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SMALL UNITARY REPRESENTATIONS OF THE DOUBLE COVER OF SL(m)

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ABSTRACT. The irreducible unitary representations of the double cover $\widetilde{\mathrm{SL}(m)}$ of the real group $\mathrm{SL}(m)$, with infinitesimal character $\frac{1}{2}\rho$, which are *small* in the sense that their annihilator in the universal enveloping algebra is maximal, are expressed as Langlands quotients of generalized principal series. In the case where m is even we show that there are four such representations and in the case where m is odd there is just one. The representations' smallness allows them to be written as a sum of virtual representations, leading to a character formula for their K-types. We investigate the place of these small representations in the orbit method and, in the case of $\widetilde{\mathrm{SL}(2l+1)}$, show that the representation is attached to a nilpotent coadjoint orbit.The K-type spectrum for the Langlands quotients is explicitly determined and shown to be multiplicity free.

1. INTRODUCTION

In the late 1960's it was naively hoped that the unitary representations of the real reductive Lie groups could be more of less completely classified by means of the orbit picture of Kirillov and Kostant (see [1] and [2]). For a real reductive group G, the philosophy of coadjoint orbits in its purest form suggests that unitary representations of G are closely related to the orbits of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. Another form of the correspondence from orbits to representations, due mostly to Duflo [3] (and the approach we shall follow), says that for G a reductive Lie group having maximal compact subgroup K with complexification $K_{\mathbb{C}}$, that a unitary representation of G, attached to a nilpotent orbit, corresponds with a $K_{\mathbb{C}}$ admissible orbit datum. A number of celebrated counter-examples showed that this hope was unduly optimistic. To mention one, the same unitary representation may arise from each of several admissible orbit data. Indeed, Torasso observed that for the double cover of SL(3) the orbit method overestimates the number of irreducible representations attached to an orbit [4].

When it comes to unipotent representations, the orbit method reaches its limit. As a step towards better understanding the situation, so we may eventually repair the method, this paper carries out a thorough study of small irreducible unitary representations of the nonlinear group $\widetilde{\mathrm{SL}(m)}$, the double cover of $\mathrm{SL}(m,\mathbb{R})$. The class of representations of $\widetilde{\mathrm{SL}(m)}$ we study are characterized by three properties.

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First, the representation must be *genuine* and not factor to the linear group. Second, the representation must be *small* in the sense that it has a maximal primitive ideal. Third, the representation should have infinitesimal character $\frac{1}{2}\rho$, where ρ is half the sum of the positive roots. The infinitesimal character $\frac{1}{2}\rho$ was chosen so that the generalized principal series of $\widetilde{\mathrm{SL}(m)}$ would be reducible [5]. This seemed like a good place to look for small representations. These representations form a finite set which we explicitly describe. They are known to be unitary (see theorem 4.2 of [6]) and hence are of interest. In the case of $\widetilde{\mathrm{SL}(2)}$, one of the representations is the metaplectic or quantum harmonic oscillator representation from mathematical physics.

Our small representations of $\widetilde{\mathrm{SL}(m)}$, with infinitesimal character $\frac{1}{2}\rho$, can be written as Langlands quotients of generalized principal series according to the Langlands classification theorem. There are four small Langlands quotients in the case of the double cover of $\mathrm{SL}(2l)$ and one small Langlands quotient in the case of the double cover of $\mathrm{SL}(2l+1)$. In section 2 we show:

Theorem 1. *Langlands quotients of* $\widetilde{\mathrm{SL}(2l)}$ *with infinitesimal character* $\frac{1}{2}\rho$ *having a maximal primitive ideal are* $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$, $J_{P_{max}}(\frac{1}{2}\rho)$, *and* $J_{P_{max}}(s_{(e_{2l-1}-e_{2l})}\frac{1}{2}\rho)$.

Theorem 2. *The Langlands quotient of* $\widetilde{\mathrm{SL}(2l+1)}$ *with infinitesimal character* $\frac{1}{2}\rho$ *having a maximal primitive ideal is* $J_{P_{min}}(\frac{1}{2}\rho)$.

In $\mathrm{SL}(2l)$ two of the Langlands quotients have a minimal parabolic and two have a maximal parabolic. For $\mathrm{SL}(2l+1)$ the small genuine Langlands quotient with infinitesimal character $\frac{1}{2}\rho$ is the quotient of a principal series representation. Adams and Huang proved an analogous result using Kazhdan-Patterson lifting for $\mathrm{GL}(m)$[7]. They showed that at infinitesimal character $\frac{1}{2}\rho$, there are two small irreducible representations of $\widetilde{\mathrm{GL}(2l)}$, and one small irreducible representation of $\widetilde{\mathrm{GL}(2l+1)}$.

In section 3 we determine the place of our small representations of $\widetilde{\mathrm{SL}(m)}$ in the orbit method. First, we identify the nilpotent orbit the representations are attached to. We find:

Theorem 3. *There are* 4 *genuine admissible* $\mathrm{Spin}(2l,\mathbb{C})$ *orbit data, and* 2 *genuine admissible* $\mathrm{Spin}(2l+1,\mathbb{C})$ *orbit data for* $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$.

The theorem supports the observation of Torasso mentioned above by showing that the orbit method double counts the number of small representations of $\widetilde{\mathrm{SL}(2l+1)}$ having infinitesimal character $\frac{1}{2}\rho$.

Predicting the K-types of the irreducible representations associated with a given orbit is an important special aspect of the orbit method program. Because compact groups are well understood, the structure of an infinite dimensional representation is often better understood by restricting the representation to a maximal compact subgroup. The orbit method conjectures that under certain conditions, the representations attached to the orbit may be realized as algebraic sections of an algebraic vector bundle. The conditions are satisfied for the orbit in the case of $\widetilde{\mathrm{SL}(2l+1)}$ and we explicitly determine the K types of the algebraic representation.

In section 4 we prove a character formula for Langlands quotients, of $\widetilde{\mathrm{SL}(m)}$, with infinitesimal character $\frac{1}{2}\rho$, having a maximal primitive ideal. We write the Langlands quotients as a sum of much simpler virtual representations. The K-types of these virtual representations can be explicitly determined (at least in theory) by Blattner's formula. Our computation of the character of small representations as a combination of coherent continuation of fundamental series characters agrees with Adams' result for the metaplectic representation [7].

Finally, we explicitly determine the K-type spectrum of our small representations of $\widetilde{\mathrm{SL}(m)}$ with infinitesimal character $\frac{1}{2}\rho$. Our main result shows that the K-types for the small Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ are multiplicity free:

Theorem 4. *The Langlands quotients with minimal parabolic, $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2l)}$ have K-types*

$$(\frac{1}{2}+2a_1,\dots,\frac{1}{2}+2a_{l-1},\pm(\frac{1}{2}+2a_l))$$

where $a_1 \geq a_2 \geq \cdots \geq a_l \geq 0$.

The Langlands quotients with maximal cuspidal parabolic, $J_{P_{max}}(\frac{1}{2}\rho)$ and $J_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2l)}$, have K-types

$$(\frac{3}{2}+2a_1,\dots,\frac{3}{2}+2a_{l-1},\pm(\frac{3}{2}+2a_l))$$

where $a_1 \geq a_2 \geq \cdots \geq a_l \geq 0$.

The Langlands quotient with minimal parabolic, $J_{P_{min}}(\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2l+1)}$ have K-types

$$(\frac{1}{2}+2a_1,\dots,\frac{1}{2}+2a_l)$$

where $a_1 \geq a_2 \geq \cdots \geq a_l \geq 0$.

The proof of the K-types for $\widetilde{\mathrm{SL}(2l)}$ involves realizing our representations as cohomologically induced representations of type $A_{\mathfrak{q}}(\lambda)$. To find the K-types for $\widetilde{\mathrm{SL}(2l+1)}$ involves a novel strategy where the small representation is expressed as an alternating sum of parabolically induced representations.

2. Langlands quotients for $\widetilde{\mathrm{SL}(m)}$

In this section we explicitly determine the Langlands quotients for $\widetilde{\mathrm{SL}(m)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. Unless otherwise specified G will be a connected real semisimple Lie group with maximal compact subgroup K, a Θ-invariant subgroup H, and complexified Lie algebra $\mathfrak{g}$. We write the Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$.

2.1. **The group $\widetilde{\mathrm{SL}(m)}$.** Let $\mathrm{SL}(m)$ be the group of determinant one real $m \times m$ matrices. The fundamental group of $\mathrm{SL}(m)$, $\pi_1(\mathrm{SL}(m))$, is $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ depending on whether $m = 2$ or $m \geq 3$ respectively. The covers of $\mathrm{SL}(m)$ are the quotients of the universal cover $\mathrm{SL}(m)$ by subgroups of $\pi_1(\mathrm{SL}(m))$. We note that all such subgroups are central and hence normal in the universal cover so the quotient is a group. Because $\pi_1(\mathrm{SL}(m))$ has a unique subgroup of index 2, $\mathrm{SL}(m)$ has a unique connected double cover which we denote by $\widetilde{\mathrm{SL}(m)}$.

The subgroup $K = \mathrm{SO}(m)$ in the Iwasawa decomposition contains all of the nontrivial topology of $\mathrm{SL}(m)$. Identifying A and N with one of their two leaves in $\widetilde{KAN}$ allows us to write $\widetilde{K}AN$ as the double cover of $\mathrm{SL}(m)$. The double cover of $\widetilde{K}$ is the group $\mathrm{Spin}(m)$.

2.2. **The genuine discrete series of $\widetilde{M}$.** To write representations of $\widetilde{\mathrm{SL}(m)}$ as Langlands quotients of generalized principal series we first determine what are the isomorphism classes of genuine discrete series for the subgroups M of cuspidal parabolics $P = MAN$.

2.2.1. *The cuspidal parabolic subgroups of* $\widetilde{\mathrm{SL}(m)}$. The Langlands decomposition describes a parabolic subgroup, P, of $\mathrm{SL}(m)$ as a product $P = MAN$. Here $A = \exp \mathfrak{a}$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{s}$. N is the unipotent radical of P, and the Levi subgroup, MA, is the centralizer of A in G. The double cover of the parabolic is written as $\widetilde{P} = \widetilde{M}AN$.

The parabolic subgroups of $\mathrm{SL}(m)$ are in one to one correspondence with the set of subsets of simple roots of $\mathfrak{sl}(m)$. The subgroup M for each parabolic consists of blocks of determinant plus or minus one, $\mathrm{SL}(n_1)^{\pm}, \mathrm{SL}(n_2)^{\pm}, \ldots, \mathrm{SL}(n_r)^{\pm}$ along the diagonal of $\mathrm{SL}(m)$, such that $n_1 + \cdots + n_r = m$ and M has determinant one. A parabolic subgroup P is *cuspidal* if M in the Langlands decomposition of P contains a compact Cartan subgroup. Since M is a product of $\mathrm{SL}(n_i)^{\pm}$, each n_i must be less than or equal to two in that case. Here we are using the fact that $\mathrm{SL}(n)$ has a compact Cartan subgroup iff n equals 1 or 2. We will denote the minimal parabolic, where $n_i = 1$ for all i, by P_{min}.

Next we describe $\widetilde{M}$ for the cuspidal parabolics.

Lemma 2.1. *Let $\widetilde{M_{min}}$ denote the group $\widetilde{M}$ for the minimal parabolic of* $\widetilde{\mathrm{SL}(m)}$. *Then $\widetilde{M_{min}}$ is a finite group of order 2^m equal to all even monomials in* $\mathrm{Spin}(m)$.

Proof. The group M consists of ± 1 along the diagonal with an even number of signs. Hence the order of M is 2^{m-1}. The preimage of a pair of negative signs in the i^{th} and j^{th} diagonal entries are $\pm e_i e_j$ in $\mathrm{Spin}(m)$. Here $e_1, \ldots, e_m$ is the standard basis for $\mathbb{R}^m$. The claim follows. □

Lemma 2.2. *Let -1 be the nontrivial element in the kernel of the projection homomorphism of* $\widetilde{\mathrm{SL}(2)}$ *to* $\mathrm{SL}(2)$. *Let M_o be the identity component of M for a nonminimal parabolic consisting of b* $\mathrm{SL}(2)^{\pm}$ *diagonal blocks in* $\widetilde{\mathrm{SL}(m)}$. *Then $\widetilde{M_o} = (\widetilde{\mathrm{SL}(2)})^b/A$, where A is the subgroup of $\{\pm 1\}^b$ having elements $(x_1, \ldots, x_b)$, with $\Pi x_i = 1$. Furthermore, assuming the* $\mathrm{SL}(2)^{\pm}$ *diagonal blocks in M are consecutive from the top, then $\widetilde{M}$ consists of 2^{m-b-1} cosets of $\widetilde{M_o}$.*

Proof. In M_o each $\mathrm{SL}(2)^{\pm}$ block has determinant one and all other diagonal entries are one. For $(\widetilde{\mathrm{SL}(2)})^b$ to be a double cover of $\mathrm{SL}(2)^b$ requires us to modulo out by the subgroup of $\{\pm 1\}^b$ of elements $(x_1, \ldots, x_b)$, with $\Pi x_i = 1$. This is isomorphic to the double cover of M_0.

Assume the $SL(2)^{\pm}$ diagonal blocks in M are consecutive from the top. The group M is the product of M_0 with the set of diagonal matrices with ± 1 along the diagonal having an even number of signs. Because minus the identity is in $\mathrm{SL}(2)$ we may choose the first b odd diagonal entries to equal 1. These 2^{m-b-1}

diagonal matrices lift to the monomials e_o together with monomials $e_{i_1}e_{i_2}\dots e_{i_{2k}}$, having $i_j \in \{2,4,\dots,2b,2b+1,2b+2,\dots,m\}$ and $i_1 < \dots < i_{2k}$ in $\mathrm{Spin}(m)$. We don't include the monomials with coefficient -1 since the kernel of the projection homomorphism of $\widetilde{M}$ to M, namely ± 1, is in $\widetilde{M}_0$. □

2.2.2. *The discrete series of* $\widetilde{\mathrm{SL}(2)}$. Harish-Chandra showed that a connected semisimple Lie group has a nonempty discrete series exactly when it contains a compact Cartan subgroup. Harish Chandra parameterized the discrete series by $\lambda \in \mathfrak{t}^*$, dominant with respect to a positive imaginary root system $\Delta^+(\mathfrak{g},\mathfrak{t})$. We will denote the discrete series with Harish Chandra parameter λ by δ_λ.

The unimodular group $\widetilde{\mathrm{SL}(2)}$ has a compact Cartan subgroup, $\mathrm{Spin}(2)$, and hence a discrete series δ_λ. Letting $\Delta^+(\mathfrak{g},\mathfrak{t}) = e_1 - e_2$ and $\lambda = \frac{n}{4}(e_1 - e_2) = (\frac{n}{4}, -\frac{n}{4})$, for a positive integer n, the discrete series $\delta_{(\frac{n}{4},-\frac{n}{4})}$ has the lowest $\widetilde{K}$ type $\frac{n}{2}+1$. With the opposite choice of positive imaginary roots, the discrete series $\delta_{(-\frac{n}{4},\frac{n}{4})}$ has the lowest $\widetilde{K}$ type $-\frac{n}{2}-1$.

2.2.3. *The genuine discrete series of* $\widetilde{M_{min}}$ *for minimal parabolics.* The discrete series of $\widetilde{M}$, associated with cuspidal parabolics, which do not descend to discrete series of the linear group M are called *genuine*. Genuine discrete series send the nontrivial element of the projection homomorphism $\widetilde{M} \longrightarrow M$ to -1 (i.e. $\delta(-1) = -1$). First, we will describe the genuine discrete series for $\widetilde{M_{min}}$, associated with the minimal parabolic. The group is finite and the discrete series are just the finite group representations of $\widetilde{M_{min}}$.

Proposition 2.3. *For* $\widetilde{\mathrm{SL}(2n+1)}$ *there are* 2^{2n} *one-dimensional nongenuine representations and one* 2^n*-dimensional genuine irreducible representation of* $\widetilde{M_{min}}$.

Proof. By Lemma 2.1 the order of $\widetilde{M_{min}}$ is 2^{2n+1}. The group $\widetilde{M_{min}}$ consists of all monomials in $\mathrm{Spin}(2n+1)$. The center of this group is $\pm e_0$ and it is not difficult to verify that there are $2^{2n}+1$ conjugacy classes consisting of each element of the center and $\pm$ each monomial $e_{i_1}\dots e_{i_{2k}}$ with $i_1 < \dots < i_{2k}$ in $\mathrm{Spin}(2n+1)$. By a standard result on finite group representations we conclude that there are $2^{2n}+1$ irreducible representations of $\widetilde{M_{min}}$ [22]. The group M_{min} is an abelian group of order 2^{2n} and so it has 2^{2n} one-dimensional representations. These representations lift to representations of $\widetilde{M_{min}}$. There is only one way to write 2^{2n+1} as a sum of $2^{2n}+1$ squares where 2^{2n} of the squares are the square of one, namely $2^{2n+1} = 1^2 + \dots + 1^2 + (2^n)^2$. It follows that $\widetilde{M_{min}}$ has 2^{2n} one-dimensional representations and one 2^n-dimensional irreducible representation. The 2^n-dimensional irreducible representation doesn't descend to a representation of M and hence must be genuine. The one-dimensional representations do descend and are not genuine. □

The corresponding result for $\widetilde{\mathrm{SL}(2n)}$ is analogous.

Proposition 2.4. *For* $\widetilde{\mathrm{SL}(2n)}$ *there are* 2^{2n-1} *nongenuine one-dimensional representations and two* 2^{n-1}*-dimensional genuine irreducible representations of* $\widetilde{M_{min}}$.

We will need to be able to distinguish the two genuine irreducible representations of $\widetilde{M_{min}}$. This is achieved below.

Proposition 2.5. *The two genuine irreducible representations of* $\widetilde{M_{min}}$ *in* $\widetilde{\mathrm{SL}(2n)}$ *are distinguished by their restriction to the center of* $\widetilde{M_{min}}$.

Proof. The center $\widetilde{Z}$ of $\widetilde{M_{min}}$ consists of the four element abelian group $\{\pm e_0, \pm e_1 \cdot \cdots \cdot e_{2n}\}$. Here e_0 is identity of $\mathrm{Spin}(m)$. There are two genuine irreducible representations, ξ_c^+, ξ_c^-, of $\widetilde{Z}$ sending $e_N = e_1 \cdots \cdot e_{2n}$ to $(\sqrt{-1})^n$ and $-(\sqrt{-1})^n$ respectively. The representations ξ_c^+ and ξ_c^- can be extended to irreducible genuine representations ξ^+ and ξ^- respectively (for example by taking an irreducible component of the induced representation). Because e_N is central in $\widetilde{M_{min}}$, by Schur's lemma, $\xi^+(e_N)$ and $\xi^-(e_N)$ are a scalar times the identity with scalar equal to $\xi_c^+(e_N)$ and $\xi_c^-(e_N)$ respectively. The representations ξ^+ and ξ^- are not isomorphic since their restrictions to $\widetilde{Z}$ are not isomorphic. By Proposition 2.4 there are exactly two genuine irreducible representations of $\widetilde{M_{min}}$, and so these representations must be ξ^+ and ξ^-. □

In the sequel we will write $\xi^\pm$ for the two genuine irreducible representations of $\widetilde{M_{min}}$ in $\widetilde{\mathrm{SL}(2n)}$. We will also sometimes write $\xi^\pm$ for the single genuine irreducible representations of $\widetilde{M_{min}}$ in $\widetilde{\mathrm{SL}(2n+1)}$ with the understanding that $\xi^+ = \xi^-$.

2.2.4. *The genuine discrete series of* $\widetilde{M}$ *for nonminimal cuspidal parabolics.* We will first establish what are the genuine discrete series of $\widetilde{\mathrm{SL}(2)}$. We may write $\mathrm{Spin}(2)$ as $\cos(\frac{t}{2}) + e_1e_2\sin(\frac{t}{2})$ for $0 \le t \le 4\pi$. The discrete series restricted to $\mathrm{Spin}(2)$, $\delta_{\pm(\frac{n}{4},-\frac{n}{4})}(\cos(\frac{t}{2}) + e_1e_2\sin(\frac{t}{2}))$, have $\widetilde{K}$ types $e^{\pm\frac{(n+2)it}{2}}, e^{\pm\frac{(n+6)it}{2}}, \ldots$. Evaluating the discrete series at -1 amounts to setting $t = 2\pi$. Then $\delta_{\pm(\frac{n}{4},-\frac{n}{4})}(-1)$ equals $e^{\pm(2+n)i\pi}, e^{\pm(n+6)i\pi}, \ldots$. It follows that n must be odd for $\delta_{\pm(\frac{n}{4},-\frac{n}{4})}$ to be genuine.

By Lemma 2.2, $\widetilde{M_o} = (\widetilde{\mathrm{SL}(2)})^b/A$ where A is the subgroup of $\{\pm 1\}^b$ having elements $(x_1, \ldots, x_b)$, with $\Pi x_i = 1$. For a discrete series of $\widetilde{M_o}$ to be genuine it should be genuine for each $\widetilde{\mathrm{SL}(2)}$ in the product, thereby sending the subgroup A to 1. A discrete series of the product of $\widetilde{\mathrm{SL}(2)}$ is the product of discrete series for each $\widetilde{\mathrm{SL}(2)}$. The genuine discrete series for the identity component of $\widetilde{M}$ has Harish-Chandra parameter $\lambda = (\frac{2l_1+1}{4}, -\frac{2l_1+1}{4}, \frac{2l_2+1}{4}, -\frac{2l_2+1}{4}, \ldots)$, for integers l_i.

