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A SIMPLE FORMULA FOR AN ANALOGUE OF CONDITIONAL WIENER INTEGRALS AND ITS APPLICATIONS

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Abstract. Let $C[0,T]$ denote the space of real-valued continuous functions on the interval $[0,T]$ and for a partition $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$, let $X_\tau : C[0,T] \to \mathbb{R}^{n+1}$ be given by $X_\tau(x) = (x(t_0), x(t_1), \cdots, x(t_n))$.

In this paper, with the conditioning function X_τ, we derive a simple formula for conditional expectations of functions defined on $C[0,T]$ which is a probability space and a generalization of Wiener space. As applications of the formula, we evaluate the conditional expectation of functions of the form

$$F_m(x) = \int_0^T (x(t))^m dt, \quad m \in \mathbb{N},$$

for $x \in C[0,T]$ and derive a translation theorem for the conditional expectation of integrable functions defined on the space $C[0,T]$.

1. Introduction and preliminaries

Let $C_0[0,T]$ be the space of continuous real-valued functions x on $[0,T]$ with $x(0) = 0$. It is well-known that the space $C_0[0,T]$ equipped with Wiener measure is a probability space. On the space, Yeh introduced an inversion formula that a conditional expectation can be found by Fourier-transform ([9]). Also, in [10, 11], he obtained very useful results including the Kac-Feynman integral equation and the conditional Cameron-Martin translation theorem using the inversion formula. But Yeh's inversion formula is very complicated in its applications when the conditioning function is vector-valued. In [6], Park and Skoug derived a simple formula for conditional Wiener integrals on $C_0[0,T]$ with the conditioning function $X : C_0[0,T] \to \mathbb{R}^n$ given by

$$X(x) = (x(t_1), \cdots, x(t_n))$$

where $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of the interval $[0,T]$. In their simple formula they expressed the conditional Wiener integral directly in terms of the ordinary Wiener integral. Using the formula, they generalized the Kac-Feynman formula and obtained a Cameron-Martin type translation theorem for conditional Wiener integrals.

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On the other hand, let $C[0,T]$ denote the space of continuous real-valued functions on the interval $[0,T]$. Im and Ryu introduced a probability measure w_φ on $(C[0,T], \mathcal{B}(C[0,T]))$ where $\mathcal{B}(C[0,T])$ denotes a Borel σ-algebra on $C[0,T]$ and φ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ([4, 7]). This measure space is a generalization of Wiener space. In [4], they introduced a translation theorem of the w_φ-integral, which corresponds to the Cameron-Martin translation theorem on Wiener space ([2]). Also, Im and Ryu evaluated the conditional w_φ-integral of various functions on $C[0,T]$ with the conditioning function $X_\tau : C[0,T] \to \mathbb{R}^{n+1}$ given by

$$X_\tau(x) = (x(t_0), x(t_1), \cdots, x(t_n)) \tag{1}$$

and derived a translation theorem of the conditional w_φ-integral when the conditioning function is $X_\tau(x) = x(t_n)$. But their processes were complicated in its proof.

In this paper, on the space $(C[0,T],\ \mathcal{B}(C[0,T]), w_\varphi)$, we derive a simple formula for the conditional w_φ-integral of functions on $C[0,T]$ with the vector-valued conditioning function X_τ given by (1). This formula expresses the conditional w_φ-integral directly in terms of the non-conditional w_φ-integral. As applications of the formula, we evaluate the conditional w_φ-integral of functions of the form

$$F_m(x) = \int_0^T (x(t))^m dt$$

for any positive integer m and using the translation theorem of w_φ-integral in [4], we derive a translation theorem for the conditional w_φ-integral of functions on $C[0,T]$ with the conditioning function X_τ.

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the set of complex numbers and that of complex numbers with positive real parts, respectively.

Now, we begin by introducing the probability space $(C[0,T], \mathcal{B}(C[0,T]), w_\varphi)$.

For a positive real T, let $C = C[0,T]$ be the space of all real-valued continuous functions on the closed interval $[0,T]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le T$, let $J_{\vec{t}} : C[0,T] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)).$$

For B_j $(j = 0, 1, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$m_\varphi\left(J_{\vec{t}}^{-1}\left(\prod_{j=0}^n B_j\right)\right) = \int_{B_0} \int_{\prod_{j=1}^n B_j} W_{n+1}(\vec{t}; u_0, u_1, \cdots, u_n) d(u_1, \cdots, u_n) d\varphi(u_0),$$

where

$$\begin{aligned} &W_{n+1}(\vec{t}; u_0, u_1, \cdots, u_n) \\ &= \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})}\right]^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}. \end{aligned} \tag{2}$$

$\mathcal{B}(C[0,T])$, the Borel σ-algebra of $C[0,T]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,T], \mathcal{B}(C[0,T]))$

such that $w_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$ ([4, 7, 8]). This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ.

By the change of variable theorem, we can easily prove the following theorem.

Theorem 1.1 ([4, Lemma 2.1])**.** *If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then we have*

$$\begin{aligned} &\int_C f(x(t_0), x(t_1), \cdots, x(t_n)) dw_\varphi(x) \\ \overset{*}{=} \quad &\int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \cdots, u_n) W_{n+1}(\vec{t}; u_0, u_1, \cdots, u_n) d(u_1, \cdots, u_n) d\varphi(u_0) \end{aligned}$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $C_0 = C_0[0, T]$ be a Wiener space with Wiener measure m_w. Then Theorem 1.1 is reduced to the well-known Wiener integration theorem if φ is a Dirac measure concentrated at 0.

Theorem 1.2 (Wiener integration theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{C}$ is a Borel measurable function, then we have*

$$\begin{aligned} &\int_{C_0} f(x(t_1), \cdots, x(t_n)) dm_w(x) \\ \overset{*}{=} \quad &\int_{\mathbb{R}^n} f(u_1, \cdots, u_n) W_{n+1}(\vec{t}; 0, u_1, \cdots, u_n) d(u_1, \cdots, u_n) \end{aligned}$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $\{e_k : k = 1, 2, \cdots\}$ be a complete orthonormal subset of $L_2[0, T]$ such that each e_k is of bounded variation. For f in $L_2[0, T]$ and x in $C[0, T]$, we let

$$(f, x) = \lim_{n \to \infty} \int_0^T \left[\sum_{k=1}^n e_k(t) \int_0^T f(s) e_k(s) ds \right] dx(t)$$

if the limit exists. (f, x) is called the Paley-Wiener-Zygmund integral of f according to x.

Let $F : C[0, T] \to \mathbb{C}$ be integrable and let X be a random vector on $C[0, T]$. Then, we have the conditional expectation $E[F|X]$ of F given X from a well-known probability theory ([5]). Further, there exists a P_X-integrable complex-valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0, T]$, where P_X is the probability distribution of X. The function ψ is called the conditional w_φ-integral of F given X and it is also denoted by $E[F|X]$.

2. A simple formula for the conditional w_φ-integral

In this section, we derive a simple formula for the conditional w_φ-integral of the functions on $C[0, T]$ with the conditioning function X_τ given by (1).

