T. Since the real roots are trivial on T^0, we have
$$T^0 \subseteq Z(M_\alpha) \subseteq T.$$
Thus, the values of the character α on T are 1 or -1 and α defines a non-trivial character of T if and only if there exists another simple real root β such that $\alpha(h_\beta) = -1$. It follows that
$$\varphi_\alpha(SL_2(\mathbb{R})) \cup \varphi_\alpha(SL_2(\mathbb{R})) \cdot h_\beta$$
is a subgroup of M_α isomorphic to G_1 and, clearly, $Z(M_\alpha)G_1 = M_\alpha$. Since the intersection of $Z(M_\alpha)$ and G_1 is contained in the center of G_1, the proposition follows. □

We are now ready to prove the main result of this section. It generalizes (to the case of quasi-split groups) what we did in the previous section with the group $Sp_4(\mathbb{R})$. See also [V, Prop. 8.3.18].

Proposition 17. *Let α be a simple real root with non-trivial restriction to T. Let δ be a character of $M = T$ and ν in $\mathfrak{a}^*$. If $\langle \nu, \alpha^\vee \rangle$ is a non-zero integer and*

$$\langle \nu, \alpha^\vee \rangle \equiv \delta(h_\alpha) + 1 \pmod 2,$$

then the principal series representations $H_K^{(\delta,\nu)}$ and $H_K^{(\delta+\alpha,\nu)}$ share a common subquotient.

Proof. The restriction of the principal series $| \,|^{s_1} sgn^{\epsilon_1} \times | \,|^{s_2} sgn^{\epsilon_2}$ of $GL_2(\mathbb{R})$ to G_1 depends on the difference $s = s_1 - s_2$. The principal series representation (now of G_1) reduces if and only if $s \in \mathbb{Z}_{\neq 0}$ and the *parity* condition

$$s \equiv \epsilon_1 - \epsilon_2 + 1 \pmod 2)$$

is satisfied. This is exactly the assumption condition on (δ, ν). Moreover, if we add 1 to both ϵ_1 and ϵ_2, then the two reducible principal series representations of G_1 share a common subquotient. Since replacing (ϵ_1, ϵ_2) by $(\epsilon_1+1, \epsilon_2+1)$ corresponds to replacing δ by $\delta + \alpha$, the proposition follows from Proposition 16 by induction in stages. □

Corollary 18. *Let G be a quasi-split, real linear group. Let $\delta_1, \ldots, \delta_m$ be all irreducible representations of $M = T$ with the same differential and the same central character for Z_G. Then the center acts by the same character on all subquotients of principal series representations $H_K^{(\delta_i,\nu)}$.*

Proof. This is a direct consequence of Propositions 17 and 15. □

Let $cl(\chi)$ be the set of all weights ϵ in $\mathfrak{t}^*$ such that there exists an irreducible representation δ of M with the infinitesimal character ϵ and the central character (for G) equal to χ. Then

Corollary 19. *Let W_ϵ be the centralizer of $\epsilon \in \mathfrak{t}^*$ in the relative Weyl group $W(G : A)$. Then*

$$Z(\mathfrak{g}, K)_\chi \subseteq (\prod_{\epsilon \in cl(\chi)} U(\mathfrak{a})^{W_\epsilon})^{W(G:A)}.$$

Proof. Of course, our task here is to sharpen the inclusion given by Corollary 13:

$$i_\chi : Z(\mathfrak{g}, K)_\chi \to (\prod_{\delta \in Cl(\chi)} U(\mathfrak{a})^{W_\delta})^{W(G:A)}.$$

To that end, fix ϵ in $\mathfrak{t}^*$, and let $\delta_1, \ldots, \delta_n$ be all the characters of $M = T$ with the differential ϵ. Consider the projection

$$pr_\epsilon : (\prod_{\delta \in Cl(\chi)} U(\mathfrak{a})^{W_\delta}) \to \prod_{j=1}^{m} U(\mathfrak{a})^{W_{\delta_i}}.$$

