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A TENSOR NORM PRESERVING UNCONDITIONALITY FOR $\mathcal{L}_p$-SPACES

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ABSTRACT. We show that, for each $n \in \mathbb{N}$, there is an n-tensor norm α (in the sense of Grothendieck) with the surprising property that the α-tensor product $\tilde{\bigotimes}_\alpha(Y_1, \ldots, Y_n)$ has local unconditional structure for each choice of n arbitrary $\mathcal{L}_{p_j}$-spaces Y_j. In fact, α is the tensor norm associated to the ideal of multiple 1-summing n-linear forms on Banach spaces.

1. INTRODUCTION

There is an intensive study on unconditionality for tensor products of Banach spaces which can be traced back to the early sixties, and includes a remarkable list of deep papers. The first results were given on Schatten classes S_p. In 1961, Gelbaum and Gil de Lamadrid proved in [19] that the canonical basis in the space $S_\infty = \ell_2 \tilde{\otimes}_\varepsilon \ell_2$ of all compact operators on the Hilbert space ℓ_2 is not unconditional. Kwapień and Pełczyński showed in [23] that S_∞ and the trace class S_1 do not even admit any unconditional basis. For general S_p the final answer came from the seminal paper [20] of Gordon and Lewis which shows that the only Schatten class that admits an unconditional basis is the class of Hilbert-Schmidt operators S_2. The really new feature of their approach was the use of probabilistic techniques, which since then has been an important tool in the subject.

The next milestone was given independently by Pisier [34] and Schütt [35] (see here Theorem 4.3). They proved that, in order to assure the existence of any unconditional basis in the tensor product $X \otimes_\alpha Y$ of two Banach spaces with unconditional bases, it is enough to check the canonical basis; here α is a tensor norm on $X \otimes Y$ in the sense that it lies between the injective norm ε and the projective norm π and moreover satisfies the metric mapping property: $\|S \otimes T : X \otimes_\alpha Y \to X \otimes_\alpha Y\| = \|S\| \, \|T\|$ for two arbitrary (bounded and linear) operators S, T on X and Y, respectively. Now, by a theorem from [23] the canonical basis in $\ell_2 \otimes_\alpha \ell_2$ is unconditional if and only if α (up to constants) equals the Hilbert-Schmidt norm (see [28] for an analogue of this for tensor products of n spaces, a result which needed an alternative proof). Hence the Pisier-Schütt theorem has the following remarkable consequence: *The only tensor norm on $\ell_2 \otimes \ell_2$ that admits an unconditional basis is the Hilbert-Schmidt norm.*

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But the study of unconditionality in tensor products is still a very active area of research. Our motivation mainly comes from the multilinear theory of Banach spaces. In particular, from a recent positive answer to a conjecture of Dineen [16], the space of m-homogeneous polynomials on an infinite dimensional Banach space never has an unconditional basis (see [11] and also [8]). In [9] the authors apply unconditionality in tensor products to give multidimensional analogues of Bohr's classical power series theorem from [5] (for more on that see [1], [3], [4], [9], [17]). Moreover, in [10] unconditionality helps to analyze Hahn-Banach type extension theorems for multilinear forms and polynomials, and finally in [12], it is used to investigate monomial expansions of holomorphic functions in infinitely many variables.

All these studies share a common *philosophy*, namely that tensor norms usually destroy unconditionality.

Following Grothendieck's "résumé" [22], we call a map α which assigns to each n-tuple of Banach spaces X_j a norm $\alpha(\cdot; X_1, \dots, X_n)$ on $\bigotimes_{j=1}^n X_j$ such that (1) $\varepsilon(\cdot; X_1, \dots, X_n) \le \alpha(\cdot; X_1, \dots, X_n) \le \pi(\cdot; X_1, \dots, X_n)$ and (2) α satisfies the metric mapping property:

$$\| \bigotimes_{j=1}^n T_j : \bigotimes_{\alpha, j=1}^n X_j \longrightarrow \bigotimes_{\alpha, j=1}^n Y_j \| = \prod_{j=1}^n \|T_j\| ,$$

a tensor norm of order n on the class of all Banach spaces (or simply an n-tensor norm).

There is a simple *unconditionality test* for n-tensor norms proved in [32, 2.3 and 2.6]: If a tensor norm α (of order n on the class of all Banach spaces) preserves unconditionality for each choice of n Banach spaces with unconditional bases, then it has to coincide (up to constants) with the injective norm ε on $\bigotimes_{j=1}^n c_0$ and with the projective norm π on $\bigotimes_{j=1}^n \ell_1$. Then we say that α fulfills the *unconditionality condition*.

From [32, 2.7] and [26, 4.2] we conclude that each α satisfying the unconditionality condition on the tensor product of Hilbert spaces equals the Hilbert-Schmidt norm (up to constants).

Here we are going to deal with the following question of J. Diestel.

Question 1.1. Is there a tensor norm (of order n on the class of all Banach spaces) that preserves unconditionality?

This problem was motivated by the work of Carne from [7] where he proved that 4 out of the 14 natural tensor norms of Grothendieck (see e.g. [14, section 27]) preserve the structure of Banach algebras. So it seems to be natural to ask whether there exist tensor norms that preserve the structure of Banach lattices (thanks to the above cited result of Pisier and Schütt this question among spaces with 1-unconditional basis is equivalent to 1.1; see Theorem 4.3 below). In [32] it is shown that none of Grothendieck's 14 norms satisfies the unconditionality condition. However, the answer to the above question for a general tensor norm still remains open. Our contribution is the following: We show that there exists a somewhat non-artificial tensor norm α, first defined in [29], which on tensor products taken from a surprisingly large class of Banach spaces (including all $\mathcal{L}_p$-spaces) inherits unconditionality – indicating that the complexity of the above question might be greater than originally expected.

Before introducing this norm let us recall that, for a finite sequence $(x_i)_{i=1}^m$ in a Banach space X and $1 \le p < \infty$, the weak-ℓ_p-norm is given by

$$\|(x_i)_{i=1}^m\|_p^\omega := \sup_{\|x^*\|\le 1} \left(\sum_{i=1}^m |x^*(x_i)|^p \right)^{\frac{1}{p}}.$$

Now, for a vector $u \in Y_1 \otimes \cdots \otimes Y_n$ in the tensor product of n Banach spaces, the norm $\alpha_1(u)$ is defined to be

$$\inf \left\{ \sum_{m=1}^M \|(\lambda_{m,i_m^1,\dots,i_m^n})_{i_m^1,\dots,i_m^n=1}^{I_m^1,\dots,I_m^n}\|_{\ell_\infty} \|(y_{m,i_m^1}^1)_{i_m^1=1}^{I_m^1}\|_1^\omega \cdots \|(y_{m,i_m^n}^n)_{i_m^n=1}^{I_m^n}\|_1^\omega \right\},$$

where the infimum is taken among all the representations

$$u = \sum_{m=1}^M \sum_{i_m^1,\dots,i_m^n=1}^{I_m^1,\dots,I_m^n} \lambda_{m,i_m^1\dots,i_m^n} y_{m,i_m^1}^1 \otimes \cdots \otimes y_{m,i_m^n}^n .$$

The adjoint tensor norm α_1^* is defined by trace duality: If $z \in Y_1 \otimes \cdots \otimes Y_n$ and all Y_j are finite dimensional, then

$$\alpha_1^*(z) := \sup |\langle z, u \rangle| ,$$

where the supremum is taken over all $u \in Y_1^* \otimes \cdots \otimes Y_n^*$ for which $\alpha_1(u) \le 1$, and if z is in the n-fold tensor product of n arbitrary Y_j, then

$$\alpha_1^*(z) := \inf \alpha_1^*(z) ,$$

the infimum taken over all $z \in M_1 \otimes \cdots \otimes M_n$ with all M_j finite dimensional subspaces of the Y_j.

The norm α_1 and its adjoint α_1^* satisfy the unconditionality condition, so they are natural test candidates for Diestel's question. Even more, from [32, 2.7] we know: *A tensor norm α satifies the unconditionality condition if and only if there is a constant $c > 0$ such that*

$$\frac{1}{c}\, \alpha_1^* \le \alpha \le c\, \alpha_1;$$

hence α_1^* and α_1 form natural bounds for all norms under question. Our aim is to show that both norms α_1 and α_1^* on surprisingly wide classes of Banach spaces have unexpectedly good stability properties, i.e., they

- are injective and projective, respectively (Section 3),
- preserve unconditionality (Section 4) , and
- inherit type (convexity) and cotype (concavity) (Section 5).

2. Preliminaries and notation

We shall use standard notation and notions from Banach space theory and the theory of tensor norms as presented, e.g., in [14], [15], [24] or [37]. By $\mathbb{K}$ we will denote both $\mathbb{R}$ and $\mathbb{C}$, X and Y will always be Banach spaces, X^* will stand for the dual of X, and $\mathcal{L}(X, Y)$ for the space of (linear and continuous) operators from

X to Y. We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are K-equivalent if there exist two constants $C, c > 0$ such that $\frac{C}{c} \leq K$ and $c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$. For the notions of (Rademacher) type and cotype we refer to [15] or [24]. As usual we write $C_q(X)$ and $T_p(X)$ for the cotype q and type p constants.

