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AFFINE GEOMETRIC CRYSTALS AND LIMIT OF PERFECT CRYSTALS

MASAKI KASHIWARA, TOSHIKI NAKASHIMA, AND MASATO OKADO

ABSTRACT. For every non-exceptional affine Lie algebra, we explicitly construct a positive geometric crystal associated with a fundamental representation. We also show that its ultra-discretization is isomorphic to the limit of certain perfect crystals of the Langlands dual affine Lie algebra.

1. INTRODUCTION

The theory of perfect crystals was introduced in [7], [8] in order to study certain physical models, called vertex models, which are associated with quantum R-matrices. A perfect crystal is a (pseudo-)crystal base of a finite-dimensional module of a quantum affine algebra $U_q(\mathfrak{g})$ (for more details, see 3.3), and it is labeled by a positive integer l, called the level (Definition 3.6). One of the most important properties of perfect crystals is as follows: for a perfect crystal B of level l and the crystal $B(\lambda)$ of the irreducible $U_q(\mathfrak{g})$-module with a dominant integral weight λ of level l as a highest weight, there exist a unique dominant integral weight μ of level l and an isomorphism of crystals:

$$B(\lambda) \cong B(\mu) \otimes B.$$

By iterating this isomorphism, we obtain the path realization of $B(\lambda)$, which plays a crucial role in solving the vertex models. In [8], for each $\mathfrak{g} = A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, perfect crystals are constructed explicitly by patching two classical crystals. It is conjectured that certain "Kirillov-Reshetikhin modules" have perfect crystal bases and all perfect crystals are obtained as a tensor product of those perfect crystal bases ([2], [3]), but, it is still far from the complete classification of perfect crystals.

Let $\{B_l\}_{l\geq 1}$ be a family of perfect crystals. If it satisfies certain conditions (Definition 3.7), there exists a *limit* B_∞ of $\{B_l\}_{l\geq 1}$. In such a case the family $\{B_l\}_{l\geq 1}$ is called a *coherent family* of perfect crystals ([6]). Let $B(\infty)$ be the crystal of the lower nilpotent subalgebras $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$. Then, similarly to perfect crystals, we have an isomorphism of crystals:

$$B(\infty) \otimes B_\infty \cong B(\infty).$$

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By iterating this isomorphism, the path realization of $B(\infty)$ is obtained. In [6], for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, explicit forms of the limit of perfect crystals are described, which are also reviewed in §3 below.

A. Berenstein and D. Kazhdan introduced the theory of geometric crystals ([1]), which is a structure on geometric objects analogous to crystals. Let I be the index set of simple roots of $\mathfrak{g}$. A geometric crystal consists of a variety X, a rational $\mathbb{C}^\times$- action $e_i\colon \mathbb{C}^\times \times X \longrightarrow X$ and rational functions $\gamma_i, \varepsilon_i\colon X \longrightarrow \mathbb{C}$ which satisfy certain conditions (see Definition 2.1). This structure resembles the one of crystals; for instance, we have the tensor product of a pair of geometric crystals similarly to the tensor product of crystals. There is a more direct and remarkable relation between crystals and geometric crystals, that is, the *ultra-discretization* functor $\mathcal{UD}$. This is a functor from the category of *positive* geometric crystals to that of crystals, where a positive geometric crystal is a geometric crystal equipped with a positive structure (see §2.4). Applying this functor, positive rational functions are transferred to piecewise linear functions by the following simple correspondence:

$$x \times y \mapsto x + y, \qquad \frac{x}{y} \mapsto x - y, \qquad x + y \mapsto \max(x, y).$$

The purpose of this paper is to construct a positive affine geometric crystal $\mathcal{V}(\mathfrak{g})$, whose ultra-discretization is isomorphic to a limit of perfect crystals $B_\infty(\mathfrak{g}^L)$, where $\mathfrak{g}^L$ is the Langlands dual of $\mathfrak{g}$.

Let G (resp. $\mathfrak{g}$) be the affine Kac-Moody group (resp. algebra) associated with a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. Let $B^\pm$ be the Borel subgroup and T the maximal torus. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ by the group morphism $\mathbb{C}^\times \to T$ induced by the simple coroot $\alpha_i^\vee$ as in §2.1. We set $Y_i(c) := y_i(c^{-1})\,\alpha_i^\vee(c) = \alpha_i^\vee(c)\,y_i(c)$. Let W (resp. $\widetilde{W}$) be the Weyl group (resp. the extended Weyl group) associated with $\mathfrak{g}$. The Schubert cell $X_w := BwB/B$ ($w = s_{i_1} \cdots s_{i_k} \in W$) is isomorphic to the variety

$$B_w^- := \left\{ Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \,|\, x_1, \cdots, x_k \in \mathbb{C}^\times \right\} \subset B^-,$$

and X_w has a natural geometric crystal structure ([1], [14]).

We choose $0 \in I$ as in [9], and let $\{\varpi_i\}_{i\in I\setminus\{0\}}$ be the set of level 0 fundamental weights. Let $W(\varpi_i)$ be the fundamental representation of $U_q(\mathfrak{g})$ with ϖ_i as an extremal weight ([9]). Let us denote its reduction at $q = 1$ by the same notation $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$-module. The module $W(\varpi_i)$ is irreducible over $U_q(\mathfrak{g})$, but its reduction at $q = 1$ is not necessarily an irreducible $\mathfrak{g}$-module. We set $\mathbb{P}(\varpi_i) := (W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$.

For any $i \in I$, define (see §3.2 for the definition of the inner product),

$$c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)}). \tag{1.1}$$

Then the translation $t(c_i^\vee \varpi_i)$ belongs to $\widetilde{W}$ (see §5.1). For a subset J of I, let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i\in J}$. For an integral weight μ, define $I(\mu) := \{j \in I \,|\, \langle h_j, \mu\rangle \geq 0\}$.

Then we have the following conjecture ([2, 3, 9]).

Conjecture 1.1. (1) *For any $i \in I \setminus \{0\}$ and any positive integer n, there exists a finite-dimensional irreducible $U_q(\mathfrak{g})$-module $W(n\varpi_i)$ with extremal weight $n\varpi_i$, called the "Kirillov-Reshetikhin module".*
(2) *The module $W(n\varpi_i)$ has a global crystal basis.*
(3) *Its crystal $B(n\varpi_i)$ is perfect if and only if $n \in c_i^\vee \mathbb{Z}$, and its level is equal to $n/c_i^\vee$.*
(4) *$\{B(nc_i^\vee \varpi_i)\}_n$ is a coherent family of perfect crystals. We denote by $B_\infty(\varpi_i)$ its limit crystal.*

Now, let us state our conjecture.

Conjecture 1.2. *For any $i \in I \setminus \{0\}$, there exist a unique variety X endowed with a positive $\mathfrak{g}$-geometric crystal structure and a rational mapping $\pi\colon X \longrightarrow \mathbb{P}(\varpi_i)$ satisfying the following properties:*

(1) *for an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^\vee \mu)$ as $\iota w \in \widetilde{W}$ with a Dynkin diagram automorphism ι and $w = s_{i_1} \cdots s_{i_k}$ (see §5.1), there exists a birational mapping $\xi\colon B_w^- \longrightarrow X$ such that ξ is a morphism of $\mathfrak{g}_{I(\mu)}$-geometric crystals and such that the composition $\pi \circ \xi\colon B_w^- \to \mathbb{P}(\varpi_i)$ coincides with $Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)\overline{u}$, where $s_{i_1} \cdots s_{i_k}$ is a reduced expression of w and $\overline{u}$ is the line including u,*
(2) *the ultra-discretization of X is isomorphic to the crystal $B_\infty(\varpi_i)$ of the Langlands dual $\mathfrak{g}^L$.*

In this paper, we construct a positive geometric crystal for some $i = 1$ and for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$, with this conjecture as a guide.

Let $W(\varpi_1)$ be the fundamental $\mathfrak{g}$-module as above ([9]) and $u_1 := u_{\varpi_1}$ an extremal weight vector of $W(\varpi_1)$ with the weight ϖ_1.

Then $t(c_1^\vee \varpi_1)$ belongs to $\widetilde{W}$ and (see §5.1) there exist $w_1 \in W$ and a Dynkin diagram automorphism ι such that $t(c_1^\vee \varpi_1) = \iota \cdot w_1$. Associated with this w_1, we define

$$X_1 := B_{w_1}^-.$$

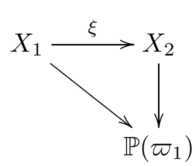
Then $I(\varpi_1) = I \setminus \{0\}$ and X_1 has a structure of a $\mathfrak{g}_{I\setminus\{0\}}$-geometric crystal.

Let us choose another extremal weight vector u_2 with extremal weight η such that $I(\eta) = I \setminus \{i_2\}$ for some $i_2 \neq 0$. Then there exists $w_2 \in W$ such that $t(c_1^\vee \eta) = \iota \cdot w_2$. We define similarly

$$X_2 := B_{w_2}^-,$$

which has a structure of a $\mathfrak{g}_{I\setminus\{i_2\}}$-geometric crystal. We have the rational mapping $X_\nu \to \mathbb{P}(\varpi_1)$ $(\nu = 1, 2)$.

Then we see (by a case-by-case calculation) that there exists a unique positive birational mapping $\xi\colon X_1 \to X_2$ such that the diagram



commutes and ξ commutes with e_i for $i \neq 0, i_2$. Moreover, ξ is an isomorphism of $\mathfrak{g}_{I\setminus\{0,i_2\}}$-geometric crystals.

Now the $\mathfrak{g}$-geometric crystal $\mathcal{V}(\mathfrak{g})$ is obtained by patching X_1 and X_2 by ξ. The relations (1)-(4) in Definition 2.1 are obvious for $(i,j) \neq (0,i_2),(i_2,0)$ and we check these relations for $(i,j)=(0,i_2),(i_2,0)$ by hand. In this paper we choose $\eta=\sigma\varpi_1$ for some Dynkin diagram automorphism σ except the case $\mathfrak{g}=A_{2n}^{(2)}$.

We then show that the ultra-discretization of $\mathcal{V}(\mathfrak{g})$ is isomorphic to the crystal $B_\infty(\varpi_1)$ of the Langlands dual $\mathfrak{g}^L$.

The organization of the paper is as follows. In §2, we review basic facts about geometric crystals following [1], [14], [15]. In §3, we shall recall the notion of the limit of perfect crystals and give examples of the limit $B_\infty(\mathfrak{g}^L)$ of perfect crystals following [6]. In §4, we present the explicit form of the fundamental representation $W(\varpi_1)$ for each affine Lie algebra $\mathfrak{g}$. In §5, we shall construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})$ associated with the fundamental representation $W(\varpi_1)$. In §6, we prove that the ultra-discretization $\mathcal{UD}(\mathcal{V}(\mathfrak{g}))$ is isomorphic to $B_\infty(\mathfrak{g}^L)$.

2. Geometric crystals

In this section, we review Kac-Moody groups and geometric crystals following [1], [4], [5], [11], [14], [16].

2.1. **Kac-Moody algebras and Kac-Moody groups.** Fix a symmetrizable generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t},\{\alpha_i\}_{i\in I},\{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I}\subset\mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I}\subset\mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee)=a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g}=\mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$ and the Chevalley generators e_i and f_i $(i\in I)$ with the usual defining relations ([5], [16]). There exists a root space decomposition $\mathfrak{g}=\bigoplus_{\alpha\in\mathfrak{t}^*}\mathfrak{g}_\alpha$. Denote the set of roots by $\Delta:=\{\alpha\in\mathfrak{t}^*\,|\,\alpha\neq 0,\ \mathfrak{g}_\alpha\neq(0)\}$. Set $Q=\sum_i\mathbb{Z}\alpha_i$, $Q_+=\sum_i\mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee=\sum_i\mathbb{Z}\alpha_i^\vee$ and $\Delta_+=\Delta\cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P\subset\mathfrak{t}^*$ be a weight lattice such that $\mathbb{C}\otimes P=\mathfrak{t}^*$, whose element is called a weight.

Define the simple reflections $s_i\in\mathrm{Aut}(\mathfrak{t})$ $(i\in I)$ by $s_i(h):=h-\alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. They induce the action of W on $\mathfrak{t}^*$ by $s_i(\lambda):=\lambda-\lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\mathrm{re}}:=\{w(\alpha_i)\,|\,w\in W,\ i\in I\}$, whose elements are called real roots.

Let G be the Kac-Moody group associated with $\mathfrak{g}$ ([16]). Let $U_\alpha:=\exp\mathfrak{g}_\alpha$ $(\alpha\in\Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha\in\Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha\in\Delta_+^{\mathrm{re}}=\Delta^{\mathrm{re}}\cap Q_+)$, *i.e.*, $U^\pm:=\langle U_{\pm\alpha}|\alpha\in\Delta_+^{\mathrm{re}}\rangle$.

For any $i\in I$, there exists a unique group homomorphism $\phi_i\colon SL_2(\mathbb{C})\to G$ such that

$$\phi_i\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)=\exp(te_i),\ \phi_i\left(\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\right)=\exp(tf_i)\qquad(t\in\mathbb{C}).$$

Set $\alpha_i^\vee(c):=\phi_i\left(\left(\begin{smallmatrix}c & 0\\ 0 & c^{-1}\end{smallmatrix}\right)\right)$, $x_i(t):=\exp(te_i)$, $y_i(t):=\exp(tf_i)$, $G_i:=\phi_i(SL_2(\mathbb{C}))$, $T_i:=\alpha_i^\vee(\mathbb{C}^\times)$ and $N_i:=N_{G_i}(T_i)$. Let T be the subgroup of G with the Lie algebra $\mathfrak{t}$, which is called a *maximal torus* in G, and let $B^\pm=U^\pm T$ be the Borel subgroup of G. Let N be the subgroup of G generated by the N_i's. Then we have the isomorphism $\phi\colon W\xrightarrow{\sim}N/T$ defined by $\phi(s_i)=N_iT/T$. An element $\overline{s}_i:=x_i(-1)y_i(1)x_i(-1)=\phi_i\left(\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\right)$ in $N_G(T)$ is a representative of $s_i\in W=N_G(T)/T$.

2.2. **Geometric crystals.** Let W be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \left\{(i_1, i_2, \cdots, i_l) \in I^l \,|\, w = s_{i_1} s_{i_2} \cdots s_{i_l}\right\},$$

where l is the length of w, *i.e.*, $R(w)$ is the set of reduced expressions of w.

Let X be a variety, $\gamma_i \colon X \to \mathbb{C}$ and $\varepsilon_i \colon X \longrightarrow \mathbb{C}$ $(i \in I)$ rational functions on X, and $e_i \colon \mathbb{C}^\times \times X \longrightarrow X$ $((c, x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action. For $w \in W$ and $\mathbf{i} = (i_1, \cdots, i_l) \in R(w)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$\begin{array}{rcl} e_{\mathbf{i}} \colon T \times X & \to & X \\ (t, x) & \mapsto & e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x). \end{array}$$

Definition 2.1. A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

(1) $\{1\} \times X \subset \mathrm{dom}(e_i)$ for any $i \in I$.
(2) $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
(3) $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$ and $\mathbf{i}$. $\mathbf{i}' \in R(w)$.
(4) $\varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x)$.

Note that the condition (3) is equivalent to the following so-called *Verma relations*:

$$\begin{array}{ll} e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} & \text{if } a_{ij} = a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^2 c_2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^2 c_2} e_i^{c_1} & \text{if } a_{ij} = -2,\ a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1^3 c_2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1 c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1^3 c_2^2} e_i^{c_1^2 c_2} e_j^{c_1^3 c_2} e_i^{c_1} & \text{if } a_{ij} = -3,\ a_{ji} = -1. \end{array}$$

Note that the last formula is different from the one in [1], [14], [15], which seems to be incorrect.

2.3. **Geometric crystal on Schubert cell.** Let $X := G/B$ be the flag variety, which is the inductive limit of finite-dimensional projective varieties. For $w \in W$, let $X_w := BwB/B \subset X$ be the Schubert cell associated with w, which has a natural geometric crystal structure ([1], [14]). For $\mathbf{i} = (i_1, \cdots, i_k) \in R(w)$, set

$$B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \cdots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_l}(c_k) \,|\, c_1, \cdots, c_k \in \mathbb{C}^\times\} \subset B^- \tag{2.1}$$

where $Y_i(c) := y_i(c^{-1}) \alpha_i^\vee(c)$. Then $B_{\mathbf{i}}^-$ is birationally isomorphic to X_w and endowed with the induced geometric crystal structure. The explicit form of the action e_i^c on $B_{\mathbf{i}}^-$ is given by

$$e_i^c(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)) = Y_{i_1}(\mathcal{C}_1) \cdots Y_{i_l}(\mathcal{C}_k)$$

where

$$\mathcal{C}_j := c_j \cdot \frac{\displaystyle\sum_{1 \le m \le j,\, i_m = i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j < m \le k,\, i_m = i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1 \le m < j,\, i_m = i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j \le m \le k,\, i_m = i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}. \tag{2.2}$$

We also have the explicit forms of the rational functions ε_i and γ_i:

$$\varepsilon_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = \sum_{1\le m\le k,\, i_m=i} \frac{1}{c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}}c_m},$$
$$\gamma_i(Y_{i_1}(c_1)\cdots Y_{i_l}(c_k)) = c_1^{a_{i_1,i}}\cdots c_k^{a_{i_k,i}}.$$

2.4. **Positive structure, ultra-discretization and tropicalization.** Let us recall the notions of positive structure and ultra-discretization/tropicalization.

The setting below is the same as in [15]. Set $R := \mathbb{C}(c)$ and define

$$\begin{array}{cccc} v\colon & R\setminus\{0\} & \longrightarrow & \mathbb{Z} \\ & f(c) & \mapsto & \deg(f(c)). \end{array}$$

Here deg is the degree of the pole at $c=\infty$. Note that for $f_1, f_2 \in R\setminus\{0\}$, we have

$$v(f_1f_2) = v(f_1)+v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1)-v(f_2). \tag{2.3}$$

We say that a non-zero rational function $f(c)\in\mathbb{C}(c)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients. If $f_1,\ f_2\in R$ are positive, then we have

$$v(f_1+f_2) = \max(v(f_1), v(f_2)). \tag{2.4}$$

Let $T\simeq(\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T,\mathbb{C}^\times)$ (resp. $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T)$) be the lattice of characters (resp. co-characters) of T. A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are positive linear combinations of characters of T.

Definition 2.2. Let $f\colon T\to T'$ be a rational mapping between two algebraic tori T and T'. We say that f is *positive*, if $\chi\circ f$ is positive for any character $\chi\colon T'\to\mathbb{C}$.

Denote by $\mathrm{Mor}^+(T,T')$ the set of positive rational mappings from T to T'.

