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# TOWARDS A UNIVERSAL SELF-NORMALIZED MODERATE DEVIATION

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Abstract. This paper is an attempt to establish a universal moderate deviation for self-normalized sums of independent and identically distributed random variables without any moment condition. The exponent term in the moderate deviation is specified when the distribution is in the centered Feller class. An application to the law of the iterated logarithm is given.

## 1. Introduction and main results

Let $X, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) nondegenerate random variables. Put

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2$$

and define the self-normalized sum by $S_n/V_n$. In contrast to the classical limit theorems, the self-normalized limit theorems enjoy much better properties with few or no moment conditions. Griffin and Kuelbs [10] obtained a self-normalized law of the iterated logarithm for all distributions in the domain of attraction of a normal or stable law. Shao [14] showed that no moment conditions are needed for a self-normalized large deviation result: *If* $EX = 0$ *or* $EX^2 = \infty$, *then for* $0 < x < 1$,

$$\lim_{n\to\infty} P(S_n/V_n \geq x\sqrt{n})^{1/n} = \exp\big\{ -\lambda(x^2)\big\}, \tag{1.1}$$

*where* $\lambda(x) = \inf_{b\geq 0} \sup_{t\geq 0} \big(tx - \ln E\exp\big\{t(2bX - b^2X^2)\big\}\big)$. It was also shown in Shao [14] that the tail probability of $S_n/V_n$ is Gaussian like when $X$ is in the domain of attraction of the normal law and sub-Gaussian like when $X$ is in the domain of attraction of a stable law. In particular, when $X$ is symmetric and in the domain of attraction of a stable law of order $\alpha$ $(0 < \alpha < 2)$,

$$\ln P(S_n/V_n \geq x_n) \sim -x_n^2\beta_\alpha \tag{1.2}$$

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for any $x_n \to \infty$ satisfying $x_n = o(\sqrt{n})$, where $\beta_\alpha$ is the solution of

$$\int_0^\infty \frac{2 - \exp\{2x - x^2/\beta\} - \exp\{-2x - x^2/\beta\}}{x^{\alpha+1}}\,dx = 0. \tag{1.3}$$

Here and in the sequel, $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. Jing, Shao and Zhou [12] refined the self-normalized deviation and proved a self-normalized saddlepoint approximation without any moment assumption. Jing, Shao and Wang [11] established a Cramér type large deviation for self-normalized sums of independent random variables under a Lindeberg type condition. For other self-normalized results we refer to Logan, Mallows, Rice and Shepp [13] for the limiting distribution of $S_n/V_n$ when $X$ is in the domain of attraction of a stable law, which has been proved also necessary by Götze and Chistyakov [8], Bentkus and Götze [1] for Berry-Esseen inequalities, Giné, Götze and Mason [6] for the necessary and sufficient condition for the asymptotic normality, Csörgő, Szyszkowicz and Wang [2, 3] for the Darling–Erdős theorem and Donsker theorem, and Shao [15, 16] for surveys on recent developments in this area.

The main aim of this paper is to establish a universal self-normalized moderate deviation without any moment condition. Results in (1.1) and (1.2) suggest that a very likely result would be

$$\ln P(S_n/V_n \ge x_n) \sim -n\lambda(x_n^2/n) \tag{1.4}$$

for $x_n \to \infty$ with $x_n = o(\sqrt{n})$. We shall prove that this is the case when $X$ is in the centered Feller class.

Before we state our main theorem, let us introduce some notation. Let $C_s$ denote the support of $X$, that is,

$$C_s = \{x : P(X \in (x-\epsilon, x+\epsilon)) > 0, \text{ for any } \epsilon > 0\}.$$

We denote the number of elements in $C_s$ by $\mathrm{Card}(C_s)$ and define $\mathrm{Card}(C_s) = \infty$ if $C_s$ does not contain a finite number of elements. The random variable $X$ is said to satisfy condition $(H1)$ if

$$(H1) \qquad C_s \cap R^+ \ne \emptyset \text{ and } C_s \cap R^- \ne \emptyset, \text{ where } R^+ = \{x : x > 0\}, R^- = \{x : x < 0\}$$

and to satisfy condition $(H2)$ if

$$(H2) \qquad EX = 0 \text{ or } EX^2 = \infty.$$

We say $X$ is in the centered Feller class, introduced by Giné and Mason [7], if there exists a sequence of norming constants $\{a_n\}_{n\ge1}$ such that for every subsequence of $\{n\}$ there exists a further subsequence $\{n'\}$ such that $S_{n'}/a_{n'}$ converges in distribution to a nondegenerate law. Giné and Mason [7] also showed that $X$ is in the centered Feller class if and only if $X \in \mathcal{F}_\theta$ for some $0 \le \theta < \infty$, where

$$\mathcal{F}_\theta = \{X : \limsup_{a\to\infty} \frac{a^2\Big\{P(|X| > a) + a^{-1}|EXI(|X| \le a)|\Big\}}{EX^2 I(|X| \le a)} = \theta\}. \tag{1.5}$$

**Theorem 1.1.** *Assume that $X$ satisfies conditions $(H1)$ and $(H2)$. Also assume that $X$ is in the centered Feller class. Then (1.4) holds for any sequence $\{x_n, n \ge 1\}$ with $x_n \to \infty$ and $x_n = o(\sqrt{n})$ as $n \to \infty$. If, in addition, $\mathrm{Card}(C_s) \ge 3$, then*

$$\lim_{n\to\infty} \frac{\ln P\big(\frac{S_n}{V_n} \ge x_n\big)}{x_n^2} = -t_0, \tag{1.6}$$

*where* $t_0 = \lim_{x\to 0^+} t_x$, *and* $(t_x, b_x)$ *satisfy the following saddlepoint equations:*

$$Eb(2X - bX^2)\exp\{tb(2X - bX^2)\} = xE\exp\{tb(2X - bX^2)\}, \tag{1.7}$$

$$E(X - bX^2)\exp\{tb(2X - bX^2)\} = 0. \tag{1.8}$$

As will be proved in Proposition 2.2, $t_0$ is a positive and finite number. Theorem 1.1 together with the subsequence method is ready to give the following law of the iterated logarithm (LIL).

**Theorem 1.2.** *Assume that* (*H*1) *and* (*H*2) *are satisfied and that* $X$ *is in the centered Feller class. Then*

$$\limsup_{n\to\infty} \frac{S_n}{V_n\sqrt{\log\log n}} = \frac{1}{\sqrt{t_0}} \ a.s. \tag{1.9}$$

*Remark* 1.1. Theorem 1.2 is related to the main result in Giné and Mason [7]. They show that the lim sup in (1.9) is finite whenever

$$\frac{S_n}{V_n} = O_p(1). \tag{1.10}$$

Later, Griffin [9] proved that (1.10) holds if and only if

$$\limsup_{a\to\infty} \frac{\big|aEXI(|X| \le a)\big|}{a^2P(|X| > a) + EX^2I(|X| \le a)} < \infty.$$

*Remark* 1.2. Note that $EX^2$ is either infinite, which means that (H2) is satisfied, or finite, which implies that the mean must be finite. So assumption (H2) indeed doesn't require any moment condition. Assumption (H1) simply avoids the case where $X$ is nonnegative or nonpositive.

*Remark* 1.3. If $X$ is symmetric and in the domain of attraction of a stable law of order $\alpha$ $(0 < \alpha < 2)$, then $t_0 = \beta_\alpha$, where $\beta_\alpha$ is the solution of (1.3). In general, when $X$ is in the domain of attraction of a stable law, $t_0$ does equal the constant $\beta(\alpha, c_1, c_2)$ given in Theorem 3.2 of Shao [14].

This paper is organized as follows. In the next two sections we will focus on some basic properties of the solution $(t_x, b_x)$ of saddlepoint equations (1.7) and (1.8), which may be of independent interest. Proofs of the main results will be postponed to Section 4.

## 2. The solution to the saddlepoint equations

This section is devoted to the study of some basic properties of the solution $(t_x, b_x)$ to the saddlepoint equations. Let

$$g(t, b; x) = tx - \ln E\exp\{t(2bX - b^2X^2)\}. \tag{2.1}$$

Our first property is that $\inf_{b\ge 0}\sup_{t\in R} g(t, b; x)$ is attained at some finite unique points $t_x > 0$ and $b_x > 0$ for $x$ in a right neighborhood of zero.

**Proposition 2.1.** *Assume that* (*H*1) *and* (*H*2) *are satisfied and* $\mathrm{Card}(C_s) \ge 3$. *Then there exists an* $\epsilon_0 > 0$, *such that for all* $x \in (0, \epsilon_0)$, $\inf_{b>0}\sup_{t\in R} g(t, b; x)$ *is attained at some finite unique points* $t_x > 0$ *and* $b_x > 0$ *and* $(t_x, b_x)$ *satisfy the saddlepoint equations* (1.7) *and* (1.8).

To prove Proposition 2.1, we need the following lemmas.

