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ON THE ASYMPTOTIC BEHAVIOUR OF THE EIGENMODES FOR ELLIPTIC PROBLEMS IN DOMAINS BECOMING UNBOUNDED

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Abstract. The aim of this work is to analyze the asymptotic behaviour of the eigenmodes of some elliptic eigenvalue problems set on domains becoming unbounded in one or several directions. In particular, in the case of a linear elliptic operator in divergence form, we prove that the sequence of the k-th eigenvalues convergences to the first eigenvalue of an elliptic problems set on the section of the domain. Moreover, an optimal rate of convergence of this sequence is given.

1. Introduction

The goal of this paper is to analyze the asymptotic behaviour of the eigenvalues and the eigenfunctions of elliptic problems set in domains which can become unbounded in one or several directions. More precisely, with the notation of Section 2, we will consider the following eigenvalue problems set on cylindrical domains $\Omega_\ell = (-\ell, \ell)^p \times \omega$:

$$-\nabla \cdot \big(A(x)\nabla u_\ell\big) = \lambda_\ell u_\ell \text{ in } \Omega_\ell, \qquad u_\ell = 0 \text{ on } \partial\Omega_\ell, \tag{1.1}$$

and the eigenvalue problem set on the section ω of Ω_ℓ, namely

$$-\nabla \cdot \big(A_{22}(X_2)\nabla u\big) = \mu u \text{ in } \omega, \qquad u = 0 \text{ on } \partial\omega. \tag{1.2}$$

When the positive parameter ℓ goes to plus infinity, the asymptotic behaviour of the unknowns of problem (1.1) will be described in terms of the solutions u, μ of problem (1.2). Indeed, under the assumptions of Section 2, we prove that the sequence of the k-th eigenvalues of problem (1.1) converges toward the first eigenvalue of (1.2) as ℓ goes to plus infinity. Moreover the convergence rate is in $\frac{1}{\ell^2}$ which is, in some sense optimal; see (1.11) below.

In Theorem 3.4, we show for block diagonal matrix A that the normalized first eigenfunctions of problem (1.1) converge on any fixed subdomain of Ω_ℓ toward a well identified first eigenfunction of (1.2) as ℓ goes to plus infinity. The case of the sequence of the k-th eigenfunction is considered in Theorem 3.8. This kind of issue, namely the approximation of the solutions of problems set on cylinders by the solution of problems set on their section, is addressed in [2], [3], [4], [5], [6], [7], [8] for some differential equations, variational inequalities or systems.

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Let us start with a simple example. Suppose that for $\ell > 0$, Ω_ℓ is the rectangle in $\mathbb{R}^2$ defined as

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1) = (-\ell, \ell) \times \omega$$

where we have set $\omega = (-1, 1)$. We denote by x_1, x_2 the coordinates in $\mathbb{R}^2$

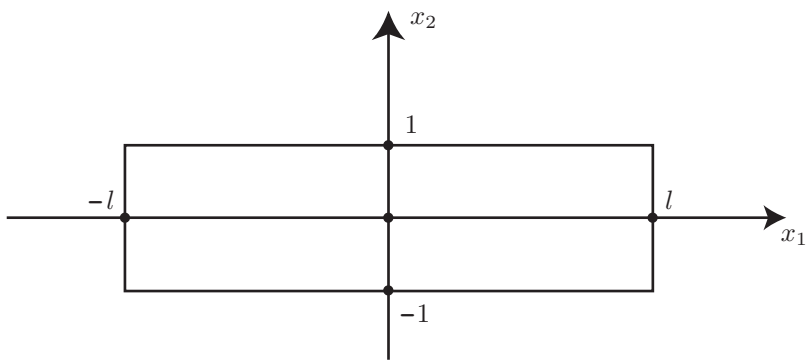

FIGURE 1.1

(see Figure 1.1). Let us then determine the eigenvalues and eigenfunctions of the Laplace operator in Ω_ℓ with Dirichlet boundary conditions. To achieve that, let us first consider the following one dimensional eigenvalue problem:

$$\begin{cases} -u'' = \lambda u & \text{in } (-a, a), \\ u(-a) = u(a) = 0. \end{cases} \tag{1.3}$$

It is clear that the general solution of the first equation of (1.3) is given by

$$u = A \sin \sqrt{\lambda}(x + a) + B \cos \sqrt{\lambda}(x + a)$$

($\lambda > 0$, $A, B \in \mathbb{R}$). In order to match the boundary conditions one has to have

$$B = 0, \qquad \sin 2a\sqrt{\lambda} = 0.$$

($A = 0$ would give the trivial solution.) Thus the eigenvalues of the problem (1.3) are given by

$$2a\sqrt{\lambda_k} = k\pi \qquad \Longleftrightarrow \qquad \lambda_k = \left(\frac{k\pi}{2a}\right)^2, \quad k = 1, 2, \ldots, \tag{1.4}$$

and the corresponding eigenfunctions by

$$u_k = A_k \sin \frac{k\pi}{2a}(x + a). \tag{1.5}$$

If one wants to normalize u_k in such a way that its L^2-norm is equal to one, one is led to choose A_k such that

$$A_k^2 \int_{-a}^{a} \sin^2 \frac{k\pi}{2a}(x + a)\, dx = 1$$

which leads to

$$u_k = \frac{1}{\sqrt{a}} \sin \frac{k\pi}{2a}(x + a).$$

Suppose now that we want to find the eigenvalues and eigenfunctions of the problem

$$\begin{cases} -\Delta u = \lambda_\ell u & \text{in } \Omega_\ell, \\ u = 0 & \text{on } \partial\Omega_\ell. \end{cases} \tag{1.6}$$

It is clear from above that

$$u = \frac{1}{\sqrt{\ell}} \sin \frac{k\pi}{2\ell}(x_1 + \ell) \sin \frac{m\pi}{2}(x_2 + 1) \tag{1.7}$$

is an eigenfunction for (1.6) corresponding to the eigenvalue

$$\left(\frac{k\pi}{2\ell}\right)^2 + \left(\frac{m\pi}{2}\right)^2, \quad k, m = 1, 2, \ldots. \tag{1.8}$$

There are no other eigenvalues. Indeed, suppose that (v, λ_v) is another couple of eigenfunction and eigenvalue for (1.6). We have

$$-\Delta v = \lambda_v v.$$

Multiplying by u given by (1.7) and integrating by parts we obtain

$$\lambda_u(v, u) = \lambda_v(v, u)$$

where λ_u is given by (1.8). Since we have assumed $\lambda_v \neq \lambda_u$ we get

$$(v, u) = 0.$$

($(\cdot, \cdot)$ is the usual $L^2(\Omega_\ell)$-scalar product.) Thus we have $(v, u) = 0$ for any u given by (1.7). By density of the span of the functions $\sin \frac{k\pi}{2a}(x + \ell)$ in $L^2(-a, a)$ we obtain

$$\int_{\Omega_\ell} v\varphi(x_1)\psi(x_2)\, dx_1\, dx_2 = 0 \quad \forall\, \varphi \in L^2(-\ell, \ell),\ \ \forall\, \psi \in L^2(-1, 1).$$

It follows easily that $v = 0$ and λ_v cannot be an eigenvalue. This proves that (1.7), (1.8) give complete sets of eigenfunctions and eigenvalues of (1.6). Let us examine what happens when ℓ goes to plus infinity. The first eigenvalue of (1.6) is given by (1.8) for $k = 1$, $m = 1$, i.e. is given by

$$\lambda_\ell^1 = \left(\frac{\pi}{2\ell}\right)^2 + \left(\frac{\pi}{2}\right)^2. \tag{1.9}$$

Clearly, it converges when $\ell \to +\infty$ towards the first eigenvalue

$$\mu^1 = \left(\frac{\pi}{2}\right)^2$$

of the problem

$$\begin{cases} -u'' = \lambda u & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega \end{cases} \tag{1.10}$$

(see (1.3), (1.4)). Moreover, the rate of convergence is in $\frac{1}{\ell^2}$. One could next expect that the k-th eigenvalue to (1.6) converges also to the k-eigenvalue to (1.10). This is not what happens. Indeed, going back to (1.8), if λ_ℓ^k denotes the k-th eigenvalue to (1.6) one sees that

$$\lambda_\ell^k = \left(\frac{k\pi}{2\ell}\right)^2 + \left(\frac{\pi}{2}\right)^2 \quad \text{for } \sqrt{3}\ell > k. \tag{1.11}$$

Thus, when $\ell \to +\infty$, one has

$$\lambda_\ell^k \longrightarrow \mu^1$$

and all the eigenvalues of (1.6) are piling up on μ^1 when $\ell \to +\infty$. We will see below that this phenomenon also happens in the general case. Of course if one allows k to depend on ℓ the behaviour of λ_ℓ^k could be different. For instance if $\sqrt{3}\ell$ is an integer

$$\lambda_\ell^{\sqrt{3}\ell} = \mu^2$$

where μ^2 is the second eigenvalue of (1.10).

