

HOCHSCHILD COHOMOLOGY AND GRADED HECKE ALGEBRAS

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We dedicate this article to Sergey Yuzvinsky on the occasion of his 70th birthday.

ABSTRACT. We develop and collect techniques for determining Hochschild cohomology of skew group algebras $S(V)\#G$ and apply our results to graded Hecke algebras. We discuss the explicit computation of certain types of invariants under centralizer subgroups, focusing on the infinite family of complex reflection groups $G(r, p, n)$ to illustrate our ideas. Resulting formulas for Hochschild two-cocycles give information about deformations of $S(V)\#G$ and, in particular, about graded Hecke algebras. We expand the definition of a graded Hecke algebra to allow a nonfaithful action of G on V , and we show that there exist nontrivial graded Hecke algebras for $G(r, 1, n)$, in contrast to the case of the natural reflection representation. We prove that one of these graded Hecke algebras is equivalent to an algebra that has appeared before in a different form.

1. INTRODUCTION

Lusztig [18] showed that the graded version of an affine Hecke algebra essentially retains its representations, which in turn determine certain representations of the corresponding group of Lie type [15]. Lusztig's graded Hecke algebra for a Coxeter group is a deformation of the skew group algebra generated by the group and the vectors upon which the group acts (in its natural reflection representation). Drinfeld [9] had considered a different type of deformation, a degenerate affine Hecke algebra. He allowed *any* finite subgroup of $GL(V)$ and used skew-symmetric forms on V to deform the skew group algebra. Ram and the first author [22] showed that Lusztig's graded Hecke algebra was a special case of Drinfeld's construction. Drinfeld's algebra thus serves as a graded Hecke algebra for arbitrary finite subgroups of $GL(V)$. One complication arises from this viewpoint when working with a wider class of groups: Drinfeld's construction for a given finite subgroup of $GL(V)$ may only yield the trivial graded Hecke algebra. This article is partly motivated by a

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desire to understand and resolve this problem. We use Hochschild cohomology as a tool to explore those deformations of skew group algebras that satisfy an expanded definition of graded Hecke algebra. To illustrate our ideas (and as an application of results), we focus on the infinite family of complex reflection groups.

Drinfeld's construction yields only the trivial algebra for the complex reflection groups $G(r, 1, n) = (\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ when $r \geq 3$ and $n \geq 4$ (see [22]). Ariki and Koike [1] defined a Hecke algebra (the Ariki-Koike algebra) for these complex reflection groups by generators and relations. Around the same time, Broué and Malle [2] defined a cyclotomic Hecke algebra for $G(r, 1, n)$ (coinciding with the Ariki-Koike algebra) and Broué, Malle, and Rouquier [4] gave a topological description using braid groups (see also [3]). Although the Hecke algebra itself enjoys fruitful study (see the survey paper by Mathas [20]), a satisfactory formulation of an affine Hecke algebra for the groups $G(r, 1, n)$ seems to be lacking. One asks when a graded Hecke algebra may be a reasonable substitute. In this article, we use Hochschild cohomology and algebraic deformation theory to address a natural question: When and what kinds of graded Hecke algebras may be defined for $G(r, 1, n)$?

Although Drinfeld's construction only yields trivial graded Hecke algebras for $G(r, 1, n)$ in general, Ram and the first author [22] constructed a novel graded Hecke algebra for these groups using ad hoc methods. Dezélee [7] independently discovered the same algebra as a subalgebra of a rational Cherednik algebra in the case $r = 2$, that is, for the Weyl group $WB_n = G(2, 1, n)$, and defined this algebra for $WD_n = G(2, 2, n)$. Dezélee [8] further realized the graded Hecke algebras of Ram and the first author, for general r , as subalgebras of rational Cherednik algebras (equivalently, graded Hecke algebras associated to $V \oplus V^*$, where V is the reflection representation). These rational Cherednik algebras are examples of symplectic reflection algebras, defined by Etingof and Ginzburg [11], who were interested in related deformations of orbifolds.

In this paper, we expand the definition of graded Hecke algebra to allow non-faithful representations of a finite group G on a finite dimensional vector space V (Definition 8.1). We construct a natural graded Hecke algebra for $G(r, 1, n)$ using a nonfaithful representation, and we show that this construction coincides with the algebra of Ram and the first author [22] (Theorem 9.8). We further give the generic form of a graded Hecke algebra associated to $G(r, 1, n)$, depending on many parameters, in (9.1). A related paper of Chmutova [6] deals with existence of these more general types of graded Hecke algebras defined for any symplectic representation of G .

A graded Hecke algebra for a group G , with a given representation on a finite dimensional vector space V , is a particular type of deformation of the skew group algebra $S(V) \# G$. Thus every graded Hecke algebra defines an element in degree 2 Hochschild cohomology $\mathrm{HH}^2(S(V) \# G)$. We characterize all possible graded Hecke algebras, both in relation to Hochschild cohomology (Theorem 8.7) and in relation to defining skew-symmetric forms (Corollary 8.17). Corollary 8.17 generalizes a result of Ram and the first author [22], first formulated by Drinfeld [9] for Coxeter groups.

There may well be other compelling deformations of $S(V) \# G$ (arising from some of the many elements of degree 2 Hochschild cohomology) that are not graded Hecke algebras. Some small examples of such deformations were given in [5, 26]. With these potential deformations in mind, as well as the immediate applications to

graded Hecke algebras in this article, we compute the relevant cohomology in two cases:

- (1) G is the monomial reflection group $G(r, p, n)$ acting via its natural reflection representation on $V = \mathbb{C}^n$ (Theorem 5.1), and
- (2) $G = G(r, p, n)$ acting on $V = \mathbb{C}^n$ via the permutation representation of its quotient symmetric group \mathfrak{S}_n (Theorem 7.7).

The graded vector space structure of Hochschild cohomology $\mathrm{HH}^*(S(V)\#G)$ can be described in terms of invariants of centralizer subgroups of G (see [12, 14]). We recall this structure in Section 3 and reformulate it in terms of semi-invariants. In Sections 5–7, we explicitly compute these semi-invariants in degree 2 using techniques of classical invariant theory and the theory of hyperplane arrangements; the full Hochschild cohomology space $\mathrm{HH}^*(S(V)\#G)$ could be computed similarly using our approach. Although we are primarily interested in applications to graded Hecke algebras and other deformations, our explicit computation of Hochschild cohomology should also be of interest to homological algebraists and to geometers studying cohomology for associated orbifolds.

Our results in case (1) above show that the lack of nontrivial graded Hecke algebras (when $r \geq 3$, $n \geq 4$, see [22]) stems from a lack of certain types of elements in $\mathrm{HH}^2(S(V)\#G)$. This conclusion motivates our computation in case (2), leading to the aforementioned nontrivial graded Hecke algebras under a nonfaithful representation of G .

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2. TERMINOLOGY AND NOTATION

We work over the complex numbers \mathbb{C} , letting $\otimes = \otimes_{\mathbb{C}}$ unless otherwise specified. Let G be a finite group and let V be the vector space \mathbb{C}^n with a representation of G . Let V^* denote the contragredient (or dual) representation. We will often need a choice of basis v_1, \dots, v_n of V and dual basis x_1, \dots, x_n of V^* .

Let R be an associative \mathbb{C} -algebra with an action of G by automorphisms. In this paper, R will always be either the symmetric algebra $S(V)$ or the tensor algebra $T(V)$. The **skew group algebra** $R\#G$ is the vector space $R \otimes \mathbb{C}G$ with multiplication

$$(r \otimes g)(s \otimes h) = r \cdot g(s) \otimes gh$$

for all $r, s \in R$ and $g, h \in G$. We abbreviate $r \otimes g$ by $r\bar{g}$ ($r \in R, g \in G$) and $r \otimes 1$ or $1 \otimes g$ simply by r or \bar{g} , respectively. Note that an element $g \in G$ acts on R by conjugation by \bar{g} : $\bar{g}r(\bar{g})^{-1} = g(r)\bar{g}(\bar{g})^{-1} = g(r)$ for all $r \in R$.

For our cohomological computations, we fix some notation involving the action of G on V . Let $V^G = \{v \in V : g(v) = v \text{ for all } g \in G\}$, the set of G -invariants in V . More generally, let $\chi : G \rightarrow \mathbb{C}^\times$ be any linear character (i.e., any group homomorphism from G to \mathbb{C}^\times) and let $V^\chi = \{v \in V : g(v) = \chi(g)v \text{ for all } g \in G\}$, the set of χ -**invariants** in V (also called **semi-invariants** with respect to χ).

For any $g \in G$, let $Z(g) = \{h \in G : gh = hg\}$ be the centralizer of g in G , and let $V^g = \{v \in V : g(v) = v\}$ be the g -invariant subspace of V . Since G is finite, we may assume G acts by isometries on V (i.e. G preserves a Hermitian form). If $h \in Z(g)$, then h preserves both V^g and its orthogonal complement $(V^g)^\perp$ and we define

$$h^\perp = h|_{(V^g)^\perp}$$

(i.e., h^\perp is the linear transformation by which h acts on the vector space $(V^g)^\perp$). We are particularly interested in semi-invariants with respect to the **Hochschild character** $\chi_g : Z(g) \rightarrow \mathbb{C}$ of g defined by

$$(2.1) \quad \chi_g(h) = \det(h^\perp)$$

for all $h \in Z(g)$. When no confusion will arise, we simply write χ instead of χ_g . Note that χ_g is a linear character. Let 1 denote the identity element of G and let I_n (or just I) denote the $n \times n$ identity matrix.

Our results involve the following complex reflection groups parametrized by positive integers r, p, n : The **full monomial group** $G(r, 1, n) < \text{GL}_n(\mathbb{C})$ is the group of all monomial $n \times n$ matrices whose nonzero entries are r -th roots of unity. The full monomial group is a finite complex reflection group acting on $V = \mathbb{C}^n$ and is isomorphic to the wreath product of the cyclic group of order r and the symmetric group:

$$G(r, 1, n) = N \cdot G(1, 1, n) \cong (\mathbb{Z}/r\mathbb{Z})^n \rtimes \mathfrak{S}_n = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n,$$

where N is the normal subgroup of diagonal matrices in $G(r, 1, n)$ and $G(1, 1, n) \leq G(r, 1, n)$ is isomorphic to the symmetric group \mathfrak{S}_n .

Let $\xi \in \mathbb{C}$ be a fixed primitive r -th root of unity. Let ξ_i be the diagonal matrix $\text{diag}\{1, \dots, 1, \xi, 1, \dots, 1\}$, with i -th entry ξ . We denote each element in $G(r, 1, n)$ by $g = \xi_1^{a_1} \dots \xi_n^{a_n} \sigma$, where $\sigma \in \mathfrak{S}_n$ and $0 \leq a_i < r$. In the special case that r is even (so $\xi^{r/2} = -1$), we write $(1, -2) := \xi_2^{r/2}(1, 2)$ for the transformation $v_2 \mapsto v_1, v_1 \mapsto -v_2$.

For any p dividing r , we consider a natural subgroup of $G(r, 1, n)$ also acting as a reflection group on V . Let $\psi(M)$ be the product of the nonzero entries of any matrix M . Then

$$G(r, p, n) = \{M \in G(r, 1, n) : \psi(M)^{r/p} = 1\};$$

that is, M in $G(r, 1, n)$ lies in $G(r, p, n)$ when the sum of the exponents of ξ in the nonzero entries of M is 0 modulo p . Several real reflection groups (Coxeter groups) are special cases of the groups $G(r, p, n)$:

- $G(1, 1, n)$ is the symmetric group \mathfrak{S}_n ,
- $G(2, 1, n)$ is the Weyl group WB_n of type B_n ,
- $G(2, 2, n)$ is the Weyl group WD_n of type D_n , and
- $G(r, r, 2)$ is the dihedral group $I_2(r)$ of order $2r$.

We briefly record some facts on centralizers and conjugacy classes in $G(r, 1, n)$. Note that $Z_{G(r,p,n)}(g) = Z_{G(r,1,n)}(g) \cap G(r,p,n)$ for any g in $G(r,p,n)$. The (generalized) cycle structure of an element g in $G(r, 1, n)$ determines its conjugacy class and the size of its centralizer $Z(g)$ in the following way. Every element g in $G(r, 1, n)$ can be written as the product of disjoint cycles of the form $\xi_i^{a_i} \xi_{i+1}^{a_{i+1}} \dots \xi_m^{a_m}(i, i + 1, \dots, m)$. We call such an element an (a, k) -cycle, where $k = m - i + 1$ and $a = a_i + a_{i+1} + \dots + a_m \pmod r$. Two elements in $G(r, 1, n)$ are conjugate if and only if they have the same number of (a, k) -cycles for each a and k (where $0 \leq a < r$ and $1 \leq k \leq n$). Let $m_{a,k}$ be the number of (a, k) -cycles for g ; the order of the centralizer of g in $G(r, 1, n)$ is

$$(2.2) \quad |Z(g)| = \prod_{a,k} m_{a,k}! \cdot k^{m_{a,k}} \cdot r^{m_{a,k}}.$$

See [22, Equation (2.5)] and [19, Appendix B, Section 3].

3. HOCHSCHILD COHOMOLOGY OF $S(V)\#G$

The **Hochschild cohomology** of a \mathbb{C} -algebra R is the graded vector space $\text{HH}^\bullet(R) = \text{Ext}^\bullet_{R^e}(R, R)$, where $R^e = R \otimes R^{op}$ acts on R by left and right multiplication. More generally, if M is an R -bimodule (equivalently a left R^e -module), then $\text{HH}^\bullet(R, M) = \text{Ext}^\bullet_{R^e}(R, M)$ so that $\text{HH}^\bullet(R) = \text{HH}^\bullet(R, R)$. For more details, see [25].

Fix a finite group G and a representation $V = \mathbb{C}^n$ of G . The Hochschild cohomology of $S(V)\#G$ is given in [12, 14] when G acts faithfully on V . In this section, we reformulate this result to aid our explicit computations and our determination of graded Hecke algebras. We state all results more generally for representations which are not necessarily faithful. No new techniques are needed for the generalization, and so we merely sketch those arguments that appear elsewhere for the faithful case.

Let \mathcal{C} be a set of representatives of the conjugacy classes of G . In the following theorem, a negative exterior power is interpreted as 0. Recall the Hochschild characters χ_g defined in (2.1).

Theorem 3.1. *There is an isomorphism of graded vector spaces*

$$\text{HH}^\bullet(S(V)\#G) \cong \bigoplus_{g \in \mathcal{C}} \left(S(V^g) \otimes \Lambda^{\bullet - \text{codim } V^g}((V^g)^*) \right)^{\chi_g}.$$

For any $g \in \mathcal{C}$, let $\text{HH}^2(g)$ be the g -component of $\text{HH}^2(S(V)\#G)$ in the theorem:

$$(3.2) \quad \text{HH}^2(g) = \left(S(V^g) \otimes \Lambda^{2 - \text{codim } V^g}((V^g)^*) \right)^{\chi_g}.$$

We call $\text{HH}^2(g)$ the set of **Hochschild semi-invariants** of g . Note $\text{HH}^2(g) = 0$ if $\text{codim } V^g > 2$, since the exterior power $2 - \text{codim } V^g$ is negative in this case.