Lemma 2.3 ([1])**.** *For any $f\in\mathrm{Mor}^+(T_1,T_2)$ and $g\in\mathrm{Mor}^+(T_2,T_3)$, the composition $g\circ f$ is well-defined and belongs to $\mathrm{Mor}^+(T_1,T_3)$.*

By Lemma 2.3, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and whose arrows are positive rational mappings.

Let $f\colon T\to T'$ be a positive rational mapping of algebraic tori T and T'. We define a map $\widehat{f}\colon X_*(T)\to X_*(T')$ by

$$\langle\chi, \widehat{f}(\xi)\rangle = v(\chi\circ f\circ\xi),$$

where $\chi\in X^*(T')$ and $\xi\in X_*(T)$.

Lemma 2.4 ([1])**.** *For any algebraic tori T_1, T_2, T_3, and positive rational mappings $f\in\mathrm{Mor}^+(T_1,T_2)$, $g\in\mathrm{Mor}^+(T_2,T_3)$, we have $\widehat{g\circ f} = \widehat{g}\circ\widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{array}{cccc} \mathcal{UD}\colon & \mathcal{T}_+ & \longrightarrow & \mathrm{Set} \\ & T & \mapsto & X_*(T) \\ & (f\colon T\to T') & \mapsto & (\widehat{f}\colon X_*(T)\to X_*(T')). \end{array}$$

Let us come back to the situation in §2.2. Hence G is a Kac-Moody group and T is its Cartan subgroup.

Definition 2.5 ([1]). Let $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ be a G-geometric crystal, T' an algebraic torus and $\theta\colon T' \to X$ a birational mapping. The mapping θ is called a *positive structure* on χ if it satisfies

(1) for any $i \in I$ the rational functions $\gamma_i \circ \theta\colon T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta\colon T' \to \mathbb{C}$ are positive,
(2) for any $i \in I$, the rational mapping $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c,t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta\colon T' \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to the positive rational mappings $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta\colon T' \to \mathbb{C}^\times$, we obtain

$$\begin{aligned} \tilde{e}_i := \mathcal{UD}(e_{i,\theta})\colon &\quad \mathbb{Z} \times X_*(T') \to X_*(T'),\\ \mathrm{wt}_i := \mathcal{UD}(\gamma_i \circ \theta), \varepsilon_i := \mathcal{UD}(\varepsilon_i \circ \theta)\colon &\quad X_*(T') \to \mathbb{Z}. \end{aligned}$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a free pre-crystal structure (see [1, 2.2]) and we denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. We thus have the following theorem:

Theorem 2.6 ([1], [14]). *For any geometric crystal* $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ *and a positive structure* $\theta\colon T' \to X$*, the associated pre-crystal* $\mathcal{UD}_{\theta,T'}(\chi)$ *is a crystal* (see [1, 2.2]).

Now, let $\mathcal{GC}^+$ be the category whose object is a triplet (χ, T', θ), where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta\colon T' \to X$ is a positive structure on χ, and the morphism $f\colon (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi\colon X_1 \longrightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1 \colon T'_1 \longrightarrow T'_2$$

is a positive rational mapping. Let $\mathcal{CR}$ be the category of crystals. Then by the theorem above, we have

Corollary 2.7. $\mathcal{UD}$ *defines a functor*

$$\begin{array}{ccc} \mathcal{UD} : \mathcal{GC}^+ & \longrightarrow & \mathcal{CR}, \\ (\chi, T', \theta) & \mapsto & X_*(T'), \\ (f\colon (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)) & \mapsto & (\widehat{f}\colon X_*(T'_1) \to X_*(T'_2)). \end{array}$$

We call the functor $\mathcal{UD}$ *"ultra-discretization"* as in [14], [15], while it is called "tropicalization" in [1]. For a crystal B, if there exists a geometric crystal χ and a positive structure $\theta\colon T' \to X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \simeq B$ as crystals, we call an object (χ, T', θ) in $\mathcal{GC}^+$ a *tropicalization* of B.

3. Limits of perfect crystals

We review the limits of perfect crystals following [6]. (See also [7], [8].)

3.1. **Crystals.** First we recall the notion of crystals, which is obtained by abstracting the combinatorial properties of crystal bases.

Definition 3.1. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned}
&\mathrm{wt}\colon B \longrightarrow P,\\
&\varepsilon_i\colon B \longrightarrow \mathbb{Z}\sqcup\{-\infty\},\quad \varphi_i\colon B \longrightarrow \mathbb{Z}\sqcup\{-\infty\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i\colon B\sqcup\{0\} \longrightarrow B\sqcup\{0\},\quad \tilde{f}_i\colon B\sqcup\{0\} \longrightarrow B\sqcup\{0\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i(0)=\tilde{f}_i(0)=0.
\end{aligned}$$

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\begin{aligned}
&\varphi_i(b)=\varepsilon_i(b)+\langle\alpha_i^\vee,\mathrm{wt}(b)\rangle,\\
&\mathrm{wt}(\tilde{e}_i b)=\mathrm{wt}(b)+\alpha_i \text{ if } \tilde{e}_i b\in B,\\
&\mathrm{wt}(\tilde{f}_i b)=\mathrm{wt}(b)-\alpha_i \text{ if } \tilde{f}_i b\in B,\\
&\tilde{e}_i b_2=b_1 \Longleftrightarrow \tilde{f}_i b_1=b_2,\\
&\varepsilon_i(b)=-\infty \Longrightarrow \tilde{e}_i b=\tilde{f}_i b=0.
\end{aligned}$$

The following tensor product structure is one of the most crucial properties of crystals.

Theorem 3.2. *Let B_1 and B_2 be crystals. Set $B_1\otimes B_2 := \{b_1\otimes b_2 \mid b_j\in B_j\ (j=1,2)\}$. Then we have*

(1) *$B_1\otimes B_2$ is a crystal,*
(2) *for $b_1\in B_1$ and $b_2\in B_2$, we have*

$$\tilde{f}_i(b_1\otimes b_2)=\begin{cases}\tilde{f}_i b_1\otimes b_2 & \text{if } \varphi_i(b_1)>\varepsilon_i(b_2),\\ b_1\otimes\tilde{f}_i b_2 & \text{if } \varphi_i(b_1)\le\varepsilon_i(b_2);\end{cases}$$

$$\tilde{e}_i(b_1\otimes b_2)=\begin{cases}b_1\otimes\tilde{e}_i b_2 & \text{if } \varphi_i(b_1)<\varepsilon_i(b_2),\\ \tilde{e}_i b_1\otimes b_2 & \text{if } \varphi_i(b_1)\ge\varepsilon_i(b_2).\end{cases}$$

Definition 3.3. Let B_1 and B_2 be crystals. A *morphism* of crystals $\psi\colon B_1 \longrightarrow B_2$ is a map $\psi\colon B_1\sqcup\{0\} \longrightarrow B_2\sqcup\{0\}$ satisfying

(1) $\psi(0)=0$, $\psi(B_1)\subset B_2$,
(2) for $b, b'\in B_1$, $\tilde{f}_i b=b'$ implies $\tilde{f}_i\psi(b)=\psi(b')$,
(3) $\mathrm{wt}(\psi(b))=\mathrm{wt}(b)$, $\varepsilon_i(\psi(b))=\varepsilon_i(b)$, $\varphi_i(\psi(b))=\varphi_i(b)$ for any $b\in B_1$.

In particular, a bijective morphism is called an *isomorphism of crystals*.

Example 3.4. (1) If (L,B) is a crystal base, then B is a crystal.
(2) For the crystal base $(L(\infty),B(\infty))$ of the subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.
(3) For $\lambda\in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by

$$\tilde{e}_i(t_\lambda)=\tilde{f}_i(t_\lambda)=0,\quad \varepsilon_i(t_\lambda)=\varphi_i(t_\lambda)=-\infty,\quad \mathrm{wt}(t_\lambda)=\lambda.$$

Definition 3.5. To a crystal B, a colored oriented graph is associated by

$$b_1 \xrightarrow{i} b_2 \Longleftrightarrow \tilde{f}_i b_1 = b_2.$$

We call this graph the *crystal graph* of B.

3.2. **Affine weights.** Let $\mathfrak{g}$ be an affine Lie algebra, and let the sets $\mathfrak{t}$, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ be as in §2.1. We take $\mathfrak{t}$ so that $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q \,|\, \langle \alpha_i^\vee, \lambda\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$

and let $\mathbf{c} \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i^\vee \subset \mathfrak{g}$ be a unique central element satisfying

$$\{h \in Q^\vee \,|\, \langle h, \alpha_i\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}.$$

We write ([4, 6.1])

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i. \tag{3.1}$$

Let (,) be the non-degenerate W-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda\rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\rm cl} := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl}\colon \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{\rm cl}$ be the canonical projection. Then we have $\mathfrak{t}^*_{\rm cl} \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}^*_0 := \{\lambda \in \mathfrak{t}^* \,|\, \langle \mathbf{c}, \lambda\rangle = 0\}$, $(\mathfrak{t}^*_{\rm cl})_0 := \mathrm{cl}(\mathfrak{t}^*_0)$. Then we have a positive-definite symmetric form on $(\mathfrak{t}^*_{\rm cl})_0$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}^*_{\rm cl}$ $(i \in I)$ be a weight such that $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{i,j}$, which is called a *fundamental weight*. We choose P so that $P_{\rm cl} := \mathrm{cl}(P)$ coincides with $\bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ and we call $P_{\rm cl}$ the *classical weight lattice*.

3.3. **Definitions of a perfect crystal and its limit.** Let $\mathfrak{g}$ be an affine Lie algebra, $P_{\rm cl}$ the classical weight lattice as above and set $(P_{\rm cl})^+_l := \{\lambda \in P_{\rm cl}\langle \mathbf{c}, \lambda\rangle = l, \langle \alpha_i^\vee, \lambda\rangle \geq 0\}$ for $l \in \mathbb{Z}_{>0}$.

Definition 3.6. We say that a crystal B is *perfect* of level l if

(1) $B \otimes B$ is connected as a crystal graph.
(2) There exists $\lambda_0 \in P_{\rm cl}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\mathrm{cl}(\alpha_i), \qquad \sharp B_{\lambda_0} = 1.$$

(3) There exists a finite-dimensional $U_q'(\mathfrak{g})$-module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$.
(4) The maps $\varepsilon, \varphi\colon B^{min} := \{b \in B \,|\, \langle c, \varepsilon(b)\rangle = l\} \longrightarrow (P^+_{\rm cl})_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b)\Lambda_i$.

Let $\{B_l\}_{l\geq 1}$ be a family of perfect crystals of level l and set $J := \{(l, b) | l > 0, b \in B_l^{min}\}$.

Definition 3.7. A crystal B_∞ with an element b_∞ is called the *limit of* $\{B_l\}_{l\geq 1}$ if

(1) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.
(2) For any $(l, b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)}\colon \quad & T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty \\ & t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty. \end{aligned}$$

(3) $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$.

As for the crystal T_λ, see Example 3.4 (3). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

The following is one of the most important properties of the limits of perfect crystals.

Proposition 3.8 ([6]). *Let* $B(\infty)$ *be the crystal as in* Example 3.4 (2). *Then we have the following isomorphism of crystals:*

$$B(\infty)\otimes B_\infty \xrightarrow{\sim} B(\infty).$$

In the rest of this section, for each affine Lie algebra $\mathfrak{g}=A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, we explicitly describe an example of $B_\infty=B_\infty(\mathfrak{g})$ following [6].

3.4. $A_n^{(1)}$ $(n\geq 2)$. The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I:=\{0,1,\cdots,n\})$ of type $A_n^{(1)}$ is

$$a_{ij}=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } i\equiv j\pm 1 \mod n+1,\\ 0 & \text{otherwise.}\end{cases}$$

We have $\mathbf{c}=\sum_{i\in I}\alpha_i^\vee$ and $\delta=\sum_{i\in I}\alpha_i$. A limit of perfect crystals is given as follows:

$$B_\infty(A_n^{(1)})=\Big\{(b_1,\cdots,b_{n+1})\in\mathbb{Z}^{n+1}\,|\,\sum_{i=1}^{n+1}b_i=0\Big\},$$

for $b=(b_1,\cdots,b_{n+1})\in B_\infty(A_n^{(1)})$. We have

$$\begin{cases}\tilde{e}_0b=(b_1-1,b_2,\cdots,b_n,b_{n+1}+1),\\ \tilde{e}_ib=(b_1,\cdots,b_i+1,b_{i+1}-1,\cdots,b_{n+1})\ (i=1,\cdots,n),\\ \tilde{f}_i=\tilde{e}_i^{-1},\end{cases}$$

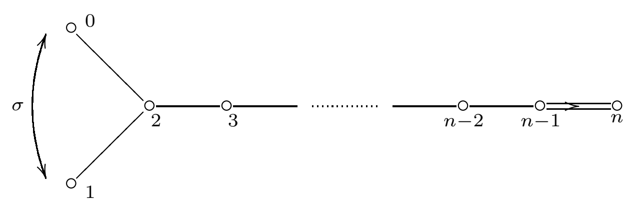
and

$$\begin{cases}\mathrm{wt}(b)=(b_{n+1}-b_1)\Lambda_0+\sum_{i=1}^n(b_i-b_{i+1})\Lambda_i,\\ \varepsilon_0(b)=b_1,\quad \varepsilon_i(b)=b_{i+1}\ (i=1,\cdots,n),\\ \varphi_0(b)=b_{n+1},\quad \varphi_i(b)=b_i\ (i=1,\cdots,n).\end{cases}$$

3.5. $B_n^{(1)}$ $(n\geq 3)$. The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I:=\{0,1,\cdots,n\})$ of type $B_n^{(1)}$ is

$$a_{ij}=\begin{cases}2 & i=j,\\ -1 & |i-j|=1 \text{ and } (i,j)\neq(0,1),(1,0),(n,n-1) \text{ or } (i,j)=(0,2),(2,0),\\ -2 & (i,j)=(n,n-1),\\ 0 & \text{otherwise.}\end{cases}$$

The Dynkin diagram is



where σ is a Dynkin diagram automorphism which we use later. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n-1} \alpha_i^\vee + \alpha_n^\vee, \qquad \delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n} \alpha_i.$$

A limit of perfect crystals is given as follows:

$$B_\infty(B_n^{(1)}) = \{(b_1, \cdots, b_n, \overline{b}_n, \cdots, b_1) \in \mathbb{Z}^{n-1} \times \left(\frac{1}{2}\mathbb{Z}\right)^2 \times \mathbb{Z}^{n-1} \,| \\ \sum_{i=1}^{n}(b_i + \overline{b}_i) = 0,\ b_n + \overline{b}_n \in \mathbb{Z}\},$$

for $b = (b_1, \cdots, b_n, \overline{b}_n, \cdots, b_1) \in B_\infty(B_n^{(1)})$. We have

$$\tilde{e}_0 b = \begin{cases} (b_1, b_2 - 1, \cdots, \overline{b}_2, \overline{b}_1 + 1) & \text{if } b_2 > \overline{b}_2, \\ (b_1 - 1, b_2, \cdots, \overline{b}_2 + 1, \overline{b}_1) & \text{if } b_2 \le \overline{b}_2, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (b_1 \cdots, b_i + 1, b_{i+1} - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} > \overline{b}_{i+1}, \\ (b_1 \cdots, \overline{b}_{i+1} + 1, \overline{b}_i - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} \le \overline{b}_{i+1}, \end{cases} \qquad (i = 1, \cdots, n-1),$$

$$\tilde{e}_n b = (b_1, \cdots, b_n + \frac{1}{2}, \overline{b}_n - \frac{1}{2}, \cdots, \overline{b}_1),$$

$$\tilde{f}_i = \tilde{e}_i^{-1},$$

and

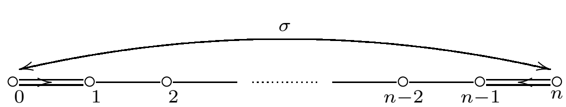
$$\begin{cases} \mathrm{wt}(b) = (\overline{b}_1 - b_1 + \overline{b}_2 - b_2)\Lambda_0 + \sum_{i=1}^{n-1}(b_i - \overline{b}_i + \overline{b}_{i+1} - b_{i+1})\Lambda_i + 2(b_n - \overline{b}_n)\Lambda_n, \\ \varepsilon_0(b) = b_1 + (b_2 - \overline{b}_2)_+,\ \varepsilon_i(b) = \overline{b}_i + (b_{i+1} - \overline{b}_{i+1})_+ \ (i = 1, \cdots, n-1), \\ \varepsilon_n(b) = 2\overline{b}_n, \varphi_0(b) = \overline{b}_1 + (\overline{b}_2 - b_2)_+,\ \varphi_i(b) = b_i + (\overline{b}_{i+1} - b_{i+1})_+ \ (i = 1, \cdots, n-1), \\ \varphi_n(b) = 2b_n. \end{cases}$$

Note that the presentation above is slightly different from the one in [6]. But we see that they are equivalent by the correspondence $\nu_n + \frac{1}{2}\nu_0 \leftrightarrow b_n$ and $\overline{\nu}_n + \frac{1}{2}\nu_0 \leftrightarrow \overline{b}_n$.

3.6. $C_n^{(1)}$) ($n \geq 2$)**.** The Cartan matrix $(a_{ij})_{i,j \in I}$ ($I := \{0, 1, \cdots, n\}$) of type $C_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i,j) \neq (1,0), (n-1,n), \\ -2 & (i,j) = (1,0), (n-1,n), \\ 0 & \text{otherwise.} \end{cases}$$

Then the Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon \alpha_i \mapsto \alpha_{n-i}$. We have

$$\mathbf{c} = \sum_{i \in I} \alpha_i^\vee, \qquad \delta = \alpha_0 + 2\sum_{i=1}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(C_n^{(1)}) = \{(b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1) \in \mathbb{Z}^{2n} | \sum_{i=1}^{n} (b_i + \overline{b}_i) \in 2\mathbb{Z}\},$$

for $b = (b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1) \in B_\infty(C_n^{(1)})$. We have

$$\tilde{e}_0 b = \begin{cases} (b_1 - 2, b_2, \cdots, \overline{b}_2, \overline{b}_1) & \text{if } b_1 > \overline{b}_1 + 1, \\ (b_1 - 1, b_2, \cdots, \overline{b}_2, \overline{b}_1 + 1) & \text{if } b_1 = \overline{b}_1 + 1, \\ (b_1, b_2, \cdots, \overline{b}_2, \overline{b}_1 + 2) & \text{if } b_1 \le \overline{b}_1, \end{cases}$$
$$\tilde{e}_i b = \begin{cases} (b_1 \cdots, b_i + 1, b_{i+1} - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} > \overline{b}_{i+1}, \\ (b_1 \cdots, \overline{b}_{i+1} + 1, \overline{b}_i - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} \le \overline{b}_{i+1}, \end{cases} (1 \le i < n),$$
$$\tilde{e}_n b = (b_1, \cdots, b_n + 1, \overline{b}_n - 1, \cdots, \overline{b}_1),$$
$$\tilde{f}_i = \tilde{e}_i^{-1},$$

and

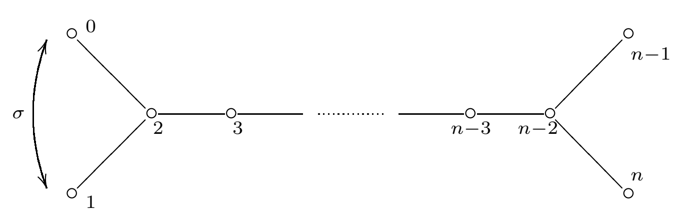
$$\begin{cases} \mathrm{wt}(b) = (\overline{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1}(b_i - \overline{b}_i + \overline{b}_{i+1} - b_{i+1})\Lambda_i + (b_n - \overline{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -\frac{1}{2}l(b) + (b_1 - \overline{b}_1)_+, \varepsilon_i(b) = \overline{b}_i + (b_{i+1} - \overline{b}_{i+1})_+ (1 \le i < n), \varepsilon_n(b) = \overline{b}_n, \\ \varphi_0(b) = -\frac{1}{2}l(b) + (\overline{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\overline{b}_{i+1} - b_{i+1})_+ (1 \le i < n), \varphi_n(b) = b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^{n}(b_i + \overline{b}_i)$.

