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A NEW CONSTRUCTION OF 6-MANIFOLDS

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Abstract. This paper provides a topological method to construct all simply-connected, spin, smooth 6-manifolds with torsion-free homology using simply-connected, smooth 4-manifolds as building blocks. We explicitly determine the invariants that classify these 6-manifolds from the intersection form and specific homology classes of the 4-manifold building blocks.

0. Introduction

The goal of this paper is to provide an explicit construction of all smooth, closed, simply-connected, and spin 6-dimensional manifolds with torsion-free homology. A spin manifold is an oriented manifold such that the second Stiefel-Whitney class, w_2, of the tangent bundle is zero. We construct these 6-dimensional manifolds using a plumbing construction on 2-disk bundles over a carefully chosen collection of smooth, simply-connected 4-dimensional manifolds. We then explicitly relate specific characteristic classes and invariants of these 4-manifolds with the invariants that classify the 6-manifold. In Section 2, we provide the existence results for the 4-manifolds that we use in the constructions.

Manifolds of dimension 6 have been completely classified by C. T. C. Wall [Wa-66] using standard algebraic topological invariants.

Theorem 0.1 (Wall). *Orientation-preserving diffeomorphism classes of simply-connected, smooth, spin, closed 6-manifolds M with torsion-free (co)homology correspond bijectively to isomorphism classes of systems of invariants consisting of*

(1) *a free Abelian group H $(= H^2(M;\mathbb{Z}))$,*
(2) *a symmetric trilinear map $\mu : H \times H \times H \to Z$ defined by $\mu(x,y,z) = x \cup y \cup z[M]$ satisfying $\mu(x,x,y) \equiv \mu(x,y,y) \pmod 2$ for all $x, y \in H$,*
(3) *a homomorphism $p_1 : H \to Z$ such that $p_1(x) \equiv 4\mu(x,x,x) \pmod{24}$ for all $x \in H$,*
(4) *a nonnegative integer $r = b_3/2$.*

Wall proved Theorem 0.1 by using surgerical methods and homotopy information associated with these surgeries. In [Wa-66], Wall also proves that every 6-manifold M is diffeomorphic to $M_0 \#_{b_3/2} S^3 \times S^3$ with M_0 having the same invariants as M

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but with $r = 0$. The manifold M_0 is called the core of M. Therefore, without loss of generality, our constructions deal with those 6-manifolds with $r = 0$.

An interesting family of simply-connected, spin 6-manifolds are homotopy complex projective 3-spaces. 6-manifolds which are homotopy equivalent to $\mathbb{C}P^3$ are called homotopy complex projective 3-spaces and each of them is generically denoted by $H\mathbb{C}P^3$. Their cohomology rings are isomorphic to that of $\mathbb{C}P^3$. The diffeomorphism types of $H\mathbb{C}P^3$'s are distinguished by their first Pontrjagin classes and they are in one to one correspondence with the set of integers. In [MY-66], Montgomery and Yang give two characterizations of these manifolds. First they show that the diffeomorphism types of $H\mathbb{C}P^3$'s are in one to one correspondence with free differentiable actions of S^1 on a homotopy S^7. It is also shown that smooth S^1-actions on a homotopy S^7 which have an S^3 as the fixed point set are in one to one correspondence with the isotopy classes of pairs (S^6, K) where K is an embedding of S^3 into S^6. These are exactly the smooth 3-knots in S^6 (or so-called Haefliger knots; see [Ha-62] and [Ha-66]). The set of Haefliger knots is also in one to one correspondence with the integers. The second characterization of $H\mathbb{C}P^3$'s is based on a surgery on these knots. If we remove a neighborhood $B^3 \times K^3$ of K from S^6, the remaining part, the knot complement, is diffeomorphic to $S^2 \times B^4$. Each $H\mathbb{C}P^3$ can be formed by taking out a neighborhood $B^3 \times S^3$ of the knot from S^6 and attaching an $S^2 \times B^4$ to the knot complement. The attaching map f from $S^2 \times S^3$ to itself is different from the ordinary one, otherwise the space after the attachment would be $S^2 \times S^4$. Montgomery and Yang do not give an explicit formula for f except for $\mathbb{C}P^3$, but they point out that the action of f on the third homotopy group of $S^2 \times S^3$ must satisfy two conditions. Let $\rho_1 : S^2 \times S^3 \to S^2$ and $\rho_2 : S^2 \times S^3 \to S^3$ be the projection maps of $S^2 \times S^3$ to S^2 and S^3, respectively. Let $\varphi : S^3 \to S^2 \times S^3$ be the inclusion map. Then the self-diffeomorphism f of $S^2 \times S^3$ must satisfy the following two conditions:

(1) $\rho_1 \circ f \circ \varphi : S^3 \to S^2$ represents a generator of $\pi_3(S^2)$ (i.e. $\rho_1 \circ f \circ \varphi$ is the Hopf fibration).

(2) $\rho_2 \circ f \circ \varphi : S^3 \to S^3$ is of degree -1.

In Section 1, we give a new way to construct $H\mathbb{C}P^3$'s. This construction is a model for the later constructions of all 6-manifolds. We take some 2-disk bundles on the 4-manifold and then close the boundary to get the $H\mathbb{C}P^3$.

When the second Betti number is greater than one, we need a surgery method which is called plumbing. Given two spheres Σ_1 and Σ_2, let N_i be the total space of the direct sum of two 2-disk bundles η_{i1} and η_{i2} over Σ_i. The plumbing on Σ_1 and Σ_2 is done by gluing N_1 and N_2 under a map that takes Σ_1 to Σ_2 and exchanges the factors of the direct sum pointwise. If Σ_1 and Σ_2 are embedded in 6-manifolds M_1 and M_2, respectively, then plumbing of M_1 and M_2 on the spheres Σ_1 and Σ_2 is done by identifying the normal neighborhoods of the spheres in the respective manifold. A more detailed description is given in Section 1. Now let us state the main theorem of the paper.

Main Theorem. *Let V be a smooth, closed, simply-connected, spin 6-manifold with torsion-free homology and $b_3(V) = 0$. Suppose that $H^2(V; \mathbb{Z})$ is isomorphic to the direct sum of n copies of $\mathbb{Z}$, each of which is generated by x_i $(1 \le i \le n)$. Also suppose that $p_1(V)x_i = 24k_i + 4\mu(x_i, x_i, x_i)$, where p_1 is the first Pontrjagin class of V and μ is the symmetric trilinear form of V. Then we can find a collection,*

$\{X_i\}$, of smooth, closed, simply-connected 4-manifolds with odd intersection forms Q_i and second cohomology classes $\alpha_{ij} \in H^2(X_i;\mathbb{Z})$ $(1 \leq i, j \leq n)$ satisfying

(1) *the signature of X_i is $8k_i + \mu(x_i, x_i, x_i)$,*
(2) *α_{ii} are primitive, characteristic and $Q_i(\alpha_{ii}, \alpha_{ii}) = \mu(x_i, x_i, x_i)$,*
(3) *if $i \neq j$, then α_{ij} has a smooth sphere representative in X_i, ¡*
(4) *$Q_i(\alpha_{ij}, \alpha_{ik}) = \mu(x_i, x_j, x_k)$,*

so that the manifold M that is constructed by closing the boundaries of the plumbed 2-disk bundles over these 4-manifolds X_i with Euler class α_{ii} is diffeomorphic to V. The plumbing of the respective bundles over X_i is done on the sphere representatives of α_{ij} in X_i and α_{ji} in X_j.

Note that $\alpha \in H^2(X;\mathbb{Z})$ is characteristic if and only if $Q(\alpha, \beta) \equiv Q(\beta, \beta)$ for all $\beta \in H^2(X;\mathbb{Z})$. If $H^2(X;\mathbb{Z})$ has no torsion, α is primitive if and only if the subspace of $H^2(X;\mathbb{Z})$ obtained by modding out the subspace generated by α has no torsion and its rank is $b_2(X) - 1$.