Sketch of a proof of Theorem 3.1. We give only an outline as the bulk of the calculation follows [12, 26]. A result of Ştefan on Hopf Galois extensions [24, Cor. 3.4] implies that there is an action of G on $\text{HH}^\bullet(S(V), S(V)\#G)$ for which

$$\text{HH}^\bullet(S(V)\#G) \cong \text{HH}^\bullet(S(V), S(V)\#G)^G.$$

Now $S(V)\#G \cong \bigoplus_{g \in G} S(V)\bar{g}$ as an $S(V)$ -bimodule. The above isomorphism of graded vector spaces may thus be rewritten as

$$(3.3) \quad \text{HH}^\bullet(S(V)\#G) \cong \left(\bigoplus_{g \in G} \text{HH}^\bullet(S(V), S(V)\bar{g}) \right)^G.$$

The action of G permutes the direct summands by conjugation on their indices $g \in G$. We may therefore rewrite the vector space structure of these G -invariants as $\bigoplus_{g \in \mathcal{C}} \text{HH}^\bullet(S(V), S(V)\bar{g})^{Z(g)}$ (one applies a transfer (trace) operator $\sum_{h \in G/Z(g)} h$ to each $Z(g)$ -invariant to obtain a G -invariant). To compute the summands $\text{HH}^\bullet(S(V), S(V)\bar{g})^{Z(g)}$, one may first use a Koszul complex (see [25, §4.5] for details of Koszul complexes) to find $\text{HH}^\bullet(S(V), S(V)\bar{g})$ and then determine the $Z(g)$ -invariants. The use of a Koszul complex yields

$$(3.4) \quad \text{HH}^\bullet(S(V)\#G) \cong \bigoplus_{g \in \mathcal{C}} \left(S(V^g)\bar{g} \otimes \Lambda^{\bullet - \text{codim } V^g}((V^g)^*) \otimes \Lambda^{\text{codim } V^g}(((V^g)^\perp)^*) \right)^{Z(g)},$$

where $Z(g)$ acts diagonally on the given tensor product of three representations. Its action on the factor $S(V^g)\bar{g}$ is by conjugation by \bar{h} ($h \in Z(g)$). The actions on the given dual vector spaces are the contragredient actions. The factor $\bigwedge^{\text{codim } V^g}(((V^g)^\perp)^*)$ is one-dimensional since $\text{codim } V^g = \dim((V^g)^\perp)^*$. We also denote this factor by $\det(((V^g)^\perp)^*)$; it is included in the above expression (3.4) only since it carries a potentially nontrivial $Z(g)$ -action.

We find the contribution to Hochschild cohomology from each $g \in G$ representing a conjugacy class by determining the $Z(g)$ -invariants of

$$(3.5) \quad S(V^g)\bar{g} \otimes \bigwedge^{\star-\text{codim } V^g}((V^g)^*) \otimes \det(((V^g)^\perp)^*).$$

Let x_1, \dots, x_m be a basis of $((V^g)^\perp)^*$. Let $s \in S(V^g) \otimes \bigwedge^{\star-\text{codim } V^g}((V^g)^*)$. Then the element $s\bar{g} \otimes x_1 \wedge \dots \wedge x_m$ of $S(V^g)\bar{g} \otimes \bigwedge^{\star-\text{codim } V^g}((V^g)^*) \otimes \det(((V^g)^\perp)^*)$ is $Z(g)$ -invariant if and only if

$$s\bar{g} \otimes x_1 \wedge \dots \wedge x_m = \det((h^{-1})^\perp) h(s)\bar{g} \otimes x_1 \wedge \dots \wedge x_m$$

for all $h \in Z(g)$. Equivalently, $h(s) = \det(h^\perp)s = \chi_g(h)s$ for all $h \in Z(g)$. □

The next lemma gives in particular a necessary condition for $\text{HH}^2(g)$ to be nonzero (cf. [12, Example 3.10]).

Lemma 3.6. *Let $g \in G$, and suppose $(S(V^g) \otimes \bigwedge^{i-\text{codim } V^g}((V^g)^*))^{\chi_g} \neq 0$ for some $i \geq 0$. If $h \in Z(g)$ and $h|_{V^g}$ is the identity map, then $\det(h) = 1$. In particular, $\det(g) = 1$.*

Proof. Let s be a nonzero element of $(S(V^g) \otimes \bigwedge^{i-\text{codim } V^g}((V^g)^*))^{\chi_g}$. If $h \in Z(g)$ and $h|_{V^g}$ is the identity, then $h(s) = s$, and so $\det(h)s = \det(h^\perp)s = \chi_g(h)s = h(s) = s$. □

We may now rewrite Theorem 3.1 for degree 2 cohomology using (3.2) and the above lemma. The condition $\det(g) = 1$ from the lemma implies $\text{codim } V^g \neq 1$ if $\text{HH}^2(g) \neq 0$, and the appearance of the exterior power $2-\text{codim } V^g$ in (3.2) implies that $\text{codim } V^g \leq 2$. Thus we have

$$(3.7) \quad \text{HH}^2(S(V)\#G) \cong \bigoplus_{g \in \mathcal{C}} \text{HH}^2(g) \cong \bigoplus_{\substack{g \in \mathcal{C} \\ \det(g)=1 \\ \text{codim } V^g \in \{0,2\}}} \text{HH}^2(g).$$

Remark 3.8. We briefly mention a generalization that we do not pursue in this paper. Let $\alpha : G \times G \rightarrow \mathbb{C}^\times$ be a two-cocycle; that is, $\alpha(g, h)\alpha(gh, \ell) = \alpha(h, \ell)\alpha(g, h\ell)$ for all $g, h, \ell \in G$. The **crossed product algebra** $S(V)\#_\alpha G$ is the vector space $S(V) \otimes \mathbb{C}G$ with multiplication $(r\bar{g})(s\bar{h}) = \alpha(g, h)r \cdot g(s)\bar{g}\bar{h}$ for all $r, s \in R, g, h \in G$. The derivation of the more general Hochschild cohomology $\text{HH}^\bullet(S(V)\#_\alpha G)$ is similar to that of $\text{HH}^\bullet(S(V)\#G)$. (See [26, Cor. 6.5], which remains valid for nonfaithful actions of G on V .) One obtains a sum as in Theorem 3.1, namely

$$\text{HH}^\bullet(S(V)\#_\alpha G) \cong \bigoplus_{g \in \mathcal{C}} \left(S(V^g) \otimes \bigwedge^{\star-\text{codim } V^g}((V^g)^*) \right)^{\chi_g^\alpha}.$$

However, the linear character χ_g^α on $Z(g)$ now depends on α :

$$\chi_g^\alpha(h) = \det(h^\perp)\alpha(g, h)(\alpha(h, g))^{-1}$$

for all $h \in Z(g)$. These potentially different characters χ_g^α can lead to different semi-invariants. In this broader setting, deformations of $S(V)\#_\alpha G$ are treated in [5, 6, 26, 27].

4. INVARIANT THEORY OF $G(r, p, n)$

The Hochschild cohomology of the skew group algebra $S(V)\#G(r, p, n)$ is derived from the invariant theory of $G(r, p, n)$ and its centralizer subgroups. In this section, we briefly record the invariant theory of $G(r, p, n)$ acting via its natural reflection representation on $V = \mathbb{C}^n$ and on the exterior algebra of derivations on V^* . We first recall some well-known facts on reflection groups.

Recall that a **reflection** is an element of $GL(V)$ of finite order which fixes a hyperplane (called the **reflecting hyperplane**) in V pointwise. Let G be a reflection group, i.e., a (finite) group $G \leq GL(V)$ generated by reflections. The collection of reflecting hyperplanes \mathcal{A} for G is called the **reflection arrangement**. For each $H \in \mathcal{A}$, let l_H in V^* be a linear form defining $H = \ker l_H$. The reflection arrangement \mathcal{A} is defined (up to a nonzero constant) by the polynomial $Q = \prod_{H \in \mathcal{A}} l_H$ in $S(V^*)$. When G is a reflection group acting on V^* , we identify $(V^*)^*$ with V and define the reflection arrangement \mathcal{A} by a polynomial Q in $S(V)$.

Remark 4.1. By the Shephard-Todd-Chevalley Theorem (see [21, Chapter 6]), the invariant ring $S(V)^G$ is a polynomial algebra generated by n algebraically independent invariants: $S(V)^G = \mathbb{C}[f_1, \dots, f_n]$, for some polynomials f_i called **basic invariants**. In fact, a set of homogeneous, algebraically independent, invariant polynomials f_1, \dots, f_n generate $S(V)^G$ if and only if the product of the degrees $\deg f_i$ is $|G|$ (for example, see [16, Theorem 3.7.5]). Hence, the invariants of a direct sum of reflection representations is the tensor product of the invariants: If G and G' are reflection groups acting on complex vector spaces V and V' with basic invariants f_1, \dots, f_n and $f'_1, \dots, f'_{n'}$, respectively, then

$$S(V \oplus V')^{G \oplus G'} = \mathbb{C}[f_1, \dots, f_n, f'_1, \dots, f'_{n'}] \cong S(V)^G \otimes S(V')^{G'}$$

The following proposition (see [23], for example, or above remark) describes the polynomials invariant under $G(r, p, n)$. Recall that the k -th elementary symmetric function in variables z_1, \dots, z_n is the polynomial $f_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} z_{i_1} \cdots z_{i_k}$.

Proposition 4.2. *Let $G = G(r, p, n)$ act on V with \mathbb{C} -basis v_1, \dots, v_n via its natural reflection representation. Then*

$$S(V)^{G(r,p,n)} = \mathbb{C}[f_1, \dots, f_n]$$

where f_i is the i -th elementary symmetric function in variables v_1^r, \dots, v_n^r for $i < n$ and $f_n = (v_1 \cdots v_n)^{r/p}$.

We identify $S(V) \otimes \wedge^k(V^*)$ with the $S(V)$ -module of k -derivations (or k -vector fields) on V^* . The following proposition (see [21]) describes the invariant derivations on V^* . The generators θ_i in the proposition are called **basic derivations**.

Proposition 4.3. *Let G be a reflection group acting on $V^* = \mathbb{C}^n$ with reflection arrangement defined by $Q \in S(V)$. Suppose $\theta_1, \dots, \theta_n$ in $(S(V) \otimes V^*)^G$ are invariant derivations whose coefficient matrix has determinant Q up to a nonzero scalar. Then*

$$\left(S(V) \otimes \wedge^k(V^*) \right)^G = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} S(V)^G (\theta_{i_1} \wedge \dots \wedge \theta_{i_k}).$$

Proof. Note that $S(V)^{\det^{-1}} = S(V)^G Q$, where \det^{-1} is the inverse of the determinant character of G acting on V^* (see [21, Example 6.40]). Solomon’s Theorem [21, Prop. 6.47] then implies that the invariant k -derivations are generated freely over $S(V)^G$ by the nonzero wedge products of the θ_i (taken k at a time). \square

We apply the above proposition to $G = G(r, p, n)$ (see [21, Appendix B]).

Proposition 4.4. *Let $G(r, p, n)$ act on $V = \mathbb{C}^n$ with \mathbb{C} -basis v_1, \dots, v_n and on V^* contragrediently with dual basis x_1, \dots, x_n . Then*

$$(S(V) \otimes \wedge(V^*))^{G(r,p,n)} = \bigoplus_{1 \leq k \leq n} \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} S(V)^{G(r,p,n)} (\theta_{i_1} \wedge \dots \wedge \theta_{i_k}),$$

where

$$\theta_j := \sum_{1 \leq i \leq n} v_i^{(j-1)r+1} \otimes x_i$$

for $j < n$ and

$$\theta_n := \begin{cases} \sum_{1 \leq i \leq n} v_i^{(n-1)r+1} \otimes x_i & \text{if } p \neq r, \\ \sum_{1 \leq i \leq n} (v_1 \cdots v_{i-1} v_{i+1} \cdots v_n)^{r-1} \otimes x_i & \text{if } p = r. \end{cases}$$

5. HOCHSCHILD COHOMOLOGY IN DEGREE 2 FOR $G(r, p, n)$

We give the Hochschild 2-cohomology $\text{HH}^2(S(V)\#G)$ explicitly when the monomial group $G = G(r, p, n)$ acts (faithfully) via its natural reflection representation on $V = \mathbb{C}^n$. The results rely on some computations postponed until Section 6. We assume $n \geq 4$ throughout this section. Results for $n = 2$ and $n = 3$ are slightly different because the pattern of relevant conjugacy classes in G differs for low values of n . The techniques from this section apply to give explicit Hochschild cohomology in the case $n = 2$ or $n = 3$, but we omit these results for the sake of brevity. In the following theorem, we use the notation $\text{HH}^2(g)$ defined in (3.2).

Theorem 5.1. *Assume $n \geq 4$. Let $G = G(r, p, n)$ act on $V = \mathbb{C}^n$ via the natural reflection representation. The Hochschild cohomology in degree 2 for the skew group algebra $S(V)\#G$ is*

$$\text{HH}^2(S(V)\#G) \cong \text{HH}^2(1) \oplus \text{HH}^2(1, 2, 3) \oplus \text{HH}^2(1, -2) \oplus \bigoplus_{i=1, \dots, \lfloor r/2 \rfloor} \text{HH}^2(\xi_1^i \xi_2^{-i})$$

where

- $\text{HH}^2(1) = \left(S(V) \otimes \wedge^2(V^*) \right)^{G(r,p,n)}$ is given explicitly in Proposition 4.4,
- $\text{HH}^2(1, 2, 3)$ is given explicitly in Proposition 6.1,
- $\text{HH}^2(1, -2) = 0$ unless $r = 2p$, in which case $\text{HH}^2(1, -2)$ is given explicitly in Proposition 6.4,
- $\text{HH}^2(\xi_1^\ell \xi_2^{-\ell}) = 0$ unless $r > 1$, $p = r$, and $\ell \neq r/2$, in which case $\text{HH}^2(\xi_1^\ell \xi_2^{-\ell})$ is given explicitly in Proposition 6.10.

Proof. We apply (3.7). If $g \in G(r, p, n)$ with $\text{codim}(V^g) = 0$, then g is the identity element 1 (as $G(r, p, n)$ acts faithfully on V). Furthermore, the Hochschild character χ for $g = 1$ should be interpreted as the trivial character of $Z(g) = G(r, p, n)$ since

$(V^g)^\perp = \{0\}$. Hence

$$\mathrm{HH}^2(1) = \left(S(V^g) \otimes \Lambda^2((V^g)^*) \right)^{G(r,p,n)},$$

which is given explicitly in Proposition 4.4.