The behaviour, when $\ell \to +\infty$, of the eigenfunctions is touchy as well. Indeed, consider the first eigenfunction of (1.6). It is given (see (1.7)) by

$$u = \frac{1}{\sqrt{\ell}} \sin \frac{\pi}{2\ell}(x_1 + \ell) \sin \frac{\pi}{2}(x_2 + 1).$$

When $\ell \to +\infty$, it converges toward 0. To obtain something converging toward the first eigenfunction to (1.10) one has to re-scale it properly. Indeed

$$\bar{u} = \sin \frac{\pi}{2\ell}(x_1 + \ell) \sin \frac{\pi}{2}(x_2 + 1) \quad \longrightarrow \quad \sin \frac{\pi}{2}(x_2 + 1)$$

when $\ell \to +\infty$, i.e. $\bar{u}$ is such that

$$\int_{\Omega_\ell} \bar{u}^2 \, dx_1 \, dx_2 = \ell,$$

and thus to expect convergence of the eigenfunctions from (1.6) toward the eigenfunctions to (1.10) some proper scaling is in order.

The convergence rate in the H^1 norm in any subdomain is then exactly in $\frac{1}{\ell^2}$. This contrasts with the nonreasonance case where the convergence is, in general, faster. For instance, if we consider the problem

$$\begin{cases} -\Delta u_\ell = f(x_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases}$$

where f is some L^2 function depending only on the x_2 variable, then the decay in the H^1 norm of $u_\ell - u_\infty$ in any subdomain is at least exponential (see [3], Proposition 2.1). Here, u_∞ is the solution to

$$\begin{cases} -u'' = f(x_2) & \text{in } (-1, 1), \\ u = 0 & \text{on } \{-1, 1\}. \end{cases}$$

These are the kinds of issues that we would like to analyze in this paper. We will do it in a general setting. The rest of the article is divided as follows. In the next section we prove convergence of the eigenvalues in the case of a general elliptic operator. In section 3 we study the convergence of the eigenfunctions. Finally we give some applications.

2. Convergence of the eigenvalues

Let us first introduce some notation that we will use throughout the paper. We denote by Ω_ℓ the open subset of $\mathbb{R}^n$ defined as

$$\Omega_\ell = (-\ell, \ell)^p \times \omega.$$

ℓ is a positive number, $1 \leq p < n$ an integer, and ω is a bounded open subset of $\mathbb{R}^{n-p}$. The points in $\mathbb{R}^n$ will be denoted by

$$x = (X_1, X_2),$$

with

$$X_1 = (x_1, \dots, x_p), \qquad X_2 = (x_{p+1}, \dots, x_n).$$

Let $A = A(X_1, X_2)$ be an $n \times n$-symmetric matrix of the type

$$A = A(X_1, X_2) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{12}^T(X_2) & A_{22}(X_2) \end{pmatrix}. \tag{2.1}$$

In (2.1), A_{11} is a $p \times p$ symmetric matrix. We will assume that A is uniformly bounded and uniformly positive definite on $\mathbb{R}^p \times \omega$; that is to say that

$$\|A(x)\| \leq \Lambda \quad \text{a.e. } x \in \mathbb{R}^p \times \omega, \tag{2.2}$$

$$A(x)\xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^p \times \omega, \quad \forall\, \xi \in \mathbb{R}^n. \tag{2.3}$$

In the above formula $\|\cdot\|$ stands for a norm on matrices, Λ, λ are some positive constants, $|\cdot|$ is the euclidean norm, and the "$\cdot$"denotes the usual euclidean scalar product. If Ω is a bounded open set of $\mathbb{R}^k$ we will denote by $H^1(\Omega)$, $H_0^1(\Omega)$ the usual spaces of functions defined as

$$H^1(\Omega) = \{\, v \in L^2(\Omega) \mid \partial_{x_i} v \in L^2(\Omega)\ \forall i = 1, \dots, k \,\}, \tag{2.4}$$

$$H_0^1(\Omega) = \{\, v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega \,\}. \tag{2.5}$$

In the above definition ∂_{x_i} denotes the partial derivation in x_i, and $\partial\Omega$ the boundary of Ω. We will always assume that $H_0^1(\Omega)$ is equipped with the norm

$$\big|\!\big|\nabla v\big|\!\big|_{2,\Omega} = \left\{ \int_\Omega |\nabla v(x)|^2\, dx \right\}^{1/2} \tag{2.6}$$

($|\nabla v|$ denotes the euclidean norm of the gradient of v).

We would like to consider the eigenvalue problem

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), \\ \displaystyle\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v\, dx = \lambda_\ell \int_{\Omega_\ell} u_\ell v\, dx \quad \forall\, v \in H_0^1(\Omega_\ell). \end{cases} \tag{2.7}$$

To be more precise, we say that λ_ℓ is an eigenvalue to (2.7) if there exists a $u_\ell \neq 0$ solution of the above system. When $\ell \to +\infty$, we expect that the limit eigenvalue problem (2.7) will be the eigenvalue problem on ω defined as

$$\begin{cases} u \in H_0^1(\omega), \\ \displaystyle\int_\omega A_{22}(X_2)\nabla_{X_2} u \cdot \nabla_{X_2} v\, dX_2 = \mu \int_\omega uv\, dX_2 \quad \forall\, v \in H_0^1(\omega). \end{cases} \tag{2.8}$$

In the above system $dX_2 = dx_{p+1} \dots dx_n$, $\nabla_{X_2} = (\partial_{x_{p+1}}, \dots, \partial_{x_n})$, i.e. ∇_{X_2} denotes the gradient in the last variables. By a "limit problem" we mean that we expect that the eigenmodes of (2.7) will converge towards the eigenmodes of (2.8) when $\ell \to +\infty$. Of course this convergence has to be taken with caution as we pointed out already in our Introduction.

Let us denote by λ_ℓ^1 and μ^1 the first eigenvalues of (2.7) and (2.8), respectively. It is well known that

$$\begin{aligned}\lambda_\ell^1 &= \operatorname{Inf}\left\{\int_{\Omega_\ell} A\nabla u\cdot\nabla u\,dx \mid u\in H_0^1(\Omega_\ell),\ \int_{\Omega_\ell} u^2\,dx = 1\right\}\\ &= \operatorname*{Inf}_{H_0^1(\Omega_\ell)\setminus\{0\}} \frac{\int_{\Omega_\ell} A\nabla u\cdot\nabla u\,dx}{\int_{\Omega_\ell} u^2\,dx},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\mu^1 &= \operatorname{Inf}\left\{\int_{\omega} A_{22}(X_2)\nabla_{X_2}u\cdot\nabla_{X_2}u\,dX_2 \mid u\in H_0^1(\omega),\ \int_{\omega} u^2\,dX_2 = 1\right\}\\ &= \operatorname*{Inf}_{H_0^1(\omega)\setminus\{0\}} \frac{\int_{\omega} A_{22}(X_2)\nabla_{X_2}u\cdot\nabla_{X_2}u\,dX_2}{\int_{\omega} u^2\,dX_2}.\end{aligned} \tag{2.10}$$

Moreover, it is well known that these eigenvalues are simple and the corresponding eigenfunctions do not change sign in Ω_ℓ or ω; see [10], or [12].

Then we have:

Theorem 2.1. *It holds that*

$$\mu^1 \le \lambda_\ell^1 \le \mu^1 + \frac{C}{\ell^2} \tag{2.11}$$

where $C = p|A_{11}|_\infty \frac{\pi^2}{4}$, $|A_{11}|_\infty = \operatorname{esssup}_{\mathbb{R}^n} |A_{11}(x)|$, $|\cdot|$ *is the norm of matrices subordinated to the euclidean norm.*

To prove this theorem we will need the following lemma:

Lemma 2.2. *Let* Ω *be a bounded open set of* $\mathbb{R}^n$. *Let* A_ε *be a family of symmetric matrices such that for* λ, Λ *positive independent of* ε *we have*

$$\lambda|\xi|^2 \le A_\varepsilon(x)\xi\cdot\xi \qquad \text{a.e. } x\in\Omega,\quad \forall\,\xi\in\mathbb{R}^n, \tag{2.12}$$

$$\|A(x)\| \le \Lambda \qquad \text{a.e. } x\in\Omega. \tag{2.13}$$

Suppose that

$$A_\varepsilon(x)\to A(x) \quad \text{a.e. } x\in\Omega \text{ as } \varepsilon\to 0. \tag{2.14}$$

Set

$$\lambda_\varepsilon = \operatorname{Inf}\left\{\int_\Omega A_\varepsilon\nabla u\cdot\nabla u\,dx \mid u\in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1\right\}, \tag{2.15}$$

$$\lambda_0 = \operatorname{Inf}\left\{\int_\Omega A\nabla u\cdot\nabla u\,dx \mid u\in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1\right\}; \tag{2.16}$$

then we have

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda_0. \tag{2.17}$$

(i.e. the first eigenvalue of the operator $-\nabla\cdot(A_\varepsilon\nabla\cdot)$ *with Dirichlet boundary conditions converges towards the first eigenvalue of* $-\nabla\cdot(A\nabla\cdot)$ *with the same boundary conditions).*

Proof of Lemma 2.2. Let u_ε be the first eigenfunction realizing the infimum of (2.15). (It can be shown that it does not change sign in Ω and is unique if we assume it to be positive; see [10].) Denote by v a fixed function in $H_0^1(\Omega)$ satisfying

$$\int_\Omega v^2\,dx = 1. \tag{2.18}$$

By (2.12), (2.13), (2.15) we have

$$\lambda\big|\big|\nabla u_\varepsilon\big|\big|_{2,\Omega}^2 \le \int_\Omega A_\varepsilon(x)\nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx = \lambda_\varepsilon \le \int_\Omega A_\varepsilon(x)\nabla v \cdot \nabla v \, dx \le C \tag{2.19}$$

where C is a constant independent of ε. Thus λ_ε is bounded and so is u_ε in $H_0^1(\Omega)$. At the expense of extracting a subsequence we can assume that

$$\lambda_\varepsilon \to \lambda^0, \tag{2.20}$$

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in} \quad H_0^1(\Omega), \qquad u_\varepsilon \to u_0 \quad \text{in} \quad L^2(\Omega). \tag{2.21}$$