**Lemma 2.1.** *Let $x > 0$ and assume* $\mathrm{Card}(C_s) \geq 3$. *Then under $(H2)$, $g(t, b; x)$ is strictly increasing in $t$ for $t \in (-\infty, \epsilon_1)$ for some $\epsilon_1 > 0$.*

*Proof.* The proof follows the same lines of that of Lemma A.2 in Jing, Shao and Zhou [12]. See the Appendix for a detailed proof. □

For $0 < x < 1$, define

$$\begin{aligned} a_{10} &:= a_{10}(b) = \frac{1}{b}(1 - \sqrt{1-x}),\ a_{20} := a_{20}(b) = \frac{1}{b}(1 + \sqrt{1-x}), \\ a_1 &:= a_1(b) = \min(a_{10}, a_{20}), \quad a_2 := a_2(b) = \max(a_{10}, a_{20}) \end{aligned}$$

and

$$U = \{b : (a_1(b), a_2(b)) \cap C_s \neq \emptyset\}.$$

**Lemma 2.2.** *Assume* $\mathrm{Card}(C_s) \geq 3$ *and $(H2)$ is satisfied. Let $x \in (0, 1)$.*

(i) *If $b \in U$, then*

$$\sup_{t \in R} g(t, b; x) = \sup_{t > 0} g(t, b; x), \tag{2.2}$$

*and the supremum is attained at some finite unique point $\tilde{t} := \tilde{t}(b, x) > 0$ which is a finite unique solution to the equation $\partial g(t, b; x)/\partial t = 0$.*

(ii) *If $b \notin U$, then*

$$\sup_{t \in R} g(t, b; x) = -\ln\left(P(X = a_1) + P(X = a_2)\right) \tag{2.3}$$

*and the supremum is attained at $\infty$.*

*Proof.* We first note that $U \neq \emptyset$ by the fact that $\bigcup_b \{b : (a_1(b), a_2(b))\} = R$. An elementary argument also yields that $U$ is an open set. (If $(H1)$ holds, then $U \cap R^+$ is also a nonvoid open set.) Let

$$h(y) = b^2 y^2 - 2by + x = b^2(y - a_1)(y - a_2).$$

Suppose $b \in U$. Since

$$(a_1, a_2) \cap C_s \neq \emptyset,$$

there must exist $W := [a_3, a_4] \subset (a_1, a_2)$ so that

(a) there exists $\delta > 0$ such that $h(y) < -\delta$ for each $y \in W$;
(b) $P(X \in W) > 0$.

Then, as $t \to \infty$,

$$\begin{aligned} g(t, a; b) &= -\ln E e^{-th(X)} \leq -\ln E e^{-th(X)} I(X \in W) \\ &\leq -\ln\{e^{t\delta} P(X \in W)\} \to -\infty, \end{aligned}$$

which combined with Lemma 2.1 shows that $\sup_{t \in R} g(t, b; x)$ is attained at some finite $\tilde{t} = \tilde{t}(b, x) > 0$. Since $g(t, b; x)$ is a differentiable function of $t$ when $t > 0$, we get that $\partial g(\tilde{t}, b; x)/\partial t = 0$. Also note that

$$\frac{\partial^2 g(t, b; x)}{\partial t^2} = -\left(\frac{E Z^2 e^{tZ}}{E e^{tZ}} - \left(\frac{E Z e^{tZ}}{E e^{tZ}}\right)^2\right) < 0, \tag{2.4}$$

where $Z = 2bX - b^2 X^2$. There is at most one solution to the equation $\partial g(t, b; x)/\partial t = 0$. Therefore $\tilde{t}$ is also unique.

Next, suppose $b \notin U$. Since $C_s$ is necessarily closed, $[a_1, a_2] \cap C_s$ contains at most two points $\{a_1, a_2\}$. Clearly, we have

$$h(y) > 0 \text{ for each } y \in C_s \setminus \{a_1, a_2\}.$$

Therefore,

$$\begin{aligned} g(t, b; x) &= -\ln E e^{-th(X)} \\ &= -\ln\Big(E e^{-th(X)} I(X \in C_s \setminus \{a_1, a_2\}) + P(X = a_1) + P(X = a_2)\Big) \\ &\nearrow -\ln\Big(P(X = a_1) + P(X = a_2)\Big) \qquad \text{as } t \to \infty. \quad \square \end{aligned}$$

**Lemma 2.3.** *Assume that* $\mathrm{Card}(C_s) \geq 3$, $R^+ \cap C_s \neq \emptyset$ *and that* $(H2)$ *is satisfied. Then for* $\epsilon_0$ *sufficiently small and* $x \in (0, \epsilon_0)$,

$$\inf_{b>0} \sup_{t \in R} g(t, b; x) = \inf_{b \in U \cap R^+} \sup_{t \in R} g(t, b; x). \tag{2.5}$$

*Proof.* We use the contradiction method. Suppose (2.5) doesn't hold for $\epsilon_0$ sufficiently small. Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $(0, 1)$ such that $x_k \to 0$ as $k \to \infty$ and

$$\inf_{b>0} \sup_{t \in R} g(t, b; x_k) = \inf_{b \in R^+ \setminus U} \sup_{t \in R} g(t, b; x_k) \tag{2.6}$$

for $k \geq 1$. Since $\mathrm{Card}(C_s) \geq 3$, there exist $c_1, c_2, c_3$ and $\delta > 0$ such that

$$p_i := P(X \in [c_i - \delta, c_i + \delta]) > 0$$

and the three intervals $[c_i - \delta, c_i + \delta], i = 1, 2, 3$ are disjoint. Lemma 2.2 implies

$$\begin{aligned} &\sup_{b \in R^+ \setminus U} \inf_{t \in R} E \exp\Big\{t(2bX - b^2X^2 - x_k)\Big\} \\ &= \exp\Big(-\inf_{b \in R^+ \setminus U} \sup_{t \in R} g(t, b; x_k)\Big) \\ &= \sup_{b \in R^+ \setminus U} P\Big(X = \frac{1}{b}(1 - \sqrt{1 - x_k})\Big) + P\Big(X = \frac{1}{b}(1 + \sqrt{1 - x_k})\Big) \\ &\leq 1 - \min(p_1, p_2, p_3) < 1. \qquad (2.7) \end{aligned}$$

To get a contradiction, it suffices to show that

$$\sup_{b \in R^+ \setminus U} \inf_{t>0} E \exp\big\{t(2bX - b^2X^2 - x_k)\big\} \geq 1. \tag{2.8}$$

Let $p = P(X > 0)$. Since $R^+ \cap C_s \neq \emptyset$, we have $p > 0$. Put $b_k = \sqrt{x_k}$. Then

$$P(6x_k/(pb_k) \leq X \leq 1/b_k) \to p \ \text{ and } P(-x_k/b_k \leq X \leq 0) \to 1 - p \tag{2.9}$$

and

$$
\begin{aligned}
&\sup_{b\in R^+\setminus U}\inf_{t>0} E\exp\big\{t(2bX-b^2X^2-x_k)\big\}\\
&= \sup_{b>0}\inf_{t>0} E\exp\big\{t(2bX-b^2X^2-x_k)\big\} \qquad\qquad [\text{by } (2.6)]\\
&\geq \inf_{t>0} E\exp\big\{t(2b_kX-b_k^2X^2-x_k)\big\}\\
&\geq \inf_{t>0}\Big\{E\exp\big\{t(2b_kX-b_k^2X^2-x_k)\big\}I(-x_k/b_k\leq X\leq 0)\\
&\qquad +E\exp\big\{t(2b_kX-b_k^2X^2-x_k)\big\}I(6x_k/(pb_k)\leq X\leq 1/b_k)\Big\}\\
&\geq \inf_{t>0}\Big\{e^{-4tx_k}P(-x_k/b_k\leq X\leq 0)+e^{5tx_k/p}P(6x_k/(pb_k)\leq X\leq 1/b_k)\Big\}\\
&= P(-x_k/b_k\leq X\leq 0)+P(6x_k/(pb_k)\leq X\leq 1/b_k)\\
&\to 1
\end{aligned}
$$

by (2.9), where in the last equality, we used the fact that $e^{-4tx_k}P(-x_k/b_k \leq X \leq 0) + e^{5tx_k/p}P(6x_k/(pb_k) \leq X \leq 1/b_k)$ is nondecreasing in $t > 0$ for $k$ large enough. This proves (2.8) and hence the lemma. □

**Lemma 2.4.** *For $0 < x < 1$, we have*

$$
\lim_{b\to\infty}\sup_{t>0} g(t,b;x)=\infty \ \textit{and}\ \lim_{b\to 0^+}\sup_{t>0} g(t,b;x)=\infty. \tag{2.10}
$$

*Proof.* Let $k$ be a positive number. Then

$$
\begin{aligned}
\sup_{t>0} g(t,b;x) &\geq g\,(k,b;x)\\
&= kx-\ln E\exp\big\{-k(b^2X^2-2bX)\big\}\\
&= kx-\ln E\exp\big\{-k(bX-1)^2+k\big\}. 
\end{aligned}\tag{2.11}
$$

It follows from Lebesgue's dominated convergence theorem that

$$
\lim_{b\to\infty} E\exp\big\{-k(bX-1)^2+k\big\}=P(X=0) \text{ and } \lim_{b\to 0^+} M(a)=1. \tag{2.12}
$$

Combining (2.11)-(2.12) gives

$$
\liminf_{b\to\infty}\sup_{t>0} g(t,b;x)\geq kx \text{ and } \liminf_{b\to 0^+}\sup_{t>0} g(t,b;x)\geq kx.
$$

This proves (2.10) by the arbitrariness of $k$. □

We are now ready to prove Proposition 2.1.