If $g \in G(r, p, n)$ with $\mathrm{codim}(V^g) = 2$, then g is conjugate to one of the following elements (see [22, Section 2B]):

$$g_b = \xi_1^a \xi_3^{-a}(1, 2, 3), \quad 0 \leq a \leq \gcd(p, 3) - 1,$$

$$g_c = \xi_1^{a+\ell} \xi_2^{-a}(1, 2), \quad \ell \neq 0 \text{ (so } r \neq 1),$$

$$g_d = \xi_1^{\ell_1} \xi_2^{\ell_2}, \quad \ell_1 \neq 0, \ell_2 \neq 0 \text{ (so } r \neq 1),$$

$$g_e = (1, 2)\xi_3^\ell, \quad \ell \neq 0,$$

$$g_f = (1, 2)\xi_3^a \xi_4^{-a}(3, 4).$$

We can further conjugate these elements into simpler forms under our assumption $n \geq 4$. Each element of the form $g_b = \xi_1^a \xi_3^{-a}(1, 2, 3)$ is conjugate to $(1, 2, 3)$ (via $(1, 2, 3)\xi_3^a \xi_4^{-a}$) in $G(r, p, n)$. Each element of the form $g_c = \xi_1^{a+\ell} \xi_2^{-a}(1, 2)$ is conjugate to $\xi_2^\ell(1, 2)$ (via $\xi_1^a(1, 2)\xi_3^{-a}$) in $G(r, p, n)$. We assume $1 = \det g_c = -\xi^\ell$, else $\mathrm{HH}^2(g_c) = 0$ by Lemma 3.6. Thus, if $\mathrm{HH}(g_c)$ is nonzero, r must be even with $\xi^\ell = -1$ and we may assume $g_c = \xi_2^\ell(1, 2) = (1, -2)$. The sets of Hochschild semi-invariants $\mathrm{HH}^2(1, 2, 3)$ and $\mathrm{HH}^2(1, -2)$ are given in Propositions 6.1 and 6.4.

By Proposition 6.8, the set of Hochschild semi-invariants $\mathrm{HH}^2(g_d)$ is zero unless $G = G(r, r, n)$. The element g_d lies in $G(r, r, n)$ only when $\ell_2 \equiv -\ell_1 \pmod r$. In the special case that $\ell_1 = \ell_2 = r/2$, $\mathrm{HH}^2(g_d)$ is zero by Proposition 6.9. Otherwise, $\mathrm{HH}^2(g_d)$ is given in Proposition 6.10.

The sets of Hochschild semi-invariants $\mathrm{HH}^2(g_e)$ and $\mathrm{HH}^2(g_f)$ are both zero by Propositions 6.11 and 6.12. \square

The Hochschild cohomology in degree 2 simplifies when G is the symmetric group $\mathfrak{S}_n = G(1, 1, n)$ and when G is the Weyl group $WB_n = G(2, 1, n)$. (See Proposition 4.2 for the explicit rings of invariants $S(V)^{\mathfrak{S}_n}$ and $S(V)^{WB_n}$ needed in the following statements.)

Corollary 5.2. *Let G be the symmetric group $\mathfrak{S}_n = G(1, 1, n)$ for $n \geq 4$ acting on V with \mathbb{C} -basis v_1, \dots, v_n (via the natural permutation representation). The Hochschild cohomology in degree 2 for the skew group algebra $S(V)\#G$ is*

$$\mathrm{HH}^2(S(V)\#G) \cong \mathrm{HH}^2(1) \oplus \mathrm{HH}^2(1, 2, 3)$$

where

$$\bullet \mathrm{HH}^2(1) = \left(S(V) \otimes \Lambda^2(V^*) \right)^{\mathfrak{S}_n} = \bigoplus_{1 \leq i_1 < i_2 \leq n} S(V)^{\mathfrak{S}_n} (\theta_{i_1} \wedge \theta_{i_2}),$$

for

$$\theta_j := \sum_{1 \leq i \leq n} v_i^{j-1} \otimes x_i,$$

$$\bullet \mathrm{HH}^2(1, 2, 3) = \mathbb{C}[f_0, f_1, \dots, f_{n-3}], \text{ where } f_0 = (v_1 + v_2 + v_3) \text{ and } f_i \text{ is the } i\text{-th elementary symmetric function in } v_4, \dots, v_n \text{ for } i > 0.$$

Corollary 5.3. *Let G be the Weyl group $WB_n = G(2, 1, n)$ for $n \geq 4$ acting on V with \mathbb{C} -basis v_1, \dots, v_n (via its natural reflection representation). The Hochschild cohomology in degree 2 for the skew group algebra $S(V)\#G$ is*

$$\mathrm{HH}^2(S(V)\#G) \cong \mathrm{HH}^2(1) \oplus \mathrm{HH}^2(1, 2, 3) \oplus \mathrm{HH}^2(1, -2)$$

where

- $\mathrm{HH}^2(1) = \left(S(V) \otimes \Lambda^2(V^*) \right)^{WB_n} = \bigoplus_{1 \leq i_1 < i_2 \leq n} S(V)^{WB_n} (\theta_{i_1} \wedge \theta_{i_2}),$

for

$$\theta_j := \sum_{1 \leq i \leq n} (v_i)^{2(j-1)+1} \otimes x_i,$$

- $\mathrm{HH}^2(1, 2, 3) = \mathbb{C}[f_0, f_1, \dots, f_{n-3}]$, where $f_0 = (v_1 + v_2 + v_3)^2$ and f_i is the i -th elementary symmetric function in v_4^2, \dots, v_n^2 for $i > 0$,
- $\mathrm{HH}^2(1, -2) = \mathbb{C}[f'_1, \dots, f'_{n-2}]$, where f'_i is the i -th elementary symmetric function in v_3^2, \dots, v_n^2 .

6. INVARIANT THEORY FOR CERTAIN CENTRALIZER SUBGROUPS OF $G(r, p, n)$

In Section 5, we gave the Hochschild cohomology of $S(V)\#G$ in degree 2, where V is the natural reflection representation of $G = G(r, p, n)$. The Hochschild cohomology can be expressed as the direct sum of certain subspaces, each subspace invariant under an action of a centralizer subgroup $Z(g)$ for some $g \in G$. In this section, we determine these invariant subspaces. We analyze the contribution $\mathrm{HH}^2(g)$ from the group elements $g \in G(r, p, n)$ with $\mathrm{codim}(V^g) = 2$ and $\det g = 1$. Recall that v_1, \dots, v_n is the given \mathbb{C} -basis for V .

Proposition 6.1. *Assume $n \geq 4$ and $G = G(r, p, n)$. Let $g = (1, 2, 3)$. Then*

$$S(V^g)^{Z(g)} = \bigoplus_{\substack{0 \leq i < r, 0 \leq j < m \\ i \equiv 3jr/p \pmod r}} \mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m] \quad f_0^i f_{n'}^j,$$

where

- $n' = n - 3$,
- $m = p/3$ if 3 divides p ; $m = p$ otherwise,
- $f_0 = v_1 + v_2 + v_3$,
- $f_{n'} = (v_4 \cdots v_n)^{r/p}$,
- f_i is the i -th elementary symmetric function in v_4^r, \dots, v_n^r for $1 \leq i \leq n' - 1$.

Let χ be the Hochschild character for g as defined in (2.1). The set of Hochschild semi-invariants for g is

$$\mathrm{HH}^2(g) = S(V^g)^\chi = \bigoplus_{\substack{0 \leq i < r, 0 \leq j < m \\ i \equiv 2+3jr/p \pmod r}} \mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m] \quad f_0^i f_{n'}^j.$$

Proof. Let $v_0 = f_0 = v_1 + v_2 + v_3$. Note that $V^g = \mathbb{C}\text{-span}\{v_0, v_4, \dots, v_n\}$ and $(V^g)^\perp \subset \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$. Let $V_A = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$ and $V_B = \mathbb{C}\text{-span}\{v_4, \dots, v_n\}$. One may use (2.2) to verify that the centralizer of g in G is

$$Z(g) = \{h_A \oplus h_B : h_A \in \langle \xi I_3, (1, 2, 3) \rangle < G(r, 1, 3), h_B \in G(r, 1, n')\} \cap G(r, p, n).$$

The centralizer $Z(g)$ includes the subgroup

$$Z' = \{I_3 \oplus h_B : h_B \in G(r, p, n')\}.$$

Since $\chi(h) = 1$ for all $h \in Z'$, $S(V^g)^\chi \subset S(V^g)^{Z'}$, and since $Z' \subset Z(g)$, we have $S(V^g)^{Z(g)} \subset S(V^g)^{Z'}$. We begin by finding the invariants of $S(V^g)$ under Z' .

The group Z' acts on V_B as the reflection group $G(r, p, n')$ with invariants

$$S(V_B)^{G(r,p,n')} = \mathbb{C}[v_4, \dots, v_n]^{G(r,p,n')} = \mathbb{C}[f_1, \dots, f_{n'}]$$

by Proposition 4.2. The group Z' fixes v_0 , so acts on V^g as the direct sum of the trivial group with $G(r, p, n')$. Thus, $S(V^g)^{Z'} = \mathbb{C}[v_0, v_4, \dots, v_n]^{Z'} = \mathbb{C}[f_0, f_1, \dots, f_{n'}]$ by Remark 4.1. Hence,

$$S(V^g)^{Z(g)} \subset \mathbb{C}[f_0, f_1, \dots, f_{n'}] \quad \text{and} \quad S(V^g)^\chi \subset \mathbb{C}[f_0, f_1, \dots, f_{n'}].$$

The polynomials $f_1, \dots, f_{n'-1}$ are invariant under $Z(g)$. But the polynomials f_0 and $f_{n'}$ pick up a scalar when $Z(g)$ acts. We introduce some characters to record and compare the actions of $Z(g)$ on f_0 and on $f_{n'}$.

Define a linear character on $Z(g)$,

$$\tau : Z(g) \rightarrow \mathbb{C},$$

by $h(f_0) = \tau(h) f_0$ for all h in $Z(g)$. Fix $h = h_A \oplus h_B \in Z(g)$ and note that $h_A = (\xi^k I_3)(1, 2, 3)^j$, for some j and k . Since $\det(h^\perp) = \xi^{2k}$ (as $(\dim V^g)^\perp = 2$), $\tau^2(h) = \xi^{2k} = \det(h^\perp) = \chi(h)$. Hence, $\chi = \tau^2$.

Recall that $\psi(M)$ is the product of the nonzero entries of M , for any matrix M . Since $M \in G(r, p, n)$ implies that $\psi(M)^{r/p} = 1$, we see that $\psi(h_B)^{r/p} = \psi(h_A)^{-r/p}$ (as $\psi(h) = \psi(h_A)\psi(h_B)$). Observe that $\psi(h_A) = \xi^{3k} = \tau(h)^3$. Then

$$h(f_{n'}) = [\psi(h_B)(v_4 \cdots v_n)]^{r/p} = \psi(h_B)^{r/p} f_{n'} = \psi(h_A)^{-r/p} f_{n'} = \tau(h)^{-3r/p} f_{n'}.$$

In summary, the action of $Z(g)$ on $f_0, \dots, f_{n'}$ is given by

$$(6.2) \quad h(f_k) = \begin{cases} \tau(h) f_k & \text{for } k = 0, \\ f_k & \text{for } 0 < k < n', \\ \tau(h)^{-3r/p} f_k & \text{for } k = n' \end{cases}$$

for all $h \in Z(g)$. In particular, for any i, j ,

$$(6.3) \quad \begin{aligned} h(f_0^i f_{n'}^j) &= \tau(h)^{i-3jr/p} f_0^i f_{n'}^j, & \text{and} \\ \chi(h) &= \tau^2(h). \end{aligned}$$

We now give an explicit description of $S(V^g)^{Z(g)}$ and $S(V^g)^\chi$ as subsets of $\mathbb{C}[f_0, \dots, f_{n'}]$. Fix some s in $\mathbb{C}[f_0, \dots, f_{n'}]$ and write s as a polynomial in f_0 and $f_{n'}$ with coefficients in $f_1, \dots, f_{n'-1}$:

$$s = \sum_{i,j \geq 0} a_{ij} f_0^i f_{n'}^j, \quad \text{where } a_{ij} \in \mathbb{C}[f_1, \dots, f_{n'-1}] \subset S(V^g)^{Z(g)}.$$

Then s is invariant (respectively χ -invariant) under $Z(g)$ exactly when each $f_0^i f_{n'}^j$ with $a_{ij} \neq 0$ is invariant (respectively χ -invariant) under $Z(g)$. The monomial $f_0^i f_{n'}^j$ is invariant under $Z(g)$ exactly when

$$\tau(h)^{i-3jr/p} = 1 \quad \text{for all } h \in Z(g)$$

by Equation (6.3). Similarly, the monomial $f_0^i f_{n'}^j$ is χ -invariant under $Z(g)$ exactly when

$$\begin{aligned} \tau(h)^2 f_0^i f_{n'}^j &= \chi(h) f_0^i f_{n'}^j = h(f_0^i f_{n'}^j) = \tau(h)^{i-3jr/p} f_0^i f_{n'}^j, & \text{i.e.,} \\ \tau(h)^{2+3jr/p-i} &= 1 & \text{for all } h \in Z(g). \end{aligned}$$

But the set $\{\tau(h) : h \in Z(g)\}$ depends on p and n .

Note that $\tau(h)$ is a power of ξ for each $h \in Z(g)$ and that $h' = \xi_1 \xi_2 \xi_3 \xi_4^{-3} \in Z(g)$ with $\tau(h') = \xi$. Hence

$$\{\tau(h) : h \in Z(g)\} = \langle \xi \rangle.$$

Thus the monomial $f_0^i f_{n'}^j$ is invariant under $Z(g)$ exactly when $\xi^{i-3jr/p} = 1$:

$$f_0^i f_{n'}^j \in S(V^g)^{Z(g)} \quad \text{if and only if} \quad i \equiv 3jr/p \pmod r.$$

In particular, f_0^r is invariant under $Z(g)$. Also, $f_{n'}^m$ is invariant under $Z(g)$ where $m = p/3$ if p is divisible by 3 and $m = p$ otherwise. So

$$\mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m] \subset S(V^g)^{Z(g)} \subset \mathbb{C}[f_0, f_1, \dots, f_{n'-1}, f_{n'}].$$

Thus, $S(V^g)^{Z(g)}$ is the free $\mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m]$ -module

$$S(V^g)^{Z(g)} = \bigoplus_{\substack{0 \leq i < r, 0 \leq j < m \\ i \equiv 3jr/p \pmod r}} \mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m] f_0^i f_{n'}^j.$$

Similarly, $f_0^i f_{n'}^j$ is χ -invariant under $Z(g)$ exactly when $\xi^{2+3jr/p-i} = 1$:

$$f_0^i f_{n'}^j \in S(V^g)^\chi \quad \text{if and only if} \quad i \equiv 2 + 3jr/p \pmod r.$$

Thus, $S(V^g)^\chi$ is the free $\mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m]$ -module

$$S(V^g)^\chi = \bigoplus_{\substack{0 \leq i < r, 0 \leq j < m \\ i \equiv 2+3jr/p \pmod r}} \mathbb{C}[f_0^r, f_1, \dots, f_{n'-1}, f_{n'}^m] f_0^i f_{n'}^j.$$

□

Proposition 6.4. *Assume $n \geq 4$, $G = G(r, p, n)$. Suppose $g = (1, -2)$. Then*

$$S(V^g)^{Z(g)} = \begin{cases} \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^{p/2}] & \text{if } p \text{ is even,} \\ \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^p] & \text{if } p \text{ is odd,} \end{cases}$$

where

- $n' = n - 2$,
- $f_{n'} = (v_3 \dots v_n)^{r/p}$,
- f_i is the i -th elementary symmetric function in v_3^r, \dots, v_n^r for $i < n'$.