Since the image of $Z(\mathfrak{g}, K)_\chi$ under i_χ is $W(G : A)$-invariant, and the stabilizer $W_\epsilon \subseteq W(G : A)$ of ϵ permutes the characters $\delta_1, \ldots, \delta_m$, the image of $Z(\mathfrak{g}, K)$ under the composition $pr_\epsilon \circ i_\chi$ lands in

$$(\prod_{j=1}^{m} U(\mathfrak{a})^{W_{\delta_i}})^{W_\epsilon}.$$

Moreover, Corollary 18 implies that the image of $Z(\mathfrak{g}, K)$ consists of m-tuples $(a_{\delta_1}, \ldots, a_{\delta_m})$ such that $a_{\delta_1} = \cdots = a_{\delta_m}$. These equations precisely characterize $U(\mathfrak{a})^{W_\epsilon}$ as a subspace

$$U(\mathfrak{a})^{W_\epsilon} \subseteq (\prod_{j=1}^{m} U(\mathfrak{a})^{W_{\delta_i}})^{W_\epsilon}$$

given by the diagonal inclusion. (See the example of $Sp_4(\mathbb{R})$.) The corollary is proved. □

Corollary 20. *Let G be a split group. Then $Z(\mathfrak{g}, K)_\chi = Z(\mathfrak{g})$ for every central character χ.*

Proof. If G is split, then $\mathfrak{t} = 0$, so every class $cl(\chi)$ consists of only one element, 0. Thus, Corollary 19 in this case simply states that

$$Z(\mathfrak{g}, K)_\chi \subseteq U(\mathfrak{a})^W \cong Z(\mathfrak{g}).$$

On the other hand, $Z(\mathfrak{g})$ is clearly contained in $Z(\mathfrak{g}, K)_\chi$. The corollary is proved. □

9. Geometry

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the complexified $\mathfrak{h}_0$. Restricting simple roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ to $\mathfrak{a}$ gives simple roots in a restricted root system denoted by $\Delta(\mathfrak{g}, \mathfrak{a})$. Possible cases (for quasi-split groups) are:

$\Delta(\mathfrak{g}, \mathfrak{h})$	A_{2n-1}	A_{2n}	D_n	E_6
$\Delta(\mathfrak{g}, \mathfrak{a})$	C_n	BC_n	B_{n-1}	F_4

The Weyl group of the restricted root system is isomorphic to $W(G : A)$. Moreover, it can be realized as a subgroup of the absolute Weyl group W as follows (see [Wr, Proposition 1.1.3.3]):

Proposition 21. *Let W_σ be the subgroup of W consisting of all w commuting with σ. (Such a w leaves $\mathfrak{a}^*$ and $\mathfrak{t}^*$ invariant.) The map $w \mapsto w|_{\mathfrak{a}^*}$ is a bijective homomorphism from W_σ onto $W(G : A)$.*

Proposition 22. *Let ϵ be in $\mathfrak{t}^*$, and let W_ϵ be the stabilizer of ϵ in $W(G : A)$. Let*

$$i_\epsilon : \mathfrak{a}^* \to \mathfrak{h}^*$$

be given by $i_\epsilon(a) = \epsilon + a$. Then the pull back via i_ϵ of $U(\mathfrak{h})^W$ is $U(\mathfrak{a})^{W_\epsilon}$.

Proof. First of all, note that $\mathfrak{h}^*$ is a Hilbert space for the W-invariant inner product. Since the involution σ preserves the inner product, the subspaces $\mathfrak{t}^*$ and $\mathfrak{a}^*$ are perpendicular. Let w be a stabilizer of $\epsilon + \mathfrak{a}^*$. Since ϵ is the unique point in $\epsilon + \mathfrak{a}^*$ closest to 0, the element w stabilizes ϵ and, therefore, it stabilizes $\mathfrak{a}^* = (\epsilon + \mathfrak{a}^*) - \epsilon$. It follows that w is in $W_\sigma = W(G : A)$. Thus the stabilizer of $\epsilon + \mathfrak{a}$ in W is equal to $W_\epsilon \subseteq W(G : A)$. □

Definition 23. Let ϵ in $\mathfrak{t}^*$ be the differential of a character of $T = M$. We denote by $V_{[\epsilon]} \subseteq \mathfrak{h}^*$ the union of W-conjugates of $i_\epsilon(\mathfrak{a}^*) = \epsilon + \mathfrak{a}^*$. Here $[\epsilon]$ denotes the set of all $W(G : A)$-conjugates of ϵ. The notation $V_{[\epsilon]}$ reflects the fact that $V_{[\epsilon]}$ depends only on $[\epsilon]$.