Let $Z_{\widetilde{M}}(\widetilde{M_o})$ be the centralizer of $\widetilde{M_0}$ in $\widetilde{M}$ for $\widetilde{\mathrm{SL}(m)}$, and $Z(\widetilde{M_o}) = Z_{\widetilde{M}}(\widetilde{M}) \cap \widetilde{M_o}$. We extend $\delta_\lambda|_{Z(\widetilde{M_o})}$ to a representation of $Z_{\widetilde{M}}(\widetilde{M_o})$. In the case where $m = 2n$, by a statement analogous to Proposition 2.5 there are precisely two genuine irreducible representations of $Z_{\widetilde{M}}(\widetilde{M_o})$ distinguished by their evaluation at the central element $e_N = (e_1 \cdot \cdots \cdot e_m)$ in $Z_{\widetilde{M}}(\widetilde{M_o})$. Although these representations are defined in terms of δ_λ we abuse notation and write these two representations of $Z_{\widetilde{M}}(\widetilde{M_o})$ as $\xi^\pm$ where $\xi^+(e_N) = \sqrt{-1}^{\,l}$ and $\xi^-(e_N) = -\sqrt{-1}^{\,l}$. In the case where $m = 2n+1$ by a statement analogous to Proposition 2.3 there is one genuine irreducible representation of $Z_{\widetilde{M}}(\widetilde{M_o})$ extending $\delta_\lambda|_{Z(\widetilde{M_o})}$. For convenience, we will denote this representation by $\xi^\pm$ with the understanding that $\xi^+ = \xi^-$.

We are now ready to describe the genuine discrete series for the disconnected group $\widetilde{M}$. Let δ_λ be a genuine discrete series for $\widetilde{M_0}$ and let $\xi^\pm$ be a genuine representation of $Z_{\widetilde{M}}(\widetilde{M_o})$. Then $\xi^\pm \otimes \delta_\lambda$ is a genuine discrete series of $Z_{\widetilde{M}}(\widetilde{M_o}) \cdot \widetilde{M_0}$.

We denote the genuine discrete series of $\widetilde{M}$ by $\delta_\lambda^\pm$ where

$$\delta_\lambda^\pm = \mathrm{Ind}_{Z_{\widetilde{M}}(\widetilde{M_o})\cdot\widetilde{M_0}}^{\widetilde{M}} \xi^\pm \otimes \delta_\lambda.$$

2.2.5. *Isomorphisms among the genuine discrete series of $\widetilde{M}$ for nonminimal cuspidal parabolics.* Due to the disconnectedness of $\widetilde{M}$ for nonminimal cuspidal parabolics there are isomorphisms among the genuine discrete series to which we need to be aware. In what follows let b be the number of $\widetilde{SL(2)}$ blocks in $\widetilde{M}$. The Harish-Chandra parameter

$$\lambda = (\frac{2l_1+1}{4}, -\frac{2l_1+1}{4}, \frac{2l_2+1}{4}, -\frac{2l_2+1}{4}, \ldots, \frac{2l_b+1}{4}, -\frac{2l_b+1}{4}),$$

where l_i is an integer, will be abbreviated $(\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_b)$ where λ_i is the positive half integer $|l_i + \frac{1}{2}|$. The genuine discrete series representations of $\widetilde{M}$ in $\widetilde{SL(m)}$ are $\delta^\pm_{(\pm\lambda_1,\pm\lambda_2,\ldots,\pm\lambda_b)} = \mathrm{Ind}_{Z_{\widetilde{M}}(\widetilde{M_o})\cdot\widetilde{M_0}}^{\widetilde{M}} \xi^\pm \otimes \delta_{(\pm\lambda_1,\pm\lambda_2,\ldots,\pm\lambda_b)}$.

First a general isomorphism statement about induced spaces.

Lemma 2.6. *Let $H \subset G$ be a subgroup of G and let π be a representation of H. Let $g \in N_G^H$, the normalizer of H in G. Then $\mathrm{Ind}_H^G \pi \cong \mathrm{Ind}_H^G g\cdot\pi$ where $(g\cdot\pi)(h) = \pi(ghg^{-1})$.*

Proof. By definition, $\mathrm{Ind}_H^G\pi$ is the space of functions on G which transform as $f(xh) = \pi(h^{-1})f(x)$ for $x \in G$ and $h \in H$. Our isomorphism will take a function f from $\mathrm{Ind}_H^G\pi$ and send it to a function f' from $\mathrm{Ind}_H^G g\cdot\pi$ where $f'(x) = f(xg^{-1})$. To show that this function is well defined note that $f'(xh) = f(xhg^{-1}) = f(xg^{-1}(ghg^{-1}))$. Since g is in the normalizer of H, $ghg^{-1} \in H$ and $f(xg^{-1}(ghg^{-1})) = \pi(gh^{-1}g^{-1})f(xg^{-1})$ which equals $(g\cdot\pi)(h^{-1})f'(x)$ as required. All steps in the above derivation are reversible so our map is a bijection. Furthermore right translation commutes with left translation by g so our map is an isomorphism. ☐

To apply Lemma 2.6 we will let $\pi = \xi^\pm \otimes \delta_{(\lambda_1,\ldots,\lambda_b)}$, the subgroup $H = Z_{\widetilde{M}}(\widetilde{M_o})\cdot\widetilde{M_0}$, and $G = \widetilde{M}$. We will need to find elements $g \in N_G^H$ and understand how they change π under conjugation.

To find the isomorphisms of genuine discrete series of $\widetilde{M}$, for a nonminimal parabolic, it is enough to consider the case when the b diagonal SL(2) blocks in M are consecutive from the top left.

Let

$J_{ev} = \{j_1, \ldots, j_{2r}\}$ be a subset of $\{2, 4, \ldots, 2b\}$,
$K_{ev} = \{k_1, \ldots, k_{b-2r}\}$ be the set $\{2, 4, \ldots, 2b\}\backslash\{j_1, \ldots, j_{2r}\}$,
$J_{odd} = \{j_1, \ldots, j_{2r-1}\}$ be a subset of $\{2, 4, \ldots, 2b\}$,
$K_{odd} = \{k_1, \ldots, k_{b-(2r-1)}\}$ be the set $\{2, 4, \ldots, 2b\}\backslash\{j_1, \ldots, j_{2r-1}\}$.

Here K_{ev} and K_{odd} are the remainder of J_{ev} and J_{odd}, respectively, in the set $\{2, 4, \ldots, 2b\}$.

Let

$$e_{J_{ev}} = e_{j_1} \ldots e_{j_{2r}},$$
$$e_{K_{ev}} = e_{k_1} \ldots e_{j_{b-2r}},$$
$$e_{J_{odd}} = e_{j_1} \ldots e_{j_{2r-1}},$$
$$e_{K_{odd}} = e_{k_1} \ldots e_{k_{b-(2r-1)}}.$$

Let $t_i = \cos(\frac{\theta_i}{2}) + e_{i-1}e_i \sin(\frac{\theta_i}{2})$ in $\mathrm{Spin}(m)$ for $1 \leq i \leq 2b$ and $0 \leq \theta_i \leq 4\pi$ and write

$$\begin{aligned} t_{J_{ev}} &= t_{j_1} \dots t_{i_{2r}}, \\ t_{K_{ev}} &= t_{k_1} \dots t_{j_{2b-2r}}, \\ t_{J_{odd}} &= t_{j_1} \dots t_{j_{2r-1}}, \\ t_{K_{odd}} &= t_{k_1} \dots t_{k_{2b-2r-1}}. \end{aligned}$$

To further simplify notation we will write the Harish-Chandra parameter $(\lambda_1, \dots, \lambda_b)$ as $(\lambda_{J_{ev}}, \lambda_{K_{ev}})$ or $(\lambda_{J_{odd}}, \lambda_{K_{odd}})$.

Lemma 2.7. *Let b be the number of copies of* $\mathrm{SL}(2)$ *in M_o inside* $\mathrm{SL}(m)$. *The result of conjugating the genuine irreducible representations of $Z_{\widetilde{M}}(\widetilde{M_o}) \cdot \widetilde{M_0}$ by $e_{J_{ev}}$ and by $e_{J_{odd}}e_{2b+1}$, when $2b < m$, is:*

$$e_{J_{ev}} \cdot (\xi^{\pm} \otimes \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}) = \xi^{\pm} \otimes \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})}$$

and

$$e_{J_{odd}}e_{2b+1} \cdot (\xi^{\pm} \otimes \delta_{(\lambda_{J_{odd}}, \lambda_{K_{odd}})}) = \xi^{\pm} \otimes \delta_{(-\lambda_{J_{odd}}, \lambda_{K_{odd}})} \quad (2b < m).$$

In the case where m is odd or m=2b, $\xi^+ = \xi^-$.

Proof. We first consider the action of $e_{J_{ev}}$ and $e_{J_{odd}}e_{2b+1}$ on the side of the tensor product involving the discrete series of M_o. We have $(e_{J_{ev}} \cdot \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})})(t_{J_{ev}}t_{K_{ev}})$ equals $\delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}(t_{J_{ev}}^{-1}t_{K_{ev}}) = \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})})(t_{J_{ev}}t_{K_{ev}})$. A similar statement holds for the action of $e_{J_{odd}}e_{2b+1}$.

The action of $e_{J_{ev}}$ and $e_{J_{odd}}e_{2b+1}$ on $\xi^{\pm}$ sends ξ^+ to itself and ξ^- to itself since $\xi^{\pm}$ are distinguished by a central element. Note that because $\xi^{\pm}$ was defined as an extension of $\delta_\lambda|_{Z(M_o)}$ the $\xi^{\pm}$ in $\xi^{\pm} \otimes \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}$ is different from the $\xi^{\pm}$ in $\xi^{\pm} \otimes \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})}$, etc. We apologize for the notation.

When m is odd then there is just a single extension of $\delta_\lambda|_{Z(M_o)}$ to $Z(M_o) = Z_M(M_o)$. In the case where $m = 2b$, $Z(M_o) = Z_M(M_o)$ so no extension is necessary. □

From Lemmas 2.6 and 2.7, we immediately have the following isomorphism statement for genuine discrete series of $\widetilde{M}$ for nonminimal cuspidal parabolics.

Proposition 2.8. *Let b be the number of copies of $\widetilde{\mathrm{SL}(2)}$ in $\widetilde{M}$ inside $\widetilde{\mathrm{SL}(m)}$. If $(\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_b)$ has an even number of negative signs or $2b < m$, then*

$$\delta^{\xi^{\pm}}_{(\lambda_1, \dots, \lambda_b)} \cong \delta^{\xi^{\pm}}_{(\pm\lambda_1, \dots, \pm\lambda_b)}.$$

In the case where m is odd or $m = 2b$, $\xi^+ = \xi^-$.

2.3. **Langlands quotients with a maximal primitive ideal.** Let $P = MAN$ be a cuspidal parabolic of G, δ_λ a discrete series for M, and ν an irreducible character on A with $\mathrm{Re}\,\nu$ weakly dominant with respect to P. We will be concerned with the case where $G = \widetilde{\mathrm{SL}(m)}$, and δ_λ is a genuine discrete series representation of $\widetilde{M}$, and $\lambda + \nu = \frac{1}{2}\rho$. Recall that the *Langlands quotient*, $J_P(\delta_\lambda \otimes \nu)$, is the largest completely reducible quotient of $\mathrm{Ind}_P(\delta_\lambda \otimes \nu)$. In the case where P is a minimal parabolic, the Langlands quotient, $J_P(\delta \otimes \nu)$, is called a *principal series* representation. For nonminimal parabolics it is called a *principal, generalized series.*

The infinitesimal character of $J_P(\delta_\lambda \otimes \nu)$ is $\lambda + \nu$. If ν is weakly anti-dominant with respect to $\widetilde{P}$, $J_P(\delta \otimes \nu)$ is the maximal submodule of $\mathrm{Ind}_{\widetilde{P}}(\delta_\lambda \otimes \nu)$ and is

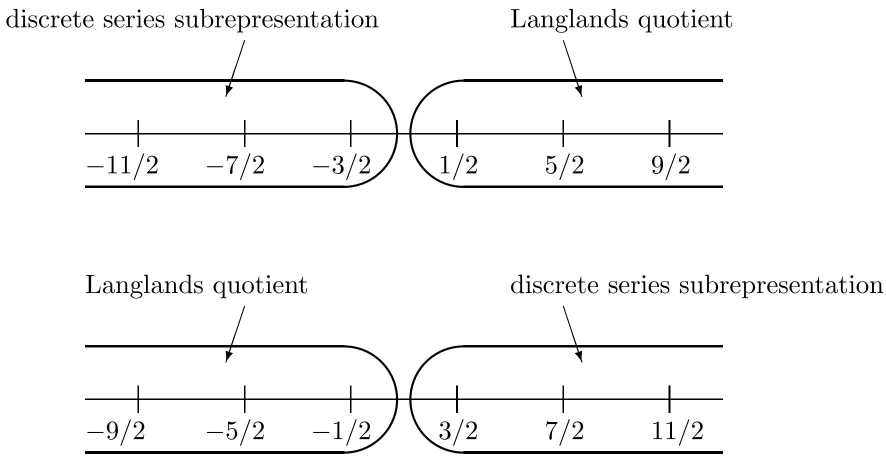

FIGURE 1. The four genuine irreducible representations of $\widetilde{\mathrm{SL}(2)}$ are shown inside of the principal series $\mathrm{Ind}_{P_{min}}(\xi^+ \otimes \frac{1}{2}\rho)$ and $\mathrm{Ind}_{P_{min}}(\xi^- \otimes \frac{1}{2}\rho)$.

called a *Langlands subrepresentation*. The parameter $\lambda \in (\mathfrak{m} \cap \mathfrak{k})^*$ and ν are called the *Harish-Chandra parameter*, and the *continuous parameter* respectively. These two pieces of data together, $\gamma = (\lambda, \nu)$, called the *Langlands parameter*, specify the Langlands quotient $J_P(\delta_\lambda \otimes \nu)$.

For the genuine discrete series with a nonminimal cuspidal parabolic, $\delta_\lambda^\pm$, we will often write the Langlands quotient as $J_P^\pm(\gamma)$. When $\xi^+ = \xi^-$ we write $J_P(\gamma)$. Note that λ doesn't distinguish between $\delta_\lambda^\pm$ so the Langlands parameter doesn't uniquely specify the Langlands quotient here.

Example 2.9. We provide an example of the Langlands quotients for $\widetilde{\mathrm{SL}(2)}$ with infinitesimal character $\frac{1}{2}\rho$. The group $\widetilde{\mathrm{SL}(2)}$ has two cuspidal parabolics, namely the minimal parabolic, $\widetilde{P_{min}}$, and the whole group, $\widetilde{P_{max}}$. Let $\xi^\pm$ be the two genuine irreducible representations of $\widetilde{M_{min}}$ and $\delta_{(\frac{1}{4},-\frac{1}{4})}$, $\delta_{(-\frac{1}{4},\frac{1}{4})}$ be the genuine irreducible representations of $\widetilde{M_{max}}$ with Harish-Chandra parameter $\pm\frac{1}{2}\rho$. There are four Langlands quotients of $\widetilde{\mathrm{SL}(2)}$ with infinitesimal character $\frac{1}{2}\rho$: the two quotients of principal series $J_{P_{min}}(\xi^+ \otimes \frac{1}{2}\rho)$ and $J_{P_{min}}(\xi^- \otimes \frac{1}{2}\rho)$, and two discrete series with Harish-Chandra parameter $\pm\frac{1}{2}\rho$. The two discrete series can be realized as subquotients in $J_{P_{min}}(\xi^\pm \otimes -\frac{1}{2}\rho)$. Figure 1 shows how the $\widetilde{K}$ types for the two principal series contain the $\widetilde{K}$ types of their Langlands quotient and submodule.

We wish to determine the Langlands quotients for $\widetilde{\mathrm{SL}(m)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal (i.e. where all simple integral roots with respect to $\frac{1}{2}\rho$ are τ invariants for our representation). Vogan's theorem on τ invariance (see Theorem 6.16 in [5] or Theorem 4.12 of [8]) provides a way to check that an integral root α is a τ invariant. However, one cannot apply the theorem directly

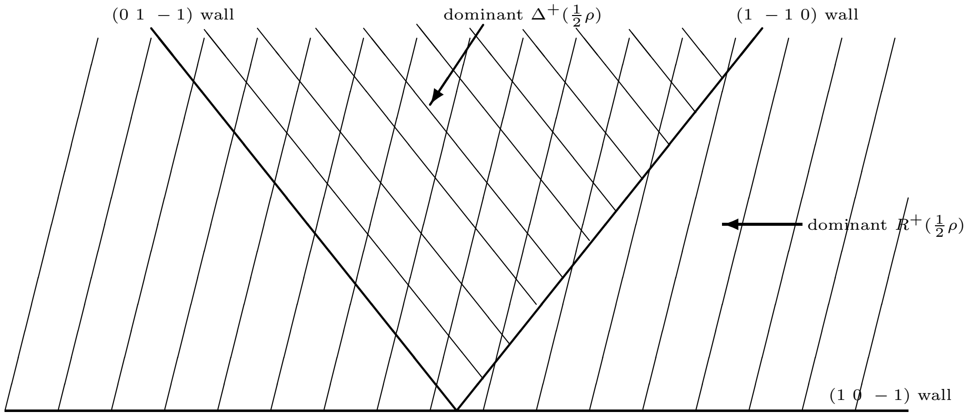

FIGURE 2. Dominant $\Delta^+(\frac{1}{2}\rho)$ and $R^+(\frac{1}{2}\rho)$ chambers in $\mathfrak{h}^*$ for $\mathfrak{sl}(3)$.

unless the integral root α is a simple root for $\Delta^+(\frac{1}{2}\rho)$. The relationship between the dominant $\Delta^+(\frac{1}{2}\rho)$ Weyl chamber and the dominant $R^+(\frac{1}{2}\rho)$ Weyl chamber in $\mathfrak{h}^*$ for $\mathfrak{sl}(3)$ is illustrated in Figure 2. Unfortunately for $\frac{1}{2}\rho = (\frac{m-1}{4}, \frac{m-3}{4}, \ldots, -\frac{m-1}{4})$ none of the integral roots of $R^+(\frac{1}{2}\rho)$ are simple roots in $\Delta^+(\frac{1}{2}\rho)$. We need to find an infinitesimal character $\frac{1}{2}\rho + \lambda$ in the dominant regular integral Weyl chamber based at $\frac{1}{2}\rho$ with respect to which all simple integral roots are simple in $\Delta^+(\frac{1}{2}\rho+\lambda)$. Here note that we are abusing notation by not differentiating the infinitesimal character with a member of its Weyl group orbit.

We propose to find the Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal, by first determining how many of them there are. It suffices to determine the number of Langlands quotients with maximal primitive ideal at an infinitesimal character in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber. This follows from the fact that irreducible members of a coherent family based at infinitesimal character $\frac{1}{2}\rho$ all have the same τ invariants.

Proposition 2.10. *Let $G = \widetilde{\mathrm{SL}(2l)}$, $l \geq 1$, and let γ be a Weyl group representative of the infinitesimal character $(p_{l-1}, p_{l-2}, \ldots, p_0, n_0, n_1, \ldots, n_{l-1})$, where*

$$p_0 = \tfrac{3}{4} + \tfrac{l}{2} \text{ and } p_i = p_0 + i,$$
$$n_0 = -\tfrac{3}{4} - \tfrac{l}{2} \text{ and } n_i = n_0 - i, \text{ for } i \in \{0, 1, \ldots, l-1\}.$$

The weight γ lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and the simple integral roots in $R^+(\gamma)$ are simple in $\Delta^+\gamma$.

Proof. γ is a representative of the infinitesimal character $\frac{1}{2}\rho + (l, -l, \ldots, l, -l)$. It is easy to check that p_i and n_i have the values given in the claim. The p_i are all positive and differ from one another by an integer. Similarly, the n_i are all negative and differ from one another by an integer. Because of the descending ordering $p_{l-1} > p_{l-2} > \cdots > p_0 > n_0 > \cdots > n_{l-1}$ of its entries, γ lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and $\{e_1 - e_2, \ldots, e_{l-1} - e_l, e_{l+1} - e_{l+2}, \ldots, e_{2l-1} - e_{2l}\}$ are simple integral roots for both $R^+(\gamma)$ and $\Delta^+(\gamma)$. □

The statement for $\widetilde{\mathrm{SL}(2l+1)}$ is less symmetric looking than Proposition 2.10 because of the odd number of entries in γ'. The proof however is no different than Proposition 2.10 except that γ' is a representative of the infinitesimal character $\frac{1}{2}\rho + (l, -(l+1), \dots, l, -(l+1), l)$.

Proposition 2.11. *Let $G = \widetilde{\mathrm{SL}(2l+1)}$, $l \geq 1$, and let γ' be a Weyl group representative of the infinitesimal character $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{l-1})$, where*

$$p'_0 = \tfrac{l}{2} \text{ and } p'_j = p'_0 + j \text{ for } j \in \{0, 1, \dots, l\},$$
$$n'_0 = -\tfrac{l+3}{2} \text{ and } n'_i = n'_0 - i, \text{ for } i \in \{0, 1, \dots, l-1\}.$$

The weight γ' lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and the simple integral roots in $R^+(\gamma')$ are simple in $\Delta^+\gamma'$.

For the purpose of counting the number of maximal primitive ideals we will study the Langlands quotients having infinitesimal character

$$(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$$

in the even case and

$$(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{l-1})$$

in the odd case. Below we will concentrate on $\widetilde{\mathrm{SL}(2l)}$ leaving the companion statements and proofs for $\widetilde{\mathrm{SL}(2l+1)}$ to the reader. The following technical lemma will help us determine which Weyl group representatives of an infinitesimal character can be Langlands parameters.

