TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 360, Number 7, July 2008, Pages 3771–3793
S 0002-9947(08)04384-5
Article electronically published on February 27, 2008

# A GENERALIZATION OF DAHLBERG'S THEOREM CONCERNING THE REGULARITY OF HARMONIC GREEN POTENTIALS

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Abstract. Let $\mathbb{G}_D$ be the solution operator for $\Delta u = f$ in $\Omega$, Tr $u = 0$ on $\partial\Omega$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$. B. E. J. Dahlberg proved that for a bounded Lipschitz domain $\Omega, \nabla\mathbb{G}_D$ maps $L^1(\Omega)$ boundedly into weak-$L^1(\Omega)$ and that there exists $p_n > 1$ such that $\nabla\mathbb{G}_D : L^p(\Omega) \to L^{p^*}(\Omega)$ is bounded for $1 < p < n, \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. In this paper, we generalize this result by addressing two aspects. First we are also able to treat the solution operator $\mathbb{G}_N$ corresponding to Neumann boundary conditions and, second, we prove mapping properties for these operators acting on Sobolev (rather than Lebesgue) spaces.

## 1. Introduction and statement of main results

As is well-known, the free-space Green function for the Laplace operator $\Delta = \sum_{j=1}^{n} \partial_j^2$ in $\mathbb{R}^n$ is given by

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \ln |x|, & \text{if } n = 2, \\ \frac{c_n}{|x|^{n-2}}, & \text{if } n \geq 3, \end{cases} \tag{1.1}$$

where $c_n = [(2-n)\omega_n]^{-1}$, and $\omega_n$ denotes the area of the unit sphere in $\mathbb{R}^n$. This allows one to solve the Poisson problem for the Laplacian in the whole space via integral operators. More specifically, the Newtonian potential

$$\Pi f(x) := \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy \tag{1.2}$$

satisfies $\Delta(\Pi f) = f$ in $\mathbb{R}^n$, at least if $f$ is well-behaved. In the case when the Poisson problem is consider in a bounded domain $\Omega \subset \mathbb{R}^n$, boundary conditions must be imposed. When the homogeneous Dirichlet boundary condition is considered, the problem reads

$$\begin{cases} \Delta u = f \text{ in } \Omega, \\ \operatorname{Tr} u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

Received by the editors May 22, 2006.
2000 *Mathematics Subject Classification.* Primary 35J05, 46E35; Secondary 42B20, 34B27.
*Key words and phrases.* Green potentials, Poisson problem, Lipschitz domain, Sobolev spaces.
The author was supported in part by NSF FRG Grant #0456306.

where Tr stands for the operator of trace on $\partial\Omega$. Paralleling the setup corresponding to the entire Euclidean space, the solution operator for (1.3) may once again be expressed in integral form, i.e.

$$u(x) = \mathbb{G}_D f(x) := \int_\Omega G_D(x,y) f(y)\, dy, \qquad x \in \Omega, \tag{1.4}$$

where $G_D(x,y)$ is the Green function on $\Omega$ corresponding to the Dirichlet boundary condition. The mapping properties one can expect of the operator $\mathbb{G}_D$ are inexorably linked to the nature of the singularity in $G_D(x,y)$ which in turn is, to a large extent, dictated by the smoothness of the boundary of the domain $\Omega$. It is folklore that

$$\partial\Omega \in C^\infty \Longrightarrow |\nabla_x G_D(x,y)| \leq C|x-y|^{1-n}, \qquad x,y \in \Omega, \tag{1.5}$$

though (1.5) may *fail* if $\partial\Omega$ contains irregularities.

Estimates such as (1.5) are important for establishing mapping properties for the first order derivatives of the Dirichlet Green operator

$$\nabla\mathbb{G}_D f(x) = \int_\Omega \nabla_x G_D(x,y) f(y)\, dy, \qquad x \in \Omega. \tag{1.6}$$

For example, if $\partial\Omega$ is sufficiently smooth (e.g., $\partial\Omega \in C^{1+\gamma}$ for some $\gamma > 0$ will do), the estimate (1.5) and the Hardy-Littlewood-Sobolev Fractional Integration Theorem (cf., e.g., [21]) yield

$$\nabla\mathbb{G}_D : L^p(\Omega) \longrightarrow L^q(\Omega), \quad 1 < p < n, \quad \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{1}{n}, \tag{1.7}$$

$$|\{x \in \Omega : |\nabla\mathbb{G}_D f(x)| > \lambda\}| \leq C\Big(\lambda^{-1}\|f\|_{L^1(\Omega)}\Big)^{\frac{n}{n-1}}, \quad \forall\, \lambda > 0. \tag{1.8}$$

The extent to which the classical estimates (1.7)-(1.8) continue to hold for less smooth domains has been studied by B.E. Dahlberg in [5]. Recall that a *Lipschitz* domain is a domain whose boundary is locally given by graphs of Lipschitz functions (considered in appropriately rotated and translated Cartesian systems of axes). Dahlberg's main result in [5] reads as follows.

**Theorem 1.1.** *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the following are true:*

(i) *There exists a finite constant $C = C(\Omega) > 0$ such that the estimate* (1.8) *holds for every $f \in L^1(\Omega)$.*
(ii) *There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the operator* (1.7) *is bounded whenever $1 < p < p_n + \varepsilon$ where $p_2 := \frac{4}{3}$ and $p_n := 3n(n+3)^{-1}$ for $n \geq 3$.*

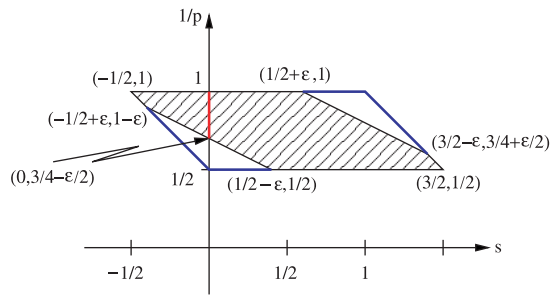
Dahlberg also proved that the choice of $p_n$ is sharp in the class of Lipschitz domains. His theorem illustrates the fact that $\nabla\mathbb{G}_D$ continues to behave like a fractional integration operator of order 1 but only for a more restricted range of indices than what the classical theory would warrant.

In this paper we provide a new, conceptually different proof, as well as an extension of Dahlberg's result. Our generalization addresses two aspects. First, we are able to treat Green operators corresponding to *Neumann* boundary conditions and, second, we prove mapping properties for these Green operators acting on *Sobolev* (rather than Lebesgue) spaces. In order to state our main results we need some more notation and a few definitions. We shall work with a family of regions

$\mathcal{P}(n,\varepsilon) \subset \mathbb{R}^2$, defined for $n \in \mathbb{N}$, $n \geq 2$ and $\varepsilon > 0$. The region $\mathcal{P}(2,\varepsilon)$ consists of points of the form $(s, \frac{1}{p})$ in $\mathbb{R}^2$ whose coordinates satisfy the inequalities:

$$\begin{cases} \frac{1}{2} < \frac{1}{p} < 1, \quad \frac{1}{2} - s < \frac{1}{p} < 2 - s, \\ -\frac{s}{2} + \frac{3}{4} - \frac{\varepsilon}{2} < \frac{1}{p} < -\frac{s}{2} + \frac{5}{4} + \frac{\varepsilon}{2}. \end{cases} \tag{1.9}$$

The region $\mathcal{P}(2,\varepsilon)$ is the shaded region in the figure below. The quadrilateral with vertices $(-\frac{1}{2}, 1)$, $(0, \frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$, and $(1,1)$ corresponds to the case when $\varepsilon = \frac{1}{2}$. When

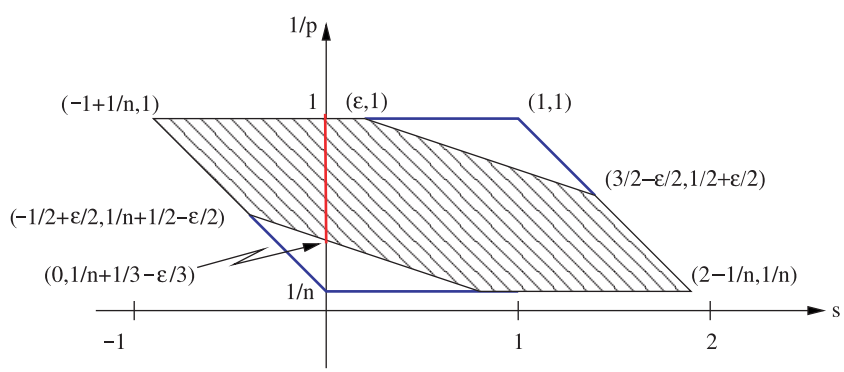



The region P(2,ε)

$n \geq 3$, the region $\mathcal{P}(n,\varepsilon)$ consists of points $(s, \frac{1}{p})$ in $\mathbb{R}^2$ whose coordinates satisfy the inequalities:

$$\begin{cases} \frac{1}{n} < \frac{1}{p} < 1, \quad \frac{1}{n} - s < \frac{1}{p} < 2 - s, \\ -\frac{s}{3} + \frac{1}{3} - \frac{\varepsilon}{2} < \frac{1}{p} < -\frac{s}{3} + \frac{\varepsilon}{3} + \frac{n+1}{n}. \end{cases} \tag{1.10}$$

The region $\mathcal{P}(n,\varepsilon)$ is the shaded region in the figure below. The quadrilateral with vertices $(-1+\frac{1}{n}, 1)$, $(0, \frac{1}{n})$, $(2-\frac{1}{n}, \frac{1}{n})$, and $(1,1)$ corresponds to the case when $\varepsilon = 1$. Let $L^p_s(\Omega)$ stand for the usual scale of Bessel-potential spaces on $\Omega$, of smoothness




The region P(n,ε)

$s \in \mathbb{R}$ and integrability $p \in (1, \infty)$. Finally, denote by $\mathbb{G}_N$ the solution operator to the Poisson problem for the Laplacian with a homogeneous Neumann boundary condition, i.e. $u = \mathbb{G}_N f$ is supposed to solve $\Delta u = f - |\Omega|^{-1}\langle f, 1\rangle$ in $\Omega$, $\partial_\nu u = 0$ on $\partial\Omega$, where $|\Omega|$ stands for the Euclidean volume of $\Omega \subset \mathbb{R}^n$.

We are now ready to state the main results of our paper.