Let $\epsilon_1, \dots \epsilon_n$ be all the $W(G:A)$-conjugates of ϵ. Let $W(\epsilon_i)$ be the stabilizer of ϵ_i in $W(G:A)$. Then Proposition 22 implies that the inclusions

$$\epsilon + \mathfrak{a}^* \subseteq \bigcup_{i=1}^{n} \epsilon_i + \mathfrak{a}^* \subseteq V_{[\epsilon]}$$

induce the following isomorphisms:

$$\mathbb{C}[V_{[\epsilon]}]^W = (\prod_{i=1}^{n} U(\mathfrak{a})^{W_{\epsilon_i}})^{W(G:A)} = U(\mathfrak{a})^{W_\epsilon}.$$

Thus the interpretation of the center in Corollary 19 can be rewritten as

Proposition 24. *Fix a central character χ. Let $cl(\chi)$ be the set of all ϵ in $\mathfrak{t}$ such that there exists a character δ of T such that the restriction of δ to Z_G is χ and the differential of δ is ϵ. Then*

$$Z(\mathfrak{g}, K)_\chi \subseteq \prod_{[\epsilon] \subseteq cl(\chi)} \mathbb{C}[V_{[\epsilon]}]^W.$$

The next, major, step is to show that the elements of the center, as regular functions on various $V_{[\epsilon]}$, coincide on every intersection

$$V_{[\epsilon]} \cap V_{[\epsilon']}.$$

This will be accomplished using the induction in stages, but this time we also need to involve the parabolic subgroups corresponding to complex roots.

10. Complex groups: $SL_2(\mathbb{C})$

If G is complex, then the Weyl group of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ and that of $\mathfrak{h}_0$ in $\mathfrak{g}_0$ are naturally isomorphic. We fix the isomorphism $\mathfrak{g} \simeq \mathfrak{g}_0 \times \mathfrak{g}_0$ such that its Cartan subalgebra $\mathfrak{h}_0 \times \mathfrak{h}_0$ can be decomposed as follows:

$$\mathfrak{h}_0 \times \mathfrak{h}_0 = \mathfrak{a} \oplus \mathfrak{t}, \quad \mathfrak{a} = \mathfrak{t} = \mathfrak{h}_0$$

where

$$\begin{cases} \mathfrak{a} = \{(x, x);\ x \in \mathfrak{h}_0\}, \\ \mathfrak{t} = \{(x, -x);\ x \in \mathfrak{h}_0\}. \end{cases}$$

Thus, every linear functional of $\mathfrak{h}_0 \times \mathfrak{h}_0$ can be written in two ways:

$$(\lambda, \mu) = \nu \oplus \delta, \quad \nu = \lambda + \mu,\ \delta = \lambda - \mu.$$

If $\delta \in \mathfrak{h}_0^*$ is a weight of a finite–dimensional representation of K, then we let

$$X(\lambda, \mu) := H_K^{(\delta,\nu)}.$$

The Frobenius reciprocity implies that $X(\lambda, \mu)$ contains a unique irreducible subquotient containing the K–type having the extremal weight δ with multiplicity one. We denote it by $\bar{X}(\lambda, \mu)$.

Let's look at the example of $SL_2(\mathbb{C})$. The group center has two elements, so the category of $(\mathfrak{g}, K)$-modules has two components

$$\mathcal{M}(\mathfrak{g}, K)_+ \text{ and } \mathcal{M}(\mathfrak{g}, K)_-$$

parameterized by two central characters, trivial $(+)$ and non-trivial $(-)$. The principal series representations are $X(\lambda, \mu)$ where λ and μ are two complex numbers such that $n = \lambda - \mu$ is an integer. If n is even, then the principal series belongs to $\mathcal{M}(\mathfrak{g}, K)_+$. If n is odd, then the principal series belongs to $\mathcal{M}(\mathfrak{g}, K)_-$.