First, we define a stochastic process $X_t(x) : C[0, T] \to \mathbb{R}$ by

$$X_t(x) = x(t) \tag{1}$$

for $t \in [0, T]$. We have the following lemma which is useful to prove several results.

Lemma 2.1. *Let W_t be the standard Wiener process given by $W_t(x) = x(t)$ on $C_0[0,T]$. Then $X_t - X_s$ and $W_t - W_s$ have the same distribution if $0 \le s \le t \le T$, that is, for any Borel subset B of $\mathbb{R}$*

$$w_\varphi(X_t - X_s \in B) = m_w(W_t - W_s \in B).$$

Moreover, if $0 \le s < t \le T$, then they are normally distributed with mean 0 and variance $t - s$. In particular, if $0 \le s_1 < s_2 \le s_3 < s_4 \le T$, then $X_{s_4} - X_{s_3}$ and $X_{s_2} - X_{s_1}$ are stochastically independent on $C[0,T]$.

Proof. Suppose that $s = t$. Then for a Borel subset B of $\mathbb{R}$

$$w_\varphi(X_t - X_s \in B) = \delta_0(B) = m_w(W_t - W_s \in B)$$

where δ_0 is Dirac measure concentrated at 0.

Now suppose that $0 < s < t$. By Theorem 1.1, we have

$$\begin{aligned} w_\varphi(X_t - X_s \in B) &= \int_C \chi_B(x(t) - x(s)) dw_\varphi(x) \\ &= \left[\frac{1}{(2\pi)^2 s(t-s)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \chi_B(u_2 - u_1) \exp\left\{-\frac{(u_1 - u_0)^2}{2s}\right. \\ &\quad \left. -\frac{(u_2 - u_1)^2}{2(t-s)}\right\} d(u_1, u_2) d\varphi(u_0) \end{aligned}$$

where χ_B denotes the indicator function concentrated on B. Let $v_2 = u_2 - u_1$ and $v_1 = u_1 - u_0$. By the change of variable and Fubini's theorems, we have

$$\begin{aligned} w_\varphi(X_t - X_s \in B) &= \left[\frac{1}{(2\pi)^2 s(t-s)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_B(v_2) \exp\left\{-\frac{v_1^2}{2s} - \frac{v_2^2}{2(t-s)}\right\} \\ &\quad d\varphi(u_0) dv_1 dv_2 \\ &= \left[\frac{1}{2\pi(t-s)}\right]^{\frac{1}{2}} \int_B \exp\left\{-\frac{v_2^2}{2(t-s)}\right\} dv_2 \\ &= m_w(W_t - W_s \in B) \end{aligned}$$

since φ is a probability measure and $W_t - W_s$ is normally distributed with mean 0, variance $t - s$. If $0 = s < t$, the proof is similar to the case $0 < s < t$.

Since $W_{s_4} - W_{s_3}$ and $W_{s_2} - W_{s_1}$ are stochastically independent on $C_0[0,T]$, $X_{s_4} - X_{s_3}$ and $X_{s_2} - X_{s_1}$ are stochastically independent on $C[0,T]$. Now the proof is completed. □

Remark 2.2. It is well-known that the Wiener process W_t is the standard Brownian motion process. On the other hand, X_t need not be a Brownian motion process since $X_0(x) = x(0)$ can take arbitrary values. Note that W_t is normally distributed with mean 0 and variance t, but X_t need not be if φ is not Dirac measure δ_0. See (2.6) of [4, p. 803] and (6) of [7, p. 4930].

For a given partition $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ and for x in $C[0,T]$, define the polygonal function $[x]$ on $[0,T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})), \quad t_{j-1} \le t \le t_j, \quad j = 1, \cdots, n.$$

Similarly, for $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, define the polygonal function $[\vec{\xi}]$ on $[0,T]$ by

$$[\vec{\xi}](t) = \xi_{j-1} + \frac{t - t_j}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}), \quad t_{j-1} \le t \le t_{j-1}, \quad j = 1, \cdots, n.$$

Then both $[x]$ and $[\vec{\xi}]$ are continuous on $[0,T]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\xi}](t_j) = \xi_j$ at each t_j.

To derive a simple formula for the conditional w_φ-integral, we begin with letting for $t_{j-1} < t < t_j$

$$\alpha_j = \frac{t_j - t}{t_j - t_{j-1}}, \quad \beta_j = \frac{t - t_{j-1}}{t_j - t_{j-1}}, \quad \Gamma_j(t) = \frac{t_j - t_{j-1}}{(t_j - t)(t - t_{j-1})}, \tag{2}$$

$$W_j(t,x) = x(t) - [x](t) \text{ for } x \in C_0[0,T] \tag{3}$$

and

$$X_j(t,x) = x(t) - [x](t) \text{ for } x \in C[0,T] \tag{4}$$

for each $j = 1, \cdots, n$.

The following lemma is useful to evaluate several conditional w_φ-integrals.

Lemma 2.3. *Let $t \in (t_{j-1}, t_j)$ for some $j \in \{1, \cdots, n\}$ and $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then*

$$\int_{C_0} f(W_j(t,x))dm_w(x) \stackrel{*}{=} \left[\frac{\Gamma_j(t)}{2\pi}\right]^{\frac{1}{2}} \int_{\mathbb{R}} f(u) \exp\left\{-\frac{\Gamma_j(t)}{2}u^2\right\}du$$

where $\Gamma_j(t)$ is given by (2). *In particular, $W_j(t,\cdot)$ is normally distributed with mean* 0 *and variance $\frac{1}{\Gamma_j(t)}$.*

Proof. For convenience, let

$$\Gamma = \left[\frac{1}{(2\pi)^2(t_j - t)(t - t_{j-1})}\right]^{\frac{1}{2}}.$$

Note that

$$\begin{aligned} W_j(t,x) &= x(t) - x(t_{j-1}) - \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t) + x(t) - x(t_{j-1})) \\ &= \alpha_j(x(t) - x(t_{j-1})) - \beta_j(x(t_j) - x(t)) \end{aligned}$$

where α_j and β_j are given by (2). Now, for $j = 2, \cdots, n$, we have

$$\begin{aligned} \int_{C_0} f(W_j(t,x))dm_w(x) &\stackrel{*}{=} \int_{\mathbb{R}^3} f(\alpha_j(u_2 - u_1) - \beta_j(u_3 - u_2)) \\ &\quad \times W_4((0, t_{j-1}, t, t_j); 0, u_1, u_2, u_3)d(u_1, u_2, u_3) \end{aligned}$$

by Theorem 1.2 where W_4 is given by (2). Let

$$v_2 = \alpha_j(u_2 - u_1)$$

and

$$v_3 = -\beta_j(u_3 - u_2).$$

Then we have by the change of variable theorem

$$\begin{aligned}
&\int_{C_0} f(W_j(t,x))dm_w(x) \\
\overset{*}{=}\ & \frac{\Gamma}{\alpha_j\beta_j}\left(\frac{1}{2\pi t_{j-1}}\right)^{\frac{1}{2}}\int_{\mathbb{R}^3} f(v_2+v_3)\exp\Big\{-\frac{1}{2t_{j-1}}u_1^2-\frac{1}{2\alpha_j^2(t-t_{j-1})}v_2^2 \\
& -\frac{1}{2\beta_j^2(t_j-t)}v_3^2\Big\}d(u_1,v_2,v_3) \\
=\ & \frac{\Gamma}{\alpha_j\beta_j}\int_{\mathbb{R}^2} f(v_2+v_3)\exp\Big\{-\frac{1}{2\alpha_j^2(t-t_{j-1})}v_2^2-\frac{1}{2\beta_j^2(t_j-t)}v_3^2\Big\}d(v_2,v_3).
\end{aligned}$$