3.7. $D_n^{(1)}$ $(n \ge 4)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I := \{0, 1, \cdots, n\})$ of type $D_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } 1 \le i, j \le n - 1 \\ & \text{or } (i, j) = (0, 2), (2, 0), (n - 2, n), (n, n - 2), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon \alpha_0 \leftrightarrow \alpha_1$ and $\sigma\alpha_i = \alpha_i$ for $i \ne 0, 1$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n-2} \alpha_i^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee, \qquad \delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(D_n^{(1)}) = \{(b_1, \cdots, b_n, \overline{b}_{n-1}, \cdots, \overline{b}_1) \in \mathbb{Z}^{2n-1} | \sum_{i=1}^{n} b_i + \sum_{1}^{n-1} \overline{b}_i = 0\},$$

for $b = (b_1, \cdots, b_n, \overline{b}_{n-1}, \cdots, \overline{b}_1) \in B_\infty(D_n^{(1)})$. We have

$$\tilde{e}_0 b = \begin{cases} (b_1, b_2 - 1, \cdots, \overline{b}_2, \overline{b}_1 + 1) & \text{if } b_2 > \overline{b}_2, \\ (b_1 - 1, b_2, \cdots, \overline{b}_2 + 1, \overline{b}_1) & \text{if } b_2 \leq \overline{b}_2, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (b_1 \cdots, b_i + 1, b_{i+1} - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} > \overline{b}_{i+1}, \\ (b_1 \cdots, \overline{b}_{i+1} + 1, \overline{b}_i - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} \leq \overline{b}_{i+1}, \end{cases} (i = 1, \cdots, n-2),$$

$$\tilde{e}_{n-1} b = (b_1, \cdots, b_{n-1} + 1, b_n - 1, \overline{b}_{n-1}, \cdots, \overline{b}_1),$$

$$\tilde{e}_n b = (b_1, \cdots, b_{n-1}, b_n + 1, \overline{b}_{n-1} - 1 \cdots, \overline{b}_1),$$

$$\tilde{f}_i = \tilde{e}_i^{-1},$$

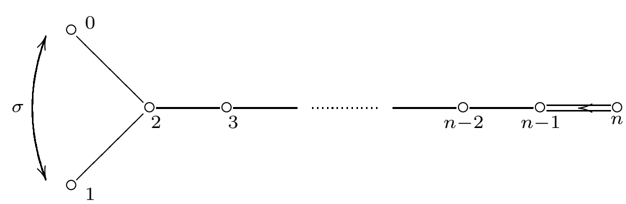
and

$$\begin{cases} \mathrm{wt}(b) = (\overline{b}_1 - b_1 + \overline{b}_2 - b_2)\Lambda_0 + \sum_{i=1}^{n-2} (b_i - \overline{b}_i + \overline{b}_{i+1} - b_{i+1})\Lambda_i \\ \qquad\qquad +(b_{n-1} - \overline{b}_{n-1} - b_n)\Lambda_{n-1} + (b_{n-1} - \overline{b}_{n-1} + b_n)\Lambda_n, \\ \varepsilon_0(b) = b_1 + (b_2 - \overline{b}_2)_+, \ \varepsilon_i(b) = \overline{b}_i + (b_{i+1} - \overline{b}_{i+1})_+ \ (i = 1, \cdots, n-2), \\ \varepsilon_{n-1}(b) = b_n + \overline{b}_{n-1}, \ \varepsilon_n(b) = \overline{b}_{n-1}, \\ \varphi_0(b) = \overline{b}_1 + (\overline{b}_2 - b_2)_+, \ \varphi_i(b) = b_i + (\overline{b}_{i+1} - b_{i+1})_+ \ (i = 1, \cdots, n-2), \\ \varphi_{n-1}(b) = b_{n-1}, \ \varphi_n(b) = b_{n-1} + b_n. \end{cases}$$

3.8. $A_{2n-1}^{(2)}$ $(n \geq 3)$**.** The Cartan matrix $(a_{ij})_{i,j \in I}$ $(I := \{0, 1, \cdots, n\})$ of type $A_{2n-1}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } 1 \leq i, j \leq n-1 \text{ or } (i,j) = (0,2), (2,0), (n, n-1), \\ -2 & (i,j) = (n-1, n), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma \colon \alpha_0 \leftrightarrow \alpha_1$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n} \alpha_i^\vee, \qquad \delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(A_{2n-1}^{(2)}) = \{(b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1) \in \mathbb{Z}^{2n} | \sum_{i=1}^{n} (b_i + \overline{b}_i) = 0\},$$

for $b=(b_1,\cdots,b_n,\overline{b}_n,\cdots,\overline{b}_1)\in B_\infty(A^{(2)}_{2n-1})$. We have

$$\tilde{e}_0b=\begin{cases}(b_1,b_2-1,\cdots,\overline{b}_2,\overline{b}_1+1) & \text{if } b_2>\overline{b}_2,\\ (b_1-1,b_2,\cdots,\overline{b}_2+1,\overline{b}_1) & \text{if } b_2\le\overline{b}_2,\end{cases}$$

$$\tilde{e}_ib=\begin{cases}(b_1\cdots,b_i+1,b_{i+1}-1,\cdots,\overline{b}_1) & \text{if } b_{i+1}>\overline{b}_{i+1},\\ (b_1\cdots,\overline{b}_{i+1}+1,\overline{b}_i-1,\cdots,\overline{b}_1) & \text{if } b_{i+1}\le\overline{b}_{i+1},\end{cases}\ (i=1,\cdots,n-1),$$

$$\tilde{e}_nb=(b_1,\cdots,b_{n-1},b_n+1,\overline{b}_n-1,\cdots,\overline{b}_1),$$

$$\tilde{f}_i=\tilde{e}_i^{-1},$$

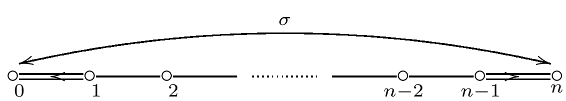
and

$$\begin{cases}\mathrm{wt}(b)=(\overline{b}_1-b_1+\overline{b}_2-b_2)\Lambda_0+\sum_{i=1}^{n-1}(b_i-\overline{b}_i+\overline{b}_{i+1}-b_{i+1})\Lambda_i+(b_n-\overline{b}_n)\Lambda_n,\\ \varepsilon_0(b)=b_1+(b_2-\overline{b}_2)_+,\varepsilon_i(b)=\overline{b}_i+(b_{i+1}-\overline{b}_{i+1})_+\ (i=1,\cdots,n-1),\\ \varepsilon_n(b)=\overline{b}_n,\varphi_0(b)=\overline{b}_1+(\overline{b}_2-b_2)_+,\varphi_i(b)=b_i+(\overline{b}_{i+1}-b_{i+1})_+(i=1,\cdots,n-1)\\ \varphi_n(b)=b_n.\end{cases}$$

3.9. $D^{(2)}_{n+1}$ $(n\ge 2)$. The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I:=\{0,1,\cdots,n\})$ of type $D^{(2)}_{n+1}$ is

$$a_{ij}=\begin{cases}2 & i=j,\\ -1 & |i-j|=1 \text{ and } (i,j)\ne(0,1),(n,n-1),\\ -2 & (i,j)=(0,1),(n,n-1),\\ 0 & \text{otherwise.}\end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon \alpha_i\leftrightarrow\alpha_{n-i}$ $(i=0,1\cdots,n)$. We have

$$\mathbf{c}=\alpha_0^\vee+\alpha_1^\vee+2\sum_{i=2}^{n-1}\alpha_i^\vee+\alpha_n^\vee,\qquad \delta=\sum_{i=0}^{n}\alpha_i.$$

A limit of perfect crystals is given as follows:

$$B_\infty(D^{(2)}_{n+1})=\{(b_1,\cdots,b_n,\overline{b}_n,\cdots,b_1)\in\mathbb{Z}^{n-1}\times\left(\frac{1}{2}\mathbb{Z}\right)^2\times\mathbb{Z}^{n-1}\big|b_n+\overline{b}_n\in\mathbb{Z}\},$$

for $b=(b_1,\cdots,b_n,\overline{b}_n,\cdots,b_1)\in B_\infty(D^{(2)}_{n+1})$. We have

$$\tilde{e}_0b=\begin{cases}(b_1-1,b_2,\cdots,\overline{b}_2,\overline{b}_1) & \text{if } b_1>\overline{b}_1,\\ (b_1,b_2,\cdots,\overline{b}_2,\overline{b}_1+1) & \text{if } b_1\le\overline{b}_1,\end{cases}$$

$$\tilde{e}_ib=\begin{cases}(b_1\cdots,b_i+1,b_{i+1}-1,\cdots,\overline{b}_1) & \text{if } b_{i+1}>\overline{b}_{i+1},\\ (b_1\cdots,\overline{b}_{i+1}+1,\overline{b}_i-1,\cdots,\overline{b}_1) & \text{if } b_{i+1}\le\overline{b}_{i+1},\end{cases}\ (1\le i<n),$$

$$\tilde{e}_nb=(b_1,\cdots,b_n+\frac{1}{2},\overline{b}_n-\frac{1}{2},\cdots,\overline{b}_1),$$

$$\tilde{f}_i=\tilde{e}_i^{-1},$$

and

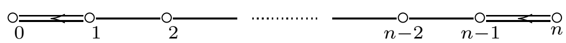
$$\begin{cases} \mathrm{wt}(b) = 2(\overline{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1}(b_i - \overline{b}_i + \overline{b}_{i+1} - b_{i+1})\Lambda_i + 2(b_n - \overline{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -l(b) + 2(b_1 - \overline{b}_1)_+, \varepsilon_i(b) = \overline{b}_i + (b_{i+1} - \overline{b}_{i+1})_+ (1 \le i < n), \varepsilon_n(b) = 2\overline{b}_n, \\ \varphi_0(b) = -l(b) + 2(\overline{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\overline{b}_{i+1} - b_{i+1})_+ (1 \le i < n),\ \varphi_n(b) = 2b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^{n}(b_i + \overline{b}_i)$. Note that the presentation above is slightly different from the one in [6]. As in §3.5, they are equivalent by the correspondence $\nu_n + \frac{1}{2}\nu_0 \leftrightarrow b_n$ and $\overline{\nu}_n + \frac{1}{2}\nu_0 \leftrightarrow \overline{b}_n$.

3.10. $A_{2n}^{(2)}$ $(n \ge 2)$. The Cartan matrix $(a_{ij})_{i,j \in I}$ $(I := \{0, 1, \cdots, n\})$ of type $A_{2n}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1 \text{ and } (i,j) \ne (0,1), (n-1,n), \\ -2 & (i,j) = (0,1), (n-1,n), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is



Note that there exists no Dynkin diagram automorphism in this case. We have

$$\mathbf{c} = \alpha_0^\vee + 2\sum_{i=1}^{n} \alpha_i^\vee, \qquad \delta = 2\sum_{i=0}^{n-1} \alpha_i + \alpha_n.$$

A limit of perfect crystals is given as follows:

$$B_\infty(A_{2n}^{(2)}) = \{(b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1) \in \mathbb{Z}^{2n}\},$$

for $(b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1) \in B_\infty(A_{2n}^{(2)})$. We have

$$\tilde{e}_0 b = \begin{cases} (b_1 - 1, b_2, \cdots, \overline{b}_2, \overline{b}_1) & \text{if } b_1 > \overline{b}_1, \\ (b_1, b_2, \cdots, \overline{b}_2, \overline{b}_1 + 1) & \text{if } b_1 \le \overline{b}_1, \end{cases}$$

$$\tilde{e}_i b = \begin{cases} (b_1 \cdots, b_i + 1, b_{i+1} - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} > \overline{b}_{i+1}, \\ (b_1 \cdots, \overline{b}_{i+1} + 1, \overline{b}_i - 1, \cdots, \overline{b}_1) & \text{if } b_{i+1} \le \overline{b}_{i+1}, \end{cases} (1 \le i < n),$$

$$\tilde{e}_n b = (b_1, \cdots, b_{n-1}, b_n + 1, \overline{b}_n - 1, \cdots, \overline{b}_1),$$

$$\tilde{f}_i = \tilde{e}_i^{-1},$$

and

$$\begin{cases} \mathrm{wt}(b) = 2(\overline{b}_1 - b_1)\Lambda_0 + \sum_{i=1}^{n-1}(b_i - \overline{b}_i + \overline{b}_{i+1} - b_{i+1})\Lambda_i + (b_n - \overline{b}_n)\Lambda_n, \\ \varepsilon_0(b) = -l(b) + 2(b_1 - \overline{b}_1)_+, \varepsilon_i(b) = \overline{b}_i + (b_{i+1} - \overline{b}_{i+1})_+ (1 \le i < n), \varepsilon_n(b) = \overline{b}_n, \\ \varphi_0(b) = -l(b) + 2(\overline{b}_1 - b_1)_+, \varphi_i(b) = b_i + (\overline{b}_{i+1} - b_{i+1})_+ (1 \le i < n),\ \varphi_n(b) = b_n, \end{cases}$$

where $l(b) := \sum_{i=1}^{n}(b_i + \overline{b}_i)$.

4. Fundamental representations

4.1. Fundamental representation $W(\varpi_1)$**.** Let $\mathfrak{g} = A_n^{(1)},\ B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)},\ A_{2n-1}^{(2)},\ D_{n+1}^{(2)},\ A_{2n}^{(2)}$, and let $\{\Lambda_i \,|\, i \in I\}$ be the set of fundamental weights as in §3. Let $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as in §3 and $a_i^\vee$ is given in (3.1).

Let $V(\varpi_1)$ be the extremal weight module over $U_q(\mathfrak{g})$ associated with ϖ_1 ([9]) and let $W(\varpi_1) \cong V(\varpi_1)/(z_1 - 1)V(\varpi_1)$ be the fundamental representation of $U'_q(\mathfrak{g})$ where z_1 is a $U'_q(\mathfrak{g})$-linear automorphism on $V(\varpi_1)$ (see [9, Sect. 5]).

By [9, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider its specialization $q = 1$ and obtain a finite-dimensional $\mathfrak{g}$-module, which will be denoted by the same notation $W(\varpi_1)$.

Now we present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$.

4.2. $A_n^{(1)}$ $(n \geq 2)$**.** The global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_{n+1}\},$$

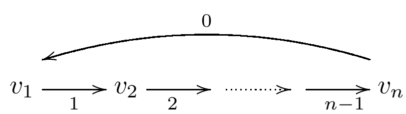
and we have

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1} \quad (i = 1 \cdots, n+1),$$

where we understand $\Lambda_{n+1} = \Lambda_0$. The explicit actions of the f_i's are

$$\begin{aligned} &f_i v_i = v_{i+1} \quad (1 \leq i \leq n), \quad f_0 v_{n+1} = v_1, \\ &f_i v_j = 0 \qquad \text{otherwise.} \end{aligned}$$

Its crystal graph is:



4.3. $B_n^{(1)}$ $(n \geq 3)$**.** The global basis of $W(\varpi_1)$ is

$$\{v_1, v_2,\ \cdots,\ v_n,\ v_0,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}}\},$$

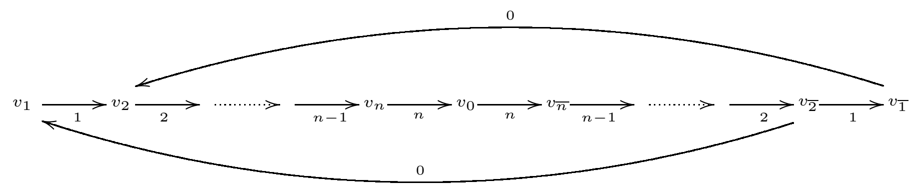
and we have

$$\begin{aligned} &\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 0, 2, n), \\ &\mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}}) = \Lambda_0 + \Lambda_1 - \Lambda_2, \\ &\mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n, \qquad \mathrm{wt}(v_0) = 0. \end{aligned}$$

The explicit forms of the actions by the f_i's are

$$\begin{aligned} &f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1), \\ &f_n v_n = v_0, \qquad f_n v_0 = 2v_{\overline{n}}, \\ &f_0 v_{\overline{2}} = v_1, \qquad f_0 v_{\overline{1}} = v_2, \\ &f_i v_j = 0 \qquad \text{otherwise.} \end{aligned}$$

Its crystal graph is



4.4. $C_n^{(1)}$ $(n \geq 2)$**.** The global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_n,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}}\},$$

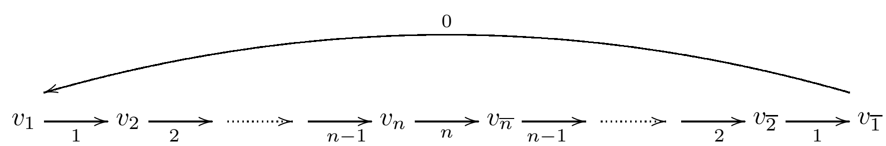
and we have

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i = 1 \cdots, n).$$

The explicit forms of the actions by the f_i's are

$$\begin{aligned} &f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1),\\ &f_n v_n = v_{\overline{n}}, \qquad f_0 v_{\overline{1}} = v_1,\\ &f_i v_j = 0 \qquad \text{otherwise.} \end{aligned}$$

Its crystal graph is



4.5. $D_n^{(1)}$ $(n \geq 4)$**.** The global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_n,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}}\},$$