Now, by definition of the first eigenvalue one has, since A_ε is also symmetric,

$$\begin{aligned} \int_\Omega A_\varepsilon(x)\nabla u_\varepsilon \cdot \nabla v \, dx &= \int_\Omega \nabla u_\varepsilon \cdot A_\varepsilon(x)\nabla v \, dx \\ &= \lambda_\varepsilon \int_\Omega u_\varepsilon v \, dx \quad \forall\, v \in H_0^1(\Omega). \end{aligned} \tag{2.22}$$

Noticing that by the Lebesgue theorem we have

$$A_\varepsilon \nabla v \to A\nabla v \quad \text{in} \quad L^2(\Omega)$$

(see (2.14)) we obtain by passing to the limit in (2.22)

$$\int_\Omega A\nabla u_0 \cdot \nabla v \, dx = \lambda^0 \int_\Omega u_0 v \, dx \quad \forall\, v \in H_0^1(\Omega). \tag{2.23}$$

Thus λ^0 is an eigenvalue of the problem (2.23). Note that $u_0 \neq 0$ since

$$1 = \int_\Omega u_\varepsilon^2 \, dx \to \int_\Omega u_0^2 \, dx = 1. \tag{2.24}$$

From (2.19) we also derive that it holds that

$$\lambda^0 \le \int_\Omega A(x)\nabla v \cdot \nabla v \, dx \quad \forall\, v \in H_0^1(\Omega), \ \int_\Omega v^2 \, dx = 1. \tag{2.25}$$

Thus λ^0 is the first eigenvalue of (2.23), i.e. $\lambda^0 = \lambda_0$. By uniqueness of the possible limits this shows (2.17) and completes the proof of the lemma. □

Remark 2.1. Since when $\varepsilon \to 0$

$$\int_\Omega u_\varepsilon^2 \, dx \to \int_\Omega u_0^2 \, dx, \qquad \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx \to \int_\Omega A\nabla u_0 \cdot \nabla u_0 \, dx, \tag{2.26}$$

it is easy to show that $u_\varepsilon \to u_0$ in $H^1(\Omega)$ and u_0 is the first eigenfunction (normalized and positive) of (2.23).

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let us first establish the lower bound in (2.11). For this consider A_ε defined by

$$A_\varepsilon = \begin{cases} A \quad \text{on} \quad \Omega_{\ell-\varepsilon}, \\ \\ \begin{pmatrix} \lambda & & 0 & \vdots & \\ & \ddots & & \vdots & 0 \\ 0 & & \lambda & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & & \vdots & A_{22} \end{pmatrix} \quad \text{on } \Omega_\ell \setminus \Omega_{\ell-\varepsilon}, \end{cases} \tag{2.27}$$

where λ is the constant in the inequality (2.3). Clearly A_ε satisfies all the assumptions of Lemma 2.2 with $\Omega = \Omega_\ell$. Define

$$\lambda^1_{\ell,\varepsilon} = \operatorname{Inf}\left\{ \int_{\Omega_\ell} A_\varepsilon \nabla u \cdot \nabla u \, dx \mid u \in H^1_0(\Omega_\ell), \ \int_{\Omega_\ell} u^2 \, dx = 1 \right\}, \tag{2.28}$$

the first eigenvalue of the operator $-\nabla \cdot (A_\varepsilon \nabla \cdot)$ with Dirichlet boundary conditions. Also denote by u_ε the function – that we can assume > 0 – where the infimum of (2.28) is achieved. We have

$$\int_{\Omega_\ell} A_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla v \, dx = \lambda^1_{\ell,\varepsilon} \int_{\Omega_\ell} u_\varepsilon v \, dx \quad \forall \, v \in H^1_0(\Omega_\ell). \tag{2.29}$$

Then denote by ϱ_ε a smooth function on $[-\ell, \ell]$ such that

$$0 \le \varrho_\varepsilon \le 1, \quad \varrho_\varepsilon = 1 \text{ on } (-\ell+\varepsilon, \ell-\varepsilon), \quad \varrho_\varepsilon(\pm\ell) = 0, \quad \varrho_\varepsilon \text{ is concave,} \tag{2.30}$$

i.e.

$$\varrho''_\varepsilon \le 0. \tag{2.31}$$

Denote by w_1 the positive function realizing the infimum of (2.10), and in (2.29) choose

$$v = w_1(X_2) \prod_{i=1}^{p} \varrho_\varepsilon(x_i). \tag{2.32}$$

We obtain (we decompose A_ε as A in (2.1) and put ε above as an upper index)

$$\begin{aligned} \int_{\Omega_\ell} \{ & A^\varepsilon_{11} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + A^\varepsilon_{12} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \\ & + A^{\varepsilon T}_{12} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v + A^\varepsilon_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \} \, dx = \lambda^1_{\ell,\varepsilon} \int_{\Omega_\ell} u_\varepsilon v \, dx. \end{aligned} \tag{2.33}$$

If we denote by Π the product in (2.32) we obtain

$$\begin{aligned} \int_{\Omega_\ell} \{ & A^\varepsilon_{11} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \Pi \cdot w_1 + A^\varepsilon_{12} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} \Pi \, w_1 \\ & + A^{\varepsilon T}_{12} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi + A^\varepsilon_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \} \, dx \\ = \lambda^1_{\ell,\varepsilon} & \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx. \end{aligned} \tag{2.34}$$

First remark that

$$\nabla_{X_1} \Pi = 0 \quad \text{on} \quad \Omega_{\ell-\varepsilon} \tag{2.35}$$

so that by (2.27) the second integral in (2.34) vanishes. Next, for the first integral we have

$$
\begin{aligned}
&\int_{\Omega_\ell} A_{11}^{\varepsilon} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \Pi w_1 \, dx \\
&\quad = \int_{\Omega_\ell \setminus \Omega_{\ell-\varepsilon}} A_{11}^{\varepsilon} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \Pi \, w_1 \, dx \\
&\quad = \lambda \int_{\Omega_\ell \setminus \Omega_{\ell-\varepsilon}} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} \Pi \, w_1 \, dx \\
&\quad = \lambda \int_{\Omega_\ell \setminus \Omega_{\ell-\varepsilon}} \nabla_{X_1} \cdot \{ u_\varepsilon \nabla_{X_1} \Pi \, w_1 \} - u_\varepsilon \prod_{j \neq i}^{p} \varrho_\varepsilon(x_j) \varrho_\varepsilon''(x_i) w_1 \, dx \\
&\quad \geq \lambda \int_{\Omega_\ell \setminus \Omega_{\ell-\varepsilon}} \nabla_{X_1} \cdot \{ u_\varepsilon \nabla_{X_1} \Pi w_1 \} \, dx
\end{aligned}
\tag{2.36}
$$

by (2.30), (2.31). But since $u_\varepsilon \nabla_{X_1} \Pi$ vanishes on the boundary of $\Omega_\ell \setminus \Omega_{\ell-\varepsilon}$ the above integral is identically equal to 0. Thus, we derive from (2.34) that it holds

$$
\begin{aligned}
&\int_{\Omega_\ell} A_{12}^{\varepsilon T} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx + \int_{\Omega_\ell} A_{22}^{\varepsilon} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx \\
&\quad \leq \lambda_{\ell,\varepsilon}^1 \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx
\end{aligned}
$$

which is also by (2.27)

$$
\begin{aligned}
&\int_{\Omega_{\ell-\varepsilon}} A_{12}^{T} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx + \int_{\Omega_\ell} A_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx \\
&\quad \leq \lambda_{\ell,\varepsilon}^1 \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx.
\end{aligned}
\tag{2.37}
$$

The second integral above can be written as

$$
\begin{aligned}
&\int_{\Omega_\ell} A_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx \\
&\quad = \int_{[-\ell,\ell]^p} \Pi \int_{\omega} A_{22} \nabla_{X_2} u_\varepsilon(X_1, X_2) \cdot \nabla w_1(X_2) \, dX_2 \, dX_1 \\
&\quad = \int_{[-\ell,\ell]^p} \Pi \int_{\omega} \mu^1 u_\varepsilon w_1 \, dX_2 \, dX_1 \\
&\quad = \mu^1 \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx
\end{aligned}
\tag{2.38}
$$

by definition of w_1 and since for a.e. X_1, $u_\varepsilon(X_1, \cdot) \in H_0^1(\omega)$. It follows that (2.37) becomes

$$
\begin{aligned}
&\int_{\Omega_{\ell-\varepsilon}} A_{12}^{T} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} w_1 \Pi \, dx + \mu^1 \int_{\Omega_\ell} u_\varepsilon w_1 \Pi \, dx \\
&\quad \leq \lambda_{\ell,\varepsilon}^1 \int_{\Omega_\ell} u_\varepsilon w_1 \Pi, \, dx.
\end{aligned}
\tag{2.39}
$$