Let $u=v_2+v_3$. Note that $\alpha_j^2(t-t_{j-1})+\beta_j^2(t_j-t)=\frac{1}{\Gamma_j(t)}$ and hence

$$1-\beta_j^2\Gamma_j(t)(t_j-t)=\alpha_j^2\Gamma_j(t)(t-t_{j-1}).$$

Again, by the change of variable theorem we have

$$\begin{aligned}
&\int_{C_0} f(W_j(t,x))dm_w(x) \\
\overset{*}{=}\ & \frac{\Gamma}{\alpha_j\beta_j}\int_{\mathbb{R}^2} f(u)\exp\Big\{-\frac{1}{2\alpha_j^2(t-t_{j-1})}(u-v_3)^2-\frac{1}{2\beta_j^2(t_j-t)}v_3^2\Big\}d(u,v_3) \\
=\ & \frac{\Gamma}{\alpha_j\beta_j}\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)\exp\Big\{-\frac{1}{2\alpha_j^2(t-t_{j-1})}u^2+\frac{\beta_j^2\Gamma_j(t)(t_j-t)}{2\alpha_j^2(t-t_{j-1})}u^2 \\
& -\frac{1}{2\alpha_j^2\beta_j^2\Gamma_j(t)(t_j-t)(t-t_{j-1})}[v_3^2-2\beta_j^2\Gamma_j(t)(t_j-t)uv_3+\beta_j^4(\Gamma_j(t))^2 \\
& \times(t_j-t)^2u^2]\Big\}dv_3du \\
=\ & [2\pi\Gamma^2\Gamma_j(t)(t_j-t)(t-t_{j-1})]^{\frac{1}{2}}\int_{\mathbb{R}} f(u)\exp\Big\{-\frac{\alpha_j^2\Gamma_j(t)(t-t_{j-1})}{2\alpha_j^2(t-t_{j-1})}u^2\Big\}du \\
=\ & \left[\frac{\Gamma_j(t)}{2\pi}\right]^{\frac{1}{2}}\int_{\mathbb{R}} f(u)\exp\Big\{-\frac{\Gamma_j(t)}{2}u^2\Big\}du
\end{aligned}$$

which is the desired result. If $j=1$, then the proof is similar to the case $j\geq 2$. □

The following theorem gives an interesting observation for the process $x(t)-[x](t)$ on $C[0,T]$. In fact, $x(t)-[x](t)$ is a Brownian bridge motion on each subinterval.

Theorem 2.4. *For each $j=1,\cdots,n$, let W_j and X_j be given by* (3) *and* (4), *respectively. Then X_j is a Brownian bridge motion process on $[t_{j-1},t_j]$. Moreover for $t\in(t_{j-1},t_j)$, both $W_j(t,\cdot)$ and $X_j(t,\cdot)$ are normally distributed with mean* 0 *and variance $\frac{1}{\Gamma_j(t)}$ where $\Gamma_j(t)$ is given by* (2).

Proof. Note that W_j is a Brownian bridge motion process ([6, Theorem 1]). By Lemma 2.1, W_j and X_j have the same distribution so that X_j is a Brownian bridge motion process on the subinterval $[t_{j-1},t_j]$ since $X_j(t_{j-1},x)=x(t_{j-1})-[x](t_{j-1})=0=x(t_j)-[x](t_j)=X_j(t_j,x)$ for $x\in C[0,T]$.

The second result immediately follows from Lemmas 2.1 and 2.3. □

By Theorem 2.4 and [6, Corollary 1], we have the following corollary.

Corollary 2.5. *The processes* $\{X_j(t,\cdot) : t_{j-1} \le t \le t_j\}$, *where* $j = 1, \cdots, n$ *are stochastically independent.*

Theorem 2.6. *Let* $Y_\tau : C[0,T] \to \mathbb{R}^{n+1}$ *be given by*

$$Y_\tau(x) = (x(t_0), x(t_1) - x(t_0), \cdots, x(t_n) - x(t_0)).$$

Then the process $\{x(t) - [x](t) : 0 \le t \le T\}$ *and* Y_τ *are independent.*

Proof. Define $Y : [0,T] \times C[0,T] \to \mathbb{R}$ by

$$Y(s,x) = X_s(x) - X_0(x) = x(s) - x(t_0)$$

where X_s and X_0 are given by (1) with replacing t by s and 0, respectively.

Fix $j \in \{1, \cdots, n\}$ and let X_j be given by (4). By Theorem 2.4, for $t_{j-1} < t < t_j$, $X_j(t,\cdot)$ is normally distributed with mean 0 and so is $Y(s,\cdot)$ for $s \in (0, t_{j-1}] \cup [t_j, T]$ by Lemma 2.1. Moreover, it is not difficult to show that the distributions of $Y(0,\cdot)$, $X_j(t_{j-1},\cdot)$ and $X_j(t_j,\cdot)$ are the Dirac measure δ_0 concentrated at 0.

First, we show that $X_j(t,\cdot)$ and $Y(s,\cdot)$ are stochastically independent for $t \in [t_{j-1}, t_j]$ and $s \in [0, t_{j-1}] \cup [t_j, T]$, and so are $X_j(t,\cdot)$ and X_0. To prove the assertions, let B_1 and B_2 be Borel subsets of $\mathbb{R}$.

Suppose that $t = t_{j-1}$ or $t = t_j$. Then we have for $s \in [0, t_{j-1}] \cup [t_j, T]$

$$\begin{aligned} w_\varphi(X_j(t,\cdot) \in B_1, Y(s,\cdot) \in B_2) &= \delta_0(B_1) w_\varphi(Y(s,\cdot) \in B_2) \\ &= w_\varphi(X_j(t,\cdot) \in B_1) w_\varphi(Y(s,\cdot) \in B_2) \end{aligned}$$

which shows the independence of $X_j(t,\cdot)$ and $Y(s,\cdot)$. If we replace $Y(s,\cdot)$ by X_0, then we have the independence of $X_j(t,\cdot)$ and X_0.

Similarly, we can prove the independence of $X_j(t,\cdot)$ and $Y(0,\cdot)$ for $t_{j-1} < t < t_j$.