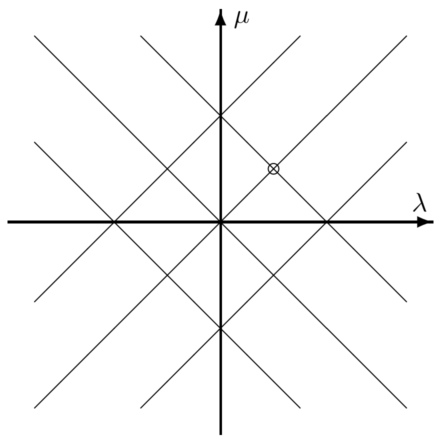


FIGURE 1

Identify $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0 \cong \mathbb{C} \oplus \mathbb{C}$ and $W \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Under these identifications W acts by changing the signs of two coordinates in $\mathbb{C} \oplus \mathbb{C}$. Here $V_{[n]}$ is the subset of infinitesimal characters of the principal series representations $X(\lambda, \mu)$ with $\lambda - \nu = n$. Then $V_{[n]}$ is the union of the line $\lambda - \mu = n$ and its W-conjugates. Figure 1 contains the union

$$V_{[0]} \cup V_{[2]}.$$

The line $\lambda - \mu = 2$ is conjugated to the line $\lambda + \mu = 2$, which intersects the line $\lambda - \mu = 0$ at $(1, 1)$, the circled point. Thus we have two principal series representations $X(1, 1)$ and $X(1, -1)$ with the same infinitesimal character. In fact, the principal series $X(1, 1)$ contains the (irreducible) principal series representation $X(1, -1)$ as a subquotient.

In fact, a more general result holds:

Lemma 25. *Let m and n be two integers such that $m \equiv n \pmod 2$. Then the principal series representations $X(\frac{m}{2}, \frac{n}{2})$ and $X(\frac{m}{2}, -\frac{n}{2})$ share a common subquotient.*

This shows that the principal series representations at the intersection points of the $V_{[n]}$'s (for n of the fixed parity) share common subquotients. Thus if we put

$$V_+ = \bigcup_{n \text{ even}} V_{[n]} \text{ and } V_- = \bigcup_{n \text{ odd}} V_{[n]},$$

then the following sharpening of Proposition 24 holds:

Proposition 26. *Let $\mathbb{C}[V_+]^W$ and $\mathbb{C}[V_-]^W$ be the algebras of W-invariant regular functions on V_+ and V_-, respectively. Then*

$$Z(\mathfrak{g}, K)_+ \subseteq \mathbb{C}[V_+]^W \text{ and } Z(\mathfrak{g}, K)_- \subseteq \mathbb{C}[V_-]^W.$$

This result can be easily generated to complex groups as follows. An infinitesimal character for G is specified by a pair (λ, μ) such that $\lambda - \mu$ is in L, the integral weight lattice in $\mathfrak{a}^*$. Let $L_\lambda = \lambda + L$ and let W_λ be the integral Weyl group for λ. Note that $W_\lambda = W_\mu$ for every μ in L_λ. Pick a basis Δ_λ for integral roots, and let L_λ^+ be the set of weights μ in L_λ such that

$$\langle \mu, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_\lambda.$$

Assume that λ is in Λ^+ and μ in Λ^-. Then the principal series representations with the infinitesimal character (λ, μ) are

$$X(\lambda, \nu)$$

where ν is a W_λ-conjugate of μ. Moreover, the principal series representation $X(\lambda, \mu)$ is irreducible and appears as a subquotient in all other $X(\lambda, \nu)$. Thus we have:

Theorem 27. *Let G be a complex group. Any central character χ corresponds to a class L_χ of the root lattice R in the weight lattice. Let V_χ be the set of all (λ, μ) such that $\lambda - \mu$ is in L_χ together with all its conjugates for the absolute Weyl group. Then*

$$Z(\mathfrak{g}, K) \subseteq \mathbb{C}[V_\chi]^W.$$

11. Cross-action of complex roots

In order to avoid discussing $SU(2,1)$ we shall assume that every complex root α is perpendicular to $\sigma(\alpha)$. For every such complex root α there exists a homomorphism

$$\varphi_\alpha : SL_2(\mathbb{C}) \to G.$$

Moreover, the homomorphisms φ_α and $\varphi_{\sigma(\alpha)}$ differ by complex conjugation. Thus, for every complex root α there is a parabolic subgroup $P_\alpha = M_\alpha A_\alpha N_\alpha$ such that A_α is the kernel of α on A and

$$M_\alpha = T \cdot \varphi_\alpha(SL_2(\mathbb{C})).$$

Notice that $P_\alpha = P_{\sigma(\alpha)}$. The following lemma is similar to Proposition 16 , so we omit the proof.

Lemma 28. *Let α be a simple complex root. Let $Z(M_\alpha)$ be the centralizer of M_α. Then*

$$M_\alpha = Z(M_\alpha) \times_{\{\pm 1\}} \varphi_\alpha(SL_2(\mathbb{C})).$$

Decompose any complex root α as

$$\alpha = \alpha^+ + \alpha^- = \frac{\alpha + \sigma(\alpha)}{2} + \frac{\alpha - \sigma(\alpha)}{2}.$$

Note that α^- can be viewed both as a character of T and as an element of $\mathfrak{t}^*$, while α^+ is an element of $\mathfrak{a}^*$. We now have the following (see [V, Prop. 8.3.18]):

Proposition 29. *Let δ be a character of T and ϵ the corresponding differential in $\mathfrak{t}^*$. Let ν be in $\mathfrak{a}^*$. Assume that*

$$m = \langle \epsilon + \nu, \alpha^\vee \rangle$$

is an integer. Define $\delta' = \delta - m\alpha^-$ and $\nu' = \nu - m\alpha^+$. Then the principal series representations $H_K^{(\delta,\nu)}$ and $H_K^{(\delta',\nu')}$ have a common subquotient.

Proof. This follows from induction in stages through the parabolic subgroup P_α. Indeed, let

$$n = \langle \epsilon, \alpha^\vee - \sigma(\alpha^\vee) \rangle.$$

Note that n is an integer, since ϵ is the differential of δ, a character of T. Thus, if we induce first to P_α, then the two induced representations, when restricted to $\varphi_\alpha(SL_2(\mathbb{C}))$, are isomorphic to $X(\frac{2m-n}{2}, \frac{n}{2})$ and $X(\frac{2m-n}{2}, -\frac{n}{2})$, which have a common subquotient. The proposition now follows easily from Lemma 28. □

12. Main theorem

Fix a character δ of T with the differential ϵ. The set of infinitesimal characters of the principal series representations $H_K^{(\delta,\nu)}$ (here ν is in $\mathfrak{a}^*$) can be identified, via the Harish-Chandra homomorphism, with the subset $\epsilon + \mathfrak{a}^*$. In fact, as we argued before, it is more natural to take the set $V_{[\epsilon]}$ of all W-conjugates of $\epsilon + \mathfrak{a}^*$.

Proposition 30. *Let G be a quasi-split, simple, linear group, but not BC_n. Let χ be a central character. Let $V_\chi = \bigcup_{\epsilon \in cl(\chi)} V_{[\epsilon]}$. Then*

$$Z(\mathfrak{g}, K)_\chi \subseteq \mathbb{C}[V_\chi]^W.$$

Proof. This has already been shown for split and complex groups (Theorem 27). It remains to deal with quasi-split groups of relative type C_n, B_n and F_4.

Lemma 31. *Let G be quasi-split but not of the type BC_n. Let ϵ and ϵ' be differentials of two characters of T. Then the integral points (for the absolute root system Δ) in the intersection*

$$w(\epsilon + \mathfrak{a}^*) \cap (\epsilon' + \mathfrak{a}^*)$$

are Zariski-dense.

This lemma will be proved in the next section.