Passing to the limit in ε – see also Remark 2.1 – we get (see Lemma 2.2)

$$
\int_{\Omega_\ell} A_{12}^{T} \nabla_{X_1} u_0 \cdot \nabla_{X_2} w_1 \, dx + \mu^1 \int_{\Omega_\ell} u_0 w_1 \, dx \leq \lambda_\ell^1 \int_{\Omega_\ell} u_0 w_1 \, dx
$$

($\Pi = \Pi_\varepsilon \to 1$ as $\varepsilon \to 0$). But since $u_0 \in H_0^1(\Omega_\ell)$ and A_{12}^T, w_1 depend only on X_2 we have

$$\int_{\Omega_\ell} A_{12}^T \nabla_{X_1} u_0 \cdot \nabla_{X_2} w_1 \, dx = \int_{\Omega_\ell} \nabla_{X_1} \cdot (u_0 A_{12} \nabla_{X_2} w_1) \, dx = 0. \tag{2.40}$$

We then derive

$$\mu^1 \int_{\Omega_\ell} u_0 w_1 \, dx \le \lambda_\ell^1 \int_{\Omega_\ell} u_0 w_1 \, dx, \tag{2.41}$$

which implies the first inequality of (2.11) since u_0, w_1 are positive (see Remark 2.1). It remains to show the second inequality of (2.11). For that let v be a smooth function in $H_0^1([-\ell,\ell]^p)$ such that

$$\int_{[-\ell,\ell]^p} v^2 \, dX_1 = 1 \tag{2.42}$$

and w_1 as above. It is clear that

$$u = v w_1 = v(X_1) w_1(X_2) \in H_0^1(\Omega_\ell) \quad \text{and} \quad \int_{\Omega_\ell} u^2 \, dx = 1. \tag{2.43}$$

Thus, by (2.9) we have

$$\begin{aligned}
\lambda_\ell^1 &\le \int_{\Omega_\ell} A \nabla(v w_1) \cdot \nabla(v w_1) \, dx \\
&= \int_{\Omega_\ell} A_{11} \nabla_{X_1} v \cdot \nabla_{X_1} v w_1^2 \, dx + \int_{\Omega_\ell} A_{12} \nabla_{X_2} w_1 \cdot \nabla_{X_1} v (v w_1) \, dx \\
&\quad + \int_{\Omega_\ell} A_{12}^T \nabla_{X_1} v \cdot \nabla_{X_2} w_1 (v w_1) \, dx \\
&\quad + \int_{\Omega_\ell} A_{22} \nabla_{X_2} w_1 \cdot \nabla_{X_2} w_1 v^2 \, dx \\
&= \int_{\Omega_\ell} A_{11} \nabla_{X_1} v \cdot \nabla_{X_1} v w_1^2 \, dx + 2 \int_{\Omega_\ell} A_{12} \nabla_{X_2} w_1 \nabla_{X_1} v (v w_1) \, dx \\
&\quad + \int_{\Omega_\ell} A_{22} \nabla_{X_2} w_1 \cdot \nabla_{X_2} w_1 v^2 \, dx.
\end{aligned} \tag{2.44}$$

One remarks that the second integral is equal to 0 since $v \in H_0^1([-\ell,\ell]^p)$. Indeed – recall that A_{12}, w_1 are depending on X_2 only – and thus we have

$$\begin{aligned}
&2 \int_{\Omega_\ell} A_{12} \nabla_{X_2} w_1 \nabla_{X_1} v \cdot v w_1 \, dx \\
&\qquad = \int_\omega \int_{[-\ell,\ell]^p} \nabla_{X_1} \cdot \{ v^2 A_{12} \nabla_{X_2} w_1 w_1 \} \, dX_1 \, dX_2 = 0.
\end{aligned} \tag{2.45}$$

Next, due to the definition of w_1 we have (see (2.42))

$$\begin{aligned}
&\int_{\Omega_\ell} A_{22} \nabla_{X_2} w_1 \cdot \nabla_{X_2} w_1 v^2 \, dx \\
&\qquad = \int_{[-\ell,\ell]^p} v^2 \int_\omega A_{22} \nabla_{X_2} w_1 \cdot \nabla_{X_2} w_1 \, dX_2 \, dX_1 \\
&\qquad = \int_{[-\ell,\ell]^p} v^2 \mu^1 = \mu^1.
\end{aligned} \tag{2.46}$$

Thus, from (2.44) we derive

$$\begin{aligned} &\lambda_\ell^1 \le \mu^1 + \int_{\Omega_\ell} A_{11}\nabla_{X_1} v \cdot \nabla_{X_1} v w_1^2\, dx \\ &\forall\, v \in H_0^1([-\ell,\ell]), \quad \int_{[-\ell,\ell]^p} v^2\, dX_1 = 1. \end{aligned} \tag{2.47}$$

It follows that

$$\begin{aligned} \lambda_\ell^1 &\le \mu^1 + \int_{[-\ell,\ell]^p} |A_{11}||\nabla_{X_1} v|^2\, dX_1 \int_\omega w_1^2\, dX_2 \\ &\le \mu^1 + |A_{11}|_\infty \int_{[-\ell,\ell]^p} |\nabla_{X_1} v|^2\, dX_1 \\ &\qquad \forall\, v \in H_0^1([-\ell,\ell]^p), \ \int_{[-\ell,\ell]^p} v^2\, dX_1 = 1. \end{aligned} \tag{2.48}$$

It is clear – see Section 1 – that the first eigenvalue for the Laplace operator with Dirichlet boundary conditions in $[-\ell,\ell]^p$ is given by

$$p\Big(\frac{\pi}{2\ell}\Big)^2. \tag{2.49}$$

Thus taking the infimum on v in (2.48) we obtain

$$\lambda_\ell^1 \le \mu^1 + \frac{p\pi^2 |A_{11}|_\infty}{4\ell^2}. \tag{2.50}$$

This completes the proof of Theorem 2.1. □

As a corollary we have

Theorem 2.3 (Convergence of the first eigenvalue)**.** *Let Ω'_ℓ denote a domain in $\mathbb{R}^n$ such that*

$$\Omega_\ell \subset \Omega'_\ell \subset \Omega_{\ell'} \tag{2.51}$$

for some ℓ'. Let λ'^1_ℓ be the first eigenvalue of $-\nabla\cdot(A\nabla\cdot)$ with Dirichlet boundary conditions on Ω'_ℓ, i.e.

$$\lambda'^1_\ell = \mathrm{Inf}\Big\{\int_{\Omega'_\ell} A\nabla u\cdot\nabla u\, dx \mid u \in H_0^1(\Omega'_\ell),\ \int_{\Omega'_\ell} u^2\, dx = 1\Big\}; \tag{2.52}$$

then it holds that

$$\lim_{\ell\to+\infty} \lambda'^1_\ell = \mu^1. \tag{2.53}$$

Proof. It is easy to see from the definition – see for instance (2.52) – that the first eigenvalue of the operator $-\nabla\cdot(A\nabla\cdot)$ with Dirichlet boundary conditions on Ω is nonincreasing when the size of Ω is increasing. Thus we have by (2.51)

$$\mu^1 \le \lambda^1_{\ell'} \le \lambda'^1_\ell \le \lambda^1_\ell \le \mu^1 + \frac{c}{\ell^2} \tag{2.54}$$

and the result follows. □

As we explained in our Introduction it is reasonable to expect that λ_ℓ^k, the k-th eigenvalue of the operator $-\nabla\cdot(A\nabla\cdot)$ with Dirichlet boundary conditions on Ω_ℓ, converges towards μ^1 as ℓ goes to plus infinity. Indeed we have

Theorem 2.4 (Convergence of the k-th eigenvalue λ_ℓ^k)**.** *With the above notation there exists a constant $C_k = C(k,p,|A_{11}|_\infty)$ such that*

$$\mu^1 \le \lambda_\ell^k \le \mu^1 + \frac{C_k}{\ell^2}. \tag{2.55}$$

Proof. Of course we can assume w.l.o.g. that $k \ge 2$. Let us split the domain Ω_ℓ in k subdomains Q_i in the x_1-direction; i.e. let us set

$$Q_i = \left(-\ell + (i-1)\frac{2\ell}{k}, -\ell + i\frac{2\ell}{k}\right) \times (-\ell,\ell)^{p-1} \times \omega, \quad i = 1,\dots,k. \tag{2.56}$$

Moreover let us denote by $\lambda_{Q_i}^1$ the first eigenvalue defined by

$$\begin{aligned} \lambda_{Q_i}^1 &= \lambda_{Q_i}^1(A) \\ &= \operatorname{Inf}\left\{\int_{Q_i} A\nabla u\cdot\nabla u\,dx \mid u \in H_0^1(Q_i),\ \int_{Q_i} u^2\,dx = 1\right\} \end{aligned} \tag{2.57}$$

and by u_i the first eigenfunction, i.e. the only positive function achieving the infimum above. Now set

$$Q = \left(-\frac{\ell}{k}, \frac{\ell}{k}\right) \times (-\ell,\ell)^{p-1} \times \omega. \tag{2.58}$$