Now suppose that $t_{j-1} < t < t_j$. If $s \in (0, t_{j-1}]$, then we have

$$\begin{aligned} \mathrm{Cov}(X_j(t,\cdot), Y(s,\cdot)) &= E[X_j(t,\cdot) Y(s,\cdot)] \\ &= E[(\alpha_j(X_t - X_{t_{j-1}}) - \beta_j(X_{t_j} - X_t))(X_s - X_0)] = 0 \end{aligned}$$

by Lemma 2.1 where α_j and β_j are given by (2). If $s \in [t_j, T]$, then we also have

$$\begin{aligned} \mathrm{Cov}(X_j(t,\cdot), Y(s,\cdot)) &= E[X_j(t,\cdot) Y(s,\cdot)] \\ &= E[\alpha_j(X_t - X_{t_{j-1}})^2 - \beta_j(X_{t_j} - X_t)^2] \\ &= \alpha_j(t - t_{j-1}) - \beta_j(t_j - t) = 0 \end{aligned}$$

by Lemma 2.1, which shows the independence of $X_j(t,\cdot)$ and $Y(s,\cdot)$. Finally, let

$$\Delta_t = \left[\frac{1}{(2\pi)^3 (t_j - t)(t - t_{j-1}) t_{j-1}} \right]^{\frac{1}{2}},$$

if $j \in \{2, \cdots, n\}$. Then we have by Theorem 1.1

$$\begin{aligned} & w_\varphi(X_j(t,\cdot) \in B_1, X_0 \in B_2) \\ = & \int_C \chi_{B_1}(X_j(t,x)) \chi_{B_2}(X_0(x)) dw_\varphi(x) \\ = & \Delta_t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \chi_{B_1}(\alpha_j(u_2 - u_1) - \beta_j(u_3 - u_2)) \chi_{B_2}(u_0) \exp\Big\{ -\frac{(u_1 - u_0)^2}{2t_{j-1}} \\ & -\frac{(u_2 - u_1)^2}{2(t - t_{j-1})} - \frac{(u_3 - u_2)^2}{2(t_j - t)} \Big\} d(u_1, u_2, u_3) d\varphi(u_0). \end{aligned}$$

For $i = 1, 2, 3$, let $v_i = u_i - u_{i-1}$. By the change of variable theorem and Fubini's theorem, we have

$$\begin{aligned}
& w_\varphi(X_j(t,\cdot) \in B_1, X_0 \in B_2) \\
= \; & \Delta_t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \chi_{B_1}(\alpha_j v_2 - \beta_j v_3) \chi_{B_2}(u_0) \exp\left\{ -\frac{v_1^2}{2t_{j-1}} - \frac{v_2^2}{2(t-t_{j-1})} - \frac{v_3^2}{2(t_j - t)} \right\} \\
& d(v_1, v_2, v_3) d\varphi(u_0) \\
= \; & \left[\Delta_t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \chi_{B_1}(\alpha_j v_2 - \beta_j v_3) \exp\left\{ -\frac{v_1^2}{2t_{j-1}} - \frac{v_2^2}{2(t-t_{j-1})} - \frac{v_3^2}{2(t_j - t)} \right\} \right. \\
& \left. d(v_1, v_2, v_3) d\varphi(u_0) \right] \left[\int_{\mathbb{R}} \chi_{B_2}(u_0) d\varphi(u_0) \right] \\
= \; & \left[\Delta_t \int_{\mathbb{R}} \int_{\mathbb{R}^3} \chi_{B_1}(\alpha_j(u_2 - u_1) - \beta_j(u_3 - u_2)) \exp\left\{ -\frac{(u_1 - u_0)^2}{2t_{j-1}} - \frac{(u_2 - u_1)^2}{2(t-t_{j-1})} \right. \right. \\
& \left. \left. - \frac{(u_3 - u_2)^2}{2(t_j - t)} \right\} d(u_1, u_2, u_3) d\varphi(u_0) \right] \left[\int_{\mathbb{R}} \chi_{B_2}(u_0) d\varphi(u_0) \right] \\
= \; & w_\varphi(X_j(t,\cdot) \in B_1) w_\varphi(X_0 \in B_2)
\end{aligned}$$

which shows the independence of $X_j(t,\cdot)$ and X_0. If $j = 1$, then we can show that $X_j(t,\cdot)$ and X_0 are independent by a similar method.

Now, for each $j = 1, \cdots, n$, the process $\{X_j(t,\cdot) : t_{j-1} \le t \le t_j\}$ is independent of Y_τ, which completes the proof. □

Remark 2.7. Since for $x \in C[0,T]$ and $t_{j-1} \le t \le t_j$

$$\begin{aligned}
X_j(t,x) &= \alpha_j(x(t) - x(t_{j-1})) - \beta_j(x(t_j) - x(t)) \\
&= \alpha_j(x(t) - x(t_0) - (x(t_{j-1}) - x(t_0))) \\
&\quad - \beta_j(x(t_j) - x(t_0) - (x(t) - x(t_0))),
\end{aligned}$$

we can also obtain Corollary 2.5 from Theorem 2.6.

Theorem 2.8. *Let $X_\tau : C[0,T] \to \mathbb{R}^{n+1}$ be given by*

$$X_\tau(x) = (x(t_0), x(t_1), \cdots, x(t_n)). \tag{5}$$

Then the process $\{x(t) - [x](t) : 0 \le t \le T\}$ and X_τ are independent.

Proof. Take $j \in \{1, \cdots, n\}$. For $s \in [0, t_{j-1}] \cup [t_j, T]$, we have

$$X_s(x) = x(s) = (x(s) - x(t_0)) + x(t_0).$$

Note that $X_j(t,x) = x(t) - [x](t), t_{j-1} \le t \le t_j$ and $(x(s) - x(t_0), x(t_0))$ are independent by Theorem 2.6. Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(u_1, u_2) = u_1 + u_2$. Since g is Borel measurable and $x(s) = g(x(s) - x(t_0), x(t_0))$, $X_j(t,\cdot)$ and X_s are independent. Now the theorem follows. □

Applying the same method used in the proof of Theorem 2 in [6, p. 383] with Problem 4 of [1, p. 216], we have the following theorem from Theorem 2.8.

Theorem 2.9. *Let $F : C[0,T] \to \mathbb{C}$ be integrable and X_τ be given by* (5) *of Theorem* 2.8. *Then for a Borel subset B of $\mathbb{R}^{n+1}$ we have*

$$\int_{X_\tau^{-1}(B)} F(x) dw_\varphi(x) = \int_B E[F(x - [x] + [\vec{\xi}])] dP_{X_\tau}(\vec{\xi}) \tag{6}$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

Remark 2.10. We emphasize that both $[x](t_0) = x(t_0)$ and $[\vec{\xi}](t_0) = \xi_0$ need not be 0. Applying the same method used in the proof of Theorem 2 in [6, p. 383], we can prove (6) using Theorem 2.6 directly.