Lemma 2.12. *With n_i, p_i as in Proposition 2.10 the inequalities $n_i \geq \frac{p_j+n_k}{2}$ and $\frac{p_j+n_k}{2} \geq p_i$ are impossible for any $i, j, k \in \{0, 1, \dots, l-1\}$.*

Proof. To prove the first inequality it suffices to show that $\frac{p_0+n_{l-1}}{2} > n_0$ since $p_0+n_{l-1} = \min\{p_j+n_k : j,k \in \{0,\dots,l-1\}\}$ and $n_0 = \max\{n_i : i \in \{0,\dots,l-1\}\}$. Indeed, $\frac{p_0+n_{l-1}}{2} = \frac{p_0+n_0-l+1}{2} = \frac{-l+1}{2} > -\frac{2l+3}{4} = n_0$.

To prove the second inequality it suffices to show that $\frac{p_{l-1}+n_0}{2} < p_0$ since $p_{l-1} + n_0 = \max\{p_j + n_k : j, k \in \{0, \dots, l-1\}\}$ and $p_0 = \min\{p_i : i \in \{0, \dots, l-1\}\}$. Indeed, $\frac{p_{l-1}+n_0}{2} = \frac{p_0+l-1+n_0}{2} = \frac{l-1}{2} < \frac{2l+3}{4} = p_0$. □

We are now ready to count the number of Langlands quotients having maximal primitive ideal. We begin with a lemma about Langlands parameters for Langlands quotients with a cuspidal parabolic.

Lemma 2.13. *Let $(a_1, \dots, a_m)$ be a Langlands parameter of a genuine Langlands quotient with cuspidal parabolic $\widetilde{P} = \widetilde{M}AN$. If the linear group M has an $\mathrm{SL}(2)$ block along the $\{j, j+1\}$ diagonal entries, then $a_j - a_{j+1} \notin \mathbb{N}$. When immediately followed by another $\mathrm{SL}(2)$ block the Langlands parameter must satisfy $\frac{a_j+a_{j+1}}{2} \geq \frac{a_{j+2}+a_{j+3}}{2}$, otherwise $\frac{a_j+a_{j+1}}{2} \geq a_{j+2}$. If no $\mathrm{SL}(2)$ block begins or ends at the $\{j, j+1\}$ diagonal entries, then $a_j \geq a_{j+1}$.*

Proof. Suppose that M has an $\mathrm{SL}(2)$ block along the $\{j, j+1\}$ diagonal entries. Then entries j and $j+1$ of the Harish-Chandra parameter are

$$(\dots, \frac{a_j - a_{j+1}}{2}, -\frac{a_j - a_{j+1}}{2}, \dots)$$

and entries j and $j+1$ of the continuous parameter, ν, are

$$(\dots, \frac{a_j + a_{j+1}}{2}, \frac{a_j + a_{j+1}}{2}, \dots).$$

We must have $a_j - a_{j+1} \notin \mathbb{N}$ for our Harish-Chandra parameter to belong to a genuine discrete series. For ν to be the continuous parameter of a Langlands quotient it must be weakly dominant with respect to $\widetilde{P}$. As a result, we must have the $(j+2)$ entry of the continuous parameter be less than or equal to the $j+1$ entry. When our SL(2) block is immediately followed by another SL(2) block this translates to the condition that $\frac{a_j+a_{j+1}}{2} \geq \frac{a_{j+2}+a_{j+3}}{2}$. One can show similarly that if the SL(2) block along the $\{j, j+1\}$ diagonal entries is immediately preceded by an SL(2) block the Langlands parameter must satisfy $\frac{a_{j-2}+a_{j-1}}{2} \geq \frac{a_j+a_{j+1}}{2}$, otherwise $a_{j-1} \geq \frac{a_j+a_{j+1}}{2}$. When our SL(2) block is not immediately followed by another SL(2) block we have $\frac{a_j+a_{j+1}}{2} \geq a_{j+2}$. □

Proposition 2.14. *Let $G = \widetilde{\mathrm{SL}(2l)}$ and let $\widetilde{P} = \widetilde{M}AN$ be a cuspidal parabolic subgroup of G. With notation as in Proposition* 2.10 *let*

$$(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$$

be the infinitesimal character of a Langlands quotient with parabolic $\widetilde{P}$. The diagonal SL(2) *blocks in the linear group M must be consecutive and centered in the matrix. For the Langlands quotient to have a maximal primitive ideal, the number of blocks, b, must equal $l-1$ or l.*

Proof. Our Langlands parameter is in the Weyl group orbit of the infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$. By Lemma 2.13, for there to be a SL(2) block in the $\{j, j+1\}$ diagonal entry of M the difference between the j and $j+1$ entry of the Langlands parameter must be a half integer. It follows that one of the two entries must be positive and the other one negative. By Lemma 2.12, if the SL(2) block in M is not immediately followed by another SL(2) block, then the $j+2$ entry of the Langlands parameter must be negative. If there is another SL(2) block further down the diagonal of M, then we would have a situation where $n_i \geq \frac{p_j+n_k}{2}$ or $\frac{p_j+n_k}{2} \geq p_i$ for some $i, j, k \in \{0, 1, \dots, l-1\}$. This is impossible by Lemma 2.12. If the SL(2) block in the $\{j, j+1\}$ diagonal entry of M is not immediately preceded by another SL(2) block, then the $j-1$ entry of the Langlands parameter must be positive. Were there to be another SL(2) block further up the diagonal of M then we again would have a situation where $n_i \geq \frac{p_j+n_k}{2}$ or $\frac{p_j+n_k}{2} \geq p_i$ for some $i, j, k \in \{0, 1, \dots, l-1\}$. It follows that the SL(2) blocks must be consecutive in M.

Suppose the SL(2) blocks lie along the j through $j+2b-1$ diagonal entries in M. Each pair of entries between j and $j+2b+1$ in the Langlands parameter contains a positive and a negative entry. Furthermore, entries 1 through $j-1$ of the Langlands parameter are descending positive numbers while entries $j+2b+2$ through $2l$ are descending negative numbers. Since there are an equal number of positive and negative entries in the Langlands parameters, the SL(2) blocks must be centered in M (i.e. $b = l - j$).

Finally we show that to have a maximal primitive ideal, then b must equal $l-1$ or l. Indeed if this were not the case, then either a simple integral root would be a real root or the positive entry of the first SL(2) block would be greater than one of the preceding positive entries (for example $(p_2, p_0, p_1, n_1, n_0, n_2)$). By an immediate corollary of Vogan's τ invariance theorem a simple integral root cannot be real if the Langlands quotient has a maximal primitive idea (see Theorem 6.16 in [5] or Theorem 4.12 of [8]). As well, the second scenario is ruled out by an easy τ invariance computation. □

Proposition 2.15. *In the notation of Proposition* 2.10*, there are four Langlands quotients of* $\widetilde{\mathrm{SL}(2l)}$ *with infinitesimal character* $(p_{l-1}, p_{l-2}, \ldots, p_0, n_0, n_1, \ldots, n_{l-1})$ *having a maximal primitive ideal.*

Proof. By Proposition 2.14 there are only two subgroups $\widetilde{M}$ we need to consider, namely when the linear group M contains $b = l-1$ and $b = l$ diagonal SL(2) blocks centered in M. In the case where $b = l-1$ by Proposition 2.8 the genuine discrete series of $\widetilde{M}$ are, up to isomorphism, $\delta^{\xi^{\pm}}_{(\lambda_1,\ldots,\lambda_b)}$, where λ_i are positive half integers. In the case where $b = l$ by Proposition 2.8 the genuine discrete series of $\widetilde{M}$ are, up to isomorphism, $\delta^{\xi^{\pm}}_{(\lambda_1,\ldots,\lambda_b)}$ and $\delta^{\xi^{\pm}}_{(\lambda_1,\ldots,-\lambda_b)}$, where λ_i are positive half integers and $\xi^+ = \xi^-$.

We first prove the proposition for $\widetilde{\mathrm{SL}(4)}$. Let M contain one SL(2) block along the second and third diagonal entries. The Langlands parameters we are considering are permutations of (p_1, p_0, n_0, n_1). Because the Harish-Chandra parameter is a positive half integer, we must have a positive and a negative entry in the second and third entries respectively of the Langlands parameter. Furthermore, by Lemma 2.12, the first entry of the Langlands parameter must be positive and the fourth entry negative. Any permutation with these restrictions is a Langlands parameter for our parabolic. However requiring the simple integral roots to be τ invariant pins things down. The root, $e_1 - e_2$ is simple and integral for the Langlands parameter (p_1, p_0, n_0, n_1) and $\Theta(e_1 - e_2) = -e_1 + e_3 \in \Delta^-(\gamma)$. Also for the other simple integral root, $\Theta(e_3 - e_4) = -e_2 + e_4 \in \Delta^-(\gamma)$. Hence $e_1 - e_2$ and $e_3 - e_4$ are τ invariants and so the Langlands quotient with Langlands parameter (p_1, p_0, n_0, n_1) has a maximal primitive ideal. It is easy to check that the three other possible Langlands parameters don't have both simple roots as τ invariants. We have two genuine Langlands quotients with this Langlands parameter, $J^{\pm}_{\widetilde{P}_{b=1}}(p_1, p_0, n_0, n_1)$.

Next we consider the case for $\widetilde{\mathrm{SL}(4)}$ where M contains two SL(2) blocks. By Proposition 2.8, up to isomorphism of the discrete series of $\widetilde{M}$, the first and second entries of the Langlands parameter must be positive and negative respectively, but the third and fourth entries can be positive or negative in any order. There are four permutations of (p_1, n_0, p_0, n_1) where positive entries can be interchanged and negative entries can be interchanged. The simple integral roots are $\pm(e_1 - e_3)$ and $\pm(e_2 - e_4)$. Suppose that $e_1 - e_3$ is simple. Then $\Theta(e_1 - e_3) = -e_2 + e_4$. Hence for $e_1 - e_3$ to be a τ invariant, $e_2 + e_4$ must be a simple root. It follows that the Langlands quotient with Langlands parameter (p_1, n_0, p_0, n_1) has τ invariants $e_1 - e_3$ and $e_2 - e_4$. If on the other hand $e_3 - e_1$ is simple, then to be a τ invariant, $e_4 - e_2$ would have to be simple. However by Lemma 2.13, (p_0, n_1, p_1, n_0) isn't a Langlands parameter because $\frac{p_0+n_1}{2} < \frac{p_1+n_0}{2}$. Switching the third and fourth entries in the above argument shows (p_1, n_0, n_1, p_0) is also a Langlands parameter with simple integral roots which are τ invariant. The two resulting Langlands quotients with maximal primitive ideal are $J_{P_{b=2}}(p_1, n_0, p_0, n_1)$ and $J_{P_{b=2}}(p_1, n_0, n_1, p_0)$.

For $\widetilde{\mathrm{SL}(2l)}$, investigation of Langlands parameters whose simple integral roots are τ invariants does not require us to look at more than two integral roots at a time, and hence the above arguments directly generalize. The four Langlands quotients with a maximal primitive ideal are

$$J^{\pm}_{\widetilde{P_{b=l-1}}}(p_{l-1}, p_{l-2}, n_0, \ldots, p_0, n_{l-2}, n_{l-1}),$$
$$J_{\widetilde{P_{b=l}}}(p_{l-1}, n_0, p_{l-2}, n_1, \ldots, p_0, n_{l-1}),$$

and

$$J_{\widetilde{P_{b=l}}}(p_{l-1}, n_0, p_{l-2}, n_1, \ldots, p_1, n_{l-2}, n_{l-1}, p_0).$$

□

The proof for $\widetilde{\mathrm{SL}(2l+1)}$ is almost identical to Proposition 2.15. The Langlands quotient with maximal primitive ideal is $J_{\widetilde{P_{b=l}}}(p'_l, p'_{l-1}, n'_0, \ldots, n'_{l-2}, p'_0, n'_{l-1})$.

Proposition 2.16. *In the notation of Proposition* 2.11*, there is one Langlands quotient of* $\widetilde{\mathrm{SL}(2l+1)}$ *with infinitesimal character*

$$(p'_l, p'_{l-1}, \ldots, p'_0, n'_0, n'_1, \ldots, n'_{l-2}, n'_{l-1})$$

having a maximal primitive ideal.

From what was said just prior to Proposition 2.10 we then have the following result.

Corollary 2.17. $\widetilde{\mathrm{SL}(2l)}$ *has four representations with maximal primitive ideals with infinitesimal character* $\frac{1}{2}\rho$*, and* $\widetilde{\mathrm{SL}(2l+1)}$ *has one representation with maximal primitive ideals with infinitesimal character* $\frac{1}{2}\rho$*.*

We now identify the Langlands quotients at infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. To do this we will need to translate our Langlands quotient in the coherent family. We use Zuckerman's exact translation functor $\Psi^{\gamma}_{\gamma+\gamma'}$ from the category of Harish-Chandra modules with regular infinitesimal character γ to the category of Harish-Chandra modules with infinitesimal character $\gamma + \gamma'$. We begin with a statement that tells us how to do this.

Lemma 2.18. *Let* G *be a semisimple group with Cartan subgroup* H *and parabolic subgroup* MAN*. Let* $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$ *be a Langlands parameter. Let* $\mathrm{Ind}_{MAN}(V_\gamma)$ *be a generalized principal series representation. Let* $\gamma' = (\lambda', \nu') \in \mathfrak{h}^*$ *be an extremal weight of a finite dimensional representation* $F^{\gamma'}$ *of* G*. Then* $\Psi^{\gamma}_{\gamma+\gamma'}\mathrm{Ind}^G_{MAN}(\gamma) = \mathrm{Ind}^G_{MAN}(\gamma + \gamma')$*.*

Proof. One has $\Psi^{\gamma}_{\gamma+\gamma'}\mathrm{Ind}^G_{MAN}(\gamma) = \mathrm{Ind}^G_{MAN}(\Psi^{\gamma}_{\gamma+\gamma'}(\gamma))$ following an argument of Zuckerman [13]. Then $\Psi^{\gamma}_{\gamma+\gamma'}(V_\gamma) = P_{\gamma'}(V_\gamma \otimes F^{\gamma'}|_{MAN}) = V_{\gamma+\gamma'}$, where $P_{\gamma'}$ is the projection functor on the category of Harish-Chandra modules . □

To simplify notation, for $\widetilde{\mathrm{SL}(2l)}$, we will denote the maximal cuspidal parabolic, $P_{b=l}$, by P_{max}.

Theorem 5. *Langlands quotients of* $\widetilde{\mathrm{SL}(2l)}$ *with infinitesimal character* $\frac{1}{2}\rho$ *having a maximal primitive ideal are* $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$*,* $J_{P_{max}}(\frac{1}{2}\rho)$*, and* $J_{P_{max}}(s_{(e_{2l-1}-e_{2l})}\frac{1}{2}\rho)$*.*

Proof. For each of the representations in the claim there exists a coherent family based at that representation. We will translate each representation to the member of the coherent family with infinitesimal character $\frac{1}{2}\rho+(l,-l,\ldots,l,-l)$. The translated representation will be a Langlands quotient from Proposition 2.15 having a maximal primitive ideal.

We start with the representations $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$. By induction-in-stages (see [13]),

$$\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-l,\ldots,l,-l)}\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\frac{1}{2}\rho)=\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\frac{1}{2}\rho+(l,-l,\ldots,l,-l)).$$

Because Ind is an exact functor $\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-l,\ldots,l,-l)}(J^{\pm}_{\widetilde{P_{min}}}(\frac{1}{2}\rho))$ is an irreducible quotient in $\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\frac{1}{2}\rho+(l,-l,\ldots,l,-l))$. However, because the continuous parameter $\nu=\frac{1}{2}\rho+(l,-l,\ldots,l,-l)$ is not weakly dominant with respect to P_{min}, it is not a Langlands quotient.

To find the irreducible quotient in $\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\frac{1}{2}\rho+(l,-l,\ldots,l,-l))$ it is helpful to write this representation in another way using induction-in-stages. Let $P_{min}=M_{min}AN$ be the Iwasawa decomposition of the minimal parabolic, and let $P_{l-1}=M'_{l-1}A'N'$ be the Iwasawa decomposition of the parabolic where M_{l-1} has $l-1$ SL(2) blocks centered along the diagonal. Letting $A''=A\cap M_{l-1}$ and $N''=N\cap M_{l-1}$, the subgroup $MA''N''$ consists of $l-1$ upper triangular 2×2 blocks centered along the diagonal. In the notation of Proposition 2.10 we write $\frac{1}{2}\rho+(l,-l,\ldots,l,-l)=(p_{l-1},n_0,p_{l-2},\ldots,n_{l-2},p_0,n_{l-1})$. We have the identity

$$\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\nu)=\mathrm{Ind}_{\widetilde{P}_{l-1}}(\mathrm{Ind}_{\widetilde{M}A''N''}(\xi^{\pm},\nu|_{\mathfrak{a}''}),\nu|_{\mathfrak{a}'})$$

where

$$\begin{aligned}\nu&=(p_{l-1},n_0,p_{l-2},n_1,\ldots,p_0,n_{l-1}),\\ \nu|_{\mathfrak{a}''}&=(0,\tfrac{n_0-p_{l-2}}{2},-\tfrac{(n_0-p_{l-2})}{2},\ldots,\tfrac{n_{l-2}-p_0}{2},-\tfrac{(n_{l-2}-p_0)}{2},0),\\ \nu|_{\mathfrak{a}'}&=(p_{l-1},\tfrac{n_0+p_{l-2}}{2},\tfrac{n_0+p_{l-2}}{2},\ldots,\tfrac{n_{l-2}+p_0}{2},\tfrac{n_{l-2}+p_0}{2},n_{l-1}).\end{aligned}$$

The irreducible quotient of $\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\nu)$ is the irreducible quotient of $\mathrm{Ind}_{\widetilde{P}_{l-1}}$ (the irreducible quotient of $\mathrm{Ind}_{\widetilde{M}A''N''}(\xi^{\pm},\nu|_{\mathfrak{a}''})$). This follows from the fact that Ind is exact, but Ind of an irreducible quotient isn't necessarily irreducible. The irreducible quotient of $\mathrm{Ind}_{\widetilde{M}A''N''}(\xi^{\pm},\nu|_{\mathfrak{a}''})$ is the genuine discrete series $\delta^{\pm}_{(p_{l-2}-n_0,\ldots,p_0-n_{l-1})}$. It follows that the Langlands quotient $J_{P_{l-1}}(p_{l-1},p_{l-2},n_0,\ldots,p_0,n_{l-2},n_{l-1})$ is the irreducible quotient of $\mathrm{Ind}_{\widetilde{P_{min}}}(\xi^{\pm},\nu)$. We know from Proposition 2.15 that this representation has a maximal primitive ideal.

Proving that the Langlands quotients having nonminimal parabolic have a maximal primitive ideal is much easier since their translation to an irreducible with infinitesimal character $\frac{1}{2}\rho+(l,-l,\ldots,l,-l)$ is a Langlands quotient. We have

$$\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-l,\ldots,l,-l)}\mathrm{Ind}_{\widetilde{P_{max}}}(\frac{1}{2}\rho)=\mathrm{Ind}_{\widetilde{P_{max}}}(\frac{1}{2}\rho+(l,-l,\ldots,l,-l)).$$

As $\frac{1}{2}\rho+(l,-l,\ldots,l,-l)$ is dominant with respect to $\widetilde{P}_l$, $\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-l,\ldots,l,-l)}J_{\widetilde{P}_l}(\frac{1}{2}\rho)$ is a Langlands quotient in $\mathrm{Ind}_{\widetilde{P_{max}}}(\frac{1}{2}\rho+(l,-l,\ldots,l,-l))$. From Proposition 2.15 we know that this Langlands quotient has a maximal primitive ideal. Similarly we find that

$$\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-l,\ldots,l,-l)}J_{\widetilde{P_{max}}}(s_{e_{2l-1}-e_{2l}}\frac{1}{2}\rho)=J_{\widetilde{P_{max}}}(s_{e_{2l-1}-e_{2l}}(\frac{1}{2}\rho+(l,-l,\ldots,l,-l))$$

which we showed in Proposition 2.15 to have a maximal primitive ideal. ☐

Theorem 6. *The Langlands quotient of* $\widetilde{\mathrm{SL}(2l+1)}$ *with infinitesimal character* $\frac{1}{2}\rho$ *having a maximal primitive ideal is* $J_{P_{min}}(\frac{1}{2}\rho)$.

Proof. The proof is analogous to the proof that $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$ has a maximal primitive ideal. We choose a coherent family based at this representation and translate to the irreducible representation having infinitesimal character $\frac{1}{2}\rho + (l, -(l+1), \ldots, l, -(l+1), l)$. Using an induction-in-stages argument we find that

$$\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+(l,-(l+1),\ldots,l,-(l+1),l)} J_{P_{min}}(\frac{1}{2}\rho) = J_{P_{max}}(p'_l, p'_{l-1}, \ldots, p'_0, n'_0, n'_1, \ldots, n'_{l-2}, n'_{l-1})$$

where $J_{P_{max}}(p'_l, p'_{l-1}, \ldots, p'_0, n'_0, n'_1, \ldots, n'_{l-2}, n'_{l-1})$ is the Langlands quotient shown to have a maximal primitive ideal in Proposition 2.16. □

3. Orbit method predictions

Because the Langlands quotients considered in section 2 have a maximal primitive ideal, they provide a good paradigm for how unitary representations can be "attached" to nilpotent coadjoint orbits. By saying that a representation is attached to a nilpotent orbit, we roughly mean that the associated variety of the annihilator of the representation is the closure of the nilpotent orbit. There may be several representations attached in this sense. The orbit method conjectures that when the orbit satisfies a certain condition on the codimension of its boundary then the set of representations attached to the orbit is parameterized by the set of admissible orbit data. With each admissible orbit datum, the orbit method gives a realization of the locally finite K-types of the attached representation.