**Theorem 1.2.** *For each $\Omega$ bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, there exists $\varepsilon = \varepsilon(\Omega) \in (0, \frac{1}{2}]$ if $n = 2$ and $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ if $n \geq 3$, such that the operators*

$$\mathbb{G}_N : L^p_{-s}(\Omega) \to L^{p^*}_{1-s}(\Omega), \tag{1.11}$$

$$\mathbb{G}_D : L^p_{-s}(\Omega) \to L^{p^*}_{1-s}(\Omega) \tag{1.12}$$

*are well-defined, linear and bounded whenever*

$$\Big(s, \tfrac{1}{p}\Big) \in \mathcal{P}(n, \varepsilon) \quad \text{and} \quad \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n}. \tag{1.13}$$

*Moreover, when $\partial\Omega \in C^1$, one can take $\varepsilon = \frac{1}{2}$ if $n = 2$ and $\varepsilon = 1$ if $n \geq 3$.*

Observe that part (ii) of Dahlberg's result corresponds to the particular case $s = 0$ in (1.12). As regards part (i), we have the following.

**Theorem 1.3.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then*

$$|\{x \in \Omega : |\nabla \mathbb{G}_N f(x)| > \lambda\}| \leq C\Big(\lambda^{-1}\|f\|_{L^1(\Omega)}\Big)^{\frac{n}{n-1}}, \quad \forall\, \lambda > 0. \tag{1.14}$$

Dahlberg's original proof of Theorem 1.1 is structured around the inequality

$$|\nabla \mathbb{G}_D f(x)| \leq C\Big(I_1 f(x) + \operatorname{dist}(x, \partial\Omega)^{-1}\mathbb{G}_D f(x)\Big), \qquad x \in \Omega, \tag{1.15}$$

which he proves that is valid whenever $f \geq 0$ in $\Omega$. Above, $I_1$ is the fractional integration operator of order 1 where, in general,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n, \tag{1.16}$$

for $0 < \alpha < n$. As is well-known,

$$I_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n), \quad 1 < p < \tfrac{n}{\alpha},\ \tfrac{1}{q} = \tfrac{1}{p} - \tfrac{\alpha}{n}, \tag{1.17}$$

and, if $1/q = 1 - \alpha/n$,

$$|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \leq C\Big(\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}\Big)^q. \tag{1.18}$$

In particular, the mapping properties of the operator $I_1$ mimic those of $\nabla\mathbb{G}_D$ in (1.7)-(1.8), and much of Dahlberg's work in [5] is directed at proving similar conclusions for the assignment $f \mapsto \operatorname{dist}(\cdot, \partial\Omega)^{-1}\mathbb{G}_D f$. In turn, this portion of his analysis is based on positivity, which limits its applicability to Dirichlet boundary conditions and to Lebesgue spaces.

In [12], D. Jerison and C. Kenig have succeeded in proving sharp results for the Poisson problem for the Dirichlet Laplacian $\Delta_D$ on Sobolev spaces in Lipschitz domains. As a corollary of their theory (most notably, the mapping properties of $\sqrt{-\Delta_D}$), they gave a new proof of (ii) in Theorem 1.1. Once again, this proof relies on maximum principles and, hence, it does not readily extend to Neumann boundary conditions. A new approach to the results of D. Jerison and C. Kenig has been devised by E. Fabes, O. Mendez and M. Mitrea in [7] where the authors have employed the method of boundary integral equations. This provides more flexibility since, in principle, this method does not differentiate between Dirichlet and Neumann boundary conditions. Subsequently, by combining the approach from [12] with the results from [7], O. Mendez and M. Mitrea have proved in [14] the version of (ii) in Theorem 1.1 corresponding to Neumann boundary conditions. This latter result has then been reproved by S. Mayboroda and M. Mitrea in [13], using

interpolation. More recently, D. Mitrea has developed in [15] a new approach to (ii) in Theorem 1.1 whose main ingredients are the mapping properties of harmonic layer potentials and embedding theorems for Sobolev spaces. Here we also want to mention K. Nyström's work in [18], where he has extended Theorem 1.1 to bounded NTA domains in $\mathbb{R}^n$, $n \geq 2$.

The approach in this paper is more akin to that in [15], whose main results are further refined as to allow an optimal range of Sobolev spaces, and are extended to boundary conditions of Neumann type; cf. Theorems 1.2-1.3. As a byproduct, a new proof of (i) in Theorem 1.1 is given as well.

## 2. Lorentz spaces

Let $(X, \mu)$ be a measure space and for a measurable function $f : X \to \mathbb{R}$ set

$$m(\lambda, f) := \mu(\{x \in X : |f(x)| > \lambda\}), \qquad \forall\, \lambda > 0, \tag{2.1}$$

and define the non-increasing rearrangement of $f$ as

$$f^*(t) := \inf\{\lambda > 0 : m(\lambda, f) \leq t\}, \qquad t > 0. \tag{2.2}$$

In particular, $m(\lambda, f) = m(\lambda, f^*)$ for every $\lambda > 0$. Finally, if $0 < p \leq \infty$, $0 < q \leq \infty$, consider

$$L^{p,q}(X) := \Big\{ f : X \to \mathbb{R} \text{ measurable} : t^{1/p} f^*(t) \in L^q((0,\infty), \tfrac{dt}{t}) \Big\}, \tag{2.3}$$

equipped with the quasi-norm

$$\|f\|_{L^{p,q}(X)} := \|t^{1/p} f^*(t)\|_{L^q((0,\infty),\frac{dt}{t})}. \tag{2.4}$$

Equivalent quasi-norms for the case when $q = \infty$ and $0 < p \leq \infty$, corresponding to weak-$L^p$ spaces, are as follows:

$$\|f\|_{L^{p,\infty}(X)} \approx \sup\{\lambda^p m(\lambda, f) : \lambda > 0\}, \tag{2.5}$$

$$\|f\|_{L^{p,\infty}(X)} \approx \sup_{0<\mu(E)<\infty} \mu(E)^{-\frac{1}{r}+\frac{1}{p}} \Big( \int_E |f|^r \, d\mu \Big)^{1/r}, \tag{2.6}$$

where $r \in (0, p)$ is fixed. The scale of Lorentz spaces contains Lebesgue spaces

$$L^{p,p}(X) = L^p(X), \quad 0 < p \leq \infty, \tag{2.7}$$

and the following inclusion is continuous:

$$L^{p,q_1}(X) \hookrightarrow L^{p,q_2}(X), \quad 0 < p \leq \infty,\ 0 < q_1 \leq q_2 \leq \infty. \tag{2.8}$$

For further reference, we also note that when $X$ is reasonable ($\sigma$-finite and non-atomic, to be exact),

$$\Big(L^{p,q}(X)\Big)^* = \begin{cases} \{0\} \ \text{ if } 0 < p < 1,\ 0 < q \leq \infty, \text{ or } p = 1 \text{ and } 1 < q < \infty, \\ L^\infty(X) \text{ when } p = 1 \text{ and } 1 < q < \infty, \\ L^{p',\infty}(X) \text{ for } 1 < p < \infty \text{ and } 0 < q \leq 1, \\ L^{p',q'}(X) \text{ if } 1 < p, q < \infty, \end{cases} \tag{2.9}$$

where, as usual, $p'$ and $q'$ are defined by $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Later on, it will be useful for us to note that, for each open set $\Omega \subset \mathbb{R}^n$ and each $\alpha > 0$,

$$|x - \cdot|^{-\alpha} \in L^{n/\alpha,\infty}(\Omega) \text{ uniformly in } x \in \mathbb{R}^n. \tag{2.10}$$

## 3. Real interpolation

Let $A_0$, $A_1$ be two quasi-normed spaces, continuously embedded into a larger, common topological vector space. The $K$-functional of $f \in A_0 + A_1$ at $t > 0$ is then defined by

$$K(t, f : A_0, A_1) := \inf\{\|f\|_{A_0} + t\|f\|_{A_1} : f = f_0 + f_1,\ f_0 \in A_0,\ f_1 \in A_1\}. \tag{3.1}$$

Then the intermediate space $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $0 < q \leq \infty$, is the space of all $f \in A_0 + A_1$ such that $\|f\|_{(A_0,A_1)_{\theta,q}} < +\infty$, where

$$\|f\|_{(A_0,A_1)_{\theta,q}} := \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, f : A_0, A_1)\right]^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0}\left[t^{-\theta} K(t, f : A_0, A_1)\right] & \text{if } q = \infty. \end{cases} \tag{3.2}$$

Using the real interpolation method between Lebesgue spaces over a measure space $(X, \mu)$ yields for $0 < p_0 < p_1 \leq \infty$

$$(L^{p_0}(X), L^{p_1}(X))_{\theta,q} = L^{p,q}(X), \quad \text{if } p_0 < q \leq \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 < \theta < 1. \tag{3.3}$$

## 4. Lipschitz domains

Recall that a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is called Lipschitz provided there exists $M > 0$ such that $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})} := \sup\{|\varphi(x') - \varphi(y')|/|x' - y'| : x', y' \in \mathbb{R}^{n-1}\} < +\infty$. An unbounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is simply the overgraph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, i.e.

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \varphi(x')\}.$$

In this case, we shall refer to $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})}$ as the Lipschitz constant of $\Omega$. Finally, $\Omega \subset \mathbb{R}^n$ is called a bounded Lipschitz domain provided its boundary $\partial\Omega$ can be covered by finitely many balls $\{B(x_i, R_i)\}_{1\leq i\leq N}$, $x_i \in \partial\Omega$, $R_i > 0$, with the property that for each $i$ there exists an unbounded Lipschitz domain $\Omega_i$ (considered in a system of coordinates which is a rotation and a translation of the original one) such that $\Omega \cap B(x_i, R_i) = \Omega_i \cap B(x_i, R_i)$, $1 \leq i \leq N$. See, e.g., the definition and comments on p. 189 in Stein's book [21].

The number and size of the above family of balls, referred to in the sequel as *coordinate balls*, along with the Lipschitz constants for the family of unbounded Lipschitz domains $\Omega_i$, $1 \leq i \leq N$, make up what we will call the Lipschitz character of $\Omega$. Throughout the paper, writing $C = C(\partial\Omega)$ will indicate that the constant $C$ depends exclusively on the Lipschitz character of $\Omega$.