The motivation behind these constructions is to study the symplectic structures on the 6-manifolds and relate them to the smooth or symplectic topology of the 4-manifolds. As a starting point, we observe that while all the exotic smooth structures on a given 4-manifold vanish when crossed with the 2-sphere S^2, it still may be the case that the symplectic exoticness is retained. This is hinted at in the early work of Ruan [Ru-94].

There was not much known about 6-dimensional symplectic topology until now. There is no general method to distinguish the symplectic structures on a smooth 6-manifold. Also it seems hard to decide whether a smooth 6-manifold is symplectic or not. In particular, despite their fairly simple topology, it is unknown which of the $H\mathbb{C}P^3$'s possess a symplectic structure, other than $\mathbb{C}P^3$ itself. With the construction given in this paper, this problem is replaced with a problem of symplectic surgery. In order to use the results of this paper to study the symplectic structures on the 6-manifolds considered in Subsections 1.2 and 1.3, first one must show that plumbing is a symplectic operation. Moreover, we must know which of the 4-manifolds that we use as the building blocks can be chosen to be symplectic. Before we start building the manifolds, we state a theorem about certain 5-manifolds that appear as the boundaries of the 6-manifolds in the intermediate steps of our constructions.

Theorem 0.2 (Duan-Liang ([DL-05])). *Assume that X is a closed, simply-connected, smooth 4-manifold and $\alpha \in H^2(X;\mathbb{Z})$ is a primitive, characteristic class. If X_α is the total space of the S^1-bundle over X with Euler class equal to α, then X_α is diffeomorphic to $\#_{b_2(X)-1} S^2 \times S^3$.*

1. The constructions

Our constructions use a certain collection of 4-manifolds as the building blocks. We establish the existence of the appropriate 4-manifolds in Section 2.

1.1. **Spin 6-manifolds with $b_2 = 1$.** Let X be a closed, simply-connected, smooth 4-manifold and let $\alpha \in H^2(X)$ be a primitive, characteristic element. A 2-disk bundle over X is characterized by its Euler class $\alpha \in H^2(X;\mathbb{Z})$. If M_α is the total space of the 2-disk bundle ζ over X with Euler class α, then ∂M_α is a circle bundle over X with its Euler class equal to α. By Theorem 0.2, ∂M_α is diffeomorphic to

$\#_{b_2(X)-1} S^2 \times S^3$. Denote the manifold that is constructed by attaching a $B^3 \times S^3$ to each component in the connected sum by M. More precisely, we get M by capping off the boundary of M_α with $\natural_{b_2(X)-1} B^3 \times S^3$, the boundary sum of $b_2(X)-1$ copies of $B^3 \times S^3$. The invariants of the closed 6-manifold M are given in the following propositions.

Proposition 1.1. *M is simply-connected.*

Proof. The homotopy equivalence between M_α and X and the simply-connectedness of X implies the simply-connectedness of M_α. The boundary of M_α is diffeomorphic to $\#_{b_2(X)-1} S^2 \times S^3$ (Theorem 0.2), which is connected and simply-connected. Notice that $\natural_{b_2(X)-1} B^3 \times S^3$ is also simply-connected. Then by Van-Kampen's theorem, M is simply-connected. □

Proposition 1.2. *For M, $b_0 = b_2 = b_4 = b_6 = 1$ and $b_1 = b_3 = b_5 = 0$.*

Proof. Since M is connected, $H_0(M;\mathbb{Z}) = \mathbb{Z}$. The abelianization of the fundamental group of M is $H_1(M;\mathbb{Z})$. The fundamental group is trivial, so $H_1(M;\mathbb{Z})$ is trivial. The manifold M is closed and oriented, therefore $H_6(M;\mathbb{Z}) = \mathbb{Z}$ and Poincare duality can be applied. This gives $H^5(M;\mathbb{Z}) = 0$. Since there are no torsion elements, $H_5(M;\mathbb{Z}) = 0$ by the Universal Coefficient theorem. Similarly, $b_2(M) = b_4(M)$. Here is a part of the Mayer-Vietoris sequence that applies to the construction:

$$
\begin{aligned}
&0 \to H_4(M_\alpha;\mathbb{Z}) \oplus H_4(\natural_{b_2(X)-1} B^3 \times S^3;\mathbb{Z}) \to H_4(M;\mathbb{Z}) \to H_3(\#_{b_2(X)-1} S^2 \times S^3;\mathbb{Z})\\
&\to H_3(M_\alpha;\mathbb{Z}) \oplus H_3(\natural_{b_2(X)-1} B^3 \times S^3;\mathbb{Z}) \to H_3(M;\mathbb{Z}) \to H_2(\#_{b_2(X)-1} S^2 \times S^3;\mathbb{Z})\\
&\to H_2(M_\alpha;\mathbb{Z}) \oplus H_2(\natural_{b_2(X)-1} B^3 \times S^3;\mathbb{Z}) \to H_2(M;\mathbb{Z}) \to 0.
\end{aligned}
\tag{1}
$$

By the homotopy equivalence of M_α and X, $H_4(M_\alpha;\mathbb{Z}) = \mathbb{Z}$, and by the homotopy equivalence of $\natural_{b_2(X)-1} B^3 \times S^3$ and a wedge of S^3's, $H_4(\natural_{b_2(X)-1} B^3 \times S^3) = 0$. Because of this fact, the map below which is induced by the inclusion map is an isomorphism:

$$H_3(\#_{b_2(X)-1} S^2 \times S^3;\mathbb{Z}) \to H_3(M_\alpha;\mathbb{Z}) \oplus H_3(\natural_{b_2(X)-1} B^3 \times S^3;\mathbb{Z}).$$

Therefore, exact sequence (1) splits into three short exact sequences. The first gives $b_4(M) = 1$ and $b_2(M) = 1$. The second part of the split sequence implies that $H_3(M;\mathbb{Z}) = 0$. □

Proposition 1.3. *$H^2(M;\mathbb{Z}) = \mathbb{Z}$ has a generator whose pullback in X is α.*

Proof. The boundary of M_α is the circle bundle over X with Euler class α. A part of the Gysin sequence for this circle bundle over X is given below:

$$0 \xrightarrow{\pi^*} H^3(\#_{b_2(X)-1} S^2 \times S^3;\mathbb{Z}) \to H^2(X;\mathbb{Z}) \xrightarrow{\cup\alpha} H^4(X;\mathbb{Z}) \xrightarrow{\pi^*} 0. \tag{2}$$

The kernel of $\cup\alpha$ is equal to the image of the second map which is injective. The image is the quotient of $H^2(X;\mathbb{Z})$ by the subspace generated by a class such that intersection of this class with α is one, because $\cup\alpha$ is surjective. The elements in the second homology of $\#_{b_2(X)-1} S^2 \times S^3$, given by the Poincare duals of the third cohomology, are capped with B^3's of $\natural_{b_2(X)-1} B^3 \times S^3$. The integer multiples of α are among the surviving cohomology elements. The injection of X into M induces the pullback of the generator x to X as α. □

Proposition 1.4. *M is spin.*

Proof. It is enough to calculate the value of $w_2(M)$ on the generator $PD(x)$ of $H_2(M;\mathbb{Z}) = \mathbb{Z}$. Since inclusion of $H_2(M_\alpha)$ to $H_2(M)$ is surjective, $w_2(M) = j^*(w_2(M_\alpha))$, where j is the inclusion map of M_α into M. Let i be the inclusion map of X into M_α. Writing $i^*(TM_\alpha) = TX \oplus \zeta$, the Whitney sum formula gives $i^*w_2(M_\alpha) \equiv w_2(X) + \alpha \pmod 2$ in $H^2(X;\mathbb{Z}_2)$. In $H_4(M;\mathbb{Z})$, $PD(x) = i_*PD(\alpha)$. Therefore, $w_2(M)PD(x) \equiv (i^*w_2(M_\alpha))PD(x) \equiv (w_2(X)+\alpha)PD(\alpha) \equiv 0 \bmod 2$. $\square$