Let χ be the Hochschild character for g as defined in (2.1). The set of Hochschild semi-invariants for g is

$$\text{HH}^2(g) = S(V^g)^\chi = \bigoplus_{\substack{0 \leq i < r \\ 2 \equiv 2(-r/p)i \pmod r}} \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r] f_{n'}^i.$$

In particular, $\text{HH}^2(g)$ is zero unless $r = 2p$.

Proof. Each element h of the centralizer $Z(g)$ breaks into a direct sum: We may write $h = h^\perp \oplus h_\perp$, where $h^\perp \in G(r, 1, 2)$ is the matrix of h acting on $(V^g)^\perp$ (with respect to the basis $\{v_1, v_2\}$ of $(V^g)^\perp$) and $h_\perp \in G(r, 1, n')$ is the matrix of h acting on V^g (with respect to the basis $\{v_3, \dots, v_n\}$ of V^g). One may use (2.2) to verify that

$$Z(g) = \{h^\perp \oplus h_\perp : h^\perp \in \langle g^\perp, \xi I_2 \rangle, h_\perp \in G(r, 1, n')\} \cap G(r, p, n).$$

The centralizer $Z(g)$ includes the subgroup Z' of G fixing $(V^g)^\perp$ pointwise:

$$Z' = \{I_2 \oplus B : B \in G(r, p, n')\} \subset Z(g).$$

The group Z' acts on $V^g = \mathbb{C}\text{-span}\{v_3, \dots, v_n\}$ as the reflection group $G(r, p, n')$. By Proposition 4.2,

$$S(V^g)^{Z'} = \mathbb{C}[v_3, \dots, v_n]^{G(r, p, n')} = \mathbb{C}[f_1, \dots, f_{n'}].$$

Since $\chi(h) = 1$ for all $h \in Z'$, $S(V^g)^\chi \subset S(V^g)^{Z'}$, and since $Z' \subset Z(g)$, $S(V^g)^{Z(g)} \subset S(V^g)^{Z'}$. Hence,

$$S(V^g)^{Z(g)} \subset \mathbb{C}[f_1, \dots, f_{n'}] \quad \text{and} \quad S(V^g)^\chi \subset \mathbb{C}[f_1, \dots, f_{n'}].$$

Observe that $f_1, \dots, f_{n'-1}$ lie in $S(V^g)^{Z(g)}$. How does $Z(g)$ act on the last polynomial, $f_{n'}$? Recall that $\psi(M)$ is the product of the nonzero entries of M , for any matrix M , and that $M \in G(r, p, n)$ implies $\psi(M)^{r/p} = 1$. For any $h \in Z(g)$, $\psi(h) = \psi(h^\perp)\psi(h_\perp)$ and

$$\begin{aligned} h(f_{n'}) &= h(v_3 \cdots v_n)^{r/p} \\ &= (\psi(h_\perp)v_3 \cdots v_n)^{r/p} \\ (6.5) \quad &= \psi(h_\perp)^{r/p} f_{n'} \\ &= \psi(h)^{r/p} \psi(h^\perp)^{-r/p} f_{n'} \\ &= \psi(h^\perp)^{-r/p} f_{n'}. \end{aligned}$$

Fix some s in $\mathbb{C}[f_1, \dots, f_{n'}]$ and write s as a polynomial in $f_{n'}$ with coefficients in $f_1, \dots, f_{n'-1}$:

$$s = \sum_{i \geq 0} a_i f_{n'}^i \quad \text{where} \quad a_i \in \mathbb{C}[f_1, \dots, f_{n'-1}] \subset S(V^g)^{Z(g)}.$$

Then s is invariant (respectively χ -invariant) under $Z(g)$ exactly when each $f_{n'}^i$ with $a_i \neq 0$ is invariant (respectively χ -invariant) under $Z(g)$.

From Equation (6.5), the polynomial $f_{n'}^i$ is invariant under $Z(g)$ exactly when

$$(6.6) \quad 1 = \psi(h^\perp)^{-ir/p} \quad \text{for all } h \in Z(g).$$

Also, the polynomial $f_{n'}^i$ is χ -invariant under $Z(g)$ exactly when

$$(6.7) \quad \begin{aligned} \det(h^\perp) f_{n'}^i &= \chi(h) f_{n'}^i = h(f_{n'}^i) = \psi(h^\perp)^{-ir/p} f_{n'}^i, \quad \text{i.e.,} \\ \det(h^\perp) &= \psi(h^\perp)^{-ir/p}, \quad \text{for all } h \in Z(g). \end{aligned}$$

Note that $Z(g)^\perp := \{h^\perp : h \in Z(g)\}$ is generated by g^\perp and $\xi I_2 = \xi_1 \xi_2$ since $n \geq 3$ implies that the centralizer $Z(g)$ contains $\xi_1 \xi_2 \xi_3^{-2} \in G(r, p, n)$. (If $n = 2$, the matrix ξI_2 need not lie in $G(r, p, n)$.) Thus, in order to establish the invariance or semi-invariance of $f_{n'}^i$, we need only check Equations (6.6) and (6.7) for g^\perp and ξI_2 . Note that $g = \xi_2^{r/2}(1, 2) \in G(r, p, n)$ implies that the integer r/p is even.

Hence $\psi(g^\perp)^{-ir/p} = (-1)^{-ir/p} = 1$, and $\det(g^\perp) = 1$. Hence, both Equations (6.6) and (6.7) are satisfied for the generator g^\perp of $Z(g)^\perp$. So, the invariance or semi-invariance of $f_{n'}^i$, depends only on i and the action of the generator ξI_2 of $Z(g)^\perp$ on $f_{n'}$.

Thus, $f_{n'}^i$ is invariant under $Z(g)$ exactly when

$$1 = \psi(\xi I_2)^{-ir/p} = (\xi^2)^{-ir/p}, \quad \text{i.e. } 0 \equiv 2ir/p \pmod r.$$

But $0 \equiv 2ir/p \pmod r$ means that i is a multiple of $p/2$ when p is even and a multiple of p when p is odd. Also, $f_{n'}^i$ is χ -invariant under $Z(g)$ exactly when

$$\xi^2 = \det(\xi I_2) = \psi(\xi I_2)^{-ir/p} = (\xi^2)^{-ir/p}, \quad \text{i.e. } 2 \equiv -2ir/p \pmod r.$$

In particular, $f_{n'}^r$ is invariant under $Z(g)$. Hence

$$S(V^g)^{Z(g)} = \begin{cases} \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^{p/2}] & \text{if } p \text{ is even,} \\ \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^p] & \text{if } p \text{ is odd,} \end{cases}$$

and

$$\text{HH}^2(g) = S(V^g)^\chi = \bigoplus_{\substack{0 \leq i < r \\ 2 \equiv 2(-r/p)i \pmod r}} \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r] f_{n'}^i.$$

□

Proposition 6.8. *Assume $n \geq 2$, $r \geq 2$, $G = G(r, p, n)$. Let $g = \xi_1^{\ell_1} \xi_2^{\ell_2}$, where $\ell_i \neq 0$. The set of Hochschild semi-invariants for g is trivial,*

$$\text{HH}^2(g) = S(V^g)^\chi = 0,$$

unless $G = G(r, r, n)$.

Proof. Note that $V^g = \mathbb{C}\text{-span}\{v_3, \dots, v_n\}$ and $(V^g)^\perp = \mathbb{C}\text{-span}\{v_1, v_2\}$. Let $h = \xi_1 \xi_2^{p-1} \in Z(g)$. Then $h|_{V^g} = 1$ and $\det h^\perp = \xi^p$. Lemma 3.6 implies that if $\text{HH}^2(g)$ is nonzero, then $\xi^p = 1$ and so $p = r$. □

Proposition 6.9. *Assume $n \geq 2$, $r \geq 2$ (r even). Let $G = G(r, r, n)$. Let $g = \xi_1^{r/2} \xi_2^{r/2}$. The set of Hochschild semi-invariants for g is trivial:*

$$\text{HH}^2(g) = S(V^g)^\chi = 0.$$

Proof. Note that $V^g = \mathbb{C}\text{-span}\{v_3, \dots, v_n\}$ and $(V^g)^\perp = \mathbb{C}\text{-span}\{v_1, v_2\}$. Let $h = (1, 2)$. Then $h \in Z(g)$ with $h|_{V^g} = 1$. But $\det h^\perp = -1 \neq 1$. By Lemma 3.6, $\text{HH}^2(g)$ is zero. □

Proposition 6.10. *Assume $n \geq 3$, $r \geq 2$. Let $G = G(r, r, n)$. Let $g = \xi_1^\ell \xi_2^{-\ell}$, where $\ell \neq r/2$. Then*

$$S(V^g)^{Z(g)} = \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r],$$

where

- $n' = n - 2$,
- $f_{n'} = v_3 \cdots v_n$,
- f_i is the i -th elementary symmetric function in v_3^r, \dots, v_n^r for $i < n'$.

Let χ be the Hochschild character for g as defined in (2.1). The set of Hochschild semi-invariants for g is

$$\text{HH}^2(g) = S(V^g)^\chi = \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r] f_{n'}^{r-1}.$$

Proof. Note that $V^g = \mathbb{C}\text{-span}\{v_3, \dots, v_n\}$ and $(V^g)^\perp = \mathbb{C}\text{-span}\{v_1, v_2\}$. Let $s \in \text{HH}^2(g)$ be a Hochschild semi-invariant for g . Consider the subgroup of $Z(g)$,

$$Z' = \{I_2 \oplus B : B \in G(r, r, n')\}.$$

Since $\det h^\perp = 1$ for all $h \in Z'$, s is invariant under Z' , that is, $s \in S(V^g)^{Z'} = \mathbb{C}[v_3, \dots, v_n]^{Z'}$. But Z' acts on the vector space V^g as $G(r, r, n')$ with respect to the basis $\{v_3, \dots, v_n\}$ of V^g . By Proposition 4.2,

$$S(V^g)^{Z'} = \mathbb{C}[v_3, \dots, v_n]^{G(r, r, n')} = \mathbb{C}[f_1, \dots, f_{n'}].$$

Hence,

$$\text{HH}^2(g) = \{\text{Hochschild semi-invariants for } g\} \subset \mathbb{C}[f_1, \dots, f_{n'}].$$

One may use (2.2) to verify that

$$Z(g) = \{A \oplus B \in G(r, r, n) : A \text{ is a } 2 \times 2 \text{ diagonal matrix}\}.$$

Let s be any element of $\mathbb{C}[f_1, \dots, f_{n'}]$ and write s as a polynomial in $f_{n'}$ with coefficients in $f_1, \dots, f_{n'-1}$:

$$s = \sum_{i \geq 0} a_i f_{n'}^i \quad \text{where each } a_i \in \mathbb{C}[f_1, \dots, f_{n'-1}].$$

Note that $f_1, \dots, f_{n'-1}$ are invariant under $Z(g)$. Hence s is invariant (respectively χ -invariant) under $Z(g)$ exactly when each $f_{n'}^i$ with $a_i \neq 0$ is invariant (respectively χ -invariant) under $Z(g)$.

Observe that

$$h(f_{n'}) = \chi(h)^{-1} f_{n'} = (\det h^\perp)^{-1} f_{n'}$$

for any $h \in Z(g)$ since $p = r$. Hence, $f_{n'}^i$ is invariant under $Z(g)$ exactly when

$$1 = (\det h^\perp)^{-i} \quad \text{for all } h \in Z(g).$$

Similarly, $f_{n'}$ is χ -invariant under $Z(g)$ exactly when

$$\begin{aligned} \det(h^\perp) &= (\det h^\perp)^{-i} \quad \text{for all } h \in Z(g), & \text{i.e.,} \\ 1 &= (\det h^\perp)^{1+i} \quad \text{for all } h \in Z(g). \end{aligned}$$

As $n \geq 3$, $\{\det h^\perp : h \in Z(g)\} = \langle \xi \rangle$. Hence, $f_{n'}^i$ is invariant under $Z(g)$ exactly when i is a multiple of r , and $f_{n'}^i$ is χ -invariant under $Z(g)$ exactly when $-1 \equiv i \pmod r$. Thus,

$$S(V^g)^{Z(g)} = \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r]$$

and

$$\text{HH}^2(g) = S(V^g)^\chi = \mathbb{C}[f_1, \dots, f_{n'-1}, f_{n'}^r] f_{n'}^{r-1}.$$

□

Proposition 6.11. *Assume $r \geq 2$, $G = G(r, p, n)$. Let $g = (1, 2)_{\xi_3}^\ell$. The set of Hochschild semi-invariants for g is zero:*

$$\text{HH}^2(g) = S(V^g)^\chi = 0.$$

Proof. Note $V^g = \mathbb{C}\text{-span}\{v_1 + v_2, v_4, v_5, \dots, v_n\}$ and $(V^g)^\perp = \mathbb{C}\text{-span}\{v_1 - v_2, v_3\}$. Let $h = (1, 2) \in Z(g)$. Then $h|_{V^g} = 1$ while $\det h^\perp = -1$, so Lemma 3.6 implies that $\text{HH}^2(g) = 0$. □

Proposition 6.12. *Assume $n \geq 4$. Let $g = (1, 2)\xi_3^a \xi_4^{-a}(3, 4)$. The set of Hochschild semi-invariants for g is zero:*

$$\text{HH}^2(g) = S(V^g)^x = 0.$$

Proof. Note that $V^g = \mathbb{C}\text{-span}\{v_1 + v_2, v_3 + \xi^{-a}v_4, v_5, \dots, v_n\}$ and that $(V^g)^\perp = \mathbb{C}\text{-span}\{v_2 - v_1, v_3 - \xi^{-a}v_4\}$. Let $h = (1, 2) \in Z(g)$. Then $h|_{V^g} = 1$ while $\det h^\perp = -1$, so Lemma 3.6 implies that $\text{HH}^2(g) = 0$. \square

7. HOCHSCHILD 2-COHOMOLOGY FOR NONFAITHFUL ACTION OF $G(r, p, n)$

Let $G = G(r, 1, n)$, with $n \geq 3$. We define a *nonfaithful* action of G on $V = \mathbb{C}^n$ and determine the Hochschild 2-cohomology for the skew group algebra $S(V)\#G$. This nonfaithful action will yield nontrivial graded Hecke algebras in Section 9.

