

TOWARDS A UNIVERSAL SELF-NORMALIZED MODERATE DEVIATION

BING-YI JING, QI-MAN SHAO, AND WANG ZHOU

ABSTRACT. This paper is an attempt to establish a universal moderate deviation for self-normalized sums of independent and identically distributed random variables without any moment condition. The exponent term in the moderate deviation is specified when the distribution is in the centered Feller class. An application to the law of the iterated logarithm is given.

1. INTRODUCTION AND MAIN RESULTS

Let X, X_1, X_2, \dots be independent and identically distributed (i.i.d.) nondegenerate random variables. Put

$$S_n = \sum_{i=1}^n X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2$$

and define the self-normalized sum by S_n/V_n . In contrast to the classical limit theorems, the self-normalized limit theorems enjoy much better properties with few or no moment conditions. Griffin and Kuelbs [10] obtained a self-normalized law of the iterated logarithm for all distributions in the domain of attraction of a normal or stable law. Shao [14] showed that no moment conditions are needed for a self-normalized large deviation result: *If $EX = 0$ or $EX^2 = \infty$, then for $0 < x < 1$,*

$$(1.1) \quad \lim_{n \rightarrow \infty} P(S_n/V_n \geq x\sqrt{n})^{1/n} = \exp\{-\lambda(x^2)\},$$

where $\lambda(x) = \inf_{b \geq 0} \sup_{t \geq 0} (tx - \ln E \exp\{t(2bX - b^2X^2)\})$. It was also shown in Shao [14] that the tail probability of S_n/V_n is Gaussian like when X is in the domain of attraction of the normal law and sub-Gaussian like when X is in the domain of attraction of a stable law. In particular, when X is symmetric and in the domain of attraction of a stable law of order α ($0 < \alpha < 2$),

$$(1.2) \quad \ln P(S_n/V_n \geq x_n) \sim -x_n^2 \beta_\alpha$$

Received by the editors February 7, 2006 and, in revised form, August 14, 2006.

2000 *Mathematics Subject Classification*. Primary 60F10, 60F15; Secondary 60G50.

Key words and phrases. Moderate deviation, large deviation, self-normalized sums, the law of the iterated logarithm.

The first author was supported in part by Hong Kong RGC CERG No. HKUST6117/02P and DAG05/06.SC01.

The second author was partially supported by the National Science Foundation under Grant No. DMS-0103487 and HKUST DAG 05/06 Sc27 and RGC CERG No. 602206.

The third author was partially supported by the grants R-155-000-035-112 and R-155-050-055-133/101 at the National University of Singapore.

for any $x_n \rightarrow \infty$ satisfying $x_n = o(\sqrt{n})$, where β_α is the solution of

$$(1.3) \quad \int_0^\infty \frac{2 - \exp\{2x - x^2/\beta\} - \exp\{-2x - x^2/\beta\}}{x^{\alpha+1}} dx = 0.$$

Here and in the sequel, $a_n \sim b_n$ means $\lim_{n \rightarrow \infty} a_n/b_n = 1$. Jing, Shao and Zhou [12] refined the self-normalized deviation and proved a self-normalized saddlepoint approximation without any moment assumption. Jing, Shao and Wang [11] established a Cramér type large deviation for self-normalized sums of independent random variables under a Lindeberg type condition. For other self-normalized results we refer to Logan, Mallows, Rice and Shepp [13] for the limiting distribution of S_n/V_n when X is in the domain of attraction of a stable law, which has been proved also necessary by Götze and Chistyakov [8], Bentkus and Götze [1] for Berry-Esseen inequalities, Giné, Götze and Mason [6] for the necessary and sufficient condition for the asymptotic normality, Csörgő, Szyszkowicz and Wang [2, 3] for the Darling–Erdős theorem and Donsker theorem, and Shao [15, 16] for surveys on recent developments in this area.

The main aim of this paper is to establish a universal self-normalized moderate deviation without any moment condition. Results in (1.1) and (1.2) suggest that a very likely result would be

$$(1.4) \quad \ln P(S_n/V_n \geq x_n) \sim -n\lambda(x_n^2/n)$$

for $x_n \rightarrow \infty$ with $x_n = o(\sqrt{n})$. We shall prove that this is the case when X is in the centered Feller class.

Before we state our main theorem, let us introduce some notation. Let C_s denote the support of X , that is,

$$C_s = \{x : P(X \in (x - \epsilon, x + \epsilon)) > 0, \text{ for any } \epsilon > 0\}.$$

We denote the number of elements in C_s by $\text{Card}(C_s)$ and define $\text{Card}(C_s) = \infty$ if C_s does not contain a finite number of elements. The random variable X is said to satisfy condition (H1) if

$$(H1) \quad C_s \cap R^+ \neq \emptyset \text{ and } C_s \cap R^- \neq \emptyset, \text{ where } R^+ = \{x : x > 0\}, R^- = \{x : x < 0\}$$

and to satisfy condition (H2) if

$$(H2) \quad EX = 0 \text{ or } EX^2 = \infty.$$

We say X is in the centered Feller class, introduced by Giné and Mason [7], if there exists a sequence of norming constants $\{a_n\}_{n \geq 1}$ such that for every subsequence of $\{n\}$ there exists a further subsequence $\{n'\}$ such that $S_{n'}/a_{n'}$ converges in distribution to a nondegenerate law. Giné and Mason [7] also showed that X is in the centered Feller class if and only if $X \in \mathcal{F}_\theta$ for some $0 \leq \theta < \infty$, where

$$(1.5) \quad \mathcal{F}_\theta = \left\{ X : \limsup_{a \rightarrow \infty} \frac{a^2 \left\{ P(|X| > a) + a^{-1} |EXI(|X| \leq a)| \right\}}{EX^2 I(|X| \leq a)} = \theta \right\}.$$

Theorem 1.1. *Assume that X satisfies conditions (H1) and (H2). Also assume that X is in the centered Feller class. Then (1.4) holds for any sequence $\{x_n, n \geq 1\}$ with $x_n \rightarrow \infty$ and $x_n = o(\sqrt{n})$ as $n \rightarrow \infty$. If, in addition, $\text{Card}(C_s) \geq 3$, then*

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{\ln P\left(\frac{S_n}{V_n} \geq x_n\right)}{x_n^2} = -t_0,$$

where $t_0 = \lim_{x \rightarrow 0^+} t_x$, and (t_x, b_x) satisfy the following saddlepoint equations:

$$(1.7) \quad Eb(2X - bX^2) \exp \{tb(2X - bX^2)\} = xE \exp \{tb(2X - bX^2)\},$$

$$(1.8) \quad E(X - bX^2) \exp \{tb(2X - bX^2)\} = 0.$$

As will be proved in Proposition 2.2, t_0 is a positive and finite number. Theorem 1.1 together with the subsequence method is ready to give the following law of the iterated logarithm (LIL).

Theorem 1.2. *Assume that (H1) and (H2) are satisfied and that X is in the centered Feller class. Then*

$$(1.9) \quad \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{\log \log n}} = \frac{1}{\sqrt{t_0}} \text{ a.s.}$$

Remark 1.1. Theorem 1.2 is related to the main result in Giné and Mason [7]. They show that the lim sup in (1.9) is finite whenever

$$(1.10) \quad \frac{S_n}{V_n} = O_p(1).$$

Later, Griffin [9] proved that (1.10) holds if and only if

$$\limsup_{a \rightarrow \infty} \frac{|aEXI(|X| \leq a)|}{a^2P(|X| > a) + EX^2I(|X| \leq a)} < \infty.$$

Remark 1.2. Note that EX^2 is either infinite, which means that (H2) is satisfied, or finite, which implies that the mean must be finite. So assumption (H2) indeed doesn't require any moment condition. Assumption (H1) simply avoids the case where X is nonnegative or nonpositive.

Remark 1.3. If X is symmetric and in the domain of attraction of a stable law of order α ($0 < \alpha < 2$), then $t_0 = \beta_\alpha$, where β_α is the solution of (1.3). In general, when X is in the domain of attraction of a stable law, t_0 does equal the constant $\beta(\alpha, c_1, c_2)$ given in Theorem 3.2 of Shao [14].

This paper is organized as follows. In the next two sections we will focus on some basic properties of the solution (t_x, b_x) of saddlepoint equations (1.7) and (1.8), which may be of independent interest. Proofs of the main results will be postponed to Section 4.

2. THE SOLUTION TO THE SADDLEPOINT EQUATIONS

This section is devoted to the study of some basic properties of the solution (t_x, b_x) to the saddlepoint equations. Let

$$(2.1) \quad g(t, b; x) = tx - \ln E \exp \{t(2bX - b^2X^2)\}.$$

Our first property is that $\inf_{b \geq 0} \sup_{t \in R} g(t, b; x)$ is attained at some finite unique points $t_x > 0$ and $b_x > 0$ for x in a right neighborhood of zero.

Proposition 2.1. *Assume that (H1) and (H2) are satisfied and $\text{Card}(C_s) \geq 3$. Then there exists an $\epsilon_0 > 0$, such that for all $x \in (0, \epsilon_0)$, $\inf_{b > 0} \sup_{t \in R} g(t, b; x)$ is attained at some finite unique points $t_x > 0$ and $b_x > 0$ and (t_x, b_x) satisfy the saddlepoint equations (1.7) and (1.8).*

To prove Proposition 2.1, we need the following lemmas.