Proposition 1.5. *$Q(\alpha,\alpha) = \mu(x,x,x)$, where x is the generator of $H^2(M;\mathbb{Z})$.*

Proof. The Poincare dual of $x \in H^2(M;\mathbb{Z})$ is $[i_*X]$, where i is the inclusion map. In $H^2(X;\mathbb{Z})$, $\alpha = i^*x$. $[X] = i^*PD(x)$ in $H_4(X;\mathbb{Z})$. $\mu(x,x,x) = (x \cup x \cup x)[M] = (x\cup x)(x \cap [M]) = (x \cup x)[PD(x)] = (x\cup x)[i_*X] = (i^*\alpha \cup i^*\alpha)[i_*X] = (\alpha\cup\alpha)[X] = Q(\alpha,\alpha)$. $\square$

Proposition 1.6. *$p_1(M) = p_1(X) + \alpha \cup \alpha$, where p_1 is the first Pontrjagin class of M.*

Proof. To see this equality, let's consider the Whitney sum formula for Pontrjagin classes. The first Pontrjagin class of M, $p_1(M)$, is given as the second Chern class $c_2(TM \otimes \mathbb{C})$ of the complexification of the tangent bundle of M. Since there is no contribution to the fourth cohomology of M from $\natural_{b_2(X)-1} B^3 \times S^3$, it is safe to make the calculations in M_α. The Whitney sum formula for the second Chern number of $TM_\alpha \otimes \mathbb{C}$ is $c_2(TM_\alpha \otimes \mathbb{C}) = c_2(TX \otimes \mathbb{C}) + c_1(TX\otimes\mathbb{C})c_1(\nu X|_{M_\alpha} \otimes \mathbb{C}) + c_2(\nu X|_{M_\alpha}\otimes\mathbb{C})$. The term which is the product of the first Chern classes has order two in the cohomology ([MS-75], p. 175). The cohomology of X is torsion-free, thus the term consisting of the product of c_1's is 0. The second Chern class of the complexification of the 2-disk bundle over X with the Euler class α is $\alpha\cup\alpha$ ([MS-75]). Consequently, $p_1(M)x = p_1(M)(PD(x)) = p_1(X)(PD(x)) + \alpha\cup\alpha(PD(x)) = p_1(X)[X] + Q(\alpha,\alpha)$. By the Hirzebruch signature theorem for 4-manifolds, this is equal to $3\sigma(X) + m$. $\sigma(X) = 8k + m$ implies that $p_1(M)x = 24k + 4m$. $\square$

Using the propositions given above, we prove the theorem below. This theorem is a special case of the main theorem where $b_2 = 1$.

Theorem 1.7. *Let V be any closed, simply-connected, spin 6-manifold with torsion-free homology and $b_3(V) = 0$. Assume that $H^2(V;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ which is generated by x. Also assume that $p_1(V)x = 24k + 4\mu(x,x,x)$, where p_1 is the first Pontrjagin class of V and μ is the symmetric trilinear form of V. Then we can find a smooth, closed, simply-connected 4-manifold X with an odd intersection form Q and a second cohomology class $\alpha \in H^2(X;\mathbb{Z})$ satisfying*

(1) *the signature of X is $8k + \mu(x,x,x)$,*
(2) *α is primitive, characteristic and $Q(\alpha,\alpha) = \mu(x,x,x)$,*

so that the manifold M that is constructed by the method described above is diffeomorphic to V.

Proof. The existence of an appropriate 4-manifold is established below in Lemma 2.1. Hence, to prove the theorem, we must show that the constructed manifold M has the same invariants as V.

The manifold M is simply connected by Proposition 1.1. The manifold M is spin by Proposition 1.4. As shown above in Proposition 1.2, the Betti numbers of M and V are the same. Since $Q(\alpha, \alpha) = \mu(x, x, x)$, cohomology rings are isomorphic by Proposition 1.5. The first Pontrjagin classes of M and V coincide by Proposition 1.6. By Wall's theorem (Theorem 0.1) M is diffeomorphic to V. $\square$

Example 1.8. The family of homotopy complex projective 3-spaces forms an interesting collection of 6-manifolds. Some information about these manifolds can be found in the introduction. $H\mathbb{C}P^3$'s are distinguished by their first Pontrjagin classes and parametrized by $\mathbb{Z}$. Namely, for all $k \in \mathbb{Z}$, the first Pontrjagin class of the corresponding $H\mathbb{C}P^3$ evaluated on the second cohomology element is $24k + 4$. Take X such that $\sigma(X) = 8k + 1$ and $\alpha \in H^2(X; \mathbb{Z})$ such that $\alpha \cup \alpha = 1$. When we apply the construction, we end up with an $H\mathbb{C}P^3$ with its first Pontrjagin class evaluated on x equal to $24k + 4$. For example, if X is $\mathbb{C}P^2$ and α is h where h generates $H^2(\mathbb{C}P^2; \mathbb{Z})$, we get $\mathbb{C}P^3$. Here $b_2 - 1 = 0$, hence M_α is the Hopf fibration over $\mathbb{C}P^2$ and ∂M_α is S^5. Gluing the 6-disk to the boundary gives $\mathbb{C}P^3$.

Example 1.9. The smooth quintic hypersurface Q in $\mathbb{C}P^4$, which is the zero set of a degree 5 polynomial, has $b_3 = 204$. Q_0, the core of Q, has its second cohomology group $H^2(Q_0; \mathbb{Z})$ isomorphic to $\mathbb{Z}$. Let's denote the generator of this group by L. $\mu(L, L, L) = 5$ and $p_1(Q_0)L = -100$. We can construct Q_0 by the construction given above by taking a 4-manifold with signature equal to -35 and α with self intersection 5. For example we can choose X as $\mathbb{C}P^2 \#_{36} \overline{\mathbb{C}P^2}$ and α as $7h + 3e_1 + 3e_2 + \Sigma_3^{36} e_i$. This choice is not unique. In fact there are infinitely many choices. Another choice for X is the degree 5 hypersurface in $\mathbb{C}P^3$ which is diffeomorphic to $\#_9 \mathbb{C}P^2 \#_{44} \overline{\mathbb{C}P^2}$.

1.2. **Spin 6-manifolds with $b_2 = 2$.** In this subsection, we apply a construction similar to that in Subsection 1.1 in order to get the simply-connected, spin, torsion-free, smooth 6-manifolds with $b_2 = 2$ and $b_3 = 0$. Let X_i^4 $(i = 1, 2)$ be closed, simply-connected, smooth 4-manifolds, let Q_i be their respective intersection forms and let $\alpha_{ij} \in H^2(X_i, \mathbb{Z})$ be primitive elements where $j = 1, 2$ and α_{ii} are characteristic. Let M_i be the total space of the 2-disk bundle ζ_i over X_i with Euler class α_{ii}. Suppose that the Poincare dual of each cohomology class, α_{ij} $(i \neq j)$, can be represented by an embedded sphere Σ_{ij} in X_i. Our 6-manifold M is formed by gluing M_1 and M_2 over a neighborhood of the Σ_{ij} in M_i by a diffeomorphism explained below and then capping off the boundary.

Let ν_{ij} be the normal 2-disk bundle of the sphere Σ_{ij} in the 4-manifold X_i, and let ζ_{ij}, with total space N_{ij}, be the restriction of ζ_i to the total space of ν_{ij}. M_1 and M_2 are attached along the total spaces N_{12} and N_{21}. Since N_{ij} is the tubular neighborhood of Σ_{ij} in M_i, it is also the total space of the B^4-bundle η_{ij} over Σ_{ij}.