To establish notation, we will let G denote a real semisimple Lie group with Lie algebra $\mathfrak{g}_0$. $G_{\mathbb{C}}$ will be the complexification of G with Lie algebra $\mathfrak{g}$. K will be a maximal compact subgroup of G with complexification $K_{\mathbb{C}}$. Let $\Theta_{\mathbb{C}}$ be a Cartan involution on $\mathfrak{g}$ fixing $\mathfrak{k}$, the Lie algebra of $K_{\mathbb{C}}$. If $Z = X + \sqrt{-1}\ Y$ in $\mathfrak{g}$, then $\Theta_{\mathbb{C}}(Z) = -\overline{(\sigma Z)}^t = -\overline{X}^t - \sqrt{-1}\ \overline{Y}^t$, where σ is a complex conjugation with respect to the real form (i.e. $\sigma(X + \sqrt{-1}\ Y) = X - \sqrt{-1}\ Y$). Let $\mathfrak{k}$ and $\mathfrak{s}$ be the 1 and -1 eigenspace of $\Theta_{\mathbb{C}}$. In the case where $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$ and the real form is $\mathfrak{sl}(m)$, then $\Theta_{\mathbb{C}} Z = -Z^t$. We have $\mathfrak{k}$ and $\mathfrak{s}$ equal to the skew symmetric and symmetric matrices in $\mathfrak{sl}(m, \mathbb{C})$ respectively.

3.1. Attaching a nilpotent orbit.

We outline below how one finds the nilpotent orbit "attached" to our Langlands quotients, having a maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$.

Let $I \subset U(\mathfrak{g})$ be a primitive ideal. Then the quotient ring $U(\mathfrak{g})/I$ is a finitely generated $U(\mathfrak{g})$-module. The natural grading on $U(\mathfrak{g})$ defines a filtration on $U(\mathfrak{g})/I$ and makes the associated graded algebra $\mathrm{gr}(U(\mathfrak{g})/I)$. Regarding $S(\mathfrak{g})$ as an algebra of polynomial functions on $\mathfrak{g}$ we define the *associated variety* of I to be $\mathcal{V}(I) = \{\lambda \in \mathfrak{g}^* | p(\lambda) = 0 \text{ whenever } p \in \mathrm{gr}\, I\}$.

The orbit of an element of $\mathfrak{g}^*$ under the action of $Ad^*(G_{\mathbb{C}})$ is called a *coadjoint nilpotent orbit* or *nilpotent orbit* if its closure is a cone. Formally,

$$\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* |\ t\lambda \in G_{\mathbb{C}} \cdot \lambda \text{ for all } t \in \mathbb{C}^{\times}\}.$$

Because of the identification of $\mathfrak{g}$ with $\mathfrak{g}*$, we will not distinguish between nilpotent coadjoint orbits and nilpotent adjoint orbits.

Let $I \subset U(\mathfrak{g})$ be a primitive ideal. The associated variety $\mathcal{V}(I)$ is the closure of a single coadjoint nilpotent orbit in $\mathfrak{g}^*$ (Corollary 4.7 of [10]). If the primitive ideal I is maximal and has infinitesimal character λ, then we write $\mathcal{O}_{\mathbb{C}}(\lambda)$ for the nilpotent orbit whose closure is $\mathcal{V}(I)$. We wish to identify $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ for $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$ in the case where I is a maximal primitive ideal.

One ingredient for identifying the nilpotent orbit is the Springer correspondence between nilpotent orbits and Weyl group representations. The nilpotent orbits for $\mathfrak{sl}(m, \mathbb{C})$ are in one to one correspondence with the partitions of m. To a partition of $m = [m_1, \ldots, m_d]$ we associate the nilpotent orbit

$$\mathcal{O}_{[m_1,\ldots,m_d]} = \mathrm{SL}(m, \mathbf{C}) \cdot \begin{bmatrix} J_{m_1} & & \\ & \ddots & \\ & & J_{m_d} \end{bmatrix},$$

where

$$J_{m_i} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{bmatrix}_{m_i \times m_i}.$$

The irreducible representations of the symmetric group on m letters, S_m, are in one to one correspondence with the partitions of m (partitions of m correspond to the conjugacy classes of S_m which in turn correspond to the irreducible representations of S_m).

The Weyl group for $\mathfrak{sl}(m, \mathbb{C})$ is S_m. The integral Weyl group with respect to $\frac{1}{2}\rho$ is $S_l \times S_l$ and $S_{l+1} \times S_l$ for $m = 2l$ and $m = 2l + 1$ respectively. This group acts on a coherent family of virtual $(\mathfrak{g}, K)$ modules by the *coherent continuation representation*, defined below. Given a coherent family Φ based at an irreducible $(\mathfrak{g}, K)$ module, X, with regular infinitesimal character γ, then $s_\alpha\Phi(\lambda) = \Phi(s_\alpha\lambda)$ for $\alpha \in R^+(\gamma)$ a simple integral root and $\lambda \in \gamma + \Lambda$. If X has a maximal primitive ideal, then $s_\alpha\Phi(\gamma) = -\Phi(\gamma)$.

We wish to extend this representation of the integral Weyl group to the entire Weyl group. There is no well defined representation of the Weyl group on the coherent family of $(\mathfrak{g}, K)$ modules for $\widetilde{\mathrm{SL}}(m)$. There is however an integral Weyl group equivariant map between the virtual $(\mathfrak{g}, K)$ modules having a maximal primitive ideal and the symmetric algebra, $S^d(\mathfrak{h}^*)$, where d is the number of positive integral roots (Theorem 4.2.2 of [19]). The integral Weyl group acts by the sign representation on $\prod_{\alpha_i \in R^+\gamma} \alpha_i \in S^d(\mathfrak{h}^*)$. This characterizes a Weyl group representation on $S^d(\mathfrak{h}^*)$ with conjugacy classes $[n, n]^t$ or $[n+1, n]^t$ by Young's characterization of the representations of the symmetric group [17]. By the Springer Correspondence this Weyl group representation corresponds to the nilpotent orbit $\mathcal{O}_{[n,n]^t}$ or $\mathcal{O}_{[n+1,n]^t}$. Rossmann shows that this is the nilpotent orbit $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ [18].

3.2. $K_{\mathbb{C}}$ **orbits.** In connection with determining admissible orbit data we will be concerned with action of $K_{\mathbb{C}}$ on $(\mathfrak{g}/\mathfrak{k})^*$ (i.e. $K_{\mathbb{C}}$ *orbits*). The corresponding nilpotent cone is $\mathcal{N}^*_{\mathfrak{k}} = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$. For $\lambda \in \mathfrak{g}^*$, the intersection $\mathcal{O}_{\mathbb{C}}(\lambda) \cap (\mathfrak{g}/\mathfrak{k})^*$ is a finite union of $K_{\mathbb{C}}$ orbits (see Corollary 5.20 in [10]). We will need to explicitly determine the $K_{\mathbb{C}}$ orbits associated with $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$. The Kostant-Sekiguchi correspondence relates $K_{\mathbb{C}}$ orbits on nilpotent elements in $\mathfrak{s}$ to G orbits on nilpotent elements in $\mathfrak{g}_o$. By finding the decomposition of $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho) \cap \mathfrak{g}_o$ into G orbits, we may use the Kostant-Sekiguchi correspondence to determine the $K_{\mathbb{C}}$ orbits.

To find the SL(m) nilpotent orbits we begin by first finding the GL(m) nilpotent orbits. We note that adjoint orbits of a linear group and its double cover coincide so we will not write $\widetilde{\mathrm{SL}(m)}$ orbits.

From the theorem on rational canonical form we have the following.

Proposition 3.1. *The* GL(m) *nilpotent orbits correspond to the partitions of* m*. To a partition of* $m = [m_1, \ldots, m_d]$ *we associate the nilpotent orbit*

$$\mathcal{O}_{[m_1,\ldots,m_d]} = \mathrm{GL}(m) \cdot \begin{bmatrix} J_{m_1} & & \\ & \ddots & \\ & & J_{m_d} \end{bmatrix},$$

where

$$J_{m_i} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{bmatrix}_{m_i \times m_i}.$$

Each GL(m) nilpotent orbit may decompose as a sum of SL(m) nilpotent orbits. For example,

$$\mathrm{GL}(2)\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{SL}(2)\cdot\begin{bmatrix} & 1 \\ & \end{bmatrix} \cup \mathrm{SL}(2)\cdot\begin{bmatrix} & -1 \\ & \end{bmatrix}.$$

The next statement indicates that this example generalizes to SL$(2n)$.

Proposition 3.2. *The* SL$(2n+1)$ *nilpotent orbits are the same as the* GL$(2n+1)$ *nilpotent orbits. The* GL$(2n)$ *nilpotent orbits split as a union of at most two* SL$(2n)$ *nilpotent orbits. If the rows of the partition are all even, then there are two* SL(m) *orbits, otherwise there is just one.*

Proof. Let $X \in \mathrm{GL}(2n+1)$ and J be a nilpotent matrix in Jordan canonical form. Conjugating J by X and by $\frac{X}{\sqrt[2n+1]{\det X}}$ are equivalent and $\frac{X}{\sqrt[2n+1]{\det X}} \in \mathrm{SL}(2n+1)$. For the second part of the claim define $\mathrm{GL}(2n)^+ = \{x \in \mathrm{GL}(2n) : \det X > 0\}$ and $\mathrm{GL}(2n)^- = \{x \in \mathrm{GL}(2n) : \det X < 0\}$. Indeed, the action of $\mathrm{GL}(2n)^+$ and SL$(2n)$ coincide by the above argument. Further we have

$$\mathrm{GL}(2n)^- = \mathrm{GL}(2n)^+ \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

It follows that the action of $\mathrm{GL}(2n)^-$ and $\mathrm{SL}(2n)\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ coincide. For the last part of the claim one must determine whether $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \cdot J \in$ $\mathrm{SL}(2n) \cdot J$. This we leave to the reader. □

Let $J_{[m_1,\ldots,m_d]}$ be a nilpotent element of $\mathrm{SL}(m)$ in canonical Jordan form. An immediate consequence of Proposition 3.2 is:

Corollary 3.3. *The* $\mathrm{SL}(2n)$ *nilpotent orbits are* $\mathrm{SL}(2n)\cdot J_{[m_1,\ldots,m_d]}$ *and*

$$\mathrm{SL}(2n)\cdot\left(\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix} J_{[m_1,\ldots,m_d]} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\right)$$

for all partitions $[m_1,\ldots,m_d]$ *of* $2n$. *The* $\mathrm{SL}(2n+1)$ *nilpotent orbits are* $\mathrm{SL}(2n+1)\cdot J_{[m_1,\ldots,m_d]}$ *for all partitions* $[m_1,\ldots,m_d]$ *of* $2n+1$.

Next we use the Kostant-Sekiguchi correspondence [10] to find the corresponding $K_{\mathbb{C}}$ orbits. According to the Kostant-Sekiguchi Correspondence, given a homomorphism $\phi:\mathfrak{sl}(2,\mathbb{C})\to\mathfrak{g}$ respecting notions of complex conjugation in $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}$ and respecting Cartan involution, there is a one to one correspondence between G nilpotent orbits and $K_{\mathbb{C}}$ nilpotent orbits given by

$$G\cdot\phi\begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}\longleftrightarrow K_{\mathbb{C}}\cdot\phi\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{-1}}{2}\\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2}\end{bmatrix}.$$

An important example is for $\mathfrak{g}=\mathfrak{sl}(2)$. Here ϕ is the identity map and the $\mathrm{SL}(2)$ orbits

$$\mathrm{SL}(2)\cdot\begin{bmatrix} & 1\\ & \end{bmatrix}\cup\ \mathrm{SL}(2)\cdot\begin{bmatrix} & -1\\ & \end{bmatrix}$$

correspond to the $\mathrm{SO}(2,\mathbb{C})$ orbits

$$\mathrm{SO}(2,\mathbb{C})\cdot\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{-1}}{2}\\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2}\end{bmatrix}\cup\ \mathrm{SO}(2,\mathbb{C})\cdot\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{-1}}{2}\\ \frac{\sqrt{-1}}{2} & -\frac{1}{2}\end{bmatrix}.$$

Corollary 3.4. *For* $\widetilde{\mathrm{SL}(2l)}$,

$$\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)\cap(\mathfrak{g}/\mathfrak{k})^*=K_{\mathbb{C}}\cdot\frac{1}{2}\begin{bmatrix} 1 & -i & & & & & \\ -i & -1 & & & & & \\ & & 1 & -i & & & \\ & & -i & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & -i\\ & & & & & -i & -1\end{bmatrix}$$

$$\cup\ K_{\mathbb{C}}\cdot\frac{1}{2}\begin{bmatrix} 1 & i & & & & & \\ i & -1 & & & & & \\ & & 1 & -i & & & \\ & & -i & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & -i\\ & & & & & -i & -1\end{bmatrix}.$$

For $\widetilde{\mathrm{SL}(2l+1)}$,

$$\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)\cap(\mathfrak{g}/\mathfrak{k})^*=K_{\mathbb{C}}\cdot\frac{1}{2}\begin{bmatrix} 1 & -i & & & & & & \\ -i & -1 & & & & & & \\ & & 1 & -i & & & & \\ & & -i & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & -i & \\ & & & & & -i & -1 & \\ & & & & & & & 0\end{bmatrix}.$$

Proof. This follows easily from the Kostant-Sekiguchi correspondence and Corollary 3.3. ∎

To find $K_{\mathbb{C}}^x$ we write the centralizer of a nonzero nilpotent element in $\mathfrak{k}$ as the direct sum of a nilpotent ideal and a reductive subalgebra. Let x be a nonzero nilpotent element of $\mathfrak{g}$, and $\{x, y, h\}$ a standard $\mathfrak{sl}(2)$ triple containing x. We have $K_{\mathbb{C}}^x = \{g \in K_{\mathbb{C}} : \mathrm{Ad}(g)x = x\}$, with Levi subgroup, $K_{\mathbb{C}}^{\mathfrak{sl}(2,\mathbb{C})}$, of elements of $K_{\mathbb{C}}$ commuting with the image of $\mathfrak{sl}(2)$. We can write $K_{\mathbb{C}}^x = K_{\mathbb{C}}^{\mathfrak{sl}(2,\mathbb{C})} \cdot U^x$ as a semidirect product of $K_{\mathbb{C}}^{\mathfrak{sl}(2,\mathbb{C})}$ and the unipotent radical, U^x, as in [17] (Lemma 3.7.3). To find U^x we let $\mathfrak{u}$ be the nilpotent subalgebra of $\mathfrak{k}$ consisting of the sum of the positive eigenspaces of $\mathrm{ad}(h)$. We let $U = \exp \mathfrak{u}$ and U^x be the centralizer of x in U. The Levi decomposition of $K_{\mathbb{C}}^x$ can be written as $\mathfrak{k}_{\mathbb{C}}^x = \mathfrak{k}_{\mathbb{C}}^{\mathfrak{sl}(2)} \oplus \mathfrak{u}^x$. The reductive part of the Levi decomposition, $\mathfrak{k}_{\mathbb{C}}^{\mathfrak{sl}(2)}$ in $\mathfrak{so}(2l+\epsilon, \mathbb{C})$, is a diagonal embedding of $\mathfrak{so}(l, \mathbb{C})$. We will denote this diagonal subalgebra by $\mathfrak{so}(l, \mathbb{C})$.

Lemma 3.5. *The identity component of $\tilde{K}_{\mathbb{C}}^{\mathfrak{sl}(2)}$ in $\mathrm{SO}(2l+\epsilon, \mathbb{C})$, $\epsilon \in \{0, 1\}$, is isomorphic to its double cover. The full group can be realized as $\{\pm 1, \pm e_1 e_2\} \cdot \mathrm{SO}(l, \mathbb{C})$.*

Proof. The identity component of the linear group with Lie algebra $\mathfrak{so}(l, \mathbb{C})$ consists of two copies of $\mathrm{SO}(l, \mathbb{C})$. In the case where $\epsilon = 1$ the lower diagonal entry is 1. We denote this diagonal group by $\mathrm{SO}(l, \mathbb{C})$. In the double cover, the nontrivial element in the projection homomorphism onto the linear group, -1, is multiplied in both copies giving 1. The first statement in the claim follows from this. The linear group with Lie algebra $\mathfrak{so}(l, \mathbb{C})$ is the semidirect product of the matrices with -1 along the first two diagonal entries and 1 along all other diagonal entries, with $\mathrm{SO}(l, \mathbb{C})$. Identifying $\mathrm{SO}(l, \mathbb{C})$ with its double cover, we have $\tilde{K}_{\mathbb{C}}^{\mathfrak{sl}(2)} = \{\pm 1, \pm e_1 e_2\} \cdot \mathrm{SO}(l, \mathbb{C})$. □

We will write $\mathrm{O}(l, \mathbb{C})$ for the group $K_{\mathbb{C}}^{\mathfrak{sl}(2)}$.

3.3. **Admissible $K_{\mathbb{C}}$ orbits.** Let $x \in \mathcal{N}^*$ and define $\gamma(x)$ to be the character by which $K_{\mathbb{C}}^x$ acts on top degree differential forms at x:

$$\gamma(x) : K_{\mathbb{C}}^x \to \mathbb{C}^\times, \quad \gamma(x)(k) = \det(\mathrm{Ad}^*(k))|_{(\mathfrak{k}/\mathfrak{k}^x)^*}.$$

A representation ξ of $K_{\mathbb{C}}^x$ is called *admissible* if the differential of ξ is $\frac{1}{2} d\gamma(x)$. We have $\det Ad^*(k)|_{(\mathfrak{k}/\mathfrak{k}^x)^*} = (\det \mathrm{Ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x})^{-1}$ and $\mathrm{tr}\,\mathrm{ad}^*(k)|_{(\mathfrak{k}/\mathfrak{k}^x)^*} = -\mathrm{tr}\,\mathrm{ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x}$. Hence the test for admissibility is

$$d\xi(k) = -\frac{1}{2} \mathrm{ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x} \text{ for all } k \in \mathfrak{k}^x.$$

Lemma 3.6. *For $\mathfrak{k} = \mathfrak{so}(m, \mathbb{C})$, $\mathrm{tr}\,\mathrm{ad}k|_{\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^x} = 0$ for all $k \in \mathfrak{k}_{\mathbb{C}}^x$.*

Proof. We have the Levi decomposition $\mathfrak{so}(2l+\epsilon, \mathbb{C})^x = \mathfrak{so}(l, \mathbb{C}) \oplus u^x$ for $\epsilon \in \{0, 1\}$. For $l > 2$, $\mathfrak{so}(l, \mathbb{C})$ is semisimple so its one dimensional representation $d\gamma$ must be zero. It follows that $\mathrm{ad}(k)$ is traceless since ad-nilpotent elements are. One can check directly that the claim holds true for the case $l = 1$ and $l = 2$. □

It follows from Lemma 3.6 that an irreducible representation of $\widetilde{\mathrm{SO}(m, \mathbb{C})}^x$ is admissible if its differential is zero.

Lemma 3.7. *Let π be an irreducible algebraic representation of $K_{\mathbb{C}}^x = \mathrm{O}(l, \mathbb{C}) \cdot U^x$. Then U^x acts by the trivial representation.*

Proof. Let $\pi : K_{\mathbb{C}}^x \to GL(V)$. Then $\pi(U^x)$ is a unipotent algebraic group in $GL(V)$. By Engel's theorem there exists a $v \in V$ fixed by U^x. Because U^x is a normal subgroup, V^{U^x} is a nonzero $K_{\mathbb{C}}^x$ invariant subspace of V. By the irreducibility of π, $V^{U^x} = V$, so U^x acts trivially. □

Theorem 7. *There are* 4 *genuine admissible* $\mathrm{Spin}(2l, \mathbb{C})$ *orbit data, and* 2 *genuine admissible* $\mathrm{Spin}(2l+1, \mathbb{C})$ *orbit data for* $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$.

Proof. Let π be an irreducible representation of $K_{\mathbb{C}}^x$. By Lemma 3.7, π acts by the trivial representation on U^x. For π to be admissible, by Lemma 3.6, its differential must be zero. This forces π to be the trivial representation on $\mathrm{SO}(l, \mathbb{C})$. Hence we may think of π as an irreducible representation of $\{\pm 1, \pm e_1 e_2\}$. There are two such genuine representations, π^+ and π^- defined by $\pi^+(e_1 e_2) = \sqrt{-1}$ and $\pi^-(e_1 e_2) = -\sqrt{-1}$. Because there are two $\mathrm{Spin}(2l, \mathbb{C})$ orbits, $\mathcal{O}_{[l,l]^t}$ has 4 admissible orbit data $(\pi^\pm, x)$ and $(\pi^\pm, x')$. Because there is one $\mathrm{Spin}(2l+1, \mathbb{C})$ orbit $\mathcal{O}_{[l,l]^t}$ has 2 admissible orbit data $(\pi^\pm, x)$. □

3.4. **Orbit method prediction of K-types.** The following statement lies at the heart of attaching representations to nilpotent orbits [10] (Conjecture 12.1).