*Proof of Proposition* 2.1. Lemmas 2.1-2.4 imply that $\inf_{b>0}\sup_{t\in R} g(t,b;x)$ is attained at some finite points $b_x \in U$ and $t_x := \tilde{t}(b_x,x) > 0$ for all $x \in (0,\epsilon_0)$ when $\epsilon_0$ is sufficiently small. When $b \in U$, by Lemma 2.2(i), we have

$$
\frac{\partial g(\tilde{t},b;x)}{\partial t}=x-\frac{EZe^{\tilde{t}Z}}{Ee^{\tilde{t}Z}}=0, \qquad \text{where } Z=2bX-b^2X^2. \tag{2.13}
$$

Note that the assumption $\mathrm{Card}(C_s) \geq 3$ implies that $Z$ is nondegenerate; thus (A.2) is true. It then follows from the implicit function theorem that $\tilde{t}(b,x)$ is a differentiable function in some neighborhood $U^*(b)$ of $b$ (also a differentiable function in some neighborhood of $x$). We can also guarantee that $U^*(b) \subset U$.

Hence $\sup_{t\in R} g(t,b;x)$ is also a differentiable function in some neighborhood of $b_x$. So $b_x$ satisfies the equation $dg(\tilde{t},b;x)/db=0$, i.e.,

$$E(X-bX^2)\exp\left\{\tilde{t}\left(2bX-b^2X^2\right)\right\}=0. \tag{2.14}$$

It follows from (2.13) and (2.14) that $b_x$ and $t_x$ are the solutions to the equations

$$EZe^{tZ}=xEe^{tZ}, \qquad EXe^{tZ}=bEX^2e^{tZ}.$$

Now we will show the uniqueness of $(t_x,b_x)$. Suppose $(t'_x,b'_x)$ is another point such that $g(t'_x,b'_x;x)=\inf_{b>0}\sup_{t\in R} g(t,b;x)$. Recall that

$$g(t,b;x)=-\ln E\exp\left\{t(2bX-b^2X^2-x)\right\}.$$

We must have

$$\begin{aligned}
E\exp\left\{t_x(2b_xX+b_x^2X^2-x)\right\} &= \sup_{b>0} E\exp\left\{t_x(2bX+b^2X^2-x)\right\}\\
&\geq E\exp\left\{t_x(2b'_xX-b'^2_xX^2-x)\right\}\\
&\geq \inf_{t>0} E\exp\left\{t(2b'_xX-b'^2_xX^2-x)\right\}\\
&= E\exp\left\{t'_x(2b'_xX-b'^2_xX^2-x)\right\}.
\end{aligned} \tag{2.15}$$

If $t_x\neq t'_x$, then

$$E\exp\left\{t_x(2b'_xX-b'^2_xX^2-x)\right\}>\inf_{t>0} E\exp\left\{t(2b'_xX-b'^2_xX^2-x)\right\} \tag{2.16}$$

by the fact that $E\exp\left\{t(2bX-b^2X^2-x)\right\}$ is a strictly convex function of $t$ for each fixed $b$ and $2bX-b^2X^2-x$ is not identically equal to 0. Combining (2.15) and (2.16) gives

$$E\exp\left\{t_x(2b_xX+b_x^2X^2-x)\right\}>E\exp\left\{t'_x(2b'_xX-b'^2_xX^2-x)\right\},$$

which contradicts our assumption. Hence

$$t_x=t'_x. \tag{2.17}$$

Define

$$f(b,s)=E\exp\left\{s(\frac{2}{b}X-X^2-\frac{x}{b^2})\right\}.$$

So we have

$$f(b_x,s_x)=f(b'_x,s'_x)=\sup_{b>0}\inf_{s>0} f(b,s),$$

where $s_x=t_xb_x^2, s'_x=t_xb'^2_x$. Noting that $f(b,s)$ is a strictly convex function of $s$ for each fixed $b$, similar to the proof of (2.17), we get $s_x=s'_x$. Hence $b_x=b'_x$. This completes the proof of uniqueness. □

From now on, we assume $x\in(0,\epsilon_0)$ so that $\sup_{b>0}\inf_{t>0} E\exp\left\{t(2bX–b^2X^2-x)\right\}$ is attained at the unique point $(t_x,b_x)$. The next proposition shows that, as a function of $x\in(0,\epsilon_0)$, $E\exp\left\{t_x(2b_xX-b_xX^2-x)\right\}$ is differentiable.

**Proposition 2.2.** *Assume that* $(H1)$ *and* $(H2)$ *are satisfied and* $\mathrm{Card}(C_s)\geq 3$. *Then,* $E\exp\left\{t_x(2b_xX-b_x^2X^2-x)\right\}$ *is a differentiable function of* $x$ *when* $x\in(0,\epsilon_0)$.

The next two lemmas are needed to prove the proposition. We also assume that the condition of Proposition 2.2 is satisfied.

**Lemma 2.5.** $\forall x_0\in(0,\epsilon_0)$, *$b_x$ is a continuous function of $x$ in a neighborhood of* $x_0$.

*Proof.* Let $g(b,x) = \inf_{t>0} E\exp\{t(2bX - b^2X^2 - x)\}$. By (2.10), we have

$$\lim_{b\to\infty} g(b,x) = 0 \text{ and } \lim_{b\to 0} g(b,x) = 0.$$

Hence there exists some $\delta_0' > 0$ such that if $x \in [x_0 - \delta_0', x_0 + \delta_0']$, then $b_x \in [L, U]$, where both $L$ and $U$ are positive constants depending only on $x_0$ and $\delta_0'$. We can also choose $L$ and $U$ so that if $b < L$ or $b > U$, then

$$|g(b,x_0) - g(b_{x_0}, x_0)| \geq \frac{1}{2} g(b_{x_0}, x_0). \tag{2.18}$$

Now we claim that $\exists \delta_0 > 0$, $\forall 0 < \delta \leq \delta_0$, $\exists 0 < \epsilon < g(b_{x_0}, x_0)$ such that

$$\{b : |g(b,x_0) - g(b_{x_0}, x_0)| \leq \epsilon\} \subset (b_{x_0} - \delta, b_{x_0} + \delta). \tag{2.19}$$

Otherwise, $\exists$ a sufficiently small $\delta > 0$, $\forall \epsilon > 0$,

$$\{b : |g(b,x_0) - g(b_{x_0}, x_0)| \leq \epsilon\} \cap (b_{x_0} - \delta, b_{x_0} + \delta)^c \neq \emptyset.$$

So we will have a sequence $\{b_m\}_{m=1}^{\infty}$ such that $|b_m - b_{x_0}| \geq \delta$ for each $m \geq 1$ and $\lim_{m\to\infty} g(b_m, x_0) = g(b_{x_0}, x_0)$. (2.18) implies $\exists m_0 > 0$ such that if $m \geq m_0$, $b_m \in [L, U]$. Without loss of generality, we assume $b_m \to b_{x_0}'$ as $m \to \infty$. The continuity of $g(b,x)$ gives

$$g(b_{x_0}', x_0) = g(b_{x_0}, x_0). \tag{2.20}$$

Since $|b_{x_0}' - b_{x_0}| > \delta$, (2.20) contradicts the fact that $b_{x_0}$ is unique. Therefore, we have (2.19). Note that

$$\begin{aligned} g(b,x) &= \inf_{t>0} E\exp\{t(2bX - b^2X^2 - x)\} \\ &=: E\exp\{\tilde{t}(x,b)(2bX - b^2X^2 - x)\}, \end{aligned}$$

where $\tilde{t}(x,b)$ is the unique solution to the following equation:

$$E(2bX - b^2X^2)\exp\{t(2bX - b^2X^2)\} = xE\exp\{t(2bX - b^2X^2)\},$$

for each fixed $b$ and $x$. The implicit function theorem shows that $\tilde{t}(x,b)$ is a differentiable function of $x$ and $b$. Hence we can find some $0 < \delta_1 \leq \min(\delta_0, \delta_0')$ such that if $x \in [x_0 - \delta_1, x_0 + \delta_1]$,

$$|g(b,x) - g(b,x_0)| \leq \frac{\epsilon}{2} \tag{2.21}$$

uniformly in $b \in [L, U]$, and

$$|g(b_x, x) - g(b_{x_0}, x_0)| \leq \frac{\epsilon}{2}. \tag{2.22}$$

Combining (2.21) and (2.22) yields

$$|g(b_x, x_0) - g(b_{x_0}, x_0)| \leq \epsilon. \tag{2.23}$$

It follows from (2.19) and (2.23) that

$$|b_x - b_{x_0}| < \delta,$$

which shows the continuity of $b_x$ in some neighborhood of $x_0$. □

**Lemma 2.6.** *$t_x$ is an increasing and continuous function of $x$ in some neighborhood of $x_0$ for $x_0 \in (0, \epsilon_0)$.*