To glue M_1 and M_2 along N_{12} and N_{21}, we need them to be diffeomorphic. Since $\pi_2(SO(3)) = \mathbb{Z}_2$, there are only two topologically distinct 4-disk bundles over S^2, determined by their second Stiefel-Whitney class. Thus, for the neighborhoods to be diffeomorphic, we need $w_2(\eta_{12}) = w_2(\eta_{21})$. Since the Euler class of ζ_i is α_{ii}, we have that $w_2(\eta_{ij}) = Q_i(\alpha_{ij}, \alpha_{ii}) + Q_i(\alpha_{ij}, \alpha_{ij}) \mod 2$. However, since α_{ii} are characteristic, we have that $w_2(\eta_{ij}) = 0$. This implies that the bundles η_{ij} are trivial and N_{ij} are diffeomorphic to $S^2 \times B^4$.

To perform our plumbing construction, we need a stronger condition. The normal bundle η_{ij} of the sphere Σ_{ij} in M_i can be decomposed as the direct sum of two

2-disk bundles ([Be-96]). The first is the normal bundle ν_{ij} of the sphere Σ_{ij} in the 4-manifold X_i and the second is the restriction of the 2-disk bundle ζ_i over X_i restricted to Σ_{ij}. In other words, $\eta_{ij} = \nu_{ij} \oplus \zeta_i|_{\Sigma_{ij}}$.

In order to mimic the plumbing of 2-disk bundles over surfaces, we glue N_{ij} by a diffeomorphism, switching the first and the second bundles of the decomposition of η_{ij} over Σ_{ij} with the total space N_{ij}. That is, we identify the normal bundle ν_{12} of Σ_{12} in X_1 to the restriction of the 2-disk bundle ζ_2 over X_2 to Σ_{21} and vice versa. All the identifications are made in an orientation-preserving way. In order to do this, however, we need the 2-disk bundles that are identified to be homotopic. This then places two conditions on the cohomology classes α_{ij}. These conditions are $Q_1(\alpha_{11}, \alpha_{12}) = Q_2(\alpha_{21}, \alpha_{21})$ and $Q_1(\alpha_{12}, \alpha_{12}) = Q_2(\alpha_{21}, \alpha_{22})$. Therefore, identifying N_{12} in M_1 and N_{21} in M_2 by pointwise identification of the fibers of ν_{12} with $\zeta_2|_{\Sigma_{21}}$ and ν_{21} with $\zeta_1|_{\Sigma_{12}}$ is a well-defined operation. Let us denote the product manifold by M_{12}.

Proposition 1.10. *M_{12} is simply-connected.*

Proof. The homotopy equivalence between M_i and X_i and the simply-connectedness of X_i implies the simply-connectedness of M_i. Note that $S^2 \times B^4$ is also simply-connected. Since the gluing space for the plumbing operation is path-connected, by Van-Kampen's theorem, M_{12} is simply-connected. □

Proposition 1.11. *For M_{12}, $b_0 = 1$, $b_2 = b_2(X_1) + b_2(X_2) - 1$, $b_4 = 2$ and $b_1 = b_3 = b_5 = b_6 = 0$.*

Proof. We know that M_{12} is connected, so $H_0(M_{12}; \mathbb{Z}) = \mathbb{Z}$. The abelianization of the fundamental group is $H_1(M_{12}; \mathbb{Z})$. The fundamental group is trivial, so $H_1(M_{12}; \mathbb{Z})$ is trivial. Writing the Mayer-Vietoris sequence of this step enables us to determine the Betti numbers:

$$
\begin{aligned}
0 &\to H_6(M_1;\mathbb{Z}) \oplus H_6(M_2;\mathbb{Z}) \to H_6(M_{12};\mathbb{Z}) \to H_5(S^2 \times B^4;\mathbb{Z}) \\
&\to H_5(M_1;\mathbb{Z}) \oplus H_5(M_2;\mathbb{Z}) \to H_5(M_{12};\mathbb{Z}) \to H_4(S^2 \times B^4;\mathbb{Z}) \\
&\to H_4(M_1;\mathbb{Z}) \oplus H_4(M_2;\mathbb{Z}) \to H_4(M_{12};\mathbb{Z}) \to H_3(S^2 \times B^4;\mathbb{Z}) \\
&\to H_3(M_1;\mathbb{Z}) \oplus H_3(M_2;\mathbb{Z}) \to H_3(M_{12};\mathbb{Z}) \to H_2(S^2 \times B^4;\mathbb{Z}) \\
&\to H_2(M_1;\mathbb{Z}) \oplus H_2(M_2;\mathbb{Z}) \to H_2(M_{12};\mathbb{Z}) \to 0.
\end{aligned} \tag{3}
$$

When we place the known values in sequence (3), it is straightforward to get $b_2(M_{12}) = b_2(X_1) + b_2(X_2) - 1$, $b_4(M_{12}) = 2$, $b_0(M_{12}) = 1$ and $b_1(M_{12}) = b_3(M_{12}) = b_5(M_{12}) = b_6(M_{12}) = 0$. □

By Theorem 0.2, ∂M_i is diffeomorphic to $\#_{b_2(X_i)-1} S^2 \times S^3$. We now claim that the boundary of M_{12} is again diffeomorphic to a connected sum of a number of $S^2 \times S^3$'s. More precisely, ∂M_{12} is diffeomorphic to $\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3$. To see this, we write the relative homology sequence for the pair $(M_{12}, \partial M_{12})$:

$$
\begin{aligned}
0 &\to H_5(M_{12};\mathbb{Z}) \to H_5(M_{12}, \partial M_{12};\mathbb{Z}) \to H_4(\partial M_{12};\mathbb{Z}) \\
&\to H_4(M_{12};\mathbb{Z}) \to H_4(M_{12}, \partial M_{12};\mathbb{Z}) \to H_3(\partial M_{12};\mathbb{Z}) \\
&\to H_3(M_{12};\mathbb{Z}) \to H_3(M_{12}, \partial M_{12};\mathbb{Z}) \to H_2(\partial M_{12};\mathbb{Z}) \\
&\to H_2(M_{12};\mathbb{Z}) \to H_2(M_{12}, \partial M_{12};\mathbb{Z}) \to H_1(\partial M_{12};\mathbb{Z}) \\
&\to H_1(M_{12};\mathbb{Z}) \to H_1(M_{12}, \partial M_{12};\mathbb{Z}) \to \widetilde{H}_0(\partial M_{12};\mathbb{Z}) = 0.
\end{aligned} \tag{4}
$$

By Poincare-Lefschetz duality and the Universal Coefficient theorem, we see that $H_2(M_{12}, \partial M_{12})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and $H_3(M_{12}, \partial M_{12}) = H_3(M_{12})$. The fourth homology of M_{12} consists of the manifolds X_1 and X_2, so the map $H_4(\partial M_{12}; \mathbb{Z}) \to H_4(M_{12}; \mathbb{Z})$ is the zero map. Now, using Poincare duality for ∂M_{12}, it is clear that the primitive classes that we are using for the plumbing are reflected as essential second homology classes in the boundary. By using sequence (4), we find the Betti numbers of ∂M_{12} to be $b_0 = b_5 = 1$, $b_1 = b_4 = 0$ and $b_2 = b_3 = b_2(X_1) + b_2(X_2) - 3$. By the classification of simply-connected, spin, smooth 5-manifolds ([Sm-62]), the boundary is diffeomorphic to $\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3$.

Once we know the boundary is diffeomorphic to $\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3$, it is easy to obtain the desired manifold M by capping off the boundary of M_{12} with $\natural_{b_2(X_1)+b_2(X_2)-3} B^3 \times S^3$. The invariants of M are calculated in the following propositions.