Conjecture 3.8. *Suppose X is an irreducible unipotent Harish-Chandra module and $\mathcal{O}$ a nilpotent coadjoint orbit with $\mathcal{V}(\mathrm{gr}\,\mathrm{Ann}X) = \bar{\mathcal{O}}$ and assume the codimension of $\partial\bar{\mathcal{O}}$ in $\bar{\mathcal{O}}$ is at least* 4. *Then there is an element $x \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$ and an admissible representation π of the stabilizer $K_{\mathbb{C}}^x$ such that, as a representation of $K_{\mathbb{C}}$,*

$$X \cong Ind_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}}(\pi).$$

We wish to use Conjecture 3.8 to determine the K-types of our Langlands quotient with maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$ attached to $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$. We first check whether $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ satisfies the codimension hypothesis of the conjecture.

Lemma 3.9. *The codimension of the boundary of the complex nilpotent orbit $\mathcal{O}_{[l,l]^t}$ of $\mathfrak{sl}(2l, \mathbb{C})$ is* 2 *and the codimension of the boundary of the complex nilpotent orbit $\mathcal{O}_{[l+1,l]^t}$ of $\mathfrak{sl}(2l+1, \mathbb{C})$ is* 4.

Proof. We use the formula of Corollary 6.1.4 in [17] to compute the dimension of a complex nilpotent orbit of $\mathfrak{sl}(m)$. The boundary of $\mathcal{O}_{[l,l]^t}$ is $\mathcal{O}_{[l+1,l-1]^t}$. It has codimension 2. The boundary of $\mathcal{O}_{[l+1,l]^t}$ is $\mathcal{O}_{[l+2,l-1]^t}$. It has codimension 4. □

The codimension condition of Conjecture 3.8 is satisfied for the nilpotent orbit $\mathcal{O}_{[l+1,l]^t}$ of $\mathfrak{sl}(2l+1, \mathbb{C})$. Then the algebraic representation in Conjecture 3.8 is our Langlands quotient $J_{P_{\min}}(\frac{1}{2}\rho)|_K$ from section 2 thought of as a representation of $K_{\mathbb{C}}$. We note that $\mathcal{O}_{[l+1,l]^t}$ has two admissible orbits but $\widetilde{\mathrm{SL}(m)}$ has only one Langlands quotient with maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$. We thus have an example where the orbit method overestimates the number of irreducible representations attached to an orbit. Indeed Torasso observed this for the case of $\widetilde{\mathrm{SL}(3)}$ [4].

In hopes that one of the algebraic representations given in the conjecture is our Langlands quotient, we will proceed to determine the $K_{\mathbb{C}}$ types of the algebraic representation. To investigate the K-types of $Ind_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}}(\pi)$ it will be convenient to use the transitivity property of induction. Let $\{x, y, h\}$ be a $\mathfrak{sl}(2,\mathbb{C})$ triple and $\mathfrak{k}^x$ as defined before Lemma 3.5. By Lemma 3.8.4 in [17], the sum of the nonnegative $ad(h)$ weight spaces is a parabolic subalgebra of $\mathfrak{k}$ containing $\mathfrak{k}^x$. This parabolic subalgebra is $\mathfrak{gl}(l,\mathbb{C}) \oplus \mathfrak{u}$. Here $\mathfrak{gl}(l,\mathbb{C})$ is a diagonal subalgebra of $\mathfrak{k}$ containing $\mathfrak{o}(l,\mathbb{C})$. At the group level we write the parabolic subgroup as $\widetilde{\mathrm{GL}(l,\mathbb{C})}\cdot U$. Because the unipotent groups U and U^x play no role in the representation, we will omit them from the notation. By the transitivity property of induction we have

$$\mathrm{Ind}_{\widetilde{K^x_{\mathbb{C}}}}^{\widetilde{K_{\mathbb{C}}}}(\pi) = \mathrm{Ind}_{\widetilde{\mathrm{GL}(l,\mathbb{C})}}^{\widetilde{K_{\mathbb{C}}}} \mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}} \pi.$$

To analyze the K-types we will use the Borel-Weil Theorem [13]. We see that to determine the K-types of $\mathrm{Ind}_{K^x_{\mathbb{C}}}^{K_{\mathbb{C}}}(\pi^{\pm})$ it suffices to determine the irreducible representations ξ_λ of $\widetilde{\mathrm{GL}(l,\mathbb{C})}$ in $\mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}}\pi^{\pm}$.

Proposition 3.10. *For admissible representations $\pi^{\pm}$ of $K^x_{\mathbb{C}}$ the K-types of $\mathrm{Ind}_{K^x_{\mathbb{C}}}^{K_{\mathbb{C}}}(\pi^+)$ are $(\frac{1}{2} + 2a_1, \dots, \frac{1}{2} + 2a_l)$ and the K-types of $\mathrm{Ind}_{K^x_{\mathbb{C}}}^{K_{\mathbb{C}}}(\pi^-)$ are $(\frac{3}{2} + 2a_1, \dots, \frac{3}{2} + 2a_l)$ for $a_1 \geq \dots \geq a_l \geq 0$.*

Proof. The $\det^{\frac{1}{2}}$ cover of $\mathrm{GL}(l,\mathbb{C})$, is $\widetilde{\mathrm{GL}(l,\mathbb{C})} = \{(g,z) \in \mathrm{GL}(l,\mathbb{C}) \times \mathbb{C}^\times \,|\, \det g = z^2\}$. The subgroup $\widetilde{\mathrm{O}(l,\mathbb{C})} = \{(x,y) | \det x = z^2\}$ has 2 genuine characters: $\det^{\frac{1}{2}}$ and $\det^{\frac{3}{2}}$. We note that $\det^{\frac{1}{2}} = \pi^+$ and $\det^{\frac{3}{2}} = \pi^-$ in the statement of the proposition.

We have the identity $\mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}}(\det^{\frac{1}{2}})^a \cong (\det^{\frac{1}{2}})^a \otimes \mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}} 1$ for $a \in \{1,3\}$. $\mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}} 1$ consists of algebraic functions on $\widetilde{\mathrm{GL}(l,\mathbb{C})}$ containing an $\widetilde{O(l,\mathbb{C})}$ fixed vector. By Helgason's theorem on spherical representations [20] (Theorem 4.12) the K-types of $\mathrm{Ind}_{\widetilde{O(l,\mathbb{C})}}^{\widetilde{\mathrm{GL}(l,\mathbb{C})}} 1$ are $(2a_1, \dots, 2a_l)$. Twisting by $\det^{\frac{1}{2}}$ adds $(\frac{1}{2}, \dots, \frac{1}{2})$ and twisting by $\det^{\frac{3}{2}}$ adds $(\frac{3}{2}, \dots, \frac{3}{2})$, proving the claim. □

4. A character formula

Here we prove a character formula for Langlands quotients, of $\widetilde{\mathrm{SL}(m)}$, with infinitesimal character $\frac{1}{2}\rho$, having a maximal primitive ideal. We write the Langlands quotients of section 2 as a sum of much simpler virtual representations. In deriving the character formula we first determine the lowest K-types of the Langlands quotients under consideration. For the Langlands quotients of principal series we use the fact that the lowest K type of these representations are the highest weight of fine representations of the maximal compact subgroup. For the Langlands quotients with maximal parabolic we find the lowest K-type directly. We then turn to certain sums of virtual characters which we prove to be irreducible and having a maximal primitive ideal. By matching lowest K-types we prove the character formula.

4.1. **Lowest K-types.** In section 2 we showed that there are four Langlands quotients of $\widetilde{\mathrm{SL}(2l)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. For $\widetilde{\mathrm{SL}(2l+1)}$ we showed that there is a single Langlands quotient with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. We maintain the same notation as in section 2 except that we suppose that G is split (the group G being split allows us to consider a real root space decomposition of $\mathfrak{g}$). Our aim in this section is to give a character formula for the K-types of these Langlands quotients. We turn now to the determination of their lowest K-types.

We can assign a nonnegative real number to each $\pi \in \hat{K}$, roughly the length of the highest weight of π. The *lowest K-types* of a nonzero $(\mathfrak{g}, K)$ module, X, are the irreducible representations of K, with nonzero multiplicity in X, minimal with respect to this "norm" on $\hat{K}$.

We begin by determining the lowest K-types for those Langlands quotients induced from a minimal parabolic subgroup. To do this will involve a discussion of fine representations of K as presented in [12].

For each positive root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ let $\varphi_\alpha : \mathfrak{sl}(2) \to \mathfrak{g}$ be an injection so that $H_\alpha = \varphi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$, $\varphi_\alpha(-{}^tX) = \theta\varphi_\alpha(X)$, and $X_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ lies in the α root space of $\mathfrak{a}$ in $\mathfrak{g}$. Let $Z_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$. Note that the bracket relations on H_α, $X_{\pm\alpha}$ and the relation $\theta X_\alpha = X_{-\alpha}$ only determines $X_{\pm\alpha}$, and hence Z_α, up to a sign. We say that an irreducible representation μ of K is *fine* if $\mu(iZ_\alpha)$ has eigenvalues between -1 and 1 for each positive root α. (We abuse notation and denote the differential of μ as μ.)

Proposition 4.1. *Let $G = \widetilde{\mathrm{SL}(m)}$ for $m = 2n$ or $m = 2n+1$. The fine representations of $K = \mathrm{Spin}(m)$ have highest weight $(m_1, m_2, \ldots, m_n)$ with $|m_1| \leq 1$.*

Proof. For each root α of $\mathfrak{sl}(m)$ we determine which representations μ of $\mathfrak{so}(m)$ evaluated at iZ_α take eigenvalues between -1 and 1. We will see below that we only need to consider the root $\alpha = e_1 - e_2$. The roots $e_1 - e_2$ and $e_2 - e_1$ of $\mathfrak{sl}(m)$ have the unit matrices E_{12}, E_{21} respectively as their root vectors. Together with their bracket, $[E_{12}, E_{21}]$, these root vectors span a copy of $\mathfrak{sl}(2)$ sitting inside of $\mathfrak{sl}(m)$. $Z_{e_1-e_2}$ equals $\pm(E_{12} - E_{21})$ and spans a copy of $\mathfrak{so}(2)$ sitting inside of $\mathfrak{k}$.

We would like to extend the action of the Weyl group of $\mathfrak{sl}(m)$ to Z_α, and establish their conjugacy under the Weyl group. We identify the analytic Weyl group of $\widetilde{\mathrm{SL}(m)}$ with the analytic Weyl group of the linear group. The analytic Weyl group of $\mathrm{SL}(m)$ is the normalizer of $\mathfrak{h}$ in $\mathrm{SO}(m)$ modulo the centralizer of $\mathfrak{h}$ in $\mathrm{SO}(m)$. This consists of all monomial matrices (i.e. matrices with one nonzero entry in every row and column) in $\mathrm{SO}(m)$, modulo all diagonal matrices with entries ± 1 having an even number of negative signs. Coset representatives are $n \times n$ permutation matrices with the nonzero entry in the first column negated if the permutation matrix has determinant minus one. The Weyl group acts on $\mathfrak{h}$ by conjugation. Because $E_{12} - E_{21}$ is not in the Cartan subalgebra of $\mathfrak{sl}(2)$ the Weyl group action on $E_{12} - E_{21}$ is only well defined if we identify $E_{12} - E_{21}$ with $E_{21} - E_{12}$. This sign problem occurs because different coset representatives of the identity element of the Weyl group map $E_{12} - E_{21}$ to different signs of $E_{12} - E_{21}$. For example in $\mathfrak{sl}(3, \mathbb{R})$ the identity matrix and the diagonal matrix with 1 in the first diagonal entry and -1 in the last two diagonal entries map $E_{12} - E_{21}$ to $E_{12} - E_{21}$ and $E_{21} - E_{12}$ respectively. We can extend the action to $Z_{e_1-e_2}$ however since $Z_{e_1-e_2}$ is defined only up to a sign. In this way we get a well defined action of the Weyl group on Z_α for every positive root α. The Weyl group permutes Z_α according to the formula $wZ_\alpha w^{-1} = Z_{w\alpha}$, where w is an element of the Weyl group. To see this one can first use the bracket relations on H_α, $X_{\pm\alpha}$, together with the formula $wH_\alpha w^{-1} = H_{w\alpha}$, to show that $wX_\alpha w^{-1}$ is a root vector of $w\alpha$ ($wX_\alpha w^{-1}$ makes sense since X_α is only defined up to a sign). Each root space being one dimensional implies that the Weyl group maps Z_α into a multiple of $Z_{w\alpha}$. Then $wZ_\alpha w^{-1} = Z_{w\alpha}$ follows from the Weyl group consisting of monomial matrices with entries ± 1. Because the Weyl group acts transitively on the roots of $\mathfrak{sl}(m)$ the Z_α are all conjugate to one another by an element of the Weyl group. This indicates that to check the fineness of a representation μ of $\mathfrak{so}(m)$ it suffices to check that the eigenvalues of $\mu(iZ_{e_1-e_2})$ are between -1 and 1.

If $\lambda = (m_1, m_2, \ldots, m_n)$ is a highest weight of a representation μ of $\mathfrak{so}(m)$, then for $m = 2n$, the lowest weight is $(-m_1, -m_2, \ldots, -m_n)$ and for $m = 2n+1$ the lowest weight is $(-m_1, -m_2, \ldots, -m_{n-1}, m_n)$. These weights are in the $\mathfrak{so}(m)$ Weyl group orbit of the highest weight of μ. By the theorem of highest weight, the weights of μ are $\lambda - \sum \alpha$ for positive roots α. It is important to note that the weights of μ have a smaller e_1 coefficient than that of the highest weight. Since $iZ_{e_1-e_2}$ is i times an element of the Cartan subalgebra of $\mathfrak{so}(m)$, the weights of μ applied to $(iZ_{e_1-e_2})$ are the eigenvalues of $\mu(iZ_{e_1-e_2})$. Applying a weight $\beta = (l_1, \ldots, l_n)$ to $iZ_{e_1-e_2}$ picks off the first coefficient up to a sign. In other words, $\beta(iZ_{e_1-e_2}) = l_1 e_1(iZ_{e_1-e_2}) = \pm l_1$ is an eigenvalue of $\mu(iZ_{e_1-e_2})$. This implies that the e_1 coefficient of the highest and lowest weights of μ give a bound on the size of the eigenvalues of $\mu(iZ_{e_1-e_2})$, namely the eigenvalues necessarily have absolute value less than m_1. □

Taking into account the dominant analytically integral forms for $\mathrm{Spin}(m)$ we immediately get the following result.

Corollary 4.2. *The fine representations of* $\mathrm{Spin}(2n)$ *have highest weights:*

$$(1,0,\ldots,0), (1,1,0,\ldots,0), \ldots, (1,\ldots,1,0,0), (1,\ldots,\pm 1),$$

and

$$(\frac{1}{2},\ldots,\frac{1}{2}), (\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2}).$$

The fine representations of $\mathrm{Spin}(2n+1)$*, have highest weights*

$$(1,0,\ldots,0), (1,1,0,\ldots,0), \ldots, (1,\ldots,1) \text{ and } (\frac{1}{2},\ldots,\frac{1}{2}).$$

Let $\mathrm{Ind}_{MAN}(\delta \otimes \nu)$ be a generalized principal series of a semisimple Lie group G. Its K-types are independent of ν since we have the equality $\mathrm{Ind}_{MAN}^{KAN}(\delta \otimes \nu)|_K = \mathrm{Ind}_{M\cap K}^{K}(\delta)$. Let $A(\delta)$ be the set of lowest K-types of $\mathrm{Ind}_{MAN}(\delta \otimes \nu)$.

For G a split group with minimal P and $\delta \in \hat{M}$, then $A(\delta)$ consists of fine representations of K [12]. It follows that the lowest K type of the principal series of $\widetilde{\mathrm{SL}(m)}$ are fine representations. A generalized principal series can have many irreducible constituents, known as *Langlands subquotients*. The lowest K-types of these quotients is precisely the set of lowest K-types of the generalized principal series [5].

We now determine the lowest K-types of the Langlands quotients having a minimal parabolic.

Lemma 4.3. *With notation as below Proposition* 2.5 *let* μ^+ *and* μ^- *be fine representations of* $\widetilde{K} = \mathrm{Spin}(2l)$ *whose restriction to* $\widetilde{M}$ *contains* ξ^+ *and* ξ^- *respectively. Then* μ^+ *and* μ^- *have highest weights* $(\frac{1}{2},\ldots,\frac{1}{2})$ *and* $(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$ *respectively.*

Proof. Because μ^+ and μ^- are fine representations of $\widetilde{K}$ their highest weight must be among the set of l highest weights given in Corollary 4.2. The restriction of μ^+ and μ^- to $\widetilde{M}$ contains a genuine irreducible representations so μ^+ and μ^- must send -1 to the scalar matrix -1. This rules out the possibility that they have highest weight with integer coefficients. It remains to determine which of the half integer highest weight representations contains ξ^+ and which contains ξ^-. Recall that ξ^+ and ξ^+ are differentiated by their evaluation at the central element e_I. We have $\xi^+(e_I) = \sqrt{-1}^{\,l}$ and $\xi^-(e_I) = -\sqrt{-1}^{\,l}$. Indeed, $\mu_{(\frac{1}{2},\ldots,\frac{1}{2})}(e_I) = (\exp\sqrt{-1}\frac{\pi}{2})^{\,l} = \sqrt{-1}^{\,l} = \xi^+(e_I)$ and $\mu_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}(e_I) = (\exp\sqrt{-1}\frac{\pi}{2})^{\,l-2} = -\sqrt{-1}^{\,l} = \xi^-(e_I)$ proving the claim. □

From Lemma 4.3 we immediately have the following statements.

Corollary 4.4. *The Langlands quotients $J^+_{P_{min}}(\frac{1}{2}\rho)$ and $J^-_{P_{min}}(\frac{1}{2}\rho)$ for $\widetilde{\mathrm{SL}(2l)}$ have lowest K-types $(\frac{1}{2},\ldots,\frac{1}{2})$ and $(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$, respectively.*

Corollary 4.5. *The lowest K type for the Langlands quotient $J_{P_{min}}(\frac{1}{2}\rho)$ of $\widetilde{\mathrm{SL}(2l+1)}$ is $(\frac{1}{2},\ldots,\frac{1}{2})$.*

Next we determine the lowest K-types for the Langlands quotients of $\widetilde{\mathrm{SL}(2l)}$ having nonminimal parabolic. Let α_0 be the simple root $\alpha_0 = e_{2l-1} - e_{2l}$ and s_{α_0} the simple reflection with respect to α_0. We showed in section 2 that $J_{P_l}(\frac{1}{2}\rho)$ and $J_{P_l}(s_{\alpha_0}\frac{1}{2}\rho)$ have a maximal cuspidal parabolic subgroup in $\widetilde{\mathrm{SL}(2l)}$. Let $\Delta^+_M(\frac{1}{2}\rho)$ be the positive roots in $\Delta^+(\frac{1}{2}\rho)$ which lie in M (i.e. the imaginary roots) and ρ_M half the sum of the roots in $\Delta^+_M(\frac{1}{2}\rho)$. We have the following result.

Proposition 4.6. *Let $\widetilde{P} = \widetilde{M}AN$ be a maximal cuspidal parabolic subgroup of $\widetilde{\mathrm{SL}(2l)}$. Let $\gamma = (\lambda,\nu)$ be a Langlands parameter. For $\lambda = (\lambda_1,\ldots,\lambda_l)$ and $\rho_M = (\rho_1,\ldots,\rho_l)$, the lowest K type for the generalized principal series $\mathrm{Ind}_{\widetilde{P}}(\lambda,\nu)$ has extremal weight $(\lambda_1+\rho_1,\ldots,\lambda_l+\rho_l)$.*

Proof. For determining K-types it suffices to restrict the generalized principal series to $\widetilde{K}$. We have $\mathrm{Ind}_{\widetilde{P}}(\lambda,\nu)|_{\widetilde{K}} = \mathrm{Ind}^{\widetilde{K}}_{\widetilde{M}\cap\widetilde{K}}(\delta_\lambda|_{\widetilde{M}\cap\widetilde{K}})$. The discrete series $\delta_{(\lambda_1,\ldots,\lambda_l)}$ of $\widetilde{M}$ has $M\cap K$ types with highest weight $(\lambda_{k_1},\ldots,\lambda_{k_l})$ where $\lambda_{k_i} = \lambda_i + \rho_i + \mathrm{sign}(\rho_i)2k_i$, $1 \le i \le l$, for integers $k_i \ge 0$. We have $\delta_\lambda|_{\widetilde{M}\cap\widetilde{K}} = \sum_{k_1,\ldots,k_l\ge 0}(\lambda_{k_1},\ldots,\lambda_{k_l})$. Then,

$$\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}\Big(\sum_{k_1,\ldots,k_l\ge 0}(\lambda_{k_1},\ldots,\lambda_{k_l})\Big) = \sum_{k_1,\ldots,k_l\ge 0}\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}(\lambda_{k_1},\ldots,\lambda_{k_l}).$$

To prove the proposition it then suffices to show that $\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}(\lambda_{k_1},\ldots,\lambda_{k_l})$ has the lowest K type with extremal weight $(\lambda_{k_1},\ldots,\lambda_{k_l})$. By Frobenius reciprocity, the irreducible representation of $\widetilde{K}$ with extremal weight $(a_1,\ldots,a_l)$ is not in $\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}(\lambda_{k_1},\ldots,\lambda_{k_l})$ if $a_i < \bar{\lambda}_i$ for $1 \le i \le l$.