All information needed on unconditionality in Banach spaces will be given at the beginning of section 4.

Recall that for $1 \leq p \leq \infty$ and $\lambda > 1$ a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space if, for every finite dimensional subspace $E \subset X$ there exists another finite dimensional subspace F containing E and such that there exists an isomorphism $v : F \longrightarrow \ell_p^{\dim F}$ with $\|v\|\|v^{-1}\| < \lambda$. We say that X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda > 1$. Clearly, $L_p(\mu)$ is the basic example of an $\mathcal{L}_p$-space. Finally, recall that every $\mathcal{L}_p$-space has type $\min(p, 2)$ and cotype $\max(p, 2)$.

For all needed information on the theory of Banach operator ideals we refer to [14], [15], [33] or [37]. We write (Π_p, π_p) for the Banach operator ideal of p-summing operators, and (Γ_p, γ_p) for the ideal of all p-factorable operators. Recall that an operator T between Banach spaces belongs to the Banach operator ideal $(\Gamma_p^{inj}, \gamma_p^{inj})$, the injective hull of (Γ_p, γ_p), iff it factorizes through a subspace of some $\mathcal{L}_p$-space, and T belongs to the surjective hull $(\Gamma_p^{sur}, \gamma_p^{sur})$ iff it factorizes through a quotient of such a space. In particular, for a finite dimensional Banach space Y we have that

$$\gamma_1^{inj}(Y) := \gamma_1^{inj}(\mathrm{id}_Y) = \inf \|v\|\|v^{-1}\|\,,$$

where the infimum is taken with respect to all subspaces U of some ℓ_1^n and all linear bijections $v : Y \to U$; for the transposed norm $\gamma_\infty^{sur}(Y)$, subspaces of ℓ_1^n's have to be replaced by quotients of ℓ_∞^n's. Note that

$$\gamma_1^{inj}(Y) = \gamma_\infty^{sur}(Y^*)\,.$$

There is no general reference for tensor norms of order n on tensor products of Banach spaces (see the introduction), though one can find the definition and some properties in [18]. All abstract theory on such tensor norms that we are going to use comes as a straightforward generalization of the bilinear case for which we refer to [14]. If α is a tensor norm of order n, then we will use the notation $\bigotimes_{\alpha,j=1}^n Y_j$ or $\bigotimes_\alpha(Y_1, \dots, Y_n)$, and write the symbol $\tilde{\bigotimes}$ for the corresponding completions. The only tensor norms we are interested in are the projective norm π and the injective norm ε, as well as both norms α_1 and α_1^* defined above. Note that all of them are finitely generated tensor norms α; i.e. for $z \in \bigotimes(Y_1, \dots, Y_n)$ we have

$$\alpha(z) = \inf \alpha(z; M_1, \dots, M_n)\,,$$

where the infimum is taken over all finite dimensional subspaces M_j of Y_j for which $z \in \bigotimes(M_1, \dots, M_n)$.

In order to manage α_1 and α_1^* in our context, we need to relate them to the theory of *multiple summing operators*, a class of n-linear operators between Banach spaces which has been recently developed in a series of papers by Bombal, Villanueva and the second author (see [6], [26], [27] [29], [30], [31]).

An n-linear operator $T : X_1 \times \cdots \times X_n \longrightarrow Y$ between Banach spaces is *multiple p-summing*, and we write $T \in \Pi_p^n(X_1, \dots, X_n; Y)$, if there exists a constant

$K > 0$ such that, for every choice of finite sequences $(x^j_{i_j})_{i_j=1}^{m_j}$ in X_j, the following inequality holds:

$$\left(\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|T(x^1_{i_1},\dots,x^n_{i_n})\|^p \right)^{\frac{1}{p}} \leq K \prod_{j=1}^{n} \|(x^j_{i_j})_{i_j=1}^{m_j}\|_p^{\omega} ;$$

the best of all such constants K is called the *multiple p-summing norm* of T and denoted by $\pi_p(T)$.

Clearly, the class of multiple p-summing operators (Π_p^n, π_p) is a maximal operator ideal (of n-linear operators, in the sense of [18]). It is also injective; that is, if $i : Y \hookrightarrow Z$ is an isometric embedding, then $T : X_1 \times \cdots \times X_n \longrightarrow Y$ is multiple p-summing if and only if iT is and, in this case, $\pi_p(T) = \pi_p(iT)$. Trivially, a linear operator $T : X \to Y$ is multiple p-summing if and only if it is p-summing (in the usual sense), and in this case the corresponding norms coincide. Moreover, as shown in the above references, the class of multiple p-summing operators shares most of the crucial properties of its linear analogue.

Finally, we recall the duality relations connecting the class Π_p^n with the tensor norms α_1 and α_1^* (in both cases the identifications are the natural ones): For each choice of Banach spaces $X_1, \dots, X_n$,

$$\left(\tilde{\bigotimes}_{\alpha_1}(X_1,\dots,X_n) \right)^* = \Pi_1^n(X_1,\dots,X_n;\mathbb{K}) \,,$$

and if all spaces are finite dimensional,

$$\bigotimes_{\alpha_1^*}(X_1^*,\dots,X_n^*) = \Pi_1^n(X_1,\dots,X_n;\mathbb{K}) \,.$$

3. Injectivity and projectivity

In this section we will prove that α_1^* is injective within the class of subspaces of arbitrary $\mathcal{L}_p$-spaces, $1 \leq p < \infty$. More precisely, define within all Banach spaces the subclass

$$\mathcal{C} = \{Y : Y \text{ is a subspace of an } \mathcal{L}_1\text{-space or } Y \text{ has type } 2\} \,,$$

and note that this class in particular contains all subspaces of $\mathcal{L}_p$-spaces, $1 \leq p < \infty$. The following theorem is the main result of this section.

Theorem 3.1. *Let $Y_1, \dots, Y_n$ be arbitrary Banach spaces, $Y_1 \in \mathcal{C}$ and U_1 a subspace of Y_1. Then*

$$\tilde{\bigotimes}_{\alpha_1^*}(U_1, Y_2, \dots, Y_n) \hookrightarrow \tilde{\bigotimes}_{\alpha_1^*}(Y_1,\dots,Y_n)$$

is an isomorphic embedding.

The proof will be reduced to the finite dimensional case; more precisely, it is a consequence of the following inequality entirely formulated for finite dimensional spaces.

Proposition 3.2. *Let $Y_1, \dots, Y_n$ be finite dimensional Banach spaces, and U_1 a subspace of Y_1. Then for each $z \in \bigotimes(U_1, Y_2, \dots, Y_n)$,*

$$\|z\|_{\otimes_{\alpha_1^*}(U_1,Y_2,\dots,Y_n)} \leq \min(\gamma_1^{inj}(Y_1), \sqrt{2}\,T_2(Y_1))\ \|z\|_{\otimes_{\alpha_1^*}(Y_1,\dots,Y_n)}.$$

Before we start with the proof of this proposition, let us, for the sake of completeness, show how the fact that α_1^* is finitely generated implies Theorem 3.1. Take $z \in \bigotimes(U_1, Y_2, \dots, Y_n)$, the Y_j and U_1 now as in the theorem. Let $M_j \subset Y_j$ be finite dimensional subspaces such that $z \in \bigotimes(U_1 \cap M_1, M_2, \dots, M_n)$. Then

$$\begin{aligned}\|z\|_{\otimes_{\alpha_1^*}(U_1,Y_2,\dots,Y_n)} &\leq \|z\|_{\otimes_{\alpha_1^*}(U_1\cap M_1,M_2,\dots,M_n)} \\ &\leq \min(\gamma_1^{inj}(M_1), \sqrt{2}\,T_2(M_1))\ \|z\|_{\otimes_{\alpha_1^*}(M_1,\dots,M_n)}.\end{aligned}$$

Taking the infimum over all possible M_j gives the conclusion.

For the proof of the proposition we will check the following two inequalities separately: For each $z \in \bigotimes(U_1, Y_2, \dots, Y_n)$ (the Y_j and U_1 now again finite dimensional)
Case 1: $\|z\|_{\otimes_{\alpha_1^*}(U_1,Y_2,\dots,Y_n)} \leq \gamma_1^{inj}(Y_1)\ \|z\|_{\otimes_{\alpha_1^*}(Y_1,\dots,Y_n)}$,
Case 2: $\|z\|_{\otimes_{\alpha_1^*}(U_1,Y_2,\dots,Y_n)} \leq \sqrt{2}\,T_2(Y_1)\ \|z\|_{\otimes_{\alpha_1^*}(Y_1,\dots,Y_n)}$.