*Proof.* Recall that $b_x$ and $t_x$ satisfy the following two equations:

$$\frac{EZe^{tZ}}{Ee^{tZ}} = x, \tag{2.24}$$

$$\frac{E(2X - 2bX^2)e^{tZ}}{Ee^{tZ}} = 0, \tag{2.25}$$

where $Z = 2bX - b^2X^2$. Write

$$\begin{aligned} a_1(b,t) &= \frac{EZ^2e^{tZ}}{Ee^{tZ}} - \Big(\frac{EZe^{tZ}}{Ee^{tZ}}\Big)^2, \\ a_2(b,t) &= \frac{EZ(2X-2bX^2)e^{tZ}}{Ee^{tZ}}, \\ a_3(b,t) &= \frac{E(2X-2bX^2)^2e^{tZ}}{Ee^{tZ}}t - 2\frac{EX^2e^{tZ}}{Ee^{tZ}}. \end{aligned}$$

Since $b_x$ is a supremum point of $g(b,x)$, we have

$$a_3(b_x, t_x) \le 0.$$

Also

$$a_1(b_x, t_x) > 0$$

because $2b_xX - b_x^2X^2$ is nondegenerate. If $a_3(b_{x_0}, t_{x_0}) < 0$ or $a_2(b_{x_0}, t_{x_0}) \neq 0$, then

$$\left.\frac{dt_x}{dx}\right|_{x=x_0} = \frac{a_3(b_{x_0}, t_{x_0})}{a_1(b_{x_0}, t_{x_0})a_3(b_{x_0}, t_{x_0}) - a_2^2(b_{x_0}, t_{x_0})} \ge 0$$

by the implicit function theorem. So in this case, $t_x$ is increasing and continuous at $x = x_0$.

If $a_3(b_{x_0}, t_{x_0}) = 0$ and $a_2(b_{x_0}, t_{x_0}) = 0$, then there exists some neighborhood of $x_0$ such that $a_3(b_x, t_x) < 0$ or $a_2(b_x, t_x) \neq 0$ when $x \in U\backslash\{x_0\}$. Otherwise, we find some sequence $\{x_n\}_{n=1}^\infty$ which goes to $x_0$ as $n \to \infty$ such that

$$a_3(b_{x_n}, t_{x_n}) = 0 \ \text{ and } \ a_2(b_{x_n}, t_{x_n}) = 0.$$

Since $a_3(b,t)$ and $a_2(b,t)$ are infinitely differentiable functions of $b$ and $t$,

$$a_3(b,t) \equiv 0 \ \text{ and } \ a_2(b,t) \equiv 0 \tag{2.26}$$

when $(b,t)^T$ belongs to some neighborhood $U_1$ of $(b_{x_0}, t_{x_0})^T$. Noting that $(b_{x_0}, t_{x_0})^T$ satisfies (2.25), we can conclude that

$$E(2X - 2bX^2)e^{tZ} \equiv 0$$

when $(b,t)^T \in U_1$. Hence

$$E\exp\big\{t_{x_0}(2bX - b^2X^2)\big\} \equiv E\exp\big\{t_{x_0}(2b_{x_0}X - b_{x_0}^2X^2)\big\} \tag{2.27}$$

when $b$ is in some neighborhood of $b_{x_0}$. But (2.27) contradicts the uniqueness of $b_{x_0}$. So

$$\frac{dt_x}{dx} \ge 0 \ \text{ if } \ x \in U \setminus \{x_0\},$$

which, combined with Lemma 2.5 and the fact that $\tilde{t}(x,b)$ determined by (2.24) is a continuous function of $x$ and $b$, implies $t_x$ is increasing and continuous at $x = x_0$. $\square$

*Proof of Proposition* 2.2. Let

$$e(x) = \int_{x_0}^{x} (-t_y) E \exp\big\{t_y(2b_y X - b_y^2 X^2 - y)\big\} dy + E \exp\big\{t_{x_0}(2b_{x_0} X - b_{x_0}^2 X^2 - x_0)\big\}.$$

Then

$$\frac{de(x)}{dx} = (-t_x) E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\}.$$

We also know that $E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\}$ is differentiable a.e. in $(0, \epsilon_0)$, and

$$\frac{d}{dx} E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\} = -t_x E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\} \quad \text{a.e.}$$

So

$$e(x) = E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\} \quad \text{a.e.}$$

Since both $e(x)$ and $E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\}$ are continuous functions of $x$ by Lemmas 2.5-2.6,

$$e(x) = E \exp\big\{t_x(2b_x X - b_x^2 X^2 - x)\big\},$$

which gives the final assertion. □

## 3. Convergence of $b_x$ and $t_x$

**Proposition 3.1.** *Assume that (H1) and (H2) are satisfied and that* $\mathrm{Card}(C_s) \geq 3$. *Then*

$$\lim_{x \to 0^+} b_x = 0. \tag{3.1}$$

*Proof.* We will show that for any sequence $\{x_n\}_{n=1}^{\infty}$ decreasing to 0, it is impossible that

$$\lim_{n \to \infty} b_{x_n} = \infty, \tag{3.2}$$

$$\lim_{n \to \infty} b_{x_n} = b_0, \tag{3.3}$$

where $b_0$ is some positive constant. Now we suppose (3.2). Then

$$\begin{aligned} \inf_{t>0} E \exp\big\{t(2b_{x_n} X - b_{x_n}^2 X^2 - x_n)\big\} &= \inf_{t>0} E \exp\big\{-t(b_{x_n} X - 1)^2 + t(1 - x_n)\big\} \\ &\leq E \exp\big\{-c(b_{x_n} X - 1)^2 + t_0(1 - x_n)\big\} \\ &\to P(X = 0), \end{aligned}$$

as $n \to \infty$, where $c$ is any positive constant. But the above result contradicts Lemma 8.1 of Shao [14].

It remains to exclude (3.3). So we assume (3.3). Equation (1.7) gives

$$\frac{E(2X - b_{x_n} X^2) \exp\big\{s_n(2X - b_{x_n} X^2)\big\}}{E \exp\big\{s_n(2X - b_{x_n} X^2)\big\}} = \frac{x_n}{b_n}, \tag{3.4}$$

where $s_n = t_{x_n} b_{x_n}$. Lemma 2.6, combined with (3.3), shows $\lim_{n\to\infty} s_n = s_0$, where $s_0 = t_0 b_0$. Letting $n \to \infty$ in (3.4) yields

$$2EX \exp\big\{s_0(2X - b_0 X^2)\big\} = b_0 E X^2 \exp\big\{s_0(2X - b_0 X^2)\big\},$$

which contradicts another equation (1.8)

$$E(X - b_{x_n} X^2) \exp\big\{s_n(2X - b_{x_n} X^2\big\} = 0.$$ □

**Proposition 3.2.** *Assume that (H1), (H2) and* (1.5) *are satisfied and that* $\mathrm{Card}(C_s) \geq 3$*. Then*

$$\lim_{x \to 0^+} t_x = t_0 > 0. \tag{3.5}$$

*Proof.* Convergence of $t_x$ follows from Lemma 2.6. So we need to show $t_0 > 0$. Otherwise, suppose $t_0 = 0$. Write $Z_x = 2b_xX - b_x^2X^2$ and $D_x = t_x^{-1/3}$. Then

$$\begin{aligned} & \left| Eb_xXe^{t_xZ_x}I(|b_xX| \leq D_x) - Eb_xXI(|b_xX| \leq D_x) \right| \\ \leq \quad & \epsilon_x E(b_xX)^2 e^{t_xZ_x} I(|b_xX| \leq D_x), \end{aligned}$$

where $\epsilon_x \to 0$ as $x \to 0^+$. Since $X \in \mathcal{F}_\theta$ for some $\theta < \infty$, we have for $x$ sufficiently close to 0,

$$\begin{aligned} & Eb_xXe^{t_xZ_x}I(|b_xX| \leq D_x) \\ & \leq \quad Eb_xXI(|b_xX| \leq D_x) + \epsilon_x E(b_xX)^2 e^{t_xZ_x} I(|b_xX| \leq D_x) \\ & \leq \quad \frac{\theta+1}{D_x} E(b_xX)^2 I(|b_xX| \leq D_x) + \epsilon_x E(b_xX)^2 e^{t_xZ_x} I(|b_xX| \leq D_x) \\ & \leq \quad \Big( \frac{\theta + \epsilon^{'}}{D_x} e^{t_x(2D_x + D_x^2)} + \epsilon_x \Big) E(b_xX)^2 e^{t_xZ_x} I(|b_xX| \leq D_x) \\ & < \quad (1/4) E(b_xX)^2 e^{t_xZ_x} I(|b_xX| \leq D_x). \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} & 2Eb_xXe^{t_xZ_x}I(|b_xX| > D_x) \\ < \quad & E(b_xX)^2 e^{t_xZ_x} I(|b_xX| > D_x) \end{aligned} \tag{3.7}$$

for $x$ sufficiently small.