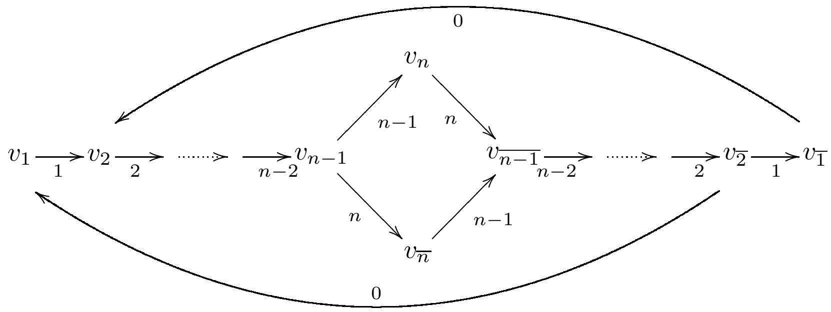
and we have

$$\begin{aligned} &\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 2, n-1),\\ &\mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}}) = \Lambda_0 + \Lambda_1 - \Lambda_2,\\ &\mathrm{wt}(v_{n-1}) = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n-1}}) = \Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n. \end{aligned}$$

The explicit forms of the actions by the f_i's are

$$\begin{aligned} &f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1),\\ &f_n v_n = v_{\overline{n-1}}, \qquad f_n v_{n-1} = v_{\overline{n}},\\ &f_0 v_{\overline{2}} = v_1, \qquad f_0 v_{\overline{1}} = v_2,\\ &f_i v_j = 0 \qquad \text{otherwise.} \end{aligned}$$

Its crystal graph is



4.6. $A^{(2)}_{2n-1}$ $(n \geq 3)$. The global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_n,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}}\},$$

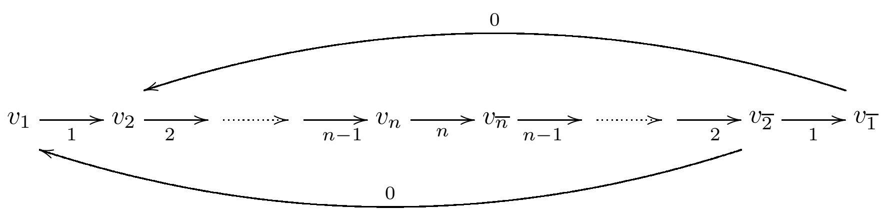
and we have

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 2),$$
$$\mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}}) = \Lambda_0 + \Lambda_1 - \Lambda_2.$$

The explicit forms of the actions by the f_i's are

$$f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1),$$
$$f_n v_n = v_{\overline{n}},$$
$$f_0 v_{\overline{2}} = v_1, \qquad f_0 v_{\overline{1}} = v_2,$$
$$f_i v_j = 0 \qquad \text{otherwise.}$$

Its crystal graph is



4.7. $D^{(2)}_{n+1}$ $(n \geq 2)$. The global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_n,\ v_0,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}},\ \phi\},$$

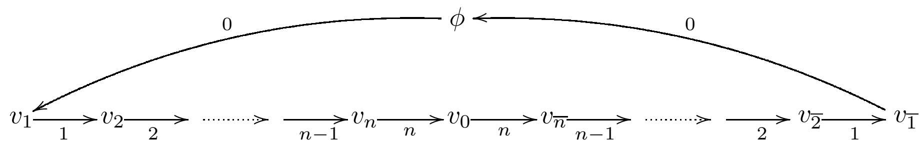
and we have

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 0, 1, n),$$
$$\mathrm{wt}(v_1) = \Lambda_1 - 2\Lambda_0, \qquad \mathrm{wt}(v_{\overline{1}}) = 2\Lambda_0 - \Lambda_1,$$
$$\mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n,$$
$$\mathrm{wt}(v_0) = 0, \qquad \mathrm{wt}(\phi) = 0.$$

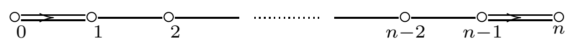
The explicit forms of the actions by the f_i's are

$$f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1),$$
$$f_n v_n = v_0, \qquad f_n v_0 = 2 v_{\overline{n}},$$
$$f_0 v_{\overline{1}} = \phi, \qquad f_0 \phi = 2 v_1,$$
$$f_i v_j = 0, \quad f_i \phi = 0 \qquad \text{otherwise.}$$

Its crystal graph is



4.8. $A_{2n}^{(2)\,\dagger}$ $(n \geq 2)$**.** We take the Cartan data transposed from that in §3.10. Then the Dynkin diagram is



In this case, we denote this type by $A_{2n}^{(2)\,\dagger}$ in order to distinguish it from the one in §3.10. Then the global basis of $W(\varpi_1)$ is

$$\{v_1,\ v_2, \cdots,\ v_n,\ v_0,\ v_{\overline{n}}, \cdots,\ v_{\overline{2}},\ v_{\overline{1}}\},$$

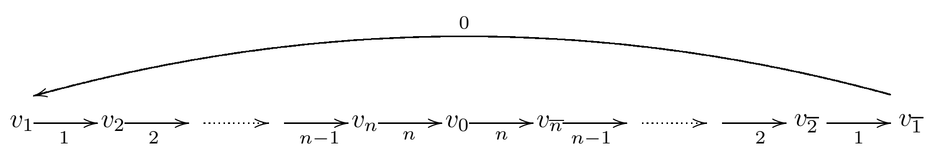
and we have

$$\begin{aligned}
&\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i = 1, \cdots, n-1),\\
&\mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_0) = 0, \qquad \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n.
\end{aligned}$$

The explicit forms of the actions by the f_i's are

$$\begin{aligned}
&f_i v_i = v_{i+1}, \qquad f_i v_{\overline{i+1}} = v_{\overline{i}} \qquad (i = 1, \cdots, n-1),\\
&f_n v_n = v_{\overline{0}}, \qquad f_n v_0 = 2v_{\overline{n}},\\
&f_0 v_{\overline{1}} = v_1,\\
&f_i v_j = 0, \quad \text{otherwise.}
\end{aligned}$$

Its crystal graph is



5. Affine geometric crystals

In this section, we shall construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})$, which is realized in the fundamental representation $W(\varpi_1)$.

5.1. **Translation** $t(\tilde{\varpi}_1)$**.** For $\xi_0 \in (\mathfrak{t}^*_{\mathrm{cl}})_0$, let $t(\xi_0)$ be as in [9, Sect.4]:

$$t(\xi_0)(\lambda) := \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\mathrm{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of ξ, and it is well-defined.

Let $c_i^\vee$ be as in (1.1). Then $t(m\varpi_i)$ belongs to $\widetilde{W}$ if and only if $m \in c_i^\vee\mathbb{Z}$. Setting $\widetilde{\varpi}_i := c_i^\vee \varpi_i$ $(i \in I)$ ([10]), $t(\widetilde{\varpi}_1)$ is expressed as follows (see *e.g.* [12]):

$$t(\widetilde{\varpi}_1) = \begin{cases} \iota(s_n s_{n-1} \cdots s_2 s_1) & A_n^{(1)} \text{ case,} \\ \iota(s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & B_n^{(1)},\ A_{2n-1}^{(2)} \text{ cases,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & C_n^{(1)},\ D_{n+1}^{(2)} \text{ cases,} \\ \iota(s_1 \cdots s_n)(s_{n-2} \cdots s_2 s_1) & D_n^{(1)} \text{ case,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & A_{2n}^{(2)\dagger} \text{ case,} \end{cases}$$

where ι is the Dynkin diagram automorphism

$$\iota = \begin{cases} \sigma & \mathfrak{g} = A_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}, \\ \alpha_0 \leftrightarrow \alpha_1 \text{ and } \alpha_{n-1} \leftrightarrow \alpha_n & \mathfrak{g} = D_n^{(1)}. \end{cases}$$

Now, we know that each $t(\widetilde{\varpi}_1)$ is in the form w_1 or $\iota \cdot w_1$ for some $w_1 \in W$, *e.g.*, $w_1 = s_n \cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1 \cdots s_n)(s_{n-1} \cdots s_1)$ for $B_n^{(1)}$, *etc.*, $\ldots$.

In the case $\mathfrak{g} = A_{2n}^{(2)\dagger}$, $\eta := \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n$ is a unique weight of $W(\varpi_1)$ which satisfies $\langle \alpha_i^\vee, \eta \rangle \geq 0$ for $i \neq n$. For this η we have

$$t(\eta) = (s_n s_{n-1} \cdots s_1)(s_0 s_1 \cdots s_{n-1}) =: w_2, \tag{5.1}$$

which will be used later.

5.2. **Affine geometric crystals.** Let σ be the Dynkin diagram automorphism as in §3.4–3.9 and $w_1 = s_{i_1} \cdots s_{i_k}$ be as in the previous subsection. Set

$$\mathcal{V}(\mathfrak{g}) := \{ v(x_1, \cdots, x_k) := Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) v_1 \mid x_1, \cdots, x_k \in \mathbb{C}^\times \} \subset W(\varpi_1). \tag{5.2}$$

Since the vector v_1 is a highest weight vector in $W(\varpi_1)$ as a $\mathfrak{g}_0$-module, $\mathcal{V}(\mathfrak{g})$ has a G_0-geometric crystal structure. Moreover $(\mathbb{C}^\times)^k \to \mathcal{V}(\mathfrak{g})$ is a birational morphism. We shall define a G-geometric crystal structure on $\mathcal{V}(\mathfrak{g})$ by using the Dynkin diagram automorphism σ except for $A_{2n}^{(2)}$. This σ induces an automorphism of $W(\varpi_1)$, which is also denoted by $\sigma \colon W(\varpi_1) \longrightarrow W(\varpi_1)$. In the subsequent subsections, we shall show the following theorems by case-by-case arguments.

Theorem 5.1. (1) *Case* $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$. *For* $x = (x_1, \cdots, x_k) \in (\mathbb{C}^\times)^k$, *there exist a unique* $y = (y_1, \cdots, y_k) \in (\mathbb{C}^\times)^k$ *and a positive rational function* $a(x)$ *such that*

$$v(y) = a(x)\sigma(v(x)), \quad \varepsilon_{\sigma(i)}(v(y)) = \varepsilon_i(v(x)) \quad \text{if } i, \sigma(i) \neq 0. \tag{5.3}$$

(2) *Case* $\mathfrak{g} = A_{2n}^{(2)\dagger}$. *Associated with* w_1 *and* w_2 *as in the previous section, we define*

$$\mathcal{V}(\mathfrak{g}) := \{ v_1(x) = Y_0(x_0) Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times \},$$
$$\mathcal{V}_2(\mathfrak{g}) := \{ v_2(y) = Y_n(y_n) \cdots Y_1(y_1) Y_0(y_0) Y_1(\overline{y}_1) \cdots Y_{n-1}(\overline{y}_{n-1}) v_{\overline{n}} \mid y_i, \overline{y}_i \in \mathbb{C}^\times \}.$$

For any $x \in (\mathbb{C}^\times)^{2n}$ *there exist a unique* $y \in (\mathbb{C}^\times)^{2n}$ *and a rational function* $a(x)$ *such that* $v_2(y) = a(x) v_1(x)$.

Now, using this theorem, we define the rational mapping

(5.4)
$$\begin{array}{llll} \overline{\sigma}\colon \mathcal{V}(\mathfrak{g}) \longrightarrow \mathcal{V}(\mathfrak{g}), & & \overline{\sigma}\colon \mathcal{V}(\mathfrak{g}) \longrightarrow \mathcal{V}_2(\mathfrak{g}), & \\ v(x) \mapsto v(y) & (\mathfrak{g}\neq A_{2n}^{(2)\,\dagger}), & v_1(x) \mapsto v_2(y) & (\mathfrak{g}= A_{2n}^{(2)\,\dagger}). \end{array}$$

Theorem 5.2. *The rational mapping $\overline{\sigma}$ is birational. If we define*

$$(5.5)\qquad \begin{cases} e_0^c := \overline{\sigma}^{-1}\circ e^c_{\sigma(0)}\circ\overline{\sigma}, \quad \varepsilon_0 := \varepsilon_{\sigma(0)}\circ\overline{\sigma}, \quad \gamma_0 := \gamma_{\sigma(0)}\circ\overline{\sigma}, & \text{for } \mathfrak{g}\neq A_{2n}^{(2)\,\dagger},\\ e_0^c := \overline{\sigma}^{-1}\circ e^c_{0}\circ\overline{\sigma}, \quad \varepsilon_0 := \varepsilon_{0}\circ\overline{\sigma}, \quad \gamma_0 := \gamma_{0}\circ\overline{\sigma}, & \text{for } \mathfrak{g}= A_{2n}^{(2)\,\dagger}, \end{cases}$$

then $(\mathcal{V}(\mathfrak{g}), \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is an affine $\mathfrak{g}$-geometric crystal.

Remark. In the case $\mathfrak{g} = A_{2n}^{(2)\,\dagger}$, $\mathcal{V}_2(\mathfrak{g})$ has a $\mathfrak{g}_{I\setminus\{n\}}$-geometric crystal structure. Thus, $e_0, \gamma_0, \varepsilon_0$ are well defined on $\mathcal{V}_2(\mathfrak{g})$.

The following lemma is obvious and it shows Theorem 5.2 partially.

Lemma 5.3. *Suppose that $\mathfrak{g}\neq A_{2n}^{(2)\,\dagger}$. If there exists $\overline{\sigma}$ as above and*

$$(5.6)\qquad e^c_{\sigma(i)} = \overline{\sigma}\circ e_i^c\circ\overline{\sigma}^{-1}, \quad \gamma_i = \gamma_{\sigma(i)}\circ\overline{\sigma}, \quad \varepsilon_i = \varepsilon_{\sigma(i)}\circ\overline{\sigma},$$

for $i\neq\sigma^{-1}(0), 0$, then we obtain

(1)
$$\begin{array}{l} e_0^{c_1}e_i^{c_2} = e_i^{c_2}e_0^{c_1} \quad \text{if } a_{0i}=a_{i0}=0,\\ e_0^{c_1}e_i^{c_1c_2}e_0^{c_2} = e_i^{c_2}e_0^{c_1c_2}e_i^{c_2} \quad \text{if } a_{0i}=a_{i0}=-1,\\ e_0^{c_1}e_i^{c_1^2c_2}e_0^{c_1c_2}e_i^{c_2} = e_i^{c_2}e_0^{c_1c_2}e_i^{c_1^2c_2}e_0^{c_1} \quad \text{if } a_{0i}=-2,\ a_{i0}=-1, \end{array}$$

(2) $\gamma_0(e_i^c(v(x))) = c^{a_{i0}}\gamma_0(v(x))$ *and* $\gamma_i(e_0^c(v(x))) = c^{a_{0i}}\gamma_i(v(x))$,
(3) $\varepsilon_0(e_0^c(v(x))) = c^{-1}\varepsilon_0(v(x))$.

Proof. For example, we have

$$\begin{aligned} \gamma_0(e_i^c(v(x))) &= \gamma_{\sigma(0)}(\overline{\sigma}e_i^c\overline{\sigma}^{-1}(\overline{\sigma}(v(x))))\\ &= \gamma_{\sigma(0)}(e^c_{\sigma(i)}(\overline{\sigma}(v(x)))) = c^{a_{\sigma(i),\sigma(0)}}\gamma_{\sigma(0)}(\overline{\sigma}(v(x)))\\ &= c^{a_{i,0}}\gamma_0(v(x)), \end{aligned}$$

where we use $a_{\sigma(0),\sigma(i)} = a_{0i}$ in the last equality. The other assertions are obtained similarly. □

In the rest of this section, we shall prove Theorems 5.1 and 5.2 in each case.

5.3. $A_n^{(1)}$**-case.** We have $w_1 := s_n s_{n-1}\cdots s_2 s_1$, and

$$\mathcal{V}(A_n^{(1)}) := \{Y_n(x_n)\cdots Y_2(x_2)Y_1(x_1)v_1 \,\big|\, x_i\in\mathbb{C}^\times\} \subset W(\varpi_1).$$

Since $\exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$, $v(x) = Y_n(x_n)\cdots Y_2(x_2)Y_1(x_1)v_1$ is explicitly written as

$$v(x) = v(x_1,\cdots,x_n) = \left(\sum_{i=1}^n x_i v_i\right) + v_{n+1}.$$

Let $\sigma\colon \alpha_k \mapsto \alpha_{k+1}$ ($k \in I$) be the Dynkin diagram automorphism for $A_n^{(1)}$, which gives rise to the automorphism $\sigma\colon W(\varpi_1) \to W(\varpi_1)$. We have $\sigma v_i = v_{i+1}$. Then, we obtain

$$\sigma(v(x)) = v_1 + \sum_{i=1}^{n} x_i v_{i+1}.$$

Then the equation $v(y) = a(x)\sigma(v(x))$, *i.e.*

$$\sum_{i=1}^{n} y_i v_i + v_{n+1} = a(x)(v_1 + \sum_{i=1}^{n} x_i v_{i+1}),$$

is solved by

$$a(x) = \frac{1}{x_n}, \quad y_1 = \frac{1}{x_n}, \quad y_i = \frac{x_{i-1}}{x_n}\ (i = 2, \cdots, n), \tag{5.7}$$

that is,

$$\overline{\sigma}(v(x_1 \cdots, x_n)) = v(\frac{1}{x_n}, \frac{x_1}{x_n}, \cdots, \frac{x_{n-1}}{x_n}). \tag{5.8}$$

The A_n-geometric crystal structure on $\mathcal{V}(A_n^{(1)})$ induced from the one on $B^-_{w_1}$ is given by:

$$e_i^c(v(x_1, \cdots, x_n)) = v(x_1, \cdots, cx_i, \cdots, x_n) \quad (i = 1, \cdots, n), \tag{5.9}$$

(5.10)

$$\gamma_1(v(x)) = \frac{x_1^2}{x_2}, \quad \gamma_i(v(x)) = \frac{x_i^2}{x_{i-1}x_{i+1}} \quad (i = 2, \cdots, n-1), \quad \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}},$$

$$\varepsilon_i(v(x)) = \frac{x_{i+1}}{x_i} \quad (i = 1, \cdots, n-1), \quad \varepsilon_n(v(x)) = \frac{1}{x_n}. \tag{5.11}$$

Then we have

$$\varepsilon_{i+1}(\overline{\sigma}(v(x))) = \begin{cases} \frac{x_{i+1}}{x_i} & \text{if } i = 1, \cdots, n-2, \\ \frac{x_n}{x_{n-1}} & \text{if } i = n-1, \end{cases}$$

which implies $\varepsilon_{\sigma(i)}(\overline{\sigma}(v(x))) = \varepsilon_i(v(x))$, and then we have completed the proof of Theorem 5.1 in this case.