Proof of Case 1. Two simple lemmas on multiple 1-summing operators are needed. The first one is a straightforward reformulation of the definition of the multiple 1-summing norm. □

Lemma 3.3. *Let $X_1, \dots, X_n$ be finite dimensional Banach spaces, and $T : X_1 \times \cdots \times X_n \longrightarrow Y$ a multilinear operator. Then*

$$\pi_1(T) = \sup \sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \|T(u_1(e_{i_1}), \dots, u_n(e_{i_n}))\|\ ,$$

where the sup is taken over all operators $u_j : \ell_\infty^{m_j} \to X_j$ with norm ≤ 1.

The next lemma, a first application of the preceding one, will be crucial.

Lemma 3.4. *Let $X_1, \dots, X_n$ be finite dimensional. Then for each j and each $T : X_1 \times \cdots \times X_n \longrightarrow \mathbb{K}$,*

$$\pi_1(T) \leq \pi_1(T_j) \leq \gamma_\infty^{sur}(X_j)\,\pi_1(T),$$

where $T_j : X_1 \times \cdots \times X_n \longrightarrow X_j^*$ is the associated $(n-1)$-linear operator.

Proof. The first estimate is an observation from [14, Proposition 2.5]. For the second estimate we assume without loss of generality that $j = 1$. Fix a quotient mapping $q : \ell_\infty^k \to U_1$ and a bijection $v : U_1 \to X_1$. Then we deduce from the preceding lemma that

$$\begin{aligned}
\pi_1(T_1) &\leq \|v^{-1}\| \pi_1(v^* T_1) \\
&= \|v^{-1}\| \sup_{\|u_k\|\leq 1} \sum_{i_2,\ldots,i_m=1}^{m_2,\ldots,m_n} \|v^* T_1(u_2(e_{i_2}),\ldots,u_n(e_{i_n}))\|_{U_1^*} \\
&= \|v^{-1}\| \sup_{\|u_k\|\leq 1} \sum_{i_2,\ldots,i_m=1}^{m_2,\ldots,m_n} \|q^* v^* T_1(u_2(e_{i_2}),\ldots,u_n(e_{i_n}))\|_{\ell_1^k} \\
&= \|v^{-1}\| \sup_{\|u_k\|\leq 1} \sum_{i_1,\ldots,i_m=1}^{m_1,\ldots,m_n} |T(vq(e_{i_1}),u_2(e_{i_2}),\ldots,u_n(e_{i_n}))| \\
&\leq \|v^{-1}\| \|v\| \pi_1(T)\,.
\end{aligned}$$

Taking the infimum over all possible v we obtain the desired result. ☐

Now we are prepared to prove **Case 1:** Take $z \in \bigotimes(U_1, Y_2, \ldots, Y_n)$ (the Y_j and U_1 finite dimensional), and let $T_z : U_1^* \times Y_2^* \times \cdots \times Y_n^* \longrightarrow \mathbb{K}$ be the associated multilinear form. Then by the preceding lemma,

$$\begin{aligned}
\|z\|_{\bigotimes_{\alpha_1^*}(U_1,Y_2,\ldots,Y_n)} &= \pi_1(T_z : U_1^* \times Y_2^* \times \cdots \times Y_n^* \longrightarrow \mathbb{K}) \\
&\leq \pi_1((T_z)_1 : Y_2^* \times \cdots \times Y_n^* \longrightarrow U_1) \\
&= \pi_1((T_z)_1 : Y_2^* \times \cdots \times Y_n^* \longrightarrow Y_1) \\
&\leq \gamma_\infty^{sur}(Y_1^*) \pi_1(T_z : Y_1^* \times \cdots \times Y_n^* \longrightarrow \mathbb{K}) \\
&= \gamma_1^{inj}(Y_1) \|z\|_{\bigotimes_{\alpha_1^*}(Y_1,\ldots,Y_n)}\,,
\end{aligned}$$

the conclusion.

Proof of Case 2. Here the main tool will be Maurey's Extension Theorem (see e.g. [15, Theorem 12.22] or [37, Theorem 13.13]), but we only need a reformulation of a special case of this result in terms of projective tensor products. ☐

Lemma 3.5. *Let Y be a finite dimensional Banach space, U a subspace and $m \in \mathbb{N}$. Then for each $z \in U \otimes \ell_\infty^m$,*

$$\|z\|_{U \otimes_\pi \ell_\infty^m} \leq \sqrt{2}\; T_2(Y) \|z\|_{Y \otimes_\pi \ell_\infty^m}\,.$$

Proof. Recall that the dual of $Y \otimes_\pi \ell_\infty^m$ can be identified isometrically with $\mathcal{L}(Y, \ell_1^m)$. Hence, by the Hahn-Banach theorem it suffices to show that each operator $T : U \to \ell_1^m$ has an extension $\tilde{T} : Y \to \ell_1^m$ satisfying $\|\tilde{T}\| \leq \sqrt{2}\; T_2(Y)\|T\|$. But this is a very special case of the above cited result of Maurey. ☐

Moreover, we need to see the norm α_1 as a so-called traced tensor norm (in the spirit of [14]; see also Remark 6.2). By the very definition of the norm α_1 it is not difficult to see that if we, for n finite dimensional Banach spaces Y_j, define the sequence of surjections

$$Q_m : \left(\bigotimes_{j=1,\pi}^{n} \left(Y_j^* \otimes_\varepsilon \ell_1^m \right) \right) \otimes_\pi \bigotimes_{j=1,\varepsilon}^{n} \ell_\infty^m \twoheadrightarrow \bigotimes_{j=1}^{n} Y_j^*$$

$$\bigotimes (y_j^* \otimes e_{i_j}) \otimes \bigotimes e_{k_j} \to \left(\prod_{j=1}^{n} \delta_{i_j,k_j} \right) \bigotimes y_j^*,$$

we have that, for $z \in \bigotimes_{j=1}^n Y_j^*$,

$$\alpha_1(z) = \inf_m \inf\{\|w\| : Q_m(w) = z\}.$$

Dualizing we obtain that, for $w \in \bigotimes_{j=1}^n Y_j$,

$$\alpha_1^*(w) = \sup_m \|I_m(w)\| \,, \tag{1}$$

where I_m is the injection

$$I_m = Q_m^* : \bigotimes_{j=1}^{n} Y_j \hookrightarrow \left(\bigotimes_{j=1,\varepsilon}^{n} \left(Y_j \otimes_\pi \ell_\infty^m \right) \right) \otimes_\varepsilon \bigotimes_{j=1,\pi}^{n} \ell_1^m.$$

Now it is obvious that the preceding lemma combined with the formula from (1) leads directly to a **proof of Case 2**. This completes the proof of the proposition, and by what was said before also the proof of the theorem.

As a trivial consequence we obtain that α_1^* is *injective* (up to isomorphisms) among all subspaces of $\mathcal{L}_p$'s with $1 \leq p < \infty$:

Corollary 3.6. *If, for every $1 \leq j \leq n$, Y_j is an $\mathcal{L}_{p_j}$-space with $1 \leq p_j < \infty$ and $U_j \subset Y_j$, then*

$$\tilde{\bigotimes}_{\alpha_1^*}(U_1, \ldots, U_n) \hookrightarrow \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$$

is an isomorphic embedding.

We finish this section with a pair of remarks.

Remark 3.7. (1) Knowing that $\alpha_1^* = \varepsilon$ on tensor products of $\mathcal{L}_\infty$-spaces it is trivial to see that we cannot expect Corollary 3.6 for $p = \infty$.
(2) Clearly, the norm α_1 has the corresponding *projective* behavior.

4. Unconditionality

A sequence $(e_i)_{i\in I}$ (the index set I a subset of $\mathbb{N}$) of non-zero vectors in a Banach space X is said to be a K-unconditional basic sequence if

$$\left\| \sum_{i\in I} \epsilon_i \mu_i e_i \right\| \leq K \left\| \sum_{i\in I} \mu_i e_i \right\|$$

for every choice of scalars $\epsilon_i, \mu_i \in \mathbb{K}$ with $|\epsilon_i| \leq 1$. By $\mathrm{ub}((e_i)_{i\in I}; X)$, the unconditional basis constant of $(e_i)_{i\in I}$, we denote the best of such constants K. If X has an unconditional basis (a Schauder basis which is K-unconditional for some K), we write $\mathrm{ub}(X)$ for the unconditional basis constant of X, that is, the infimum over all possible unconditional basis constants $\mathrm{ub}((e_i)_{i\in I}; X)$. A dense unconditional basic sequence is an unconditional basis. For finite dimensional Banach spaces $Y_1, \dots, Y_n$ with fixed bases $(e_i^j)_{i=1}^{\dim Y_j}$ we put

$$\mathrm{ub}_{\mathrm{mon}} \left(\bigotimes\nolimits_\alpha (Y_1, \dots, Y_n) \right) := \mathrm{ub} \left((e_{i_1, \dots, i_n}); \bigotimes\nolimits_\alpha (Y_1, \dots, Y_n) \right) ,$$

where the unconditional basis constant on the right is taken with respect to the monomial basis formed by the tensors $e_{i_1,\dots,i_n} := e^1_{i_1} \otimes \dots \otimes e^n_{i_n}$, $1 \leq i_j \leq \dim Y_j$, $1 \leq j \leq n$.