Combining (3.6) and (3.7), we have

$$EZ_xe^{t_xZ_x} < 0$$

for $x$ sufficiently close to 0, but this contradicts the saddlepoint equation

$$E(Z_x - x)e^{t_xZ_x} = 0. \qquad \square$$

We shall also need the following lemma.

**Lemma 3.1.** *Assume that (H1), (H2) and* (1.5) *are satisfied and that* $\mathrm{Card}(C_s) \geq 3$*. Write* $Z_x = 2b_xX - b_x^2X^2$*. Then*

$$Ee^{t_xZ_x} = 1 + o(x), \tag{3.8}$$

$$E(b_xX)^2 e^{t_xZ_x} = x(1 + o(x)), \tag{3.9}$$

*as* $x \to 0^+$.

*Proof.* From the proof of Proposition 2.2, we see that

$$\frac{d}{dx} E \exp\big\{ t_x(Z_x - x) \big\} = -t_x E \exp\big\{ t_x(Z_x - x) \big\}.$$

Hence

$$E\exp\{t_x(Z_x - x)\} = \exp\Big(-\int_0^x t_y dy\Big),$$

which combined with Proposition 3.1 gives

$$\begin{aligned} Ee^{t_x Z_x} &= E\exp\{t_x(Z_x - x)\}e^{xt_x} \\ &= \exp\Big(\int_0^x (t_x - t_y)dy\Big) \\ &= \exp(o(x)) = 1 + o(x). \end{aligned}$$

By Equations (1.7) and (1.8),

$$\begin{aligned} E(b_x X)^2 e^{t_x Z_x} &= xEe^{t_x Z_x} \\ &= x(1 + o(x)). \quad \square \end{aligned}$$

## 4. Proof of main theorems

Write

$$K(x) = x^{-2}EX^2 I(|X| \le x) \quad \text{for} \quad x > 0,$$

$$d(t) = \inf\{s \ge a_0 + 1 : K(s) \le 1/t\}, \quad \text{where} \quad a_0 = \inf\{x \ge 1 : K(x) > 0\}.$$

4.1. **Proof of Theorem 1.1.** If $\mathrm{Card}(C_s) \le 2$, then $X$ takes only two possible values. In this case, (1.4) is a direct consequence of Theorem 3.1 of Shao [14]. If $\mathrm{Card}(C_s) \ge 3$, then (1.4) follows from (1.6). In fact, by Lemma 3.1 (with $t_n = t_{x_n^2/n}$ and $b_n = b_{x_n^2/n}$)

$$\begin{aligned} n\lambda(x_n^2/n) &= t_n x_n^2 - n\ln E\exp\Big(t_n(2b_n X - b_n^2 X^2)\Big) \\ &= t_n x_n^2 - n\ln(1 + o(x_n^2/n)) = t_n x_n^2 + o(x_n^2) = t_0 x_n^2 + o(x_n^2). \end{aligned}$$

Thus we only need to prove (1.6).

**Upper bound proof.** Let $\delta$ be an arbitrarily small but fixed constant, $B$ an arbitrarily large but fixed constant. Both $\delta$ and $B$ will be specified later. Then

$$\begin{aligned} P\left(\frac{S_n}{V_n} \ge x_n\right) &\le P\left(\frac{S_n}{V_n} \ge x_n, 0 \le V_n \le \frac{1}{\delta}a_n\right) + P\left(\frac{S_n}{V_n} \ge x_n, V_n > \frac{1}{\delta}a_n\right) \\ &:= T_1 + T_2, \end{aligned} \tag{4.1}$$

where $a_n$ is some number depending on $n$. We deal with $T_2$ first. Let $z_n$ be some positive number which will be defined later. By the Cauchy-Schwarz inequality, $|\sum_{i=1}^n X_i I(|X_i| \ge z_n)| \le V_n(\sum_{i=1}^n I(|X_i| \le z_n))^{1/2}$ and hence

$$\begin{aligned} T_2 &\le P\Big(\sum_{i=1}^n X_i I(|X_i| \le z_n) \ge \frac{x_n}{2\delta}a_n\Big) + P\Big(\sum_{i=1}^n X_i I(|X_i| \ge z_n) \ge \frac{x_n}{2}V_n\Big) \\ &\le P\Big(\sum_{i=1}^n X_i I(|X_i| \ge z_n) \ge \frac{x_n}{2\delta}a_n\Big) + P\Big(\sum_{i=1}^n I(|X_i| \le z_n) \ge \frac{x_n^2}{4}\Big) \\ &:= T_{21} + T_{22}. \end{aligned}$$

For $T_{21}$, Bernstein's inequality gives

(4.2)
$$T_{21} \le 2\exp\Big(-\frac{n\Big(x_n a_n/(2n\delta) - EXI(|X| \le z_n)\Big)^2}{2\mathrm{Var}XI(|X| \le z_n) + \frac{4}{3}z_n\Big(x_n a_n/(2n\delta) - EXI(|X| \le z_n)\Big)}\Big).$$

Applying the Chernoff large deviation to the binomial random variable $B(n,p)$, it follows that for all $a > 0$,

$$P\big(B(n,p) > an\big) \le \Big(\frac{ep}{a}\Big)^{an}.$$

Therefore

$$T_{22} \le \Big(\frac{12P(|X| \ge z_n)}{x_n^2/n}\Big)^{x_n^2/4}. \tag{4.3}$$

In the following, we will analyze the exponent in (4.2) and the ratio in (4.3), respectively. Let

$$z_n := \inf\Big\{x \ge a_0 + 1 : (2x^2)^{-1}EX^2 I(|X| \le x) \le \frac{\delta^2}{B^2}\frac{x_n^2}{n}\Big\}. \tag{4.4}$$

Then $z_n = d\Big(1/(2\frac{\delta^2}{B^2}\frac{x_n^2}{n})\Big) \to \infty$ as $n \to \infty$.

Define $a_n = x_n z_n/B$. Hence for $n$ sufficiently large,

$$\begin{aligned}
\frac{x_n a_n}{2n\delta} - EXI(|X| \le z_n) \quad &\ge \quad \frac{1}{B\delta}\Big(\frac{1}{2} - \frac{4\theta\delta^3}{B}\Big)\frac{x_n^2}{n}z_n,\\
\frac{n\Big(x_n a_n/(2n\delta) - EXI(|X| \le z_n)\Big)^2}{2\mathrm{Var}XI(|X| \le z_n)} \quad &\ge \quad \frac{B}{4\delta^3}\Big(\frac{1}{2} - \frac{4\theta\delta^3}{B}\Big)^2 x_n^2,\\
\frac{n\Big(x_n a_n/(2n\delta) - EXI(|X| \le z_n)\Big)}{4z_n/3} \quad &\ge \quad \frac{3}{4B\delta}\Big(\frac{1}{2} - \frac{4\theta\delta^3}{B}\Big)x_n^2,\\
\frac{12P(|X| \ge z_n)}{x_n^2/n} \quad &\le \quad 30\theta\frac{\delta^2}{B^2}.
\end{aligned}$$

Hence we can select suitable $B$ and $\delta$ so that $B\delta$ is very small, which ensures that $T_{21}$ and $T_{22}$ are negligible relative to $\exp(-t_n x_n^2)$. So we have

$$\frac{\ln T_2}{t_n x_n^2} \to -\infty. \tag{4.5}$$

Now let's deal with $T_1$. Let $u_n = z_n/(\delta B x_n^2)$. Noting that $a_n/(\delta x_n^2) = u_n x_n$ and that $0 \le c \le v \le h$ implies

$$v \ge (v^2 + ch)/(c + h),$$

we have for $s > 0$,

$$\begin{aligned}
T_1 &= P\Big(\frac{S_n}{V_n} \ge x_n, 0 \le V_n \le \frac{1}{\delta} a_n\Big) \\
&\le \sum_{i=0}^{[x_n^2]} P\Big(\frac{S_n}{x_n} \ge V_n, \frac{i a_n}{\delta x_n^2} \le V_n \le \frac{(i+1)a_n}{\delta x_n^2}\Big) \\
&\le \sum_{i=0}^{[x_n^2]} P\Big((2i+1)u_n S_n \ge V_n^2 + i(i+1)u_n^2 x_n^2\Big) \\
&\le \sum_{i=0}^{[x_n^2]} \Big(E\exp\Big\{s\big((2i+1)u_n X - X^2 - i(i+1)u_n^2 x_n^2/n\big)\Big\}^n\Big) \\
&= \sum_{i=0}^{[x_n^2]} \Big(E\exp\Big\{s\big((2i+1)u_n X - X^2 - (i+1/2)^2 u_n^2 x_n^2/n\big) + (s/4)u_n^2 x_n^2/n\Big\}^n\Big).
\end{aligned}$$