Although we restrict our attention to $G(r, 1, n)$ in this section, the same results generally hold for $G(r, p, n)$ (under the nonfaithful action). In fact, Propositions 7.3 and 7.6 and Theorem 7.7 below hold verbatim when we replace $G(r, 1, n)$ by $G(r, p, n)$ and replace $n \geq 3$ by $n \geq 5$. We simply intersect each centralizer $Z(g)$ in $G(r, 1, n)$ with $G(r, p, n)$ throughout the proofs. We leave the details to the reader. (For $p \geq 2$ and low values of n , Remark 7.4 does not apply as the centralizers exhibit a different structure.)

Recall that G is the group product $N \cdot G(1, 1, n)$, where N is the normal subgroup of diagonal matrices in G and $G(1, 1, n) \leq G$ is isomorphic to \mathfrak{S}_n . Define a representation

$$(7.1) \quad \rho : G \rightarrow \text{GL}(V)$$

by composing the quotient map $G \rightarrow \mathfrak{S}_n$ with the permutation representation of \mathfrak{S}_n on V (permuting the fixed basis v_1, \dots, v_n).

The next lemma describes the centralizer of a diagonal element. Given a diagonal matrix $g = \xi^{a_1}I_{n_1} \oplus \dots \oplus \xi^{a_k}I_{n_k}$, rename the basis vectors v_1, \dots, v_n of V so that $v_1^{(1)}, \dots, v_{n_1}^{(1)}$ are the first n_1 basis vectors, $v_1^{(2)}, \dots, v_{n_2}^{(2)}$ are the next n_2 basis vectors, etc., and decompose V as $V_1 \oplus \dots \oplus V_k$ with $V_i = \mathbb{C}\text{-span}\{v_1^{(i)}, \dots, v_{n_i}^{(i)}\}$. One may use (2.2) to verify the following lemma.

Lemma 7.2. *Let g' be a diagonal matrix in $G = G(r, 1, n)$ and suppose n_1, \dots, n_k are the multiplicities of the diagonal entries. Then g' is conjugate in G to some $g = \xi^{a_1}I_{n_1} \oplus \dots \oplus \xi^{a_k}I_{n_k}$ and $Z(g)$ acts on $V = V_1 \oplus \dots \oplus V_k$ as the direct sum of reflection groups,*

$$Z(g) = \{M_1 \oplus \dots \oplus M_k : M_i \in G(r, 1, n_i)\} = G(r, 1, n_1) \oplus \dots \oplus G(r, 1, n_k).$$

Contribution from g with $\text{codim } V^g = 0$.

We first find the contribution to the Hochschild 2-cohomology of $S(V)\#G(r, 1, n)$ from g with $\text{codim } V^g = 0$. The set of g in $G = G(r, 1, n)$ with $\text{codim}(V^g) = 0$ under the nonfaithful representation ρ is exactly the subgroup N of diagonal matrices in $G(r, 1, n)$. Below, we use the notation $\text{HH}^2(g)$ defined in (3.2) and we identify $S(V) \otimes \wedge^k(V^*)$ with the $S(V)$ -module of k -derivations on V^* .

Proposition 7.3. *Let $G = G(r, 1, n)$ act on $V = \mathbb{C}^n$ via the nonfaithful representation ρ . Let $g' \in G$ be a diagonal matrix. Then g' is conjugate to some $g \in G$ whose set of Hochschild semi-invariants $\text{HH}^2(g)$ is the set of derivation 2-forms invariant under a direct sum of symmetric groups.*

Explicitly: Suppose n_1, \dots, n_k are the multiplicities of the diagonal entries of g' and decompose V as $V_1 \oplus \dots \oplus V_k$ with each $V_i \cong \mathbb{C}^{n_i}$ as above. For $i = 1, \dots, k$, let $\theta_1^{(i)}, \dots, \theta_{n_i}^{(i)} \in S(V_i) \otimes V_i^*$ be basic derivations for the permutation action of the symmetric group \mathfrak{S}_{n_i} on V_i . Then $\text{HH}^2(g)$ is generated by the wedge products of the $\theta_j^{(i)}$ taken two at a time:

$$\text{HH}^2(g) = \left(S(V) \otimes \Lambda^2(V^*) \right)^{Z(g)} = \bigoplus_{\substack{i \leq i' \\ j < j' \text{ when } i=i'}} S(V)^{Z(g)} (\theta_j^{(i)} \wedge \theta_{j'}^{(i')}).$$

Also,

$$S(V)^{Z(g)} = \bigoplus_{i=1}^k S(V_i)^{\mathfrak{S}_{n_i}}.$$

Remark 7.4. We may easily construct explicit basic invariants and basic derivations for the last proposition using Propositions 4.2 and 4.4. Let $x_1^{(i)}, \dots, x_{n_i}^{(i)}$ be the basis of V_i^* dual to the basis $v_1^{(i)}, \dots, v_{n_i}^{(i)}$ of V_i . Let $f_j^{(i)}$ be the j -th elementary symmetric function in $v_1^{(i)}, \dots, v_{n_i}^{(i)}$. Then

$$S(V)^{Z(g)} = \mathbb{C}[f_1^{(1)}, \dots, f_{n_1}^{(1)}, \dots, f_1^{(k)}, \dots, f_{n_k}^{(k)}].$$

We may also explicitly define $\theta_1^{(i)}, \dots, \theta_{n_i}^{(i)} \in S(V_i) \otimes V_i^*$:

$$\theta_j^{(i)} := \sum_{1 \leq l \leq n_i} (v_l^{(i)})^{(j-1)r+1} \otimes x_l^{(i)}.$$

Proof of Proposition 7.3. By Lemma 7.2, g' is conjugate to some diagonal g with

$$Z(g) = G(r, 1, n_1) \oplus \dots \oplus G(r, 1, n_k).$$

Under the representation ρ , g acts as the identity and $V^g = V$. Thus, the Hochschild character $\chi : Z(g) \rightarrow \mathbb{C}$ (given by $h \mapsto \det h^\perp$) is trivial and $\text{HH}^2(g)$ is just the set of derivation 2-forms invariant under $Z(g)$,

$$\text{HH}^2(g) = \left(S(V^g) \otimes \Lambda^{2-\text{codim } V^g} ((V^g)^*) \right)^X = \left(S(V) \otimes \Lambda^2(V^*) \right)^{Z(g)}.$$

Each $G(r, 1, n_i)$ acts on V_i as the symmetric group $\mathfrak{S}_{n_i} \cong G(1, 1, n_i)$ under the representation ρ . Thus the block diagonal group $Z(g)$ acts as the direct sum of symmetric groups, and we simply find the polynomials and derivation 2-forms invariant under the permutation action of $\mathfrak{S}_{n_1} \oplus \dots \oplus \mathfrak{S}_{n_k}$ on $V_1 \oplus \dots \oplus V_k$.

By Remark 4.1, the ring of invariant polynomials is just the tensor product of the corresponding rings $S(V_i)^{\mathfrak{S}_{n_i}}$ of symmetric polynomials. The underlying arrangement of reflecting hyperplanes is given by the product of the polynomials defining the subarrangements. More precisely, we define the reflection arrangement (after identifying $(V^*)^*$ with V) by the polynomial

$$Q = Q_1 \cdots Q_k \in S(V),$$

where each $Q_i \in S(V_i)$ is the Vandermonde determinant ($Q_i = \prod_{j_1 < j_2} x_{j_1}^{(i)} - x_{j_2}^{(i)}$) for \mathfrak{S}_{n_i} acting on V_i^* (see Section 4). As the $\theta_j^{(i)}$ are basic derivations, the coefficient matrix of $\{\theta_1^{(i)}, \dots, \theta_{n_i}^{(i)}\}$ has determinant Q_i up to a nonzero scalar (by Theorem 4.3). Thus $\theta_1^{(1)} \wedge \dots \wedge \theta_{n_k}^{(k)}$ is a nonzero scalar multiple of the form $(Q_1 \cdots Q_k) x_1 \wedge \dots \wedge x_n = Q x_1 \wedge \dots \wedge x_n$. By Proposition 4.3, the derivations $\theta_j^{(i)}$

wedged together two at a time then generate the invariant derivation 2-forms over the ring of invariant polynomials. \square

Contribution from g with $\text{codim } V^g = 2$.

We now determine the contribution to the Hochschild 2-cohomology of the skew group algebra $S(V)\#G(r, 1, n)$ (under the nonfaithful representation ρ defined in (7.1)) from g with $\text{codim } V^g = 2$.

Remark 7.5. Suppose $g \in G = G(r, 1, n)$ with $\text{codim}(V^g) = 2$ under the nonfaithful action of G on V given by ρ . Then for $n \geq 3$, g is conjugate to a diagonal matrix times either a 3-cycle or the product of two 2-cycles (in case $n \geq 4$). In fact, we may assume g is conjugate to a diagonal matrix times the permutation

$$(1, 2, 3) \quad \text{or} \quad (1, 2)(3, 4).$$

If g is the product of a diagonal matrix with $(1, 2)(3, 4)$, then the set of Hochschild semi-invariants for g is zero, i.e. $\text{HH}^2(g) = 0$: Suppose $g = \xi_1^{a_1} \cdots \xi_n^{a_n}(1, 2)(3, 4)$ and define $h = \xi_1^{a_1} \xi_2^{a_2}(1, 2) \in Z(g)$. Then $\det(h^\perp) = -1 \neq 1$ (under the nonfaithful action given by ρ) and hence $\text{HH}^2(g) = 0$ by Lemma 3.6.

In the following proposition, we give explicit invariants by decomposing V into subspaces. Let $V_A = \mathbb{C}\text{-span}\{v_1, v_2, v_3\}$ and $V_B = \mathbb{C}\text{-span}\{v_4, \dots, v_n\}$. If g' is the product of a diagonal matrix and a 3-cycle, then g' is conjugate to $g = g_A \oplus g_B$ where $g_A \in G(r, 1, 3)$ is a 3×3 diagonal matrix times the 3-cycle $(1, 2, 3)$ and $g_B = \xi^{a_1} I_{n_1} \oplus \cdots \oplus \xi^{a_k} I_{n_k} \in G(r, 1, n - 3)$. (Note that n_1, \dots, n_k are simply the multiplicities of the diagonal entries of g_B .) We apply Lemma 7.2 to g_B acting on V_B : Rename the basis vectors v_4, \dots, v_n of V_B so that $v_1^{(1)}, \dots, v_{n_1}^{(1)}$ are the first n_1 basis vectors, $v_1^{(2)}, \dots, v_{n_2}^{(2)}$ are the next n_2 basis vectors, etc.; decompose V_B as $V_1 \oplus \cdots \oplus V_k$ with $V_i = \mathbb{C}\text{-span}\{v_1^{(i)}, \dots, v_{n_i}^{(i)}\}$.

Proposition 7.6. *Assume $n \geq 3$. Let $G = G(r, 1, n)$ act on $V \cong \mathbb{C}^n$ via the nonfaithful representation ρ , and let g' be the product of a diagonal matrix with a 3-cycle. Let $f_l^{(i)}$ be the l -th elementary symmetric function in the elements $v_j^{(i)}$ defined above. Let $f_0 = v_1 + v_2 + v_3$. Then g' is conjugate to g with*

$$\text{HH}^2(g) = S(V^g)^\chi = S(V^g)^{Z(g)} = \mathbb{C}[f_0, f_1^{(1)}, \dots, f_{n_1}^{(1)}, \dots, f_1^{(k)}, \dots, f_{n_k}^{(k)}].$$

Proof. As in the above paragraph, let $g = g_A \oplus g_B$. The action of $Z(g)$ on $V = V_A \oplus V_B$ (and on $V^g = V_A^g \oplus V_B^g$) decomposes as a direct sum:

$$Z(g) = Z_A \oplus Z_B$$

where $Z_A = Z_{G(r,1,3)}(g_A)$ and $Z_B = Z_{G(r,1,n-3)}(g_B)$. In fact, $Z_A = \langle \xi I_3, (1, 2, 3) \rangle$. Under the representation ρ , g acts on V as the permutation $(1, 2, 3)$. Hence $V^g = \mathbb{C}\text{-span}\{v_1 + v_2 + v_3, v_4, \dots, v_n\}$ and $(V^g)^\perp = \mathbb{C}\text{-span}\{v_1 - v_2, v_1 - v_3\}$. The subgroup $Z(g)$ acts on $(V^g)^\perp$ as the group $\langle (1, 2, 3) \rangle$ with determinant 1 under the representation ρ . Thus, the Hochschild character $\chi : Z(g) \rightarrow \mathbb{C}$ (given by $h \mapsto \det h^\perp$) is trivial (the diagonal matrices act trivially):

$$S(V^g)^\chi = S(V^g)^{Z(g)}.$$

Note that $S(V_A^g) = \mathbb{C}[v_1 + v_2 + v_3]$ and Z_A acts trivially on $v_1 + v_2 + v_3$. Hence,

$$S(V_A^g)^{Z_A} = \mathbb{C}[v_1 + v_2 + v_3].$$

We apply Lemma 7.2 and Remark 7.4 to the diagonal matrix $g_B \in G(r, 1, n - 3)$ acting on V_B :

$$S(V_B^g)^{Z_B} = \mathbb{C}[f_1^{(1)}, \dots, f_{n_1}^{(1)}, \dots, f_1^{(k)}, \dots, f_{n_k}^{(k)}].$$

Hence, by Remark 4.1,

$$S(V^g)^{Z(g)} \cong S(V_A^g)^{Z_A} \otimes S(V_B^g)^{Z_B} \cong \mathbb{C}[f_0, f_1^{(1)}, \dots, f_{n_1}^{(1)}, \dots, f_1^{(k)}, \dots, f_{n_k}^{(k)}].$$

□

Putting it together, nonfaithful case.

We now record the Hochschild 2-cohomology of $S(V)\#G$ where $G = G(r, 1, n)$ acts on V via the nonfaithful representation ρ . We combine the contribution to cohomology from $g \in G$ with $\text{codim } V^g = 0$ and the contribution from $g \in G$ with $\text{codim } V^g = 2$.

Theorem 7.7. *Assume $n \geq 3$. Let $G = G(r, 1, n)$ act on $V = \mathbb{C}^n$ via the non-faithful representation ρ . The Hochschild cohomology in degree 2 for the skew group algebra $S(V)\#G$ is*

$$\text{HH}^2(S(V)\#G) \cong \bigoplus_{g \in C} \text{HH}^2(g) \oplus \bigoplus_{g' \in C'} \text{HH}^2(g')$$

where

- C is a set of representatives of those conjugacy classes in G containing diagonal matrices, and $\text{HH}^2(g)$ is given in Proposition 7.3,
- C' is a set of representatives of those conjugacy classes in G containing the product of a diagonal matrix in G with a 3-cycle, and $\text{HH}^2(g')$ is given in Proposition 7.6.