It is well-known that for each Lipschitz domain $\Omega \subset \mathbb{R}^n$ (bounded or unbounded), there is a canonical surface measure $d\sigma$, with respect to which the outward unit normal, $\nu$, is well-defined at almost every boundary point. For $1 \leq p \leq \infty$, we denote by $L^p(\partial\Omega)$ the corresponding Lebesgue space of measurable functions which are $p$-th power integrable with respect to $d\sigma$ on $\partial\Omega$, and by $L^p_1(\partial\Omega)$ the $L^p$-based Sobolev space of order one on $\partial\Omega$. That is,

$$f \in L^p_1(\partial\Omega) \Longleftrightarrow f \in L^p(\partial\Omega) \ \text{ and } \ \partial_{\tau_{jk}} f \in L^p(\partial\Omega) \text{ for } j, k = 1, ..., n, \tag{4.1}$$

where, if $\nu_1, ..., \nu_n$ are the components of the outward unit normal $\nu$,

$$\partial_{\tau_{jk}} := \nu_k \partial_j - \nu_j \partial_k, \qquad j, k = 1, ..., n. \tag{4.2}$$

We equip this space with the natural norm, i.e.

$$\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)}.$$

The scale of Besov spaces $B^{p,q}_s(\partial\Omega)$, $0 < s < 1$, $1 < p, q < \infty$, can then be introduced via real interpolation, i.e.

$$B^{p,q}_s(\partial\Omega) := \left(L^p(\partial\Omega), L^p_1(\partial\Omega)\right)_{s,q}, \qquad 0 < s < 1, \quad 1 < p, q < \infty. \tag{4.3}$$

Besov spaces with a negative amount of smoothness can then be introduced via duality:

$$B^{p,q}_{-s}(\partial\Omega) := \left(B^{p',q'}_s(\partial\Omega)\right)^* \tag{4.4}$$

where $0 < s < 1$, $1 < p, q < \infty$, and $1/p + 1/p' = 1/q + 1/q' = 1$.

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$ (bounded or unbounded), for some fixed, sufficiently large $\kappa = \kappa(\partial\Omega) > 0$ we set

$$\gamma(x) := \{y \in \Omega : \, |x - y| \leq \kappa \, \mathrm{dist}\,(y, \partial\Omega)\}, \qquad x \in \partial\Omega. \tag{4.5}$$

Then if $u$ is defined in $\Omega$, $\mathcal{N}(u)$, the nontangential maximal function of $u$, is defined at boundary points by

$$\mathcal{N}(u)(x) := \sup\{|u(y)| : \, y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{4.6}$$

**Lemma 4.1.** *Let $\Omega$ be a Lipschitz domain in $\mathbb{R}^n$ and assume that $0 < p < \infty$, $0 < q \leq \infty$. Then for every function $u : \Omega \to \mathbb{R}$,*

$$\mathcal{N}(u) \in L^{p,q}(\partial\Omega) \Longrightarrow u \in L^{\frac{np}{n-1},q}(\Omega), \tag{4.7}$$

*plus a natural estimate.*

*Proof.* To this end, fix $u$ such that $\mathcal{N}(u) \in L^{p,q}(\partial\Omega)$ for some $0 < p < \infty$, $0 < q \leq \infty$, and also fix $\lambda > 0$. We claim that

$$|\{|u| > \lambda\}| \leq C\,\sigma(\{\mathcal{N}(u) > \lambda\})^{n/(n-1)}, \tag{4.8}$$

where $|E|$, $\sigma(F)$ stand, respectively, for the $n$-dimensional Lebesgue measure of $E \subset \mathbb{R}^n$ and the surface measure of $F \subset \partial\Omega$. The proof of (4.8) requires working with "tent" regions

$$T(\mathcal{O}) := \Omega \setminus \Big[\bigcup_{x \in \partial\Omega \setminus \mathcal{O}} \gamma(x)\Big] \tag{4.9}$$

associated with arbitrary open subsets $\mathcal{O}$ of $\partial\Omega$. The idea is that (4.8) is going to be a consequence of the inclusion

$$\{|u| > \lambda\} \subseteq T(\{\mathcal{N}(u) > \lambda\}), \tag{4.10}$$

itself a consequence of (4.6) and (4.9), used in concert with a general fact, to the effect that

$$|T(\mathcal{O})| \leq C\,\sigma(\mathcal{O})^{n/(n-1)}, \quad \forall\, \mathcal{O} \text{ open subset of } \partial\Omega. \tag{4.11}$$

In turn, (4.11) is seen by decomposing $\mathcal{O}$ into a disjoint union of Whitney cubes $\{Q_k\}_k$ (considering $\partial\Omega$ as a space of homogeneous type), so that $T(\mathcal{O}) \subset \cup_k T(c\,Q_k)$

for some constant $c = c(\partial\Omega) > 0$, and then writing

$$\begin{aligned} |T(\mathcal{O})| &\leq \sum_k |T(c\,Q_k)| \leq C \sum_k \sigma(Q_k)^{n/(n-1)} \\ &\leq C\Big[\sum_k \sigma(Q_k)\Big]^{n/(n-1)} = C\,\sigma(\mathcal{O})^{n/(n-1)}. \end{aligned} \tag{4.12}$$

Hence (4.8) holds. Consequently, by (2.1),

$$m(\lambda, u) \leq C\, m(\lambda, \mathcal{N}(u))^{\frac{n-1}{n}}, \qquad \forall\, \lambda > 0. \tag{4.13}$$

In particular, for every $t > 0$,

$$\{\lambda :\, m(\lambda, \mathcal{N}(u)) \leq t\} \subseteq \{\lambda :\, m(\lambda, u) \leq C\, t^{\frac{n}{n-1}}\}, \tag{4.14}$$

which, by (2.2), entails

$$u^*(s) \leq \mathcal{N}(u)^*((s/C)^{\frac{n-1}{n}}), \qquad \forall\, s > 0, \tag{4.15}$$

and, further,

$$\begin{aligned} \int_0^\infty \Big[s^{\frac{n-1}{np}} u^*(s)\Big]^q \frac{ds}{s} &\leq \int_0^\infty \Big[s^{\frac{n-1}{np}} \mathcal{N}(u)^*((s/C)^{\frac{n}{n-1}})\Big]^q \frac{ds}{s} \\ &\leq C \int_0^\infty \Big[t^{1/p} \mathcal{N}(u)^*(t)\Big]^q \frac{dt}{t} \\ &= C\|\mathcal{N}(u)\|^q_{L^{p,q}(\partial\Omega)}, \end{aligned} \tag{4.16}$$

where in the second inequality we have made the change of variables $t = (s/C)^{\frac{n-1}{n}}$, and the last step uses (2.4). □

## 5. Bessel-potential spaces

With $\mathcal{F}$ denoting the Fourier transform in $\mathbb{R}^n$, for $s \in \mathbb{R}$ and $1 < p < \infty$ the Bessel-potential spaces $L^p_s(\mathbb{R}^n)$ are defined as

$$L^p_s(\mathbb{R}^n) := \Big\{ f \in S'(\mathbb{R}^n) :\, \|f\|_{L^p_s(\mathbb{R}^n)} := \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}f)\|_{L^p(\mathbb{R}^n)} < +\infty\} \tag{5.1}$$

where $S'(\mathbb{R}^n)$ stands for the space of tempered distributions in $\mathbb{R}^n$. As is well-known (see, e.g., [19]), the following embeddings hold:

$$L^{p_1}_{s_1}(\mathbb{R}^n) \hookrightarrow L^{p_2}_{s_2}(\mathbb{R}^n) \quad \text{if } \ s_1 \geq s_2 \text{ and } \tfrac{1}{p_2} - \tfrac{s_2}{n} \geq \tfrac{1}{p_1} - \tfrac{s_1}{n}. \tag{5.2}$$

Next, for an arbitrary open subset $\Omega$ of $\mathbb{R}^n$, we denote by $D'(\Omega)$ the space of distributions in $\Omega$ and by $\mathrm{R}_\Omega f \in D'(\Omega)$ the restriction of a distribution $f \in S'(\mathbb{R}^n)$ to $\Omega$. For $1 < p < \infty$ and $s \in \mathbb{R}$ we then set

$$\begin{aligned} &L^p_s(\Omega) := \{f \in D'(\Omega) :\, \exists\, F \in L^p_s(\mathbb{R}^n) \text{ such that } \mathrm{R}_\Omega F = f\}, \\ &\|f\|_{L^p_s(\Omega)} := \inf\{\|F\|_{L^p_s(\mathbb{R}^n)} :\, F \in L^p_s(\mathbb{R}^n),\ \mathrm{R}_\Omega F = f\}, \qquad f \in L^p_s(\Omega). \end{aligned} \tag{5.3}$$

As a corollary, the restriction operator

$$\mathrm{R}_\Omega : L^p_s(\mathbb{R}^n) \longrightarrow L^p_s(\Omega), \quad 1 < p < \infty, \quad s \in \mathbb{R}, \tag{5.4}$$

is well-defined, linear, bounded and onto, and the following embedding holds:

$$L^{p_1}_{s_1}(\Omega) \hookrightarrow L^{p_2}_{s_2}(\Omega) \quad \text{if } \ s_1 \geq s_2 \text{ and } \tfrac{1}{p_2} - \tfrac{s_2}{n} \geq \tfrac{1}{p_1} - \tfrac{s_1}{n}. \tag{5.5}$$

Also, for each $p \in (1,\infty)$, $s \in \mathbb{R}$,

$$f \in L^p_s(\Omega) \Longrightarrow \partial_j f \in L^p_{s-1}(\Omega),\ 1 \le j \le n. \tag{5.6}$$

For the remainder of this section, assume that $\Omega$ is a Lipschitz domain in $\mathbb{R}^n$. In this case, it has been proved in [20] that there exists a universal linear extension operator, i.e. a mapping $\mathrm{Ext} : D'(\Omega) \to S'(\mathbb{R}^n)$ such that

$$\mathrm{Ext} : L^p_s(\Omega) \longrightarrow L^p_s(\mathbb{R}^n) \tag{5.7}$$

boundedly for each $1 < p < \infty$, $s \in \mathbb{R}$, and for which

$$\mathrm{R}_\Omega \circ \mathrm{Ext} = \mathrm{I}, \quad \text{the identity operator.} \tag{5.8}$$

In particular, $L^p_0(\Omega) = L^p(\Omega)$ and, if $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $1 < p < \infty$, stands for the classical Sobolev space of functions whose derivatives of order $\le k$ lie in $L^p(\Omega)$, then

$$L^p_k(\Omega) = W^{k,p}(\Omega) \quad \text{whenever} \ \ k \in \mathbb{N},\ 1 < p < \infty. \tag{5.9}$$

Also, as a consequence of the fact that the scale $L^p_s(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$, is stable under complex interpolation, we also have

$$[L^{p_1}_{s_1}(\Omega), L^{p_2}_{s_2}(\Omega)]_\theta \quad = \quad L^p_s(\Omega), \tag{5.10}$$

whenever $1 < p_j < \infty$, $s_j \in \mathbb{R}$, $j \in \{1,2\}$, $\theta \in (0,1)$ and $1/p = (1-\theta)/p_1 + \theta/p_2$, $s = (1-\theta)s_1 + \theta s_2$. In (5.10), $[\cdot\,,\cdot]_\theta$ stands for the complex interpolation bracket; cf., e.g., [1].