Now, we define e_0^c, γ_0 and ε_0 by

$$e_0^c := \overline{\sigma}^{-1} \circ e_1^c \circ \overline{\sigma}, \quad \gamma_0 := \gamma_1 \circ \overline{\sigma}, \quad \varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}. \tag{5.12}$$

Their explicit forms are

$$e_0^c(v(x)) = v(\frac{x_1}{c}, \frac{x_2}{c}, \cdots, \frac{x_n}{c}), \tag{5.13}$$

$$\gamma_0(v(x)) = \frac{1}{x_1 x_n}, \quad \varepsilon_0(v(x)) = x_1. \tag{5.14}$$

Thus, we can check (5.6) easily, and then Lemma 5.3 reduces the proof of Theorem 5.2 to the statements:

$$e_0^{c_1} e_n^{c_1 c_2} e_0^{c_2} = e_n^{c_2} e_0^{c_1 c_2} e_n^{c_1}, \tag{5.15}$$

$$\gamma_0(e_n^c(v(x))) = c^{-1}\gamma_0(v(x)), \quad \gamma_n(e_0^c(v(x))) = c^{-1}\gamma_n(v(x)). \tag{5.16}$$

These are immediate from (5.9)–(5.14). Thus, we obtain Theorem 5.2.

5.4. $B_n^{(1)}$**-case.** We have $w_1 = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$, and

$$\mathcal{V}(B_n^{(1)}) := \{v(x) = Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)$ as in §4.3 that $y_i(c^{-1}) = \exp(c^{-1} f_i)$ on $W(\varpi_1)$ can be written as:

$$\exp(c^{-1} f_i) = \begin{cases} 1 + c^{-1} f_i & i \neq n, \\ 1 + c^{-1} f_n + \dfrac{1}{2c^2} f_n^2 & i = n. \end{cases}$$

Therefore, we have

$$v(x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) = \left(\sum_{i=1}^{n} \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^{n} x_{i-1} v_{\overline{i}} \right) + v_{\overline{1}},$$

$$\text{where} \quad \xi_i(x) := \begin{cases} x_1 \overline{x}_1 & i = 1, \\ \dfrac{x_{i-1}\overline{x}_{i-1} + x_i \overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \dfrac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_1 = v_{\overline{1}}$, $\sigma v_{\overline{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^{n} \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^{n} x_{i-1} v_{\overline{i}} \right) + x_1 \overline{x}_1 v_{\overline{1}}.$$

The equation $v(y) = a(x)\sigma(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n-1})$ has a unique solution:

$$a(x) = \frac{1}{x_1 \overline{x}_1}, \quad y_i = a(x) x_i = \frac{x_i}{x_1 \overline{x}_1} \quad (1 \leq i \leq n), \qquad (5.17)$$
$$\overline{y}_i = a(x) \overline{x}_i = \frac{\overline{x}_i}{x_1 \overline{x}_1} \quad (1 \leq i < n).$$

Hence we have the rational mapping:

$$\overline{\sigma}(v(x)) := v(y) = v \left(\frac{x_1}{x_1 \overline{x}_1}, \frac{x_2}{x_1 \overline{x}_1}, \cdots, \frac{x_n}{x_1 \overline{x}_1}, \frac{\overline{x}_{n-1}}{x_1 \overline{x}_1}, \cdots, \frac{\overline{x}_1}{x_1 \overline{x}_1} \right). \qquad (5.18)$$

By the explicit form of $\overline{\sigma}$ in (5.18), we have $\overline{\sigma}^2 = \mathrm{id}$, which means that the morphism $\overline{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i, $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 in this case is completed.

Now, we set $e_0^c := \overline{\sigma} \circ e_1^c \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$.

The explicit forms of the e_i's, ε_i's and γ_i's are:

$$\begin{aligned}
e_0^c:&\quad x_1 \mapsto && x_1\frac{cx_1\bar{x}_1+x_2\bar{x}_2}{c(x_1\bar{x}_1+x_2\bar{x}_2)}, \qquad x_i \mapsto \frac{x_i}{c}\ (2\le i\le n),\\
&\quad \bar{x}_1 \mapsto && \bar{x}_1\frac{x_1\bar{x}_1+x_2\bar{x}_2}{cx_1\bar{x}_1+x_2\bar{x}_2}, \qquad \bar{x}_i \mapsto \frac{\bar{x}_i}{c}\ (2\le i\le n-1),\\
e_i^c:&\quad x_i \mapsto && x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i \mapsto \overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
&\quad x_j \mapsto && x_j, \quad \overline{x}_j \mapsto \overline{x}_j\ (j\neq i) \qquad (1\le i< n-1),\\
e_{n-1}^c:&\quad x_{n-1} \mapsto && x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n^2}{x_{n-1}\overline{x}_{n-1}+x_n^2}, \quad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n^2)}{cx_{n-1}\overline{x}_{n-1}+x_n^2},\\
&\quad x_j \mapsto && x_j, \quad \overline{x}_j \mapsto \overline{x}_j\ (j\neq n-1),\\
e_n^c:&\quad x_n \mapsto && cx_n, \qquad x_j \mapsto x_j \quad \overline{x}_j \mapsto \overline{x}_j\ (j\neq n),
\end{aligned}$$

$$\begin{aligned}
\varepsilon_0(v(x)) &= \frac{x_1\overline{x}_1+x_2\overline{x}_2}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1}\left(1+\frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right),\\
\varepsilon_i(v(x)) &= \frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right)\ (2\le i\le n-2),\\
\varepsilon_{n-1}(v(x)) &= \frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right),\ \varepsilon_n(v(x)) = \frac{x_{n-1}}{x_n},
\end{aligned}$$

$$\begin{aligned}
\gamma_0(v(x)) &= \frac{1}{x_2\overline{x}_2},\ \gamma_1(v(x)) = \frac{(x_1\overline{x}_1)^2}{x_2\overline{x}_2},\\
\gamma_i(v(x)) &= \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2\le i\le n-2),\\
\gamma_{n-1}(v(x)) &= \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \quad \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}.
\end{aligned}$$

Since $\sigma(i)=i$ for $i\neq 0,1$, the condition (5.6) in Lemma 5.3 can be easily seen by (5.18) and by the explicit form of e_i, γ_i and ε_i $(i\in I)$. Thus, in order to prove Theorem 5.2, it suffices to show that

$$e_0^{c_1}e_1^{c_2} = e_1^{c_2}e_0^{c_1}, \tag{5.19}$$

$$\gamma_0(e_1^c(v(x))) = \gamma_0(v(x)), \quad \gamma_1(e_0^c(v(x))) = \gamma_1(v(x)). \tag{5.20}$$

It follows from the explicit formula above that

$$\begin{aligned}
e_0^{c_1}e_1^{c_2}(v(x)) &= e_1^{c_2}e_0^{c_1}(v(x))\\
&= v\left(x_1\frac{c_1c_2x_1\overline{x}_1+x_2\overline{x}_2}{c_1(x_1\overline{x}_1+x_2\overline{x}_2)}, \frac{x_2}{c_1}, \cdots, \frac{\overline{x}_2}{c_1}, \overline{x}_1\frac{c_2(x_1\overline{x}_1+x_2\overline{x}_2)}{c_1c_2x_1\overline{x}_1+x_2\overline{x}_2}\right),
\end{aligned}$$

which implies (5.19). We get (5.20) immediately from the formula above and we complete the proof of Theorem 5.2 for $B_n^{(1)}$.

5.5. $C_n^{(1)}$**-case.** We have $w_1 = s_0 s_1 \cdots s_n s_{n-1} \cdots s_1$ and

$$\mathcal{V}(C_n^{(1)}) := \{v(x) = Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

Due to the explicit description of $W(\varpi_1)$ in §4.4, we obtain $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$. Hence we have

$$\begin{aligned} &v(x_0, x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) \\ &= \left(\sum_{i=1}^n \xi_i v_i\right) + \left(\sum_{i=1}^n x_{i-1} v_{\bar{i}}\right) \text{ where } \xi_i := \begin{cases} \frac{x_0 + x_1\overline{x}_1}{x_0} & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases} \end{aligned}$$

The explicit forms of $\varepsilon_i(x)$ $(1 \le i \le n)$ are:

$$\begin{aligned} &\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \ (1 \le i \le n-2), \\ &\varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1 + \frac{x_n}{x_{n-1}\overline{x}_{n-1}}\right), \ \varepsilon_n(v(x)) = \frac{x_{n-1}^2}{x_n}. \end{aligned}$$

Since $\sigma v_{i+1} = v_{\overline{n-i}}$ and $\sigma v_{\overline{i+1}} = v_{n-i}$ $(0 \le i < n)$, we have

$$\sigma(v(x)) = \left(\sum_{i=1}^n x_{n-i} v_i\right) + \left(\sum_{i=1}^n \xi_{n-i+1} v_{\bar{i}}\right).$$

We obtain a unique solution of the equations $v(y) = a(x)\sigma(v(x))$, $\varepsilon_{n-1}(v(y)) = \varepsilon_1(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n})$:

$$\begin{aligned} a(x) &= \frac{\overline{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}, \\ y_0 &= x_n\left(\frac{\overline{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}\right)^2, \\ y_i &= \left(\overline{x}_{n-i-1} + \frac{x_{n-i}\overline{x}_{n-i}}{x_{n-i-1}}\right)\left(\frac{\overline{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}\right) \quad (1 \le i < n), \\ y_{n-1} &= \left(1 + \frac{x_1\overline{x}_1}{x_0}\right)\left(\frac{\overline{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}\right), \\ y_n &= x_0\left(\frac{\overline{x}_{n-1}}{x_n} + \frac{1}{x_{n-1}}\right)^2, \\ \overline{y}_{n-1} &= \frac{(x_{n-1}\overline{x}_{n-1} + x_n)x_0x_1\overline{x}_1}{(x_0 + x_1\overline{x}_1)x_{n-1}x_n}, \\ \overline{y}_i &= \frac{(x_{n-1}\overline{x}_{n-1} + x_n)x_{n-i-1}x_{n-i}\overline{x}_{n-i}}{(x_{n-i-1}\overline{x}_{n-i-1} + x_{n-i}\overline{x}_{n-i})x_{n-1}x_n} \quad (1 \le i < n-1). \end{aligned}$$

Now we have the rational mapping $\overline{\sigma}\colon \mathcal{V}(C_n^{(1)}) \longrightarrow \mathcal{V}(C_n^{(1)})$ defined by $v(x) \mapsto v(y)$. By the above explicit from of y, we have $\overline{\sigma}^2 = \mathrm{id}$, which means that $\overline{\sigma}$ is birational.

By direct calculations, we have $\varepsilon_{n-i}(v(y)) = \varepsilon_i(v(x))$ $(1 \le i \le n-1)$, and we complete the proof of Theorem 5.1 for $C_n^{(1)}$.

Let us define $e_0^c := \overline{\sigma} \circ e_n^c \circ \overline{\sigma}$ $(\overline{\sigma}^2 = \mathrm{id})$, $\gamma_0 := \gamma_n \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_n \circ \overline{\sigma}$. The explicit forms of e_i, γ_i and ε_0 are

$$
\begin{aligned}
e_0^c\colon \quad & x_0 \mapsto \quad x_0\frac{(cx_0+x_1\bar{x}_1)^2}{c(x_0+x_1\bar{x}_1)^2}, \qquad x_i \mapsto x_i\frac{cx_0+x_1\bar{x}_1}{c(x_0+x_1\bar{x}_1)}\ (1 \le i \le n-1),\\
& x_n \mapsto \quad x_n\frac{(cx_0+x_1\bar{x}_1)^2}{c^2(x_0+x_1\bar{x}_1)^2}, \qquad \bar{x}_i \mapsto \bar{x}_i\frac{cx_0+x_1\bar{x}_1}{c(x_0+x_1\bar{x}_1)}\ (1 \le i \le n-1),\\
e_i^c\colon \quad & x_i \mapsto \quad x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \qquad \overline{x}_i \mapsto \overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto \quad x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne i), \qquad\qquad (1 \le i < n-1),\\
e_{n-1}^c\colon \quad & x_{n-1} \mapsto \quad x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n}{x_{n-1}\overline{x}_{n-1}+x_n}, \qquad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n)}{cx_{n-1}\overline{x}_{n-1}+x_n},\\
& x_j \mapsto \quad x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-1),\\
e_n^c\colon \quad & x_n \mapsto \quad cx_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n),
\end{aligned}
$$

$$
\begin{aligned}
&\gamma_0(v(x)) = \frac{x_0^2}{(x_1\overline{x}_1)^2}, \quad \gamma_1(v(x)) = \frac{(x_1\overline{x}_1)^2}{x_0x_2\overline{x}_2},\\
&\gamma_i = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}} \quad (2 \le i \le n-2),\\
&\gamma_{n-1}(v(x)) = \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n}, \quad \gamma_n(v(x)) = \frac{x_n^2}{(x_{n-1}\overline{x}_{n-1})^2},\\
&\varepsilon_0(v(x)) = \frac{1}{x_0}\left(1+\frac{x_1\overline{x}_1}{x_0}\right)^2.
\end{aligned}
$$

Let us check the condition (5.6) in Lemma 5.3. The following are useful for this purpose:

$$y_i\overline{y}_i = a(x)^2x_{n-i}\overline{x}_{n-i}, \quad y_0 = a(x)^2x_n, \tag{5.21}$$

$$a(v(y)) = a(\overline{\sigma}(v(x))) = \frac{1}{a(v(x))}. \tag{5.22}$$

Using these we can easily check the two conditions $\gamma_i = \gamma_{\sigma(i)} \circ \overline{\sigma}$ and $\varepsilon_i = \varepsilon_{\sigma(i)} \circ \overline{\sigma}$. The condition $e_{\sigma(i)}^c = \overline{\sigma} \circ e_i^c \circ \overline{\sigma}^{-1}$ for $i = 2, \cdots, n-2$ is also immediate from (5.21) and (5.22). Next let us see the case $i = 1, n-1$. We have

$$a(e_{n-1}^c(v(y))) = \frac{y_n + cy_{n-1}\overline{y}_{n-1}}{y_{n-1}y_n} \cdot \frac{x_1\overline{x}_1 + x_0}{cx_1\overline{x}_1 + x_0} = \frac{1}{a(v(x))}.$$

Using this, we can get $e_{n-i}^c = \overline{\sigma} \circ e_1^c \circ \overline{\sigma}^{-1}$ and then $e_1^c = \overline{\sigma} \circ e_{n-1}^c \circ \overline{\sigma}^{-1}$ since $\overline{\sigma}^2 = \mathrm{id}$. Now, it remains to show that

$$e_0^{c_1}e_n^{c_2} = e_n^{c_2}e_0^{c_1}, \quad \gamma_0(e_n^c(v(x))) = \gamma_0(v(x)), \quad \gamma_n(e_0^c(v(x))) = \gamma_n(v(x)).$$

They easily follow from the explicit form of e_0^c. Thus, the proof of Theorem 5.2 in this case is completed.

5.6. $D_n^{(1)}$**-case.** We have $w_1 = s_1 s_2 \cdots s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_2 s_1$, and

$$\mathcal{V}(D_n^{(1)}) := \{v(x) = Y_1(x_1) \cdots Y_{n-1}(x_{n-1}) Y_n(x_n) Y_{n-2}(\bar{x}_{n-2}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit form of $W(\varpi_1)$ in §4.5 that $y_i(c^{-1}) = \exp(c^{-1} f_i) = 1 + c^{-1} f_i$ on $W(\varpi_1)$. Thus, we have

$$v(x) = \left(\sum_{i=1}^{n-1} \xi_i(x) v_i\right) + x_n v_n + \left(\sum_{i=2}^{n} x_{i-1} v_{\bar{i}}\right) + v_{\bar{1}},$$

$$\text{where} \quad \xi_i(x) := \begin{cases} x_1 \overline{x}_1 & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i \overline{x}_i}{x_{i-1}} & i \neq 1, n-1, \\ \frac{x_{n-2}\overline{x}_{n-2} + x_{n-1} x_n}{x_{n-2}} & i = n-1. \end{cases}$$

Since $\sigma v_1 = v_{\bar{1}}$, $\sigma v_{\bar{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^{n} \xi_i(x) v_i\right) + x_n v_n + \left(\sum_{i=1}^{n-1} x_{i-1} v_{\bar{i}}\right) + \xi_1 v_{\bar{1}}.$$

Then the equation $v(y) = a(x)\sigma(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n-2})$ has the following unique solution:

$$\begin{aligned} a(x) = \frac{1}{x_1 \overline{x}_1}, \quad y_i &= a(x) x_i = \frac{x_i}{x_1 \overline{x}_1} \quad (1 \le i \le n), \\ \overline{y}_i &= a(x) \overline{x}_i = \frac{\overline{x}_i}{x_1 \overline{x}_1} \quad (1 \le i \le n-2). \end{aligned} \tag{5.23}$$

We define the rational mapping $\overline{\sigma} \colon \mathcal{V}(D_n^{(1)}) \longrightarrow \mathcal{V}(D_n^{(1)})$ by

$$\overline{\sigma}(v(x)) = v\left(\frac{x_1}{x_1\overline{x}_1}, \frac{x_2}{x_1\overline{x}_1}, \cdots, \frac{x_n}{x_1\overline{x}_1}, \frac{\overline{x}_{n-2}}{x_1\overline{x}_1}, \cdots, \frac{\overline{x}_1}{x_1\overline{x}_1}\right). \tag{5.24}$$

It is immediate from (5.24) that $\overline{\sigma}^2 = \mathrm{id}$, which implies the birationality of the morphism $\overline{\sigma}$. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i, $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $D_n^{(1)}$ is completed.

Now, we set $e_0^c := \overline{\sigma} \circ e_1^c \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$. The explicit forms of e_i, ε_i and γ_i are:

$$
\begin{aligned}
e_0^c\colon \quad & x_1 \mapsto x_1 \frac{c x_1\bar{x}_1 + x_2\bar{x}_2}{c(x_1\bar{x}_1 + x_2\bar{x}_2)}, \qquad x_i \mapsto \frac{x_i}{c}\ (2 \le i \le n),\\
& \bar{x}_1 \mapsto \bar{x}_1 \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{c x_1\bar{x}_1 + x_2\bar{x}_2}, \qquad \bar{x}_i \mapsto \frac{\bar{x}_i}{c}\ (2 \le i \le n-2),\\
e_i^c\colon \quad & x_i \mapsto x_i \frac{c x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i \mapsto \overline{x}_i \frac{c(x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1})}{c x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne i), \qquad (1 \le i \le n-3),\\
e_{n-2}^c\colon \quad & x_{n-2} \mapsto x_{n-2} \frac{c x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n}{x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n},\\
& \overline{x}_{n-2} \mapsto \overline{x}_{n-2} \frac{c(x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n)}{c x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-2),\\
e_{n-1}^c\colon \quad & x_{n-1} \mapsto c x_{n-1}, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-1),\\
e_n^c\colon \quad & x_n \mapsto c x_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n),
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon_0(v(x)) = \frac{x_1\overline{x}_1 + x_2\overline{x}_2}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1}\left(1 + \frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right),\\
&\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \quad (2 \le i \le n-3),\\
&\varepsilon_{n-2}(v(x)) = \frac{x_{n-3}}{x_{n-2}}\left(1 + \frac{x_{n-1}x_n}{x_{n-2}\overline{x}_{n-2}}\right), \quad \varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}},\\
&\varepsilon_n(v(x)) = \frac{x_{n-2}}{x_n},
\end{aligned}
$$

$$
\begin{aligned}
\gamma_0(v(x)) &= \frac{1}{x_2\overline{x}_2},\ \gamma_1(v(x)) = \frac{(x_1\overline{x}_1)^2}{x_2\overline{x}_2},\\
\gamma_i(v(x)) &= \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2 \le i \le n-3),\\
\gamma_{n-2}(v(x)) &= \frac{(x_{n-2}\overline{x}_{n-2})^2}{x_{n-3}\overline{x}_{n-3}x_{n-1}x_n}, \quad \gamma_{n-1}(v(x)) = \frac{x_{n-1}^2}{x_{n-2}\overline{x}_{n-2}},\\
\gamma_n(v(x)) &= \frac{x_n^2}{x_{n-2}\overline{x}_{n-2}}.
\end{aligned}
$$

By these formulas, we can show Theorem 5.2 for $D_n^{(1)}$ similarly to the one for $B_n^{(1)}$.