A Banach space X with a 1-unconditional basis $(e_i)_{i \in I}$ is a Banach lattice with the pointwise order with respect to that basis. In this sense we will use the notions of 2-convexity and 2-concavity (defined for Banach lattices; see e.g. [15] or [24]). As usual we write $M^{(2)}(X)$ for the 2-convexity and $M_{(2)}(X)$ for the 2-concavity constant of X. Recall the duality relations $M^{(2)}(X) = M_{(2)}(X^*)$, $M_{(2)}(X) = M^{(2)}(X^*)$, and that X is 2-concave if and only if X has cotype 2. We will also use the notions of 2-concavity and 2-convexity for spaces X with an unconditional basis (e_i) since a renorming of X is possible which makes (e_i) 1-unconditional (see e.g. [24, 1.c.]).

A Banach space X has local unconditional structure or simply *lust* (as defined by Gordon and Lewis in [20]) whenever there exists a constant $\Lambda \geq 1$ with the property that, for every finite dimensional subspace $E \subset X$, there exists a Banach space Y with unconditional basis and operators $u \in \mathcal{L}(E, Y), v \in \mathcal{L}(Y, X)$ such that vu is the canonical inclusion and such that $\|u\| \|v\| \,\mathrm{ub}(Y) \leq \Lambda$; the best such Λ is denoted by $\mathrm{lust}(X)$.

Of course, the class of all spaces with *lust* contains all spaces with an unconditional basis, but many spaces more, for example every $\mathcal{L}_p$-space ($1 \leq p \leq \infty$) or every Banach lattice has lust.

The aim of this section is to prove that the tensor norm α_1^* behaves nicely with respect to unconditional bases. More precisely, if

$$\mathcal{C}' = \{ Y : Y \text{ is a subspace of an } \mathcal{L}_1 \text{ or } Y^* \text{ has cotype } 2\}$$

(note the similarity with $\mathcal{C}$), we are going to show the following.

Theorem 4.1. *Let $Y_1, \dots, Y_n$ be Banach spaces in $\mathcal{C}'$ with unconditional bases. Then*

$$\tilde{\bigotimes}_{\alpha_1^*} (Y_1, \dots, Y_n)$$

has unconditional basis.

This result will be an immediate consequence of the following estimate on finite dimensional spaces.

Proposition 4.2. *For n finite dimensional Banach spaces Y_j with 1-unconditional bases $(e_i^j)_i$ and each $0 \leq k \leq n$ we have*

$$\mathrm{ub}_{\mathrm{mon}} \left(\bigotimes\nolimits_{\alpha_1^*} (Y_1, \dots, Y_n) \right) \leq 2^{n+1} K_G^{n-k} \prod_{j=1}^{k} \gamma_1^{inj}(Y_j) \prod_{j=k+1}^{n} M^{(2)}(Y_j);$$

here K_G denotes Grothendieck's constant (read $\prod_{j=1}^0 = 1$ and $\prod_{j=n+1}^n = 1$).

Before we start preparing the proof of this proposition let us indicate how it implies the theorem: Without loss of generality we assume that the bases (e_i^j) of the Y_j are 1-unconditional, that $Y_1, \ldots, Y_k$ are subspaces of an $\mathcal{L}_1$ and that $Y_{k+1}^*, \ldots, Y_n^*$ have cotype 2. Define Y_j^m to be the span of the first m basis vectors in Y_j. Then by the proposition,

$$\sup_m \mathrm{ub}_{\mathrm{mon}}\left(\bigotimes_{\alpha_1^*}(Y_1^m, \ldots, Y_n^m)\right) < \infty\,.$$

Since all $\bigotimes_{\alpha_1^*}(Y_1^m, \ldots, Y_n^m)$ are 1-complemented in $\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$, and the span of all $e_{i_1,\ldots,i_n}$ is dense in $\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$, the conclusion follows.

For the proof of the proposition we again need some preparation. The main tool will be the so-called Gordon-Lewis property. A Banach space Y has the Gordon-Lewis property if every 1-summing operator $T : Y \longrightarrow \ell_2$ is 1-factorable, and

$$\mathrm{gl}(Y) := \sup_{\pi_1(T)\leq 1} \gamma_1(T)$$

is then called the Gordon-Lewis constant of Y. A crucial fact for this paper is that for any (finite dimensional) Banach space Y,

$$\mathrm{gl}(Y) \leq \gamma_1^{inj}(Y)\,, \tag{2}$$

which easily follows from the extension property of 2-summing operators (see e.g. [15, 4.15]).

It can also be found in [15, 17.7] that for every (finite dimensional) Banach space Y,

$$\mathrm{gl}(Y) \leq \mathrm{lust}(Y) \leq \mathrm{ub}(Y)\,, \tag{3}$$

though the reciprocal is not true. However, for α-tensor products of Banach spaces with unconditional bases, things can be considerably simplified thanks to the following theorem of Pisier [34] and Schütt [35] already mentioned in the introduction (for our n-linear version, see [8, Remark 1]):

Theorem 4.3. *Let $Y_1, \ldots, Y_n$ be Banach spaces with unconditional bases and α an n-tensor norm. Then*

$$\mathrm{ub}_{\mathrm{mon}}\left(\tilde{\bigotimes}_\alpha(Y_1, \ldots, Y_n)\right) \leq 2^{n+1}\,\mathrm{gl}\left(\tilde{\bigotimes}_\alpha(Y_1, \ldots, Y_n)\right)\,.$$

We need three more lemmata of independent interest.

Lemma 4.4. *Let $Y_1, \ldots, Y_n$ be finite dimensional Banach spaces. Then*

$$\gamma_1^{inj}\left(\bigotimes_{\alpha_1^*}(Y_1, \ldots, Y_n)\right) \leq \prod_{k=1}^n \gamma_1^{inj}(Y_j)\,,$$

or equivalently, for finite dimensional spaces $X_1, \ldots, X_n$,

$$\gamma_1^{inj}(\Pi_1^n(X_1, \ldots, X_n; \mathbb{K})) \leq \prod_{k=1}^n \gamma_\infty^{sur}(X_j)\,.$$

Proof. We prove the second statement and assume without loss of generality that all duals X_j^* are isometric subspaces of some $\ell_1^{k_j}$. Using Lemma 3.4 (in the equalities) and the fact that Π_1^{n-1} is injective (in the inclusions) we have the following chain:

$$\begin{aligned}
\Pi_1^n(X_1,\dots,X_n;\mathbb{K}) &= \Pi_1^{n-1}(X_2,\dots,X_n;X_1^*)\\
&\subset \Pi_1^{n-1}(X_2,\dots,X_n;\ell_1^{k_1})\\
&= \Pi_1^n(\ell_\infty^{k_1},X_2,\dots,X_n;\mathbb{K})\\
&= \Pi_1^{n-1}(\ell_\infty^{k_1},X_3,\dots,X_n;X_2^*)\\
&\subset \Pi_1^{n-1}(\ell_\infty^{k_1},X_3,\dots,X_n;\ell_1^{k_2})\\
&= \Pi_1^n(\ell_\infty^{k_1},\ell_\infty^{k_2},X_3,\dots,X_n;\mathbb{K})\\
&= \dots \quad \dots \quad \dots\\
&\subset \Pi_1^n(\ell_\infty^{k_1},\dots,\ell_\infty^{k_n};\mathbb{K})\,.
\end{aligned}$$

Since by [30, Proposition 3.1],

$$\Pi_1^n(\ell_\infty^{k_1},\dots,\ell_\infty^{k_n};\mathbb{K}) = (\ell_\infty^{k_1}\otimes_\varepsilon\cdots\otimes_\varepsilon\ell_\infty^{k_n})^* = \ell_1^{k_1}\otimes_\pi\cdots\otimes_\pi\ell_1^{k_n} = \ell_1^{\prod k_j}\,,$$

the proof is complete. □

Observe that Theorem 4.3, inequality (2), and Lemma 4.4 prove Proposition 4.2 in the case $k=n$. The general case again needs some more preparation. The ideas below are inspired by the thesis of I. Schütt [36].

Lemma 4.5. *Let $X_1,\dots,X_n$ be finite dimensional Banach spaces, $1\le j\le n$ an index for which X_j has a 1-unconditional basis, and $T: X_1\times\ldots\times X_n\to\mathbb{K}$ an n-linear form. Then*

$$\frac{1}{K_G M_{(2)}(X_j)}\pi_1(T)\le\sup_\lambda\pi_1(T_\lambda)\le\pi_1(T)\,,$$

where the sup is taken over all diagonal operators $d_\lambda:\ell_2^{\dim X_j}\to X_j$ of norm ≤ 1 and $T_\lambda := T(\dots,d_\lambda,\dots): X_1\times\cdots\times\ell_2^{\dim X_j}\times\cdots\times X_n\to\mathbb{K}$.