Denote by $\lambda_Q^1 = \lambda_Q^1(A)$ the first eigenvalue defined by (2.57) where Q_i is replaced by Q. Performing a translation in x_1 in the integral

$$\int_{Q_i} A\nabla u\cdot\nabla u\,dx$$

it is easy to see that

$$\lambda_{Q_i}^1(A) = \lambda_Q^1(\tilde{A}) \tag{2.59}$$

where $\tilde{A}(x) = A(x_1 + (2i-1)\frac{\ell}{k} - \ell, x_2, \dots, x_n)$. Since

$$\Omega_{\ell/k} \subset Q \subset \Omega_\ell$$

it follows from (2.54) that it holds that

$$\mu^1 \le \lambda_{Q_i}^1(A) = \lambda_Q^1(\tilde{A}) \le \lambda_{\ell/k}^1 \le \mu^1 + \frac{pk^2\pi^2|A_{11}|_\infty}{4\ell^2}. \tag{2.60}$$

Suppose now that u_i is extended by 0 in $\Omega_\ell\backslash Q_i$. Denote by u_ℓ^j the j-th eigenfunction corresponding to λ_ℓ^j (our numbering of the eigenvalues is such that the λ_ℓ^j are not necessarily pairwise distinct). Since one has more unknowns than equations, it is possible to find $\alpha_1,\dots,\alpha_k$ not all equal to zero such that

$$u = \sum_{i=1}^k \alpha_i u_i \tag{2.61}$$

satisfies

$$(u, u_\ell^j) = \sum_{i=1}^k \alpha_i (u_i, u_\ell^j) = 0 \quad \forall\, j = 1,\dots,k-1. \tag{2.62}$$

$(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega_\ell)$. It is well known – see [14] – that

$$\lambda_\ell^1 \le \lambda_\ell^k = \operatorname*{Inf}_{\substack{v\in H_0^1(\Omega_\ell)\setminus\{0\}\\ v\perp u_\ell^1,\dots,u_\ell^{k-1}}} \left\{ \int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx \Big/ \int_{\Omega_\ell} v^2\,dx \right\}. \tag{2.63}$$

Thus from (2.62) we get

$$\mu^1 \le \lambda_\ell^k \le \int_{\Omega_\ell} A\nabla u\cdot\nabla u\,dx \Big/ \int_{\Omega_\ell} u^2\,dx. \tag{2.64}$$

Since the u_i have disjoint supports it follows that

$$\begin{aligned}\mu^1 \le \lambda_\ell^k &\le \sum_{i=1}^k \alpha_i^2 \int_{Q_i} A\nabla u_i\cdot\nabla u_i\,dx \Big/ \sum_{i=1}^k \alpha_i^2 \int_{Q_i} u_i^2\,dx\\ &= \sum_{i=1}^k \alpha_i^2 \lambda_{Q_i}^1(A) \Big/ \sum_{i=1}^k \alpha_1^2\end{aligned}$$

(recall that u_i is such that $\int_{Q_i} u_i^2\,dx = 1$). Going back to (2.60) we obtain

$$\mu^1 \le \lambda_\ell^k \le \mu^1 + \frac{pk^2\pi^2|A_{11}|_\infty}{4\ell^2} \tag{2.65}$$

which completes the proof of the theorem. □

3. Convergence of the eigenfunctions

As we have seen in the Introduction the convergence of the eigenfunctions depends strongly on the scaling of them – recall that they are defined up to a multiplicative constant. To further stress this point we consider the case where the eigenfunctions are from separated variables, i.e. the case when

$$A = \begin{pmatrix} A_{11}(X_1) & 0\\ 0 & A_{22}(X_2)\end{pmatrix}. \tag{3.1}$$

A satisfying the assumptions of the preceding section, in particular for some constants $\lambda, \Lambda > 0$, we have

$$\lambda|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda|\xi|^2 \quad \text{a.e. } x\in\mathbb{R}^n,\ \forall\,\xi\in\mathbb{R}^n. \tag{3.2}$$

Let us set

$$Q_\ell = (-\ell,\ell)^p \tag{3.3}$$

and denote by λ_ℓ the first eigenvalue of the operator $-\nabla_{X_1}\cdot(A_{11}(X_1)\nabla_{X_1}\cdot)$ on Q_ℓ with Dirichlet boundary conditions, i.e.

$$\lambda_\ell = \operatorname{Inf}\left\{ \int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}u\cdot\nabla_{X_1}u\,dX_1 \mid u\in H_0^1(Q_\ell),\ \int_{Q_\ell} u^2\,dX_1 = 1\right\}. \tag{3.4}$$

Due to (3.2) one has for $u\in H_0^1(Q_\ell)$

$$\begin{aligned}\lambda\int_{Q_\ell} |\nabla_{X_1}u|^2\,dX_1 &\le \int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}u\cdot\nabla_{X_1}u\,dX_1\\ &\le \Lambda\int_{Q_\ell} |\nabla_{X_1}u|^2\,dX_1.\end{aligned} \tag{3.5}$$

Taking the infimum in u this leads (see (2.49)) to

$$p\frac{\lambda\pi^2}{4\ell^2} \le \lambda_\ell \le p\frac{\Lambda\pi^2}{4\ell^2}. \tag{3.6}$$

We denote by u_ℓ the first eigenfunction corresponding to λ_ℓ, positive and such that

$$\int_{Q_\ell} u_\ell^2\, dX_1 = \ell^p. \tag{3.7}$$

We introduce v_ℓ as

$$v_\ell(y) = u_\ell(\ell y) \quad \text{a.e.} \quad y \in Q_1 = (-1,1)^p. \tag{3.8}$$

Then we have:

Proposition 3.1. *$\ell^2\lambda_\ell$ is the first eigenvalue of the operator*

$$-\nabla\cdot(A_{11}(\ell y)\nabla\cdot) \tag{3.9}$$

on Q_1 with Dirichlet boundary conditions and v_ℓ is the first normalized eigenfunction, i.e. we have in particular

$$\begin{cases} \displaystyle\int_{Q_1} A_{11}(\ell y)\nabla v_\ell\cdot\nabla v\, dy = \ell^2\lambda_\ell \int_{Q_1} v_\ell v\, dy \quad \forall\, v \in H_0^1(Q_1), \\ v_\ell \in H_0^1(Q_1), \int_{Q_1} v_\ell^2\, dy = 1. \end{cases} \tag{3.10}$$

(∇ is the gradient in the y-variable.)

Proof. Clearly $v_\ell \in H_0^1(Q_1)$. Moreover we have

$$\int_{Q_1} v_\ell^2(y)\, dy = \int_{Q_1} u_\ell^2(\ell y)\, dy = \int_{Q_\ell} u_\ell^2(X_1)\frac{dX_1}{\ell^p} = 1. \tag{3.11}$$

By definition of λ_ℓ and u_ℓ we have

$$\begin{aligned} \lambda_\ell &= \frac{\int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}u_\ell\cdot\nabla_{X_1}u_\ell\, dX_1}{\int_{Q_\ell} u_\ell^2\, dX_1} \\ &= \int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}u_\ell\cdot\nabla_{X_1}u_\ell\frac{dX_1}{\ell^p} \\ &\le \int_{Q_\ell} A_{11}(X_1)\nabla v\cdot\nabla v\frac{dX_1}{\ell^p} \qquad \forall\, v \in H_0^1(Q_\ell), \int_{Q_\ell} v^2\, dX_1 = \ell^p. \end{aligned} \tag{3.12}$$

One notices that by (3.8)

$$\nabla v_\ell(y) = \ell\nabla_{X_1}u_\ell(\ell y). \tag{3.13}$$

Moreover for every $w \in H_0^1(Q_1)$ with $\int_{Q_1} w^2\, dy = 1$ we have

$$v(X_1) = w\Big(\frac{X_1}{\ell}\Big) \in H_0^1(Q_\ell), \qquad \nabla_{X_1}v(X_1) = \frac{1}{\ell}\nabla w\Big(\frac{X_1}{\ell}\Big)$$

and

$$\int_{Q_\ell} v^2(X_1)\, dX_1 = \ell^p. \tag{3.14}$$

From these remarks we derive for every $w \in H_0^1(Q_1)$, $\int_{Q_1} w^2\, dy = 1$:

$$\lambda_\ell = \frac{1}{\ell^2}\int_{Q_1} A_{11}(\ell y)\nabla v_\ell\cdot\nabla v_\ell\, dy \le \frac{1}{\ell^2}\int_{Q_1} A_{11}(\ell y)\nabla w\cdot\nabla w\, dy. \tag{3.15}$$

This shows that

$$\ell^2\lambda_\ell \text{ is the first eigenvalue of } -\nabla\cdot(A_{11}(\ell y)\nabla\cdot) \text{ on } Q_1, \tag{3.16}$$

$$v_\ell \text{ is the first eigenfunction of } -\nabla\cdot(A_{11}(\ell y)\nabla\cdot) \text{ on } Q_1 \tag{3.17}$$

for the Dirichlet boundary conditions, and this completes the proof of the proposition. □

We now study the asymptotic behaviour of v_ℓ. For this we suppose that

$$\lim_{|X_1|\to+\infty} A_{11}(X_1) = A_\infty \tag{3.18}$$

where A_∞ is a constant $p\times p$ matrix. Then we have

Proposition 3.2. *Under the assumption* (3.18) *it holds that*

$$\ell^2\lambda_\ell \to \lambda_\infty, \qquad v_\ell \to v_\infty \text{ in } H_0^1(Q_1) \tag{3.19}$$

where λ_∞ is the first eigenvalue, and v_∞ the first normalized eigenfunction of the operator $-\nabla\cdot(A_\infty\nabla\cdot)$ with Dirichlet boundary conditions on Q_1, i.e. in particular

$$\begin{cases} \displaystyle\int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v\,dy = \lambda_\infty\int_{Q_1} v_\infty v\,dy \quad \forall\, v\in H_0^1(Q_1),\\ v_\infty\in H_0^1(Q_1), \int_{Q_1} v_\infty^2\,dy = 1. \end{cases} \tag{3.20}$$