Bessel-potential spaces in $\Omega$ are related to Besov spaces on $\partial\Omega$ via the trace operator. This maps

$$\mathrm{Tr} : L^p_s(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega), \quad 1 < p < \infty, \quad \tfrac{1}{p} < s < \tfrac{1}{p} + 1, \tag{5.11}$$

in a bounded fashion, and has a bounded, linear right-inverse (regarded as an extension operator).

We continue to assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Later on, we shall also need to work with the normal derivative of a function $u$ in the Bessel-potential spaces $L^p_s(\Omega)$, where $p \in (1,\infty)$, $1/p < s < 1 + 1/p$. As opposed to the ordinary trace (5.11), a peculiarity of the normal derivative operator is that this depends not only on the function $u$ itself but on the choice of an extension of $\Delta u$, as a distribution (in $L^p_{s-2}(\Omega)$), to a functional acting on $L^{p'}_{2-s}(\Omega)$ (and thus in $\Big(L^{p'}_{2-s}(\Omega)\Big)^*$), where $\frac{1}{p} + \frac{1}{p'} = 1$. More specifically, if $C^\infty_0(\Omega)$ is the space of test functions in $\Omega$, we define

$$\begin{aligned} &\partial_\nu : \{(u,f) \in L^p_s(\Omega) \oplus \Big(L^{p'}_{2-s}(\Omega)\Big)^* :\ f|_{C^\infty_0(\Omega)} = \Delta u\} \longrightarrow B^{p,p}_{s-1-\frac{1}{p}}(\partial\Omega), \\ &\langle \partial_\nu(u,f), \varphi\rangle := \langle f, \Phi\rangle + \langle \nabla u, \nabla\Phi\rangle, \\ &\forall\, \varphi \in B^{p',p'}_{1+\frac{1}{p}-s}(\partial\Omega) \ \text{ and } \Phi \in L^{p'}_{2-s}(\Omega) \ \text{ such that } \mathrm{Tr}\,\Phi = \varphi. \end{aligned} \tag{5.12}$$

Observe that the pairings in (5.12) are well-defined. In particular, the last one uses the fact that

$$\Big(L^p_{s-1}(\Omega)\Big)^* = L^{p'}_{1-s}(\Omega), \qquad 1 < p < \infty, \quad \tfrac{1}{p} < s < 1 + \tfrac{1}{p}. \tag{5.13}$$

A more detailed discussion, proofs and further references on these matters can be found in, e.g., [1], [12], [7] [19], [20] and [22].

## 6. Hardy spaces

We start by briefly recalling the definition of Hardy spaces in $\mathbb{R}^n$. Fix $\psi$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n}\psi\neq 0$, and for $t>0$, $x\in\mathbb{R}^n$, let $\psi_t(x):=t^{-n}\psi\left(\frac{x}{t}\right)$. Then, the radial maximal function of $f\in\mathcal{S}'(\mathbb{R}^n)$ is

$$(M_\psi f)(x):=\sup_{t>0}|(f*\psi_t)(x)|,\quad x\in\mathbb{R}^n. \tag{6.1}$$

For $0<p<\infty$, the Hardy spaces are then defined as

$$H^p(\mathbb{R}^n):=\{f\in\mathcal{S}'(\mathbb{R}^n):\ M_\psi f\in L^p(\mathbb{R}^n)\}. \tag{6.2}$$

We remark that $H^p(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ for $1<p<\infty$. However, if $p\leq 1$ the Hardy and Lebesgue spaces differ. In particular, $H^1(\mathbb{R}^n)\subsetneqq L^1(\mathbb{R}^n)$.

Weak Hardy spaces are defined in a similar manner. Concretely, for $0<p<\infty$,

$$H^{p,\infty}(\mathbb{R}^n):=\{f\in\mathcal{S}'(\mathbb{R}^n):\ M_\psi f\in L^{p,\infty}(\mathbb{R}^n)\}. \tag{6.3}$$

Since $L^p(\mathbb{R}^n)\hookrightarrow L^{p,\infty}(\mathbb{R}^n)$ for all $0<p<\infty$, we have that $H^p(\mathbb{R}^n)\hookrightarrow H^{p,\infty}(\mathbb{R}^n)$, for $0<p<\infty$. When $1<p<\infty$, weak Lebesgue and weak Hardy spaces coincide, i.e., $H^{p,\infty}(\mathbb{R}^n)=L^{p,\infty}(\mathbb{R}^n)$. We also want to mention here that $M_\psi f\leq C\,\mathcal{M}f$, where $\mathcal{M}$ is the classical Hardy-Littlewood maximal operator (cf. Theorem 2, pp. 62-63 in [21]). Since the latter operator maps $L^1(\mathbb{R}^n)$ boundedly into $L^{1,\infty}(\mathbb{R}^n)$, it follows that

$$L^1(\mathbb{R}^n)\hookrightarrow H^{1,\infty}(\mathbb{R}^n). \tag{6.4}$$

Assume now that $\Omega\subset\mathbb{R}^n$ is the domain lying above the graph of the Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$. Then, for $\frac{n-1}{n}<p<\infty$, we define Hardy spaces on $\partial\Omega$ using an appropriate change of coordinates, i.e.

$$f\in H^p(\partial\Omega)\overset{def}{\Longleftrightarrow} f(x',\varphi(x'))\sqrt{1+|\nabla\varphi(x')|^2}\in H^p(\mathbb{R}^{n-1}). \tag{6.5}$$

For $\frac{n-1}{n}<p<\infty$ we also set

$$f\in H^{p,\infty}(\partial\Omega)\overset{def}{\Longleftrightarrow} f(x',\varphi(x'))\sqrt{1+|\nabla\varphi(x')|^2}\in H^{p,\infty}(\mathbb{R}^{n-1}) \tag{6.6}$$

so that, by recalling (6.4),

$$L^1(\partial\Omega)\hookrightarrow H^{1,\infty}(\partial\Omega). \tag{6.7}$$

We next record a basic interpolation result.

**Proposition 6.1.** *Let $\Omega$ be an unbounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$(H^{p_0}(\partial\Omega),H^{p_1}(\partial\Omega))_{\theta,\infty}=H^{p,\infty}(\partial\Omega) \tag{6.8}$$

*whenever $\frac{n-1}{n}<p_0,p_1<\infty$, $0<\theta<1$, and $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.*

*Proof.* When $\Omega=\mathbb{R}^n_+$, in which case $\partial\Omega\equiv\mathbb{R}^{n-1}$, formula (6.8) is due to C. Fefferman, N. Rivière, Y. Sagher; see [8]. The case of a Lipschitz boundary then follows from this and (6.6)-(6.5). □

The definitions and results described so far have natural analogues in the context of local Hardy spaces (of the sort introduced by D. Goldberg in [10]). More specifically, $h^p(\mathbb{R}^n)$ are defined as in (6.2) with $M_\psi$ replaced by the truncated radial maximal operator. The latter is defined as in (6.1), this time taking the supremum over $1>t>0$ instead of $t>0$. Consequently, $H^p(\mathbb{R}^n)\subset h^p(\mathbb{R}^n)$ for every $p>0$.

Next, let $\Omega \subset \mathbb{R}^n$ be a fixed, arbitrary bounded Lipschitz domain. On its boundary, local Hardy spaces $h^p(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$, can be defined by lifting their Euclidean counterpart via a standard localization procedure (involving a smooth partition of unity subordinate to a covering of $\partial\Omega$ with coordinate balls as in §4) and a change of variables of the form (6.5). Weak local Hardy spaces $h^{p,\infty}(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$, can also be introduced in a similar fashion, and the analogues of (6.7) and (6.8) continue to hold in this setting. That is,

$$L^1(\partial\Omega) \hookrightarrow h^{1,\infty}(\partial\Omega) \tag{6.9}$$

and, for $\frac{n-1}{n} < p_0, p_1 < \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$(h^{p_0}(\partial\Omega), h^{p_1}(\partial\Omega))_{\theta,\infty} = h^{p,\infty}(\partial\Omega). \tag{6.10}$$

Finally, we introduce the Hardy-Sobolev spaces of order one

$$h_1^p(\mathbb{R}^n) := \{f \in h^p(\mathbb{R}^n) : \partial_j f \in h^p(\mathbb{R}^n),\ j = 1, \dots, n\}, \tag{6.11}$$

as well as their weak version

$$h_1^{p,\infty}(\mathbb{R}^n) := \{f \in h^{p,\infty}(\mathbb{R}^n) : \partial_j f \in h^{p,\infty}(\mathbb{R}^n),\ j = 1, \dots, n\}. \tag{6.12}$$

**Proposition 6.2.** *For any $0 < p_0, p_1 < \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,*

$$(h_1^{p_0}(\mathbb{R}^n), h_1^{p_1}(\mathbb{R}^n))_{\theta,\infty} = h_1^{p,\infty}(\mathbb{R}^n). \tag{6.13}$$

*Proof.* Consider the lifting operator $J_1$ defined on the Fourier transform side as

$$\mathcal{F}(J_1 f)(\xi) := (1 + |\xi|^2)^{-\frac{1}{2}} (\mathcal{F}f)(\xi), \quad \xi \in \mathbb{R}^n. \tag{6.14}$$

It is proved in [23] that $J_1 : h^p(\mathbb{R}^n) \to h_1^p(\mathbb{R}^n)$ is an isomorphism for all $0 < p < \infty$; hence, by interpolation, we obtain that for any $0 < p_0, p_1 < \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $J_1 : h^{p,\infty}(\mathbb{R}^n) \to (h_1^{p_0}(\mathbb{R}^n), h_1^{p_1}(\mathbb{R}^n))_{\theta,\infty}$ is an isomorphism. Therefore, it remains to show that $J_1(h^{p,\infty}(\mathbb{R}^n)) = h_1^{p,\infty}(\mathbb{R}^n)$. This, in turn, is proved much as Theorem 7 in [4]. □

If now $\Omega$ is the region in $\mathbb{R}^n$ above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, then for $\frac{n-1}{n} < p < \infty$ we set

$$f \in h_1^p(\partial\Omega) \overset{def}{\iff} f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \in h_1^p(\mathbb{R}^{n-1}), \tag{6.15}$$

$$f \in h_1^{p,\infty}(\partial\Omega) \overset{def}{\iff} f(x', \varphi(x'))\sqrt{1 + |\nabla\varphi(x')|^2} \in h_1^{p,\infty}(\mathbb{R}^{n-1}). \tag{6.16}$$

The definitions of $h_1^p(\partial\Omega)$, $h_1^{p,\infty}(\partial\Omega)$, $\frac{n-1}{n} < p < \infty$, in the case when $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, are similar except that, as in the case of $h^p(\partial\Omega)$, a localization process is involved. Also, much as before, the following corollary is true.