Now if $\epsilon + \nu$ and $\epsilon' + \nu'$ are two integral points such that $w(\epsilon + \nu) = \epsilon' + \nu'$, write $w = s_{\alpha_n} \cdot \ldots \cdot s_{\alpha_1}$ as a product of reflections. Let δ be any character of T with differential ϵ. Let $H_K^{(\delta_1,\nu_1)}$ be the principal series obtained from $H_K^{(\delta,\nu)}$ by the cross action of s_{α_1} if α_1 is complex or the usual intertwining operator corresponding to s_{α_1} if α_1 is real. In this way we can construct a sequence of induced representations

$$H_K^{(\delta,\nu)}, H_K^{(\delta_1,\nu_1)}, \ldots, H_K^{(\delta',\nu')}$$

such that δ' has the differential ϵ' and any two consecutive representations share a common subquotient. It follows that for any element of the center, the corresponding polynomial functions of $V_{[\epsilon]}$ and $V_{[\epsilon']}$ coincide on a Zariski-dense set in the intersection $V_{[\epsilon]} \cap V_{[\epsilon']}$. The proposition is proved. □

Theorem 32. *Let χ be a central character for G. Then the component $Z(\mathfrak{g}, K)_\chi$ is isomorphic to the subalgebra of*

$$\prod_{[\epsilon] \subseteq cl(\chi)} \mathbb{C}[V_{[\epsilon]}]^W$$

consisting of all infinite tuples $(f_{[\epsilon]})_{[\epsilon] \subseteq cl(\chi)}$ such that $f_{[\epsilon]}$ is a regular function on $V_{[\epsilon]}$ and such that $f_{[\epsilon]}$ and $f_{[\epsilon']}$ coincide on the intersection of $V_{[\epsilon]}$ and $V_{[\epsilon']}$. In other words, $Z(\mathfrak{g}, K)_\chi$ is isomorphic to the algebra of W-invariant regular functions on $\bigcup_{[\epsilon] \subseteq cl(\chi)} V_{[\epsilon]}$.

Proof. Proposition 30 shows the inclusion. Conversely, let $(f_{[\epsilon]})_{[\epsilon]\subseteq cl(\chi)}$ be a family such that $f_{[\epsilon]}$ and $f_{[\epsilon']}$ coincide on the intersection of $V_{[\epsilon]}$ and $V_{[\epsilon']}$ for every pair $[\epsilon]$ and $[\epsilon']$. We want to construct an element of the center $Z(\mathfrak{g}, K)$ which maps to $(f_{[\epsilon]})_{[\epsilon]\subseteq cl(\chi)}$. To do so, we shall use the description of the center given by Theorem 7. Let γ be a K-type. Since the set of $\mathfrak{t}$-weights ϵ appearing in γ is finite, by Proposition 22 there exists an element f_γ in $U(\mathfrak{h})^W$ whose restriction to $V_{[\epsilon]}$ is equal to $f_{[\epsilon]}$, for every $\mathfrak{t}$-weight ϵ of γ. Let z_γ be the corresponding element of the center of the enveloping algebra $Z(\mathfrak{g}) \cong U(\mathfrak{h})^W$. It remains to show that the family (z_γ) satisfies the conditions of Theorem 7. But this is easy. Indeed, for any two K-types γ and γ' there exists an element f in $U(\mathfrak{h})^W$ such that $f = f_{[\epsilon]}$ on $V_{[\epsilon]}$ for all weights ϵ in both γ and γ'. Let z in $Z(\mathfrak{g}) \cong U(\mathfrak{h})^W$ correspond to f. Then

$$z_\gamma + Z(\mathfrak{g})/Z(\mathfrak{g}) \cap U(\mathfrak{g})I_\gamma = z + Z(\mathfrak{g})/Z(\mathfrak{g}) \cap U(\mathfrak{g})I_\gamma$$

and

$$z_{\gamma'} + Z(\mathfrak{g})/Z(\mathfrak{g}) \cap U(\mathfrak{g})I_{\gamma'} = z + Z(\mathfrak{g})/Z(\mathfrak{g}) \cap U(\mathfrak{g})I_{\gamma'}.$$

It follows that the condition involving $z_\gamma y - y z_{\gamma'}$ is trivially satisfied since z_γ and $z_{\gamma'}$ can be represented by the same element z in the center of the enveloping algebra $Z(\mathfrak{g})$. The theorem is proved. □

13. Geometric lemma

It remains to check Lemma 31. Due to the lack of better ideas we do this on a case by case basis.