Proof. Taking $v = v_\ell$ in the first equation of (3.10) and using (3.6), (3.2) we get

$$\lambda\int_{Q_1}|\nabla v_\ell|^2\,dy \le \int_{Q_1} A_{11}(\ell y)\nabla v_\ell\cdot\nabla v_\ell\,dy = \ell^2\lambda_\ell \le p\frac{\Lambda}{4}\pi^2. \tag{3.21}$$

Thus $\ell^2\lambda_\ell$ is bounded and v_ℓ is bounded in $H_0^1(Q_1)$, and up to a subsequence we can assume that for some λ_∞, v_∞ it holds that

$$\ell^2\lambda_\ell \to \lambda_\infty, \tag{3.22}$$

$$v_\ell \rightharpoonup v_\infty \text{ in } H_0^1(Q_1), \qquad v_\ell \to v_\infty \text{ in } L^2(Q_1). \tag{3.23}$$

We would like to identify these limits. First by (3.18) we have

$$A_{11}(\ell y)\nabla v \to A_\infty\nabla v \quad \text{in } L^2(Q_1) \tag{3.24}$$

for any $v\in H_0^1(Q_1)$. Thus passing to the limit in (3.10) – recall that A_{11} is symmetric – we obtain

$$\begin{cases} \displaystyle\int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v\,dy = \lambda_\infty\int_{Q_1} v_\infty v\,dy \quad \forall\, v\in H_0^1(Q_1),\\ v_\infty\in H_0^1(Q_1), \int_{Q_1} v_\infty^2\,dy = 1. \end{cases} \tag{3.25}$$

Thus v_∞ is an eigenfunction of $-\nabla\cdot(A_\infty\nabla\cdot)$. We also have by (3.10)

$$\int_{Q_1} A_{11}(\ell y)\nabla v_\ell\cdot\nabla v_\ell\,dy = \ell^2\lambda_\ell \to \lambda_\infty. \tag{3.26}$$

It then follows

$$
\begin{aligned}
&\int_{Q_1} A_{11}(\ell y)\nabla(v_\ell - v_\infty)\cdot\nabla(v_\ell - v_\infty)\,dy \\
&\quad = \int_{Q_1} A_{11}(\ell y)\nabla v_\ell\cdot\nabla v_\ell\,dy - 2\int_{Q_1} A_{11}(\ell y)\nabla v_\infty\cdot\nabla v_\ell \\
&\qquad + \int_{Q_1} A_{11}(\ell y)\nabla v_\infty\cdot\nabla v_\infty\,dy \\
&\quad \to \lambda_\infty - 2\int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v_\infty\,dy + \int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v_\infty\,dy \\
&\quad = \lambda_\infty - \int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v_\infty\,dy.
\end{aligned}
\tag{3.27}
$$

It results from (3.25) – with $v = v_\infty$ – that this last quantity is 0. It then follows that

$$
v_\ell \to v_\infty \quad \text{in} \quad H_0^1(Q_1). \tag{3.28}
$$

Going back to (3.15) we have

$$
\ell^2\lambda_\ell \le \int_{Q_1} A_{11}(\ell y)\nabla w\cdot\nabla w\,dy \quad \forall\, w\in H_0^1(Q_1), \int_{Q_1} w^2\,dy = 1. \tag{3.29}
$$

Passing to the limit we get

$$
\begin{aligned}
\lambda_\infty &= \int_{Q_1} A_\infty\nabla v_\infty\cdot\nabla v_\infty\,dy \\
&\le \int_{Q_1} A_\infty\nabla w\cdot\nabla w\,dy \quad \forall\, w\in H_0^1(Q_1), \int_{Q_1} w^2\,dy = 1.
\end{aligned}
\tag{3.30}
$$

This shows that λ_∞ is the first eigenvalue of the operator $-\nabla\cdot(A_\infty\nabla\cdot)$ and v_∞ its first normalized (positive) eigenfunction. Since λ_∞, v_∞ are uniquely determined we have obtained that (3.28), (3.22) hold for every subsequence and thus all the sequences converge. This completes the proof of the proposition. □

In addition we can show:

Proposition 3.3. *Under the assumptions of Proposition* 3.2 *it holds that*

$$
|v_\ell - v_\infty|_{\infty,Q_1} \to 0 \tag{3.31}
$$

when $\ell\to+\infty$. $|\cdot|_{\infty,Q_1}$ *denotes the usual* $L^\infty(Q_1)$*-norm.*

Proof. First it is clear from the usual regularity results for elliptic equations that v_∞ is a smooth function. Moreover, combining (3.10) and (3.20) we have in a weak sense

$$
-\nabla\cdot(A_{11}(\ell y)\nabla v_\ell) - \ell^2\lambda_\ell v_\ell = 0 = -\nabla\cdot(A_\infty\nabla v_\infty) - \lambda_\infty v_\infty.
$$

This implies

$$
\begin{aligned}
&-\nabla\cdot(A_{11}(\ell y)\nabla(v_\ell - v_\infty)) - \ell^2\lambda_\ell(v_\ell - v_\infty) \\
&\quad = \nabla\cdot([A_{11}(\ell y) - A_\infty]\nabla v_\infty) - (\lambda_\infty - \ell^2\lambda_\ell)v_\infty.
\end{aligned}
$$

Since $\ell^2\lambda_\ell$ is bounded from below independently of ℓ, using the well known L^∞-estimates of Stampacchia (see [13] or [15]) we obtain

$$|v_\ell - v_\infty|_{\infty,Q_1} \le C\big\{\big|[A_{11}(\ell y)-A_\infty]|\nabla v_\infty|\big|_{p,Q_1} + |\lambda_\infty - \ell^2\lambda_\ell||v_\infty|_{p,Q_1} + |v_\ell - v_\infty|_{2,Q_1}\big\}$$

where C is a constant independent of ℓ, and $|\cdot|_{p,Q_1}$ denotes the usual $L^p(Q_1)$-norm, $p > n$. The results then follow from the Lebesgue theorem and Proposition 3.2. □

We can now state a convergence theorem for the eigenfunctions. We have

Theorem 3.4 (Convergence of the eigenfunction)**.** *Let A be a symmetric matrix satisfying* (3.1), (3.2), (3.18). *Let $u_{1,\ell}$ be the first eigenfunction (positive) and normalized by*

$$\int_{\Omega_\ell} u_{1,\ell}^2\,dx = \ell^p, \tag{3.32}$$

corresponding to λ_ℓ^1 (see (2.9)*). Let w_1 be the first positive eigenfunction realizing the infimum of* (2.10)*. Then for any $\ell_0 > 0$ we have*

$$|u_{1,\ell} - v_\infty(0)w_1|_{\infty,\Omega_{\ell_0}} \to 0 \tag{3.33}$$

when $\ell \to +\infty$. $|\cdot|_{\infty,\Omega_{\ell_0}}$ is the usual $L^\infty(\Omega_{\ell_0})$-norm, and v_∞ has been defined in (3.20).

Proof. We claim that

$$u_{1,\ell} = u_\ell w_1 \tag{3.34}$$

where u_ℓ has been defined above (3.7). It is indeed clear from (3.7) that

$$u_\ell w_1 \in H^1(\Omega_\ell), \qquad \int_{\Omega_\ell} (u_\ell w_1)^2\,dx = \ell^p. \tag{3.35}$$

Moreover for $v \in H_0^1(\Omega_\ell)$ it holds that

$$\begin{aligned}\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx &= \int_{\Omega_\ell} A_{11}\nabla_{X_1}v\cdot\nabla_{X_1}v\,dx + \int_{\Omega_\ell} A_{22}\nabla_{X_2}v\cdot\nabla_{X_2}v\,dx\\ &= \int_\omega\int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}v\cdot\nabla_{X_1}v\,dX_1\,dX_2\\ &\quad + \int_{Q_\ell}\int_\omega A_{22}(X_2)\nabla_{X_2}v\cdot\nabla_{X_2}v\,dX_2\,dX_1.\end{aligned}$$

Using the definitions of λ_ℓ, μ^1, respectively (and the fact that $v(\cdot,X_2)\in H_0^1(Q_\ell)$ a.e. X_2) we get

$$\begin{aligned}\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx &\ge \int_\omega \lambda_\ell\int_{Q_\ell} v^2\,dX_1\,dX_2 + \int_{Q_\ell}\mu^1\int_\omega v^2\,dX_2\,dX_1\\ &= (\lambda_\ell+\mu^1)\int_{Q_\ell} v^2\,dX_1\,dX_2.\end{aligned} \tag{3.36}$$