**Corollary 6.3.** *For any bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$ the following hold:*

(i) $(h_1^{p_0}(\partial\Omega), h_1^{p_1}(\partial\Omega))_{\theta,\infty} = h_1^{p,\infty}(\partial\Omega)$ *whenever* $\frac{n-1}{n} < p_0, p_1 < \infty$, $0 < \theta < 1$, *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$;

(ii) $h_1^{p,\infty}(\partial\Omega) = \{f \in h^{p,\infty}(\partial\Omega) : \partial_{\tau_{jk}} f \in h^{p,\infty}(\partial\Omega),\ 1 \leq j,k \leq n\}$, *for* $\frac{n-1}{n} < p < \infty$.

## 7. Layer potentials

We now recall the definitions and summarize the properties of layer potentials which are most relevant for us in the sequel. To begin with, recall the Newtonian potential operator from (1.2) for which the following result is well-known.

**Proposition 7.1.** *Let $\zeta_1$, $\zeta_2 \in C_0^\infty(\mathbb{R}^n)$. Then, for each $1 < p < \infty$ and $s \in \mathbb{R}$, the doubly truncated Newtonian potential*

$$\zeta_1 \Pi \zeta_2 : L_s^p(\mathbb{R}^n) \longrightarrow L_{2+s}^p(\mathbb{R}^n) \tag{7.1}$$

*is well-defined and bounded.*

**Corollary 7.2.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $\zeta_1$, $\zeta_2 \in C_0^\infty(\mathbb{R}^n)$ are two fixed functions which are identically equal to one in a neighborhood of $\bar{\Omega}$. Then the operator*

$$\begin{aligned} &\Pi_\Omega^D f := \mathrm{R}_\Omega\Big[(\zeta_1 \Pi \zeta_2)(\mathrm{Ext}\, f)\Big], \\ &\Pi_\Omega^D : L_s^p(\Omega) \longrightarrow L_{2+s}^p(\Omega), \end{aligned} \tag{7.2}$$

*is well-defined and bounded for each $p \in (1,\infty)$, $s \in \mathbb{R}$, and satisfies*

$$\Delta\Big[\Pi_\Omega^D f\Big] = f \quad \text{in} \quad \Omega, \qquad \forall\, f \in L_s^p(\Omega). \tag{7.3}$$

*Likewise, the operator*

$$\begin{aligned} &\Pi_\Omega^N f := \mathrm{R}_\Omega\Big[(\zeta_1 \Pi \zeta_2)(f \circ \mathrm{R}_\Omega)\Big], \\ &\Pi_\Omega^N : \Big(L_{-s}^{p'}(\Omega)\Big)^* \longrightarrow L_{2+s}^p(\Omega), \end{aligned} \tag{7.4}$$

*is well-defined and bounded for each $p, p' \in (1,\infty)$ with $1/p + 1/p' = 1$, and each $s \in \mathbb{R}$. Moreover, it satisfies*

$$\Delta\Big[\Pi_\Omega^N f\Big] = f\Big|_{C_0^\infty(\Omega)} \quad \text{as distributions in} \quad \Omega, \qquad \forall\, f \in \Big(L_{-s}^{p'}(\Omega)\Big)^*. \tag{7.5}$$

*Proof.* This is direct consequence of Proposition 7.1, (5.4), and (5.7)-(5.8). □

Next, we discuss the single and double layer potential operators which act on an arbitrary function $\psi : \partial\Omega \to \mathbb{R}$ according to

$$\mathcal{S}\psi(x) := \int_{\partial\Omega} \Gamma(x-y)\psi(y)\, d\sigma_y, \qquad x \in \Omega, \tag{7.6}$$

$$\mathcal{D}\psi(x) := \int_{\partial\Omega} \partial_{\nu_y}[\Gamma(x-y)]\psi(y)\, d\sigma_y, \qquad x \in \Omega, \tag{7.7}$$

where $\Gamma$ has been introduced in (1.1). The boundary trace of (7.7) and the normal derivative of (7.6) are, respectively, given by

$$\mathrm{Tr}\Big[\mathcal{D}\psi\Big] = \big(\tfrac{1}{2}I + K\big)\,\psi, \tag{7.8}$$

$$\partial_\nu \mathcal{S}\psi = \big(-\tfrac{1}{2}I + K^*\big)\,\psi, \tag{7.9}$$

$$\mathrm{Tr}\Big[\mathcal{S}\psi\Big] = S\psi, \tag{7.10}$$

where $K$ is the principal-value operator

$$K\psi(x) := p.v.\int_{\partial\Omega} \partial_{\nu_y}[\Gamma(x-y)]\psi(y)\,d\sigma_y, \quad \text{for a.e. } x \in \partial\Omega, \tag{7.11}$$

$K^*$ is the formal adjoint of (7.11), and

$$S\psi(x) := \int_{\partial\Omega} \Gamma(x-y)\psi(y)\,d\sigma_y, \quad \text{for } \ x \in \partial\Omega. \tag{7.12}$$

**Proposition 7.3.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, for each $1 < p < \infty$ and $0 < s < 1$, the following operators are bounded:*

$$\mathcal{S} : B^{p,p}_{-s}(\partial\Omega) \longrightarrow L^p_{-s+1+\frac{1}{p}}(\Omega), \tag{7.13}$$

$$\mathcal{D} : B^{p,p}_{s}(\partial\Omega) \longrightarrow L^p_{s+\frac{1}{p}}(\Omega), \tag{7.14}$$

$$K : B^{p,p}_{s}(\partial\Omega) \longrightarrow B^{p,p}_{s}(\partial\Omega), \tag{7.15}$$

$$K^* : B^{p,p}_{-s}(\partial\Omega) \longrightarrow B^{p,p}_{-s}(\partial\Omega). \tag{7.16}$$

*Furthermore, for each $\frac{n-1}{n} < p < \infty$, the operators*

$$S : h^{p,\infty}(\partial\Omega) \to h^{p,\infty}_1(\partial\Omega), \tag{7.17}$$

$$K : h^{p,\infty}_1(\partial\Omega) \to h^{p,\infty}_1(\partial\Omega), \tag{7.18}$$

$$K^* : h^{p,\infty}(\partial\Omega) \to h^{p,\infty}(\partial\Omega), \tag{7.19}$$

*are bounded as well.*

*Proof.* The operators (7.13)-(7.15) have been discussed in [7], whereas the boundedness of the operator (7.16) follows from that of (7.15) by duality. Next, it is well-known (cf., e.g., the discussion in [6]) that the operators

$$S : h^p(\partial\Omega) \to h^p_1(\partial\Omega), \quad K : h^p_1(\partial\Omega) \to h^p_1(\partial\Omega), \quad K^* : h^p(\partial\Omega) \to h^p(\partial\Omega) \tag{7.20}$$

are bounded for every $\frac{n-1}{n} < p < \infty$. Then the fact that (7.17)-(7.19) are well-defined and bounded follows from this and (6.10), Corollary 6.3. □

For each parameter $\varepsilon > 0$ we define the following two regions in $\mathbb{R}^2$:

$$\begin{aligned} \mathcal{R}^1_\varepsilon := {} & \text{the set of points inside the hexagon with vertices at} \\ & (0,0),\ (\tfrac{1}{2}+\varepsilon, 0),\ (1, \tfrac{1}{2}-\varepsilon),\ (1,1),\ (\tfrac{1}{2}-\varepsilon, 1),\ (0, \tfrac{1}{2}+\varepsilon), \end{aligned} \tag{7.21}$$

and

$$\begin{aligned} \mathcal{R}^2_\varepsilon := {} & \text{the set of points inside the hexagon with vertices at} \\ & (0,0),\ (\varepsilon, 0),\ (1, \tfrac{1}{2}-\tfrac{\varepsilon}{2}),\ (1,1),\ (1-\varepsilon, 1),\ (0, \tfrac{1}{2}+\tfrac{\varepsilon}{2}). \end{aligned} \tag{7.22}$$

**Proposition 7.4.** *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, there exists some number $\varepsilon = \varepsilon(\partial\Omega) \in (0, \frac{1}{2}]$ for $n = 2$, and $\varepsilon = \varepsilon(\partial\Omega) \in (0, 1]$ for $n \geq 3$, such that the operators*

$$\tfrac{1}{2}I + K : B^{p,p}_s(\partial\Omega) \longrightarrow B^{p,p}_s(\partial\Omega), \tag{7.23}$$

$$-\tfrac{1}{2}I + K^* : \{f \in B^{p',p'}_{-s}(\partial\Omega) : \langle f, 1\rangle = 0\} \to \{f \in B^{p',p'}_{-s}(\partial\Omega) : \langle f, 1\rangle = 0\}, \tag{7.24}$$

*are isomorphisms whenever $(s, \frac{1}{p}) \in \mathcal{R}^1_\varepsilon$ for $n = 2$, and whenever $(s, \frac{1}{p}) \in \mathcal{R}^2_\varepsilon$ for $n \geq 3$ (as usual, $1/p + 1/p' = 1$).*

*Proof.* The invertibility of the operators (7.23)-(7.24) has been proved in [16] for $n = 2$ and in [7] for $n \geq 3$. □

In the last part of this section, we briefly elaborate on the nature of the Green functions and Green operators corresponding to the Poisson problem for the Laplacian with homogeneous Dirichlet and Neumann boundary condition. First consider (1.3) and recall that the solution may be expressed in the form of an integral operator as in (1.4). The integral kernel of this operator satisfies, for each fixed (pole) $x \in \Omega$, the following problem:

$$\begin{cases} \Delta_y G_D(x, y) = \delta_x(y), \quad y \in \Omega, \\ \operatorname{Tr} G_D(x, \cdot) = 0 \text{ on } \partial\Omega, \end{cases} \tag{7.25}$$

where $\delta_x$ is the Dirac distribution with mass at $x$. Using layer potentials, one can express the Green function $G_D$ in the form

$$G_D(x, y) = \Gamma(x - y) - \mathcal{D}\left[\left(\tfrac{1}{2}I + K\right)^{-1}[\operatorname{Tr}\Gamma(x - \cdot)]\right](y), \qquad x, y \in \Omega, \tag{7.26}$$

and the Green operator $\mathbb{G}_D$ as

$$\mathbb{G}_D(f) = \Pi^D_\Omega f - \mathcal{D}\left[\left(\tfrac{1}{2}I + K\right)^{-1}\left(\operatorname{Tr}\Pi^D_\Omega f\right)\right], \tag{7.27}$$

assuming that all inverses exist.