By iteration we also see that whenever all X_j have 1-unconditional bases, then

$$\frac{1}{K_G^n\prod_{j=1}^n M_{(2)}(X_j)}\pi_1(T)\le\sup_{\lambda^j}\pi_1(T(d_{\lambda^1},\dots,d_{\lambda^n}))\le\pi_1(T)\,.$$

Proof. From Lemma 3.3 we know that

$$\pi_1(T)=\sup_{\|u_j:\ell_\infty^{m_j}\to X_j\|\le 1}\pi_1(T(\mathrm{id},\dots,u_j,\dots,\mathrm{id}))\,.$$

Take such a u_j. Then by the Maurey-Rosenthal factorization theorem (use e.g. [24, II,1.d.12] and [13, 4.2]) there is a factorization

$$u_j = d_\lambda\, v$$

with $\|v:\ell_\infty^{m_j}\to\ell_2^{\dim X_j}\|\le K_G M_{(2)}(X_j)$ and $\|d_\lambda:\ell_2^{\dim X_j}\to X_j\|\le 1$. Hence

$$\pi_1(T(\mathrm{id},\dots,u_j,\dots,\mathrm{id}))\le K_G M_{(2)}(X_j)\pi_1(T(\mathrm{id},\dots,d_\lambda,\dots,\mathrm{id}))\,,$$

the conclusion. □

Lemma 4.6. *Let $X_1, \dots, X_n$ be finite dimensional Banach spaces with* 1*-unconditional bases. Then for each $1 \le j \le n$,*

$$\begin{aligned}&\mathrm{ub}_{\mathrm{mon}}\left(\Pi_1^n(X_1,\dots,X_n;\mathbb{K})\right)\\ &\le K_G\, M_{(2)}(X_j)\,\mathrm{ub}_{\mathrm{mon}}\left(\Pi_1^n(X_1,\dots,\ell_2^{\dim X_j},\dots,X_n;\mathbb{K})\right).\end{aligned}$$

Proof. Put $X = \Pi_1^n(X_1,\dots,X_n;\mathbb{K})$ and $Y = \Pi_1^n(X_1,\dots,\ell_2^{m_j},\dots,X_n;\mathbb{K})$ where $m_j = \dim X_j$. Then for each possible choice of scalars $\mu_{i_1,\dots,i_n}, \varepsilon_{i_1,\dots,i_n} \in \mathbb{K}$ with $|\varepsilon_{i_1,\dots,i_n}| \le 1$ we have

$$\begin{aligned}&\left\|\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \varepsilon_{i_1,\dots,i_n}\mu_{i_1,\dots,i_n} e_{i_1,\dots,i_n}\right\|_X\\ &\overset{4.5}{\le} K_G\, M_{(2)}(X_j) \sup_\lambda \left\|\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \varepsilon_{i_1,\dots,i_n}\lambda_{i_j}\mu_{i_1,\dots,i_n} e_{i_1,\dots,i_n}\right\|_Y\\ &\le K_G\, M_{(2)}(X_j)\,\mathrm{ub}_{\mathrm{mon}}(Y) \sup_\lambda \left\|\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \lambda_{i_j}\mu_{i_1,\dots,i_n} e_{i_1,\dots,i_n}\right\|_Y\\ &\overset{4.5}{\le} K_G\, M_{(2)}(X_j)\,\mathrm{ub}_{\mathrm{mon}}(Y) \left\|\sum_{i_1,\dots,i_n=1}^{m_1,\dots,m_n} \mu_{i_1,\dots,i_n} e_{i_1,\dots,i_n}\right\|_X,\end{aligned}$$

the desired estimate. □

Now we just have to collect our results in order to obtain the **proof of Proposition 4.2**:

$$\begin{aligned}&\mathrm{ub}_{\mathrm{mon}}\left(\bigotimes\nolimits_{\alpha_1^*}(Y_1,\dots,Y_n)\right)\\ &\overset{4.6}{\le} K_G^{n-k} \prod_{j=k+1}^{n} M_{(2)}(Y_j^*)\,\mathrm{ub}_{\mathrm{mon}}(\Pi_1^n(Y_1^*,\dots,Y_k^*,\ell_2^{\dim Y_{k+1}},\dots,\ell_2^{\dim Y_n};\mathbb{K}))\\ &\overset{4.3}{\le} 2^{n+1}K_G^{n-k} \prod_{j=k+1}^{n} M^{(2)}(Y_j)\,\mathrm{gl}(\Pi_1^n(Y_1^*,\dots,Y_k^*,\ell_2^{\dim Y_{k+1}},\dots,\ell_2^{\dim Y_n};\mathbb{K}))\\ &\overset{(2)}{\le} 2^{n+1}K_G^{n-k} \prod_{j=k+1}^{n} M^{(2)}(Y_j)\gamma_1^{inj}(\Pi_1^n(Y_1^*,\dots,Y_k^*,\ell_2^{\dim Y_{k+1}},\dots,\ell_2^{\dim Y_n};\mathbb{K}))\\ &\overset{4.4}{\le} 2^{n+1}K_G^{n-k} \prod_{j=k+1}^{n} M^{(2)}(Y_j) \prod_{j=1}^{k}\gamma_1^{inj}(Y_j),\end{aligned}$$

where the latter inequality uses the well-known fact that finite dimensional Hilbert spaces are isometric subspaces of appropriate ℓ_1^n's.

By what was said before, this completes the proof of Theorem 4.1.

Remark 4.7. Reasoning dually one can obtain the analogous result for α_1; that is, $\tilde{\bigotimes}_{\alpha_1}(Y_1, \ldots, Y_n)$ has unconditional basis whenever each Y_j has unconditional basis and, in addition, satifies one of the following two properties: Y_j^* is a subspace of an $\mathcal{L}_1$-space or Y_j has cotype 2.

Within the class of all $\mathcal{L}_p$-spaces, Theorem 4.1 can be improved considerably (in a sense our most far-reaching result).

Corollary 4.8. *Let $Y_1, \ldots, Y_n$ be $\mathcal{L}_{p_j}$-spaces, $1 \leq p_j \leq \infty$. Then*

$$\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$$

has lust.

Proof. Let us take a finite dimensional subspace $G \subset \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$. We can suppose without loss of generality that $G \subset \bigotimes_{j=1}^n E_j$, for some finite dimensional subspaces $E_j \subset Y_j$. Now, since Y_j is an $\mathcal{L}_{p_j,\lambda_j}$-space, we can find finite dimensional subspaces $E_j \subset F_j \subset Y_j$ and operators $u_j : F_j \longrightarrow \ell_{p_j}^{\dim(F_j)}$ with $\|u_j\|\|u_j^{-1}\| < \lambda_j$. By Proposition 4.2, we know that

$$\mathrm{ub}\left(\bigotimes_{\alpha_1^*}(F_1, \ldots, F_n)\right) \leq K \prod_{j=1}^n \lambda_j \,,$$

where K only depends on the p_j's. Consider the embeddings

$$G \hookrightarrow \bigotimes_{\alpha_1^*}(F_1, \ldots, F_n) \hookrightarrow \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)\,;$$

obviously, it is enough to show that the norm $\alpha_1^*(\cdot; Y_1, \ldots, Y_n)$, restricted to $\bigotimes_{j=1}^n F_j$, and the norm $\alpha_1^*(\cdot; F_1, \ldots, F_n)$ are $K'\left(\prod_{j=1}^n \lambda_j\right)$-equivalent, where K' depends only on the p_j's.

Let us suppose without loss of generality that $p_1 = \cdots = p_k = \infty$ and $1 \leq p_j < \infty$ for $j > k$. Clearly, by the injectivity of $\ell_\infty^{\dim(F_j)}$, we have that, when restricted to $\bigotimes_{j=1}^n F_j$, the norms $\alpha_1^*(\cdot; Y_1, \ldots, Y_n)$ and $\alpha_1^*(\cdot, F_1, \ldots, F_k, Y_{k+1}, \ldots, Y_n)$ are $\left(\prod_{j=1}^k \lambda_j\right)$-equivalent.

Now, by Proposition 3.2, there is a K'', depending only on $p_{k+1}, \ldots, p_n$, such that the norm $\alpha_1^*(\cdot, F_1, \ldots, F_k, Y_{k+1}, \ldots, Y_n)$ restricted to $\bigotimes_{j=1}^n F_j$ and the norm $\alpha_1^*(\cdot, F_1, \ldots, F_n)$ are $K''\left(\prod_{j=k+1}^n \lambda_j\right)$-equivalent; this concludes the proof. □

5. 2-convexity and 2-concavity

The main result of this section is that α_1^* preserves 2-convexity (and 2-concavity for subspaces of $\mathcal{L}_1$-spaces). As a consequence we will obtain that α_1^* preserves cotype 2 within the class of all $\mathcal{L}_p$-spaces.