For a function $F : C[0,T] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$ and $X_\tau^\lambda(x) = X_\tau(\lambda^{-\frac{1}{2}}x)$. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ-integral and (6), we have

$$E[F^\lambda|X_\tau^\lambda](\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] \tag{7}$$

for $P_{X_\tau^\lambda}$-a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, where $P_{X_\tau^\lambda}$ is the probability distribution of X_τ^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. If $E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$ has the analytic extension $J_\lambda^*(F)(\vec{\xi})$ on $\mathbb{C}_+$ as a function of λ, then it is denoted by

$$E^{anw_\lambda}[F|X_\tau](\vec{\xi}) = J_\lambda^*(F)(\vec{\xi})$$

for $\vec{\xi} \in \mathbb{R}^{n+1}$. Moreover, if for a non-zero real q, $E^{anw_\lambda}[F|X_\tau](\vec{\xi})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is denoted by

$$E^{anf_q}[F|X_\tau](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_\tau](\vec{\xi}).$$

3. Evaluations of conditional w_φ-integrals

Throughout this section, let X_τ be given by (5) and P_{X_τ} denote the probability distribution of X_τ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Now, we evaluate conditional w_φ-integrals of functions on $C[0,T]$ as applications of (6) in Theorem 2.9.

Theorem 3.1. *Let $F_m(x) = \int_0^T (x(t))^m dt$ $(m \in \mathbb{N})$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then F_m is w_φ-integrable. Moreover, $E[F_m|X_\tau](\vec{\xi})$ exists for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$ and it is given by*

$$E[F_m|X_\tau](\vec{\xi}) = \sum_{j=1}^{n} \sum_{k=0}^{M} \frac{m!}{2^k k!(m-2k)!} \int_{t_{j-1}}^{t_j} ([\vec{\xi}](t))^{m-2k} \left[\frac{1}{\Gamma_j(t)}\right]^k dt$$

where $\Gamma_j(t)$ is given by (2) *and $M = \frac{m}{2}$ if m is even and $M = \frac{m-1}{2}$ if m is odd.*

Proof. First, we show that F_m is w_φ-integrable. Indeed, we have

$$\begin{aligned}
\int_C |F_m(x)| dw_\varphi(x) &\le \int_0^T \int_C |x(t)|^m dw_\varphi(x) dt \\
&= \int_{(0,T]} \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |u_1|^m \exp\left\{-\frac{(u_1 - u_0)^2}{2t}\right\} du_1 d\varphi(u_0) dt \\
&= \int_{(0,T]} \left(\frac{1}{2\pi t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |v + u_0|^m \exp\left\{-\frac{v^2}{2t}\right\} dv d\varphi(u_0) dt
\end{aligned}$$

by the change of variable theorem and Theorem 1.1, where $v = u_1 - u_0$. By the binomial expansion, we have

$$\begin{aligned}
&\int_C |F_m(x)| dw_\varphi(x) \\
\leq\ & \int_{(0,T]} \left[\sum_{l=0}^{m} \binom{m}{l} \left(\frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} |v|^l |u_0|^{m-l} \exp\left\{ -\frac{v^2}{2t} \right\} dv d\varphi(u_0) \right] dt \\
=\ & 2 \int_{(0,T]} \left[\sum_{l=0}^{m} \binom{m}{l} \left[\left(\frac{1}{2\pi t} \right)^{\frac{1}{2}} \int_0^\infty v^l \exp\left\{ -\frac{v^2}{2t} \right\} dv \right] \left[\int_{\mathbb{R}} |u_0|^{m-l} d\varphi(u_0) \right] \right] dt.
\end{aligned}$$

Now, for $l = 0, 1, \cdots, m$, we have

$$\begin{aligned}
\int_{\mathbb{R}} |u_0|^{m-l} d\varphi(u_0) &= \int_{|u_0| \leq 1} |u_0|^{m-l} d\varphi(u_0) + \int_{|u_0|>1} |u_0|^{m-l} d\varphi(u_0) \\
&\leq \int_{|u_0| \leq 1} d\varphi(u_0) + \int_{|u_0|>1} |u_0|^m d\varphi(u_0) \\
&\leq \int_{|u_0| \leq 1} d\varphi(u_0) + \int_{\mathbb{R}} |u_0|^m d\varphi(u_0) \equiv K < \infty
\end{aligned}$$

by assumption. Let $u = \frac{v^2}{2t}$. Then $v = (2tu)^{\frac{1}{2}}$ and hence by the change of variable theorem it follows that

$$\begin{aligned}
\int_C |F_m(x)| dw_\varphi(x) &\leq K \left(\frac{1}{\pi} \right)^{\frac{1}{2}} \int_{(0,T]} \left[\sum_{l=0}^{m} \binom{m}{l} (2t)^{\frac{l}{2}} \int_0^\infty u^{\frac{l+1}{2}-1} \exp\{-u\} du \right] dt \\
&= K \left(\frac{1}{\pi} \right)^{\frac{1}{2}} \sum_{l=0}^{m} \binom{m}{l} \frac{(2T)^{\frac{l}{2}+1}}{l+2} \Gamma\left(\frac{l+1}{2} \right) < \infty
\end{aligned}$$

where Γ denotes the gamma function. Now, by Theorem 2.9, we have for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned}
&E[F_m | X_\tau](\vec{\xi}) \\
&\quad = \int_0^T \int_C (x(t) - [x](t) + [\vec{\xi}](t))^m dw_\varphi(x) dt \\
&\quad = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_C (X_j(t,x) + [\vec{\xi}](t))^m dw_\varphi(x) dt \\
&\quad = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_C \left[\sum_{l=0}^{m} \binom{m}{l} (X_j(t,x))^l ([\vec{\xi}](t))^{m-l} \right] dw_\varphi(x) dt \\
&\quad = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\sum_{l=0}^{m} \binom{m}{l} ([\vec{\xi}](t))^{m-l} \left[\frac{\Gamma_j(t)}{2\pi} \right]^{\frac{1}{2}} \int_{\mathbb{R}} u^l \exp\left\{ -\frac{\Gamma_j(t)}{2} u^2 \right\} du \right] dt
\end{aligned}$$

by Lemma 2.3 and Theorem 2.4, where $\Gamma_j(t)$ and X_j are given by (2) and (4), respectively. Let $M = \frac{m}{2}$ if m is even and $M = \frac{m-1}{2}$ if m is odd. Then we have

$$\begin{aligned} E[F_m|X_\tau](\vec{\xi}) &= 2\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left[\sum_{k=0}^{M}\binom{m}{2k}([\vec{\xi}](t))^{m-2k}\left[\frac{\Gamma_j(t)}{2\pi}\right]^{\frac{1}{2}}\int_0^\infty u^{2k}\right. \\ &\quad \left.\times\exp\left\{-\frac{\Gamma_j(t)}{2}u^2\right\}du\right]dt \\ &= \left(\frac{1}{\pi}\right)^{\frac{1}{2}}\sum_{j=1}^{n}\sum_{k=0}^{M}\binom{m}{2k}\Gamma\left(\frac{2k+1}{2}\right)\int_{t_{j-1}}^{t_j}([\vec{\xi}](t))^{m-2k}\left[\frac{2}{\Gamma_j(t)}\right]^k dt \\ &= \sum_{j=1}^{n}\sum_{k=0}^{M}\frac{m!}{2^k k!(m-2k)!}\int_{t_{j-1}}^{t_j}([\vec{\xi}](t))^{m-2k}\left[\frac{1}{\Gamma_j(t)}\right]^k dt \end{aligned}$$

which is the desired result. □

Remark 3.2. The integrand in the result of Theorem 3.1 is a polynomial of degree m with respect to t so that the integral can always be evaluated.