Similar considerations apply in the case of the Poisson problem for the Laplacian with a homogeneous Neumann boundary condition:

$$\begin{cases} \Delta u = f\Big|_{C^\infty_0(\Omega)} \text{ as distributions in } \Omega, \\ \langle \operatorname{Tr} u, 1\rangle = 0, \quad \langle f, 1\rangle = 0, \\ \partial_\nu(u, f) = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{7.28}$$

Here, it is assumed that $u$ and $f$ belong to certain suitable spaces (in particular, $f$ is a functional, acting on a space which contains $C^\infty_0(\Omega)$) which make it possible to use the formalism (5.12). Then, the solution operator $\mathbb{G}_N : f \mapsto u$ for the problem (7.28) can be formally expressed as

$$\mathbb{G}_N f(x) = \int_\Omega G_N(x, y) f(y)\, dy, \qquad x \in \Omega, \tag{7.29}$$

where $G_N(x,y)$ is the Neumann function on $\Omega$, i.e. for each fixed (pole) $x \in \Omega$, it satisfies

$$\begin{cases} \Delta_y G_N(x,y) = \delta_x(y), & y \in \Omega, \\ \partial_\nu G_N(x,\cdot) = 1/\sigma(\partial\Omega) \text{ on } \partial\Omega, \end{cases} \tag{7.30}$$

where $\sigma(\partial\Omega)$ stands for the surface measure of $\partial\Omega$. The Neumann function $G_N$ can then be expressed as

$$G_N(x,y) = \Gamma(x-y) - \mathcal{S}\left\{\left(-\tfrac{1}{2}I + K^*\right)^{-1}\left[\partial_\nu\Gamma(x-\cdot)\Big|_{\partial\Omega} - \frac{1}{\sigma(\partial\Omega)}\right]\right\}(y) \tag{7.31}$$

for $x, y \in \Omega$, provided the inverse operator exists. An equivalent expression for $\mathbb{G}_N$ using layer potentials, which we will need in the sequel, is

$$\mathbb{G}_N f = \Pi_\Omega^N f - \mathcal{S}\left\{\left(-\tfrac{1}{2}I + K^*\right)^{-1}\left[\partial_\nu(\Pi_\Omega^N f, f)\right]\right\} - c \quad \text{in} \quad \Omega, \tag{7.32}$$

where the constant $c = c(\Omega, f) \in \mathbb{R}$ is uniquely determined by the condition that

$$\langle \mathrm{Tr}\,[\mathbb{G}_N f], 1\rangle = 0. \tag{7.33}$$

## 8. The Cauchy-Clifford operator

In this section we recall the Cauchy operator in Clifford analysis and review its main properties. These will be used in §10 for the proof of Theorem 1.3.

Recall that the Clifford algebra $\mathcal{A}_n$ is the unitary, associative (typically non-commutative) algebra freely generated by $n$ imaginary units, $e_1, \ldots, e_n$ which anti-commute. More specifically, with $\delta_{jk}$ denoting the Kronecker symbol, it is assumed that

$$e_j \cdot e_k + e_k \cdot e_j = -2\delta_{jk} e_0, \qquad 1 \le j, k \le n, \tag{8.1}$$

where $e_0$ stands for the multiplicative unit and "dot" denotes the Clifford algebra multiplication.

As is customary, we shall embed $\mathbb{R}^n$ into $\mathcal{A}_n$ by identifying a point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with the Clifford element $\sum_{i=1}^n x_i e_i \in \mathcal{A}_n$. In particular, any $\mathbb{R}^n$-valued function can be naturally viewed as taking values in $\mathcal{A}_n$. We shall also use the fact that there is a natural concept of norm and conjugation in $\mathcal{A}_n$ such that for each $x \in \mathbb{R}^n$

$$\bar{x} = -x, \quad \bar{x} \cdot x = x \cdot \bar{x} = |x|^2. \tag{8.2}$$

Next recall the Dirac operator $D := \sum_{j=1}^n e_j \partial_j$ and observe that $D^2 = -\Delta$ and that $-D\Gamma(x) = c_n \frac{x}{|x|^n}$ if $x \neq 0$ (where $c_n$ is a purely dimensional constant). Given a $\mathcal{A}_n$-valued function $u$, we write $Du$ and $uD$ for the action of $D$ on $u$, considered from the left and from the right, respectively. Now, if $\Omega \subset \mathbb{R}^n$ is a reasonable domain with outward unit normal $\nu = (\nu_1, ..., \nu_n) \equiv \nu_1 e_1 + \cdots + \nu_n e_n$, then for any two $\mathcal{A}_n$-valued functions $u$, $v$ defined in $\Omega$ which behave well near $\partial\Omega$, we have the following integration by parts identity:

$$\int_{\partial\Omega} u \cdot \nu \cdot v \, d\sigma = \int_\Omega [(uD) \cdot v + u \cdot (Dv)] \, dx \tag{8.3}$$

where, as before, $d\sigma$ stands for the surface measure on $\partial\Omega$. The Cauchy Clifford singular integral operator acting on a $\mathcal{A}_n$-valued function $f$ is defined as

$$\mathcal{C}f(x) := c_n \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \cdot \nu(y) \cdot f(y)\, d\sigma(y), \quad x \notin \partial\Omega. \tag{8.4}$$

Three important properties of this operator, themselves immediate corollaries of (8.3), are as follows. First, the Cauchy operator $\mathcal{C}$ reproduces the null-solutions of $D$ (acting from the left). Specifically,

$$\mathcal{C}(u|_{\partial\Omega}) = u \ \text{ in } \Omega \text{ if } Du = 0 \text{ in } \Omega. \tag{8.5}$$

Second,

$$\text{if } Du = 0 \text{ in } \Omega_- := \mathbb{R}^n \setminus \bar{\Omega}, \text{ then } \mathcal{C}(u|_{\partial\Omega}) = 0 \text{ in } \Omega. \tag{8.6}$$

Third, if we set $\Omega_+ := \Omega$, then the following Plemelj type jump-relation holds:

$$\mathcal{C}f|_{\partial\Omega_\pm} = (\pm\tfrac{1}{2}I + C)f, \tag{8.7}$$

where

$$Cf(x) := c_n\, p.v. \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \cdot \nu(y) \cdot f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{8.8}$$

and, as before, $p.v.$ indicates that the integral is taken in the principal value sense. More detailed accounts of these matters can be found in, e.g., [2], [9], [17].

## 9. Proof of Theorem 1.2

The starting point in the proof of the boundedness of the operator (1.11) is the observation that

$$\text{the region of points } (s, \tfrac{1}{p}) \text{ such that (1.11) is bounded is convex.} \tag{9.1}$$

Indeed, this is a simple consequence of (5.10).

Next consider the integral representation formula (7.32). The goal is to show that there exists $\varepsilon > 0$ with the following significance. Recall the region $\mathcal{P}(n,\varepsilon)$ introduced in (1.9)-(1.10) and, for an arbitrary pair $(s, \frac{1}{p}) \in \mathcal{P}(n,\varepsilon)$, consider $1 < p^*, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. In this setting, for each $f \in (L^{p'}_s(\Omega))^*$ which satisfies $\langle f, 1\rangle = 0$, we wish to show that the right-hand side of (7.32) is well-defined and contained in $L^{p^*}_{1-s}(\Omega)$.

To this end, we first remark that, by (7.4) and (5.5),

$$\Pi^N_\Omega f \in L^p_{2-s}(\Omega) \hookrightarrow L^{p^*}_{1-s}(\Omega). \tag{9.2}$$

This takes care of the first term in the right-hand side of (7.32). Treating the second term in the right-hand side of (7.32) amounts to proving that the operator

$$\mathcal{S} \circ (-\tfrac{1}{2}I + K^*)^{-1} \left[\partial_\nu(\Pi^N_\Omega \cdot, \cdot)\right] : \Big\{ g \in \Big(L^{p'}_s(\Omega)\Big)^* : \langle g, 1\rangle = 0 \Big\} \longrightarrow L^{p^*}_{1-s}(\Omega) \tag{9.3}$$

is well-defined and bounded. To prove this claim, fix two parameters, $q \in (1,\infty)$ and $\alpha \in \mathbb{R}$, and attempt to factor the operator (9.3) into

$$\left\{ g \in \left( L^{p'}_{s}(\Omega) \right)^{*} : \langle g, 1 \rangle = 0 \right\} \tag{9.4}$$

$$\Big\downarrow \left( \Pi^{N}_{\Omega}, \mathrm{I} \right)$$

$$\left\{ (w,g) \in L^{p}_{2-s}(\Omega) \oplus (L^{p'}_{s}(\Omega))^{*} : g\big|_{C^{\infty}_{0}(\Omega)} = \Delta w,\ \langle g, 1 \rangle = 0 \right\}$$

$$\Big\downarrow$$

$$\left\{ (w,g) \in L^{q}_{\alpha}(\Omega) \oplus (L^{q'}_{2-\alpha}(\Omega))^{*} : g\big|_{C^{\infty}_{0}(\Omega)} = \Delta w,\ \langle g, 1 \rangle = 0 \right\}$$

$$\Big\downarrow \partial_{\nu}$$

$$\left\{ \psi \in B^{q,q}_{\alpha-1-\frac{1}{q}}(\partial\Omega) : \langle \psi, 1 \rangle = 0 \right\}$$

$$\Big\downarrow \mathcal{S} \circ (-\tfrac{1}{2} I + K^{*})^{-1}$$

$$L^{q}_{\alpha}(\Omega)$$

$$\Big\downarrow$$

$$L^{p^{*}}_{1-s}(\Omega)$$

where, as before, "I" stands for the identity operator.