Lemma 2.1. *Let $x > 0$ and assume $\text{Card}(C_s) \geq 3$. Then under (H2), $g(t, b; x)$ is strictly increasing in t for $t \in (-\infty, \epsilon_1)$ for some $\epsilon_1 > 0$.*

Proof. The proof follows the same lines of that of Lemma A.2 in Jing, Shao and Zhou [12]. See the Appendix for a detailed proof. \square

For $0 < x < 1$, define

$$a_{10} := a_{10}(b) = \frac{1}{b}(1 - \sqrt{1-x}), \quad a_{20} := a_{20}(b) = \frac{1}{b}(1 + \sqrt{1-x}),$$

$$a_1 := a_1(b) = \min(a_{10}, a_{20}), \quad a_2 := a_2(b) = \max(a_{10}, a_{20})$$

and

$$U = \{b : (a_1(b), a_2(b)) \cap C_s \neq \emptyset\}.$$

Lemma 2.2. *Assume $\text{Card}(C_s) \geq 3$ and (H2) is satisfied. Let $x \in (0, 1)$.*

(i) *If $b \in U$, then*

$$(2.2) \quad \sup_{t \in R} g(t, b; x) = \sup_{t > 0} g(t, b; x),$$

and the supremum is attained at some finite unique point $\tilde{t} := \tilde{t}(b, x) > 0$ which is a finite unique solution to the equation $\partial g(t, b; x)/\partial t = 0$.

(ii) *If $b \notin U$, then*

$$(2.3) \quad \sup_{t \in R} g(t, b; x) = -\ln \left(P(X = a_1) + P(X = a_2) \right)$$

and the supremum is attained at ∞ .

Proof. We first note that $U \neq \emptyset$ by the fact that $\bigcup_b \{b : (a_1(b), a_2(b))\} = R$. An elementary argument also yields that U is an open set. (If (H1) holds, then $U \cap R^+$ is also a nonvoid open set.) Let

$$h(y) = b^2 y^2 - 2by + x = b^2(y - a_1)(y - a_2).$$

Suppose $b \in U$. Since

$$(a_1, a_2) \cap C_s \neq \emptyset,$$

there must exist $W := [a_3, a_4] \subset (a_1, a_2)$ so that

- (a) there exists $\delta > 0$ such that $h(y) < -\delta$ for each $y \in W$;
- (b) $P(X \in W) > 0$.

Then, as $t \rightarrow \infty$,

$$\begin{aligned} g(t, a; b) &= -\ln Ee^{-th(X)} \leq -\ln Ee^{-th(X)} I(X \in W) \\ &\leq -\ln\{e^{t\delta} P(X \in W)\} \rightarrow -\infty, \end{aligned}$$

which combined with Lemma 2.1 shows that $\sup_{t \in R} g(t, b; x)$ is attained at some finite $\tilde{t} = \tilde{t}(b, x) > 0$. Since $g(t, b; x)$ is a differentiable function of t when $t > 0$, we get that $\partial g(\tilde{t}, b; x)/\partial t = 0$. Also note that

$$(2.4) \quad \frac{\partial^2 g(t, b; x)}{\partial t^2} = -\left(\frac{EZ^2 e^{tZ}}{Ee^{tZ}} - \left(\frac{EZ e^{tZ}}{Ee^{tZ}} \right)^2 \right) < 0,$$

where $Z = 2bX - b^2 X^2$. There is at most one solution to the equation $\partial g(t, b; x)/\partial t = 0$. Therefore \tilde{t} is also unique.

Next, suppose $b \notin U$. Since C_s is necessarily closed, $[a_1, a_2] \cap C_s$ contains at most two points $\{a_1, a_2\}$. Clearly, we have

$$h(y) > 0 \text{ for each } y \in C_s \setminus \{a_1, a_2\}.$$

Therefore,

$$\begin{aligned} g(t, b; x) &= -\ln Ee^{-th(X)} \\ &= -\ln \left(Ee^{-th(X)} I(X \in C_s \setminus \{a_1, a_2\}) + P(X = a_1) + P(X = a_2) \right) \\ &\nearrow -\ln \left(P(X = a_1) + P(X = a_2) \right) \quad \text{as } t \rightarrow \infty. \quad \square \end{aligned}$$

Lemma 2.3. *Assume that $\text{Card}(C_s) \geq 3$, $R^+ \cap C_s \neq \emptyset$ and that (H2) is satisfied. Then for ϵ_0 sufficiently small and $x \in (0, \epsilon_0)$,*

$$(2.5) \quad \inf_{b>0} \sup_{t \in R} g(t, b; x) = \inf_{b \in U \cap R^+} \sup_{t \in R} g(t, b; x).$$

Proof. We use the contradiction method. Suppose (2.5) doesn't hold for ϵ_0 sufficiently small. Then there exists a sequence $\{x_k\}_{k=1}^\infty$ in $(0, 1)$ such that $x_k \rightarrow 0$ as $k \rightarrow \infty$ and

$$(2.6) \quad \inf_{b>0} \sup_{t \in R} g(t, b; x_k) = \inf_{b \in R^+ \setminus U} \sup_{t \in R} g(t, b; x_k)$$

for $k \geq 1$. Since $\text{Card}(C_s) \geq 3$, there exist c_1, c_2, c_3 and $\delta > 0$ such that

$$p_i := P(X \in [c_i - \delta, c_i + \delta]) > 0$$

and the three intervals $[c_i - \delta, c_i + \delta], i = 1, 2, 3$ are disjoint. Lemma 2.2 implies

$$\begin{aligned} &\sup_{b \in R^+ \setminus U} \inf_{t \in R} E \exp \left\{ t(2bX - b^2X^2 - x_k) \right\} \\ &= \exp \left(- \inf_{b \in R^+ \setminus U} \sup_{t \in R} g(t, b; x_k) \right) \\ &= \sup_{b \in R^+ \setminus U} P \left(X = \frac{1}{b}(1 - \sqrt{1 - x_k}) \right) + P \left(X = \frac{1}{b}(1 + \sqrt{1 - x_k}) \right) \\ (2.7) \quad &\leq 1 - \min(p_1, p_2, p_3) < 1. \end{aligned}$$

To get a contradiction, it suffices to show that

$$(2.8) \quad \sup_{b \in R^+ \setminus U} \inf_{t>0} E \exp \left\{ t(2bX - b^2X^2 - x_k) \right\} \geq 1.$$

Let $p = P(X > 0)$. Since $R^+ \cap C_s \neq \emptyset$, we have $p > 0$. Put $b_k = \sqrt{x_k}$. Then

$$(2.9) \quad P(6x_k/(pb_k) \leq X \leq 1/b_k) \rightarrow p \text{ and } P(-x_k/b_k \leq X \leq 0) \rightarrow 1 - p$$

and

$$\begin{aligned}
 & \sup_{b \in R^+ \setminus U} \inf_{t > 0} E \exp \{t(2bX - b^2X^2 - x_k)\} \\
 &= \sup_{b > 0} \inf_{t > 0} E \exp \{t(2bX - b^2X^2 - x_k)\} \quad [\text{by (2.6)}] \\
 &\geq \inf_{t > 0} E \exp \{t(2b_kX - b_k^2X^2 - x_k)\} \\
 &\geq \inf_{t > 0} \left\{ E \exp \{t(2b_kX - b_k^2X^2 - x_k)\} I(-x_k/b_k \leq X \leq 0) \right. \\
 &\quad \left. + E \exp \{t(2b_kX - b_k^2X^2 - x_k)\} I(6x_k/(pb_k) \leq X \leq 1/b_k) \right\} \\
 &\geq \inf_{t > 0} \left\{ e^{-4tx_k} P(-x_k/b_k \leq X \leq 0) + e^{5tx_k/p} P(6x_k/(pb_k) \leq X \leq 1/b_k) \right\} \\
 &= P(-x_k/b_k \leq X \leq 0) + P(6x_k/(pb_k) \leq X \leq 1/b_k) \\
 &\rightarrow 1
 \end{aligned}$$

by (2.9), where in the last equality, we used the fact that $e^{-4tx_k} P(-x_k/b_k \leq X \leq 0) + e^{5tx_k/p} P(6x_k/(pb_k) \leq X \leq 1/b_k)$ is nondecreasing in $t > 0$ for k large enough. This proves (2.8) and hence the lemma. \square

Lemma 2.4. *For $0 < x < 1$, we have*

$$(2.10) \quad \lim_{b \rightarrow \infty} \sup_{t > 0} g(t, b; x) = \infty \text{ and } \lim_{b \rightarrow 0^+} \sup_{t > 0} g(t, b; x) = \infty.$$

Proof. Let k be a positive number. Then

$$\begin{aligned}
 \sup_{t > 0} g(t, b; x) &\geq g(k, b; x) \\
 &= kx - \ln E \exp \{ -k(b^2X^2 - 2bX) \} \\
 (2.11) \quad &= kx - \ln E \exp \{ -k(bX - 1)^2 + k \}.
 \end{aligned}$$

It follows from Lebesgue’s dominated convergence theorem that

$$(2.12) \quad \lim_{b \rightarrow \infty} E \exp \{ -k(bX - 1)^2 + k \} = P(X = 0) \text{ and } \lim_{b \rightarrow 0^+} M(a) = 1.$$

Combining (2.11)-(2.12) gives

$$\liminf_{b \rightarrow \infty} \sup_{t > 0} g(t, b; x) \geq kx \text{ and } \liminf_{b \rightarrow 0^+} \sup_{t > 0} g(t, b; x) \geq kx.$$

This proves (2.10) by the arbitrariness of k . \square

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Lemmas 2.1-2.4 imply that $\inf_{b > 0} \sup_{t \in R} g(t, b; x)$ is attained at some finite points $b_x \in U$ and $t_x := \tilde{t}(b_x, x) > 0$ for all $x \in (0, \epsilon_0)$ when ϵ_0 is sufficiently small. When $b \in U$, by Lemma 2.2(i), we have

$$(2.13) \quad \frac{\partial g(\tilde{t}, b; x)}{\partial t} = x - \frac{EZ e^{\tilde{t}Z}}{E e^{\tilde{t}Z}} = 0, \quad \text{where } Z = 2bX - b^2X^2.$$

Note that the assumption $\text{Card}(C_s) \geq 3$ implies that Z is nondegenerate; thus (A.2) is true. It then follows from the implicit function theorem that $\tilde{t}(b, x)$ is a differentiable function in some neighborhood $U^*(b)$ of b (also a differentiable function in some neighborhood of x). We can also guarantee that $U^*(b) \subset U$.