Since

$$\begin{aligned}
&E\exp\Big\{s\big((2i+1)u_n X - X^2 - (i+1/2)^2 u_n^2 x_n^2/n\big)\Big\} \\
&\quad= E\exp\Big\{\frac{s}{(i+1/2)^2 u_n^2}\Big(\frac{2}{(i+1/2)u_n} X - \frac{1}{(i+1/2)^2 u_n^2} X^2 - x_n^2/n\Big)\Big\} \\
&\quad\le \sup_{b\ge 0} E\exp\Big\{\frac{s}{b^2}\big(2bX - b^2X^2 - x_n^2/n\big)\Big\},
\end{aligned}$$

we have

$$T_1 \le (1+[x_n^2])\Big(E\exp\Big\{t_n[2b_nX - b_n^2X^2 - \frac{x_n^2}{n}] + t_n b_n^2 u_n^2 x_n^2/n\Big\}\Big)^n, \tag{4.6}$$

where $t_n$ and $b_n$ are chosen such that

$$E\exp\Big\{t_n(2b_nX - b_n^2X^2 - x_n^2/n)\Big\} := \sup_{b\ge 0}\inf_{t\ge 0} E\exp\Big\{t(2bX - b^2X^2 - x_n^2/n)\Big\},$$

or $t_n := t_{x_n^2/n}, b_n := b_{x_n^2/n}$. Recall that $t_n$ and $b_n$ satisfy

$$\frac{E(b_nX)^2 e^{t_n(2b_nX - b_n^2X^2)}}{Ee^{t_n(2b_nX - b_n^2X^2)}} = \frac{x_n^2}{n}.$$

Hence for $\epsilon$ close enough to 0, we have

$$\begin{aligned}
& E(b_nX)^2 I(|b_nX| \le \epsilon) \\
\le\ & E(b_nX)^2 e^{t_n(2b_nX - b_n^2X^2)} I(|b_nX| \le \epsilon) e^{t_n(2\epsilon+\epsilon^2)} \\
\le\ & 2x_n^2/n,
\end{aligned}$$

where we used Lemma 3.1 in the last inequality. Hence

$$d\Big(\frac{\epsilon^2}{2}\frac{n}{x_n^2}\Big) \le \frac{\epsilon^2}{b_n}. \tag{4.7}$$

Combining Lemma 3.1 of Griffin and Kuelbs [10] and (4.7), we have for $n$ sufficiently large,

$$\begin{aligned} d\left(\frac{B^2 n}{2\delta^2 x_n^2}\right)\Big/\frac{\epsilon^2}{b_n} &\le d\left(\frac{B^2 n}{2\delta^2 x_n^2}\right)\Big/ d\left(\frac{\epsilon^2}{2}\frac{n}{x_n^2}\right) \\ &\le \Big(\frac{B^2(1+\theta)}{\epsilon^2\delta^2}\Big)^{\frac{1+\theta}{2}}, \end{aligned}$$

i.e.

$$b_n z_n \le \epsilon^2 \Big(\frac{B^2(1+\theta)}{\epsilon\delta^2}\Big)^{\frac{1+\theta}{2}}. \tag{4.8}$$

Combining (4.1), (4.5), (4.6), (4.8), Proposition 2.2 and Proposition 3.2 gives that, $\forall \epsilon_0 > 0$,

$$\frac{\ln T_1}{x_n^2} \le -t_0 + \epsilon_0$$

for $n$ sufficiently large. Therefore

$$\limsup_{n\to\infty} \frac{\ln P(\frac{S_n}{V_n} \ge x_n)}{x_n^2} \le -t_0. \quad \square$$

**Lower bound proof.** By the basic equality $|xy| = \inf_{b\ge 0}(2b)^{-1}(b^2x^2 + y^2)$, we have

$$\begin{aligned} P\left(\frac{S_n}{V_n} \ge x_n\right) &= P\Big(S_n \ge \inf_{b\ge 0}\frac{1}{2b}(b^2V_n^2 + x_n^2)\Big) \\ &= P\Big(\sup_{b\ge 0}\sum_{i=1}^n (2bX_i - b^2X_i^2 - \frac{x_n^2}{n}) \ge 0\Big) \\ &\ge P\Big(\sum_{i=1}^n (2b_nX_i - b_n^2X_i^2 - \frac{x_n^2}{n}) \ge 0\Big). \end{aligned}$$

Tilting methods give

$$\begin{aligned} &P\Big(\sum_{i=1}^n (2b_nX_i - b_n^2X_i^2 - \frac{x_n^2}{n}) \ge 0\Big) \\ =& \Big(E\exp\big\{t_n(Z_n - \frac{x_n^2}{n})\big\}\Big)^n \int_0^\infty e^{-t_n\sigma(t_n)y\sqrt{n}}dF_n(y), \end{aligned}$$

where $Z_n = 2b_nX - b_n^2X^2$,

$$\sigma^2(t_n) = \frac{EZ_n^2e^{t_nZ_n}}{Ee^{t_nZ_n}} - \Big(\frac{EZ_ne^{t_nZ_n}}{Ee^{t_nZ_n}}\Big)^2,$$

$F_n(y)$ is the distribution function of the random variable $\sum_{i=1}^n(\eta_i - E\eta_i)/\sqrt{n\sigma^2(t_n)}$, and $\eta_1, \eta_2, \ldots, \eta_n$ are i.i.d. r.v.'s with distribution function

$$V(x) = \frac{1}{Ee^{t_nZ_n}}\int_{-\infty}^x e^{t_ny}dP(Z_n \le y).$$

By Lemma 4.2 in Section 4.3, we have

$$\begin{aligned}
& \int_0^\infty \exp\big\{-t_n\sigma(t_n)y\sqrt{n}\big\}dF_n(y) \\
\geq\ & \int_0^2 \exp\big\{-t_n\sigma(t_n)y\sqrt{n}\big\}dF_n(y) \\
\geq\ & \exp\big\{-2t_n\sigma(t_n)\sqrt{n}\big\}P\Big(0 \leq \sum_{i=1}^n(\eta_i - E\eta_i)/\sqrt{n\sigma^2(t_n)} \leq 2\Big) \\
\geq\ & 2^{-1}\Big(\Phi(2)-\Phi(0)\Big)\exp\big\{-2t_n\sigma(t_n)\sqrt{n}\big\}
\end{aligned}$$

for $n$ sufficiently large. Hence

$$\begin{aligned}
& P\left(\frac{S_n}{V_n} \geq x_n\right) \\
\geq\ & 2^{-1}\Big(\Phi(2)-\Phi(0)\Big)\exp\big\{-2t_n\sigma(t_n)\sqrt{n}\big\}\Big(E\exp\big\{t_n(Z_n - \frac{x_n^2}{n})\big\}\Big)^n. \qquad (4.9)
\end{aligned}$$

Now we will prove

$$t_n\sigma(t_n)\sqrt{n} = o(x_n^2) \qquad (4.10)$$

as $n \to \infty$.

Let $\epsilon$ be some small positive constant. Then since $X \in \mathcal{F}_\theta$,

$$\begin{aligned}
\epsilon^2P(|b_nX| \geq \epsilon) &\leq 2\theta E(b_nX)^2I(|b_nX| \leq \epsilon) \\
&\leq 2\theta E(b_nX)^2e^{t_n(Z_n+4\epsilon)}I(|b_nX| \leq \epsilon) \qquad (4.11)\\
&\leq c\frac{x_n^2}{n}, \qquad (4.12)
\end{aligned}$$

where (4.12) follows from Lemma 3.1 and $c$ is some positive constant. Therefore (4.11) and (4.12) imply

$$\begin{aligned}
& t_n^2EZ_n^2e^{t_nZ_n} \\
=\ & t_n^2EZ_n^2e^{t_nZ_n}I(|b_nX| \leq \epsilon) + t_n^2EZ_n^2e^{t_nZ_n}I(|b_nX| > \epsilon) \\
\leq\ & (2+\epsilon)^2t_n^2E(b_nX)^2e^{t_nZ_n}I(|b_nX| \leq \epsilon) + eP(|b_nX| > \epsilon) \\
\leq\ & c_1\frac{x_n^2}{n}
\end{aligned}$$

for some constant $c_1 > 0$. So

$$\begin{aligned}
t_n\sigma(t_n)\sqrt{n} &\leq \Big(\frac{nt_n^2EZ_n^2e^{t_nZ_n}}{Ee^{t_nZ_n}}\Big)^{\frac{1}{2}} \\
&\leq \sqrt{c_1}x_n/\sqrt{Ee^{t_nZ_n}} \\
&= o(x_n^2),
\end{aligned}$$

where the last equality is from Lemma 3.1. So we completed the proof of (4.10). Noting that Lemma 3.1 gives, $\forall\epsilon_1 > 0$,

$$E\exp\big\{t_n(Z_n - \frac{x_n^2}{n})\big\} \geq \exp\big\{(-t_0-\epsilon_1)\frac{x_n^2}{n}\big\}$$

for $n$ sufficiently large, we have

$$\liminf_{n\to\infty} \frac{\ln P(S_n/V_n \ge x_n)}{x_n^2} \ge -t_0 - \epsilon_1$$

by (4.9) and (4.10). The arbitrariness of $\epsilon_1$ completes the lower bound proof. $\square$

4.2. **Proof of Theorem 1.2.** When $\mathrm{Card}(C_s) \le 2$, the result is due to Theorem 5.1 of Shao [14]. When $\mathrm{Card}(C_s) \ge 3$, following the proof of Theorem 1.1 (similar to Proposition 5.1 in Shao [14], see Appendix), one can prove that under the condition of Theorem 1.1, for any $0 < \varepsilon < 1/2$, there exists $\tau > 1$ such that

$$P\Big(\max_{n\le k\le \tau n} \frac{S_k}{V_k} \ge x_n\Big) \le \exp\Big(-(1-\varepsilon)t_0 x_n^2\Big). \tag{4.13}$$

The rest of the proof is similar to that of Theorem 5.1 in Shao [14] by the subsequence method. The details are omitted here.