13.1. **Quasi-split** D_n**.** Let Φ be a root system of type D_n realized in $\mathbb{R}^n$ in the usual way. In particular, $e_1 - e_2$, $e_2 - e_3, ..., e_{n-1} - e_n$ and $e_{n-1} + e_n$ are simple roots. The automorphism σ fixes all simple roots except the last two:

$$\sigma(e_{n-1} - e_n) = e_{n-1} + e_n.$$

Then $\mathfrak{h}^*$ is identified with $\mathbb{C}^n$ and the set of integral weights is equal the set of $(m_1/2, \ldots, m_n/2)$ in $\mathbb{C}^n$ such that $m_1, \ldots, m_n$ are integers of the same parity. We have

$$\begin{cases} \mathfrak{a}^* = \{(x_1, \ldots, x_{n-1}, 0)\}, \\ \mathfrak{t}^* = \{(0, \ldots, 0, x_n)\}. \end{cases}$$

The integral weights of $\mathfrak{t}$ are given by half integers $x_n = m/2$. Two weights ϵ and ϵ' given by $m/2$ and $m'/2$, respectively, belong to the same class if $m \equiv m' \pmod 2$. The set $\epsilon + \mathfrak{a}^*$ is given by the equation $x_n = m/2$. If w is in the absolute Weyl group and $w(e_n) = \pm e_k$, then $w(\epsilon' + \mathfrak{a}^*)$ is given by an equation $x_k = \pm m'/2$. Clearly, the set of all elements in $\mathbb{C}^n$ satisfying both equations contains a Zariski-dense set of integral points if m and m' have the same parity.

13.2. **Quasi-split** A_{2n-1}**.** Fix the standard realization of the A_{2n-1} root system. The involution σ is given by

$$\sigma(e_i) = -e_{2n-i}.$$

In fact, it will be more convenient to permute the indices so that $\sigma(e_{2i-1}) = -e_{2i}$ and $\sigma(e_{2i}) = -e_{2i-1}$.

In particular, we identify $\mathfrak{h}^*$ with (an affine-hyperplane) in the subspace of $\mathbb{C}^{2n}$ consisting of $2n$-tuples $(z_1, \ldots, z_{2n})$ such that $\sum_i z_i$, is a fixed integer. Then

$$\begin{cases} \mathfrak{a}^* = \{(x_1, -x_1, \ldots, x_n, -x_n)\}, \\ \mathfrak{t}^* = \{(y_1, y_1 \ldots, y_n, y_n)\}. \end{cases}$$

An integral weight in $\mathfrak{t}$ is given by half-integers $y_i = m_i/2$. Put $\alpha_i = e_{2i-1} - e_{2i}$. Then $\epsilon + \mathfrak{a}^*$ is given by the equations

$$\langle \alpha_i, x\rangle = m_i.$$

Let w be in $W = S_{2n}$, and put $\beta_i = w(\alpha_i)$. The intersection

$$\epsilon + \mathfrak{a}^* \cap w(\epsilon' + \mathfrak{a}^*)$$

is given by taking together the equations involving α_i and β_i. In order to understand these equations we shall build a graph as follows. There is one vertex for each root α_i and β_j. Connect two vertices by 0, 1 or 2 edges, equal to the value of the product $\langle \beta_j, \alpha_i\rangle$. Since the Weyl group simply permutes the basis vectors e_i, each β_i is either equal to α_j for some j, or there are precisely two α_j such that $\langle \beta_i, \alpha_j\rangle = \pm 1$. It follows that the graph consists of loops, each with an even number of vertices corresponding to α's and β's alternatively. In fact, if $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ is a loop, then