Proposition 1.12. *M is simply-connected.*

Proof. The manifold $\natural_{b_2(X_1)+b_2(X_2)-3} B^3 \times S^3$ is simply-connected. By Proposition 1.10, M_{12} is also simply-connected. Since the gluing space for the operation, $\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3$, is path-connected, by Van-Kampen's theorem, M is simply-connected. □

Proposition 1.13. *For M, $b_0 = b_6 = 1$, $b_4 = b_2 = 2$ and $b_1 = b_3 = b_5 = 0$.*

Proof. M is connected, so $H_0(M; \mathbb{Z}) = \mathbb{Z}$. The abelianization of the fundamental group is $H_1(M; \mathbb{Z})$. The fundamental group is trivial, so $H_1(M; \mathbb{Z})$ is trivial. The manifold M is closed oriented, therefore $H_6(M; \mathbb{Z}) = \mathbb{Z}$ and Poincare duality can be applied. This gives $H^5(M; \mathbb{Z}) = 0$. Since there is no torsion element, by the Universal Coefficient theorem, $H_5(M; \mathbb{Z}) = 0$. Similarly, $b_2(M) = b_4(M)$.

The calculation of the remaining Betti numbers of M is done by writing the Mayer-Vietoris sequence of the steps in the construction. There are two steps.

The first step of the construction is attaching M_1 to M_2 by the map explained before Proposition 1.10, the one that plumbs α_{12} to α_{21}. In Proposition 1.11, the Betti numbers of M_{12} are given as $b_2(M_{12}) = b_2(X_1) + b_2(X_2) - 1$, $b_4(M_{12}) = 2$, $b_0(M_{12}) = 1$ and $b_1(M_{12}) = b_3(M_{12}) = b_5(M_{12}) = b_6(M_{12}) = 0$.

The second step is closing the boundary of M_{12}. We calculate the Betti numbers as $b_0(M) = b_6(M) = 1$, $b_1(M) = b_3(M) = b_5(M) = 0$, and $b_2(M) = b_4(M) = 2$ by using the following Mayer-Vietoris sequence:

(5)
$$
\begin{aligned}
0 &\to H_4(M_{12}; \mathbb{Z}) \oplus H_4(\natural_{b_2(X_1)+b_2(X_2)-3} B^3 \times S^3; \mathbb{Z}) \to H_4(M; \mathbb{Z}) \\
&\to H_3(\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3; \mathbb{Z}) \to H_3(M_{12}; \mathbb{Z}) \oplus H_3(\natural_{b_2(X_1)+b_2(X_2)-3} B^3 \times S^3; \mathbb{Z}) \\
&\to H_3(M; \mathbb{Z}) \to H_2(\#_{b_2(X_1)+b_2(X_2)-3} S^2 \times S^3; \mathbb{Z}) \\
&\to H_2(M_{12}; \mathbb{Z}) \oplus H_2(\natural_{b_2(X_1)+b_2(X_2)-3} B^3 \times S^3; \mathbb{Z}) \to H_2(M; \mathbb{Z}) \to 0.
\end{aligned}
$$
□

The next proposition shows that M is spin.

Proposition 1.14. *M is spin.*

Proof. The proof is similar to the proof of Proposition 1.4, but in this case there are two homology classes on which $w_2(M)$ must be evaluated. Both of these cohomology classes are induced by the inclusion of X_i into M. Pulling back everything into the respective X_i, it turns out that $w_2(M)x_i \equiv w_2(X_i)x_i + w_2(\zeta_i)x_i$. Remember that ζ_i is the 2-disk bundle over the 4-manifold X_i. In this sum, $w_2(X_j)x_i$ and $w_2(\zeta_j)x_i$ $(i \neq j)$ are counted due to the fact that X_i overlaps X_j only on α_{ij} which is already included in x_i. Hence $w_2(X_i)x_i$ and $w_2(\zeta_2)x_i$ are already included in $w_2(X_1)x_i$ and $w_2(\zeta_1)x_i$, respectively, and they are omitted. Therefore, $w_2(M)x_i \equiv w_2(X_i)x_i + w_2(\zeta_i)x_i \equiv w_2(X_i)x_i + Q_i(\alpha_{ii}, \alpha_{ii}) \equiv 0$. □

As calculated in Proposition 1.13, $H^2(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This group is generated by the pushforwards of the cohomology elements α_{11} and α_{22} by the inclusion maps. We denote these two generators x_1 and x_2, respectively. The intersection form is determined by the values of $\mu(x_1, x_1, x_1)$, $\mu(x_1, x_1, x_2)$, $\mu(x_1, x_2, x_2)$ and $\mu(x_2, x_2, x_2)$. The embedding map of X_i into M is given by the composition of the inclusion maps of $f_i : X_i \to M_i$ and $g_i : M_{12} \to M$. Using the exact sequence (3) above, we see that the inclusions $H^4(M_i) \to H^4(M_{12})$ are injective. The strategy for calculating the intersection numbers of M is to focus on the intersection of these representatives in M. By transversality, we see that $PD(x_1)$ and $PD(x_2)$ in $H_4(M;\mathbb{Z})$ intersect each other (and themselves) in a 2-dimensional subspace.

Note that the representatives of the Poincare duals to the second cohomology classes x_1 and x_2 are the embedded copies of the 4-manifolds X_1 and X_2, respectively. This can be seen from the following argument. The Poincare dual of x_i, $PD(x_1)$, is in $H_4(M;\mathbb{Z})$. From the homology calculations above, $H_4(M;\mathbb{Z})$ is generated by X_1 and X_2. This means that $PD(x_1)$ is a linear combination of X_1 and X_2, say $sX_1 + tX_2$. The normal bundle of X_1 in M has the Euler class α_{11}, which lies completely in X_1. The second cohomology class x_1 is the image of α_{11} in M, therefore the evaluation of x_1 on X_2 is restricted to $X_1 \pitchfork X_2$ (the transversal intersection of X_1 and X_2 in M). This evaluation occurs within X_1, hence $t = 0$. Since $PD(x_1)$ is primitive (due to the fact that x_1 is primitive in the second cohomology), $s = 1$ and $PD(x_1) = X_1$. Similarly, $PD(x_2) = X_2$.

Our construction clearly gives that $PD(x_1) \pitchfork PD(x_2) = X_1 \pitchfork X_2$ is the representing (smooth) surface of α_{12} in X_1 and the representing (smooth) surface of α_{21} in X_2. Moreover, X_i has the normal bundle with the Euler class α_{ii} in M. As one of the consequences of the Thom isomorphism theorem ([BT-82], [Br-93], p. 382), X_i intersects itself on the surface representing α_{ii}. From these considerations, we may conclude the following proposition.

Proposition 1.15. *Let M be the 6-manifold constructed above with $b_2(M) = 2$. Suppose that $H^2(M;\mathbb{Z})$ is generated by x_1 and x_2. Then $\mu(x_1, x_1, x_1) = Q_1(\alpha_{11}, \alpha_{11})$, $\mu(x_1, x_1, x_2) = Q_1(\alpha_{11}, \alpha_{12})$, $\mu(x_1, x_2, x_2) = Q_2(\alpha_{21}, \alpha_{22})$ and $\mu(x_2, x_2, x_2) = Q_2(\alpha_{22}, \alpha_{22})$.*

Proof. (1) $\mu(x_i, x_i, x_i) = PD(x_i) \pitchfork PD(x_i) \pitchfork PD(x_i) = X_i \pitchfork X_i \pitchfork X_i$. The last X_i intersects each of the other two on α_{ii}; therefore the intersection of the three is equal to $Q_1(\alpha_{ii}, \alpha_{ii})$.

(2) $\mu(x_i, x_i, x_j) = PD(x_i) \pitchfork PD(x_i) \pitchfork PD(x_j) = X_i \pitchfork X_i \pitchfork X_j$ $(i \neq j)$. The first X_i intersects the second X_i on α_{ii} and intersects X_j on α_{ij}. As a result, these three intersect each other on the projection of α_{ij} on α_{ii} in X_i, which is nothing but $Q_i(\alpha_{ii}, \alpha_{ij})$. □

Since μ is a symmetric trilinear form, this proposition determines it completely ([OV-95]). The last piece of information we need to know about M is the linear form on the second cohomology (i.e. the first Pontrjagin class evaluated on the second cohomology elements) which we calculate now.