Now one also has

$$\begin{aligned}\int_{Q_\ell} A\nabla(u_\ell w_1)\cdot\nabla(u_\ell w_1)\,dx &= \int_\omega\int_{Q_\ell} A_{11}(X_1)\nabla_{X_1}u_\ell\cdot\nabla_{X_1}u_\ell\cdot w_1^2\,dx\\ &\quad+\int_{Q_\ell}\int_\omega A_{22}(X_2)\nabla_{X_2}w_1\cdot\nabla_{X_2}w_1u_\ell^2\,dx\\ &= (\lambda_\ell+\mu^1)\int_{\Omega_\ell}(u_\ell w_1)^2\,dx.\end{aligned}$$

This shows that

$$\lambda_\ell^1=\lambda_\ell+\mu^1,\qquad u_{1,\ell}=u_\ell w_1.$$

Now using (3.8) we have

$$\begin{aligned}u_{1,\ell}(x)-v_\infty(0)w_1 &= u_\ell(X_1)w_1(X_2)-v_\infty(0)w_1(X_2)\\ &= \left\{v_\ell\left(\frac{X_1}{\ell}\right)-v_\infty(0)\right\}w_1(X_2)\\ &= \left\{v_\ell\left(\frac{X_1}{\ell}\right)-v_\infty\left(\frac{X_1}{\ell}\right)+v_\infty\left(\frac{X_1}{\ell}\right)-v_\infty(0)\right\}w_1(X_2)\end{aligned}$$

and thus

$$\begin{aligned}&|u_{1,\ell}(x)-v_\infty(0)w_1|_{\infty,\Omega_{\ell_0}}\\ &\qquad\le |w_1|_{\infty,\omega}\left\{|v_\ell-v_\infty|_{\infty,Q_1}+\left|v_\infty\left(\frac{X_1}{\ell}\right)-v_\infty(0)\right|_{\infty,Q_{\ell_0}}\right\}.\end{aligned}$$

The result then follows from (3.31) and from the continuity of v_∞. □

Remark 3.1. The scaling of the eigenfunctions is a very delicate issue. For instance (3.32) is not enough to insure the convergence of $u_{1,\ell}$ toward w_1; one has to further scale by $v_\infty(0)$ a constant depending on the behaviour of the operator at ∞. From (3.34) we always have

$$u_{1,\ell}=v_\ell\left(\frac{X_1}{\ell}\right)w_1(X_2).$$

Now the behaviour of v_ℓ is not so easy to control. For instance (see (3.10)) if A_{11} is periodic, then v_ℓ is the solution of a homogenization problem for which only weak convergence is known (see [9]).

We now consider the case of the k-th eigenfunctions. For that we denote by $\lambda_{k,\ell}$ the k-th eigenvalue of the operator $-\nabla_{X_1}\cdot(A_{11}(X_1)\nabla_{X_1}\cdot)$ on Q_ℓ with Dirichlet boundary conditions. We denote by u_ℓ^k a corresponding eigenfunction chosen such that

$$u_\ell^k\perp u_\ell^j\quad\forall\, j<k,\qquad \int_{Q_\ell}(u_\ell^k)^2\,dX_1=\ell^p. \tag{3.37}$$

Arguing as in the Introduction it is not difficult to see that the eigenvalues of the operator $-\nabla\cdot(A\nabla\cdot)$ with Dirichlet boundary conditions on Ω_ℓ are given by

$$\lambda_{k,\ell}+\mu^m \tag{3.38}$$

where the μ^m are the eigenvalues of the operator $-\nabla_{X_2}(A_{22}\nabla_{X_2}\cdot)$ with Dirichlet boundary conditions on ω.

We can first show:

Proposition 3.5. *Assuming* (3.2) *we have*

$$p\frac{\lambda\pi^2}{4\ell^2} \le \lambda_{k,\ell} \le p\frac{\Lambda k^2\pi^2}{4\ell^2}. \tag{3.39}$$

Proof. The proof is similar to the proof of Theorem 2.4 and we only sketch it. We split the domain Q_ℓ in k subdomain Q_i in the x_1-direction by setting

$$Q_i = \left(-\ell + (i-1)\frac{2\ell}{k}, -\ell + i\frac{2\ell}{k}\right) \times (-\ell,\ell)^{p-1}, \quad i = 1,\dots,k. \tag{3.40}$$

We set

$$\lambda_{1,Q_i}(A_{11}) = \mathrm{Inf}\left\{\int_{Q_i} A_{11}\nabla_{X_1}u\cdot\nabla_{X_1}u\,dx \mid u\in H_0^1(Q_i),\ \int_{Q_i} u^2\,dX_1 = 1\right\}. \tag{3.41}$$

We also define

$$Q = \left(-\frac{\ell}{k}, \frac{\ell}{k}\right) \times (-\ell,\ell)^{p-1} \tag{3.42}$$

and denote by $\lambda_{1,Q}(A_{11})$ the first eigenvalue defined by (3.41) where Q_i is replaced by Q. It is easy to show (compare to (2.59)) that

$$\lambda_{1,Q_i}(A_{11}) = \lambda_{1,Q}(\tilde{A}_{11}) \tag{3.43}$$

where $\tilde{A}_{11} = A_{11}(x_1 + (2i-1)\frac{\ell}{k} - i, x_2,\dots,x_p)$. Since

$$Q_{\frac{\ell}{k}} \subset Q \tag{3.44}$$

it holds that (see (3.6))

$$\lambda_{1,Q_i}(A_{11}) = \lambda_{1,Q}(\tilde{A}_{11}) \le \frac{p\Lambda k^2\pi^2}{4\ell^2}. \tag{3.45}$$

Then one proceeds as in the proof of (2.60) to get (3.39). □

As a consequence of (3.38), (3.39), for ℓ large enough, a k-th eigenvalue of the operator $-\nabla\cdot(A\nabla\cdot)$ with Dirichlet boundary condition on Ω_ℓ is given by

$$\lambda_\ell^k = \lambda_{k,\ell} + \mu^1 \tag{3.46}$$

and a k-th normalized eigenfunction is given by

$$u_{k,\ell} = u_\ell^k(X_1)\cdot w_1(X_2); \tag{3.47}$$

recall definition (3.37).

Let us set

$$v_\ell^k(y) = u_\ell^k(\ell y), \quad y\in Q_1. \tag{3.48}$$

Then we have

Proposition 3.6. *$\ell^2\lambda_{k,\ell}$ is a k-th eigenvalue of the operator*

$$-\nabla\cdot(A_{11}(\ell y)\nabla\cdot) \tag{3.49}$$

on Q_1 with Dirichlet boundary conditions and v_ℓ^k is a k-th normalized eigenfunction.

Proof. This follows from the fact that v_ℓ^k satisfies (3.10) with λ_ℓ replaced by $\lambda_{k,\ell}$ and from the equalities

$$\begin{aligned}\int_{Q_1} v_\ell^k(y)v_\ell^j(y)\,dy &= \int_{Q_1} u_\ell^k(\ell y)u_\ell^j(\ell y)\,dy \\ &= \int_{Q_\ell} u_\ell^k(X_1)u_\ell^j(X_1)\,\frac{dX_1}{\ell^p} = \delta_{k,j}.\end{aligned} \tag{3.50}$$

□

Proposition 3.7. *Under the assumption* (3.18) *there is a subsequence of* ℓ *such that*

$$\ell^2\lambda_{k,\ell} \to \lambda_{k,\infty}, \qquad v_\ell^k \to v_\infty^k \quad \text{in} \quad H_0^1(Q_1) \tag{3.51}$$

where $\lambda_{k,\infty}$ *is a* k*-th eigenvalue, and* v_∞^k *a* k*-normalized eigenfunction of the operator* $-\nabla(A_\infty\nabla\cdot)$ *with Dirichlet boundary conditions on* Q_1.

Proof. One proceeds by induction on k. For $k = 1$ the result has already been proven. Assuming this holds for $k-1$ we have to establish it for k. First it follows from (3.39) that $\ell^2\lambda_{k,\ell}$ and v_ℓ^k are bounded. One can then extract a subsequence and pass to the limit as in Proposition 3.2. The fact that v_∞^k is a k-th eigenfunction follows from (3.50) by passing to the limit in ℓ. □

Remark 3.2. If $\lambda_{k,\infty}$ is simple, the whole sequence of eigenvalues or eigenfunctions converges.