5.7. $A_{2n-1}^{(2)}$**-case.** We have $w_1 = s_1 s_2 \cdots s_n s_{n-1} \cdots s_2 s_1$, and

$$\mathcal{V}(A_{2n-1}^{(2)}) := \{v(x) := Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

In this case, $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$, and we have

$$v(x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) = \left(\sum_{i=1}^{n} \xi_i v_i\right) + \left(\sum_{i=2}^{n} x_{i-1} v_{\overline{i}}\right) + v_{\overline{1}},$$

$$\text{where} \quad \xi_i := \begin{cases} x_1\overline{x}_1 & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_1 = v_{\overline{1}}$, $\sigma v_{\overline{1}} = v_1$ and $\sigma v_k = v_k$ otherwise, we have

$$\sigma(v(x)) = v_1 + \left(\sum_{i=2}^{n} \xi_i v_i\right) + \left(\sum_{i=2}^{n} x_{i-1} v_{\overline{i}}\right) + x_1\overline{x}_1 v_{\overline{1}}.$$

Solving $v(y) = a(x)\sigma(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n-1})$, we obtain a unique solution:

(5.25)

$$a(x) = \frac{1}{x_1\overline{x}_1}, \quad y_i = a(x)x_i = \frac{x_i}{x_1\overline{x}_1}, \quad \overline{y}_i = a(x)\overline{x}_i = \frac{\overline{x}_i}{x_1\overline{x}_1} \quad (1 \leq i \leq n-1),$$
$$y_n = \frac{x_n}{(x_1\overline{x}_1)^2}.$$

Then we have

$$\overline{\sigma}(v(x)) = v\left(\frac{x_1}{x_1\overline{x}_1}, \frac{x_2}{x_1\overline{x}_1}, \cdots, \frac{x_n}{(x_1\overline{x}_1)^2}, \frac{\overline{x}_{n-1}}{x_1\overline{x}_1}, \cdots, \frac{\overline{x}_1}{x_1\overline{x}_1}\right). \tag{5.26}$$

By the explicit form of $\overline{\sigma}$ in (5.26), we have $\overline{\sigma}^2 = \mathrm{id}$, which means that the morphism $\overline{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i, $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $A^{(2)}_{2n-1}$ is completed.

Now, we set $e^c_0 := \overline{\sigma} \circ e^c_1 \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$. The explicit forms of e_i, ε_i and γ_i are:

$$\begin{aligned}
e^c_0\colon \quad & x_1 \mapsto x_1\frac{cx_1\bar{x}_1 + x_2\bar{x}_2}{c(x_1\bar{x}_1 + x_2\bar{x}_2)}, \quad x_i \mapsto \frac{x_i}{c}\ (2 \leq i \leq n-1), \quad x_n \mapsto \frac{x_n}{c^2}, \\
& \bar{x}_1 \mapsto \bar{x}_1\frac{x_1\bar{x}_1 + x_2\bar{x}_2}{cx_1\bar{x}_1 + x_2\bar{x}_2}, \quad \bar{x}_i \mapsto \frac{\bar{x}_i}{c}\ (2 \leq i \leq n-1), \\
e^c_i\colon \quad & x_i \mapsto x_i\frac{cx_i\overline{x}_i + x_{i+1}\bar{x}_{i+1}}{x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i \mapsto \overline{x}_i\frac{c(x_i\overline{x}_i + x_{i+1}\bar{x}_{i+1})}{cx_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}, \\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq i) \qquad (1 \leq i < n-1), \\
e^c_{n-1}\colon \quad & x_{n-1} \mapsto x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1} + x_n}{x_{n-1}\overline{x}_{n-1} + x_n}, \quad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1} + x_n)}{cx_{n-1}\overline{x}_{n-1} + x_n}, \\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq n-1), \\
e^c_n\colon \quad & x_n \mapsto cx_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq n).
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_0(v(x)) = \frac{x_1\overline{x}_1 + x_2\overline{x}_2}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1}\left(1 + \frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right), \\
&\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \quad (2 \leq i \leq n-2), \\
&\varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1 + \frac{x_n}{x_{n-1}\overline{x}_{n-1}}\right), \quad \varepsilon_n(v(x)) = \frac{x^2_{n-1}}{x_n},
\end{aligned}$$

$$\gamma_0(v(x)) = \frac{1}{x_2\overline{x}_2}, \quad \gamma_1(v(x)) = \frac{(x_1\overline{x}_1)^2}{x_2\overline{x}_2},$$
$$\gamma_i = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}} \quad (2 \le i \le n-2),$$
$$\gamma_{n-1}(v(x)) = \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n}, \quad \gamma_n(v(x)) = \frac{x_n^2}{(x_{n-1}\overline{x}_{n-1})^2}.$$

We can show Theorem 5.2 for $A^{(2)}_{2n-1}$ similarly to the one for $B^{(1)}_n$.

5.8. $D^{(2)}_{n+1}$**-case.** We have $w_1 = s_0s_1\cdots s_ns_{n-1}\cdots s_2s_1$ and

$$\mathcal{V}(D^{(2)}_{n+1}) := \{v(x) := Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)v_1 \bigm| x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)$ as in §4.7 that on $W(\varpi_1)$:

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \ne 0, n, \\ 1 + c^{-1}f_i + \frac{1}{2c^2}f_i^2 & i = 0, n. \end{cases}$$

Then we have

$$v(x) = \left(\sum_{i=1}^n \xi_i(x)v_i\right) + x_nv_0 + x_0\phi + \left(\sum_{i=2}^n x_{i-1}v_{\overline{i}}\right) + x_0^2v_{\overline{1}}$$

where

$$\xi_i(x) := \begin{cases} \frac{x_0^2 + x_1\overline{x}_1}{x_0^2} & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \ne 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

Since $\sigma v_{i+1} = v_{\overline{n-i}}$ and $\sigma v_{\overline{i+1}} = v_{n-i}$ $(0 \le i < n)$ and $\sigma\colon v_0 \leftrightarrow \phi$, we also have

$$\sigma(v(x)) = \left(\sum_{i=1}^{n-1} x_{n-i}v_i\right) + x_0^2v_n + \left(\sum_{i=1}^n \xi_{n-i+1}v_{\overline{i}}\right) + x_n\phi + x_0v_0.$$

Solving $v(y) = a(x)\sigma(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n})$, we get a unique solution:

$$a(x) = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}x_n^2},$$
$$y_0 = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}x_n},$$
$$y_i = \frac{(x_{n-i-1}\overline{x}_{n-i-1} + x_{n-i}\overline{x}_{n-i})(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_{n-i-1}x_{n-1}x_n^2} \quad (1 \le i < n),$$
$$y_{n-1} = \frac{(x_0^2 + x_1\overline{x}_1)(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_0^2x_{n-1}x_n^2},$$
$$y_n = \frac{x_0(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_{n-1}x_n^2},$$
$$\overline{y}_i = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)x_{n-i-1}x_{n-i}\overline{x}_{n-i}}{(x_{n-i-1}\overline{x}_{n-i-1} + x_{n-i}\overline{x}_{n-i})x_{n-1}x_n^2} \quad (1 \le i \le n-2),$$
$$\overline{y}_{n-1} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)x_0^2x_1\overline{x}_1}{(x_0^2 + x_1\overline{x}_1)x_{n-1}x_n^2}.$$

Then we have the rational mapping $\overline{\sigma}\colon \mathcal{V}(D^{(2)}_{n+1}) \longrightarrow \mathcal{V}(D^{(2)}_{n+1})$ defined by $v(x) \mapsto v(y)$. The explicit forms of ε_i $(1 \le i \le n)$ are as follows:

$$\varepsilon_1(v(x)) = \frac{x_0^2}{x_1}\left(1 + \frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right), \quad \varepsilon_n(v(x)) = \frac{x_{n-1}}{x_n},$$
$$\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \ (2 \le i \le n-2),$$
$$\varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1 + \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right).$$

Then we get easily that $\varepsilon_{n-i}(v(y)) = \varepsilon_i(v(x))$ $(1 \le i \le n-1)$, which finishes the proof of Theorem 5.1 for $D^{(2)}_{n+1}$.

Let us define $e_0^c := \overline{\sigma} \circ e_n^c \circ \overline{\sigma}$ $(\overline{\sigma}^2 = \mathrm{id})$, $\gamma_0 := \gamma_n \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_n \circ \overline{\sigma}$. The explicit forms of e_i, γ_i and ε_0 are

$$\begin{aligned}
e_0^c\colon \quad & x_0 \mapsto \quad x_0\frac{c^2x_0^2 + x_1\bar{x}_1}{c(x_0^2 + x_1\bar{x}_1)}, \qquad x_i \mapsto x_i\frac{c^2x_0^2 + x_1\bar{x}_1}{c^2(x_0^2 + x_1\bar{x}_1)} \ (1 \le i \le n),\\
& \bar{x}_i \mapsto \quad \bar{x}_i\frac{c^2x_0^2 + x_1\bar{x}_1}{c^2(x_0^2 + x_1\bar{x}_1)} \ (1 \le i \le n-1),\\
e_i^c\colon \quad & x_i \mapsto \quad x_i\frac{cx_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}, \qquad \overline{x}_i \mapsto \overline{x}_i\frac{c(x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto \quad x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne i), \qquad\qquad (1 \le i < n-1),\\
e_{n-1}^c\colon \quad & x_{n-1} \mapsto \quad x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}\overline{x}_{n-1} + x_n^2}, \quad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1} + x_n^2)}{cx_{n-1}\overline{x}_{n-1} + x_n^2},\\
& x_j \mapsto \quad x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-1),\\
e_n^c\colon \quad & x_n \mapsto \quad cx_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n),
\end{aligned}$$

$$\begin{aligned}
&\gamma_0(v(x)) = \frac{x_0^2}{x_1\overline{x}_1}, \quad \gamma_1(v(x)) = \frac{(x_1\overline{x}_1)^2}{x_0^2x_2\overline{x}_2},\\
&\gamma_i(v(x)) = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}} \ (2 \le i \le n-2),\\
&\gamma_{n-1}(v(x)) = \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \ \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}},\\
&\varepsilon_0(v(x)) = \frac{x_0^2 + x_1\overline{x}_1}{x_0^3}.
\end{aligned}$$

Then similarly to the case $C^{(1)}_n$, we can show Theorem 5.2 for $D^{(2)}_{n+1}$.

5.9. **$A^{(2)\,\dagger}_{2n}$-case.** As in the beginning of this section, we have $w_1 = s_0s_1\cdots s_ns_{n-1}\cdots s_2s_1$ and

$$\mathcal{V}(A^{(2)\,\dagger}_{2n}) := \{v_1(x) = Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)v_1 \bigm| x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

By the explicit description of $W(\varpi_1)$ as in §4.8 on $W(\varpi_1)$ we have:

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \ne n,\\ 1 + c^{-1}f_n + \frac{1}{2c^2}f_n^2 & i = n. \end{cases}$$

Then we have

$$v_1(x) = \left(\sum_{i=1}^{n} \xi_i(x) v_i\right) + x_n v_0 + \left(\sum_{i=1}^{n} x_{i-1} v_{\overline{i}}\right)$$
$$\text{where} \quad \xi_i(x) := \begin{cases} \frac{x_0 + x_1\overline{x}_1}{x_0} & i=1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i=n. \end{cases}$$

Next, for $w_2 = s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$ we set

$$\mathcal{V}_2(A_{2n}^{(2)\,\dagger}) := \{v_2(y) = Y_n(y_n)\cdots Y_1(y_1)Y_0(y_0)Y_1(\overline{y}_1)\cdots Y_{n-1}(\overline{y}_{n-1})v_{\overline{n}} \,\big|\, y_i, \overline{y}_i \in \mathbb{C}^\times\}.$$

Then we have

$$v_2(y) = \left(\sum_{i=1}^{n-1} y_i v_i\right) + y_n^2 v_n + y_n v_0 + \left(\sum_{i=1}^{n} \eta_i(y) v_{\overline{i}}\right)$$
$$\text{where } \eta_i(y) := \begin{cases} \frac{y_0 + y_1\overline{y}_1}{y_1} & i=1, \\ \frac{y_{i-1}\overline{y}_{i-1} + y_i\overline{y}_i}{y_i} & i \neq 1, n, \\ \frac{y_{n-1}\overline{y}_{n-1} + y_n^2}{y_n^2} & i=n. \end{cases}$$

For $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y = (y_0, \cdots, \overline{y}_1) \in (\mathbb{C}^\times)^{2n}$ and $a(x)$ such that $v_2(y) = a(x)v_1(x)$. They are given by

$$\begin{aligned}
a(x) &= \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}x_n^2}, \\
y_0 &= a(x)^2 x_0 = \frac{x_0(x_{n-1}\overline{x}_{n-1} + x_n^2)^2}{(x_{n-1}x_n^2)^2}, \\
y_1 &= a(x)\xi_1(x) = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)(x_0 + x_1\overline{x}_1)}{x_0 x_{n-1} x_n^2}, \\
y_i &= a(x)\xi_i(x) = \frac{(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i)(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_{i-1}x_{n-1}x_n^2} \quad (1 < i < n), \\
y_n &= a(x)x_n = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}x_n}, \\
\overline{y}_1 &= a(x)\frac{x_0 x_1 \overline{x}_1}{x_0 + x_1\overline{x}_1} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)x_0 x_1\overline{x}_1}{(x_0 + x_1\overline{x}_1)x_{n-1}x_n^2}, \\
\overline{y}_i &= \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)x_{i-1}x_i\overline{x}_i}{(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i)x_{n-1}x_n^2} \quad (1 < i \leq n-1).
\end{aligned}$$

They define a rational mapping $\overline{\sigma}\colon \mathcal{V}(A^{(2)\dagger}_{2n}) \longrightarrow \mathcal{V}_2(A^{(2)\dagger}_{2n})$ $(v_1(x) \mapsto v_2(y))$.
The inverse $\overline{\sigma}^{-1}\colon \mathcal{V}_2(A^{(2)\dagger}_{2n}) \longrightarrow \mathcal{V}(A^{(2)\dagger}_{2n})$ $(v_2(y) \mapsto v_1(x))$ is given by

$$
\begin{aligned}
&a(y) := \frac{y_0y_1}{y_0+y_1\overline{y}_1} \ (= a(x)),\\
&x_0 = a(y)^{-1}\frac{y_0+y_1\overline{y}_1}{y_1},\\
&x_i = a(y)^{-1}\frac{y_i\overline{y}_i+y_{i+1}\overline{y}_{i+1}}{y_{i+1}} \quad (1 \le i \le n-2),\\
&x_{n-1} = a(y)^{-1}\frac{y_{n-1}\overline{y}_{n-1}+y_n^2}{y_n^2},\\
&x_n = a(y)^{-1}y_n,\\
&\overline{x}_i = a(y)^{-1}\frac{y_i\overline{y}_iy_{i+1}}{y_i\overline{y}_i+y_{i+1}\overline{y}_{i+1}} \quad (1 \le i \le n-2),\\
&\overline{x}_{n-1} = a(y)^{-1}\frac{y_{n-1}\overline{y}_{n-1}y_n^2}{y_{n-1}\overline{y}_{n-1}+y_n^2},
\end{aligned}
$$

which means that the morphism $\overline{\sigma}\colon \mathcal{V}(A^{(2)\dagger}_{2n}) \longrightarrow \mathcal{V}_2(A^{(2)\dagger}_{2n})$ is birational. Thus, we obtain Theorem 5.1(2).