In the following example, we evaluate $E[F_m|X_\tau], m = 1, 2, 3$, as applications of Theorem 3.1.

Example 3.3. Let $F_1(x) = \int_0^T x(t)dt$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|d\varphi(u) < \infty$. Then for P_{X_τ}-a.e. $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} E[F_1|X_\tau](\vec{\xi}) &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left[\frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}}(t - t_{j-1}) + \xi_{j-1}\right]dt \\ &= \frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(\xi_j + \xi_{j-1}) \end{aligned}$$

which can also be obtained by an application of Corollary 4.5 in [4].

Let $F_2(x) = \int_0^T (x(t))^2 dt$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$. Then for P_{X_τ}-a.e. $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} E[F_2|X_\tau](\vec{\xi}) &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\left[\frac{(t_j - t)(t - t_{j-1})}{t_j - t_{j-1}} + \left[\frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}}(t - t_{j-1}) + \xi_{j-1}\right]^2\right]dt \\ &= \frac{1}{6}\sum_{j=1}^{n}(t_j - t_{j-1})(t_j - t_{j-1} + 2\xi_j^2 + 2\xi_j\xi_{j-1} + 2\xi_{j-1}^2) \end{aligned}$$

which is the result given by Corollary 4.10 of [4].

Let $F_3(x) = \int_0^T (x(t))^3 dt$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^3 d\varphi(u) < \infty$. Then for P_{X_τ}-a.e. $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} E[F_3|X_\tau](\vec{\xi}) &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left[\frac{3(t_j - t)(t - t_{j-1})}{t_j - t_{j-1}} \left[\frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} (t - t_{j-1}) + \xi_{j-1} \right] \right. \\ &\quad \left. + \left[\frac{\xi_j - \xi_{j-1}}{t_j - t_{j-1}} (t - t_{j-1}) + \xi_{j-1} \right]^3 \right] dt \\ &= \frac{1}{2} \sum_{j=1}^n \xi_{j-1} (t_j - t_{j-1})^2 + \frac{1}{4} \sum_{j=1}^n (t_j - t_{j-1})(\xi_j^3 + \xi_j^2 \xi_{j-1} + \xi_j \xi_{j-1}^2 \\ &\quad + \xi_{j-1}^3) + 3 \sum_{j=1}^n \frac{\xi_j - \xi_{j-1}}{(t_j - t_{j-1})^2} \int_{t_{j-1}}^{t_j} (t_j - t)(t - t_{j-1})^2 dt \\ &= \frac{1}{4} \sum_{j=1}^n (t_j - t_{j-1}) [(t_j - t_{j-1})(\xi_j + \xi_{j-1}) + \xi_j^3 + \xi_j^2 \xi_{j-1} + \xi_j \xi_{j-1}^2 \\ &\quad + \xi_{j-1}^3] \end{aligned}$$

from the integration by parts formula.

Corollary 3.4. *Under the assumptions and notations given as in Theorem* 3.1, $E^{anw_\lambda}[F_m|X_\tau](\vec{\xi})$ *exists for* $\lambda \in \mathbb{C}_+$ *and for* $\vec{\xi} \in \mathbb{R}^{n+1}$. *Moreover, for a non-zero real* q, $E^{anf_q}[F_m|X_\tau](\vec{\xi})$ *exists for* $\vec{\xi} \in \mathbb{R}^{n+1}$ *and it is given by*

$$E^{anf_q}[F_m|X_\tau](\vec{\xi}) = \sum_{j=1}^n \sum_{k=0}^M \left(\frac{i}{q} \right)^k \frac{m!}{2^k k!(m-2k)!} \int_{t_{j-1}}^{t_j} ([\vec{\xi}](t))^{m-2k} \left[\frac{1}{\Gamma_j(t)} \right]^k dt.$$

Proof. For $\lambda > 0$, we have

$$E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])] = \sum_{j=1}^n \sum_{k=0}^M \frac{1}{\lambda^k} \frac{m!}{2^k k!(m-2k)!} \int_{t_{j-1}}^{t_j} ([\vec{\xi}](t))^{m-2k} \left[\frac{1}{\Gamma_j(t)} \right]^k dt$$

by an application of Theorem 3.1, where $M = \frac{m}{2}$ if m is even and $M = \frac{m-1}{2}$ if m is odd. Now, we have the corollary immediately. □

Now, we generalize the result in Example 4 of [6], which is also considered by Chang and Chang in [3].

Theorem 3.5. *Let* $F(x) = \exp\{\int_0^T x(t)dt\}$ *for* $x \in C[0,T]$ *and suppose that* F *is* w_φ*-integrable. Then, for a.e.* $y \in C[0,T]$, *we have*

$$\lim_{\|\tau\| \to 0} E[F|X_\tau](X_\tau(y)) = F(y).$$

Proof. For a.e. $y \in C[0,T]$, we have

$$\begin{aligned} E[F|X_\tau](X_\tau(y)) &= \int_C F(x - [x] + [y]) dw_\varphi(x) \\ &= \exp\left\{ \frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1})(y(t_{j-1}) + y(t_j)) \right\} \\ &\quad \times \int_C \exp\left\{ \int_0^T (x(t) - [x](t)) dt \right\} dw_\varphi(x) \end{aligned}$$

by Theorem 2.9 and the same process used in Example 3.3. Letting $\|\tau\| \to 0$, we have

$$\begin{aligned}\lim_{\|\tau\|\to 0} E[F|X_\tau](X_\tau(y)) &= \lim_{\|\tau\|\to 0} \exp\left\{\frac{1}{2}\sum_{j=1}^{n}(t_j - t_{j-1})(y(t_{j-1}) + y(t_j))\right\} \\ &\quad \times \int_C \exp\left\{\int_0^T (x(t) - [x](t))dt\right\} dw_\varphi(x) = F(y)\end{aligned}$$

because $\lim_{\|\tau\|\to 0}(x(t) - [x](t)) = 0$ for $x \in C[0,T]$. □

4. A translation theorem for the conditional w_φ-integral

In this section, we derive a translation theorem for the conditional w_φ-integral, which is a generalization of [6, Theorem 4]. To derive the theorem, we need a translation theorem for the w_φ-integral on $C[0,T]$.

Theorem 4.1 ([4, Theorem 3.1])**.** *Let h be in $C[0,T]$ and of bounded variation. Let $\alpha \in \mathbb{R}$ and let $x_0(t) = \int_0^t h(s)ds + \alpha$ for $0 \le t \le T$ and let φ_α be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B) = \varphi(B + \alpha)$ for $B \in \mathcal{B}(\mathbb{R})$. Moreover, let F be a measurable function on $C[0,T]$. Then, $F(x + x_0)$ is also measurable on $C[0,T]$ and we have*

$$\int_C F(x)dw_\varphi(x) \overset{*}{=} \exp\left\{-\frac{1}{2}\|h\|_2^2\right\} \int_C F(x + x_0)\exp\{-(h,x)\}dw_{\varphi_\alpha}(x),$$

where $\overset{*}{=}$ means that if one side exists, then both sides exist and they are equal.