Hence $\sup_{t \in R} g(t, b; x)$ is also a differentiable function in some neighborhood of b_x . So b_x satisfies the equation $dg(\tilde{t}, b; x)/db = 0$, i.e.,

$$(2.14) \quad E(X - bX^2) \exp \{ \tilde{t}(2bX - b^2X^2) \} = 0.$$

It follows from (2.13) and (2.14) that b_x and t_x are the solutions to the equations

$$EZ e^{tZ} = x E e^{tZ}, \quad EX e^{tZ} = b EX^2 e^{tZ}.$$

Now we will show the uniqueness of (t_x, b_x) . Suppose (t'_x, b'_x) is another point such that $g(t'_x, b'_x; x) = \inf_{b > 0} \sup_{t \in R} g(t, b; x)$. Recall that

$$g(t, b; x) = -\ln E \exp \{ t(2bX - b^2X^2 - x) \}.$$

We must have

$$(2.15) \quad \begin{aligned} E \exp \{ t_x(2b_x X + b_x^2 X^2 - x) \} &= \sup_{b > 0} E \exp \{ t_x(2bX + b^2 X^2 - x) \} \\ &\geq E \exp \{ t_x(2b'_x X - b_x'^2 X^2 - x) \} \\ &\geq \inf_{t > 0} E \exp \{ t(2b'_x X - b_x'^2 X^2 - x) \} \\ &= E \exp \{ t'_x(2b'_x X - b_x'^2 X^2 - x) \}. \end{aligned}$$

If $t_x \neq t'_x$, then

$$(2.16) \quad E \exp \{ t_x(2b'_x X - b_x'^2 X^2 - x) \} > \inf_{t > 0} E \exp \{ t(2b'_x X - b_x'^2 X^2 - x) \}$$

by the fact that $E \exp \{ t(2bX - b^2X^2 - x) \}$ is a strictly convex function of t for each fixed b and $2bX - b^2X^2 - x$ is not identically equal to 0. Combining (2.15) and (2.16) gives

$$E \exp \{ t_x(2b_x X + b_x^2 X^2 - x) \} > E \exp \{ t'_x(2b'_x X - b_x'^2 X^2 - x) \},$$

which contradicts our assumption. Hence

$$(2.17) \quad t_x = t'_x.$$

Define

$$f(b, s) = E \exp \left\{ s \left(\frac{2}{b} X - X^2 - \frac{x}{b^2} \right) \right\}.$$

So we have

$$f(b_x, s_x) = f(b'_x, s'_x) = \sup_{b > 0} \inf_{s > 0} f(b, s),$$

where $s_x = t_x b_x^2, s'_x = t_x b_x'^2$. Noting that $f(b, s)$ is a strictly convex function of s for each fixed b , similar to the proof of (2.17), we get $s_x = s'_x$. Hence $b_x = b'_x$. This completes the proof of uniqueness. \square

From now on, we assume $x \in (0, \epsilon_0)$ so that $\sup_{b > 0} \inf_{t > 0} E \exp \{ t(2bX - b^2X^2 - x) \}$ is attained at the unique point (t_x, b_x) . The next proposition shows that, as a function of $x \in (0, \epsilon_0)$, $E \exp \{ t_x(2b_x X - b_x X^2 - x) \}$ is differentiable.

Proposition 2.2. *Assume that (H1) and (H2) are satisfied and $\text{Card}(C_s) \geq 3$. Then, $E \exp \{ t_x(2b_x X - b_x^2 X^2 - x) \}$ is a differentiable function of x when $x \in (0, \epsilon_0)$.*

The next two lemmas are needed to prove the proposition. We also assume that the condition of Proposition 2.2 is satisfied.

Lemma 2.5. $\forall x_0 \in (0, \epsilon_0)$, b_x is a continuous function of x in a neighborhood of x_0 .

Proof. Let $g(b, x) = \inf_{t>0} E \exp \{t(2bX - b^2X^2 - x)\}$. By (2.10), we have

$$\lim_{b \rightarrow \infty} g(b, x) = 0 \text{ and } \lim_{b \rightarrow 0} g(b, x) = 0.$$

Hence there exists some $\delta'_0 > 0$ such that if $x \in [x_0 - \delta'_0, x_0 + \delta'_0]$, then $b_x \in [L, U]$, where both L and U are positive constants depending only on x_0 and δ'_0 . We can also choose L and U so that if $b < L$ or $b > U$, then

$$(2.18) \quad |g(b, x_0) - g(b_{x_0}, x_0)| \geq \frac{1}{2}g(b_{x_0}, x_0).$$

Now we claim that $\exists \delta_0 > 0, \forall 0 < \delta \leq \delta_0, \exists 0 < \epsilon < g(b_{x_0}, x_0)$ such that

$$(2.19) \quad \{b : |g(b, x_0) - g(b_{x_0}, x_0)| \leq \epsilon\} \subset (b_{x_0} - \delta, b_{x_0} + \delta).$$

Otherwise, \exists a sufficiently small $\delta > 0, \forall \epsilon > 0,$

$$\{b : |g(b, x_0) - g(b_{x_0}, x_0)| \leq \epsilon\} \cap (b_{x_0} - \delta, b_{x_0} + \delta)^c \neq \emptyset.$$

So we will have a sequence $\{b_m\}_{m=1}^\infty$ such that $|b_m - b_{x_0}| \geq \delta$ for each $m \geq 1$ and $\lim_{m \rightarrow \infty} g(b_m, x_0) = g(b_{x_0}, x_0)$. (2.18) implies $\exists m_0 > 0$ such that if $m \geq m_0, b_m \in [L, U]$. Without loss of generality, we assume $b_m \rightarrow b'_{x_0}$ as $m \rightarrow \infty$. The continuity of $g(b, x)$ gives

$$(2.20) \quad g(b'_{x_0}, x_0) = g(b_{x_0}, x_0).$$

Since $|b'_{x_0} - b_{x_0}| > \delta,$ (2.20) contradicts the fact that b_{x_0} is unique. Therefore, we have (2.19). Note that

$$\begin{aligned} g(b, x) &= \inf_{t>0} E \exp \{t(2bX - b^2X^2 - x)\} \\ &=: E \exp \{\tilde{t}(x, b)(2bX - b^2X^2 - x)\}, \end{aligned}$$

where $\tilde{t}(x, b)$ is the unique solution to the following equation:

$$E(2bX - b^2X^2) \exp \{t(2bX - b^2X^2)\} = xE \exp \{t(2bX - b^2X^2)\},$$

for each fixed b and x . The implicit function theorem shows that $\tilde{t}(x, b)$ is a differentiable function of x and b . Hence we can find some $0 < \delta_1 \leq \min(\delta_0, \delta'_0)$ such that if $x \in [x_0 - \delta_1, x_0 + \delta_1],$

$$(2.21) \quad |g(b, x) - g(b, x_0)| \leq \frac{\epsilon}{2}$$

uniformly in $b \in [L, U],$ and

$$(2.22) \quad |g(b_x, x) - g(b_{x_0}, x_0)| \leq \frac{\epsilon}{2}.$$

Combining (2.21) and (2.22) yields

$$(2.23) \quad |g(b_x, x_0) - g(b_{x_0}, x_0)| \leq \epsilon.$$

It follows from (2.19) and (2.23) that

$$|b_x - b_{x_0}| < \delta,$$

which shows the continuity of b_x in some neighborhood of $x_0.$ □

Lemma 2.6. t_x is an increasing and continuous function of x in some neighborhood of x_0 for $x_0 \in (0, \epsilon_0).$

Proof. Recall that b_x and t_x satisfy the following two equations:

$$(2.24) \quad \frac{EZ e^{tZ}}{E e^{tZ}} = x,$$

$$(2.25) \quad \frac{E(2X - 2bX^2)e^{tZ}}{E e^{tZ}} = 0,$$

where $Z = 2bX - b^2X^2$. Write

$$\begin{aligned} a_1(b, t) &= \frac{EZ^2 e^{tZ}}{E e^{tZ}} - \left(\frac{EZ e^{tZ}}{E e^{tZ}} \right)^2, \\ a_2(b, t) &= \frac{EZ(2X - 2bX^2)e^{tZ}}{E e^{tZ}}, \\ a_3(b, t) &= \frac{E(2X - 2bX^2)^2 e^{tZ}}{E e^{tZ}} t - 2 \frac{EX^2 e^{tZ}}{E e^{tZ}}. \end{aligned}$$

Since b_x is a supremum point of $g(b, x)$, we have

$$a_3(b_x, t_x) \leq 0.$$

Also

$$a_1(b_x, t_x) > 0$$

because $2b_x X - b_x^2 X^2$ is nondegenerate. If $a_3(b_{x_0}, t_{x_0}) < 0$ or $a_2(b_{x_0}, t_{x_0}) \neq 0$, then

$$\left. \frac{dt_x}{dx} \right|_{x=x_0} = \frac{a_3(b_{x_0}, t_{x_0})}{a_1(b_{x_0}, t_{x_0})a_3(b_{x_0}, t_{x_0}) - a_2^2(b_{x_0}, t_{x_0})} \geq 0$$

by the implicit function theorem. So in this case, t_x is increasing and continuous at $x = x_0$.