Lemma 5.1. *Let $X_1, \dots, X_n$ be k_j-dimensional Banach spaces with 1-unconditional bases $(e_i^j)_{i=1}^{k_j}$. Then for each n-linear form $T : X_1 \times \cdots \times X_n \to \mathbb{K}$,*

$$\frac{1}{(\sqrt{2}\,K_G)^n \prod_{j=1}^n M_{(2)}(X_j)}\,\pi_1(T)$$

$$\leq \sup_{\lambda^j} \left(\sum_{i_1,\dots,i_n=1}^{k_1,\dots,k_n} \left|T(e_{i_1}^1, \dots, e_{i_n}^n)\lambda_{i_1}^1 \cdots \lambda_{i_n}^n\right|^2 \right)^{\frac{1}{2}} \leq 2^{\frac{n}{2}}\pi_1(T)\,,$$

where the sup is taken over all diagonal operators $d_{\lambda^j} : \ell_2^{k_j} \to X_j$ with norm ≤ 1.

The main ingredient of the proof is the Hilbert case $X_j = \ell_2^{k_j}$:

$$\frac{1}{2^{\frac{n}{2}}}\pi_1(T) \leq \left(\sum_{i_1,\dots,i_n=1}^{k_1,\dots,k_n} |T(e_{i_1}, \dots, e_{i_n})|^2 \right)^{\frac{1}{2}} \leq 2^{\frac{n}{2}}\pi_1(T)\,, \tag{4}$$

which is a finite dimensional version of [26, Theorem 4.2] (and [25, Proposition 5.6]).

Proof. From Lemma 4.5 we have that

$$\frac{1}{K_G^n \prod_{j=1}^n M_{(2)}(X_j)}\pi_1(T) \leq \sup_{\lambda_j} \pi_1(T(d_{\lambda^1}, \dots, d_{\lambda^n})) \leq \pi_1(T)\,,$$

and by (4) that

$$\frac{1}{2^{\frac{n}{2}}}\pi_1(T(d_{\lambda^1}, \dots, d_{\lambda^n})) \leq \left(\sum_{i_1,\dots,i_n=1}^{k_1,\dots,k_n} \left|T(e_{i_1}, \dots, e_{i_n})\lambda_{i_1}^1 \cdots \lambda_{i_n}^n\right|^2 \right)^{\frac{1}{2}}$$

$$\leq 2^{\frac{n}{2}}\pi_1(T(d_{\lambda^1}, \dots, d_{\lambda^n}))\,,$$

which gives the conclusion. ∎

The following estimate is crucial.

Proposition 5.2. *Let $Y_1, \dots, Y_n$ be finite dimensional spaces with 1-unconditional bases $(f_i^j)_i$. Let Y be the usual renorming of $\bigotimes_{\alpha_1^*}(Y_1, \dots, Y_n)$ that makes the monomial basis 1-unconditional, i.e.,*

$$\| \sum_{i_1,\dots,i_n} \alpha_{i_1,\dots,i_n} f_{i_1,\dots,i_n} \|_Y = \sup_{|\epsilon_{i_1,\dots,i_n}| \leq 1} \alpha_1^* \left(\sum_{i_1,\dots,i_n} \epsilon_{i_1,\dots,i_n}\alpha_{i_1,\dots,i_n} f_{i_1,\dots,i_n} \right).$$

Then

$$M^{(2)}(Y) \leq 2^{2n+1} K_G^{2n} \left(\prod_{j=1}^n M^{(2)}(Y_j) \right)^2 .$$

Proof. By Proposition 4.2 we know that on $\bigotimes(Y_1, \dots, Y_n)$,

$$\alpha_1^* \leq \|\cdot\|_Y \leq 2^{n+1} K_G^n \prod_{j=1}^n M_{(2)}(Y_j^*)\alpha_1^*.$$

Let $(e_i^j)_i \subset Y_j^*$ be the dual basis of $(f_i^j)_i$, and take a finite sequence (T_k) in $\Pi_1^n(Y_1^*,\dots,Y_n^*) = \bigotimes_{\alpha_1^*}(Y_1,\dots,Y_n)$. Then by the preceding lemma,

$$\begin{aligned}
&\left\|\left(\sum_k |T_k|^2\right)^{\frac{1}{2}}\right\|_Y \\
&\le 2^{n+1} K_G^n \prod_{j=1}^n M_{(2)}(Y_j^*)\pi_1\left(\left(\sum_k |T_k|^2\right)^{\frac{1}{2}}\right) \\
&\le 2^{\frac{3n}{2}+1} K_G^{2n} \left(\prod_{j=1}^n M_{(2)}(Y_j^*)\right)^2 \sup_{\lambda^j}\left(\sum_{k,i_1,\dots,i_n} \left|T_k(e^1_{i_1},\dots,e^n_{i_n})\lambda^1_{i_1}\cdots\lambda^n_{i_n}\right|^2\right)^{\frac{1}{2}} \\
&\le 2^{\frac{3n}{2}+1} K_G^{2n} \left(\prod_{j=1}^n M^{(2)}(Y_j)\right)^2 \left(\sum_k \sup_{\lambda^{k;j}} \sum_{i_1,\dots,i_n} \left|T_k(e^1_{i_1},\dots,e^n_{i_n})\lambda^{k;1}_{i_1}\cdots\lambda^{k;n}_{i_n}\right|^2\right)^{\frac{1}{2}} \\
&\le 2^{2n+1} K_G^{2n} \left(\prod_{j=1}^n M^{(2)}(Y_j)\right)^2 \left(\sum_k \pi_1(T_k)^2\right)^{\frac{1}{2}} \\
&\le 2^{2n+1} K_G^{2n} \left(\prod_{j=1}^n M^{(2)}(Y_j)\right)^2 \left(\sum_k \|T_k\|_Y^2\right)^{\frac{1}{2}},
\end{aligned}$$

which is the desired inequality. □

We now prove the main result of this section (recall from Section 4 how we use the notion of convexity for a space with an unconditional basis with constant > 1).

Theorem 5.3. *Let $Y_1,\dots,Y_n$ be Banach spaces all of which have unconditional bases. Then $\tilde{\bigotimes}_{\alpha_1^*}(Y_1,\dots,Y_n)$ is 2-convex provided all Y_j are 2-convex.*

Proof. By renorming the spaces we can assume without loss of generality that each Y_j has a 1-unconditional basis, and again we denote by Y the space $\tilde{\bigotimes}_{\alpha_1^*}(Y_1,\dots,Y_n)$ renormed as in Proposition 5.2. Let us denote by Y_j^N the subspace of Y_j spanned by the first N basis vectors; moreover abbreviate

$$K_n = 2^{2n+1} K_G^{2n} \left(\prod_{j=1}^n M^{(2)}(Y_j)\right)^2.$$

Let us take $z_1,\dots,z_m \in Y$, and denote by M the span of these vectors endowed with the norm inherited from Y. If we fix $\varepsilon > 0$ it is easy to see that one can find an $N \in \mathbb{N}$ such that the canonical projection $P_N : Y \longrightarrow (\bigotimes(Y_1^N,\dots,Y_n^N), \|\cdot\|_Y)$ satisfies that $\|z - P_N(z)\| \le \frac{\epsilon}{2mK_n}\|z\|$ for every $z \in M$.

In fact, if we consider a basis $(f_j)_{j=1}^s$ of M and an $\varepsilon' > 0$, and we consider the constant C that makes M isomorphic to ℓ_1^s, that is, $\sum_j |\alpha_j| \le C\|\sum_j \alpha_j f_j\|$ for every $z = \sum_j \alpha_j f_j \in M$, and we choose $N \in \mathbb{N}$ in such a way that $\|P_N(f_j) - f_j\| \le \frac{\varepsilon'}{C}$

for every j, then, for $z = \sum_j \alpha_j f_j$,

$$\|z - P_N(z)\| \leq \sum_j |\alpha_j| \|f_j - P_N(f_j)\| \leq \varepsilon' \|z\|.$$

Since the norm inherited on $\bigotimes(Y_1^N, \ldots, Y_n^N)$ by the norm of Y coincides exactly with the norm given in Proposition 5.2, we can now estimate as follows:

$$\begin{aligned}
\left\| \left(\sum_k |z_k|^2 \right)^{\frac{1}{2}} \right\| &\leq \left\| \left(\sum_k |P_N(z_k)|^2 \right)^{\frac{1}{2}} \right\| + \left\| \left(\sum_k |z_k - P_N(z_k)|^2 \right)^{\frac{1}{2}} \right\| \\
&\leq \left\| \left(\sum_k |P_N(z_k)|^2 \right)^{\frac{1}{2}} \right\| + \sum_k \|z_k - P_N(z_k)\| \\
&\leq \left\| \left(\sum_k |P_N(z_k)|^2 \right)^{\frac{1}{2}} \right\| + \frac{\epsilon}{2} \\
&\overset{5.2}{\leq} K_n \left(\sum_k \|P_N(z_k)\|^2 \right)^{\frac{1}{2}} + \frac{\epsilon}{2} \\
&\leq K_n \left(\sum_k \|z_k\|^2 \right)^{\frac{1}{2}} + \epsilon \,,
\end{aligned}$$

the conclusion $\square$

In the next section we will show that, for instance, $\ell_4 \tilde{\otimes}_{\alpha_1^*} \ell_4$ fails to have type 2. Hence, 2-convexity in the preceding result cannot be replaced by type 2.