The actions of e_i $(i = 0, 1, \cdots, n-1)$ on $v_2(y)$ are induced from the ones on $Y_{\mathbf{i_2}}(y)$ $(\mathbf{i_2} = (n, \cdots, 1, 0, 1, \cdots, n-1))$ since $e_iv_{\overline{n}} = 0$ for $i = 0, 1, \cdots, n-1$. We also get $\gamma_i(v_2(y))$ and $\varepsilon_i(v_2(y))$ from the ones for $Y_{\mathbf{i_2}}(y)$ where $v_2(y) = \overline{\sigma}(v_1(x))$:

$$
\begin{aligned}
&e_0^c\colon y_0 \mapsto cy_0,\\
&e_1^c\colon y_1 \mapsto y_1\frac{cy_1\overline{y}_1+y_0}{y_1\overline{y}_1+y_0}, \quad \overline{y}_1 \mapsto \overline{y}_1\frac{c(y_1\overline{y}_1+y_0)}{cy_1\overline{y}_1+y_0},\\
&e_i^c\colon y_i \mapsto y_i\frac{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}}, \quad \overline{y}_i \mapsto \overline{y}_i\frac{c(y_i\overline{y}_i+y_{i-1}\overline{y}_{i-1})}{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}} \qquad (i = 2, \cdots, n-1),\\
&\gamma_0(v_2(y)) = \frac{y_0^2}{(y_1\overline{y}_1)^2}, \quad \gamma_1(v_2(y)) = \frac{(y_1\overline{y}_1)^2}{y_0y_2\overline{y}_2},\\
&\gamma_i(v_2(y)) = \frac{(y_i\overline{y}_i)^2}{y_{i-1}\overline{y}_{i-1}y_{i+1}\overline{y}_{i+1}} \ (i{=}2, \cdots, n{-}2), \ \gamma_{n-1}(v_2(y)) {=} \frac{(y_{n-1}\overline{y}_{n-1})^2}{y_{n-2}\overline{y}_{n-2}y_n^2},\\
&\varepsilon_0(v_2(y)) = \frac{y_1^2}{y_0}, \ \varepsilon_1(v_2(y)) = \frac{y_2}{y_1}\left(1+\frac{y_0}{y_1\overline{y}_1}\right),\\
&\varepsilon_{n-1}(v_2(y)) = \frac{y_n^2}{y_{n-1}}\left(1+\frac{y_{n-2}\overline{y}_{n-2}}{y_{n-1}\overline{y}_{n-1}}\right),\\
&\varepsilon_i(v_2(y)) = \frac{y_{i+1}}{y_i}\left(1+\frac{y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i}\right) \quad (i = 2 \cdots, n-1).
\end{aligned}
$$

The explicit forms of $\varepsilon_i(v_1(x))$ and $\gamma_i(v_1(x))$ $(1 \le i \le n)$ are also induced from the ones for $Y_{\mathbf{i_1}}(x) := Y_0(x_0) \cdots Y_1(\overline{x}_1)$, and we define $\varepsilon_0(v_1(x)) := \varepsilon_0(v_2(y))$ and

$\gamma_0(v_1(x)) := \gamma_0(v_2(y))$ $(v_2(y) := \overline{\sigma}(v_1(x)))$:

$$\varepsilon_0(v_1(x)) = \frac{1}{x_0}\left(1+\frac{x_1\overline{x}_1}{x_0}\right)^2,$$
$$\varepsilon_i(v_1(x)) = \frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \quad (1\le i\le n-2),$$
$$\varepsilon_{n-1}(v_1(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right), \quad \varepsilon_n(v_1(x)) = \frac{x_{n-1}}{x_n},$$
$$\gamma_0(v_1(x)) = \frac{x_0^2}{(x_1\overline{x}_1)^2},\ \gamma_1(v_1(x)) = \frac{(x_1\overline{x}_1)^2}{x_0x_2\overline{x}_2},$$
$$\gamma_i(v_1(x)) = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2\le i\le n-2),$$
$$\gamma_{n-1}(v_1(x)) = \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \quad \gamma_n(v_1(x)) = \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}.$$

For $i=0$, we define $e_0^c(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c\circ\overline{\sigma}(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c(v_2(y))$. Then we get

$$e_0^c\colon x_0 \mapsto x_0\frac{(cx_0+x_1\bar{x}_1)^2}{c(x_0+x_1\bar{x}_1)^2}, \qquad x_i\mapsto x_i\frac{cx_0+x_1\bar{x}_1}{c(x_0+x_1\bar{x}_1)}\ (1\le i\le n),$$
$$\bar{x}_i\mapsto \bar{x}_i\frac{cx_0+x_1\bar{x}_1}{c(x_0+x_1\bar{x}_1)}\ (1\le i\le n-1),$$
$$e_i^c\colon x_i\mapsto x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i\mapsto\overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}\ (1\le i< n-1),$$
$$e_{n-1}^c\colon x_{n-1}\mapsto x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n^2}{x_{n-1}\overline{x}_{n-1}+x_n^2}, \qquad \overline{x}_{n-1}\mapsto\overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n^2)}{cx_{n-1}\overline{x}_{n-1}+x_n^2},$$
$$e_n^c\colon x_n\mapsto cx_n,$$

where we omit the unchanged parts. In order to prove Theorem 5.2, it suffices to show the following:

$$e_i^c = \overline{\sigma}^{-1}\circ e_i^c\circ\overline{\sigma},\ \gamma_i=\gamma_i\circ\overline{\sigma},\ \varepsilon_i=\varepsilon_i\circ\overline{\sigma}\ (i\neq 0,n), \tag{5.27}$$
$$e_0^{c_1}e_n^{c_2} = e_n^{c_2}e_0^{c_1}, \tag{5.28}$$
$$\gamma_n(e_0^c(v_1(x))) = \gamma_n(v_1(x)), \quad \gamma_0(e_n^c(v_1(x))) = \gamma_0(v_1(x)), \tag{5.29}$$
$$\varepsilon_0(e_0^c(v_1(x))) = c^{-1}\varepsilon_0(v_1(x)), \tag{5.30}$$

which are immediate from the above formulae. Let us show (5.27). Set $v_2(y) := \overline{\sigma}(v_1(x))$ and $v_1(x') := \overline{\sigma}^{-1}(e_i^c(v_2(y)))$ for $i=2,\cdots,n-2$. Then $x'_j = x_j$ and $\overline{x}'_j=\overline{x}_j$ for $j\neq i-1,i$, and we have

$$a(v_2(y)) = a(v_1(x)),$$
$$x'_i = \frac{1}{a(v_2(y))}\left(\overline{y}_{i+1}+\frac{cy_i\overline{y}_i}{y_{i+1}}\right) = x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},$$
$$\overline{x}'_i = \frac{1}{a(v_2(y))}\frac{cy_i\overline{y}_iy_{i+1}}{cy_i\overline{y}_i+y_{i+1}\overline{y}_{i+1}} = \overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},$$
$$x'_{i-1} = \frac{1}{a(v_2(y))}\left(\overline{y}_i+\frac{cy_{i-1}\overline{y}_{i-1}}{y_i}\right) = x_{i-1},$$
$$\overline{x}'_{i-1} = \frac{1}{a(v_2(y))}\frac{cy_{i-1}\overline{y}_{i-1}y_i}{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}} = \overline{x}_{i-1},$$

where the formula $y_i\overline{y}_i = a(v_1(x))^2 x_i\overline{x}_i$ is useful to obtain these results. Therefore we have $e_i^c = \overline{\sigma}^{-1} \circ e_i^c \circ \overline{\sigma}$ for $i = 2, \cdots, n-2$. The others are obtained similarly.

6. Ultra-discretization of affine geometric crystals

In this section, we shall see that the limit B_∞ of perfect crystals as in §3 is isomorphic to the ultra-discretization of the geometric crystal obtained in the previous section.

Let $\mathcal{V}(\mathfrak{g})$ be the affine geometric crystal for $\mathfrak{g}$ as in the previous section. Let $m = \dim\mathcal{V}(\mathfrak{g})$. Then due to their explicit forms in the previous section, we have the following simple positive structure on $\mathcal{V}(\mathfrak{g})$:

$$\begin{aligned} \Theta\colon (\mathbb{C}^\times)^m \quad &\longrightarrow \quad \mathcal{V}(\mathfrak{g}), \\ x \quad &\mapsto \quad v(x). \end{aligned} \tag{6.1}$$

Therefore, we obtain an affine crystal $\mathcal{UD}(\mathcal{V}(\mathfrak{g})) = (\mathcal{B}(\mathfrak{g}), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$.

Theorem 6.1. *For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}$ and $A_{2n}^{(2)\dagger}$, let us denote its Langlands dual by $\mathfrak{g}^L$. Then we have the following isomorphism of crystals:*

$$\mathcal{UD}(\mathcal{V}(\mathfrak{g})) \cong B_\infty(\mathfrak{g}^L). \tag{6.2}$$

Here $L\colon A_n^{(1)} \leftrightarrow A_n^{(1)}$, $L\colon B_n^{(1)} \leftrightarrow A_{2n-1}^{(2)}$, $L\colon C_n^{(1)} \leftrightarrow D_{n+1}^{(2)}$, $L\colon D_n^{(1)} \leftrightarrow D_n^{(1)}$ and $L\colon A_{2n}^{(2)} \leftrightarrow A_{2n}^{(2)\dagger}$.

In the following subsections, we shall show Theorem 6.1 in each case.

The following formula is useful for this purpose:

$$\mathcal{UD}\Big(\frac{cx+y}{x+y}\Big) = \max(c+x, y) - \max(x, y). \tag{6.3}$$

Note that, if $c = 1$,

$$\max(c+x, y) - \max(x, y) = \begin{cases} 1 & x \geq y, \\ 0 & x < y. \end{cases}$$

6.1. $A_n^{(1)}$ $(n \geq 2)$**.** The crystal structure of $\mathcal{UD}(\mathcal{V}(A_n^{(1)}))$ is given as follows: $\mathcal{B}(A_n^{(1)}) = \mathbb{Z}^n$ and for $x = (x_1, \cdots, x_n) \in \mathbb{Z}^n$, we have

$$\begin{aligned} &\tilde{e}_0(x) = (x_1 - 1, x_2 - 1, \cdots, x_n - 1), \\ &\tilde{e}_i(x) = (\cdots, x_i + 1, \cdots) \quad (1 \leq i \leq n), \\ &\mathrm{wt}_0(x) = -x_1 - x_n, \quad \mathrm{wt}_1(x) = 2x_1 - x_2, \\ &\mathrm{wt}_i(x) = -x_{i-1} + 2x_i - x_{i+1} \quad (i = 2, \cdots, n-1), \quad \mathrm{wt}_n(x) = 2x_n - x_{n-1}, \\ &\varepsilon_0(x) = x_1, \quad \varepsilon_i(x) = x_{i+1} - x_i \quad (i = 1, \cdots, n-1), \quad \varepsilon_n(x) = -x_n. \end{aligned}$$

Comparing with the result in §3.4 and the above formulae, we easily see that the following map gives an isomorphism of $A_n^{(1)}$-crystals:

$$\begin{aligned} \mu\colon \mathcal{B}(A_n^{(1)}) = \mathbb{Z}^n \quad &\longrightarrow \quad B_\infty(A_n^{(1)}), \\ (x_1, \cdots, x_n) \quad &\mapsto \quad (b_1, \cdots, b_n, b_{n+1}) = (x_1, x_2 - x_1, \cdots, x_n - x_{n-1}, -x_n). \end{aligned}$$

Thus we have proved Theorem 6.1 for $A_n^{(1)}$.

6.2. $B_n^{(1)}$ $(n \geq 3)$**.** Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(B_n^{(1)})) = (\mathcal{B}(B_n^{(1)}), \{\tilde{e}_i\}, \{\mathrm{wt}_i\}, \{\varepsilon_i\})$. Due to the formula in §5.4, we have

$$\mathcal{B}(B_n^{(1)}) = \mathbb{Z}^{2n-1}$$

and for

$$x = (x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) \in \mathcal{B}(B_n^{(1)}),$$

we have

$$\begin{aligned}
&\tilde{e}_0(x) = \begin{cases} (x_1, x_2 - 1, \cdots, x_n - 1, \cdots, \overline{x}_2 - 1, \overline{x}_1 - 1) & \text{if } x_1 + \overline{x}_1 \geq x_2 + \overline{x}_2, \\ (x_1 - 1, x_2 - 1, \cdots, x_n - 1, \cdots, \overline{x}_2 - 1, \overline{x}_1) & \text{if } x_1 + \overline{x}_1 < x_2 + \overline{x}_2, \end{cases} \\
&\tilde{e}_i(x) = \begin{cases} (x_1, \cdots, x_i + 1, \cdots, \overline{x}_i, \cdots, \overline{x}_1) & \text{if } x_i + \overline{x}_i \geq x_{i+1} + \overline{x}_{i+1}, \\ (x_1, \cdots, x_i, \cdots, \overline{x}_i + 1, \cdots, \overline{x}_1) & \text{if } x_i + \overline{x}_i < x_{i+1} + \overline{x}_{i+1}, \end{cases} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \cdots, n-2), \\
&\tilde{e}_{n-1}(x) = \begin{cases} (x_1, \cdots, x_{n-1} + 1, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) & \text{if } x_{n-1} + \overline{x}_{n-1} \geq 2x_n, \\ (x_1, \cdots, x_{n-1}, x_n, \overline{x}_{n-1} + 1, \cdots, \overline{x}_1) & \text{if } x_{n-1} + \overline{x}_{n-1} < 2x_n, \end{cases} \\
&\tilde{e}_n(x) = (x_1, \cdots, x_{n-1}, x_n + 1, \overline{x}_{n-1}, \cdots, \overline{x}_1), \\
&\mathrm{wt}_0(x) = -(x_2 + \overline{x}_2), \quad \mathrm{wt}_1(x) = 2(x_1 + \overline{x}_1) - (x_2 + \overline{x}_2), \\
&\mathrm{wt}_i(x) = -(x_{i-1} + \overline{x}_{i-1}) + 2(x_i + \overline{x}_i) - (x_{i+1} + \overline{x}_{i+1}) \quad (i = 2, \cdots, n-2), \\
&\mathrm{wt}_{n-1}(x) = -(x_{n-2} + \overline{x}_{n-2}) + 2(x_{n-1} + \overline{x}_{n-1}) - 2x_n, \\
&\mathrm{wt}_n(x) = 2x_n - (x_{n-1} + \overline{x}_{n-1}), \\
&\varepsilon_0(x) = \max(x_1 + \overline{x}_1, x_2 + \overline{x}_2) - x_1 = \overline{x}_1 + (x_2 + \overline{x}_2 - x_1 - \overline{x}_1)_+, \\
&\varepsilon_1(x) = (x_2 + \overline{x}_2 - x_1 - \overline{x}_1)_+ - x_1, \\
&\varepsilon_i(x) = x_{i-1} - x_i + (x_{i+1} + \overline{x}_{i+1} - x_i - \overline{x}_i)_+ \ (i = 2, \cdots, n-2), \\
&\varepsilon_{n-1}(x) = x_{n-2} - x_{n-1} + (2x_n - x_{n-1} - \overline{x}_{n-1})_+, \quad \varepsilon_n(x) = x_{n-1} - x_n.
\end{aligned}$$

Comparing the result in §3.8 and the above formulae, it is easy to see that the following map gives an isomorphism of $A_{2n-1}^{(2)}$-crystals:

$$\begin{array}{rccc} \mu\colon & \mathcal{B}(B_n^{(1)}) & \longrightarrow & B_\infty(A_{2n-1}^{(2)}) \\ & (x_1, \cdots, x_n, \cdots, \overline{x}_1) & \mapsto & (b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1), \end{array}$$

where

$$\begin{aligned}
&b_1 = \overline{x}_1, \quad b_i = \overline{x}_i - \overline{x}_{i-1} \ (i = 2, \cdots, n-1), \quad b_n = x_n - \overline{x}_{n-1}, \\
&\overline{b}_i = x_{i-1} - x_i \ (i = 2, \cdots, n), \ \overline{b}_1 = -x_1.
\end{aligned}$$

We have proved Theorem 6.1 for $B_n^{(1)}$.

6.3. $C_n^{(1)}$ $(n \geq 2)$**.** Let us see the crystal structure of $\mathcal{UD}(\mathcal{V}(C_n^{(1)})) = (\mathcal{B}(C_n^{(1)}), \{\tilde{e}_i\}, \{\mathrm{wt}_i\}, \{\varepsilon_i\})$. We have $\mathcal{B}(C_n^{(1)}) = \mathbb{Z}^{2n}$ and for $x = (x_0, x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) \in \mathcal{B}(C_n^{(1)})$, we have

$$\tilde{e}_0(x) = \begin{cases} (x_0+1, x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_2, \overline{x}_1) & \text{if } x_0 \geq x_1 + \overline{x}_1, \\ (x_0-1, \cdots, x_{n-1}-1, x_n-2, \overline{x}_{n-1}-1, \cdots, \overline{x}_2-1, \overline{x}_1-1) & \\ & \text{if } x_0 < x_1 + \overline{x}_1, \end{cases}$$

$$\tilde{e}_i(x) = \begin{cases} (x_0, \cdots, x_i+1, \cdots, \overline{x}_i, \cdots, \overline{x}_1) & \text{if } x_i + \overline{x}_i \geq x_{i+1} + \overline{x}_{i+1}, \\ (x_0, \cdots, x_i, \cdots, \overline{x}_i+1, \cdots, \overline{x}_1) & \text{if } x_i + \overline{x}_i < x_{i+1} + \overline{x}_{i+1}, \end{cases}$$
$$(i = 1, \cdots, n-2),$$

$$\tilde{e}_{n-1}(x) = \begin{cases} (x_0, \cdots, x_{n-1}+1, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) & \text{if } x_{n-1} + \overline{x}_{n-1} \geq x_n, \\ (x_0, \cdots, x_{n-1}, x_n, \overline{x}_{n-1}+1, \cdots, \overline{x}_1) & \text{if } x_{n-1} + \overline{x}_{n-1} < x_n, \end{cases}$$

$$\tilde{e}_n(x) = (x_0, \cdots, x_{n-1}, x_n+1, \overline{x}_{n-1}, \cdots, \overline{x}_1),$$

$$\begin{aligned} &\mathrm{wt}_0(x) = 2x_0 - 2(x_1 + \overline{x}_1), \quad \mathrm{wt}_1(x) = 2(x_1 + \overline{x}_1) - (x_0 + x_2 + \overline{x}_2), \\ &\mathrm{wt}_i(x) = -(x_{i-1} + \overline{x}_{i-1}) + 2(x_i + \overline{x}_i) - (x_{i+1} + \overline{x}_{i+1}) \ (i = 2, \cdots, n-2) \\ &\mathrm{wt}_{n-1}(x) = -(x_{n-2} + \overline{x}_{n-2}) + 2(x_{n-1} + \overline{x}_{n-1}) - x_n, \\ &\mathrm{wt}_n(x) = 2x_n - 2(x_{n-1} + \overline{x}_{n-1}), \\ &\varepsilon_0(x) = -x_0 + 2(x_1 + \overline{x}_1 - x_0)_+, \\ &\varepsilon_i(x) = x_{i-1} - x_i + (x_{i+1} + \overline{x}_{i+1} - x_i - \overline{x}_i)_+ \ (i = 1, \cdots, n-2), \\ &\varepsilon_{n-1}(x) = x_{n-2} - x_{n-1} + (x_n - x_{n-1} - \overline{x}_{n-1})_+, \quad \varepsilon_n(x) = 2x_{n-1} - x_n. \end{aligned}$$

Comparing the result in §3.9 and the above formulae, we see that the following map gives an isomorphism of $D_{n+1}^{(2)}$-crystals:

$$\begin{array}{rcl} \mu\colon \mathcal{B}(C_n^{(1)}) & \longrightarrow & B_\infty(D_{n+1}^{(2)}) \\ (x_0, x_1, \cdots, x_n, \cdots, \overline{x}_1) & \mapsto & (b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1), \end{array}$$

where

$$\begin{aligned} &b_1 = \overline{x}_1, \quad b_i = \overline{x}_i - \overline{x}_{i-1} \ (i = 2, \cdots, n-1), \quad b_n = \frac{1}{2}x_n - \overline{x}_{n-1}, \\ &\overline{b}_n = x_{n-1} - \frac{1}{2}x_n, \quad \overline{b}_i = x_{i-1} - x_i \ (i = 1, \cdots, n-1). \end{aligned}$$

We have proved Theorem 6.1 for $C_n^{(1)}$.