If $a_3(b_{x_0}, t_{x_0}) = 0$ and $a_2(b_{x_0}, t_{x_0}) = 0$, then there exists some neighborhood of x_0 such that $a_3(b_x, t_x) < 0$ or $a_2(b_x, t_x) \neq 0$ when $x \in U \setminus \{x_0\}$. Otherwise, we find some sequence $\{x_n\}_{n=1}^\infty$ which goes to x_0 as $n \rightarrow \infty$ such that

$$a_3(b_{x_n}, t_{x_n}) = 0 \quad \text{and} \quad a_2(b_{x_n}, t_{x_n}) = 0.$$

Since $a_3(b, t)$ and $a_2(b, t)$ are infinitely differentiable functions of b and t ,

$$(2.26) \quad a_3(b, t) \equiv 0 \quad \text{and} \quad a_2(b, t) \equiv 0$$

when $(b, t)^T$ belongs to some neighborhood U_1 of $(b_{x_0}, t_{x_0})^T$. Noting that $(b_{x_0}, t_{x_0})^T$ satisfies (2.25), we can conclude that

$$E(2X - 2bX^2)e^{tZ} \equiv 0$$

when $(b, t)^T \in U_1$. Hence

$$(2.27) \quad E \exp \{t_{x_0}(2bX - b^2X^2)\} \equiv E \exp \{t_{x_0}(2b_{x_0}X - b_{x_0}^2X^2)\}$$

when b is in some neighborhood of b_{x_0} . But (2.27) contradicts the uniqueness of b_{x_0} . So

$$\frac{dt_x}{dx} \geq 0 \quad \text{if} \quad x \in U \setminus \{x_0\},$$

which, combined with Lemma 2.5 and the fact that $\tilde{t}(x, b)$ determined by (2.24) is a continuous function of x and b , implies t_x is increasing and continuous at $x = x_0$. □

Proof of Proposition 2.2. Let

$$e(x) = \int_{x_0}^x (-t_y)E \exp \{t_y(2b_yX - b_y^2X^2 - y)\} dy + E \exp \{t_{x_0}(2b_{x_0}X - b_{x_0}^2X^2 - x_0)\}.$$

Then

$$\frac{de(x)}{dx} = (-t_x)E \exp \{t_x(2b_xX - b_x^2X^2 - x)\}.$$

We also know that $E \exp \{t_x(2b_xX - b_x^2X^2 - x)\}$ is differentiable a.e. in $(0, \epsilon_0)$, and

$$\frac{d}{dx}E \exp \{t_x(2b_xX - b_x^2X^2 - x)\} = -t_xE \exp \{t_x(2b_xX - b_x^2X^2 - x)\} \text{ a.e.}$$

So

$$e(x) = E \exp \{t_x(2b_xX - b_x^2X^2 - x)\} \text{ a.e.}$$

Since both $e(x)$ and $E \exp \{t_x(2b_xX - b_x^2X^2 - x)\}$ are continuous functions of x by Lemmas 2.5-2.6,

$$e(x) = E \exp \{t_x(2b_xX - b_x^2X^2 - x)\},$$

which gives the final assertion. □

3. CONVERGENCE OF b_x AND t_x

Proposition 3.1. *Assume that (H1) and (H2) are satisfied and that $\text{Card}(C_s) \geq 3$. Then*

$$(3.1) \quad \lim_{x \rightarrow 0^+} b_x = 0.$$

Proof. We will show that for any sequence $\{x_n\}_{n=1}^\infty$ decreasing to 0, it is impossible that

$$(3.2) \quad \lim_{n \rightarrow \infty} b_{x_n} = \infty,$$

$$(3.3) \quad \lim_{n \rightarrow \infty} b_{x_n} = b_0,$$

where b_0 is some positive constant. Now we suppose (3.2). Then

$$\begin{aligned} \inf_{t>0} E \exp \{t(2b_{x_n}X - b_{x_n}^2X^2 - x_n)\} &= \inf_{t>0} E \exp \{-t(b_{x_n}X - 1)^2 + t(1 - x_n)\} \\ &\leq E \exp \{-c(b_{x_n}X - 1)^2 + t_0(1 - x_n)\} \\ &\rightarrow P(X = 0), \end{aligned}$$

as $n \rightarrow \infty$, where c is any positive constant. But the above result contradicts Lemma 8.1 of Shao [14].

It remains to exclude (3.3). So we assume (3.3). Equation (1.7) gives

$$(3.4) \quad \frac{E(2X - b_{x_n}X^2) \exp \{s_n(2X - b_{x_n}X^2)\}}{E \exp \{s_n(2X - b_{x_n}X^2)\}} = \frac{x_n}{b_n},$$

where $s_n = t_{x_n}b_{x_n}$. Lemma 2.6, combined with (3.3), shows $\lim_{n \rightarrow \infty} s_n = s_0$, where $s_0 = t_0b_0$. Letting $n \rightarrow \infty$ in (3.4) yields

$$2EX \exp \{s_0(2X - b_0X^2)\} = b_0EX^2 \exp \{s_0(2X - b_0X^2)\},$$

which contradicts another equation (1.8)

$$E(X - b_{x_n}X^2) \exp \{s_n(2X - b_{x_n}X^2)\} = 0. \quad \square$$

Proposition 3.2. *Assume that (H1), (H2) and (1.5) are satisfied and that $\text{Card}(C_s) \geq 3$. Then*

$$(3.5) \quad \lim_{x \rightarrow 0^+} t_x = t_0 > 0.$$

Proof. Convergence of t_x follows from Lemma 2.6. So we need to show $t_0 > 0$. Otherwise, suppose $t_0 = 0$. Write $Z_x = 2b_x X - b_x^2 X^2$ and $D_x = t_x^{-1/3}$. Then

$$\begin{aligned} & \left| E b_x X e^{t_x Z_x} I(|b_x X| \leq D_x) - E b_x X I(|b_x X| \leq D_x) \right| \\ & \leq \epsilon_x E (b_x X)^2 e^{t_x Z_x} I(|b_x X| \leq D_x), \end{aligned}$$

where $\epsilon_x \rightarrow 0$ as $x \rightarrow 0^+$. Since $X \in \mathcal{F}_\theta$ for some $\theta < \infty$, we have for x sufficiently close to 0,

$$\begin{aligned} & E b_x X e^{t_x Z_x} I(|b_x X| \leq D_x) \\ & \leq E b_x X I(|b_x X| \leq D_x) + \epsilon_x E (b_x X)^2 e^{t_x Z_x} I(|b_x X| \leq D_x) \\ & \leq \frac{\theta + 1}{D_x} E (b_x X)^2 I(|b_x X| \leq D_x) + \epsilon_x E (b_x X)^2 e^{t_x Z_x} I(|b_x X| \leq D_x) \\ & \leq \left(\frac{\theta + \epsilon'}{D_x} e^{t_x(2D_x + D_x^2)} + \epsilon_x \right) E (b_x X)^2 e^{t_x Z_x} I(|b_x X| \leq D_x) \\ (3.6) \quad & < (1/4) E (b_x X)^2 e^{t_x Z_x} I(|b_x X| \leq D_x). \end{aligned}$$

On the other hand,

$$(3.7) \quad \begin{aligned} & 2E b_x X e^{t_x Z_x} I(|b_x X| > D_x) \\ & < E (b_x X)^2 e^{t_x Z_x} I(|b_x X| > D_x) \end{aligned}$$

for x sufficiently small.

Combining (3.6) and (3.7), we have

$$E Z_x e^{t_x Z_x} < 0$$

for x sufficiently close to 0, but this contradicts the saddlepoint equation

$$E(Z_x - x)e^{t_x Z_x} = 0. \quad \square$$

We shall also need the following lemma.

Lemma 3.1. *Assume that (H1), (H2) and (1.5) are satisfied and that $\text{Card}(C_s) \geq 3$. Write $Z_x = 2b_x X - b_x^2 X^2$. Then*

$$(3.8) \quad E e^{t_x Z_x} = 1 + o(x),$$

$$(3.9) \quad E (b_x X)^2 e^{t_x Z_x} = x(1 + o(x)),$$

as $x \rightarrow 0^+$.