It remains to show that $\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}(\lambda_{k_1},\ldots,\lambda_{k_l})$ has K type $(\lambda_{k_1},\ldots,\lambda_{k_l})$. This follows easily from Frobenius reciprocity. □

Corollary 4.7. *With notation as in Proposition 4.6 the lowest K-types of $J_{P_{max}}(\frac{1}{2}\rho)$ and $J_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho)$ for $\widetilde{\mathrm{SL}(2l)}$ have highest weight $(\frac{3}{2},\ldots,\frac{3}{2})$ and $(\frac{3}{2},\ldots,-\frac{3}{2})$ respectively.*

Proof. The generalized principal series $\mathrm{Ind}_{P_{max}}(\frac{1}{2}\rho)$ and $\mathrm{Ind}_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho)$ have a unique lowest K type since P_{max} doesn't contain any real roots. Hence, the claim will follow by showing that they have lowest K-types $(\frac{3}{2},\ldots,\frac{3}{2})$ and $(\frac{3}{2},\ldots,-\frac{3}{2})$ respectively. For the Langlands parameter $(\lambda,\nu)=\frac{1}{2}\rho$, we have $\lambda=(\frac{1}{2},\ldots,\frac{1}{2})$ and $\rho_M=(1,\ldots,1)$. Similarly for the Langlands parameter $(\lambda,\nu)=s_{\alpha_0}\frac{1}{2}\rho$, we have $\lambda=(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$ and $\rho_M=(1,\ldots,1,-1)$. The claim then follows immediately from Proposition 4.6. □

4.2. **Characters of virtual representations.** By a *virtual representation* we will mean a formal finite combination of irreducible representations with integer coefficients. Here we will express the Langlands quotients having maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$ as a sum of virtual representations. There is a theory of coherent families of characters of virtual $(\mathfrak{g},K)$ modules paralleling the theory of coherent families of virtual $(\mathfrak{g},K)$ modules. We will often blur the distinction between characters of virtual representations, $\Theta(\Psi,\gamma)$, and the virtual representations themselves.

Let Ψ be the standard positive roots for $\mathfrak{sl}(2l)$ or $\mathfrak{sl}(2l+1)$ and define $w_0=s_{(e_1-e_2)}s_{(e_3-e_4)}\cdot\cdots\cdot s_{(e_{2l-1}-e_{2l})}$ and $s_{\alpha_0}=s_{(e_{2l-1}-e_{2l})}$. If $\gamma=(\lambda,\nu)\in\mathfrak{h}^*$ is strictly dominant for Ψ, then $\Theta(\Psi,\gamma)$ is the character of the Langlands quotient with Langlands parameter γ. Letting Φ be a coherent family based at this irreducible representation, and μ be the extremal weight of a finite dimensional representation of G, then $\Theta(\Psi,\gamma+\mu)$ is the character of the virtual $(\mathfrak{g},K)$ module $\Phi(\gamma+\mu)$ [5]. We will abbreviate various sums of virtual characters as follows:

$$\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}=\tfrac{1}{l!}\textstyle\sum_{w\in W(R^+(\frac{1}{2}\rho))}(-1)^w\Theta(\Psi,w\cdot\frac{1}{2}\rho);$$
$$\Theta_{(\frac{1}{2},\ldots,\frac{1}{2})}=\tfrac{1}{l!}\textstyle\sum_{w\in W(R^+(w_0\frac{1}{2}\rho))}(-1)^w\Theta(\Psi,w\cdot w_0\frac{1}{2}\rho);$$
$$\Theta_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}=\tfrac{1}{l!}\textstyle\sum_{w\in W(R^+(s_{\alpha_0}w_0\frac{1}{2}\rho))}(-1)^w\Theta(s_{\alpha_0}\Psi,w\cdot w_0s_{\alpha_0}\frac{1}{2}\rho);$$
$$\Theta_{(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})}=\tfrac{1}{l!}\textstyle\sum_{w\in W(R^+(s_{\alpha_0}\frac{1}{2}\rho))}(-1)^w\Theta(s_{\alpha_0}\Psi,w\cdot s_{\alpha_0}\frac{1}{2}\rho).$$

We have the following result about lowest K-types of a sum of virtual representations. Note that the lowest K type of a virtual representation may have negative multiplicity.

Lemma 4.8. *With notation as above,*

(1) $\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$ *has the lowest* K *type* $(\frac{3}{2},\ldots,\frac{3}{2})$ *with multiplicity* 1.
(2) $\Theta_{(\frac{1}{2},\ldots,\frac{1}{2})}$ *has the lowest* K *type* $(\frac{1}{2},\ldots,\frac{1}{2})$ *with multiplicity* ±1.
(3) $\Theta_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$ *has the lowest* K *type* $(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$ *with multiplicity* ±1.
(4) $\Theta_{(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})}$ *has the lowest* K *type* $(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})$ *with multiplicity* 1.

Proof. Let λ be an element in the integral Weyl group orbit of $\frac{1}{2}\rho$. We have $\Theta(\Psi,\lambda)|_{\widetilde{K}}=\mathrm{Ind}_{\widetilde{M}\cap\widetilde{K}}^{\widetilde{K}}\Theta_M(\Psi_M,\lambda|_{\mathfrak{t}^*})|_{\widetilde{M}\cap\widetilde{K}}$. Let Ψ_M^j be the root $\pm(e_{2j-1}-e_{2j})$ in Ψ_M so $\Psi_M=\{\Psi_M^1,\ldots,\Psi_M^l\}$. Also let $\lambda|_{\mathfrak{t}^*}=(\lambda_1,\ldots,\lambda_l)$. Restricting $\Theta_M(\Psi_M,\lambda|_{\mathfrak{t}^*})$ to the maximal torus $\widetilde{T}$ gives us $\Theta_M(\Phi_M,\lambda|_{\mathfrak{t}^*})|_{\widetilde{T}}=(\Theta(\Psi_M^1,\lambda_1),\ldots,\Theta(\Psi_M^l,\lambda_l))$. For $\widetilde{\mathrm{SL}(2)}$ the K-types of $\Theta(\Psi,\lambda)$ are $\lambda+\rho+\mathrm{sign}(\rho)(2k)$ for nonnegative integers k. It follows that the K-types of $\Theta(\Psi_M^j,\lambda_j)$ are $(\lambda_j+\rho_M^j+\mathrm{sign}(\rho_M^j)2k_j)$ for nonnegative integers k_j. Using Frobenius reciprocity we find that $\Theta(\Psi,\lambda)$ has the lowest K type with extremal weight $\lambda+\rho_M$. Furthermore, among the sum of virtual representations $\sum_{w\in W(R^+(\lambda))}-1^w\Theta(*,w\lambda)$, the virtual representation $\Theta(*,\lambda)$ corresponding to $w=1$ has the lowest K type. It follows that $\sum_{w\in W(R^+(\lambda))}-1^w\Theta(*,w\lambda)$ has the lowest K type $\lambda + \rho_M$. Applying this to the sums of virtual representations in the claim gives the desired lowest K type.

Note that there are $l!$ elements in the integral Weyl group orbit, $W(R^+(\frac{1}{2}\rho)) \cdot (\frac{1}{2}\rho)$, all with the same sign, whose restriction to $\mathfrak{t}^*$ equals $\frac{1}{2}\rho|_{\mathfrak{t}^*}$. Hence, the lowest K type of $\sum_{w \in W(R^+(\frac{1}{2}\rho))}(-1)^w \Theta(\Psi, w \cdot \frac{1}{2}\rho)$ has multiplicity $\pm l!$. The multiplicity of the lowest K type of the other sums of virtual representations is found analogously. Since the lowest K type of $\sum_{w \in W(R^+(\frac{1}{2}\rho))}(-1)^w \Theta(\Psi, w \cdot \frac{1}{2}\rho)$ lies in the discrete series character $\Theta(\Psi, \frac{1}{2}\rho)$ it must have positive multiplicity $l!$. Similarly the lowest K type for $\sum_{w \in W(R^+(s_{\alpha_0}\frac{1}{2}\rho))}(-1)^w \Theta(s_{\alpha_0}\Psi, w \cdot s_{\alpha_0}\frac{1}{2}\rho)$ must have positive multiplicity $l!$. It turns out that the lowest K-types of the other sums of virtual representations in the claim have positive multiplicity $l!$ (see Theorem 8). □

A definition of a τ invariant says that if $\Theta(\gamma)$ is an irreducible character, then a simple root $\alpha \in R^+(\gamma)$ is a τ invariant iff $\Theta(s_\alpha \gamma) = -\Theta(\gamma)$ [11]. If $\Theta(\Psi, \gamma)$ is a virtual character, a simple root $\alpha \in R^+(\gamma)$ is a τ invariant iff $\Theta(\Psi, s_\alpha \gamma) = -\Theta(\Psi, \gamma)$. Indeed it is clear that if each genuine representation in $\Theta(\Psi, \gamma)$ has α in its τ invariant, then $\Theta(\Psi, s_\alpha \gamma) = -\Theta(\Psi, \gamma)$. To show the converse we note that we can't have a situation whereby the virtual representation equals $X - Y$, and $s_\alpha X = Y$ since if $\alpha \notin \tau(X)$, then $s_\alpha X = X +$ (nonzero representation).

The sums of virtual representations were constructed to have the property that s_α acts by the sign representation, for every simple integral root. This give the following lemma.

Lemma 4.9. *The sums of virtual representations of Lemma* 4.8 *have a maximal primitive ideal.*

We can now derive the character formulas for Langlands quotients of $\widetilde{\mathrm{SL}(2l)}$ with infinitesimal character $\frac{1}{2}\rho$, having a maximal primitive ideal.

Theorem 8. *With the notation introduced above Lemma* 4.8*,*

(1) $J_{P_{max}}(\frac{1}{2}\rho) = \Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$.
(2) $J^+_{P_{min}}(\frac{1}{2}\rho) = \Theta_{(\frac{1}{2},\ldots,\frac{1}{2})}$.
(3) $J^-_{P_{min}}(\frac{1}{2}\rho) = \Theta_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$.
(4) $J_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho) = \Theta_{(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})}$.

Proof. First we show that $\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$ is an irreducible representation. This sum of virtual representations has infinitesimal character $\frac{1}{2}\rho$ and by Lemma 4.9 has a maximal primitive ideal. It must therefore be an integer linear combination of the four Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. In Lemma 4.8 we showed that $\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$ has the lowest K type $(\frac{3}{2},\ldots,\frac{3}{2})$ with multiplicity 1. Because the four Langlands quotients under consideration have the lowest K-types $(\frac{3}{2},\ldots,\frac{3}{2})$, $(\frac{1}{2},\ldots,\frac{1}{2})$, $(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})$, and $(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$, our sum of virtual representations must be irreducible and equal to the Langlands quotient with the lowest K type $(\frac{3}{2},\ldots,\frac{3}{2})$, namely $J_{P_{max}}(\frac{1}{2}\rho)$. The same argument shows that $J_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho) = \Theta_{(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})}$.

We use a coherent continuation argument to show that the other two sums of virtual representations are in fact irreducible representations. Consider a coherent family based at the irreducible character $\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$. If $\mu = (-\frac{1}{2},\frac{1}{2},\ldots,-\frac{1}{2},\frac{1}{2})$ is the extremal weight of a finite dimensional representation of $\widetilde{\mathrm{SL}(2l)}$, then $\frac{1}{2}\rho + \mu = w_0\frac{1}{2}\rho$

is dominant with respect to the integral roots with respect to $\frac{1}{2}\rho$. Using properties of translation functors on virtual characters [8] (Theorem 4.6) and [5] (Lemmas 5.3 − 5.5) we have $\Psi^{\frac{1}{2}\rho}_{\frac{1}{2}\rho+\mu}(\Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}) = \Theta_{(\frac{1}{2},\ldots,\frac{1}{2})}$. The translated irreducible representation is necessarily irreducible. Similarly we have $\Psi^{s_{\alpha_0}\frac{1}{2}\rho}_{s_{\alpha_0}\frac{1}{2}\rho+\mu}(\Theta_{(\frac{3}{2},\ldots,\frac{3}{2},-\frac{3}{2})}) = \Theta_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$. The dominance of $s_{\alpha_0}\frac{1}{2}\rho + \mu$ also proves the irreducibility of $\Theta_{(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})}$. Finally, matching the lowest K-types proves the claim. □

For $\widetilde{\mathrm{SL}(2l+1)}$, we have the following character formula for the Langlands quotient with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal.

Theorem 9. *With the notation introduced above Lemma* 4.8*,*

$$J_{P_{min}}(\tfrac{1}{2}\rho) = \Theta_{(\frac{1}{2},\ldots,\frac{1}{2})}.$$

Proof. The proof is identical to the proof that $J_{P_{max}}(\frac{1}{2}\rho) = \Theta_{(\frac{3}{2},\ldots,\frac{3}{2})}$ in Theorem 8. □

The character formula in Theorems 8 and 9 together with Blattner's formula can be used to determine the K-types of our Langlands quotients of $\widetilde{\mathrm{SL}(m)}$ [21]. Applying Blattner's formula, however, quickly becomes intractable for large m so in section 5 we determine the K-types using the tools of cohomological induction.

5. K-TYPE SPECTRUM FOR $\widetilde{\mathrm{SL}(m)}$

5.1. **K-types for $\widetilde{\mathrm{SL}(2n)}$.** We identify the four small Langlands quotients of $\widetilde{\mathrm{SL}(2n)}$ with infinitesimal character $\frac{1}{2}\rho$ as cohomologically induced representations of type $A_q(\lambda)$. If $(\mathfrak{g}, K)$ is a reductive pair and $(\mathfrak{l}, L\cap K)$ is a Levi pair built from a Θ stable parabolic subalgebra of $\mathfrak{g}$, *cohomological induction* refers to to the functor $\mathcal{L}_j$ passing from $(\mathfrak{l}, L\cap K)$ modules to $(\mathfrak{g}, K)$ modules.

To put us in a setting where we can use cohomological induction, we identify a Levi subgroup of $\widetilde{\mathrm{SL}(2n)}$ inside $\mathrm{GL}(n,\mathbb{C})$ and use this identification to write down unitary characters on the Levi subgroup. To start, consider the natural inclusion of $\mathrm{GL}(n,\mathbb{C})$ in $\mathrm{GL}(2n)$ given by the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Under this inclusion we have the relation $\det_{\mathbb{R}} = |\det_{\mathbb{C}}|^2$ between real and complex determinants. The image of $\mathrm{GL}(n,\mathbb{C})$ is precisely the centralizer of $J = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix}$. Its intersection with $\mathrm{SL}(2n)$ is a Levi subgroup in $\mathrm{SL}(2n)$ isomorphic to

$$L = \{g \in \mathrm{GL}(n,\mathbb{C}) \mid |\det_{\mathbb{C}}(g)| = 1\}.$$

On L there exist unitary characters $\phi_p = \det_{\mathbb{C}}^p$, for $p \in \mathbb{Z}$. Let

$$\widetilde{L} = \{(g,z) \in L \times \mathbb{C}^* \mid \det_{\mathbb{C}}(g) = z^2\}$$

be the square root of the determinant double cover of L. Projection on the second factor defines a unitary character $\phi_{\frac{1}{2}}$ of $\widetilde{L}$. The square of this character descends to L where it can be identified with ϕ_1. By defining $\phi_{\frac{p}{2}} = \phi_{\frac{1}{2}}^p$ for $p \in \mathbb{Z}$ we get unitary characters of $\widetilde{L}$ descending to L for $p \in 2\mathbb{Z}$. In this way, $\widetilde{L}$ has twice as many characters as L.

The spin double cover of $\mathrm{SO}(2n)$ induces a double cover $\widetilde{\mathrm{U}(n)}$ of $\mathrm{U}(n)$. This is isomorphic to the square root of the determinant double cover just considered. It follows that the preimage of L in $\widetilde{\mathrm{SL}(2n)}$ is isomorphic to the square root of the determinant $\widetilde{L}$.

Let $\mathfrak{u}$ be the sum of the positive eigenspaces of $\mathrm{ad}(iJ)$. There is a theta-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ in the complexified Lie algebra of $\widetilde{\mathrm{SL}(2n)}$ with Levi factor $\widetilde{L}$.

Below we will realize half of the small Langlands quotients, with infinitesimal character $\frac{1}{2}\rho$ (for the other half we need a different Levi subgroup, L', defined before Proposition 5.4), as cohomologically induced representations, $\mathcal{L}_*(\phi_{\frac{p}{2}})$, from $\widetilde{L}$ to $\widetilde{\mathrm{SL}(2n)}$ via $\mathfrak{q}$. Our first task is to determine for which p our cohomologically induced representations, $\mathcal{L}_*(\phi_{\frac{p}{2}})$, have infinitesimal character $\frac{1}{2}\rho$.

Proposition 5.1. *The cohomologically induced representation $\mathcal{L}_*(\phi_{\frac{p}{2}})$ has infinitesimal character $\frac{1}{2}\rho$ for $p = -2n-1$ and $p = -2n+1$. These representations are unitary representations of type $A_q(\lambda)$.*

Proof. Writing $\mathfrak{l} = \{A + \imath B \mid A, B \in \mathfrak{gl}(n) \text{ and } \mathrm{tr}(A) = 0\}$, the differential of $\phi_{\frac{p}{2}}$ on $\mathfrak{l}$ is $d\phi_{\frac{p}{2}}(A + \imath B) = \frac{p}{2} tr_{\mathbb{C}}(A + \imath B) = \frac{p}{2} tr(A) + \frac{\imath p}{2} tr(B)$. Writing elements of $\mathfrak{l}$ in $\mathfrak{sl}(2n)$ as $\mathfrak{l}_o$, we have $\mathfrak{l}_o = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$, where $A, B \in \mathfrak{gl}(n)$ with $\mathrm{trace}(A) = 0$. Let $\mathfrak{a}_0$ be a maximum abelian subspace of $\mathfrak{q}_0$ and $\mathfrak{t}_0$ a maximal toral subalgebra of $\mathfrak{sl}(2n)$. An element of the θ stable Cartan subalgebra $\mathfrak{h}_o = \mathfrak{a}_o + \mathfrak{t}_o$ of $\mathfrak{l}_o$ is conjugate to a diagonal matrix with $a_1 + \imath b_1, \ldots, a_n + \imath b_n$ along the first n diagonal entries and $a_1 - \imath b_1, \ldots, a_n - \imath b_n$ along the last n diagonal entries. If e_i is the linear functional that evaluates the i^{th} coordinate, for $i \in \{1, 2, \ldots, 2n\}$ with $\sum_{i=1}^{2n} e_i = 0$, the trace on $\mathfrak{a}_o$ is zero while the the trace on $\mathfrak{t}_o$ can be written as

$$\frac{1}{2}(e_1 + \cdots + e_n - e_{n+1} - \cdots - e_{2n}) = (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}).$$

It follows that $d\Phi_{\frac{p}{2}} = (\frac{p}{4}, \ldots, \frac{p}{4}, -\frac{p}{4}, \ldots, -\frac{p}{4})$. This is the weight of the one dimensional representation $d\phi_{\frac{p}{2}}$.

The infinitesimal character of $\mathcal{L}_*(\phi_{\frac{p}{2}})$ is given by adding $\rho(\Delta^+(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{h})) + \rho(\mathfrak{u})$, which equals $(\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, -\frac{2n-1}{2})$, to the highest weight of $d\phi_{\frac{p}{2}}$. It follows that the infinitesimal character of $d\Phi_{\frac{p}{2}}$ is $(\frac{p+4n-2}{4}, \frac{p+4n-6}{4}, \ldots, -\frac{p+4n-2}{4})$. Setting this equal to $\frac{1}{2}\rho = (\frac{2n-1}{4}, \frac{2n-3}{4}, \ldots, -\frac{2n-1}{4})$ there are only two solutions for p resulting from writing the odd entries of $\frac{1}{2}\rho$ first, followed by the even entries, and vice versa. Permuting the entries of $(\frac{2n-1}{4}, \frac{2n-3}{4}, \ldots, -\frac{2n-1}{4})$ so that the odd entries are written first, followed by the even entries, we find that $p = -2n+1$. In the other case we find that $p = -2n-1$.

Because the infinitesimal character of $\phi_{\frac{p}{2}}$ plus $\rho(\mathfrak{u})$ is positive with respect to $\Delta(\mathfrak{u})$, the cohomologically induced representations $\mathcal{L}_j(\phi_{\frac{p}{2}})$ vanish except for $j = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$ (see Theorems 5.99 and 9.68 of [13]). The analytically integral member $\lambda = (\frac{p}{4}, \ldots, \frac{p}{4}, -\frac{p}{4}, \ldots, -\frac{p}{4})$ of $\mathfrak{h}^*$ is orthogonal to all members of $\Delta(\mathfrak{l}) = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$ and thus by definition $\mathcal{L}_*(\phi_{\frac{p}{2}}) = A_{\mathfrak{q}}(\lambda)$ for $\lambda = (\frac{p}{4}, \ldots, \frac{p}{4}, -\frac{p}{4}, \ldots, -\frac{p}{4})$. Further, because λ is real on $\mathfrak{t}_o$ and imaginary on $\mathfrak{a}_o$, $A_{\mathfrak{q}}(\lambda)$ is a unitary representation (Corollary 9.70 of [13]). ☐

Next we prove that our representations have a maximal primitive ideal. We begin with the following technical result.