6.4. $D_n^{(1)}$ $(n \geq 4)$**.** Let us see the crystal structure of $\mathcal{U}D(\mathcal{V}(D_n^{(1)})) = (\mathcal{B}(D_n^{(1)}), \{\tilde{e}_i\}, \{\mathrm{wt}_i\}, \{\varepsilon_i\})$. We have $\mathcal{B}(D_n^{(1)}) = \mathbb{Z}^{2n-2}$ and for $x = (x_1, \cdots, x_n, \overline{x}_{n-2}, \cdots, \overline{x}_1) \in \mathcal{B}(D_n^{(1)})$, we have

$$
\begin{aligned}
&\tilde{e}_0(x) = \begin{cases} (x_1, x_2-1, \cdots, x_n-1, \cdots, \overline{x}_2-1, \overline{x}_1-1) & \text{if } x_1+\overline{x}_1 \geq x_2+\overline{x}_2, \\ (x_1-1, x_2-1, \cdots, x_n-1, \cdots, \overline{x}_2-1, \overline{x}_1) & \text{if } x_1+\overline{x}_1 < x_2+\overline{x}_2, \end{cases} \\
&\tilde{e}_i(x) = \begin{cases} (x_1, \cdots, x_i+1, \cdots, \overline{x}_i, \cdots, \overline{x}_1) & \text{if } x_i+\overline{x}_i \geq x_{i+1}+\overline{x}_{i+1}, \\ (x_1, \cdots, x_i, \cdots, \overline{x}_i+1, \cdots, \overline{x}_1) & \text{if } x_i+\overline{x}_i < x_{i+1}+\overline{x}_{i+1}, \end{cases} \\
&\hfill (i = 1, \cdots, n-3), \\
&\tilde{e}_{n-2}(x) = \begin{cases} (x_1, \cdots, x_{n-2}+1, \cdots, \overline{x}_{n-2}, \cdots, \overline{x}_1) & \text{if } x_{n-2}+\overline{x}_{n-2} \geq x_{n-1}+x_n, \\ (x_1, \cdots, x_{n-2}, \cdots, \overline{x}_{n-2}+1, \cdots, \overline{x}_1) & \text{if } x_{n-2}+\overline{x}_{n-2} < x_{n-1}+x_n, \end{cases} \\
&\tilde{e}_{n-1}(x) = (x_1, \cdots, x_{n-1}+1, x_n, \cdots, \overline{x}_1), \\
&\tilde{e}_n(x) = (x_1 \cdots, x_{n-1}, x_n+1, \cdots, \overline{x}_1), \\
&\mathrm{wt}_0(x) = -(x_2+\overline{x}_2), \quad \mathrm{wt}_1(x) = 2(x_1+\overline{x}_1) - (x_2+\overline{x}_2), \\
&\mathrm{wt}_i(x) = -(x_{i-1}+\overline{x}_{i-1}) + 2(x_i+\overline{x}_i) - (x_{i+1}+\overline{x}_{i+1}) \quad (i = 2, \cdots, n-3), \\
&\mathrm{wt}_{n-2}(x) = -(x_{n-3}+\overline{x}_{n-3}) + 2(x_{n-2}+\overline{x}_{n-2}) - (x_{n-1}+x_n), \\
&\mathrm{wt}_{n-1}(x) = -(x_{n-2}+\overline{x}_{n-2}) + 2(x_{n-1}+\overline{x}_{n-1}), \quad \mathrm{wt}_n(x) = 2x_n - (x_{n-2}+\overline{x}_{n-2}), \\
&\varepsilon_0(x) = \max(x_1+\overline{x}_1, x_2+\overline{x}_2) - x_1 = \overline{x}_1 + (x_2+\overline{x}_2-x_1-\overline{x}_1)_+, \\
&\varepsilon_1(x) = (x_2+\overline{x}_2-x_1-\overline{x}_1)_+ - x_1, \\
&\varepsilon_i(x) = x_{i-1} - x_i + (x_{i+1}+\overline{x}_{i+1}-x_i-\overline{x}_i)_+ \ (i = 2, \cdots, n-3), \\
&\varepsilon_{n-2}(x) = x_{n-3} - x_{n-2} + (x_{n-1}+x_n-x_{n-2}-\overline{x}_{n-2})_+, \\
&\varepsilon_{n-1}(x) = x_{n-2} - x_{n-1}, \quad \varepsilon_n(x) = x_{n-2} - x_n.
\end{aligned}
$$

Then the following map gives an isomorphism of $D_n^{(1)}$-crystals:

$$
\begin{array}{rccc}
\mu\colon & \mathcal{B}(D_n^{(1)}) & \longrightarrow & B_\infty(D_n^{(1)}) \\
& (x_1, \cdots, x_{n-1}, x_n, \overline{x}_{n-2}, \cdots, \overline{x}_1) & \mapsto & (b_1, \cdots, b_n, \overline{b}_{n-1}, \cdots, \overline{b}_1),
\end{array}
$$

where

$$
\begin{aligned}
&b_1 = \overline{x}_1, \quad b_i = \overline{x}_i - \overline{x}_{i-1} \ (i = 2, \cdots, n-2), \quad b_{n-1} = x_{n-1} - \overline{x}_{n-2}, \\
&b_n = x_n - x_{n-1}, \quad \overline{b}_{n-1} = x_{n-2} - x_n, \quad \overline{b}_i = x_{i-1} - x_i \ (i = 2, \cdots, n-2), \\
&\overline{b}_1 = -\sum_{i=1}^{n} b_i - \sum_{i=2}^{n-1} \overline{b}_i = -x_1.
\end{aligned}
$$

We have proved Theorem 6.1 for $D_n^{(1)}$.

6.5. $A_{2n-1}^{(2)}$ $(n \geq 3)$**.** The crystal structure of $\mathcal{U}D(\mathcal{V}(A_{2n-1}^{(2)})) = (\mathcal{B}(A_{2n-1}^{(2)}), \{\tilde{e}_i\}, \{\mathrm{wt}_i\}, \{\varepsilon_i\})$ is given by: $\mathcal{B}(A_{2n-1}^{(2)}) = \mathbb{Z}^{2n-1}$ and for $x = (x_1, \cdots, x_n, \overline{x}_{n-1}, \cdots, \overline{x}_1) \in$

$\mathcal{B}(A^{(2)}_{2n-1})$, we have

$$\tilde{e}_0(x)=\begin{cases}(x_1,x_2-1,\cdots,x_n-2,\cdots,\overline{x}_2-1,\overline{x}_1-1) & \text{if } x_1+\overline{x}_1\geq x_2+\overline{x}_2,\\ (x_1-1,x_2-1,\cdots,x_n-2,\cdots,\overline{x}_2-1,\overline{x}_1) & \text{if } x_1+\overline{x}_1< x_2+\overline{x}_2,\end{cases}$$

$$\tilde{e}_i(x)=\begin{cases}(x_1,\cdots,x_i+1,\cdots,\overline{x}_i,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i\geq x_{i+1}+\overline{x}_{i+1},\\ (x_1,\cdots,x_i,\cdots,\overline{x}_i+1,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i< x_{i+1}+\overline{x}_{i+1},\end{cases}$$
$$(i=1,\cdots,n-2),$$

$$\tilde{e}_{n-1}(x)=\begin{cases}(x_1,\cdots,x_{n-1}+1,x_n,\overline{x}_{n-1},\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}\geq x_n,\\ (x_1,\cdots,x_{n-1},x_n,\overline{x}_{n-1}+1,\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}< x_n,\end{cases}$$

$$\tilde{e}_n(x)=(x_1,\cdots,x_{n-1},x_n+1,\overline{x}_{n-1},\cdots,\overline{x}_1),$$
$$\mathrm{wt}_0(x)=-(x_2+\overline{x}_2),\quad \mathrm{wt}_1(x)=2(x_1+\overline{x}_1)-(x_2+\overline{x}_2),$$
$$\mathrm{wt}_i(x)=-(x_{i-1}+\overline{x}_{i-1})+2(x_i+\overline{x}_i)-(x_{i+1}+\overline{x}_{i+1})\quad (i=2,\cdots,n-2),$$
$$\mathrm{wt}_{n-1}(x)=-(x_{n-2}+\overline{x}_{n-2})+2(x_{n-1}+\overline{x}_{n-1})-x_n,$$
$$\mathrm{wt}_n(x)=2x_n-2(x_{n-1}+\overline{x}_{n-1}),$$
$$\varepsilon_0(x)=\max(x_1+\overline{x}_1,x_2+\overline{x}_2)-x_1=\overline{x}_1+(x_2+\overline{x}_2-x_1-\overline{x}_1)_+,$$
$$\varepsilon_1(x)=(x_2+\overline{x}_2-x_1-\overline{x}_1)_+-x_1,$$
$$\varepsilon_i(x)=x_{i-1}-x_i+(x_{i+1}+\overline{x}_{i+1}-x_i-\overline{x}_i)_+\ (i=2,\cdots,n-2),$$
$$\varepsilon_{n-1}(x)=x_{n-2}-x_{n-1}+(x_n-x_{n-1}-\overline{x}_{n-1})_+,\quad \varepsilon_n(x)=2x_{n-1}-x_n.$$

Then the following map gives an isomorphism of $B^{(1)}_n$-crystals:

$$\begin{array}{rcl}\mu\colon \mathcal{B}(A^{(2)}_{2n-1}) & \longrightarrow & B_\infty(B^{(1)}_n)\\ (x_1,\cdots,x_n,\cdots,\overline{x}_1) & \mapsto & (b_1,\cdots,b_n,\overline{b}_n,\cdots,\overline{b}_1),\end{array}$$

where

$$b_1=\overline{x}_1,\quad b_i=\overline{x}_i-\overline{x}_{i-1}\ (i=2,\cdots,n-1), b_n=\frac{1}{2}x_n-\overline{x}_{n-1},$$
$$\overline{b}_n=x_{n-1}-\frac{1}{2}x_n,\quad \overline{b}_i=x_{i-1}-x_i\ (i=2,\cdots,n-1),\ \overline{b}_1=-x_1.$$

We have proved Theorem 6.1 for $A^{(2)}_{2n-1}$.

6.6. $D^{(2)}_{n+1}$ $(n\geq 2)$**.** Let us see the crystal structure of $\mathcal{U}D(\mathcal{V}(D^{(2)}_{n+1}))=(\mathcal{B}(D^{(2)}_{n+1})$ $\{\tilde{e}_i\},\{\mathrm{wt}_i\},\{\varepsilon_i\})$. We have $\mathcal{B}(D^{(2)}_{n+1})=\mathbb{Z}^{2n}$ and for $x=(x_0,x_1,\cdots,x_n,\overline{x}_{n-1},\cdots,\overline{x}_1)$ $\in\mathcal{B}(D^{(2)}_{n+1})$, we have

$$\tilde{e}_0(x)=\begin{cases}(x_0+1,x_1,\cdots,x_n,\overline{x}_{n-1},\cdots,\overline{x}_2,\overline{x}_1) & \text{if } 2x_0\geq x_1+\overline{x}_1,\\ (x_0,x_1-1,\cdots,x_n-1,\overline{x}_{n-1}-1,\cdots,\overline{x}_1-1) & \text{if } 2x_0+1= x_1+\overline{x}_1,\\ (x_0-1,\cdots,x_{n-1}-2,x_n-2,\overline{x}_{n-1}-2,\cdots,\overline{x}_2-2,\overline{x}_1-2) & \\ & \text{if } 2x_0+1< x_1+\overline{x}_1,\end{cases}$$

where we use the formula

$$\max(2+x,y)-\max(x,y)=\begin{cases}2 & x\geq y,\\ 1 & x+1=y \quad (x,y\in\mathbb{Z}),\\ 0 & x+1<y,\end{cases}$$

$$\tilde{e}_i(x)=\begin{cases}(x_1,\cdots,x_i+1,\cdots,\overline{x}_i,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i\geq x_{i+1}+\overline{x}_{i+1},\\ (x_1,\cdots,x_i,\cdots,\overline{x}_i+1,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i< x_{i+1}+\overline{x}_{i+1},\end{cases}$$
$$(i=1,\cdots,n-2),$$
$$\tilde{e}_{n-1}(x)=\begin{cases}(x_1,\cdots,x_{n-1}+1,x_n,\overline{x}_{n-1},\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}\geq 2x_n,\\ (x_1,\cdots,x_{n-1},x_n,\overline{x}_{n-1}+1,\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}< 2x_n,\end{cases}$$
$$\tilde{e}_n(x)=(x_1,\cdots,x_{n-1},x_n+1,\overline{x}_{n-1},\cdots,\overline{x}_1),$$
$$\mathrm{wt}_0(x)=2x_0-(x_1+\overline{x}_1),\quad \mathrm{wt}_1(x)=2(x_1+\overline{x}_1)-(2x_0+x_2+\overline{x}_2),$$
$$\mathrm{wt}_i(x)=-(x_{i-1}+\overline{x}_{i-1})+2(x_i+\overline{x}_i)-(x_{i+1}+\overline{x}_{i+1})\quad (i=2,\cdots,n-2),$$
$$\mathrm{wt}_{n-1}(x)=-(x_{n-2}+\overline{x}_{n-2})+2(x_{n-1}+\overline{x}_{n-1})-2x_n,$$
$$\mathrm{wt}_n(x)=2x_n-(x_{n-1}+\overline{x}_{n-1}),$$
$$\varepsilon_0(x)=-x_0+(x_1+\overline{x}_1-2x_0)_+,\quad \varepsilon_1(x)=2x_0-x_1+(x_2+\overline{x}_2-x_1-\overline{x}_1)_+,$$
$$\varepsilon_i(x)=x_{i-1}-x_i+(x_{i+1}+\overline{x}_{i+1}-x_i-\overline{x}_i)_+\ (i=2,\cdots,n-2),$$
$$\varepsilon_{n-1}(x)=x_{n-2}-x_{n-1}+(2x_n-x_{n-1}-\overline{x}_{n-1})_+,\quad \varepsilon_n(x)=x_{n-1}-x_n.$$

Then the following map gives an isomorphism of $C_n^{(1)}$-crystals:

$$\begin{array}{rcl}\mu\colon \mathcal{B}(D_{n+1}^{(2)}) & \longrightarrow & B_\infty(C_n^{(1)})\\ (x_0,x_1,\cdots,x_n,\cdots,\overline{x}_1) & \mapsto & (b_1,\cdots,b_n,\overline{b}_n,\cdots,\overline{b}_1),\end{array}$$

where

$$b_1=\overline{x}_1,\ b_i=\overline{x}_i-\overline{x}_{i-1}\ (i=2,\cdots,n),\ \overline{b}_i=x_{i-1}-x_i\ (i=2,\cdots,n),\ \overline{b}_1=2x_0-x_1.$$

We have proved Theorem 6.1 for $D_{n+1}^{(2)}$.

6.7. $A_{2n}^{(2)\dagger}$ **($n\geq 2$).** Let $\mathcal{V}(A_{2n}^{(2)\dagger})$ be the affine $A_{2n}^{(2)\dagger}$-geometric crystal as in §5.9. We shall see that the crystal structure of $\mathcal{UD}(\mathcal{V}(A_{2n}^{(2)\dagger}))=(\mathcal{B}(A_{2n}^{(2)\dagger}),\{\tilde{e}_i\},\{\mathrm{wt}_i\},\{\varepsilon_i\})$ is given by: $\mathcal{B}(A_{2n}^{(2)\dagger})=\mathbb{Z}^{2n}$ and for $x=(x_0,x_1,\cdots,x_n,\overline{x}_{n-1},\cdots,\overline{x}_1)\in\mathcal{B}(A_{2n}^{(2)\dagger})$, we have

$$\tilde{e}_0(x)=\begin{cases}(x_0+1,x_1,\cdots,x_n,\overline{x}_{n-1},\cdots,\overline{x}_2,\overline{x}_1) & \text{if } x_0\geq x_1+\overline{x}_1,\\ (x_0-1,x_1-1\cdots,x_{n-1}-1,x_n-1,\overline{x}_{n-1}-1,\cdots,\overline{x}_1-1) & \\ & \text{if } x_0< x_1+\overline{x}_1,\end{cases}$$
$$\tilde{e}_i(x)=\begin{cases}(x_1,\cdots,x_i+1,\cdots,\overline{x}_i,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i\geq x_{i+1}+\overline{x}_{i+1},\\ (x_1,\cdots,x_i,\cdots,\overline{x}_i+1,\cdots,\overline{x}_1) & \text{if } x_i+\overline{x}_i< x_{i+1}+\overline{x}_{i+1},\end{cases}$$
$$(i=1,\cdots,n-2),$$
$$\tilde{e}_{n-1}(x)=\begin{cases}(x_1,\cdots,x_{n-1}+1,x_n,\overline{x}_{n-1},\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}\geq 2x_n,\\ (x_1,\cdots,x_{n-1},x_n,\overline{x}_{n-1}+1,\cdots,\overline{x}_1) & \text{if } x_{n-1}+\overline{x}_{n-1}< 2x_n,\end{cases}$$
$$\tilde{e}_n(x)=(x_1,\cdots,x_{n-1},x_n+1,\overline{x}_{n-1},\cdots,\overline{x}_1),$$

$$\begin{aligned}
&\mathrm{wt}_0(x) = 2x_0 - 2(x_1 + \overline{x}_1), \quad \mathrm{wt}_1(x) = 2(x_1 + \overline{x}_1) - (x_0 + x_2 + \overline{x}_2),\\
&\mathrm{wt}_i(x) = -(x_{i-1} + \overline{x}_{i-1}) + 2(x_i + \overline{x}_i) - (x_{i+1} + \overline{x}_{i+1}) \quad (i = 2, \cdots, n-2),\\
&\mathrm{wt}_{n-1}(x) = -(x_{n-2} + \overline{x}_{n-2}) + 2(x_{n-1} + \overline{x}_{n-1}) - 2x_n,\\
&\mathrm{wt}_n(x) = 2x_n - (x_{n-1} + \overline{x}_{n-1}),\\
&\varepsilon_0(x) = -x_0 + 2(x_1 + \overline{x}_1 - x_0)_+,\\
&\varepsilon_i(x) = x_{i-1} - x_i + (x_{i+1} + \overline{x}_{i+1} - x_i - \overline{x}_i)_+ \ (i = 1, \cdots, n-2),\\
&\varepsilon_{n-1}(x) = x_{n-2} - x_{n-1} + (2x_n - x_{n-1} - \overline{x}_{n-1})_+, \quad \varepsilon_n(x) = x_{n-1} - x_n.
\end{aligned}$$

Then the following map gives an isomorphism of $A^{(2)}_{2n}$-crystals:

$$\begin{array}{rcl}
\mu\colon \mathcal{B}(A^{(2)\,\dagger}_{2n}) & \longrightarrow & B_\infty(A^{(2)}_{2n})\\
(x_0, x_1, \cdots, x_n, \cdots, \overline{x}_1) & \mapsto & (b_1, \cdots, b_n, \overline{b}_n, \cdots, \overline{b}_1),
\end{array}$$

where

$$\begin{aligned}
b_1 = \overline{x}_1, \quad b_i = \overline{x}_i - \overline{x}_{i-1} \ (i = 2, \cdots, n-1),\ b_n = x_n - \overline{x}_{n-1},\\
\overline{b}_i = x_{i-1} - x_i \ (i = 1, \cdots, n).
\end{aligned}$$

We have proved Theorem 6.1 for $A^{(2)\,\dagger}_{2n}$.

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Research Institute for Mathematical Sciences, Kyoto University, Kitashiwakawa, Sakyo-ku, Kyoto 606, Japan

E-mail address: masaki@kurims.kyoto-u.ac.jp

Department of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan

E-mail address: toshiki@mm.sophia.ac.jp

Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan

E-mail address: okado@sigmath.es.osaka-u.ac.jp