Proof. From the proof of Proposition 2.2, we see that

$$\frac{d}{dx} E \exp \{t_x(Z_x - x)\} = -t_x E \exp \{t_x(Z_x - x)\}.$$

Hence

$$E \exp \{t_x(Z_x - x)\} = \exp \left(- \int_0^x t_y dy \right),$$

which combined with Proposition 3.1 gives

$$\begin{aligned} E e^{t_x Z_x} &= E \exp \{t_x(Z_x - x)\} e^{x t_x} \\ &= \exp \left(\int_0^x (t_x - t_y) dy \right) \\ &= \exp(o(x)) = 1 + o(x). \end{aligned}$$

By Equations (1.7) and (1.8),

$$\begin{aligned} E(b_x X)^2 e^{t_x Z_x} &= x E e^{t_x Z_x} \\ &= x(1 + o(x)). \quad \square \end{aligned}$$

4. PROOF OF MAIN THEOREMS

Write

$$K(x) = x^{-2} E X^2 I(|X| \leq x) \quad \text{for } x > 0,$$

$$d(t) = \inf\{s \geq a_0 + 1 : K(s) \leq 1/t\}, \quad \text{where } a_0 = \inf\{x \geq 1 : K(x) > 0\}.$$

4.1. Proof of Theorem 1.1. If $\text{Card}(C_s) \leq 2$, then X takes only two possible values. In this case, (1.4) is a direct consequence of Theorem 3.1 of Shao [14]. If $\text{Card}(C_s) \geq 3$, then (1.4) follows from (1.6). In fact, by Lemma 3.1 (with $t_n = t_{x_n^2/n}$ and $b_n = b_{x_n^2/n}$)

$$\begin{aligned} n\lambda(x_n^2/n) &= t_n x_n^2 - n \ln E \exp \left(t_n(2b_n X - b_n^2 X^2) \right) \\ &= t_n x_n^2 - n \ln(1 + o(x_n^2/n)) = t_n x_n^2 + o(x_n^2) = t_0 x_n^2 + o(x_n^2). \end{aligned}$$

Thus we only need to prove (1.6).

Upper bound proof. Let δ be an arbitrarily small but fixed constant, B an arbitrarily large but fixed constant. Both δ and B will be specified later. Then

$$\begin{aligned} P \left(\frac{S_n}{V_n} \geq x_n \right) &\leq P \left(\frac{S_n}{V_n} \geq x_n, 0 \leq V_n \leq \frac{1}{\delta} a_n \right) + P \left(\frac{S_n}{V_n} \geq x_n, V_n > \frac{1}{\delta} a_n \right) \\ (4.1) \quad &:= T_1 + T_2, \end{aligned}$$

where a_n is some number depending on n . We deal with T_2 first. Let z_n be some positive number which will be defined later. By the Cauchy-Schwarz inequality, $|\sum_{i=1}^n X_i I(|X_i| \geq z_n)| \leq V_n (\sum_{i=1}^n I(|X_i| \geq z_n))^{1/2}$ and hence

$$\begin{aligned} T_2 &\leq P \left(\sum_{i=1}^n X_i I(|X_i| \leq z_n) \geq \frac{x_n}{2\delta} a_n \right) + P \left(\sum_{i=1}^n X_i I(|X_i| \geq z_n) \geq \frac{x_n}{2} V_n \right) \\ &\leq P \left(\sum_{i=1}^n X_i I(|X_i| \geq z_n) \geq \frac{x_n}{2\delta} a_n \right) + P \left(\sum_{i=1}^n I(|X_i| \leq z_n) \geq \frac{x_n^2}{4} \right) \\ &:= T_{21} + T_{22}. \end{aligned}$$

For T_{21} , Bernstein's inequality gives

$$(4.2) \quad T_{21} \leq 2 \exp \left(- \frac{n \left(x_n a_n / (2n\delta) - EXI(|X| \leq z_n) \right)^2}{2 \text{Var} XI(|X| \leq z_n) + \frac{4}{3} z_n \left(x_n a_n / (2n\delta) - EXI(|X| \leq z_n) \right)} \right).$$

Applying the Chernoff large deviation to the binomial random variable $B(n, p)$, it follows that for all $a > 0$,

$$P(B(n, p) > an) \leq \left(\frac{ep}{a} \right)^{an}.$$

Therefore

$$(4.3) \quad T_{22} \leq \left(\frac{12P(|X| \geq z_n)}{x_n^2/n} \right)^{x_n^2/4}.$$

In the following, we will analyze the exponent in (4.2) and the ratio in (4.3), respectively. Let

$$(4.4) \quad z_n := \inf \left\{ x \geq a_0 + 1 : (2x^2)^{-1} EX^2 I(|X| \leq x) \leq \frac{\delta^2}{B^2} \frac{x_n^2}{n} \right\}.$$

Then $z_n = d \left(1 / (2 \frac{\delta^2}{B^2} \frac{x_n^2}{n}) \right) \rightarrow \infty$ as $n \rightarrow \infty$.

Define $a_n = x_n z_n / B$. Hence for n sufficiently large,

$$\begin{aligned} \frac{x_n a_n}{2n\delta} - EXI(|X| \leq z_n) &\geq \frac{1}{B\delta} \left(\frac{1}{2} - \frac{4\theta\delta^3}{B} \right) \frac{x_n^2}{n} z_n, \\ \frac{n \left(x_n a_n / (2n\delta) - EXI(|X| \leq z_n) \right)^2}{2 \text{Var} XI(|X| \leq z_n)} &\geq \frac{B}{4\delta^3} \left(\frac{1}{2} - \frac{4\theta\delta^3}{B} \right)^2 \frac{x_n^2}{n}, \\ \frac{n \left(x_n a_n / (2n\delta) - EXI(|X| \leq z_n) \right)}{4z_n/3} &\geq \frac{3}{4B\delta} \left(\frac{1}{2} - \frac{4\theta\delta^3}{B} \right) \frac{x_n^2}{n}, \\ \frac{12P(|X| \geq z_n)}{x_n^2/n} &\leq 30\theta \frac{\delta^2}{B^2}. \end{aligned}$$

Hence we can select suitable B and δ so that $B\delta$ is very small, which ensures that T_{21} and T_{22} are negligible relative to $\exp(-t_n x_n^2)$. So we have

$$(4.5) \quad \frac{\ln T_2}{t_n x_n^2} \rightarrow -\infty.$$

Now let's deal with T_1 . Let $u_n = z_n / (\delta B x_n^2)$. Noting that $a_n / (\delta x_n^2) = u_n x_n$ and that $0 \leq c \leq v \leq h$ implies

$$v \geq (v^2 + ch) / (c + h),$$

we have for $s > 0$,

$$\begin{aligned}
 T_1 &= P\left(\frac{S_n}{V_n} \geq x_n, 0 \leq V_n \leq \frac{1}{\delta} a_n\right) \\
 &\leq \sum_{i=0}^{\lfloor x_n^2 \rfloor} P\left(\frac{S_n}{x_n} \geq V_n, \frac{ia_n}{\delta x_n^2} \leq V_n \leq \frac{(i+1)a_n}{\delta x_n^2}\right) \\
 &\leq \sum_{i=0}^{\lfloor x_n^2 \rfloor} P\left((2i+1)u_n S_n \geq V_n^2 + i(i+1)u_n^2 x_n^2\right) \\
 &\leq \sum_{i=0}^{\lfloor x_n^2 \rfloor} \left(E \exp\left\{s\left((2i+1)u_n X - X^2 - i(i+1)u_n^2 x_n^2/n\right)\right\}^n\right) \\
 &= \sum_{i=0}^{\lfloor x_n^2 \rfloor} \left(E \exp\left\{s\left((2i+1)u_n X - X^2 - (i+1/2)^2 u_n^2 x_n^2/n\right) + (s/4)u_n^2 x_n^2/n\right\}^n\right).
 \end{aligned}$$

Since

$$\begin{aligned}
 &E \exp\left\{s\left((2i+1)u_n X - X^2 - (i+1/2)^2 u_n^2 x_n^2/n\right)\right\} \\
 &= E \exp\left\{\frac{s}{(i+1/2)^2 u_n^2} \left(\frac{2}{(i+1/2)u_n} X - \frac{1}{(i+1/2)^2 u_n^2} X^2 - x_n^2/n\right)\right\} \\
 &\leq \sup_{b \geq 0} E \exp\left\{\frac{s}{b^2} (2bX - b^2 X^2 - x_n^2/n)\right\},
 \end{aligned}$$

we have

$$(4.6) \quad T_1 \leq (1 + \lfloor x_n^2 \rfloor) \left(E \exp\left\{t_n [2b_n X - b_n^2 X^2 - \frac{x_n^2}{n}] + t_n b_n^2 u_n^2 x_n^2/n\right\}\right)^n,$$

where t_n and b_n are chosen such that

$$E \exp\left\{t_n (2b_n X - b_n^2 X^2 - x_n^2/n)\right\} := \sup_{b \geq 0} \inf_{t \geq 0} E \exp\left\{t(2bX - b^2 X^2 - x_n^2/n)\right\},$$

or $t_n := t_{x_n^2/n}$, $b_n := b_{x_n^2/n}$. Recall that t_n and b_n satisfy

$$\frac{E(b_n X)^2 e^{t_n(2b_n X - b_n^2 X^2)}}{E e^{t_n(2b_n X - b_n^2 X^2)}} = \frac{x_n^2}{n}.$$

Hence for ϵ close enough to 0, we have

$$\begin{aligned}
 &E(b_n X)^2 I(|b_n X| \leq \epsilon) \\
 &\leq E(b_n X)^2 e^{t_n(2b_n X - b_n^2 X^2)} I(|b_n X| \leq \epsilon) e^{t_n(2\epsilon + \epsilon^2)} \\
 &\leq 2x_n^2/n,
 \end{aligned}$$

where we used Lemma 3.1 in the last inequality. Hence

$$(4.7) \quad d\left(\frac{\epsilon^2}{2} \frac{n}{x_n^2}\right) \leq \frac{\epsilon^2}{b_n}.$$