4.3. **Auxiliary lemmas.**

**Lemma 4.1.** *Write $Z_n = 2b_n X - b_n^2 X^2$. Then*

$$\liminf_{n\to\infty} \frac{EZ_n^2 e^{t_n Z_n}}{x_n^2/n} > c_0 > 0, \tag{4.14}$$

*where $c_0$ is some constant.*

*Proof.* First, we show that for arbitrary $\epsilon_2 > 0$,

$$\liminf_{n\to\infty} \frac{Eb_n^2 X^2 I(|b_n X| \le \epsilon_2)}{x_n^2/n} = c_2 > 0. \tag{4.15}$$

Otherwise, there exists $\epsilon_3 > 0$ such that

$$\liminf_{n\to\infty} \frac{Eb_n^2 X^2 I(|b_n X| \le \epsilon_3)}{x_n^2/n} = 0.$$

Without loss of generality, we can assume

$$\lim_{n\to\infty} \frac{Eb_n^2 X^2 I(|b_n X| \le \epsilon_3)}{x_n^2/n} = 0.$$

Since $X \in \mathcal{F}_\theta$, we have

$$\lim_{n\to\infty} \frac{P(|b_n X| > \epsilon_3)}{x_n^2/n} = 0.$$

Hence

$$\begin{aligned} & \lim_{n\to\infty} \frac{E(b_x X)^2 e^{t_x Z_x} I(|b_n X| > \epsilon_3)}{x_n^2/n} \\ \le\ & \lim_{n\to\infty} \frac{c_{2n} P(|b_n X| > \epsilon_3)}{x_n^2/n} \\ =\ & 0, \end{aligned} \tag{4.16}$$

where $c_{2n} = \sup_{|y|>\epsilon_3} y^2 \exp\{t_n(2y - y^2)\}$, whose limit is positive by Proposition 3.2 as $n \to \infty$. Combining (4.16) with (3.9), we get for $n$ sufficiently large,

$$\begin{aligned} Eb_n^2 X^2 I(|b_n X| \le \epsilon_3) &\ge e^{-t_n(2\epsilon_3+\epsilon_3^2)} E(b_x X)^2 e^{t_x Z_x} I(|b_n X| \le \epsilon_3) \\ &\ge \frac{1}{2}\frac{x_n^2}{n}, \end{aligned}$$

which contradicts our assumption. Therefore (4.15) holds. Now

$$\begin{aligned}
& \liminf_{n\to\infty} \frac{EZ_n^2 e^{t_n Z_n}}{x_n^2/n} \\
\geq \ & \liminf_{n\to\infty} \frac{EZ_n^2 e^{t_n Z_n}}{x_n^2/n} I(|b_n X| \leq \epsilon_2) \\
\geq \ & \liminf_{n\to\infty} (2-\epsilon_2)^2 \exp\big\{-t_n(2\epsilon_3+\epsilon_3^2)\big\} \frac{Eb_n^2 X^2 I(|b_n X| \leq \epsilon_2)}{x_n^2/n} \\
= \ & (2-\epsilon_2)^2 \exp\big\{-t_0(2\epsilon_2+\epsilon_2^2)\big\} c_2 \\
> \ & 0,
\end{aligned}$$

which is just (4.14). □

**Lemma 4.2.**

$$\sum_{i=1}^n (\eta_i - E\eta_i)/\sqrt{n\sigma^2(t_n)} \to N(0,1)$$

*in distribution as $n \to \infty$.*

*Proof.* It suffices to show the Lindeberg condition

$$\frac{EZ_n^2 e^{t_n Z_n} I\Big(|Z_n| \geq \delta\sqrt{n}\sigma(t_n)\Big)}{\sigma^2(t_n)} \to 0$$

holds as $n \to \infty$ for arbitrary $\delta > 0$.

Lemma 4.1 yields $\sqrt{n}\sigma(t_n) \to \infty$ as $n \to \infty$. So

$$\begin{aligned}
& \sup_y (2b_n y - b_n^2 y^2)^2 \exp\big\{t_n(2b_n y - b_n^2 y^2)\big\} I\Big(|2b_n y - b_n^2 y^2| \geq \delta\sqrt{n}\sigma(t_n)\Big) \\
= \ & \delta^2 n\sigma^2(t_n) \exp\big\{-t_n\delta\sqrt{n}\sigma(t_n)\big\}.
\end{aligned}$$

Write $\delta_n = t_n\delta\sqrt{n}\sigma(t_n)$. Hence for $n$ sufficiently large,

$$\begin{aligned}
& \frac{EZ_n^2 e^{t_n Z_n} I\Big(|Z_n| \geq \delta\sqrt{n}\sigma(t_n)\Big)}{\sigma^2(t_n)} \\
\leq \ & \delta^2 n e^{-\delta_n} P\Big(|Z_n| \geq \delta\sqrt{n}\sigma(t_n)\Big) \\
\leq \ & \delta^2 n e^{-\delta_n} P(|b_n X| \geq 1) \\
\leq \ & \delta^2 n e^{-\delta_n} c_3 \frac{x_n^2}{n} \qquad (4.17) \\
\leq \ & c_3 \delta^2 x_n^2 \exp\big\{-t_n\delta\sqrt{c_0}\sqrt{n}x_n/2\big\} \\
\to \ & 0,
\end{aligned}$$

where (4.17) is due to the Feller class assumption and the saddlepoint equations, and the last inequality follows from Lemma 4.1. The proof of asymptotic normality is complete. □

## 5. Appendix

*Proof of Lemma* 2.1. It suffices to show that $g(t,b;x)$ is strictly increasing in $t$, either

$$(I)\ \text{ for } t\in(-\infty,0]; \qquad \text{or} \qquad (II)\ \text{ for } t\in(0,\epsilon_1] \text{ for some } \epsilon_1>0.$$

We prove $(I)$ first. Let $Z=2bX-b^2X^2$. For arbitrary $t$ and $t_1$ such that $t_1<t\le 0$, we need to show that $g(t_1,b;x)<g(t,b;x)$. If $Ee^{t_1Z}=\infty$, then $g(t_1,b;x)=t_1x-\ln Ee^{t_1Z}=-\infty$, in which case $(I)$ follows straightaway. Now assume that $Ee^{t_1Z}<\infty$ below, which implies that moments of $X$ of all orders exist. Thus, $g(t,b;x)$ is differentiable in $t$ for $t\in(t_1,\infty)$. Taking derivatives gives

$$\frac{\partial g(t,b;x)}{\partial t}=x-\frac{EZe^{tZ}}{Ee^{tZ}}. \tag{A.1}$$

Observe that

$$\left.\frac{\partial g(t,b;x)}{\partial t}\right|_{t=0}=x+b^2EX^2>0$$

and

$$\frac{\partial^2 g(t,b;x)}{\partial t^2}=-\left(\frac{EZ^2e^{tZ}}{Ee^{tZ}}-\left(\frac{EZe^{tZ}}{Ee^{tZ}}\right)^2\right)<0, \tag{A.2}$$

since $Z=2bX+b^2X^2$ is nondegenerate by the assumption that $\mathrm{Card}(C_s)\ge 3$ and $EX=0$ or $EX^2=\infty$. Thus, $\frac{\partial g(t,b;x)}{\partial t}>0$ when $t\in(t_1,0]$. So $g(t,b;x)$ is strictly increasing in $(t_1,0]$. Since $t_1$ is arbitrary, we have hence proved $(I)$.

We prove $(II)$ next. If there exists some $t_2<0$ such that $Ee^{t_2Z}<\infty$, then $(II)$ follows from the fact that $\frac{\partial g(0,b;x)}{\partial t}=x+b^2EX^2>0$. It remains to prove $(II)$ under the condition that

$$Ee^{t_3Z}=\infty, \qquad \text{for all } t_3<0.$$

To show this, we choose an arbitrary $t>0$. Then from (A.1), we have

$$\frac{\partial g(t,b;x)}{\partial t} \;=\; x-\frac{\int_{-\infty}^{\infty}(2by-b^2y^2)e^{-t(by-1)^2}dF(y)}{\int_{-\infty}^{\infty}e^{-t(by-1)^2}dF(y)}. \tag{A.3}$$

The monotone convergence theorem implies

$$\lim_{t\to0^+}\int_{-\infty}^{\infty}e^{-t(by-1)^2}dF(y) \;=\; 1, \tag{A.4}$$

$$\lim_{t\to0^+}\int_{-\infty}^{\infty}y^2e^{-t(by-1)^2}dF(y) \;=\; EX^2 \qquad (\text{it could be } \infty), \tag{A.5}$$

where $t\to0^+$ means that $t\to0$ from the right side of 0.