Proof. By Equation (3.7),

$$\text{HH}^2(S(V)\#G) = \bigoplus_{\substack{g \in C \\ \text{codim } V^g = 0}} \text{HH}^2(g) \oplus \bigoplus_{\substack{g' \in C' \\ \text{codim } V^g = 2}} \text{HH}^2(g').$$

In Proposition 7.3, we determined the contribution to Hochschild cohomology from those g in G with $\text{codim } V^g = 0$, i.e., from the diagonal matrices in G . In Proposition 7.6, we determined the contribution to Hochschild cohomology from those g in G with $\text{codim } V^g = 2$, i.e., from those $g \in G$ conjugate to the product of a diagonal matrix in G and a 3-cycle. □

8. GRADED HECKE ALGEBRAS AS DEFORMATIONS OF $S(V)\#G$

In this section, we define a graded Hecke algebra associated to a finite group G and any finite dimensional representation V of G . We give an explicit connection between graded Hecke algebras and the Hochschild semi-invariants defined in Section 3. As a consequence of Theorems 5.1 and 8.7, if $G = G(r, p, n)$ with $r \geq 3, n \geq 4$, and V is its natural reflection representation, then there are no non-trivial graded Hecke algebras: The relevant Hochschild cohomology, while nonzero, is not of the required form. This nonexistence of a graded Hecke algebra was discovered by Ram and the first author [22], and now we may view their result in the context of algebraic deformation theory and Hochschild cohomology. This negative result inspired their ad hoc construction of a “different graded Hecke algebra” for $G(r, 1, n)$, which coincides with an algebra defined by Dezélee [7] in case $r = 2$.

In the next section, we show that this “different graded Hecke algebra” is in fact a graded Hecke algebra under our broader Definition 8.1, in which we allow non-faithful actions of G . This also motivates our earlier computations in Section 7 of Hochschild cohomology for a nonfaithful action of $G(r, 1, n)$ on $V = \mathbb{C}^n$: Theorems 7.7 and 8.7 imply existence of a generic graded Hecke algebra depending on many parameters, described in (9.1).

The following definition includes the symplectic reflection algebras defined by Chmutova [6] in case G acts symplectically, i.e., when there is a (not necessarily injective) group homomorphism $G \rightarrow \mathrm{Sp}(V)$. For brevity, we omit tensor symbols in expressions in the tensor algebra $T(V)$.

Definition 8.1. *Let G be a finite group with a representation $V = \mathbb{C}^n$. For each $g \in G$, choose a skew-symmetric bilinear form $a_g : V \times V \rightarrow \mathbb{C}$. Let*

$$A = (T(V)\#G) / (vw - wv - \sum_{g \in G} a_g(v, w)\bar{g}),$$

where we quotient by the ideal generated by all elements of the given form for $v, w \in V$. Consider A to be a filtered algebra in which we assign degree 1 to elements of V and degree 0 to elements of G . We call A a **graded Hecke algebra** if the associated graded algebra $\mathrm{gr}A$ is isomorphic to $S(V)\#G$. Equivalently A is isomorphic as a vector space to $S(V)\#G$ via a choice of linear section $S(V) \hookrightarrow T(V)$ of the canonical projection of $T(V)$ onto $S(V)$.

In case G is a Coxeter group and V its natural reflection representation, the graded algebra associated to the affine Hecke algebra (with respect to a different choice of filtration) defined by Lusztig [18] is a graded Hecke algebra under our definition [22, §3]. This is the origin of the term graded Hecke algebra.

By definition, graded Hecke algebras are parametrized by sets $\{a_g\}_{g \in G}$ satisfying suitable conditions. These conditions may be determined by applying [22, Lemma 1.5], valid in this more general setting when the action of G on V may not be faithful:

Lemma 8.2. *Let A be the algebra defined by a set of skew-symmetric bilinear forms $\{a_g\}_{g \in G}$ in Definition 8.1. Then A is a graded Hecke algebra if and only if*

$$(8.3) \quad \bar{h}[v, w](\bar{h})^{-1} = [h(v), h(w)], \quad \text{and}$$

$$(8.4) \quad [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

in A , for all $u, v, w \in V$ and $h \in G$ (where $[v, w] = vw - wv$).

Equations (8.3) and (8.4) are equivalent to the uniqueness of the expressions $\bar{h}vw$ and wvu when rearranged with all group elements to the right and vector space elements in alphabetical order. In case G acts faithfully on V , this lemma was used in [22] to show directly that the sets $\{a_g\}_{g \in G}$ corresponding to graded Hecke algebras form a vector space of dimension $d + \dim(\bigwedge^2(V))^G$, where d is the number of conjugacy classes of $g \in G$ such that $\mathrm{codim}(V^g) = 2$ and $\chi_g \equiv 1$ (where $\chi_g(h) = \det(h^\perp)$ for $h \in Z(g)$). This approach may be adapted to nonfaithful actions to yield Corollary 8.17 below. However, we take a somewhat different route in the next theorem in order to relate graded Hecke algebras to Hochschild cohomology, and thus to other potential deformations of $S(V)\#G$.

Definition 8.5. Let $g \in G$ and $f_g \in \text{HH}^2(g)$, i.e., f_g is a Hochschild semi-invariant of g (see (3.2)). By definition, we may write f_g as a linear combination of elements of the form $p \otimes y$, where $p \in S(V^g)$, $y \in \wedge((V^g)^*)$. We say that f_g has **degree** m if the polynomial p of highest degree in such a linear combination has degree m (with respect to fixed bases of V^g , $\wedge((V^g)^*)$). Note that the homogeneous parts of f_g are also Hochschild semi-invariants of g .

Remark 8.6. In the next theorem, we explain how to define a graded Hecke algebra from an element of Hochschild cohomology. We first outline how one identifies a Hochschild semi-invariant with a function on $S(V)^e \otimes \wedge^2(V)$ (recall $S(V)^e = S(V) \otimes S(V)^{op}$). Let \mathcal{C} denote a set of representatives of the conjugacy classes of G , as before. Via (3.7), each element of $\text{HH}^2(S(V)\#G)$ may be identified with a set $\{f_g\}_{g \in \mathcal{C}}$ of Hochschild semi-invariants (see (3.2)). Applying (3.4) as well, we regard each f_g as an element of $S(V^g)\bar{g} \otimes \wedge^2(V^*)$ by making the canonical identification of the vector space $\wedge^{2-\text{codim } V^g}((V^g)^*) \otimes \wedge^{\text{codim } V^g}(((V^g)^\perp)^*)$ with a subspace of $\wedge^2(V^*)$ and by inserting the factor \bar{g} . This allows us to consider f_g as a $Z(g)$ -invariant function on $S(V)^e \otimes \wedge^2(V)$ in the following way: Suppose f_g is a linear combination of elements of the form $p\bar{g} \otimes y$ ($p \in S(V^g)$, $y \in \wedge^2(V^*)$). Let $r, s \in S(V)$, $z \in \wedge^2(V)$. Then $f_g(r \otimes s \otimes z) \in S(V)\bar{g}$ is the corresponding linear combination of elements $y(z)rp\bar{g}s$ (see the proof of Theorem 8.7 below for details). We further apply a transfer (trace) operator to each f_g to obtain the corresponding G -invariant function f on $S(V)^e \otimes \wedge^2(V)$:

$$f = \sum_{g \in \mathcal{C}} \sum_{h \in G/Z(g)} h(f_g),$$

where $(h(f_g))(r \otimes s \otimes z) = h(f_g(h^{-1}(r \otimes s \otimes z)))$.

Theorem 8.7. Let G be a finite group with a representation $V = \mathbb{C}^n$. The parameter space of graded Hecke algebras for the pair G, V is isomorphic to the space consisting of sets $\{f_g\}_{g \in G}$ of Hochschild semi-invariants whose nonzero elements have degree 0. The defining skew-symmetric bilinear forms $\{a_g\}_{g \in G}$ of the graded Hecke algebra corresponding to $\{f_g\}_{g \in \mathcal{C}}$ are given by $a_g(v, w)\bar{g} = f_g(1 \otimes 1 \otimes v \wedge w)$ and $a_{h^{-1}gh}(v, w) = a_g(h(v), h(w))$ for all $g \in \mathcal{C}$, $h \in G$, and $v, w \in V$.

The remainder of this section is devoted to proving this theorem and its corollary. We need some technical lemmas and formulas from [5, 26, 27]. We obtain the forms a_g from the functions f_g by finding intermediary Hochschild two-cocycles μ_1 and corresponding deformations of $S(V)\#G$. First, we recall the definitions of deformations and Hochschild two-cocycles. For more details, see [13] or [10].

Let t be an indeterminate. If R is any associative \mathbb{C} -algebra (such as $R = S(V)\#G$), a **deformation of R over $\mathbb{C}[t]$** consists of the $\mathbb{C}[t]$ -module $R[t] = \mathbb{C}[t] \otimes R$ together with an associative product $*$ of the form

$$(8.8) \quad a * b = ab + \mu_1(a, b)t + \mu_2(a, b)t^2 + \dots$$

for all $a, b \in R$, where ab is the product of a and b in R and $\mu_i : R \times R \rightarrow R$ is \mathbb{C} -bilinear (extended to be $\mathbb{C}[t]$ -bilinear) for each i . (In order for $a * b$ to be in $R[t]$, this sum must in fact be finite for each pair a, b ; one may also be interested in deformations over $\mathbb{C}[[t]]$ or another extension of \mathbb{C} , but we will not need these here.)

Associativity implies conditions on the μ_i . In particular μ_1 must be a **Hochschild two-cocycle**; i.e.,

$$(8.9) \quad \mu_1(a, bc) + a\mu_1(b, c) = \mu_1(ab, c) + \mu_1(a, b)c$$

for all $a, b, c \in R$. That is, μ_1 is a representative of an element of $\text{HH}^2(R)$ obtained via the bar complex

$$(8.10) \quad \dots \xrightarrow{\delta_3} R^{\otimes 4} \xrightarrow{\delta_2} R^{\otimes 3} \xrightarrow{\delta_1} R^e \xrightarrow{m} R \rightarrow 0,$$

where $\delta_i(a_0 \otimes a_1 \otimes \dots \otimes a_{i+1}) = \sum_{j=0}^i (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_{i+1}$ and m is multiplication. This is an R^e -free resolution of R (where $R^e = R \otimes R^{op}$), and thus yields $\text{HH}^*(R) = \text{Ext}_{R^e}^*(R, R)$ upon taking cohomology of the cochain complex resulting from application of $\text{Hom}_{R^e}(R, -)$. Specifically, $\text{HH}^i(R) = \text{Ker}(\delta_{i+1}^*) / \text{Im}(\delta_i^*)$, defined via the cochain complex

$$0 \rightarrow \text{Hom}_{R^e}(R^e, R) \xrightarrow{\delta_1^*} \text{Hom}_{R^e}(R^{\otimes 3}, R) \xrightarrow{\delta_2^*} \text{Hom}_{R^e}(R^{\otimes 4}, R) \xrightarrow{\delta_3^*} \dots$$

We identify $\text{Hom}_{R^e}(R^{\otimes 4}, R) \cong \text{Hom}_{\mathbb{C}}(R^{\otimes 2}, R)$, and a straightforward calculation yields Equation (8.9) as the defining relation for elements of $\text{Ker}(\delta_3^*)$.

In order to prove Theorem 8.7, we explain how to obtain the defining skew-symmetric forms $\{a_g\}_{g \in G}$ of a graded Hecke algebra from a set of Hochschild semi-invariants $\{f_g\}_{g \in \mathcal{C}}$ whose nonzero elements have degree 0. The following comparison of the bar complex (8.10) to a Koszul complex allows us to write down corresponding Hochschild 2-cocycles μ_1 explicitly.

There is a chain map from the bar complex (8.10) for $R = S(V)$ to the Koszul complex $P_\bullet = K(\{v_i \otimes 1 - 1 \otimes v_i\}_{i=1}^n)$, where v_1, \dots, v_n is a basis of V (see [25, §4.5] for details on Koszul complexes):

$$\begin{array}{ccccccccccc} \dots & \rightarrow & S(V)^{\otimes 4} & \xrightarrow{\delta_2} & S(V)^{\otimes 3} & \xrightarrow{\delta_1} & S(V)^e & \xrightarrow{m} & S(V) & \rightarrow & 0 \\ & & \downarrow \psi_2 & & \downarrow \psi_1 & & \parallel & & \parallel & & \\ \dots & \rightarrow & S(V)^e \otimes \wedge^2(V) & \xrightarrow{d_2} & S(V)^e \otimes \wedge^1(V) & \xrightarrow{d_1} & S(V)^e & \xrightarrow{m} & S(V) & \rightarrow & 0 \end{array}$$

The differentials d_1, d_2 are $S(V)^e$ -homomorphisms given in our notation by

$$\begin{aligned} d_1(1 \otimes 1 \otimes v_i) &= v_i \otimes 1 - 1 \otimes v_i, \\ d_2(1 \otimes 1 \otimes v_i \wedge v_j) &= (v_i \otimes 1 - 1 \otimes v_i) \otimes v_j - (v_j \otimes 1 - 1 \otimes v_j) \otimes v_i, \end{aligned}$$

for $1 \leq i < j \leq n$. The vertical maps ψ_1 and ψ_2 may be given by the formulas (see [26, (4.9) and (4.10)]), where the notation is somewhat different):

(8.11)

$$\psi_1(1 \otimes v_1^{k_1} \dots v_n^{k_n} \otimes 1) = \sum_{i=1}^n \sum_{a=1}^{k_i} v_i^{k_i-a} v_{i+1}^{k_{i+1}} \dots v_n^{k_n} \otimes v_1^{k_1} \dots v_{i-1}^{k_{i-1}} v_i^{a-1} \otimes v_i,$$

(8.12)

$$\begin{aligned} &\psi_2(1 \otimes v_1^{k_1} \dots v_n^{k_n} \otimes v_1^{m_1} \dots v_n^{m_n} \otimes 1) \\ &= \sum_{1 \leq i < j \leq n} \sum_{b=1}^{m_j} \sum_{a=1}^{k_i} v_i^{k_i-a} v_{i+1}^{k_{i+1}} \dots v_{j-1}^{k_{j-1}} v_j^{k_j+m_j-b} v_{j+1}^{k_{j+1}+m_{j+1}} \dots v_n^{k_n+m_n} \\ &\quad \otimes v_1^{k_1+m_1} \dots v_{i-1}^{k_{i-1}+m_{i-1}} v_i^{m_i+a-1} v_{i+1}^{m_{i+1}} \dots v_{j-1}^{m_{j-1}} v_j^{b-1} \otimes v_i \wedge v_j. \end{aligned}$$

To obtain an explicit Hochschild two-cocycle μ_1 from a set $\{f_g\}_{g \in \mathcal{C}}$ of Hochschild semi-invariants, we apply the following proposition. This proposition appears as

part of [5, Thm. 5.4], where it is stated for arbitrary degree and arbitrary resolution P . It is valid also when G acts nonfaithfully on V .