Combining Lemma 3.1 of Griffin and Kuelbs [10] and (4.7), we have for n sufficiently large,

$$\begin{aligned} d\left(\frac{B^2n}{2\delta^2x_n^2}\right) / \frac{\epsilon^2}{b_n} &\leq d\left(\frac{B^2n}{2\delta^2x_n^2}\right) / d\left(\frac{\epsilon^2}{2} \frac{n}{x_n^2}\right) \\ &\leq \left(\frac{B^2(1+\theta)}{\epsilon^2\delta^2}\right)^{\frac{1+\theta}{2}}, \end{aligned}$$

i.e.

$$(4.8) \quad b_n z_n \leq \epsilon^2 \left(\frac{B^2(1+\theta)}{\epsilon^2\delta^2}\right)^{\frac{1+\theta}{2}}.$$

Combining (4.1), (4.5), (4.6), (4.8), Proposition 2.2 and Proposition 3.2 gives that, $\forall \epsilon_0 > 0$,

$$\frac{\ln T_1}{x_n^2} \leq -t_0 + \epsilon_0$$

for n sufficiently large. Therefore

$$\limsup_{n \rightarrow \infty} \frac{\ln P\left(\frac{S_n}{V_n} \geq x_n\right)}{x_n^2} \leq -t_0. \quad \square$$

Lower bound proof. By the basic equality $|xy| = \inf_{b \geq 0} (2b)^{-1}(b^2x^2 + y^2)$, we have

$$\begin{aligned} P\left(\frac{S_n}{V_n} \geq x_n\right) &= P\left(S_n \geq \inf_{b \geq 0} \frac{1}{2b}(b^2V_n^2 + x_n^2)\right) \\ &= P\left(\sup_{b \geq 0} \sum_{i=1}^n (2bX_i - b^2X_i^2 - \frac{x_n^2}{n}) \geq 0\right) \\ &\geq P\left(\sum_{i=1}^n (2b_nX_i - b_n^2X_i^2 - \frac{x_n^2}{n}) \geq 0\right). \end{aligned}$$

Tilting methods give

$$\begin{aligned} &P\left(\sum_{i=1}^n (2b_nX_i - b_n^2X_i^2 - \frac{x_n^2}{n}) \geq 0\right) \\ &= \left(E \exp\left\{t_n\left(Z_n - \frac{x_n^2}{n}\right)\right\}\right)^n \int_0^\infty e^{-t_n\sigma(t_n)y\sqrt{n}} dF_n(y), \end{aligned}$$

where $Z_n = 2b_nX - b_n^2X^2$,

$$\sigma^2(t_n) = \frac{EZ_n^2e^{t_nZ_n}}{Ee^{t_nZ_n}} - \left(\frac{EZ_ne^{t_nZ_n}}{Ee^{t_nZ_n}}\right)^2,$$

$F_n(y)$ is the distribution function of the random variable $\sum_{i=1}^n (\eta_i - E\eta_i) / \sqrt{n\sigma^2(t_n)}$, and $\eta_1, \eta_2, \dots, \eta_n$ are i.i.d. r.v.'s with distribution function

$$V(x) = \frac{1}{Ee^{t_nZ_n}} \int_{-\infty}^x e^{t_ny} dP(Z_n \leq y).$$

By Lemma 4.2 in Section 4.3, we have

$$\begin{aligned}
 & \int_0^\infty \exp\{-t_n\sigma(t_n)y\sqrt{n}\}dF_n(y) \\
 & \geq \int_0^2 \exp\{-t_n\sigma(t_n)y\sqrt{n}\}dF_n(y) \\
 & \geq \exp\{-2t_n\sigma(t_n)\sqrt{n}\}P\left(0 \leq \sum_{i=1}^n(\eta_i - E\eta_i)/\sqrt{n\sigma^2(t_n)} \leq 2\right) \\
 & \geq 2^{-1}(\Phi(2) - \Phi(0)) \exp\{-2t_n\sigma(t_n)\sqrt{n}\}
 \end{aligned}$$

for n sufficiently large. Hence

$$\begin{aligned}
 & P\left(\frac{S_n}{V_n} \geq x_n\right) \\
 (4.9) \quad & \geq 2^{-1}(\Phi(2) - \Phi(0)) \exp\{-2t_n\sigma(t_n)\sqrt{n}\} \left(E \exp\left\{t_n\left(Z_n - \frac{x_n^2}{n}\right)\right\}\right)^n.
 \end{aligned}$$

Now we will prove

$$(4.10) \quad t_n\sigma(t_n)\sqrt{n} = o(x_n^2)$$

as $n \rightarrow \infty$.

Let ϵ be some small positive constant. Then since $X \in \mathcal{F}_\theta$,

$$(4.11) \quad \begin{aligned} \epsilon^2 P(|b_n X| \geq \epsilon) & \leq 2\theta E(b_n X)^2 I(|b_n X| \leq \epsilon) \\ & \leq 2\theta E(b_n X)^2 e^{t_n(Z_n + 4\epsilon)} I(|b_n X| \leq \epsilon) \end{aligned}$$

$$(4.12) \quad \leq c \frac{x_n^2}{n},$$

where (4.12) follows from Lemma 3.1 and c is some positive constant. Therefore (4.11) and (4.12) imply

$$\begin{aligned}
 & t_n^2 E Z_n^2 e^{t_n Z_n} \\
 & = t_n^2 E Z_n^2 e^{t_n Z_n} I(|b_n X| \leq \epsilon) + t_n^2 E Z_n^2 e^{t_n Z_n} I(|b_n X| > \epsilon) \\
 & \leq (2 + \epsilon)^2 t_n^2 E(b_n X)^2 e^{t_n Z_n} I(|b_n X| \leq \epsilon) + e P(|b_n X| > \epsilon) \\
 & \leq c_1 \frac{x_n^2}{n}
 \end{aligned}$$

for some constant $c_1 > 0$. So

$$\begin{aligned}
 t_n\sigma(t_n)\sqrt{n} & \leq \left(\frac{nt_n^2 E Z_n^2 e^{t_n Z_n}}{E e^{t_n Z_n}}\right)^{\frac{1}{2}} \\
 & \leq \sqrt{c_1} x_n / \sqrt{E e^{t_n Z_n}} \\
 & = o(x_n^2),
 \end{aligned}$$

where the last equality is from Lemma 3.1. So we completed the proof of (4.10). Noting that Lemma 3.1 gives, $\forall \epsilon_1 > 0$,

$$E \exp\left\{t_n\left(Z_n - \frac{x_n^2}{n}\right)\right\} \geq \exp\left\{(-t_0 - \epsilon_1)\frac{x_n^2}{n}\right\}$$

for n sufficiently large, we have

$$\liminf_{n \rightarrow \infty} \frac{\ln P(S_n/V_n \geq x_n)}{x_n^2} \geq -t_0 - \epsilon_1$$

by (4.9) and (4.10). The arbitrariness of ϵ_1 completes the lower bound proof. \square

4.2. Proof of Theorem 1.2. When $\text{Card}(C_s) \leq 2$, the result is due to Theorem 5.1 of Shao [14]. When $\text{Card}(C_s) \geq 3$, following the proof of Theorem 1.1 (similar to Proposition 5.1 in Shao [14], see Appendix), one can prove that under the condition of Theorem 1.1, for any $0 < \epsilon < 1/2$, there exists $\tau > 1$ such that

$$(4.13) \quad P\left(\max_{n \leq k \leq \tau n} \frac{S_k}{V_k} \geq x_n\right) \leq \exp\left(- (1 - \epsilon)t_0 x_n^2\right).$$

The rest of the proof is similar to that of Theorem 5.1 in Shao [14] by the subsequence method. The details are omitted here.

4.3. Auxiliary lemmas.

Lemma 4.1. Write $Z_n = 2b_n X - b_n^2 X^2$. Then

$$(4.14) \quad \liminf_{n \rightarrow \infty} \frac{E Z_n^2 e^{t_n Z_n}}{x_n^2/n} > c_0 > 0,$$

where c_0 is some constant.