At this point, the strategy is to collect various conditions on the indices involved ensuring that each operator in this diagram is well-defined and bounded. Ultimately, we want to show that, as far as the indices $p$ and $s$ are concerned, these conditions amount to the membership of $(s, \frac{1}{p})$ to $\mathcal{P}(n, \varepsilon)$ for some $\varepsilon = \varepsilon(\partial\Omega) > 0$.

Turning to specifics, we first note that the first arrow in (9.4) is, thanks to Corollary 7.2, well-defined and bounded for any $1 < p < \infty$, $s \in \mathbb{R}$. By (5.5), the second arrow in the diagram is well-defined provided

$$2 - s \geq \alpha \quad \text{and} \quad \frac{1}{q} - \frac{\alpha}{n} = \frac{1}{p} - \frac{2-s}{n}. \tag{9.5}$$

From the setup of the normal derivative operators in (5.12), we see that the third arrow in (9.4) is well-defined whenever

$$-1 < \alpha - 1 - \tfrac{1}{q} < 0. \tag{9.6}$$

Next, by (7.24) and (7.13), there exists $\varepsilon > 0$ such that the fourth arrow in (9.4) is well-defined and bounded if

$$\Big(-\alpha + 1 + \frac{1}{q}, 1 - \frac{1}{q}\Big) \in \mathcal{R}^1_\varepsilon \quad \text{for} \quad n = 2, \tag{9.7}$$

$$\Big(-\alpha + 1 + \frac{1}{q}, 1 - \frac{1}{q}\Big) \in \mathcal{R}^2_\varepsilon \quad \text{for} \quad n = 3. \tag{9.8}$$

Finally, the last arrow in (9.4) is well-defined granted that

$$\alpha \geq 1 - s \quad \text{and} \quad \frac{1}{p^*} - \frac{1-s}{n} = \frac{1}{q} - \frac{\alpha}{n}, \tag{9.9}$$

once again by (5.5). To summarize, given $1 < p < n$ and $s \in \mathbb{R}$, the operator (9.3) is bounded provided $q, \alpha$ can be chosen such that (9.5)-(9.9) are satisfied. There remains to express this condition on $p$ and $s$ in a more direct form, without having to involve the auxiliary parameters $q$, $\alpha$.

With this in mind, fix $\varepsilon > 0$ as given by Propositions 7.3 and 7.4. In order to simplify the exposition, we find it useful to set $t := 2 - s$. Then the first conditions in (9.5) and (9.9) are equivalent to

$$t - 1 \leq \alpha \leq t. \tag{9.10}$$

As for the last condition in (9.9), this is a consequence of (9.5) and the definition of $p^*$. Next, based on Proposition 7.4, we see that (9.7)-(9.8) hold if and only if

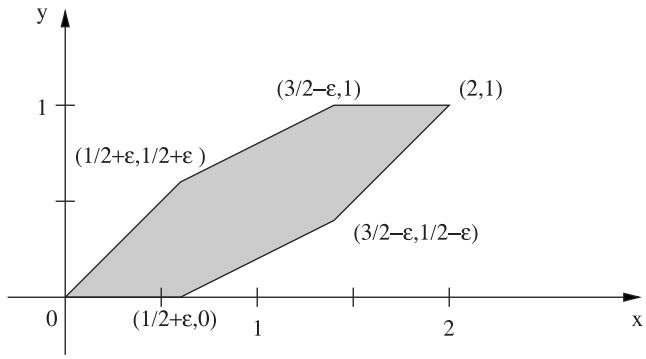
$$(\alpha, \tfrac{1}{q}) \in \mathrm{H}^1_\varepsilon \quad \text{for} \quad n = 2, \qquad \text{and} \qquad (\alpha, \tfrac{1}{q}) \in \mathrm{H}^2_\varepsilon \quad \text{for} \quad n \geq 3, \tag{9.11}$$

where the hexagons $\mathrm{H}^1_\varepsilon$ and $\mathrm{H}^2_\varepsilon$ are depicted below.

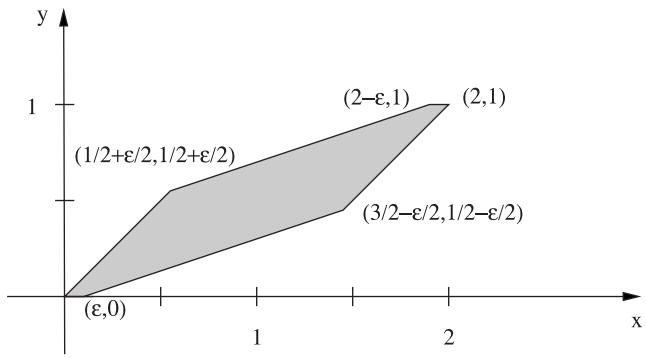
Let us also observe that (9.6) is equivalent to the membership of the point $(\alpha, \frac{1}{q})$ to the region lying in between the lines $y = x$ and $y = x - 1$ and that this latter condition is automatically satisfied if (9.11) holds.

Finally, the last condition in (9.5) is equivalent to the requirement that $(\alpha, \frac{1}{q})$ lies on the line passing through $(t, \frac{1}{p})$ and having slope $\frac{1}{n}$. To sum up, we are looking for the convex hull of the set of points $(t, \frac{1}{p})$ for which a point $(\alpha, \frac{1}{q})$ can be found such that:

(i) $(\alpha, \frac{1}{q})$ lies on the line of slope $\frac{1}{n}$ passing through $(t, \frac{1}{p})$;
(ii) $(\alpha, \frac{1}{q})$ lies at most one unit (measured horizontally) to the left of $(t, \frac{1}{p})$;
(iii) (9.11) is satisfied.




The region $H_\varepsilon^1$

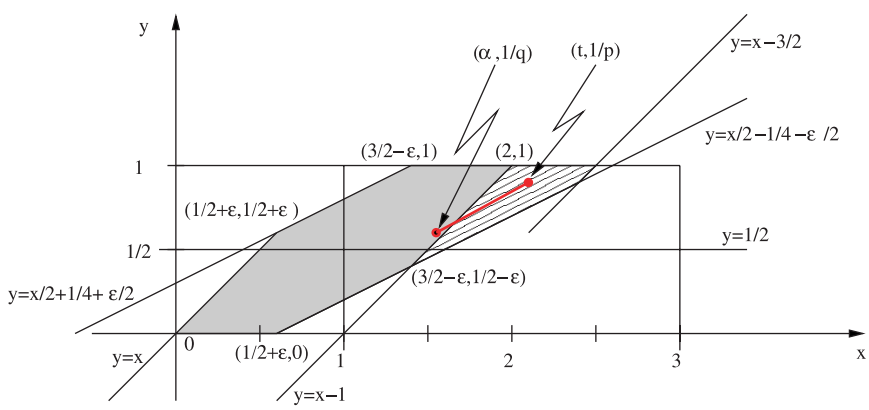



The region $H_\varepsilon^2$

Given that $H_\varepsilon^1$ and $H_\varepsilon^2$ have different shapes, in order to continue our analysis of conditions (i)-(iii), we find it convenient to treat the cases $n = 2$ and $n \geq 3$ separately.

**Case 1.** $n = 2$

Let us consider the figure below.

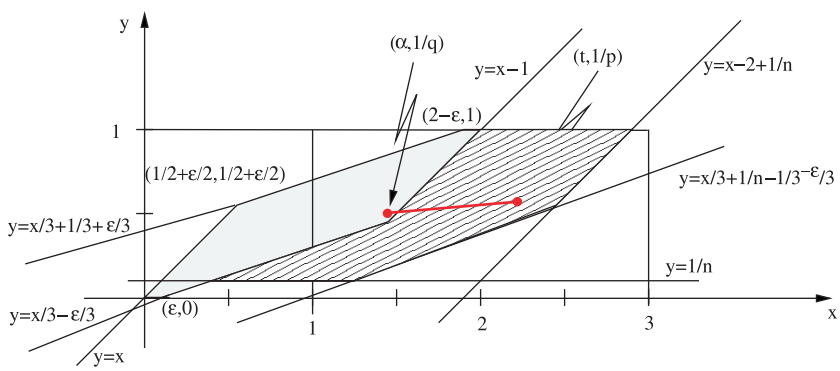



If the point $(t, \frac{1}{p})$ is taken inside the region bounded by the lines $y = 1$, $y = x - 1$, $y = \frac{1}{2}$, $y = \frac{1}{2}x - \frac{1}{4} - \frac{\varepsilon}{2}$, and $y = x - \frac{3}{2}$, then there exists a point $(\alpha, \frac{1}{q})$ satisfying

(i)-(iii) above. If $(t, \frac{1}{p})$ is taken inside the region bounded by the lines $y = 1$, $y = \frac{1}{2}x + \frac{1}{4} + \frac{\varepsilon}{2}$, $y = x$, $y = \frac{1}{2}$, and $y = x - 1$, we can take $\alpha = t$, $q = p$, and again (i)-(iii) above are satisfied. Taking the convex hull of these points yields the region bounded by the lines $y = 1$, $y = \frac{1}{2}x + \frac{1}{4} + \frac{\varepsilon}{2}$, $y = x$, $y = \frac{1}{2}$, $y = \frac{1}{2}x - \frac{1}{4} - \frac{\varepsilon}{2}$, and $y = x - \frac{3}{2}$. Finally, applying the transformation $(x, y) \mapsto (2 - x, y)$ gives $\mathcal{P}(2, \varepsilon)$.