Let $E_{w_\varphi}[F|X_\tau]$ and $E_{w_{\varphi_\alpha}}[F|X_\tau]$ denote the conditional w_φ-integral and the conditional w_{φ_α}-integral of F given X_τ, respectively. In the following theorem we derive a translation theorem for the conditional w_φ-integral using Theorem 4.1.

Theorem 4.2. *Let x_0 and φ_α be given as in Theorem 4.1. Moreover, let F be defined and w_φ-integrable on $C[0,T]$. Then we have for P_{X_τ}-a.e. $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$*

$$\begin{aligned}E_{w_\varphi}[F|X_\tau](\vec{\xi}) &= \exp\left\{\sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}\left[(\xi_j - \xi_{j-1})\right.\right. \\ &\quad \left.\left. -\frac{1}{2}(x_0(t_j) - x_0(t_{j-1}))\right]\right\} E_{w_{\varphi_\alpha}}[F(x_0 + \cdot)J|X_\tau](\vec{\xi} - \vec{x}_0)\end{aligned}$$

where $J(x) = \exp\{-\frac{1}{2}\|h\|_2^2\}\exp\{-(h,x)\}$ for $x \in C[0,T]$ and $\vec{x}_0 = (x_0(t_0), x_0(t_1), \cdots, x_0(t_n))$.

Proof. By Theorems 2.9 and 4.1, we have for P_{X_τ}-a.e. $\vec{\xi} = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned}E_{w_\varphi}[F|X_\tau](\vec{\xi}) &= \int_C F(x - [x] + [\vec{\xi}])dw_\varphi(x) \\ &= \int_C F(x_0 + x - [x] + [\vec{\xi} - \vec{x}_0])J(x)dw_{\varphi_\alpha}(x).\end{aligned}$$

But we have

$$\exp\{-(h,x)\} = \exp\{-(h, x - [x] + [\vec{\xi} - \vec{x}_0]) - (h, [x] - [\vec{\xi} - \vec{x}_0])\}$$

and

$$\begin{aligned}
& \exp\{(h,[\vec{\xi}-\vec{x}_0])\} \\
= \; & \exp\Big\{\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} h(t)d[\vec{\xi}-\vec{x}_0](t)\Big\} \\
= \; & \exp\Big\{\sum_{j=1}^{n}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}((\xi_j-\xi_{j-1})-(x_0(t_j)-x_0(t_{j-1})))\Big\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
& E_{w_\varphi}[F|X_\tau](\vec{\xi}) \\
= \; & \int_C F(x_0+x-[x]+[\vec{\xi}-\vec{x}_0])J(x-[x]+[\vec{\xi}-\vec{x}_0])\exp\{-(h,[x])\} \\
& \times\exp\Big\{\sum_{j=1}^{n}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}((\xi_j-\xi_{j-1})-(x_0(t_j)-x_0(t_{j-1})))\Big\}dw_{\varphi_\alpha}(x) \\
= \; & E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_\tau](\vec{\xi}-\vec{x}_0)\exp\Big\{\sum_{j=1}^{n}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}((\xi_j-\xi_{j-1}) \\
& -(x_0(t_j)-x_0(t_{j-1})))\Big\}\int_C \exp\{-(h,[x])\}dw_{\varphi_\alpha}(x)
\end{aligned}$$

since $x-[x]$ and $[x]$ are independent by Theorem 2.8. Again, by independence (Lemma 2.1), we have

$$\begin{aligned}
& \int_C \exp\{-(h,[x])\}dw_{\varphi_\alpha}(x) \\
= \; & \int_C \exp\Big\{-\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} h(t)d[x](t)\Big\}dw_{\varphi_\alpha}(x) \\
= \; & \int_C \exp\Big\{-\sum_{j=1}^{n}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}(x(t_j)-x(t_{j-1}))\Big\}dw_{\varphi_\alpha}(x) \\
= \; & \prod_{j=1}^{n}\Big[\Big[\frac{1}{2\pi(t_j-t_{j-1})}\Big]^{\frac{1}{2}}\int_{\mathbb{R}}\exp\Big\{-\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}v_j-\frac{1}{2(t_j-t_{j-1})}v_j^2\Big\}dv_j\Big] \\
= \; & \exp\Big\{\sum_{j=1}^{n}\frac{(x_0(t_j)-x_0(t_{j-1}))^2}{2(t_j-t_{j-1})}\Big\}
\end{aligned}$$

so that we have

$$\begin{aligned}
E_{w_\varphi}[F|X_\tau](\vec{\xi}) \; = \; & \exp\Big\{\sum_{j=1}^{n}\frac{x_0(t_j)-x_0(t_{j-1})}{t_j-t_{j-1}}\Big[(\xi_j-\xi_{j-1}) \\
& -\frac{1}{2}(x_0(t_j)-x_0(t_{j-1}))\Big]\Big\}E_{w_{\varphi_\alpha}}[F(x_0+\cdot)J|X_\tau](\vec{\xi}-\vec{x}_0)
\end{aligned}$$

which is the desired result. □

Letting $\alpha = 0$ in Theorem 4.2, we have the following corollary from the theorem since $\varphi_\alpha = \varphi$.

Corollary 4.3. *Under the assumptions and notations given as in Theorem* 4.2 *with one exception* $\alpha = 0$*, we have* $x_0(t) = \int_0^t h(s)ds$ *for* $t \in [0,T]$ *and*

$$\begin{aligned} E_{w_\varphi}[F|X_\tau](\vec{\xi}) &= \exp\Bigg\{\sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}}\Bigg[(\xi_j - \xi_{j-1}) \\ &\quad -\frac{1}{2}(x_0(t_j) - x_0(t_{j-1}))\Bigg]\Bigg\} E_{w_\varphi}[F(x_0 + \cdot)J|X_\tau](\vec{\xi} - \vec{x}_0). \end{aligned}$$

For notational convenience, let E_{m_w} denote the (conditional) Wiener integral on $C_0[0,T]$.

Under the assumption that V is a non-negative continuous function on $\mathbb{R}$ satisfying the condition

$$\int_{\mathbb{R}} V(\xi_1) \exp\Big\{-\frac{\xi_1^2}{2T}\Big\} d\xi_1 < \infty$$

for every $T > 0$, we have for a.e. $\xi_1 \in \mathbb{R}$

$$\begin{aligned} &E_{w_\varphi}\Big[\exp\Big\{-\int_0^T V(x(s))ds\Big\}\Big| x(t_0) = 0, x(T) = \xi_1\Big] \\ &= E_{m_w}\Big[\exp\Big\{-\int_0^T V(x(s))ds\Big\}\Big| x(T) = \xi_1\Big] \end{aligned} \tag{1}$$

and for a.e. $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned} &E_{w_\varphi}\Big[\exp\Big\{-\int_0^T V(x(s))ds\Big\}\Big| x(t_0) = 0, x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n\Big] \\ &= E_{m_w}\Big[\exp\Big\{-\int_0^T V(x(s))ds\Big\}\Big| x(t_1) = \xi_1, \cdots, x(t_n) = \xi_n\Big] \end{aligned} \tag{2}$$

by Theorems 2.4 and 2.9.