Proof. First, we show that for arbitrary $\epsilon_2 > 0$,

$$(4.15) \quad \liminf_{n \rightarrow \infty} \frac{E b_n^2 X^2 I(|b_n X| \leq \epsilon_2)}{x_n^2/n} = c_2 > 0.$$

Otherwise, there exists $\epsilon_3 > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{E b_n^2 X^2 I(|b_n X| \leq \epsilon_3)}{x_n^2/n} = 0.$$

Without loss of generality, we can assume

$$\lim_{n \rightarrow \infty} \frac{E b_n^2 X^2 I(|b_n X| \leq \epsilon_3)}{x_n^2/n} = 0.$$

Since $X \in \mathcal{F}_\theta$, we have

$$\lim_{n \rightarrow \infty} \frac{P(|b_n X| > \epsilon_3)}{x_n^2/n} = 0.$$

Hence

$$(4.16) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{E (b_n X)^2 e^{t_n Z_n} I(|b_n X| > \epsilon_3)}{x_n^2/n} \\ & \leq \lim_{n \rightarrow \infty} \frac{c_{2n} P(|b_n X| > \epsilon_3)}{x_n^2/n} \\ & = 0, \end{aligned}$$

where $c_{2n} = \sup_{|y| > \epsilon_3} y^2 \exp\{t_n(2y - y^2)\}$, whose limit is positive by Proposition 3.2 as $n \rightarrow \infty$. Combining (4.16) with (3.9), we get for n sufficiently large,

$$\begin{aligned} E b_n^2 X^2 I(|b_n X| \leq \epsilon_3) & \geq e^{-t_n(2\epsilon_3 + \epsilon_3^2)} E (b_n X)^2 e^{t_n Z_n} I(|b_n X| \leq \epsilon_3) \\ & \geq \frac{1}{2} \frac{x_n^2}{n}, \end{aligned}$$

which contradicts our assumption. Therefore (4.15) holds. Now

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \frac{EZ_n^2 e^{t_n Z_n}}{x_n^2/n} \\
 & \geq \liminf_{n \rightarrow \infty} \frac{EZ_n^2 e^{t_n Z_n}}{x_n^2/n} I(|b_n X| \leq \epsilon_2) \\
 & \geq \liminf_{n \rightarrow \infty} (2 - \epsilon_2)^2 \exp \{ -t_n(2\epsilon_3 + \epsilon_3^2) \} \frac{Eb_n^2 X^2 I(|b_n X| \leq \epsilon_2)}{x_n^2/n} \\
 & = (2 - \epsilon_2)^2 \exp \{ -t_0(2\epsilon_2 + \epsilon_2^2) \} c_2 \\
 & > 0,
 \end{aligned}$$

which is just (4.14). □

Lemma 4.2.

$$\sum_{i=1}^n (\eta_i - E\eta_i) / \sqrt{n\sigma^2(t_n)} \rightarrow N(0, 1)$$

in distribution as $n \rightarrow \infty$.

Proof. It suffices to show the Lindeberg condition

$$\frac{EZ_n^2 e^{t_n Z_n} I(|Z_n| \geq \delta \sqrt{n}\sigma(t_n))}{\sigma^2(t_n)} \rightarrow 0$$

holds as $n \rightarrow \infty$ for arbitrary $\delta > 0$.

Lemma 4.1 yields $\sqrt{n}\sigma(t_n) \rightarrow \infty$ as $n \rightarrow \infty$. So

$$\begin{aligned}
 & \sup_y (2b_n y - b_n^2 y^2)^2 \exp \{ t_n(2b_n y - b_n^2 y^2) \} I(|2b_n y - b_n^2 y^2| \geq \delta \sqrt{n}\sigma(t_n)) \\
 & = \delta^2 n \sigma^2(t_n) \exp \{ -t_n \delta \sqrt{n}\sigma(t_n) \}.
 \end{aligned}$$

Write $\delta_n = t_n \delta \sqrt{n}\sigma(t_n)$. Hence for n sufficiently large,

$$\begin{aligned}
 & \frac{EZ_n^2 e^{t_n Z_n} I(|Z_n| \geq \delta \sqrt{n}\sigma(t_n))}{\sigma^2(t_n)} \\
 & \leq \delta^2 n e^{-\delta_n} P(|Z_n| \geq \delta \sqrt{n}\sigma(t_n)) \\
 & \leq \delta^2 n e^{-\delta_n} P(|b_n X| \geq 1) \\
 (4.17) \quad & \leq \delta^2 n e^{-\delta_n} c_3 \frac{x_n^2}{n} \\
 & \leq c_3 \delta^2 x_n^2 \exp \{ -t_n \delta \sqrt{c_0} \sqrt{n} x_n / 2 \} \\
 & \rightarrow 0,
 \end{aligned}$$

where (4.17) is due to the Feller class assumption and the saddlepoint equations, and the last inequality follows from Lemma 4.1. The proof of asymptotic normality is complete. □

5. APPENDIX

Proof of Lemma 2.1. It suffices to show that $g(t, b; x)$ is strictly increasing in t , either

$$(I) \text{ for } t \in (-\infty, 0]; \quad \text{or} \quad (II) \text{ for } t \in (0, \epsilon_1] \text{ for some } \epsilon_1 > 0.$$

We prove (I) first. Let $Z = 2bX - b^2X^2$. For arbitrary t and t_1 such that $t_1 < t \leq 0$, we need to show that $g(t_1, b; x) < g(t, b; x)$. If $Ee^{t_1Z} = \infty$, then $g(t_1, b; x) = t_1x - \ln Ee^{t_1Z} = -\infty$, in which case (I) follows straightaway. Now assume that $Ee^{t_1Z} < \infty$ below, which implies that moments of X of all orders exist. Thus, $g(t, b; x)$ is differentiable in t for $t \in (t_1, \infty)$. Taking derivatives gives

$$(A.1) \quad \frac{\partial g(t, b; x)}{\partial t} = x - \frac{EZe^{tZ}}{Ee^{tZ}}.$$

Observe that

$$\left. \frac{\partial g(t, b; x)}{\partial t} \right|_{t=0} = x + b^2EX^2 > 0$$

and

$$(A.2) \quad \frac{\partial^2 g(t, b; x)}{\partial t^2} = - \left(\frac{EZ^2e^{tZ}}{Ee^{tZ}} - \left(\frac{EZe^{tZ}}{Ee^{tZ}} \right)^2 \right) < 0,$$

since $Z = 2bX + b^2X^2$ is nondegenerate by the assumption that $\text{Card}(C_s) \geq 3$ and $EX = 0$ or $EX^2 = \infty$. Thus, $\frac{\partial g(t, b; x)}{\partial t} > 0$ when $t \in (t_1, 0]$. So $g(t, b; x)$ is strictly increasing in $(t_1, 0]$. Since t_1 is arbitrary, we have hence proved (I).

We prove (II) next. If there exists some $t_2 < 0$ such that $Ee^{t_2Z} < \infty$, then (II) follows from the fact that $\frac{\partial g(0, b; x)}{\partial t} = x + b^2EX^2 > 0$. It remains to prove (II) under the condition that

$$Ee^{t_3Z} = \infty, \quad \text{for all } t_3 < 0.$$

To show this, we choose an arbitrary $t > 0$. Then from (A.1), we have

$$(A.3) \quad \frac{\partial g(t, b; x)}{\partial t} = x - \frac{\int_{-\infty}^{\infty} (2by - b^2y^2)e^{-t(by-1)^2} dF(y)}{\int_{-\infty}^{\infty} e^{-t(by-1)^2} dF(y)}.$$

The monotone convergence theorem implies

$$(A.4) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} e^{-t(by-1)^2} dF(y) = 1,$$

$$(A.5) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} y^2 e^{-t(by-1)^2} dF(y) = EX^2 \quad (\text{it could be } \infty),$$

where $t \rightarrow 0^+$ means that $t \rightarrow 0$ from the right side of 0.

If $EX^2 = \infty$, then

$$(A.6) \quad \begin{aligned} & \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} (2by - b^2y^2) e^{-t(by-1)^2} dF(y) \\ & \leq \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} \left(4 + \frac{b^2y^2}{4} - b^2y^2 \right) e^{-t(by-1)^2} dF(y) \\ & = 4 - \frac{3}{4}b^2EX^2 \\ & = -\infty. \end{aligned}$$

If $EX^2 < \infty$, then noting $|ye^{-t(by-1)^2}| \leq |y|$ for $t > 0$, we can use Lebesgue's dominated convergence theorem to get

$$(A.7) \quad \lim_{t \rightarrow 0^+} \int_{-\infty}^{\infty} (2by) e^{-t(by-1)^2} dF(y) = 2bEX = 0.$$

Combining (A.3)-(A.7) gives

$$\lim_{t \rightarrow 0^+} \frac{\partial g(t, b; x)}{\partial t} > 0.$$

Noting that $g(t, b; x)$ is right continuous at $t = 0$, we conclude (II). \square

Proof of Equation (4.13). Let $\eta = \varepsilon/20$. Clearly,

$$(A.8) \quad P\left(\max_{n \leq k \leq \tau n} \frac{S_k}{V_k} \geq x_n\right) \\ \leq P\left(\frac{S_n}{V_n} \geq (1 - 5\eta)x_n\right) + P\left(\max_{n < k \leq \tau n} \frac{S_k - S_n}{V_k} \geq 5\eta x_n\right).$$

By Theorem 1.1, we have

$$(A.9) \quad P\left(\frac{S_n}{V_n} \geq (1 - 5\eta)x_n\right) \leq \exp(-(1 - \varepsilon/2)t_0 x_n^2),$$

provided that n is sufficiently large.