Lemma 5.2. *Let $I \subset U(\mathfrak{g})$ be a two sided ideal. If Z is a $U(\mathfrak{g})/I$ module, then $\mathcal{L}_j(Z)$ is a $U(\mathfrak{g})/I$ module.*

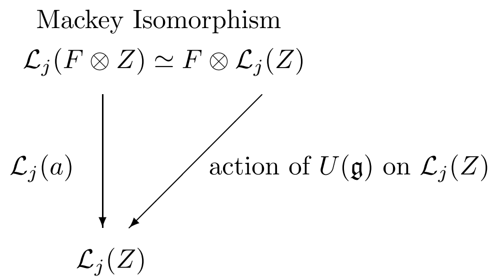

FIGURE 3. Commutative diagram

Proof. Let $F \subset U(\mathfrak{g})$ be an $\mathrm{Ad}(G)$ invariant finite dimensional representation. The action of $U(\mathfrak{g})$ defines a map $a : F \otimes Z \longrightarrow Z$ given by the $\mathrm{ad} \otimes Z$ action of $\mathfrak{g}$ on $F \otimes Z$. For the generalized Verma module $W = U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} Z^\sharp$, we have the isomorphism $(P^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j(F \otimes W) \simeq F \otimes (P^{\mathfrak{g},K}_{\mathfrak{g},L\cap K})_j(W)$ by the Mackey Isomorphism in Lemma 2.99 of [13] and the identification of the Bernstein functor Π and $P^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}$ (Proposition 2.69 in [13]). This gives the isomorphism $\mathcal{L}_j(F \otimes Z) \simeq F \otimes \mathcal{L}_j(Z)$ in Figure 3. The commutative diagram in Figure 3 follows from a generalization of $\mathfrak{g}$ to $U(\mathfrak{g})$ in Proposition 3.77 of [13]. Since Ad acts locally finite on $U(\mathfrak{g})$ we may take $F \subset I$. By the commutativity of Figure 3, $I \cdot Z = 0$ implies that the map $a = 0$. Hence $\mathcal{L}_j(a) = 0$ and F kills $\mathcal{L}_j(Z)$ for any $F \subset I$ proving the claim. □

Proposition 5.3. *The representations $A_{\mathfrak{q}}(\lambda)$, for $\lambda = (\frac{p}{4}, \ldots, \frac{p}{4}, -\frac{p}{4}, \ldots, -\frac{p}{4})$, with $p = -2n - 1$ and $p = -2n + 1$ are small.*

Proof. In what follows Dim will refer to Gelfand Kirillov dimension and dim will refer to the usual vector space dimension. Since $\mathbb{C}_\lambda$ is a $U(\mathfrak{g})/\mathrm{Ann}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda)$ module, by Lemma 5.2, $\mathcal{L}_*(\mathbb{C}_\lambda)$ is a $U(\mathfrak{g})/\mathrm{Ann}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda)$ module. It follows that $\mathrm{Ann}(L_*(\mathbb{C}_\lambda)) \subset \mathrm{Ann}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda)$ and so

$$\mathrm{Dim}(U(\mathfrak{g})/\mathrm{Ann}(L_*(\mathbb{C}_\lambda))) \leq \mathrm{Dim}(U(\mathfrak{g})/\mathrm{Ann}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda)).$$

By Corollary 4.7 of [9] and the fact that $\mathcal{L}_*(\mathbb{C}_\lambda)$ has finite length, $2\mathrm{Dim}(\mathcal{L}_*(\mathbb{C}_\lambda)) = \mathrm{Dim}(U(\mathfrak{g})/\mathrm{Ann}(\mathcal{L}_*(\mathbb{C}_\lambda)))$. Furthermore, by Proposition 6.5 of [14], $2\mathrm{Dim}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda) = \mathrm{Dim}(U(\mathfrak{g})/\mathrm{Ann}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda)))$. By Lemma 2.3 of [9] together with the fact that the Gelfand Kirillov dimension of a finite dimensional representation is zero, we have $\mathrm{Dim}(U(\mathfrak{g}) \otimes_{\overline{\mathfrak{q}}} \mathbb{C}_\lambda) = \dim(\mathfrak{g}/\overline{\mathfrak{q}}) = \dim(\mathfrak{u}) = n^2$.

The Gelfand Kirillov dimension of the minimal representation with infinitesimal character γ is $r - r_\gamma$, where r is the number of positive roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ and r_γ is the number of integral positive roots with respect to γ. For $\gamma = \frac{1}{2}\rho$ we have $r = n(2n-1)$ and $r_\gamma = n(n-1)$ so the dimension of the representation with infinitesimal character $\frac{1}{2}\rho$ is n^2. Hence, $\mathrm{Dim}(L_*(\mathbb{C}_\lambda))$ is less than or equal to the Gelfand Kirillov dimension of the minimal representation with infinitesimal character $\frac{1}{2}\rho$ proving the claim. □

To identify the remaining two out of four small Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ of $\widetilde{\mathrm{SL}(2n)}$ we use another Levi subgroup, $\widetilde{L}'$ of $\widetilde{\mathrm{SL}(2n)}$. In $\mathrm{SL}(2n)$, L' is the centralizer of $J' = \begin{bmatrix} & & & I_{n-1} \\ & & 1 & \\ & -1 & & \\ I_{n-1} & & & \end{bmatrix}$, and $\widetilde{L}'$ is the square root of the determinant double cover of L'. We denote the Lie algebra of $\widetilde{L}$ by $\mathfrak{l}'$. Let the nilradical $\mathfrak{u}' = ad(\imath J')$, and the theta stable parabolic $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}'$. Following the proofs of Proposition 5.1 and Proposition 5.3 we have the following.

Proposition 5.4. *The cohomologically induced representation $\mathcal{L}_*(\phi_{\frac{p}{2}})$ on $\widetilde{L}'$ has infinitesimal character $\frac{1}{2}\rho$ for $p = -2n-1$ and $p = -2n+1$. These representations, for $\lambda = (\frac{p}{4}, \dots, \frac{p}{4}, -\frac{p}{4}, \dots, -\frac{p}{4})$, are small unitary representations of type $A_q(\lambda)$.*

Next we identify the K-types for our $A_{\mathfrak{q}}(\lambda)$ representations.

Theorem 10. *The K-types of $\mathcal{L}_*(\phi_{\frac{p}{2}})$ on $\widetilde{L}$ have highest weights*

$$(\frac{1}{2} + 2a_1, \dots, \frac{1}{2} + 2a_n) \quad \text{for}\, p = -2n-1,$$

and

$$(\frac{3}{2} + 2a_1, \dots, \frac{3}{2} + 2a_n) \quad \text{for}\, p = -2n+1,$$

with $a_1 \geq \cdots \geq a_n \geq 0$.

The K-types of $\mathcal{L}_*(\phi_{\frac{p}{2}})$ on $\widetilde{L'}$ have highest weights

$$(\frac{1}{2} + 2a_1, \dots, \frac{1}{2} + 2a_{n-1}, -\frac{1}{2} - 2a_n) \quad \text{for}\, p = -2n-1,$$

and

$$(\frac{3}{2} + 2a_1, \dots, \frac{3}{2} + 2a_{n-1}, -\frac{3}{2} - 2a_n) \quad \text{for}\, p = -2n+1,$$

with $a_1 \geq \cdots \geq a_n \geq 0$.

Proof. We first determine the K-types of $\mathcal{L}_*(\phi_{\frac{p}{2}})$ on the Levi factor $\widetilde{L}$, for $p = -2n-1$ or $p = -2n+1$. Because $\phi_{\frac{p}{2}}$ is one dimensional, the hypothesis for Theorem 5.64 in [13] is satisfied. Letting $Z = \phi_{\frac{p}{2}}$, the theorem states that for any finite dimensional $(\mathfrak{k}, K)$ module V,

$$\sum_{j=0}^{S}(-1)^j \dim\mathrm{Hom}_K(\mathcal{L}_j(Z), V)$$

$$= \sum_{j=0}^{S}(-1)^j \sum_{n=0}^{\infty} \dim\mathrm{Hom}_{L\cap K}(S^n(\mathfrak{u}\cap\mathfrak{p}) \otimes Z^\sharp, H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, V)), \tag{1}$$

where $S = \dim_{\mathbb{C}}(\mathfrak{u}\cap\mathfrak{k})$. The terms on the right side of equation (1) are zero for n sufficiently large and so commuting Hom with a finite sum we can replace the right hand side of equation (1) with

$$\sum_{j=0}^{S}(-1)^j \dim\mathrm{Hom}_{L\cap K}(S(\mathfrak{u}\cap\mathfrak{p}) \otimes Z^\sharp, H^j(\overline{\mathfrak{u}}\cap\mathfrak{k}, V)). \tag{2}$$

Only the top cohomology is nonzero by the following result.

Lemma 5.5. *The irreducible representations of $L\cap K$ dominant for K appear in $H^j(\overline{\mathfrak{u}}\cup\mathfrak{k}, V)$ only for $j = \dim_{\mathbb{C}}(\mathfrak{u}\cup\mathfrak{k})$.*

Proof. Let λ be dominant in the positive root system

$$\Delta_1^+(\mathfrak{k}, \mathfrak{t}) = (\Delta^+(\mathfrak{k}, \mathfrak{t}) - \Delta(\mathfrak{u}\cap\mathfrak{k})) \cup \Delta(\overline{\mathfrak{u}}\cap\mathfrak{k}).$$

By Kostant's theorem, the $L\cap K$ types of $H^j(\overline{\mathfrak{u}}\cup\mathfrak{k}, V_\lambda)$ have highest weight $w(\lambda+\delta)-\delta$, for w an element of length j in the reflection group generated by reflections s_α, where α is a simple root in $\Delta(\overline{\mathfrak{u}} \cap \mathfrak{k})$. $w(\lambda+\delta)-\delta$ is dominant with respect to $\Delta^+(\mathfrak{k},\mathfrak{t})$ iff $w(\lambda+\delta)$ lies in the dominant Weyl chamber of $\Delta^+(\mathfrak{k},\mathfrak{t})$. $W(\mathfrak{k},\mathfrak{t})$ acts simply transitively on the dominant Weyl chamber so there exists a unique $w_0 \in W(\mathfrak{k},\mathfrak{t})$ sending the Weyl chamber containing $\lambda+\delta$ to the dominant chamber. Because the simple roots of $\Delta^+(\mathfrak{k},\mathfrak{t})$ and $\Delta_1^+(\mathfrak{k},\mathfrak{t})$ differ exactly by the reflection of the simple roots in $\Delta(\mathfrak{u}\cap\mathfrak{k})$, w_0 must be a product of these $\dim_{\mathbb{C}}(\mathfrak{u}\cap\mathfrak{k})$ simple reflections as required. □

The $\mathcal{L}_j(Z)$ in the right hand side of equation (1) all vanish except for the top one, $\mathcal{L}_S(Z)$. The fact that $\frac{1}{2}\rho$ is dominant in $\Delta(\mathfrak{u})$ allows us to apply Theorem 5.99 in [13]. Equation (1) now simplifies to

$$\begin{aligned}&\dim\mathrm{Hom}_K(\mathcal{L}_j(Z),V)\\ &\quad=\dim\mathrm{Hom}_{L\cap K}(S(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\sharp, H^S(\overline{\mathfrak{u}}\cap\mathfrak{k},V)).\end{aligned}$$

Lemma 5.6. $\mathfrak{sp}(2n,\mathbb{R})/\mathfrak{u}(n) \simeq \mathfrak{u}\cap\mathfrak{p}$ *as representations of* $\mathfrak{u}(n)$.

Proof. We show that $\mathfrak{sp}(2n,\mathbb{R})/\mathfrak{u}(n)$ and $\mathfrak{u}\cap\mathfrak{p}$ are isomorphic as vector spaces and have the same highest weight under the adjoint representation of $\mathfrak{u}(n)$. Indeed, $\dim_{\mathbb{R}}\mathfrak{u}(n) = n^2$ and $\dim_{\mathbb{R}}\mathfrak{sp}(2n,\mathbb{R}) = n(n+1)+n^2$ so $\dim_{\mathbb{R}}\mathfrak{sp}(2n,\mathbb{R})/\mathfrak{u}(n) = n(n+1)$. $\mathfrak{u}\cap\mathfrak{p}$ is the space of $n\times n$ complex matrices which has complex dimension $n(n+1)/2$ and real dimension $n(n+1)$ as required. Furthermore, $u\in\mathfrak{u}(n)$ acts on $\mathfrak{u}\cap\mathfrak{p}$ by $u\cdot X = uXu^t$ with highest weight $(2,0,\dots,0)$. By Schmid's theorem on the decomposition of the symmetric algebra, $S^k(\mathfrak{sp}(2n,\mathbb{R})/\mathfrak{u}(n))$ is the sum of representations of $U(n)$ with highest weight $(2a_1,\dots,2a_n)$ with $a_1\geq\dots\geq a_n\geq 0$ and $\sum_{i=1}^n a_i = k$ [15]. In the special case of $k=1$, $\mathfrak{sp}(2n,\mathbb{R})/\mathfrak{u}(n)$ has highest weight $(2,0,\dots,0)$, proving the lemma. □

Next we find the lowest weight of $\mathcal{L}_*(\phi_{\frac{p}{2}})$. $\bigwedge^{top}\mathfrak{u}$ is a 1 dimensional representation of $L\cap K$ with unique weight $2\delta(\mathfrak{u})$ relative to $\mathfrak{h}$. Hence if Z has highest weight μ, then $Z^\sharp$ has highest weight $\mu+2\delta(\mu)$. $Z=\phi_{\frac{p}{2}}$ has highest weight $(\frac{p}{2},\dots,\frac{p}{2})$ and $2\delta(\mathfrak{u}) = (2n,\dots,2n)$, so $Z^\sharp$ has highest weight $(\frac{p}{2}+2n,\dots,\frac{p}{2}+2n)$. By Lemma 5.6 and Schmid's theorem on the decomposition of the symmetric algebra, the representations of $L\cap K\simeq U(n)$ appearing in the symmetric algebra, $S(\mathfrak{u}\cap\mathfrak{p})$, are those with highest weights $(2a_1,\dots,2a_n)$ with the a_i decreasing nonnegative integers. These highest weights are shifted by the highest weight of $Z^\sharp$. It follows that $S(\mathfrak{u}\cap\mathfrak{p})\otimes Z^\sharp$ has highest weights $(\frac{p}{2}+2n,\dots,\frac{p}{2}+2n)+(2a_1,\dots,2a_n)$ with $a_1\geq\dots\geq a_n\geq 0$. We need one final lemma.

Lemma 5.7. *If* V *has highest weight* μ*, then* $H^S(\overline{\mathfrak{u}}\cap\mathfrak{k},V)$ *has highest weight* $\mu+2\delta(\mathfrak{u}\cap\mathfrak{k})$.

Proof. We have $H^S(\overline{\mathfrak{u}}\cap\mathfrak{k},V) = H_0(\overline{\mathfrak{u}}\cap\mathfrak{k},V\otimes\bigwedge^S(\overline{\mathfrak{u}}\cap\mathfrak{k})^*)$ by Corollary 3.13 in [13]. This equals $H_0(\overline{\mathfrak{u}}\cap\mathfrak{k},V)\otimes\bigwedge^S(\overline{\mathfrak{u}}\cap\mathfrak{k})^*$ since $\bigwedge^S(\overline{\mathfrak{u}}\cap\mathfrak{k})^*$ is a trivial $(\overline{\mathfrak{u}}\cap\mathfrak{k})^*$ module. Furthermore, $H_0(\overline{\mathfrak{u}}\cap\mathfrak{k},V)\simeq V^{\mathfrak{u}\cap\mathfrak{k}}$ by Lemma 4.82 in [13]. Hence $V^{\mathfrak{u}\cap\mathfrak{k}}\otimes\bigwedge^{top}(\mathfrak{u}\cap\mathfrak{k})$ has highest weight $\mu+2\delta(\mathfrak{u}\cap\mathfrak{k})$ as claimed. □

Combining the results above we see that $\mathcal{L}_*(Z)$ has the lowest K type equal to the highest weight of $Z^\sharp$, minus $2\delta(\overline{\mathfrak{u}}\cap\mathfrak{k})$. We have $2\delta(\overline{\mathfrak{u}}\cap\mathfrak{k}) = (n-1,\dots,n-1)$ so $\mathcal{L}_*(Z)$ has the lowest K type $(\frac{p}{2}+2n,\dots,\frac{p}{2}+2n)-(n-1,\dots,n-1)$ equaling $(\frac{p}{2}+n+1,\dots,\frac{p}{2}+n+1)$. Letting $p=-2n-1$, $\mathcal{L}_*(Z)$ has the lowest K type

$(\frac{1}{2},\ldots,\frac{1}{2})$, and letting $p=-2n+1$, $\mathcal{L}_*(Z)$ has the lowest K type $(\frac{3}{2},\ldots,\frac{3}{2})$. The claim follows immediately for the Levi $\widetilde{L}$.

The same argument as above gives the proof of the analogous result for $\mathcal{L}_*(\phi_{\frac{p}{2}})$ on $\widetilde{L'}$. □

5.2. **K-types for** $\widetilde{\mathrm{SL}(2n+1)}$**.** To find the K type spectrum of the single small representation of $\widetilde{\mathrm{SL}(2n+1)}$ with infinitesimal character $\frac{1}{2}\rho$, we will write this representation as an alternating sum of parabolically induced representations

$$\sum_{j=0}^{n}(-1)^j(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\wedge^j\mathbb{C}^n))\otimes|\det|^t. \tag{3}$$

Let the Levi, L, in $\mathrm{SL}(2n+1)$, be the centralizer of $\begin{bmatrix}1 & \\ & J\end{bmatrix}$ where $J=\begin{bmatrix} & I_n\\ I_n & \end{bmatrix}$. The Levi subalgebra, $\mathfrak{l}$, is embedded in $\mathfrak{sl}(2n+1)$ as $\mathrm{A}\in\mathfrak{gl}(2n)$ in the $2n\times 2n$ lower diagonal and $-\mathrm{Tr}(A)$ in the upper 1×1 diagonal. L is isomorphic to $\mathrm{GL}(n,\mathbb{C})$ and $\widetilde{L}$ is isomorphic to the square root of the determinant double cover of L described in section 5.1. Let $\phi_{\frac{p}{2}}$ be the unitary character on $\widetilde{L}$ defined in section 5.1. For integers j between 0 and n we have the finite dimensional representations, $\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n$, of $\widetilde{L}$. We begin with the following two technical results.

Lemma 5.8. *The cohomologically induced representation* $\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)$ *of* $\widetilde{\mathrm{GL}(2n)}$ *has infinitesimal character* $\frac{1}{2}\rho+(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{2n-j})$ *for* $p=-2n-1$. *The Langlands parameter in this case is*

$$(\frac{2n-3}{4},\frac{2n-7}{4},\ldots,-\frac{2n-1}{4},\frac{2n-1}{4},\frac{2n-5}{4},\ldots,-\frac{2n-3}{4})+(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{2n-j}).$$

Proof. Writing $\mathfrak{l}=\{A+\imath B\,|\,A,B\in\mathfrak{gl}(n)\}$, the differential of $\phi_{\frac{p}{2}}$ on $\mathfrak{l}$ is

$$d\phi_{\frac{p}{2}}(A+\imath B)=\frac{p}{2}tr_{\mathbb{C}}(A+\imath B)=\frac{p}{2}tr(A)+\frac{\imath p}{2}tr(B).$$

Writing elements of $\mathfrak{l}$ in $\mathfrak{gl}(2n)$ as $\mathfrak{l}_\circ$, we have $\mathfrak{l}_\circ=\begin{bmatrix}A & B\\ -B & A\end{bmatrix}$. An element of the diagonal Cartan subalgebra $\mathfrak{h}_\circ=\mathfrak{a}_\circ+\mathfrak{t}_\circ$ of $\mathfrak{l}_\circ$ is conjugate to a matrix with $a_1+\imath b_1,\ldots,a_n+\imath b_n$ along the first n diagonal entries and $a_1-\imath b_1,\ldots,a_n-\imath b_n$ along the last n diagonal entries. If e_i is the linear functional that evaluates the i^{th} coordinate, for $i\in\{1,2,\ldots,2n\}$ the trace on $\mathfrak{a}_\circ$ can be written as $\frac{1}{2}(e_1+\cdots+e_{2n})=(\frac{1}{2},\ldots,\frac{1}{2})$, while the trace on $\mathfrak{t}_\circ$ can be written as $\frac{1}{2}(e_1+\cdots+e_n-e_{n+1}-\cdots-e_{2n})=(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2})$. It follows that up to adding a scalar (namely, a multiple of $(\frac{1}{2},\ldots,\frac{1}{2})$) that $d\Phi_{\frac{p}{2}}=(\frac{p}{4},\ldots,\frac{p}{4},-\frac{p}{4},\ldots,-\frac{p}{4})$. This is the highest weight of the one dimensional representation $d\phi_{\frac{p}{2}}$. Furthermore, $\mathfrak{l}=\mathfrak{gl}(n,\mathbb{C})$ acts on $\bigwedge^j\mathbb{C}^n$ with highest weight $(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{n-j})$. Taken as a representation of l_0 in $\mathfrak{gl}(2n)$, $\bigwedge^j\mathbb{C}^n$ has highest weight $(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{2n-j})$. It follows that the highest weight of $\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n$ is $(\frac{p}{4},\ldots,\frac{p}{4},-\frac{p}{4},\ldots,-\frac{p}{4})+(\underbrace{1,\ldots,1}_{j},\underbrace{0,\ldots,0}_{2n-j})$.

The infinitesimal character of $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$ is given by adding $\rho(\Delta^+(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{h}))$ $+ \rho(\mathfrak{u})$, which equals $(\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, -\frac{2n-1}{2})$, to the highest weight of $d\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n$. Therefore, the infinitesimal character of $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$ is $(\frac{p+4n-2}{4}, \frac{p+4n-6}{4}, \ldots, -\frac{p+4n-2}{4}) + (\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2n-j})$. Analogous to the calculation done in Proposition 5.1 we set this equal to the infinitesimal character $\frac{1}{2}\rho + (\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2n-j})$ and solve for p. There are precisely two solutions for p resulting from writing the odd entries of $\frac{1}{2}\rho$ first followed by the even entries and vice versa. Permuting the entries of $(\frac{2n-1}{4}, \frac{2n-3}{4}, \ldots, -\frac{2n-1}{4})$ so that the odd entries are written first followed by the even entries we find that $p = -2n+1$. In the other case we find that $p = -2n-1$. For what follows we will only be interested in the case where $p = -2n-1$. □

Lemma 5.9. *The infinitesimal character of* $(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}} \mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)) \otimes |\det|^t$, *is* $\frac{1}{2}\rho$ *for* $t = -\frac{2n+1}{4}$ *and* $p = -2n-1$. *The Langlands parameter is*

$$(\frac{n}{2} - j, \frac{n}{2}, \frac{n}{2} - 1, \ldots, \widehat{\frac{n}{2} - j}, \ldots, -\frac{n}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}),$$

where the entry $\widehat{\frac{n}{2} - j}$ *is omitted. Hence, for* $p = -2n-1$ *and* $t = -\frac{2n+1}{4}$, *the virtual representation*

$$\sum_{j=0}^{n} (-1)^j (\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}} \mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)) \otimes |\det|^t$$

has infinitesimal character $\frac{1}{2}\rho$.