Proposition 8.13. *Let $R = S(V)\#G$. Let $f : S(V)^e \otimes \wedge^2(V) \rightarrow R$ be a function (on the degree 2 term of the above Koszul complex) representing an element of $\mathrm{HH}^2(S(V), R)^G$. Under the isomorphism $\mathrm{HH}^2(S(V), R)^G \cong \mathrm{HH}^2(R)$, f is mapped to the function $\mu_1 \in \mathrm{Hom}_{\mathbb{C}}(R^{\otimes 2}, R) \cong \mathrm{Hom}_{R^e}(R^{\otimes 4}, R)$ whose action on the degree 2 term of the bar complex (8.10) is given by*

$$\mu_1(r\bar{g} \otimes s\bar{h}) = ((f \circ \psi_2)(1 \otimes r \otimes g(s) \otimes 1))\overline{gh},$$

for all $r, s \in S(V)$ and $g, h \in G$.

Our final tool for proving Theorem 8.7 is [27, Thm. 3.2], valid when G acts nonfaithfully on V ; we also record it here for convenience. Consider $S(V)\#G$ to be a graded algebra where elements of V have degree 1 and elements of G have degree 0. In [27], a graded Hecke algebra is defined over $\mathbb{C}[t]$: It is a quotient of $T(V)\#G[t]$ by the ideal generated by all $vw - wv - \sum_{g \in G} a_g(v, w)t\bar{g}$, whose associated graded $\mathbb{C}[t]$ -algebra is isomorphic to $S(V)\#G[t]$. To obtain our Definition 8.1, simply substitute any nonzero complex number for t .

Proposition 8.14. *Up to isomorphism, the (nontrivial) graded Hecke algebras over $\mathbb{C}[t]$ are precisely the deformations of $S(V)\#G$ over $\mathbb{C}[t]$ for which $\deg \mu_i = -2i$ ($i \geq 1$).*

Proof of Theorem 8.7. Suppose $\{f_g\}_{g \in G}$ is a set of Hochschild semi-invariants and that each nonzero f_g has degree 0. As in Remark 8.6, let f be the associated function on $S(V)^e \otimes \wedge^2(V)$. Let μ_1 be the corresponding Hochschild 2-cocycle of $S(V)\#G$, given by Proposition 8.13. Let $v, w \in V$. By Remark 8.6, Proposition 8.13, and formula (8.12), $\mu_1(v \otimes w) = (f \circ \psi_2)(1 \otimes v \otimes w \otimes 1)$ is an element of the subalgebra $\mathbb{C}G$ of $S(V)\#G$. Similarly, $\mu_1(w \otimes v)$ is an element of $\mathbb{C}G$. Define scalars $a_g(v, w)$ by setting

$$(8.15) \quad \mu_1(v \otimes w) - \mu_1(w \otimes v) = \sum_{g \in G} a_g(v, w)\bar{g}.$$

As μ_1 is bilinear, each resulting function $a_g : V \times V \rightarrow \mathbb{C}$ is bilinear. By definition, a_g is skew-symmetric for each $g \in G$. We claim that the set $\{a_g\}_{g \in G}$ defines a graded Hecke algebra. Let A be the corresponding quotient given in Definition 8.1. By Lemma 8.2, it suffices to verify (8.3) and (8.4) for A .

By Definition 8.1 and Equation (8.15), $[v, w] = \mu_1(v \otimes w) - \mu_1(w \otimes v)$ lies in A , for all $v, w \in V$. Abuse notation and write $\psi_2(v \otimes w) = \psi_2(1 \otimes v \otimes w \otimes 1)$. Apply definition (8.12) of ψ_2 to our chosen basis for V to see that $\psi_2(v_i \otimes v_j) = 1 \otimes 1 \otimes v_i \wedge v_j$ if $i < j$ and 0 otherwise. After expressing all vectors in terms of this basis, we find

$$h(\psi_2(v \otimes w) - \psi_2(w \otimes v)) = \psi_2(h(v) \otimes h(w)) - \psi_2(h(w) \otimes h(v))$$

for all $v, w \in V$, $h \in G$. As f is also G -invariant, it follows that

$$\mu_1(v \otimes w) - \mu_1(w \otimes v) = (f \circ \psi_2)(v \otimes w) - (f \circ \psi_2)(w \otimes v)$$

is G -invariant as a function on $v, w \in V$, and thus (8.3) holds in A .

Next we claim that the Jacobi identity (8.4) is a direct consequence of the Hochschild 2-cocycle condition (8.9). The left side of (8.4) may be rewritten by

replacing the innermost bracket $[v, w]$ in the first term by $\mu_1(v \otimes w) - \mu_1(w \otimes v)$, and similarly for each of the other two terms. We obtain

$$\begin{aligned} & u\mu_1(v \otimes w) - u\mu_1(w \otimes v) - \mu_1(v \otimes w)u \\ & + \mu_1(w \otimes v)u + v\mu_1(w \otimes u) - v\mu_1(u \otimes w) - \mu_1(w \otimes u)v + \mu_1(u \otimes w)v \\ & + w\mu_1(u \otimes v) - w\mu_1(v \otimes u) - \mu_1(u \otimes v)w + \mu_1(v \otimes u)w. \end{aligned}$$

Substitutions from the six permutations of (8.9) in which $\{a, b, c\} = \{u, v, w\}$ yield

$$\begin{aligned} & \mu_1(uv \otimes w) - \mu_1(u \otimes vw) + \mu_1(u \otimes wv) - \mu_1(uw \otimes v) \\ & + \mu_1(vw \otimes u) - \mu_1(v \otimes wu) + \mu_1(w \otimes vu) - \mu_1(wv \otimes u) \\ & + \mu_1(v \otimes uw) - \mu_1(vu \otimes w) + \mu_1(wu \otimes v) - \mu_1(w \otimes uv). \end{aligned}$$

But this expression is zero as $S(V)$ is commutative. We have thus shown that (8.4) holds in A . This concludes the proof that a set of Hochschild semi-invariants whose nonzero elements have degree 0 gives rise to a graded Hecke algebra.

We now argue that every graded Hecke algebra arises in this way. Note that if a set $\{a_g\}_{g \in G}$ of skew-symmetric bilinear forms defines a graded Hecke algebra, then so does $\{\alpha a_g\}_{g \in G}$ for any fixed scalar α . We may therefore consider the related graded Hecke algebra over $\mathbb{C}[t]$, where the indeterminate t takes the place of the arbitrary scalar α . By Proposition 8.14, a (nontrivial) graded Hecke algebra over $\mathbb{C}[t]$ is a deformation of $S(V)\#G$ such that the associated Hochschild 2-cocycle μ_1 satisfies $\deg \mu_1 = -2$, and more generally $\deg(\mu_i) = -2i$ ($i \geq 1$). Equation (8.8) and Definition 8.1 (with a_g replaced by ta_g) force relationship (8.15) between the defining forms ta_g and the cocycle μ_1 . As μ_1 is a Hochschild 2-cocycle, Equation (3.7) implies the existence of a set of Hochschild semi-invariants $\{f_g\}_{g \in \mathcal{C}}$ and a corresponding function f (see Remark 8.6) such that $\psi_2^*(f)$ is cohomologous to μ_1 . In particular, there exists an $S(V)^e$ -homomorphism $\beta : S(V)^{\otimes 3} \rightarrow S(V)\#G$ such that $f \circ \psi_2 = \mu_1 + \beta \circ \delta_2$. Identify β with a \mathbb{C} -linear function from $S(V)$ to $S(V)\#G$, and note that

$$\beta(\delta_2(1 \otimes v \otimes w \otimes 1)) = v\beta(w) - \beta(vw) + \beta(v)w$$

for all $v, w \in V$. Thus for all $i < j$, we have

$$(f \circ \psi_2)(v_i \otimes v_j) = \mu_1(v_i \otimes v_j) - \mu_1(v_j \otimes v_i) + v_i\beta(v_j) + \beta(v_i)v_j - v_j\beta(v_i) - \beta(v_j)v_i$$

(since $\psi_2(v_j \otimes v_i) = 0$ when $i < j$). Thus $(f \circ \psi_2)(v_i \otimes v_j)$ is a sum of the element $\mu_1(v_i \otimes v_j) - \mu_1(v_j \otimes v_i)$ in $\mathbb{C}G$ and the element $v_i\beta(v_j) + \beta(v_i)v_j - v_j\beta(v_i) - \beta(v_j)v_i$ in the ideal (V) of $S(V)\#G$. Now $(f \circ \psi_2)(v_i \otimes v_j) = f(1 \otimes 1 \otimes v_i \wedge v_j)$, and the function f is discussed in Remark 8.6; in particular it is determined by these values since it is an $S(V)^e$ -homomorphism. Since $S(V)\#G \cong \mathbb{C}G \oplus S(V)$ as a vector space and the homogeneous parts of f are also G -invariant functions on $S(V)^e \otimes \bigwedge^2(V)$, we have $f = f' + f''$ where $(f' \circ \psi_2)(v_i \otimes v_j) = \mu_1(v_i \otimes v_j) - \mu_1(v_j \otimes v_i)$ and $f'' \circ \psi_2 = \beta \circ \delta_2$ is a coboundary. By Remark 8.6, f' itself corresponds to a set of Hochschild semi-invariants, $\{f'_g\}_{g \in \mathcal{C}}$. As $\deg \mu_1 = -2$ and $\deg \psi_2 = -2$, each f'_g is either 0 or has degree 0. Thus $\{f'_g\}_{g \in \mathcal{C}}$ is the desired set of Hochschild semi-invariants. (Alternatively, we could simply prove that our map from sets $\{f_g\}_{g \in \mathcal{C}}$ of Hochschild semi-invariants (whose nonzero elements have degree 0) to sets $\{a_g\}_{g \in G}$ of skew-symmetric forms defining graded Hecke algebras is *surjective* by classifying the latter directly from Lemma 8.2; cf. [22, Thm. 1.9].)

The formula for the a_g in terms of the f_g follows from our calculations above: If $i < j$, then

$$\begin{aligned} \sum_{g \in G} a_g(v_i, v_j) \bar{g} &= [v_i, v_j] = \mu_1(v_i \otimes v_j) - \mu_1(v_j \otimes v_i) \\ &= (f \circ \psi_2)(v_i \otimes v_j) = f(1 \otimes 1 \otimes v_i \wedge v_j). \end{aligned}$$

Now examine the coefficient of each \bar{g} ($g \in G$) separately, and apply (8.3). The resulting formula extends from pairs v_i, v_j of basis elements to all $v, w \in V$ by linearity. \square

Remark 8.16. We point out an interesting consequence of the Hochschild 2-cocycle condition (8.9) in this context: Let μ_1 and A be as in the first paragraph of the proof of Theorem 8.7. We showed that the Jacobi identity (8.4) holds in A as a direct consequence of the Hochschild 2-cocycle condition (8.9) for μ_1 .

The following corollary characterizes graded Hecke algebras without reference to Hochschild cohomology. Recall that $\chi_g(h) = \det(h^\perp)$ for $h \in Z(g)$, where $h^\perp = h|_{(V^g)^\perp}$, and \mathcal{C} denotes a set of representatives of the conjugacy classes of G .

Corollary 8.17. *Let d be the number of conjugacy classes of $g \in G$ such that $\text{codim } V^g = 2$ and $\chi_g \equiv 1$.*

- (i) *The sets $\{a_g\}_{g \in G}$ corresponding to graded Hecke algebras form a vector space of dimension*

$$d + \sum_{\substack{g \in \mathcal{C} \\ V^g = V}} \dim(\Lambda^2(V))^{Z(g)}.$$

- (ii) *All graded Hecke algebras arise in the following way: For each $g \in \mathcal{C}$ satisfying $\text{codim } V^g = 2$ and $\chi_g \equiv 1$, define a skew-symmetric bilinear form $a_g : V \times V \rightarrow \mathbb{C}$ by setting $a_g(w_1, w_2)$ equal to an arbitrary scalar for a fixed basis $\{w_1, w_2\}$ of $(V^g)^\perp$, $a_g(V^g, V) = 0$, and $a_{h^{-1}gh}(v, w) = a_g(h(v), h(w))$ for all $h \in G, v, w \in V$. For each $g \in \mathcal{C}$ satisfying $V^g = V$, define a_g by any choice of $Z(g)$ -invariant linear functional on $\Lambda^2(V)$, and $a_{h^{-1}gh}(v, w) = a_g(h(v), h(w))$ for all $h \in G, v, w \in V$.*

Proof. By (3.7), if $\text{HH}^2(g)$ is nonzero, then $\text{codim } V^g \in \{0, 2\}$. We apply Theorem 8.7 in each case. If $\text{codim } V^g = 0$ (i.e. $V^g = V$), the Hochschild semi-invariants of g that are either 0 or of degree 0 form a vector space of dimension $\dim(\Lambda^2(V))^{Z(g)}$ (see (3.2)). Theorem 8.7, Remark 8.6, and (8.3) then give the indicated form of the corresponding functions a_g . If $\text{codim } V^g = 2$, a nonzero scalar is a Hochschild semi-invariant for g if and only if $\chi_g \equiv 1$ (by definition). Again by Theorem 8.7, Remark 8.6, and (8.3), the values of the corresponding forms a_g are precisely those given in (ii). (Alternatively the corollary may be proven directly, without using Hochschild cohomology; cf. [22, Thm. 1.9]). \square

9. GRADED HECKE ALGEBRAS FOR $G(r, 1, n)$

Assume $n \geq 3$. Let $G = G(r, 1, n)$ act on $V = \mathbb{C}^n$ nonfaithfully as the symmetric group via ρ defined in (7.1). In this section, we find all graded Hecke algebras corresponding to this action. Similar results hold for $G(r, p, n)$. We prove that in a special case, these algebras are isomorphic to algebras that appeared in [22], and in [7] when $r = 2$.