If $EX^2=\infty$, then

$$\begin{aligned}
&\lim_{t\to0^+}\int_{-\infty}^{\infty}\left(2by-b^2y^2\right)e^{-t(by-1)^2}dF(y)\\
\le\ &\lim_{t\to0^+}\int_{-\infty}^{\infty}\left(4+\frac{b^2y^2}{4}-b^2y^2\right)e^{-t(by-1)^2}dF(y)\\
=\ &4-\frac{3}{4}b^2EX^2\\
=\ &-\infty. \qquad \text{(A.6)}
\end{aligned}$$

If $EX^2 < \infty$, then noting $|ye^{-t(by-1)^2}| \le |y|$ for $t > 0$, we can use Lebesgue's dominated convergence theorem to get

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} (2by)\, e^{-t(by-1)^2} dF(y) = 2bEX = 0. \tag{A.7}$$

Combining (A.3)-(A.7) gives

$$\lim_{t\to 0^+} \frac{\partial g(t,b;x)}{\partial t} > 0.$$

Noting that $g(t,b;x)$ is right continuous at $t=0$, we conclude $(II)$. □

*Proof of Equation* (4.13). Let $\eta = \varepsilon/20$. Clearly,

$$\begin{aligned} & P\left(\max_{n\le k\le \tau n} \frac{S_k}{V_k} \ge x_n\right) \\ \le\ & P\left(\frac{S_n}{V_n} \ge (1-5\eta)x_n\right) + P\left(\max_{n<k\le \tau n} \frac{S_k - S_n}{V_k} \ge 5\eta x_n\right). \end{aligned} \tag{A.8}$$

By Theorem 1.1, we have

$$P\left(\frac{S_n}{V_n} \ge (1-5\eta)x_n\right) \le \exp\left(-(1-\varepsilon/2)t_0 x_n^2\right), \tag{A.9}$$

provided that $n$ is sufficiently large.

Below we estimate the second term on the right hand side of (A.8). Similar to (4.3) and (4.4) in Shao [14], let

$$\begin{aligned} l(x) &= EX^2 I\{|X| \le x\}, \quad b = \inf\{x \ge 1 : l(x) > 0\}, \\ z_n &= \inf\{s : s \ge b+1, \frac{l(s)}{s^2} \le \frac{16(t_0 + t_0^2)x_n^2}{\eta^2 n}\}. \end{aligned} \tag{A.10}$$

By an elementary argument and the assumption that $x_n^2 = o(n)$, it is plain to see that

$$z_n \to \infty \ \text{ and } \ n\, l(z_n) = 16(t_0 + t_0^2)x_n^2\, z_n^2/\eta^2 \ \text{ for every } n \text{ sufficiently large.} \tag{A.11}$$

Since $X$ is in the centered Feller class, there exists $C > 0$ such that for all $x > b+1$,

$$P(|X| > x) \le Cx^{-2} l(x) \tag{A.12}$$

and

$$|EXI(|X| \le x)| \le Cx^{-1} l(x). \tag{A.13}$$

Write

$$
\begin{aligned}
&P\left(\max_{n<k\le\tau n}\frac{S_k-S_n}{V_k}\ge 5\eta x_n\right)\\
&\le\quad P\left(\max_{n<k\le\tau n}\frac{\sum_{i=n+1}^k X_iI\{|X_i|\le z_n\}}{V_k}\ge 4\eta x_n\right)\\
&\qquad+P\left(\max_{n<k\le\tau n}\frac{\sum_{i=n+1}^k |X_i|I\{|X_i|\ge z_n\}}{V_k}\ge \eta x_n\right)\\
&\le\quad P\left(\max_{n<k\le\tau n}\sum_{i=n+1}^k X_iI\{|X_i|\le z_n\}\ge 2\eta x_n\sqrt{nl(z_n)}\right)\\
&\qquad+P(V_n\le\sqrt{nl(z_n)}/2)+P\left(\sum_{i=n+1}^{[\tau n]} I\{|X_i|\ge z_n\}\ge(\eta x_n)^2\right).
\end{aligned}
$$

Note that for independent nonnegative random variables $\{\xi_i\}$ with finite second moments, we have (see, e.g., Lemma 2.1 of Einmahl and Mason [5])

$$
P(\sum_{i=1}^n \xi_i\le x)\le\exp\left(-\frac{(\mu-x)^2}{2\sigma^2}\right) \tag{A.14}
$$

for $0<x<\mu$, where $\mu=\sum_{i=1}^n E\xi_i$ and $\sigma^2=\sum_{i=1}^n E\xi_i^2$. Using (A.14), we have

$$
\begin{aligned}
&P(V_n\le\sqrt{nl(z_n)}/2)\\
&\le\quad P(\sum_{i=1}^n X_i^2I(|X_i|\le z_n)\le nl(z_n)/4)\\
&\le\quad \exp\left(-\frac{(0.75nl(z_n))^2}{2nEX^4I(|X|\le z_n)}\right)\\
&\le\quad \exp\left(-\frac{0.75^2nl(z_n)}{2z_n^2}\right)\\
&\le\quad \exp\left(-4.5t_0x_n^2\right)\qquad\text{by (A.11).}
\end{aligned}
$$

It is known that for a binomial random variable $B(n,p)$ with parameters $n$ and $p$ (see, e.g., [Dudley [4], p.16])

$$
P(B(n,p)>a\,n)\le\left(\frac{e\,p}{a}\right)^{a\,n}\quad\text{for } a>0.
$$

Hence we have

$$
\begin{aligned}
&P\left(\sum_{i=n+1}^{[\tau n]} I\{|X_i| \geq z_n\} \geq (\eta x_n)^2\right) \\
&\leq \left(\frac{3(\tau-1)nP(|X| \geq z_n)}{(\eta x_n)^2}\right)^{(\eta x_n)^2} \\
&\leq \left(\frac{3(\tau-1)nCz_n^{-2}l(z_n)}{(\eta x_n)^2}\right)^{(\eta x_n)^2} \qquad \text{by (A.12)} \\
&\leq \left(3(\tau-1)16(t_0+t_0^2)/\eta^2\right)^{(\eta x_n)^2} \qquad \text{by (A.11)} \\
&\leq \exp(-2t_0x_n^2),
\end{aligned}
$$

as long as $\tau$ is very close to one.

By (A.13) and (A.11), if $\tau - 1 > 0$ is chosen to be sufficiently small,

$$\sum_{i=n+1}^{[\tau n]} |EX_iI\{|X_i| \leq z_n\}| \leq C(\tau-1)nl(z_n)/z_n \leq \frac{1}{2}\eta x_n\sqrt{nl(z_n)}$$

and

$$\sum_{i=n+1}^{[\tau n]} \mathrm{Var}X_iI\{|X_i| \leq z_n\} \leq (\tau-1)nl(z_n) \leq \frac{1}{8}\eta n x_n^2 l(z_n).$$

Therefore, by the Ottaviani maximum inequality and the Bernstein exponential inequality,

$$
\begin{aligned}
&P\left(\max_{n<k\leq\tau n}\sum_{i=n+1}^{k} X_iI\{|X_i| \leq z_n\} \geq 2\eta x_n\sqrt{nl(z_n)}\right) \\
&\leq 2P\left(\sum_{i=n+1}^{[\tau n]} X_iI\{|X_i| \leq z_n\} - EX_iI\{|X_i| \leq z_n\} \geq \eta x_n\sqrt{nl(z_n)}\right) \\
&\leq 2\exp\left(-\frac{\eta^2x_n^2nl(z_n)}{2((\tau-1)nl(z_n)+\eta x_n\sqrt{nl(z_n)}z_n)}\right) \\
&\leq 2\exp\left(-\frac{\eta^2x_n^2nl(z_n)}{4(\tau-1)nl(z_n)}\right) + 2\exp\left(-\frac{\eta x_n\sqrt{nl(z_n)}}{4z_n}\right) \\
&\leq 2\exp(-\eta^2x_n^2/(4(\tau-1))) + \exp(-4t_0x_n^2) \qquad \text{by (A.11)} \\
&\leq 4\exp(-4t_0x_n^2)
\end{aligned}
$$

provided $(\tau - 1) > 0$ is sufficiently small. Putting together the above inequalities yields

$$P\left(\max_{n<k\leq\tau n}\frac{S_k-S_n}{V_k} \geq 5\eta x_n\right) \leq 6\exp\left(-2t_0x_n^2\right). \tag{A.15}$$

This proves (4.13), by (A.9), (A.8) and (A.15). □

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