Remark 5.4. Again one can reason dually and obtain that, if $Y_1, \ldots, Y_n$ are Banach spaces with unconditional bases, then $\tilde{\bigotimes}_{\alpha_1}(Y_1, \ldots, Y_n)$ is 2-concave provided all Y_j are 2-concave.

Theorem 5.5. *Let $Y_1, \ldots, Y_n$ be subspaces of $\mathcal{L}_1$-spaces. Then the tensor product $\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$ is a subspace of an $\mathcal{L}_1$-space; in particular, it has cotype* 2.

This is an immediate consequence of the injectivity of α_1^* (see Section 1) and the following facts: the projective tensor product of $\mathcal{L}_1$-spaces is again an $\mathcal{L}_1$-space, and the norms π and α_1^* are equivalent in the tensor product of such spaces. Alternatively, the result may also be easily obtained from Lemma 4.4.

We finish this section with the case of $\mathcal{L}_p$-spaces.

Corollary 5.6. *Let each Y_j be an $\mathcal{L}_{p_j}$-space.*

(1) *If all $1 \leq p_j \leq 2$, then $\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$ has cotype* 2.

(2) *If all $2 \leq p_j < \infty$, then $\left(\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)\right)^*$ has cotype* 2.

Proof. (1) is included in Theorem 5.5. To see (2) it is enough to show that, for any finite dimensional subspace $G \subset \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$, there exists another finite dimensional subspace $G \subset F \subset \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$ such that $C_2(F^*)$ is uniformly bounded (by, let us say, M). Indeed, if this is the case, then for $x_1^*, \ldots, x_m^* \in \left(\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)\right)^*$ we find vectors $x_1, \ldots, x_m$ in the unit ball of $\tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$

such that $\|x_i^*\| \leq |x_i^*(x_i)| + \frac{\epsilon}{\sqrt{n}}$. Now, if G is the space generated by $\{x_1, \ldots, x_m\}$, F chosen according to the assumption, and ϕ_i is the restriction of x_i^* to F, then

$$\left(\sum_{i=1}^m \|x_i^*\|^2\right)^{\frac{1}{2}} \leq \epsilon + \left(\sum_{i=1}^m |x_i^*(x_i)|^2\right)^{\frac{1}{2}} \leq \epsilon + \left(\sum_{i=1}^m \|\phi_i\|^2\right)^{\frac{1}{2}}$$

$$\leq \epsilon + M\left(\int_0^1 \left\|\sum_{i=1}^m r_i(t)\phi_i\right\|^2 dt\right)^{\frac{1}{2}} \leq \epsilon + M\left(\int_0^1 \left\|\sum_{i=1}^m r_i(t)x_i^*\right\|^2 dt\right)^{\frac{1}{2}}.$$

Finally, fix a finite dimensional $G \subset \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n)$. Reasoning as in Corollary 4.8 we can assume that without loss of generality

$$G \subset \bigotimes_{\alpha_1^*}(\ell_{p_1}^{m_1}, \ldots, \ell_{p_n}^{m_n}) \subset \tilde{\bigotimes}_{\alpha_1^*}(Y_1, \ldots, Y_n).$$

To conclude the proof it is enough to notice that, thanks to Proposition 5.2, $F = \bigotimes_{\alpha_1^*}(\ell_{p_1}^{m_1}, \ldots, \ell_{p_n}^{m_n})$ satisfies that $C_2(F^*)$ is uniformly bounded. □

In statement (2) we cannot cancel the dual, and replace cotype 2 by type 2 (which would be a stronger result); see again the comment after Theorem 5.3.

6. The case of two spaces

The results we have obtained are also new for the case of two spaces, but in this case we even can say a bit more.

Theorem 6.1. *If X, Y are Banach spaces with 1-unconditional bases, at least one of them 2-convex, then*

$$X\tilde{\otimes}_{\alpha_1^*}Y$$

has an unconditional basis.

Proof. It is enough to control $\mathrm{gl}(\Pi_1^2(X^*, Y^*; \mathbb{K}))$ for finite dimensional spaces X, Y (Theorem 4.3). By Lemma 4.6, we can reduce the problem to the case $\Pi_1^2(X^*, \ell_2^k; \mathbb{K})$. Now, by Lemma 3.4 we have that

$$\mathrm{gl}(\Pi_1^2(X^*, \ell_2^k; \mathbb{K})) = \mathrm{gl}(\Pi_1(X^*, \ell_2^k)).$$

But [36, Satz 3.2.1] shows that $\Pi_1(Z, \ell_2)$ has lust for any Banach space Z with unconditional basis, which gives our conclusion. □

Before we go on, let us briefly describe the 2-tensor norms α_1 and α_1^* in terms of their associated maximal Banach operator ideals. Recall from [14, 17.3] that a finitely generated tensor norm α and a maximal Banach operator ideal $(\mathcal{A}, \mathbf{A})$ are associated (notation: $\mathcal{A} \sim \alpha$) whenever for all finite dimensional Banach spaces M, N the equality

$$\mathcal{A}(M, N) = M^* \otimes_\alpha N$$

holds isometrically.

Remark 6.2. (a) The 2-tensor norm α_1 is associated with the smallest maximal Banach operator containing the quasi-Banach operator ideal $(\Gamma_\infty \circ \Gamma_1, \gamma_\infty \circ \gamma_1)$, the ideal of all compositions TS with $S \in \Gamma_1$ and $T \in \Gamma_\infty$ quasi-normed by $(\gamma_\infty \circ \gamma_1)(T \circ S) = \inf \gamma_\infty(T)\gamma_1(S)$. In the notation of [14, 29.8] this means that

$$\alpha_1 \sim (\Gamma_\infty \otimes \Gamma_1, \gamma_\infty \otimes \gamma_1),$$

where the latter ideal stands for the smallest maximal Banach operator ideal which contains $(\Gamma_\infty \circ \Gamma_1, \gamma_\infty \circ \gamma_1)$. By the very definition of α_1 and the notation from [14, section 29], the calculus of traced tensor norms, we have

$$\begin{aligned}\alpha_1 &= \varepsilon \otimes_{\ell_1} (\varepsilon \otimes_{c_0} \varepsilon) = \varepsilon \otimes_{\ell_1} w_1 \\ &= \varepsilon \otimes_{\ell_1} (\varepsilon \otimes w_1) = (\varepsilon \otimes_{\ell_1} \varepsilon) \otimes w_1 = w_\infty \otimes w_1,\end{aligned}$$

so that the proof follows as a consequence of [14, 29.8]. In the case $n = 2$ this somewhat formalizes the ideas from the end of section 3. Note that $(\Gamma_\infty \circ \Gamma_1, \gamma_\infty \circ \gamma_1)$ is not normed, which explains why in the definition of α_1, representations with double sums are necessary.
(b) The 2-tensor norm α_1^* is associated with the quotient ideal $(\Gamma_1^{-1} \circ \Pi_1, \gamma_1^{-1} \circ \pi_1)$ defined in the following way: $T : X \longrightarrow Y$ is in $\Gamma_1^{-1} \circ \Pi_1$ if and only if $ST \in \Pi_1$ for any $S : Y \longrightarrow Z$ in Γ_1, and $\left(\gamma_1^{-1} \circ \pi_1\right)(ST) = \sup_{\gamma_1(S) \le 1} \pi_1(ST)$. In short:

$$\alpha_1^* \sim (\Gamma_1^{-1} \circ \Pi_1, \gamma_1^{-1} \circ \pi_1)\,.$$

All this may be seen directly from Lemma 3.3 and Lemma 3.4, or a bit more formally from (a) and [14, 29.8, 25.7]. In particular, we have that for two Banach spaces X and Y the equality

$$\Pi_1^2(X, Y^*) = \Gamma_1^{-1} \circ \Pi_1(X, Y^*) = (X \otimes_{\alpha_1} Y)^*$$

holds isometrically (compare also the representation theorem for maximal Banach operator ideals, [14, 17.5]).