Proof. For $p = -2n-1$, the cohomologically induced representation $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$ of $\widetilde{\mathrm{GL}(2n)}$ has, by Lemma 5.8, the Langlands parameter

$$(\frac{2n-3}{4}, \frac{2n-7}{4}, \ldots, -\frac{2n-1}{4}, \frac{2n-1}{4}, \frac{2n-5}{4}, \ldots, -\frac{2n-3}{4}) + (\underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2n-j}).$$

Inside $\mathfrak{sl}(2n+1)$, the parabolically induced representation

$$\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}} \mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$$

has the Langlands parameter

$$(0, \frac{2n-3}{4}, \frac{2n-7}{4}, \ldots, -\frac{2n-1}{4}, \frac{2n-1}{4}, \frac{2n-5}{4}, \ldots, -\frac{2n-3}{4}) + (0, \underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2n-j}).$$

To get an infinitesimal character $\frac{1}{2}\rho = (\frac{n}{2}, \frac{n-1}{2}, \ldots, -\frac{n}{2})$, for $\mathfrak{sl}(2n+1)$ we must twist the induced representation by a character $|\det|^t$ with weight $\frac{t}{2n+1}(-2n, 1, \ldots, 1)$. Adding this weight to the Langlands parameter of the induced representation, and setting $t = -\frac{2n+1}{4}$, gives the Langlands parameter

$$(\frac{n}{2}, \frac{n-2}{2}, \frac{n-4}{2}, \ldots, -\frac{n}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}) + (-j, \underbrace{1, \ldots, 1}_{j}, \underbrace{0, \ldots, 0}_{2n-j}).$$

This equals $(\frac{n}{2}-j, \frac{n}{2}, \frac{n}{2}-1, \ldots, \widehat{\frac{n}{2}-j}, \ldots, -\frac{n}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2})$ as required. The integer sum of representations of infinitesimal character $\frac{1}{2}\rho$ has infinitesimal character $\frac{1}{2}\rho$ proving the claim. ☐

To show that the virtual representation has a maximal primitive ideal we will need the following.

Lemma 5.10. *$\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$, for $p=-2n-1$ and $0 \le j \le n$, are irreducible, small representations of $\widetilde{\mathrm{GL}(2n)}$.*

Proof. $\mathcal{L}_*(\phi_{\frac{p}{2}})$, for $p=-2n-1$, is a small representation of $\widetilde{\mathrm{GL}(2n)}$, with infinitesimal character $\frac{1}{2}\rho$ [6]. Since $\frac{1}{2}\rho$ is a regular infinitesimal character, there is a unique coherent family based at $\mathcal{L}_*(\phi_{\frac{p}{2}})$. The representations $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$, for $j>0$, have infinitesimal character $\frac{1}{2}\rho + (\underbrace{1,\ldots,1}_{j}, \underbrace{0,\ldots,0}_{2n-j})$ in a cone of the weight lattice based at $\frac{1}{2}\rho$. The weight, $\frac{1}{2}\rho + (\underbrace{1,\ldots,1}_{j}, \underbrace{0,\ldots,0}_{2n-j})$, is dominant and regular with respect to the positive integral roots, $R^+(\frac{1}{2}\rho)$, and hence $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \wedge^j \mathbb{C}^n)$ are irreducible members of the coherent family of $\mathcal{L}_*(\phi_{\frac{p}{2}})$. The dominance and regularity of these weights also implies that they have the same τ invariants as $\mathcal{L}_*(\phi_{\frac{p}{2}})$ and hence are small representations. ☐

For what follows, let Ψ be the standard positive roots, $e_i - e_j$ for $1 \le i < j \le 2n+1$, of $\mathfrak{sl}(2n+1)$. The Weyl group of $\mathfrak{sl}(2n+1)$ is isomorphic to the permutation group S_{2n+1} and permutes the $2n+1$ linear functionals $e_i \in \mathfrak{h}^*$. Let w_1 be the Weyl group element sending the odd functionals $e_1, e_3, \ldots, e_{2n+1}$ to the first $n+1$ functionals, $e_1, e_2, \ldots e_{n+1}$ and the even functionals $e_2, e_4, \ldots, e_{2n}$ to the last n functionals $e_{n+2}, e_{n+3}, \ldots, e_{2n+1}$. Let $w_1\Psi$ be the new positive root system resulting by changing the ordering of the e_i. For example in $\mathfrak{sl}(3)$ the positive roots of $w_1\Psi$ are $e_1-e_2, e_1-e_3, e_3-e_2$ and $\frac{1}{2}\rho$ is $(\frac{1}{2}, -\frac{1}{2}, 0)$. Under such a root system the odd entries of the infinitesimal character of $\frac{1}{2}\rho$ are written before the even ones. Writing the $n+1$ odd entries before the n even entries for $\frac{1}{2}\rho$ the integral Weyl group $W(R^+(\frac{1}{2}\rho))$ transitively permutes the first $n+1$ entries and the last n entries and is isomorphic to the group $S_{n+1} \times S_n$.

Proposition 5.11. *The virtual representation*

$$\sum_{j=0}^{n} (-1)^j (\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}} \mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)) \otimes |\det|^t$$

for $p=-2n-1$ and $t=-\frac{2n+1}{4}$ is small.

Proof. To show that our virtual representation is small we must show that the simple roots of the integral Weyl group, $W(R^+(\frac{1}{2}\rho))$, of $\mathfrak{sl}(2n+1)$, acts on the virtual representation by the sign representation (see equation (6) below for definition). Inside S_{n+1}, S_n can be realized as a subgroup of index $n+1$ permuting the last n entries in an $n+1$ tuple. In this way, we may realize $S_n \times S_n$ as a subgroup of $S_{n+1} \times S_n$ having index $n+1$ permuting the last $2n$ entries in a $2n+1$ tuple. Let A be the following set of $n+1$ permutations (each permutation written as a product of simple transpositions), $A = \{1, (12), (12)(23), \ldots, (12)(23)\ldots(n(n+1))\}$. The alternating sum

$$\sum_{j=0}^{n}(-1)^j(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^{j} \mathbb{C}^n)) \otimes |\det|^t$$

can be written as a sum over the elements of A

$$\sum_{a\in A}(-1)^{l(a)}(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^{l(a)} \mathbb{C}^n)) \otimes |\det|^t, \tag{4}$$

where $l(a)$ is the length of the Weyl group element a. The element $(1,2)(2,3),\ldots,$ $(j,j+1)$ in A, sends the Langlands parameter

$$(\frac{n}{2}, \frac{n}{2}-1, \ldots, -\frac{n}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2})$$

to

$$(\frac{n}{2}-j, \frac{n}{2}, \frac{n}{2}-1, \ldots, \widehat{\frac{n}{2}-j}, \ldots, -\frac{n}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}),$$

where the entry $\widehat{\frac{n}{2}-j}$ is omitted. This is the Langlands parameter of

$$(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^{j} \mathbb{C}^n)) \otimes |\det|^t$$

by Lemma 5.9. Writing equation (4) as an alternating sum of characters of continued fundamental series for $\widetilde{\mathrm{SL}(2n+1)}$, we have

$$\sum_{a\in A}(-1)^{l(a)}\Theta(w_1\Psi, a(\frac{1}{2}\rho)). \tag{5}$$

We are required to show that an element w of the integral Weyl group acts on expression (5) by the sign representation, i.e.

$$\sum_{a\in A}(-1)^{l(a)}\Theta(w_1\Psi, aw(\frac{1}{2}\rho)) = (-1)^{l(w)}\sum_{a\in A}(-1)^{l(a)}\Theta(w_1\Psi, a(\frac{1}{2}\rho)). \tag{6}$$

If w is an element of $S_n \times S_n$ inside $S_{n+1} \times S_n$, then

$$\Theta(w_1\Psi, aw(\frac{1}{2}\rho)) = (-1)^{l(w)}\Theta(w_1\Psi, a(\frac{1}{2}\rho)),$$

for all $a \in A$, since $\mathcal{L}_*(\phi_{\frac{p}{2}})$ is a small representation of $\widetilde{\mathrm{GL}(2n)}$. It remains to show that the elements of A act by the sign representation. For this it suffices to show that the simple reflection $w = s_{(e_1-e_2)}$ acts by the sign representation. Observe that the action of $w = s_{(e_1-e_2)}$ on expression (5) permutes the first two terms of the alternating sum. For the rest of the terms in expression (5) we have

$$\Theta(w_1\Psi, aw(\frac{1}{2}\rho)) = -\Theta(w_1\Psi, a(\frac{1}{2}\rho)).$$

This follows from the identity $as_{(e_1-e_2)} = s_{(e_2-e_3)}a$ for all $a \in A - \{1, (12)\}$. Hence $\Theta(w_1\Psi, as_{(e_1-e_2)}(\frac{1}{2}\rho)) = \Theta(w_1\Psi, s_{(e_2-e_3)}a(\frac{1}{2}\rho))$ for all $a \in A - \{1, (12)\}$. By Lemma 5.10, $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^{l(a)} \mathbb{C}^n)$ are small representations of $\widetilde{\mathrm{GL}(2n)}$, for $a \in A$. It follows that $\Theta(w_1\Psi, s_{(e_2-e_3)}(a\frac{1}{2}\rho)) = -\Theta(w_1\Psi, (a\frac{1}{2}\rho))$ for $a \in A - \{1, (12)\}$, completing the proof. □

Next we turn to the task of determining the K-types of the alternating sum of parabolically induced representations in expression (3). $Z = \phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}$ is an irreducible $(\mathfrak{l}, \mathrm{L} \cap \mathrm{K})$ module and $h_{\delta(\mathfrak{u})}$ acts by a scalar in Z so the hypothesis of theorem 5.64 in [13] is satisfied. Lemma 5.5 holds so only the top cohomology in equation (2) is nonzero. $\phi_{\frac{p}{2}}$ has highest weight $(\frac{p}{2}, \dots, \frac{p}{2})$. To find the highest weight of $Z = \phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n$ we will need the following result of Zelobenko (see page 232, section 79 of [16]).

Lemma 5.12. *Let μ be an irreducible representation of* $\mathrm{U}(n)$ *with highest weight $(\mu_1, \mu_2, \dots, \mu_n)$. Then $\mu \otimes \bigwedge^j \mathbb{C}^n$ is the sum of all irreducible representations of highest weight $(\mu_1 + \epsilon_1, \dots, \mu_n + \epsilon_n)$, where ϵ_i is* 0 *or* 1 *and $\sum_{i=1}^n \epsilon_i = j$, such that $(\mu_1 + \epsilon_1, \dots, \mu_n + \epsilon_n)$ is dominant.*

$\phi_{\frac{p}{2}}$ is a representation of $\mathrm{L} \cap \mathrm{K} = \mathrm{U}(n)$ so by Lemma 5.12, the highest weight of Z is $(\frac{p}{2}+\epsilon_1, \dots, \frac{p}{2}+\epsilon_n)$, where ϵ_i is 0 or 1 and $\sum_{i=1}^n \epsilon_i = j$, such that $(\frac{p}{2}+\epsilon_1, \dots, \frac{p}{2}+\epsilon_n)$ is dominant. Following the arguments below Lemmas 5.6 and 5.7 we have the following result.

Proposition 5.13. *The* $\mathrm{Pin}(2n)$ *types of $\mathcal{L}_*(\phi_{\frac{p}{2}} \otimes \bigwedge^j \mathbb{C}^n)$, for $p = -2n-1$, are the dominant weights $(\frac{1}{2}+2a_1+\epsilon_1, \dots, \frac{1}{2}+2a_n+\epsilon_n)$, where ϵ_i is* 0 *or* 1 *and $\sum_{i=1}^n \epsilon_i = j$.*

To find the $\mathrm{Spin}(2n+1)$ types of the alternating series in expression (3) we need the following key lemma, which is a consequence of the Weyl character formula.

Lemma 5.14. *Let K be a compact connected group containing a maximal torus T. Let Δ^+ be a positive root system of T in K. Let S be a subset of Δ^+ and $|S|$ be the order of S. Let μ be a dominant weight and F an irreducible representation of K. Let* $\mathrm{mult}_F(\gamma)$ *be the multiplicity of the weight γ in the restriction $F|_T$. Then,*

$$\sum_{S \subset \Delta^+} (-1)^{|S|} \, \mathrm{mult}_F(\mu + \sum_{\alpha \in S} \alpha) = \begin{cases} 1 & \text{if } F \text{ has highest weight } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let F have highest weight λ. By the Weyl character formula, the character of F is

$$\mathrm{char}(F) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.$$

Equivalently,

$$\mathrm{char}(F) \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}. \tag{7}$$

The multiplicity of e^{μ} on the left hand side of equation (7) is given by the expression

$$\sum_{S \subset \Delta^+} (-1)^{|S|} \, \mathrm{mult}_F(\mu + \sum_{\alpha \in S} \alpha).$$

On the other hand, the multiplicity of e^{μ} on the right hand side of equation (7) is $(-1)^{l(w)}$ when $e^{\mu} = e^{w(\lambda+\rho)-\rho}$, and zero otherwise. Since μ is a dominant weight, this can only occur when $w = 1$, and $\mu = \lambda$ is the highest weight of F, proving the claim. □

Removing the requirement that F be an irreducible representation we have the following statement.

Corollary 5.15. *Let K be a compact connected group containing a maximal torus T. Let Δ^+ be a positive root system of T in K. Let S be a subset of Δ^+ and $|S|$ be the order of S. Let μ be a dominant weight and F a representation of K. Let $\mathrm{mult}_F(\gamma)$ be the multiplicity of the weight γ in the restriction $F|_T$. Then,*

$$\sum_{S\subset\Delta^+} (-1)^{|S|}\,\mathrm{mult}_F(\mu + \sum_{\alpha\in S}\alpha)$$

is the multiplicity of the irreducible representation of highest weight μ in F.

We are now ready to determine the K-types of the alternating series in expression (3).

Theorem 11. *The K-types for the small irreducible representation of $\widetilde{\mathrm{SL}(2n+1)}$ with infinitesimal character $\frac{1}{2}\rho$ are $(\frac{1}{2}+2a_1, \frac{1}{2}+2a_2, \ldots, \frac{1}{2}+2a_n)$, with $a_1 \geq a_2 \cdots \geq a_n \geq 0$.*

Proof. By Lemma 5.9 and Proposition 5.11 the virtual representation

$$\sum_{j=0}^{n}(-1)^j(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n))\otimes|\det|^t$$

has infinitesimal character $\frac{1}{2}\rho$ and is small. It follows that it is a multiple of the unique irreducible small representation of $\widetilde{\mathrm{SL}(2n+1)}$ with infinitesimal character $\frac{1}{2}\rho$ of Theorem 6. The restriction to $\mathrm{Spin}(2n+1)$ of each term

$$(\mathrm{Ind}_{\widetilde{\mathrm{GL}(2n)}}^{\widetilde{\mathrm{SL}(2n+1)}}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n))\otimes|\det|^t$$

is

$$\mathrm{Ind}_{\mathrm{Pin(2n)}}^{\mathrm{Spin}(2n+1)}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)|_{\mathrm{Pin(2n)}}.$$

Let E be an irreducible representation of $\mathrm{Spin}(2n+1)$. We will show that the multiplicity of E in the virtual representation

$$\sum_{j=0}^{n}(-1)^j\mathrm{Ind}_{\mathrm{Pin(2n)}}^{\mathrm{Spin}(2n+1)}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)|_{\mathrm{Pin(2n)}}$$

is 1 if the highest weight of E is $(\frac{1}{2}+2a_1, \ldots, \frac{1}{2}+2a_n)$ and 0 otherwise. By Frobenius reciprocity,

$$\mathrm{mult}(E, \sum_{j=0}^{n}(-1)^j\mathrm{Ind}_{\mathrm{Pin(2n)}}^{\mathrm{Spin}(2n+1)}\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)|_{\mathrm{Pin(2n)}})$$

is equivalent to

$$\sum_{j=0}^{n}(-1)^j\dim\mathrm{Hom}_{\mathrm{Spin}(2n)}(E, \mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)|_{\mathrm{Pin}(2n)}). \tag{8}$$

By Proposition 5.13 the $\mathrm{Pin}(2n)$ types of $\mathcal{L}_*(\phi_{\frac{p}{2}}\otimes\bigwedge^j\mathbb{C}^n)$, for $p=-2n-1$, are the dominant weights $(\frac{1}{2}+2a_1+\epsilon_1, \ldots, \frac{1}{2}+2a_n+\epsilon_n)$, where ϵ_i is 0 or 1 and $\sum_{i=1}^n\epsilon_i = j$. We write $W_{(\frac{1}{2}+2a_1+\epsilon_1,\ldots,\frac{1}{2}+2a_n+\epsilon_n)}$ for the representation of $\mathrm{Pin}(2n)$

with highest weight $(\frac{1}{2}+2a_1+\epsilon_1,\ldots,\frac{1}{2}+2a_n+\epsilon_n)$. We may then write equation (8) as

$$\sum_{j=0}^{n}(-1)^j \dim \mathrm{Hom}_{\mathrm{Spin}(2n)}(E, \bigoplus_{a_1\geq\cdots\geq a_n\geq 0}\bigoplus_{\sum\epsilon_i=j} W_{(\frac{1}{2}+2a_1+\epsilon_1,\ldots,\frac{1}{2}+2a_n+\epsilon_n)}).$$

Passing the two summations through Hom and using Frobenius reciprocity again we have

$$\sum_{a_1\geq\cdots\geq a_n\geq 0}\sum_{j=0}^{n}(-1)^j\sum_{\sum\epsilon_i=j}\mathrm{mult}(W_{(\frac{1}{2}+2a_1+\epsilon_1,\ldots,\frac{1}{2}+2a_n+\epsilon_n)}, E|_{\mathrm{Spin}(2n)}). \tag{9}$$

The standard positive roots $\Delta^+(\mathfrak{so}(2n+1))$ consist of the standard positive roots of $\Delta^+(\mathfrak{so}(2n))$ plus the roots $\{e_1,\ldots,e_n\}$. Following the notation from Corollary 5.15, let S be a subset of $\{e_1,\ldots,e_n\}$ and rewrite equation (9) as

$$\sum_{a_1\geq\cdots\geq a_n\geq 0}\sum_{S\subset\{e_1,\ldots,e_n\}}(-1)^{|S|}\mathrm{mult}(W_{(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)+\sum_{\alpha\in S}\alpha}, E|_{\mathrm{Spin}(2n)}). \tag{10}$$

For a fixed $S\subset\{e_1,\ldots,e_n\}$, by Corollary 5.15, the multiplicity of the representation $W_{(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)+\sum_{\alpha\in S}\alpha}$ in $E|_{\mathrm{Spin}(2n)}$ equals

$$\sum_{T\subset\Delta^+(\mathfrak{so}(2n))}(-1)^{|T|}\mathrm{mult}_E((\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)+\sum_{\alpha\in S}\alpha+\sum_{\beta\in T}\beta) \tag{11}$$

provided that $(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)+\sum_{\alpha\in S}$ is a dominant weight in $\mathfrak{so}(2n)$. Substituting expression (11) into expression (10) gives

$$\sum_{a_1\geq\cdots\geq a_n\geq 0}\sum_{A\subset\Delta^+(\mathfrak{so}(2n+1))}(-1)^{|T|}\mathrm{mult}_E(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)+\sum_{\gamma\in A}\gamma. \tag{12}$$

For a fixed $a_1,\ldots,a_n$ by Lemma 5.14 this is precisely 1 if E has highest weight $(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)$ and 0 otherwise. This proves our claim about the K-types. The irreducibility of the representation follows from the K-types being multiplicity free. □

Using the results of section 5 together with the lowest K-types for the Langlands quotients determined in section 4.1 we prove that the K-types of our small unitary representations are multiplicity free and have the K-types given below.

Theorem 12. *The Langlands quotients with minimal parabolic, $J^{\pm}_{P_{min}}(\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2n)}$ have K-types*

$$(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_{n-1},\pm(\frac{1}{2}+2a_n))$$

where $a_1\geq a_2\geq\cdots\geq a_n\geq 0$.

The Langlands quotients with maximal cuspidal parabolic, $J_{P_{max}}(\frac{1}{2}\rho)$ and $J_{P_{max}}(s_{\alpha_0}\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2n)}$, have K-types

$$(\frac{3}{2}+2a_1,\ldots,\frac{3}{2}+2a_{n-1},\pm(\frac{3}{2}+2a_n))$$

where $a_1\geq a_2\geq\cdots\geq a_n\geq 0$.

The Langlands quotient with minimal parabolic, $J_{P_{min}}(\frac{1}{2}\rho)$ *of* $\widetilde{\mathrm{SL}(2n+1)}$*, has* K*-types*

$$(\frac{1}{2}+2a_1,\ldots,\frac{1}{2}+2a_n)$$

where $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.

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