Below we estimate the second term on the right hand side of (A.8). Similar to (4.3) and (4.4) in Shao [14], let

$$(A.10) \quad l(x) = EX^2 I\{|X| \leq x\}, \quad b = \inf\{x \geq 1 : l(x) > 0\}, \\ z_n = \inf\left\{s : s \geq b + 1, \frac{l(s)}{s^2} \leq \frac{16(t_0 + t_0^2)x_n^2}{\eta^2 n}\right\}.$$

By an elementary argument and the assumption that $x_n^2 = o(n)$, it is plain to see that

$$(A.11) \quad z_n \rightarrow \infty \quad \text{and} \quad n l(z_n) = 16(t_0 + t_0^2)x_n^2 z_n^2 / \eta^2 \quad \text{for every } n \text{ sufficiently large.}$$

Since X is in the centered Feller class, there exists $C > 0$ such that for all $x > b + 1$,

$$(A.12) \quad P(|X| > x) \leq Cx^{-2}l(x)$$

and

$$(A.13) \quad |EXI(|X| \leq x)| \leq Cx^{-1}l(x).$$

Write

$$\begin{aligned}
& P\left(\max_{n < k \leq \tau n} \frac{S_k - S_n}{V_k} \geq 5\eta x_n\right) \\
& \leq P\left(\max_{n < k \leq \tau n} \frac{\sum_{i=n+1}^k X_i I\{|X_i| \leq z_n\}}{V_k} \geq 4\eta x_n\right) \\
& \quad + P\left(\max_{n < k \leq \tau n} \frac{\sum_{i=n+1}^k |X_i| I\{|X_i| \geq z_n\}}{V_k} \geq \eta x_n\right) \\
& \leq P\left(\max_{n < k \leq \tau n} \sum_{i=n+1}^k X_i I\{|X_i| \leq z_n\} \geq 2\eta x_n \sqrt{nl(z_n)}\right) \\
& \quad + P(V_n \leq \sqrt{nl(z_n)}/2) + P\left(\sum_{i=n+1}^{\lceil \tau n \rceil} I\{|X_i| \geq z_n\} \geq (\eta x_n)^2\right).
\end{aligned}$$

Note that for independent nonnegative random variables $\{\xi_i\}$ with finite second moments, we have (see, e.g., Lemma 2.1 of Einmahl and Mason [5])

$$(A.14) \quad P\left(\sum_{i=1}^n \xi_i \leq x\right) \leq \exp\left(-\frac{(\mu - x)^2}{2\sigma^2}\right)$$

for $0 < x < \mu$, where $\mu = \sum_{i=1}^n E\xi_i$ and $\sigma^2 = \sum_{i=1}^n E\xi_i^2$. Using (A.14), we have

$$\begin{aligned}
& P(V_n \leq \sqrt{nl(z_n)}/2) \\
& \leq P\left(\sum_{i=1}^n X_i^2 I(|X_i| \leq z_n) \leq nl(z_n)/4\right) \\
& \leq \exp\left(-\frac{(0.75nl(z_n))^2}{2nEX^4I(|X| \leq z_n)}\right) \\
& \leq \exp\left(-\frac{0.75^2nl(z_n)}{2z_n^2}\right) \\
& \leq \exp(-4.5t_0x_n^2) \quad \text{by (A.11)}.
\end{aligned}$$

It is known that for a binomial random variable $B(n, p)$ with parameters n and p (see, e.g., [Dudley [4], p.16])

$$P(B(n, p) > an) \leq \left(\frac{ep}{a}\right)^{an} \quad \text{for } a > 0.$$

Hence we have

$$\begin{aligned}
 & P\left(\sum_{i=n+1}^{[\tau n]} I\{|X_i| \geq z_n\} \geq (\eta x_n)^2\right) \\
 & \leq \left(\frac{3(\tau-1)nP(|X| \geq z_n)}{(\eta x_n)^2}\right)^{(\eta x_n)^2} \\
 & \leq \left(\frac{3(\tau-1)nCz_n^{-2}l(z_n)}{(\eta x_n)^2}\right)^{(\eta x_n)^2} \quad \text{by (A.12)} \\
 & \leq (3(\tau-1)16(t_0+t_0^2)/\eta^2)^{(\eta x_n)^2} \quad \text{by (A.11)} \\
 & \leq \exp(-2t_0x_n^2),
 \end{aligned}$$

as long as τ is very close to one.

By (A.13) and (A.11), if $\tau-1 > 0$ is chosen to be sufficiently small,

$$\sum_{i=n+1}^{[\tau n]} |EX_i I\{|X_i| \leq z_n\}| \leq C(\tau-1)nl(z_n)/z_n \leq \frac{1}{2}\eta x_n \sqrt{nl(z_n)}$$

and

$$\sum_{i=n+1}^{[\tau n]} \text{Var}X_i I\{|X_i| \leq z_n\} \leq (\tau-1)nl(z_n) \leq \frac{1}{8}\eta^2 x_n^2 l(z_n).$$

Therefore, by the Ottaviani maximum inequality and the Bernstein exponential inequality,

$$\begin{aligned}
 & P\left(\max_{n < k \leq \tau n} \sum_{i=n+1}^k X_i I\{|X_i| \leq z_n\} \geq 2\eta x_n \sqrt{nl(z_n)}\right) \\
 & \leq 2P\left(\sum_{i=n+1}^{[\tau n]} X_i I\{|X_i| \leq z_n\} - EX_i I\{|X_i| \leq z_n\} \geq \eta x_n \sqrt{nl(z_n)}\right) \\
 & \leq 2\exp\left(-\frac{\eta^2 x_n^2 nl(z_n)}{2((\tau-1)nl(z_n) + \eta x_n \sqrt{nl(z_n)}z_n)}\right) \\
 & \leq 2\exp\left(-\frac{\eta^2 x_n^2 nl(z_n)}{4(\tau-1)nl(z_n)}\right) + 2\exp\left(-\frac{\eta x_n \sqrt{nl(z_n)}}{4z_n}\right) \\
 & \leq 2\exp(-\eta^2 x_n^2/(4(\tau-1))) + \exp(-4t_0x_n^2) \quad \text{by (A.11)} \\
 & \leq 4\exp(-4t_0x_n^2)
 \end{aligned}$$

provided $(\tau-1) > 0$ is sufficiently small. Putting together the above inequalities yields

$$(A.15) \quad P\left(\max_{n < k \leq \tau n} \frac{S_k - S_n}{V_k} \geq 5\eta x_n\right) \leq 6\exp(-2t_0x_n^2).$$

This proves (4.13), by (A.9), (A.8) and (A.15). \square

ACKNOWLEDGEMENT

We thank an anonymous referee for insightful comments.

REFERENCES

1. V. Bentkus and F. Götze. The Berry-Esseen bound for Student's statistic. *Ann. Probab.* **24** (1996), 491-503. MR1387647 (97f:62021)
2. M. Csörgő, B. Szyszkowicz and Q. Wang. Darling–Erdős theorem for self-normalized sums. *Ann. Probab.* **31** (2003), 676-692. MR1964945 (2004a:60051)
3. M. Csörgő, B. Szyszkowicz and Q. Wang. Donsker's theorem for self-normalized partial sums processes. *Ann. Probab.* **31** (2003), 1228-1240. MR1988470 (2004h:60031)
4. R.M. Dudley. *Uniform Central Limit Theorems*. Cambridge Studies in Advanced Mathematics, Vol. **63**. Cambridge University Press, Cambridge, 1999. MR1720712 (2000k:60050)
5. U. Einmahl and D.M. Mason. Some universal results on the behavior of increments of partial sums. *Ann. Probab.* **24** (1996), 1388-1407. MR1411499 (97m:60034)
6. E. Giné, F. Götze and D. M. Mason. When is the Student t -statistic asymptotically standard normal? *Ann. Probab.* **25** (1997), 1514-1531. MR1457629 (98j:60033)
7. E. Giné and D. M. Mason. On the LIL for self-normalized sums of IID random variables. *J. Theoret. Probab.* **11** (1998), 351-370. MR1622575 (99e:60082)
8. F. Götze and G. P. Chistyakov. Limit distributions of studentized means. *Ann. Probab.* **32** (2004), 28-77. MR2040775 (2005f:60055)
9. P. S. Griffin. Tightness of the Student t -statistic. *Electron. Comm. Probab.* **7** (2002), 171-180. MR1937903 (2003i:60037)
10. P. S. Griffin and J. Kuelbs. Self-normalized laws of the iterated logarithm. *Ann. Probab.* **17** (1989), 1571-1601. MR1048947 (91k:60036)
11. B.-Y. Jing, Q. M. Shao and Q. Wang. Self-normalized Cramér-type large deviations for independent random variables. *Ann. Probab.* **31** (2003), 2167–2215. MR2016616 (2004k:60069)
12. B.-Y. Jing, Q. M. Shao and W. Zhou. Saddlepoint approximation for Student's t -statistic with no moment conditions. *Ann. Statist.* **32** (2004), 2679-2711. MR2153999 (2006a:62030)
13. B. F. Logan, C. L. Mallows, S. O. Rice and L. A. Shepp. Limit distributions of self-normalized sums. *Ann. Probab.* **1** (1973), 788-809. MR0362449 (50:14890)
14. Q. M. Shao. Self-normalized large deviations. *Ann. Probab.* **25** (1997), 285-328. MR1428510 (98b:60056)
15. Q. M. Shao. Recent developments in self-normalized limit theorems. In *Asymptotic Methods in Probability and Statistics* (editor B. Szyszkowicz), pp. 467 - 480. Elsevier Science, 1998. MR1661499 (99j:60063)
16. Q. M. Shao. Recent progress on self-normalized limit theorems. *Probability, Finance and Insurance*. Edited by T.L. Lai, H. Yang, and S.P. Yung, World Scientific, 2004. MR2189198 (2006i:60056)

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,
HONG KONG

E-mail address: `majing@ust.hk`

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,
HONG KONG – AND – DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON
97403

E-mail address: `maqmshao@ust.hk`

DEPARTMENT OF STATISTICS AND APPLIED PROBABILITY, NATIONAL UNIVERSITY OF SINGAPORE,
SINGAPORE 117546

E-mail address: `stazw@nus.edu.sg`