Then we have

Theorem 3.8 (Convergence of the k-th eigenfunctions)**.** *Under the above assumptions for any* $\ell_0 > 0$ *there exists a subsequence of* ℓ *such that*

$$|u_{k,\ell} - v_\infty^k(0)w_1|_{\infty,\Omega_{\ell_0}} \to 0.$$

Proof. This follows the steps of Proposition 3.3 and Theorem 3.4. □

4. A nonreasonance case

The estimate (2.11) of Theorem 2.1 will allow us to prove a convergence result for invertible linear elliptic operators with zero order terms. More precisely, let the matrix $A = A(x)$ be given by (2.1) and satisfying (2.2), (2.3). For $\mathcal{O} = \Omega_\ell$ or ω, we denote by $H^{-1}(\mathcal{O})$ the dual space of $H_0^1(\mathcal{O})$. Also let $a_\infty \in L^\infty(\omega)$, $f \in H^{-1}(\omega)$ and, for all $\ell > 0$, $a_\ell \in L^\infty(\omega)$. We consider the problems

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), \\ -\nabla\cdot\big(A(x)\nabla u_\ell\big) = a_\ell(X_2)u_\ell + f \quad \text{in } H^{-1}(\Omega_\ell), \end{cases} \tag{4.1}$$

and

$$\begin{cases} u_\infty \in H_0^1(\omega), \\ -\nabla\cdot\big(A_{22}(X_2)\nabla u_\infty\big) = a_\infty(X_2)u_\infty + f \quad \text{in } H^{-1}(\omega). \end{cases} \tag{4.2}$$

Defining λ_ℓ^1 and μ^1 as in (2.9), (2.10), we have

Theorem 4.1. *We assume that, for some μ satisfying*

$$\mu < \mu^1, \tag{4.3}$$

there holds

$$a_\ell \le \mu, \quad a_\infty \le \mu \quad a.e \text{ in } \omega, \forall \ell > 0. \tag{4.4}$$

Then for all $\ell_0, r > 0$, there exists a constant C such that for all $\ell > 0$,

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_{\ell_0}} \le C|a_\ell - a_\infty|_{\infty,\omega} + \frac{C}{\ell^r}. \tag{4.5}$$

We first prove

Lemma 4.2. *Under the above assumptions, the problems* (4.1) *and* (4.2) *admit a unique solution.*

Proof. We only prove the lemma for (4.1). By (4.4) and (2.11), for all $v \in H_0^1(\Omega_\ell)$, it holds that

$$\begin{aligned}\int_{\Omega_\ell} a_\ell(X_2)v^2\,dx &\le \mu \int_{\Omega_\ell} v^2\,dx \le \frac{\mu}{\lambda_\ell^1}\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx \\ &\le \frac{\mu}{\mu^1}\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx.\end{aligned} \tag{4.6}$$

Hence with (4.3), (2.3),

$$\begin{aligned}\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx - \int_{\Omega_\ell} a_\ell(X_2)v^2\,dx &\ge (1-\frac{\mu}{\mu^1})\int_{\Omega_\ell} A\nabla v\cdot\nabla v\,dx \\ &\ge c_1\int_{\Omega_\ell}|\nabla v|^2\,dx\end{aligned}$$

where $c_1 > 0$. We conclude using the Lax Milgram Theorem. □

Proof of Theorem 4.1. Let v_ℓ be the solution of

$$\begin{cases} v_\ell \in H_0^1(\omega), \\ -\nabla\cdot\big(A_{22}(X_2)\nabla v_\ell\big) = a_\ell(X_2)v_\ell + f \quad \text{in } H^{-1}(\omega). \end{cases} \tag{4.7}$$

For all $\varphi \in H_0^1(\Omega_\ell)$, we clearly have

$$\int_{\Omega_\ell} A\nabla v_\ell\cdot\nabla\varphi\,dx = \int_{\Omega_\ell} a_\ell(X_2)v_\ell\varphi\,dx + \langle f,\varphi\rangle.$$

Also, the solution u_ℓ of (4.1) satisfies

$$\int_{\Omega_\ell} A\nabla u_\ell\cdot\nabla\varphi\,dx = \int_{\Omega_\ell} a_\ell(X_2)u_\ell\varphi\,dx + \langle f,\varphi\rangle.$$

Subtracting this two identities and choosing $\varphi = (u_\ell - v_\ell)\varrho^2\big(\frac{X_1}{\ell_1}\big)$ where $0 < \ell_1 < \ell$ and ϱ is a smooth function satisfying

$$\varrho = 1 \quad \text{on } \left(-\frac{1}{2},\frac{1}{2}\right)^p, \quad \varrho = 0 \quad \text{outside } (-1,1)^p, \quad |\nabla_{X_1}\varrho| \le C,$$

where C is some positive constant, we also derive using (4.6),

$$\begin{aligned}\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla\{(u_\ell - v_\ell)\varrho^2\}\,dx \\ &= \int_{\Omega_\ell} a_\ell(X_2)(u_\ell - v_\ell)^2\varrho^2\,dx \\ &\le \frac{\mu}{\mu^1}\int_{\Omega_\ell} A\nabla\{(u_\ell - v_\ell)\varrho\}\cdot\nabla\{(u_\ell - v_\ell)\varrho\}\,dx.\end{aligned}$$

(We denote, for simplicity $\varrho\big(\frac{X_1}{\ell_1}\big)$ by ϱ.) Now this latter integral equals

$$\begin{aligned}\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla(u_\ell - v_\ell)\,\varrho^2\,dx \\ + 2\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla\varrho\,(u_\ell - v_\ell)\varrho\,dx + \int_{\Omega_\ell} A\nabla\varrho\cdot\nabla\varrho\,(u_\ell - v_\ell)^2\,dx.\end{aligned}$$

Therefore, now denoting $(\nabla\varrho)\big(\frac{X_1}{\ell_1}\big)$ by $\nabla\varrho$,

$$\begin{aligned}(1-\frac{\mu}{\mu^1})\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla(u_\ell - v_\ell)\,\varrho^2\,dx \\ &\le -\frac{2}{\ell_1}\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla\varrho\,(u_\ell - v_\ell)\varrho\,dx \\ &+ \frac{2\mu}{\mu^1\ell_1}\int_{\Omega_\ell} A\nabla(u_\ell - v_\ell)\cdot\nabla\varrho\,(u_\ell - v_\ell)\varrho\,dx \\ &+ \frac{\mu}{\mu^1\ell_1^2}\int_{\Omega_\ell} A\nabla\varrho\cdot\nabla\varrho\,(u_\ell - v_\ell)^2\,dx.\end{aligned}$$

With (2.3), (2.2) and the Cauchy-Schwarz inequality, we then obtain, for a new positive constant C,

$$\begin{aligned}\int_{\Omega_\ell} |\nabla(u_\ell - v_\ell)|^2\varrho^2\,dx \le \frac{C}{\ell_1}\Big(\int_{\Omega_\ell} |\nabla(u_\ell - v_\ell)|^2\varrho^2\,dx\Big)^{1/2}\Big(\int_{\Omega_{\ell_1}} (u_\ell - v_\ell)^2\,dx\Big)^{1/2} \\ + \frac{C}{\ell_1^2}\int_{\Omega_{\ell_1}} (u_\ell - v_\ell)^2\,dx.\end{aligned}$$

At this stage, we use the following "anisotropic" Poincaré inequality: there exists a constant $C(\omega)$ depending only on ω such that, for every $v \in H^1(\Omega_\ell)$, $v = 0$ on $(-\ell_1, \ell_1)^p \times \partial\omega$,

$$\int_{\Omega_{\ell_1}} v^2\,dx \le C(\omega)\int_{\Omega_{\ell_1}} |\nabla_{X_2} v|^2\,dx$$

(see [2] or [3] for a proof). Then we get

$$\int_{\Omega_{\ell_1/2}} |\nabla(u_\ell - v_\ell)|^2\,dx \le \frac{C}{\ell_1^2}\int_{\Omega_{\ell_1}} |\nabla(u_\ell - v_\ell)|^2\,dx,$$

from which we deduce arguing as in [2] or [3],

$$\big|\big|\nabla(u_\ell - v_\ell)\big|\big|_{2,\Omega_{\ell_0}} \le \frac{C}{\ell^r}. \tag{4.8}$$

Let us now approximate u_∞ by v_ℓ. Subtracting (4.2) from (4.7), we obtain, for all $\varphi \in H_0^1(\omega)$,

$$\int_\omega A_{22}(X_2)\nabla_2(v_\ell - u_\infty)\cdot\nabla_2\varphi\, dX_2 = \int_\omega (a_\ell v_\ell - a_\infty u_\infty)\varphi\, dX_2$$
$$= \int_\omega (a_\ell - a_\infty)u_\infty\varphi + \int_\omega a_\ell(v_\ell - u_\infty)\varphi\, dX_2.$$

Choosing $\varphi = v_\ell - u_\infty$, we infer with the Cauchy-Schwarz inequality,

$$\int_\omega A_{22}(X_2)\nabla_2(v_\ell - u_\infty)\cdot\nabla_2(v_\ell - u_\infty)\, dX_2$$
$$\leq |a_\ell - a_\infty|_{\infty,\omega}|u_\infty|_{2,\omega}|v_\ell - u_\infty|_{2,\omega} + \int_\omega a_\ell(v_\ell - u_\infty)^2\, dX_2.$$

We estimate this latter integral by

$$\frac{\mu}{\mu^1}\int_\omega A_{22}(X_2)\nabla_2(v_\ell - u_\infty)\cdot\nabla_2(v_\ell - u_\infty)\, dX_2$$

(see (4.6)) and use the Poincaré inequality to obtain

$$\big||\nabla(v_\ell - u_\infty)|\big|_{2,\omega} \leq C|a_\ell - a_\infty|_{\infty,\omega},$$

which implies

$$\big||\nabla(v_\ell - u_\infty)|\big|_{2,\Omega_{\ell_0}} \leq C_1|a_\ell - a_\infty|_{\infty,\omega}. \tag{4.9}$$

The theorem follows combining (4.8) and (4.9). □

We deduce immediately from Theorem 4.1

Corollary 4.3. *Under the assumption of Theorem 4.1, assume, in addition, that*

$$a_\ell(\cdot) \equiv a_\infty(\cdot) \leq \mu < \mu^1 \quad a.e \text{ in } \omega,\ \forall \ell > 0.$$

Then for every $\ell_0, r > 0$, there exists a constant C such that for all $\ell > 0$,

$$\big||\nabla(u_\ell - u_\infty)|\big|_{2,\Omega_{\ell_0}} \leq \frac{C}{\ell^r}.$$

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