Proposition 1.16. $p_1(M)x_i = 3\sigma(X_i) + Q_i(\alpha_{ii}, \alpha_{ii})$.

Proof. We may write $p_1(M)x_i = p_1(M)PD(x_i) = p_1(M)X_i$. The tangent bundle of M is restricted to X_i as the direct sum of the tangent bundle TX of X and the normal neighborhood $\nu X_i|_M$ of X_i in M. The bundle $\nu X_i|_M$ is the B^2-bundle ζ_i over X_i with Euler class α_{ii}. Consequently, as we have seen in Subsection 1.1, $p_1(M)X_i = p_1(X_i)[X_i] + \alpha_{ii} \cup \alpha_{ii}[X_i] = 3\sigma(X_i) + Q_i(\alpha_{ii}, \alpha_{ii})$. □

1.3. **Spin 6-manifolds with $b_2 = n$.** Using the methods of the previous two subsections, it is possible to construct all simply-connected, spin, torsion-free, smooth 6-manifolds with arbitrary second Betti number and arbitrary trilinear form. (Note: There are some admissibility conditions for the trilinear forms to be realized by 6-manifolds. Here, we are not going to study them. See [OV-95] and [Sc-97].) Once we choose our 4-manifolds with the suitable second homology groups and the intersection forms, whose existence is guaranteed by Lemma 2.2, the construction is quite similar to the $b_2 = 2$ case. However, not all the manifolds in the intermediate steps are simply-connected. There are some 1-dimensional homology elements that are created in the process, so we focus on the formation of these elements.

To construct a 6-manifold M with $b_2(M) = n$, first we take n 4-manifolds X_i and n primitive characteristic second cohomology classes $\alpha_{ii} \in H^2(X_i; \mathbb{Z})$. We also take primitive second cohomology classes $\alpha_{ij} \in H^2(X_i; \mathbb{Z})$ $(1 \leq i, j \leq n,\ i \neq j)$ each of which is represented by an embedded sphere. Let M_i be the total space of the 2-disk bundle ζ_i over X_i with the Euler class α_{ii}. Attach M_i to M_j by plumbing the image of the sphere representative of α_{ij} in M_i under the inclusion map of X_i on the image of the sphere representative of α_{ji} in M_j under the inclusion map of X_j. Since the sphere representatives in any X_i may not be disjoint (in fact, we need them to have nonempty intersection in general), all the plumbing must be done one by one.

We form manifolds M_{ij} inductively by doing the plumbing one by one. Here i and j must be as if ij is an index for the terms of a matrix above the diagonal. Let M_{12} be the manifold obtained by attaching M_1 to M_2 by gluing α_{12} on α_{21} as explained in Subsection 1.2. Similarly, M_{13} is the manifold obtained by attaching M_3 to M_{12} on α_{13} and α_{31}. The manifold M_{23} is the manifold obtained by attaching M_3 to M_{1n} on α_{23} and α_{32}. Then M_{ij} is the manifold obtained by plumbing M_j to the manifold formed in the last step ($M_{i(j-1)}$ or $M_{(i-1)j}$) on α_{ij} and α_{ji}. $\frac{n(n-1)}{2}$ manifolds are formed this way. The last one is $M_{(n-1)n}$.

It is similar to the last subsection to show that the Betti numbers are as follows: $b_0(M_{ij}) = b_3(M_{ij}) = b_5(M_{ij}) = b_6(M_{ij}) = 0$, $b_4(M_{ij}) = j$ and $b_4(M_{(n-1)n}) = n$. In each step we are losing a second cohomology element, so $b_2(M_{(n-1)n}) = \frac{n(n-1)}{2}$. The manifolds M_{1j} are simply-connected by Van-Kampen's theorem. As the next proposition shows, starting with M_{23}, the manifolds M_{ij} are not simply-connected since the operation used is self-plumbing in each step and this operation increases the first Betti number by one. Consequently, $b_1(M_{(n-1)n}) = \frac{(n-1)(n-2)}{2}$.

Proposition 1.17. *Let W' be a 6-manifold obtained by plumbing two embedded spheres a and b in a 6-manifold W as above. Then $b_1(W') = b_1(W) + 1$.*

Proof. We make the attachment over $\amalg_2 S^2$, the disjoint union of 2 spheres. If we remove the plumbed sphere from W', what we have is W with the two spheres removed. This manifold is connected and the last part of the Mayer-Vietoris sequence of this operation is given below:

(6)

$$\cdots \to H_1(\amalg_2 S^2 \times S^3; \mathbb{Z}) \to H_1(W'; \mathbb{Z}) \oplus H_1(S^2 \times B^4; \mathbb{Z}) \to H_1(W - \amalg_2 S^2 \times B^4; \mathbb{Z})$$
$$\to \widetilde{H}_0(\amalg_2 S^2 \times S^3; \mathbb{Z}) \to \widetilde{H}_0(W'; \mathbb{Z}) \oplus \widetilde{H}_0(S^2 \times B^4; \mathbb{Z}) \to \widetilde{H}_0(W - \amalg_2 S^2 \times B^4; \mathbb{Z}) \to 0.$$

We get the following short sequence (7) by placing the known values to the sequence (6) above:

(7)

$$0 \to H_1(W'; \mathbb{Z}) \oplus H_1(S^2 \times B^4; \mathbb{Z}) \to H_1(W - \amalg_2 S^2 \times B^4; \mathbb{Z}) \to \widetilde{H}_0(\amalg_2 S^2 \times S^3; \mathbb{Z}) \to 0.$$

Therefore, $b_1(W') = b_1(W) + 1$. □

The generator coming from the increase of the first Betti number is also one of the generators of the fundamental group of the plumbed manifold. If we remove the interior of the normal neighborhood of the plumbed sphere from the plumbed manifold, by Van Kampen's theorem, the fundamental group remains the same. This manifold is obtained from the unplumbed manifold by adding an $S^2 \times S^3 \times I$. This last one is simply-connected so no relation is introduced in this operation. Hence the fundamental group gains a free part as a consequence of the plumbing operation.

The calculation of the homology of the boundary is done using the reduced relative homology exact sequence of the pair $(M_{ij}, \partial M_{ij})$. As an instance, sequence (8) given below is the exact sequence that is associated with one of the plumbings producing M_{13}.

$$\begin{aligned} 0 &\to H_5(M_{13}; \mathbb{Z}) \to H_5(M_{13}, \partial M_{13}; \mathbb{Z}) \to H_4(\partial M_{13}; \mathbb{Z}) \\ &\to H_4(M_{13}; \mathbb{Z}) \to H_4(M_{13}, \partial M_{13}; \mathbb{Z}) \to H_3(\partial M_{13}; \mathbb{Z}) \\ &\to H_3(M_{13}; \mathbb{Z}) \to H_3(M_{13}, \partial M_{13}; \mathbb{Z}) \to H_2(\partial M_{13}; \mathbb{Z}) \\ &\to H_2(M_{13}; \mathbb{Z}) \to H_2(M_{13}, \partial M_{13}; \mathbb{Z}) \to H_1(\partial M_{13}; \mathbb{Z}) \\ &\to H_1(M_{13}; \mathbb{Z}) \to H_1(M_{13}, \partial M_{13}; \mathbb{Z}) \to \widetilde{H}_0(\partial M_{13}; \mathbb{Z}) = 0. \end{aligned} \tag{8}$$

By using Poincare-Lefschetz duality for the compact manifolds with boundary, and placing the known values in sequence (8), we find the Betti numbers of ∂M_{13} to be $b_0 = b_1 = b_4 = b_5 = 1$ and $b_2 = b_3 = b_2(X_1) + b_2(X_2) + b_2(X_3) - 6$.