By Theorems 7.7 and 8.7, the parameter space of graded Hecke algebras for G, V has dimension equal to the number of conjugacy classes of elements g in $G(r, 1, n)$ for which g is the product of a diagonal matrix with a 3-cycle. (By Proposition 7.3 and Remark 7.4, the diagonal matrices themselves do not have Hochschild semi-invariants of degree 0.) We refine the notation introduced in the text preceding Proposition 7.6 for these elements g : Choose representatives $g = g_{b, \underline{c}, \underline{m}}$, one for each conjugacy class, where

$$g_{b, \underline{c}, \underline{m}} = \xi_3^b(1, 2, 3)\xi^{c_1}I_{m_1} \oplus \cdots \oplus \xi^{c_k}I_{m_k},$$

$\underline{c} = (c_1, \dots, c_k)$, $\underline{m} = (m_1, \dots, m_k)$, and $0 \leq b, c_1, \dots, c_k \leq r - 1$ (m_1, \dots, m_k are the multiplicities of the diagonal entries other than the first three). By Corollary 8.17(ii), all graded Hecke algebras arise as follows. For each representative $g_{b, \underline{c}, \underline{m}}$, choose a scalar $\alpha_{b, \underline{c}, \underline{m}}$. The corresponding graded Hecke algebra is

$$(9.1) \quad A = (T(V)\#G)/(vw - wv - \sum_{g \in G(r, 1, n)} a_g(v, w)\bar{g}),$$

where the skew-symmetric bilinear forms a_g ($g \in G$) are given by:

- $a_g \equiv 0$ if g is not conjugate to one of the representatives $g_{b, \underline{c}, \underline{m}}$;
- $a_{g_{b, \underline{c}, \underline{m}}}(v_1 - v_2, v_2 - v_3) = \alpha_{b, \underline{c}, \underline{m}}$ and $a_{g_{b, \underline{c}, \underline{m}}}(V^{(1, 2, 3)}, V) = 0$;
- $a_{h^{-1}g_{b, \underline{c}, \underline{m}}h}(v, w) = \alpha_{g_{b, \underline{c}, \underline{m}}}(h(v), h(w))$ for all $h \in G$.

We are particularly interested in the choice $\alpha_{0, 0, n-3} = 1$ ($k = 1$) and all other $\alpha_{b, \underline{c}, \underline{m}} = 0$. Denote the resulting graded Hecke algebra by $A_{r, 1, n}$:

Definition 9.2. $A_{r, 1, n} = (T(V)\#G(r, 1, n))/(vw - wv - \sum_{g \in G(r, 1, n)} a_g(v, w)\bar{g}),$

where the skew-symmetric bilinear forms a_g ($g \in G$) are given by:

- $a_g \equiv 0$ if g is not conjugate to $(1, 2, 3)$;
- $a_{(1, 2, 3)}(v_1 - v_2, v_2 - v_3) = 1$ and $a_{(1, 2, 3)}(V^{(1, 2, 3)}, V) = 0$;
- $a_{h^{-1}(1, 2, 3)h}(v, w) = a_{(1, 2, 3)}(h(v), h(w))$ for all $h \in G$.

We claim that the defining relations $vw - wv - \sum_{g \in G} a_g(v, w)\bar{g}$ of $A_{r, 1, n}$ may be replaced by

$$(9.3) \quad v_m v_k - v_k v_m = \frac{1}{3} \sum_{\substack{1 \leq i \leq n \\ i \neq m, k}} \sum_{a, b=0}^{r-1} \bar{\xi}_m^a \bar{\xi}_k^b (\bar{\xi}_i)^{-a-b} ((\overline{m, k, i}) - (\overline{m, i, k}))$$

for all $1 \leq m < k \leq n$. To see this, first note that the group elements in this sum are precisely those that both are conjugate to $(1, 2, 3)$ and act nontrivially on v_m, v_k . For each i, a, b let h be an element of \mathfrak{S}_n such that $h^{-1}(1, 2, 3)h = \xi_m^a \xi_k^b \xi_i^{-a-b}(m, k, i)$ (or $h^{-1}(1, 2, 3)h = \xi_m^a \xi_k^b \xi_i^{-a-b}(m, i, k)$). Then apply Definition 9.2, writing v_1, v_2, v_3 in terms of the vector space decomposition $V \cong (V^{(1, 2, 3)})^\perp \oplus V^{(1, 2, 3)}$.

We will show that the graded Hecke algebra $A_{r, 1, n}$ is precisely the algebra defined by Ram and the first author [22, (5.1)] as a substitute for a (nonexistent) graded Hecke algebra corresponding to the natural reflection representation of $G(r, 1, n)$. Their algebra agrees with one defined by Dezélée [7] in the case $r = 2$. Our results

in Section 8 will then imply that these algebras of [7, 22] arise from deformations of $S(V)\#G(r, 1, n)$ over $\mathbb{C}[t]$. They were originally given in a different form, analogous to Lusztig’s definition of a graded Hecke algebra. The proof that Lusztig’s definition is a special case of Drinfeld’s definition [22, Thm. 3.5] suggested to us how to proceed.

Definition 9.4. *As in [22, (5.1)], let $H_{r,1,n}^*$ be the algebra generated by the basis v_1, \dots, v_n of V and all \bar{g} ($g \in G(r, 1, n)$) such that $\mathbb{C}G(r, 1, n)$ and $S(V)$ are subalgebras and the following additional relations hold:*

$$(9.5) \quad \bar{\xi}_i v_k = v_k \bar{\xi}_i \quad (1 \leq i, k \leq n),$$

$$(9.6) \quad \bar{s}_i v_k = v_k \bar{s}_i \quad (k \neq i, i + 1),$$

$$(9.7) \quad \bar{s}_i v_{i+1} = v_i \bar{s}_i + \sum_{a=0}^{r-1} (\bar{\xi}_i)^a (\bar{\xi}_{i+1})^{-a} \quad (1 \leq i \leq n - 1),$$

where s_i is the simple reflection $(i, i + 1)$.

A useful variant of relation (9.7) is $\bar{s}_i v_i = v_{i+1} \bar{s}_i - \sum_{a=0}^{r-1} (\bar{\xi}_i)^a (\bar{\xi}_{i+1})^{-a}$. In Dezélée’s version [7], one may choose any nonzero scalar as the coefficient of the sum in relation (9.7) above, and the proof of the following theorem may be modified accordingly.

Theorem 9.8. *Let $A_{r,1,n}$ be the graded Hecke algebra of Definition 9.2. There is an algebra isomorphism*

$$A_{r,1,n} \cong H_{r,1,n}^*.$$

The remainder of this section is devoted to the proof of this theorem. We first collect some additional relations in $H_{r,1,n}^*$ that are consequences of (9.6) and (9.7).

Lemma 9.9. *Let $1 \leq j, k, m \leq n$ with $j < k$. In $H_{r,1,n}^*$, the element $(\overline{j, k})v_m$ is equal to*

$$\begin{cases} v_m(\overline{j, k}), & \text{if } m < j \text{ or } k < m, \\ v_m(\overline{j, k}) + \sum_{a=0}^{r-1} (\bar{\xi}_m)^a (\bar{\xi}_k)^{-a} (\overline{j, m, k}) - \sum_{a=0}^{r-1} (\bar{\xi}_j)^a (\bar{\xi}_m)^{-a} (\overline{j, k, m}), & \text{if } j < m < k, \\ v_k(\overline{j, k}) - \sum_{j < i < k} \sum_{a=0}^{r-1} (\bar{\xi}_i)^a (\bar{\xi}_k)^{-a} (\overline{j, i, k}) - \sum_{a=0}^{r-1} (\bar{\xi}_j)^a (\bar{\xi}_k)^{-a}, & \text{if } m = j, \\ v_j(\overline{j, k}) + \sum_{j < i < k} \sum_{a=0}^{r-1} (\bar{\xi}_i)^a (\bar{\xi}_j)^{-a} (\overline{k, i, j}) + \sum_{a=0}^{r-1} (\bar{\xi}_j)^a (\bar{\xi}_k)^{-a}, & \text{if } m = k. \end{cases}$$

Proof. Write $(\overline{j, k})$ as a product of simple reflections:

$$(\overline{j, k}) = \bar{s}_{k-1} \bar{s}_{k-2} \cdots \bar{s}_{j+1} \bar{s}_j \bar{s}_{j+1} \cdots \bar{s}_{k-2} \bar{s}_{k-1}.$$

If $m < j$ or $k < m$, then $(\overline{j, k})v_m = v_m(\overline{j, k})$ by (9.6). If $j < m < k$, then application of (9.7) and its variation (given in the text following (9.7)) yields the stated relation.

If $m = j$, we use induction on $k - j$. If $k - j = 1$, then the desired relation is the one given in the text following (9.7). Assume the relation given in Lemma 9.9 holds when k is replaced by $k - 1$ and $m = j < k - 1$. Then

$(\overline{j, k})v_j = \overline{s_{k-1}}(\overline{j, k-1})\overline{s_{k-1}}v_j = \overline{s_{k-1}}(\overline{j, k-1})v_j\overline{s_{k-1}}$, and application of the induction hypothesis and the variation of (9.7) yields the relation stated in the theorem. The final relation is proved by a similar induction. \square

We will make a change of generators for $H_{r,1,n}^*$. For each k ($1 \leq k \leq n$), let

$$(9.10) \quad \tilde{v}_k = v_k + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < k}} (\overline{\xi}_k)^a (\overline{\xi}_j)^{-a} (\overline{k, j}),$$

where $\delta_{j < k} = 1$ if $j < k$ and 0 otherwise.

Lemma 9.11. *The algebra $H_{r,1,n}^*$ is generated by \tilde{v}_k ($1 \leq k \leq n$) and \overline{g} ($g \in G(r, 1, n)$) and is defined by the relations of $G(r, 1, n)$ together with*

$$(9.12) \quad \overline{\xi}_i \tilde{v}_k = \tilde{v}_k \overline{\xi}_i \quad \text{for all } 1 \leq i, k \leq n,$$

$$(9.13) \quad \overline{s}_i \tilde{v}_k = \tilde{v}_k \overline{s}_i \quad \text{if } k \notin \{i, i + 1\},$$

$$(9.14) \quad \overline{s}_i \tilde{v}_i = \tilde{v}_{i+1} \overline{s}_i \quad \text{for } 1 \leq i \leq n - 1,$$

$$(9.15) \quad \tilde{v}_m \tilde{v}_k - \tilde{v}_k \tilde{v}_m = \frac{1}{4} \sum_{\substack{1 \leq i \leq n \\ i \neq m, k}} \sum_{a,b=0}^{r-1} (\overline{\xi}_m)^a (\overline{\xi}_k)^b (\overline{\xi}_i)^{-a-b} ((\overline{m, k, i}) - (\overline{m, i, k})),$$

where the last relation holds for $1 \leq m < k \leq n$.

Proof. In the definition of \tilde{v}_k , both summands $\sum \sum (-1)^{\delta_{j < k}} (\overline{\xi}_k)^a (\overline{\xi}_j)^{-a} (\overline{k, j})$ and v_k are invariant under conjugation by $\overline{\xi}_i$ ($1 \leq i \leq n$) and by \overline{s}_i ($k \neq i, i + 1$), so (9.12) and (9.13) hold.

We check (9.14), using (9.7):

$$\begin{aligned} \overline{s}_i \tilde{v}_i \overline{s}_i &= \overline{s}_i v_i \overline{s}_i + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < i}} \overline{s}_i (\overline{\xi}_i)^a (\overline{\xi}_j)^{-a} (\overline{i, j}) \overline{s}_i \\ &= v_{i+1} - \sum_{a=0}^{r-1} \overline{s}_i (\overline{\xi}_i)^a (\overline{\xi}_{i+1})^{-a} + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq i+1}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < i}} (\overline{\xi}_{i+1})^a (\overline{\xi}_j)^{-a} (\overline{i+1, j}) \\ &= v_{i+1} + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq i+1}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < (i+1)}} (\overline{\xi}_{i+1})^a (\overline{\xi}_j)^{-a} (\overline{i+1, j}) \\ &= \tilde{v}_{i+1}. \end{aligned}$$

Finally, we check (9.15). Assume $m < k$. We compute the bracket $\tilde{v}_m \tilde{v}_k - \tilde{v}_k \tilde{v}_m$, first substituting from (9.10) and then applying the relations from Lemma 9.9. The

result is a cancellation of all terms having factors of v_m or v_k :

$$\begin{aligned}
& (v_m + \frac{1}{2} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{b=0}^{r-1} (-1)^{\delta_{i < m}} (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i})) \\
& \quad \times (v_k + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < k}} (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j})) \\
& - (v_k + \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < k}} (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j})) \\
& \quad \times (v_m + \frac{1}{2} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{b=0}^{r-1} (-1)^{\delta_{i < m}} (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i})) \\
& = \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < k}} v_m (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) \\
& \quad + \frac{1}{2} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{b=0}^{r-1} (-1)^{\delta_{i < m}} (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) v_k \\
& \quad + \frac{1}{4} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a, b=0}^{r-1} (-1)^{\delta_{i < m} + \delta_{j < k}} (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) \\
& \quad - \frac{1}{2} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{b=0}^{r-1} (-1)^{\delta_{i < m}} v_k (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) \\
& \quad - \frac{1}{2} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a=0}^{r-1} (-1)^{\delta_{j < k}} (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) v_m \\
& \quad - \frac{1}{4} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a, b=0}^{r-1} (-1)^{\delta_{i < m} + \delta_{j < k}} (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) \\
& = \frac{1}{4} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a, b=0}^{r-1} (-1)^{\delta_{i < m} + \delta_{j < k}} (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) \\
& \quad - \frac{1}{4} \sum_{\substack{1 \leq i \leq n \\ i \neq m}} \sum_{\substack{1 \leq j \leq n \\ j \neq k}} \sum_{a, b=0}^{r-1} (-1)^{\delta_{i < m} + \delta_{j < k}} (\bar{\xi}_k)^a (\bar{\xi}_j)^{-a} (\overline{k, j}) (\bar{\xi}_m)^b (\bar{\xi}_i)^{-b} (\overline{m, i}) \\
& \quad + \frac{1}{2} \sum_{\substack{1 \leq i \leq n \\ i \neq m, k}} \sum_{a, b=0}^{r-1} (\bar{\xi}_m)^a (\bar{\xi}_k)^b (\bar{\xi}_i)^{-a-b} ((\overline{m, k, i}) - (\overline{m, i, k})).
\end{aligned}$$

Now in the first two of the three summands above, we may cancel the terms for which $\{m, i\} \cap \{k, j\}$ is empty. Rewriting the remaining terms and combining with the last summand, we have the desired relation (9.15).

Finally, if the \tilde{v}_k are taken instead of the v_k as generators of $H_{r,1,n}^*$ together with $G(r, 1, n)$, relations (9.12)–(9.15) are equivalent to the relations for $H_{r,1,n}^*$ given in Definition 9.4. \square

Proof of Theorem 9.8. Define an algebra homomorphism from $A_{r,1,n}$ to $H_{r,1,n}^*$ by sending \bar{g} to \bar{g} ($g \in G(r, 1, n)$) and v_k to $\frac{2}{\sqrt{3}} \tilde{v}_k$ ($1 \leq k \leq n$). This map is well-defined: The relations (9.12)–(9.14) correspond to relations in $A_{r,1,n}$, and (9.15) corresponds to (9.3) in $A_{r,1,n}$. This accounts for all the relations in $A_{r,1,n}$. The map is surjective as every generator in $H_{r,1,n}^*$ lies in its image. On the other hand, we can define an inverse map similarly. Therefore $A_{r,1,n} \cong H_{r,1,n}^*$. \square

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