In [10], Yeh showed that the function U defined on $\mathbb{R} \times (0,\infty)$ by

$$\begin{aligned} &U(\xi_1, T) \\ &= \Big(\frac{1}{2\pi T}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{\xi_1^2}{2T}\Big\} E_{m_w}\Big[\exp\Big\{-\int_0^T V(x(s))ds\Big\}\Big| x(T) = \xi_1\Big] \end{aligned} \tag{3}$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned} U(\xi_1, T) = \Big(\frac{1}{2\pi T}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{\xi_1^2}{2T}\Big\} - \int_0^T \Big[\frac{1}{2\pi(T-s)}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\Big\{-\frac{(\xi_1 - \eta)^2}{2(T-s)}\Big\} \\ \times V(\eta) U(\eta, s) d\eta ds \end{aligned}$$

whose solution is given by

$$U(\xi_1, T) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi_1, T), \quad (\xi_1, T) \in \mathbb{R} \times (0,\infty),$$

where the sequence $\{U_k\}_{k=0}^{\infty}$ is defined inductively by

$$U_0(\xi_1, T) = \left(\frac{1}{2\pi T}\right)^{\frac{1}{2}} \exp\left\{-\frac{\xi_1^2}{2T}\right\},$$
$$U_{k+1}(\xi_1, T) = \int_0^T \left[\frac{1}{2\pi(T-s)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{(\xi_1-\eta)^2}{2(T-s)}\right\} V(\eta) U_k(\eta, s) d\eta ds$$

for $k = 0, 1, \cdots$. Thus $E_{w_\varphi}[\exp\{-\int_0^T V(x(s))ds\}|x(t_0) = 0, x(T) = \xi_1]$ can be found using (1) and (3).

Now, for $\xi_1, \xi_2 \in \mathbb{R}$ and $t \in (0, T]$, let

$$\begin{aligned} &U(\xi_1, \xi_2 - \xi_1, T-t) \\ = &\left[\frac{1}{2\pi(T-t)}\right]^{\frac{1}{2}} \exp\left\{-\frac{(\xi_2-\xi_1)^2}{2(T-t)}\right\} \\ &\times E_{w_\varphi}\left[\exp\left\{-\int_t^T V(x(s))ds\right\}\Big| x(t_0) = 0, x(t) = \xi_1, x(T) = \xi_2\right]. \end{aligned} \tag{4}$$

Using the same process in [6, pp. 388-389], (4) satisfies the integral equation

$$\begin{aligned} &U(\xi_1, \xi_2 - \xi_1, T-t) \\ = &\left[\frac{1}{2\pi(T-t)}\right]^{\frac{1}{2}} \exp\left\{-\frac{(\xi_2-\xi_1)^2}{2(T-t)}\right\} - \int_0^{T-t} \left[\frac{1}{2\pi(T-t-u)}\right]^{\frac{1}{2}} \\ &\times \int_{\mathbb{R}} \exp\left\{-\frac{(\xi_2-\gamma)^2}{2(T-t-u)}\right\} V(\gamma) U(\xi_1, \gamma - \xi_1, u) d\gamma du \end{aligned}$$

by (2) so that we have

$$U(\xi_1, \xi_2 - \xi_1, T-t) = \sum_{k=0}^{\infty} (-1)^k U_k(\xi_2, \xi_2 - \xi_1, T-t),$$

where

$$U_0(\xi_1, \xi_2 - \xi_1, T-t) = \left[\frac{1}{2\pi(T-t-u)}\right]^{\frac{1}{2}} \exp\left\{-\frac{(\xi_2-\xi_1)^2}{2(T-t)}\right\}$$

and

$$\begin{aligned} &U_{k+1}(\xi_1, \xi_2 - \xi_1, T-t) \\ = &\int_0^{T-t} \left[\frac{1}{2\pi(T-t-u)}\right]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left\{-\frac{(\xi_2-\gamma)^2}{2(T-t-u)}\right\} V(\gamma) U_k(\xi_1, \gamma - \xi_1, u) d\gamma du \end{aligned}$$

for $k = 0, 1, 2, \cdots$. In particular, the solution $U(\xi_1, T)$ of the Kac-Feynman integral equation is a special case, namely,

$$U(\xi_1, T) = U(0, \xi_1, T).$$

We are now ready to write out an expression for the multi-conditional expectation. Using (4.6) in [6], (1) and (2), we can write for a.e. $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$

$$\begin{aligned} & E_{w_\varphi}\left[\exp\left\{-\int_0^T V(x(s))ds\right\}\middle| x(t_0)=0, x(t_1)=\xi_1, \cdots, x(t_n)=\xi_n\right] \\ = \quad & \prod_{j=1}^{n}[2\pi(t_j-t_{j-1})]^{\frac{1}{2}} \exp\left\{\frac{(\xi_j-\xi_{j-1})^2}{2(t_j-t_{j-1})}\right\} U(\xi_{j-1}, \xi_j-\xi_{j-1}, t_j-t_{j-1}) \end{aligned}$$

where $\xi_0 = 0$ and $t_n = T$.

References

[1] R. B. Ash, *Real analysis and probability*, Academic Press, New York-London, 1972. MR0435320 (55:8280)

[2] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. Math. **45** (1944), 386-396. MR0010346 (6:5f)

[3] K. S. Chang and J. S. Chang, *Evaluation of some conditional Wiener integrals*, Bull. Korean Math. Soc. **21** (1984), no. 2, 99-106. MR768465 (86e:28018)

[4] M. K. Im and K. S. Ryu, *An analogue of Wiener measure and its applications*, J. Korean Math. Soc. **39** (2002), no. 5, 801-819. MR1920906 (2003g:28028)

[5] R. G. Laha and V. K. Rohatgi, *Probability theory*, John Wiley & Sons, New York-Chichester-Brisbane, 1979. MR534143 (80k:60001)

[6] C. Park and D. L. Skoug, *A simple formula for condtitional Wiener integrals with applications*, Pacific J. Math. **135** (1988), no. 2, 381-394. MR968620 (90c:28022)

[7] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc. **354** (2002), no. 12, 4921-4951. MR1926843 (2004b:28020)

[8] J. Yeh, *Stochastic processes and the Wiener integral*, Marcel Dekker, New York, 1973. MR0474528 (57:14166)

[9] J. Yeh, *Inversion of conditional expectations*, Pacific J. Math. **52** (1974), 631-640. MR0365644 (51:1896)

[10] J. Yeh, *Inversion of conditional Wiener integrals*, Pacific J. Math. **59** (1975), no. 2, 623-638. MR0390162 (52:10988)

[11] J. Yeh, *Transformation of conditional Wiener integrals under translation and the Cameron-Martin translation theorem*, Tohôku Math. J. **30** (1978), no. 4, 505-515. MR516883 (80e:60094)

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