In exact sequence (8), $H_1(M_{13}, \partial M_{13}; \mathbb{Z})$ is trivial, which means that the first homology element is killed inside the 6-manifold. An induction argument, starting with M_{13}, shows that all of the first and second homology elements of the boundary are inherited from the manifold itself. Hence, considering the classification of simply-connected, spin 5-manifolds ([Sm-62]), all second and third homology elements in the boundary come from a bunch of $S^2 \times S^3$'s that are connected summed to each other.

For ∂M_{13}, the boundary of the first nonsimply-connected manifold we produced, the only remaining step is to find a manifold which is connected summed to these $S^2 \times S^3$'s. Let us call this manifold N. As before, for the self-plumbing, a new first homology generator is introduced and it contributes a free generator to the fundamental group of the boundary. Thus, the manifold N has an infinite cyclic fundamental group. By a theorem of Browder ([Br-66]), we can split this manifold as $Y \cup_{X\times S^0} (X \times I)$, where X and Y are simply-connected manifolds in dimension 4 and 5, respectively. Note that Y is a cobordism of X with itself. $\widetilde{H}_i(Y) = 0$ for $i \neq 4$. If Y had a nonzero third homology element, $X \times I$ would contribute a fourth homology element to N which is bounded by the third homology element of Y. This element could only be a multiple of the fundamental class which is already in the fourth homology of Y by the inclusion of the boundary. Consequently, Y has trivial third homology. If Y had a nonzero second homology element, this element would be an element in the second homology of X and hence an element of $X \times I$. After gluing, this would be a nontrivial element in the second homology group of N. However, N has no nontrivial second homology elements. Hence by the Hurewicz theorem, Y is homotopic to X, i.e. Y is an h-cobordism. All obstructions to the h-Cobordism theorem lies in the degree two and three homology in dimension 5. Therefore, Y is homotopy equivalent to $X \times I$ and the manifold N that we are after is homotopy equivalent to $S^1 \times S^4$. By a result of Shaneson for the smooth, closed, orientable 5-manifolds with infinite cyclic fundamental group ([Sh-68], Theorem on p. 297), this manifold is diffeomorphic to $S^1 \times S^4$. In each plumbing, this new 5-manifold must have its first and fourth homology groups isomorphic to $\mathbb{Z}$. During the plumbing, the boundary changes away from those $S^1 \times S^4$'s introduced before this step. Therefore, in each step, an $S^2 \times S^3$ is replaced by an $S^1 \times S^4$.

The S^4 in the boundary, formed during the plumbing, bounds a fifth homology element in the 6-manifold, hence it dies in the 6-manifold we constructed. Closing the boundary with the boundary sum of $B^3 \times S^3$'s and $B^2 \times S^4$'s gives a closed manifold M, whose Betti numbers are given by $b_0(M) = b_6(M) = 1$, $b_2(M) = b_4(M) = n$, $b_1(M) = b_5(M) = b_3(M) = 0$.

Since α_{ii} is characteristic, M must be spin as shown in the following proposition.

Proposition 1.18. *M is spin.*

Proof. The proof is similar to the proof of Proposition 1.14, but in this case there are n homology classes on which $w_2(M)$ must be evaluated. All of these cohomology classes x_i are induced by the inclusion of X_i into M. Pulling back everything into the respective X_i, it turns out that $w_2(M)x_i \equiv \sum_j w_2(X_j)x_i + \sum_j w_2(\eta_j)x_i$. Remember that η_i is the 2-disk bundle over the 4-manifold X_i. In this sum, $w_2(X_j)x_i$ and $w_2(\eta_j)x_i$ $(i \neq j)$ can be omitted due to the fact that X_i overlaps X_j only on α_{ij} which is already included in x_i. As a result, $w_2(X_i)x_i$ and $w_2(\eta_2)x_i$ are already included in $w_2(X_1)x_i$ and $w_2(\eta_1)x_i$, respectively. Thus, $w_2(M)x_i \equiv w_2(X_i)x_i + w_2(\eta_i)x_i \equiv w_2(X_i)x_i + Q_i(\alpha_{ii}, \alpha_{ii}) \equiv 0$. □

The intersection form of M is given in the following proposition.

Proposition 1.19. *Let M be the 6-manifold constructed above with $b_2 = n$ and let μ be its trilinear form. Then $\mu(x_i, x_j, x_k) = Q_i(\alpha_{ij}, \alpha_{ik})$.*

Proof. The proof is essentially the same as in Subsection 1.2. The intersection $\mu(x_i, x_j, x_k)$ can be written as $PD(x_i) \pitchfork PD(x_j) \pitchfork PD(x_k)$ which is equal to $X_i \pitchfork X_j \pitchfork X_k$. The embedded copy of X_i intersects the embedded copy of X_j on α_{ij}. It also intersects X_k on α_{ik}. Therefore these three intersect each other on the projection of α_{ij} on α_{ik} in X_i which is nothing but $Q_i(\alpha_{ij}, \alpha_{ik})$. □

This last proposition reflects the fact that we must choose X_i and the cohomology classes such that $Q_i(\alpha_{ij}, \alpha_{ik}) = Q_j(\alpha_{ji}, \alpha_{jk})$ for all i, j, k.

The first Pontrjagin class acts on the second cohomology as follows.

Proposition 1.20. $p_1(M)x_i = 3\sigma(X_i) + Q_i(\alpha_{ii}, \alpha_{ii})$.

Proof. The proof is similar to the proof of Proposition 1.16 in Subsection 1.2. We may write $p_1(M)x_i = p_1(M)PD(x_i) = p_1(M)X_i$. The tangent bundle of M is restricted to the embedded copy of X_i as the direct sum of the tangent bundle TX of X and the normal neighborhood $\nu X_i|_M$ of X_i in M. The bundle $\nu X_i|_M$ is the B^2-bundle over X_i with Euler class α_{ii}. Therefore, as we have seen in Subsection 1.1 (the paragraph before Example 1.8), $p_1(M)X_i = p_1(X_i)[X_i] + \alpha_{ii} \cup \alpha_{ii}[X_i] = 3\sigma(X_i) + Q_i(\alpha_{ii}, \alpha_{ii})$. □

We have proved the main theorem of this paper.

Main Theorem. *Let V be a smooth, closed, simply-connected, spin 6-manifold with torsion-free homology and $b_3(V) = 0$. Suppose that $H^2(V;\mathbb{Z})$ is isomorphic to the direct sum of n copies of $\mathbb{Z}$, each of which is generated by x_i. Also suppose that $p_1(V)x_i = 24k_i + 4\mu(x_i, x_i, x_i)$, where p_1 is the first Pontrjagin class of V and μ is the symmetric trilinear form of V. Then we can find a collection $\{X_i\}$ of smooth, closed, simply-connected 4-manifolds with odd intersection forms Q_i and second cohomology classes $\alpha_{ij} \in H^2(X_i;\mathbb{Z})$ $(1 \leq i, j \leq n)$ satisfying*

1. *the signature of X_i is $8k_i + \mu(x_i, x_i, x_i)$,*
2. *α_{ii} are primitive, characteristic and $Q_i(\alpha_{ii}, \alpha_{ii}) = \mu(x_i, x_i, x_i)$,*
3. *if $i \neq j$, then α_{ij} has a smooth sphere representative in X_i,*
4. *$Q_i(\alpha_{ij}, \alpha_{ik}) = \mu(x_i, x_j, x_k)$,*

so that the manifold M that is constructed by closing the boundaries of the plumbed 2-disk bundles over these 4-manifolds X_i with Euler class α_{ii} is diffeomorphic to V. The plumbing of the respective bundles over X_i is done on the sphere representatives of α_{ij} in X_i and α_{ji} in X_j.

2. The building blocks

In this section, we prove the fact that 4-manifolds that are used as the building blocks of the constructions in Section 1 exist. We state two existence results. The first lemma is a special case of the second, however we include it here because it makes reading the proof of the latter lemma easier.