$$\alpha_1 + \cdots + \alpha_k = \beta_1 + \cdots + \beta_k.$$

Thus the intersection $\epsilon + \mathfrak{a}^* \cap w(\epsilon' + \mathfrak{a}^*)$ is empty unless

$$m_1 + \cdots + m_k = m'_1 + \cdots + m'_k$$

for every loop. Moreover, if this condition is satisfied, then we can throw away one of the equations, for example $\langle \beta_k, x\rangle = m'_k$, in each loop. We can further permute the indices so that $\alpha_i = e_{2i-1} + e_{2i}$ and $\beta_i = e_{2i} + e_{2i+1}$. Thus the intersection is given by cutting down $\epsilon + \mathfrak{a}^*$ by equations

$$\langle e_{2i} + e_{2i+1}, z\rangle = -x_i + x_{i+1} + m_i/2 + m_{i+1}/2 = m'_i.$$

Here $z = (x_1 + m_1/2, -x_1 + m_1/2, \ldots)$. These are recursive relations which determine x_i in terms of x_1 and m_i's and m'_i's. Notice, if we pick x_1 to be a half integer, then the recursive relations imply that the x_i are also half-integers. Since

$$\langle e_{2i} - e_{2i+1}, x\rangle = -x_i - x_{i+1} + m_i/2 - m_{i+1}/2 = -2x_i + m_i - m'_i$$

and

$$\langle e_{2i-1} - e_{2i}, x\rangle = 2x_i,$$

the solution, if x_1 is a half integer, is an integral weight. This takes care of the case A_{2n-1}.

13.3. **Quasi-split E_6.** We shall only provide a sketch here, but with enough details so it is reduced to a straightforward check.

Let $\alpha_1, \ldots, \alpha_6$ be the usual indexing of simple roots, for the system E_6. In particular, the root α_4 is the branch point. The involution of the Dynkin diagram σ fixes α_2 and α_4, and

$$\sigma(\alpha_1) = \alpha_6 \text{ and } \sigma(\alpha_3) = \alpha_5.$$

The simple roots of the restricted root system, which is of type F_4, are two long roots, α_2 and α_4, and two short roots, β_3 and β_1, obtained by restricting α_3 and

α_1 to $\mathfrak{a}$, respectively. Let W_σ be the subgroup of W commuting with σ. Then W_σ is the Weyl group of the reduced root system F_4. In order to show that

$$\epsilon + \mathfrak{a}^* \cap w(\epsilon' + \mathfrak{a}^*)$$

contains a Zariski-dense set of integral weights for any pair of $\mathfrak{t}$-integral weights ϵ and ϵ', it suffices to do so for only one w in every W_σ-double coset in W. Claim:

There are only three W_σ double cosets in W: the trivial one, and two others represented by the elements

$$s_{\alpha_1} \text{ and } s_{\alpha_1} s_{\alpha_3}.$$

In order to verify this claim, we shall count the single cosets in each double coset. The total number of single W_σ-cosets in W is equal to $|W|/|W_\sigma| = 45$. The number of single cosets in a double coset $W_\sigma w W_\sigma$ is equal to the index of $wW_\sigma w^{-1} \cap W_\sigma$ in W_σ. The group $wW_\sigma w^{-1} \cap W_\sigma$ is easily calculated in each of the two cases since it is a subgroup of W_σ preserving

$$w(\mathfrak{a}^*) \cap \mathfrak{a}^*.$$

If $w = s_{\alpha_1}$, this space is spanned by restricted roots perpendicular to β_1. These roots form a subsystem B_3. If $w = s_{\alpha_1} s_{\alpha_3}$, then the intersection is perpendicular to β_1 and β_2, so it is spanned by a subsystem A_2 consisting of long roots. Thus, it follows that $wW_\sigma w^{-1} \cap W_\sigma$ is equal to $W(B_3) \times W(A_1)$ and $W(A_2) \times W(A_2)$ in the two cases, respectively. The indices of the two groups in W_σ are 12 and 32, which adds to 44. This proves the claim. Thus in order to verify Lemma 31, it suffices to do so for $w = s_{\alpha_1}$ and $s_{\alpha_1} s_{\alpha_3}$. We leave the details to the reader.

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