In the case of Banach operator ideals the so-called limit order has proved to be a useful tool, e.g. to obtain counterexamples. Following [33] we define the limit order of the norm α_1^* as the infimum of all $\lambda > 0$ such that

$$\alpha_1^*(\sum_{i=1}^n e_i \otimes e_i; \ell_p^n, \ell_q^n) \le \rho n^\lambda$$

where ρ is some constant; notation: $\lambda(\alpha_1^*, p, q)$. By Lemma 3.4 and [33] we easily obtain

$$\begin{aligned}\lambda(\alpha_1^*; p, q) &= \max\{\frac{1}{p}, \frac{1}{q}\}, \quad p, q \le 2, \\ \lambda(\alpha_1^*; p, q) &= \frac{1}{p} + \frac{1}{q} - \frac{1}{2}, \quad p \le 2 \le q \text{ or } q \le 2 \le p.\end{aligned}$$

Let us compute the remaining case, that is, $p, q > 2$. We know from Lemma 5.1 that in this case, $\alpha_1^*(\sum_{i=1}^n e_i \otimes e_i; \ell_p^n, \ell_q^n)$, up to a constant, equals

$$\sup\left\{\left(\sum_{i=1}^n |\lambda_i \mu_i|^2\right)^{\frac{1}{2}} : \lambda \in B_{\ell_r}, \mu \in B_{\ell_s}\right\}, \tag{5}$$

where $\frac{1}{r} = \frac{1}{p'} - \frac{1}{2}$ and $\frac{1}{s} = \frac{1}{q'} - \frac{1}{2}$. But now it is easy to see that (5) is just $\|id : \ell_u^n \longrightarrow \ell_2^n\|$, with $\frac{1}{u} = \frac{1}{p'} + \frac{1}{q'} - 1$. Therefore, using [33, Lemma 14.4.4] we can conclude that, in this case,

$$\lambda(\alpha_1^*; p, q) = \max\{0, \frac{1}{p} + \frac{1}{q} - \frac{1}{2}\}.$$

In fact, the above arguments prove that the diagonal of $\ell_p \tilde{\otimes}_{\alpha_1^*} \ell_q$ is a (complemented) copy of c_0 if $\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}$. This justifies the remark after Theorem 5.3. We also see that α_1^* does not preserve reflexivity in general, though we know from [21] that this is true for the class of absolutely summing operators. This leads us to the following

Question 6.3. Are these two properties (type 2 and reflexivity) preserved if $\frac{1}{p}+\frac{1}{q} > \frac{1}{2}$?

We can also use the above computations to see that α_1^* does not coincide with Δ_p if $p \neq 1, 2, \infty$, where Δ_p denotes the norm that makes $\ell_p \tilde{\otimes}_{\Delta_p} \ell_p = \ell_p(\mathbb{N} \times \mathbb{N})$.

Since $\Delta_p(\sum_{i=1}^n e_i \otimes e_i; \ell_p^n, \ell_p^n) = n^{\frac{1}{p}}$, we know that $\alpha_1^* \neq \Delta_p$ in $\ell_p \otimes \ell_p$ if $2 < p < \infty$. For the other case we can reason as follows. By [2] we know that for every $1 < p < 2$, there is an operator $u_n : \ell_p^n \longrightarrow \ell_1^n$ with $\|u_n\| \leq 1$ such that the norm of $u_n \otimes u_n : \ell_p^n \otimes_{\Delta_p} \ell_p^n \longrightarrow \ell_1^n \otimes_\pi \ell_1^n$ grows to infinity with n. Therefore, as α_1^* is a tensor norm that coincides with π in $\ell_1 \otimes \ell_1$, it cannot coincide with Δ_p in $\ell_p \otimes \ell_p$.

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References

[1] L. Aizenberg, *Multidimensional analogues of Bohr's theorem on power series*, Proc. Amer. Math. Soc. **128** (1999), no. 4, 1147–1155. MR1636918 (2000i:32001)
[2] G. Bennett, *Schur multipliers*, Duke Math. J. **44** (1977), 603–639. MR0493490 (58:12490)
[3] H.P. Boas, *Majorant series*, J. Korean Math. Soc. **37** (2000), no. 2, 321–337. MR1775963 (2001j:32001)
[4] H.P. Boas and D. Khavinson, Bohr's power series theorem in several variables, Proc. Amer. Math. Soc. **125**, 10 (1997), 2975-2979. MR1443371 (98i:32002)
[5] H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. **13** (1914), 1–5.
[6] F. Bombal, D. Pérez-García, and I. Villanueva, *Multilinear extensions of Grothendieck's theorem*, Q. J. Math. **55** (2004), 441–450. MR2104683 (2005i:47032)
[7] T. K. Carne, *Tensor products and Banach algebras*, J. London. Math. Soc. **17** (1978), 480–488. MR0493403 (58:12417)
[8] A. Defant, J. C. Díaz, D. García, and M. Maestre, *Unconditional basis and Gordon-Lewis constants for spaces of polynomials*, J. Funct. Anal. **181** (2001), 119–145. MR1818112 (2002b:46030)
[9] A. Defant, D. García, and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. reine angew. Math. **557** (2003), 173–197. MR1978407 (2004d:46012)
[10] A. Defant, D. García, M. Maestre, and D. Pérez-García, *Extension of multilinear forms and polynomials from subspaces of $\mathcal{L}_1$-spaces*, Houston J. Math. **33** (2007), 839–860. MR2335739
[11] A. Defant, N. Kalton, *Unconditionality in spaces of m-homogeneous polynomials on Banach spaces*, Q. J. Math. **56** (2005), 53–64. MR2124579 (2005k:46107)
[12] A. Defant, M. Maestre, and C. Prengel, *Domains of convergence for monomial series expansions of holomorphic functions in infinitely many variables*, preprint.
[13] A. Defant, *Variants of the Maurey-Rosenthal theorem for quasi Köthe function spaces*, Positivity **5** (2001), 153–175. MR1825653 (2002a:46031)
[14] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland, 1993. MR1209438 (94e:46130)
[15] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Univ. Press, 1995. MR1342297 (96i:46001)
[16] S. Dineen, *Complex Analysis on Infinite Dimensional Banach Spaces*, Springer-Verlag, 1999. MR1705327 (2001a:46043)
[17] S. Dineen and R.M. Timoney, *Absolute bases, tensor products and a theorem of Bohr*, Studia Math. **94** (1989), 227-234. MR1019790 (91e:46006)
[18] K. Floret and S. Hunfeld, *Ultrastability of ideals of homogeneous polynomials and multilinear mappings on Banach spaces*, Proc. Amer. Math. Soc. **130** (2001), 1425–1435. MR1879966 (2002k:46051)

[19] B. Gelbaum and J. Gil de Lamadrid, *Bases in tensor products of Banach spaces*, Pacific J. Math. **11** (1961), 1281–1286. MR0147881 (26:5394)

[20] Y. Gordon and D.R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48. MR0410341 (53:14091)

[21] Y. Gordon, D.R. Lewis, and J.R. Retherford, *Banach ideals of operators with applications*, J. Funct. Anal. **14** (1973), 85–128. MR0380488 (52:1388)

[22] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1956), 1–79. MR0094682 (20:1194)

[23] S. Kwapień and A. Pełczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. **34** (1970), 43–68. MR0270118 (42:5011)

[24] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer-Verlag, 1996. MR0500056 (58:17766)

[25] M.C. Matos, *Fully absolutely summing and Hilbert-Schmidt multilinear mappings*, Collect. Math. **54** (2003), 111-136. MR1995136 (2004e:46052)

[26] D. Pérez-García, *The inclusion theorem for multiple summing operators*, Studia Math. **165** (2004), 275-290. MR2110152 (2005i:47107)

[27] D. Pérez-García, *Comparing different classes of absolutely summing multilinear operators*, Arch. Math. (Basel) **85** (2005), 258–267. MR2172384 (2006j:46055)

[28] D. Pérez-García and I. Villanueva, *Unconditional bases in tensor products of Hilbert spaces*, Math. Scand. **96** (2005), 280–288. MR2153415 (2006b:46097)

[29] D. Pérez-García and I. Villanueva, *Multiple summing operators on Banach spaces*, J. Math. Anal. Appl. **285** (2003), 86–96. MR2000141 (2004m:47039)

[30] D. Pérez-García and I. Villanueva, *Multiple summing operators on C(K) spaces*, Ark. Mat. **42** (2004), 153-171. MR2056549 (2005g:47003)

[31] D. Pérez-García and I. Villanueva, *A composition theorem for multiple summing operators*, Monatsh. Math. **146** (2005), 257–261. MR2184227 (2006g:46072)

[32] D. Pérez-García and I. Villanueva, *There is no lattice preserving natural tensor norm*, Quaest. Math. **27** (2004), 267–273. MR2109666 (2005j:46004)

[33] A. Pietsch, *Operator ideals*, North-Holland, 1980. MR582655 (81j:47001)

[34] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. **37** (1978), 3–19. MR501916 (80e:46012)

[35] C. Schütt, *Unconditionality in tensor products*, Israel J. Math. **31** (1978), 209–216. MR516148 (81b:46082)

[36] I. Schütt, *Banachräume absolut p-summierender Operatoren*, Ph.D. Thesis, Kiel, 1986.

[37] N. Tomczak-Jaegermann, *Banach–Mazur distances and finite-dimensional operator ideals*, Longman Scientific & Technical, 1989.

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