Lemma 2.1. *For all $(k, m) \in \mathbb{Z} \oplus \mathbb{Z}$, there exists a closed, simply-connected, smooth 4-manifold X with an odd intersection form Q and a primitive characteristic cohomology class $\alpha \in H^2(X;\mathbb{Z})$ such that $Q(\alpha, \alpha) = m$ and the signature, $\sigma(X)$, of X is $8k + m$.*

Proof. The strategy of the proof is to find the manifolds for $(k, 0)$ and $(0, m)$ and then give the remaining ones as the connected sums of these manifolds. Note that the signature of a connected sum is equal to the sum of signatures of the components. The characteristic class $\alpha_{k,m}$ of the resulting manifold after connected summing is taken as the sum of the characteristic classes of each component manifold.

1. $(k, m) = (0, 0)$: Take $X_{0,0} = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\alpha_{0,0} = h + e$, where h is the generator of $H^2(\mathbb{C}P^2;\mathbb{Z})$ and e is the generator of $H^2(\overline{\mathbb{C}P^2};\mathbb{Z})$. Then $Q(\alpha_{0,0}, \alpha_{0,0}) = 0 = m$ and $\sigma(X_{0,0}) = 8k + m = 0$.

(2) (a) $m = 0$, $k > 0$: Assume that $X_{k,m} = \#_{16k+1}\mathbb{C}P^2\#_{8k+1}\overline{\mathbb{C}P^2}$ and $\alpha_{k,0} = \sum_{i=1}^{16k+1} h_i + \sum_{i=1}^{k} 3e_i + \sum_{k+1}^{8k+1} e_i$, where h_i is the generator of the second cohomology group of the ith copy of $\mathbb{C}P^2$and e_i is the generator of the second cohomology group of the ith copy of $\overline{\mathbb{C}P^2}$. Then the signature of $X_{k,0}$ is $8k = 8k + m$ and $Q(\alpha_{k,m}, \alpha_{k,m}) = 0 = m$.

(b) $m = 0$, $k < 0$: Let $X_{k,m} = \#_{-8k+1}\mathbb{C}P^2\#_{-16k+1}\overline{\mathbb{C}P^2}$ and $\alpha_{k,m} = \sum_{i=1}^{-16k+1} e_i + \sum_{i=1}^{-k} 3h_i + \sum_{-k+1}^{-8k+1} h_i$, where h_i is the generator of the second cohomology group of the ith copy of $\mathbb{C}P^2$and e_i is the generator of the second cohomology group of the ith copy of $\overline{\mathbb{C}P^2}$. Then the signature of $X_{k,m}$ is $8k = 8k + m$ and $Q(\alpha_{k,m}, \alpha_{k,m}) = 0 = m$.

(3) (a) $k = 0$, $m > 0$: Take $X_{k,m} = \#_m\mathbb{C}P^2$ and $\alpha_{k,m} = \sum_{i=1}^{m} h_i$, where h_i is the generator of the second cohomology group of the ith copy of $\mathbb{C}P^2$. The signature of $X_{k,m}$ is m and $Q(\alpha_{k,m}, \alpha_{k,m}) = m$.

(b) $k = 0$, $m < 0$: Take $X_{k,m} = \#_{-m}\overline{\mathbb{C}P^2}$ and $\alpha_{k,m} = \sum_{i=1}^{-m} e_i$, where e_i is the generator of the second cohomology group of the ith copy of $\overline{\mathbb{C}P^2}$. The signature of $X_{k,m}$ is m and $Q(\alpha_{k,m}, \alpha_{k,m}) = m$.

(4) For arbitrary (k, m), let $X_{k,m}$ be chosen as the connected sum of $X_{k,0}$ and $X_{0,m}$ and $\alpha_{k,m} = \alpha_{k,0} + \alpha_{0,m}$. □

Our second existence result is given by the following lemma.

Lemma 2.2. *Given $(k, n) \in \mathbb{Z} \oplus \mathbb{Z}_+$ and a symmetric matrix $A = \{A_{ij}\} \in GL(n; \mathbb{Z})$, it is always possible to find a closed, simply-connected, smooth 4-manifold X with an odd intersection form Q and n distinct primitive second cohomology classes of X satisfying the following conditions:*

1. *one of the cohomology classes, $\alpha = \alpha_1$, is characteristic, and $Q(\alpha, \alpha) = A_{11}$,*
2. *the $(n - 1)$ remaining cohomology classes $\{\alpha_i\}_{i=2}^{n-1}$ are represented by embedded spheres,*
3. *the intersection numbers of the n cohomology classes are given by $Q(\alpha_i, \alpha_j) = A_{ij}$,*
4. *the signature, $\sigma(X)$, of X is equal to $8k + Q(\alpha, \alpha)$.*

Proof. When $n = 1$, the proof is given in Lemma 2.1. Let $n > 1$. We give the homeomorphism type of X by writing it as a connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P^2}$'s. There is a distinguished class α which is characteristic. For each α_i, we find a manifold X_i corresponding to α_i. Then we glue all X_i together according to the intersection of α_i with the other classes. We start gathering the pieces of X by considering the difference between $Q(\alpha_i, \alpha_i)$ and $Q(\alpha_i, \alpha)$. This number is always even (since α is characteristic). Let $Q(\alpha_i, \alpha) - Q(\alpha_i, \alpha_i)$ be equal to $2k_i$. The manifold X_i and the class α are formed by the following algorithm:

1. This step is to adjust the intersection with the characteristic class α. If k_i is positive, then $X_i = \mathbb{C}P^2$, $\alpha_i = h_i$ and $\alpha = (2k_i + 1)h_i$, where h_i is the generator of the second cohomology group of $\mathbb{C}P^2$. If k_i is negative, then $X_i = \overline{\mathbb{C}P^2}$, $\alpha = (2k_i + 1)e$ and $\alpha_i = e_i$. Here e_i is the generator of the second cohomology of $\overline{\mathbb{C}P^2}$.
2. Now we adjust the intersection number with other cohomology classes. If $Q(\alpha_i, \alpha_j)$ is negative, X_i and α_i stay as they are. If $Q(\alpha_i, \alpha_j)$ is positive, then add $Q(\alpha_i, \alpha_j)$ copies of $\mathbb{C}P^2$'s to X_i by connected summing, and add

the generators of second cohomologies of the new $\mathbb{C}P^2$'s to α_i. The same number of $\mathbb{C}P^2$'s are also added to X_j but α_j is not changed.

(3) To reach the self-intersection $Q(\alpha_i, \alpha_i)$ of α_i, connect sum X_i with a number of $\mathbb{C}P^2$'s or $\overline{\mathbb{C}P^2}$'s, and add the generators of the new cohomology to α_i. The number of the new manifolds that are added is determined by the self-intersection of α_i at the end of the last step.

Now we have a collection of manifolds $\{X_i\}$. The next step is to glue all these manifolds by identifying X_i and X_j on the $\mathbb{C}P^2$'s added to X_i and X_j in the second step of the algorithm above. In this manifold, the characteristic class α is given by $\sum(2k_i+1)h_i + \sum(2k_i+1)e_i$. As a result, we have a manifold with n primitive cohomology classes. The intersection numbers of these classes were adjusted except for $Q(\alpha, \alpha)$. To obtain the desired manifold, we need to adjust the self-intersection of α and the signature. We manage the self-intersection of α by adding $\mathbb{C}P^2$'s (or $\overline{\mathbb{C}P^2}$'s) if necessary. α is changed by the addition of a bunch of h's (or e's). Finally, we reach the signature by connected summing with the manifolds used in the proof of Lemma 2.1. The contribution of these manifolds to α does not change the self-intersection of α.

The projection of α_i in each $\mathbb{C}P^2$ ($\overline{\mathbb{C}P^2}$) can be represented by a sphere. This is because it is nothing but a line h in $\mathbb{C}P^2$ (exceptional sphere e in $\overline{\mathbb{C}P^2}$). Connecting these spheres linearly by thin tubes, we can represent all the α_i's by embedded spheres. □

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