**Case 2.** $n \geq 3$

The main ideas in the proof of this case are similar to the ones used for $n = 2$. This time we have to analyze the following region:




The convex hull of the set of points $(t, \frac{1}{p})$ for which a point $(\alpha, \frac{1}{q})$ can be found such that (i)-(iii) are verified is the region bounded by the lines $y = 1$, $y = \frac{1}{3}x + \frac{1}{3} + \frac{\varepsilon}{3}$, $y = x$, $y = \frac{1}{n}$, $y = \frac{1}{3}x - \frac{\varepsilon}{3} + \frac{1}{n} - \frac{1}{3}$, and $y = x - 2 + \frac{1}{n}$. Applying the transformation $(x, y) \mapsto (2 - x, y)$ to the latter region yields $\mathcal{P}(n, \varepsilon)$. This completes the proof of the boundedness of the operator (1.11) in Theorem 1.2. The treatment of the operator (1.12) is done along a similar line (cf. also [15]).

## 10. Proof of Theorem 1.3

The crux of the matter is addressed by the theorem below.

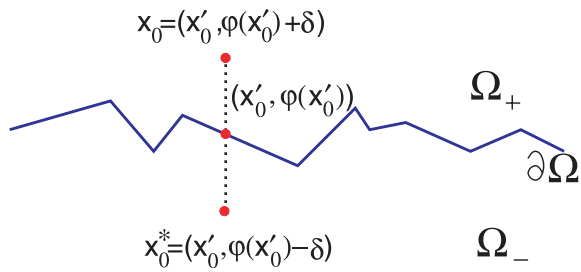
**Theorem 10.1.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, $\nabla G_D(x, \cdot)$ and $\nabla G_N(x, \cdot)$ belong to $L^{\frac{n}{n-1},\infty}(\Omega)$ uniformly in $x \in \Omega$.*

*Proof.* We complete the proof of Theorem 10.1 in 6 steps as follows.

**Step I.** $\nabla\Gamma(x - \cdot) \in L^{\frac{n}{n-1},\infty}(\Omega)$ uniformly in $x \in \Omega$. This follows readily from (2.10).

**Step II.** $\partial_\nu \Gamma(x - \cdot)\,|_{\partial\Omega}$ and $\partial_{\tau_{jk}} \Gamma(x - \cdot)\,|_{\partial\Omega}$ belong to $h^{1,\infty}(\partial\Omega)$, uniformly in $x \in \Omega$, for $j, k = 1, \ldots, n$.

The proof of this step relies on Clifford algebra techniques. Assume that $\Omega$ is the region above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ (the case of a bounded Lipschitz domain is just a minor variation of the reasoning below). Fix a point $x_0 \in \Omega_+$ and, if $x_0' \in \mathbb{R}^{n-1}$ denotes the first $n - 1$ components of $x_0$, we have that $(x_0', \varphi(x_0')) \in \partial\Omega$. Let $\delta > 0$ be such that $x_0 = (x_0', \varphi(x_0') + \delta)$ and further set $x_0^* := (x_0', \varphi(x_0') - \delta)$ as seen in the following figure.




If we now define

$$\Phi_{x_0}(x) := \frac{x-x_0}{|x-x_0|^n} - \frac{x-x_0^*}{|x-x_0^*|^n}, \tag{10.1}$$

then a direct computation shows that for $x = (x', \varphi(x')) \in \partial\Omega$

$$|\Phi_{x_0}(x)| \le c\frac{\delta}{[|x'-x_0'|^2+\delta^2]^{\frac{n}{2}}} \in L^1(\partial\Omega), \qquad \text{uniformly in } x_0. \tag{10.2}$$

We also claim that

$$\mathcal{C}(\Phi_{x_0}\,|_{\partial\Omega})(x) = \frac{x-x_0}{|x-x_0|^n} \quad \text{for } x \in \Omega_- = \mathbb{R}^n \setminus \bar{\Omega}. \tag{10.3}$$

Indeed, $\frac{x-x_0}{|x-x_0|^n} = c_n D\Gamma(x-x_0)$, so that $D(\frac{x-x_0}{|x-x_0|^n}) = c_n\, D^2\Gamma(x-x_0) = 0$ for $x \in \Omega_-$, since $D^2 = -\Delta$. Consequently, formula (8.5) gives $\mathcal{C}(\frac{\cdot-x_0}{|\cdot-x_0|^n}|_{\partial\Omega})(x) = \frac{x-x_0}{|x-x_0|^n}$ for $x \in \Omega_-$. In addition, since $D(\frac{x-x_0^*}{|x-x_0^*|^n}) = 0$ for $x \in \Omega_+$, formula (8.6) gives that $\mathcal{C}(\frac{\cdot-x_0^*}{|\cdot-x_0^*|^n}|_{\partial\Omega})(x) = 0$ for $x \in \Omega_-$. This proves (10.3).

Consider next the (non-tangential) traces on $\partial\Omega$ of both sides in (10.3). Keeping in mind that, as pointed out in (8.7), we have $\operatorname{Tr}\mathcal{C} = -\frac{1}{2}I + C$, we obtain, after multiplying each trace (in the Clifford algebra sense) from the left by $\nu$,

$$\begin{aligned} \nu(x)\cdot\left(-\tfrac{1}{2}I + C\right)(\Phi_{x_0}\,|_{\partial\Omega})(x) &= \nu(x)\cdot\frac{x-x_0}{|x-x_0|^n} \\ &= -\partial_\nu\Gamma(x-x_0) + \sum_{1\le j<k\le n}(\nu_j\partial_k - \nu_k\partial_j)\Gamma(x-x_0)e_j\cdot e_k, \end{aligned} \tag{10.4}$$

for almost every $x \in \partial\Omega$. Note that $(\nu_j\partial_k - \nu_k\partial_j) = \partial_{\tau_{jk}}$ are tangential derivatives. The integral kernel of the operator $\nu(x)\cdot C$ is $\nu(x)\cdot\frac{x-x_0}{|x-x_0|^n}$ and, thanks to the last equality in (10.4), it can be written as a linear combination of the integral kernels of $K^*$ and $\partial_{\tau_{jk}}S$, $j,k = 1,\dots,n$. From (10.2), (6.9), (10.4), (7.17), and (7.19) we may therefore conclude that $\partial_\nu\Gamma(x_0-\cdot)\,|_{\partial\Omega}$ and $\partial_{\tau_{jk}}\Gamma(x_0-\cdot)\,|_{\partial\Omega}$, $j,k = 1,\dots,n$, belong to $h^{1,\infty}(\partial\Omega)$, uniformly in $x_0 \in \Omega$, as desired.

**Step III.** $-\frac{1}{2}I + K^*$ is invertible on $h^{1,\infty}(\partial\Omega)$.

Since $-\frac{1}{2}I + K^*$ is invertible on $h^p(\partial\Omega)$ (see [6] for $1 \le p \le 2$ and [3] for $1-\epsilon < p < 1$), we use the real method of interpolation to obtain the desired conclusion, making essential use of (6.10).

**Step IV.** $\nabla\mathcal{S} : h^{1,\infty}(\partial\Omega) \to L^{\frac{n}{n-1},\infty}(\Omega)$ is a bounded operator.

By interpolation, matters are reduced to proving that for $\frac{n-1}{n} < p < \infty$, the operator $\nabla\mathcal{S} : h^p(\partial\Omega) \to L^{\frac{np}{n-1}}(\Omega)$ is bounded. To show the latter we first recall that Calderón-Zygmund theory gives that if $\frac{n-1}{n} < p < \infty$ and $f \in h^p(\partial\Omega)$, then $\mathcal{N}(\nabla\mathcal{S}f) \in L^p(\partial\Omega)$. Now the desired conclusion follows by employing Lemma 4.1 in conjunction with (2.7)-(2.8).

Combining Steps I-IV with (7.31), we obtain that $\nabla G_N(x,\cdot) \in L^{\frac{n}{n-1},\infty}(\Omega)$ uniformly in $x \in \Omega$.

**Step V.** $\Gamma(x-\cdot)\,|_{\partial\Omega}$ belongs to $h_1^{1,\infty}(\partial\Omega)$, uniformly in $x \in \Omega$.

This is an immediate consequence of Step II and Corollary 6.3.

**Step VI.** There exists $\varepsilon > 0$ such that $\frac{1}{2}I + K$ is invertible on $h_1^{p,\infty}(\partial\Omega)$, for $1-\varepsilon < p \leq 1$.

Invertibility of this operator at the level of $h_1^p(\partial\Omega)$, for $1 \leq p \leq 2$, was proved in [6], and then perturbation theory yields the extension $1-\varepsilon < p \leq 1$. Now the desired conclusion follows by invoking Corollary 6.3.

Now we are ready to finish the proof of Theorem 10.1. Indeed, for each $j \in \{1,\dots,n\}$, a direct computation gives that $\partial_j\mathcal{D}f = \sum_{k=1}^{n} \partial_k\mathcal{S}(\partial_{\tau_{jk}}f)$ for sufficiently smooth $f$. Now, combining this with (7.26), Step I, Step V, Step VI, and Step IV, we obtain that $\nabla G_D(x,\cdot) \in L^{\frac{n}{n-1},\infty}(\Omega)$ uniformly in $x \in \Omega$. □

After this preamble, we are finally ready to prove the following rephrasing of Theorem 1.3.

**Theorem 10.2.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Then the following operators are bounded:*

$$\nabla\mathbb{G}_D : L^1(\Omega) \to L^{\frac{n}{n-1},\infty}(\Omega), \tag{10.5}$$

$$\nabla\mathbb{G}_N : L^1(\Omega) \to L^{\frac{n}{n-1},\infty}(\Omega). \tag{10.6}$$

*Proof.* Making use of (2.9), we see that $\left(L^{n,1}(\Omega)\right)^* = L^{\frac{n}{n-1},\infty}(\Omega)$. Hence, by Theorem 10.1, $\nabla_y G_N(x,y) \in \left(L^{n,1}(\Omega)\right)^*$ as a function in $y$, uniformly for $x \in \Omega$. Since $G_N(x,y) = G_N(y,x)$, it follows from (7.29) that

$$(\nabla\mathbb{G}_N)^* : L^{n,1}(\Omega) \longrightarrow L^\infty(\Omega),$$

in a linear and bounded fashion. Thus,

$$\nabla\mathbb{G}_N : \left(L^\infty(\Omega)\right)^* \to \left(L^{n,1}(\Omega)\right)^* = L^{\frac{n}{n-1},\infty}(\Omega). \tag{10.7}$$

The boundedness of the operator (10.6) now follows by observing that $L^1(\Omega) \hookrightarrow \left(L^\infty(\Omega)\right)^*$. The boundedness of the operator (10.5) is proved similarly. □

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