CONTINUED FRACTIONS
WITH CIRCULAR TWIN VALUE SETS

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Abstract. We prove that if the continued fraction \( K(a_n/1) \) has circular twin value sets \( \langle V_0, V_1 \rangle \), then \( K(a_n/1) \) converges except in some very special cases. The results generalize previous work by Jones and Thron.

1. Introduction and main result

A pair \( \langle V_0, V_1 \rangle \) of sets from \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \) is called a pair of twin value sets for the continued fraction

\[
K(a_n/1) := \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \ldots}}} = \frac{a_1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \ldots}}}, \quad a_n \in \mathbb{C} \setminus \{0\}
\]

if both \( V_k \) and its complement \( V_k^c \) in \( \hat{\mathbb{C}} \) are non-empty for \( k = 0, 1 \) and

\[
a_{2n-1}/(1 + V_1) \subseteq V_0 \quad \text{and} \quad a_{2n}/(1 + V_0) \subseteq V_1 \quad \text{for} \quad n = 1, 2, 3, \ldots .
\]

Note that we do not require \( a_{2n+k} \in V_{k-1} \) for \( k = 1, 2 \) as was done in the work by Jones and Thron; see for instance their book [7, p. 64]. For given value sets we further define the corresponding element sets \( \langle E_1, E_2 \rangle \) by

\[
E_1 := \{ a \in \mathbb{C}; \ a/(1 + V_1) \subseteq V_0 \}, \quad E_2 := \{ a \in \mathbb{C}; \ a/(1 + V_0) \subseteq V_1 \}.
\]

Here, by definition, \( 0 \notin E_1 \) if \( -1 \notin \overline{V_1} \) (the closure of \( V_1 \) in \( \hat{\mathbb{C}} \)) and \( 0 \notin E_2 \) if \( -1 \notin \overline{V_0} \). The twin element sets \( \langle E_1, E_2 \rangle \) are true if \( E_k \setminus \{0\} \neq \emptyset \) for \( k = 1 \) and 2. We also say that \( \langle V_0, V_1 \rangle \) are twin value sets for \( \langle E_1, E_2 \rangle \). For convenience we shall always let \( V_2 := V_0 \), so that \( E_k = \{ a \in \mathbb{C}; \ a/(1 + V_k) \subseteq V_{k-1} \} \) for \( k = 1, 2 \).

In this paper we restrict the value sets to be closed circular domains; that is, they are closures of simply connected, open, non-empty domains on the Riemann sphere \( \hat{\mathbb{C}} \), bounded by a generalized circle. The points \( 0, -1, \infty \) are special in the classical continued fraction theory. (See (1.6).) We shall therefore distinguish between closed domains \( V \) where

- \( \infty \notin V \) (disks),
- \( \infty \) on the boundary \( \partial V \) of \( V \) (half planes),
- \( \infty \) in the interior \( V^o \) of \( V \) (complements of disks).

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We address the problem: when does $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ (i.e. all $a_{2n-1} \in E_1$ and all $a_{2n} \in E_2$) converge? By convergence we mean that the sequence of approximants $\{c_n\}$ of $K(a_n/1)$ converges to a $c \in \mathbb{C}$, where

$$c_n := S_n(0) \quad \text{and} \quad S_n(z) := \frac{a_1}{1 + \frac{a_2}{1 + \cdots + \frac{a_n}{1 + z}}},$$

(1.4)
i.e.,

$$S_n := s_0 \circ s_1 \circ s_2 \circ \cdots \circ s_n; \quad s_0(z) := z, \quad s_k(z) := \frac{a_k}{1 + z}.$$We say that the even (odd) part of $K(a_n/1)$ converges if $\{c_{2n}\}$ ($\{c_{2n+1}\}$) converges in $\mathbb{C}$. A number of papers has been written on this topic. See for instance [7, chapter 4] and the references therein. In particular, the paper [6] by Jones and Thron, published in this journal, gives a very nice and useful presentation of sufficient conditions for convergence. However, these results can be improved, as we shall show in this paper. The very special case where $0 \in \partial V_0$ and $-1 \in \partial V_1$ or vice versa still needs some extra attention, though (see Example 2.9). We shall prove:

**Theorem 1.1.** Let $\langle V_0, V_1 \rangle$ be closed circular twin value sets with corresponding element sets $(E_1, E_2)$ for the continued fraction $K(a_n/1)$. Then the following statements are true:

A. Let $V_0$ and $V_1$ be disks and $E_2^0 \neq \emptyset$. Then $K(a_n/1)$ converges to a $c \in V_0$.

B. Let $V_0$ be a disk, $V_1$ a half plane and $E_2^0 \neq \emptyset$. Then $K(a_n/1)$ converges to a $c \in V_0$.

C. Let $V_0$ be a disk and $V_1$ be the complement of a disk with respective centers $C_k$ and radii $R_k$ such that $|C_k - R_{k+1} + R_k| [1 + C_{k+1}]$ for $k = 0$ or $k = 1$ and $0 \notin \partial V_1 := \partial V_1 \cap (-1 - \partial V_0)$. Then the even part of $K(a_n/1)$ converges to a $c \in V_0$. If moreover $-1 \notin V_0 \setminus (-1 - V_0^0)$, then $K(a_n/1)$ itself converges to $c$.

D. Let $V_0$ and $V_1$ be half planes with $0, -1 \notin \partial V_1$. Then the even and odd parts of $K(a_n/1)$ converge to finite values $\in V_0$. Moreover, $K(a_n/1)$ itself converges if and only if

$$\sum_{n=1}^{\infty} |b_n| = \infty \quad \text{where} \quad b_{2n} := \frac{a_1a_3\cdots a_{2n-1}}{a_2a_4\cdots a_{2n}}, \quad b_{2n+1} := \frac{a_2a_4\cdots a_{2n}}{a_1a_3\cdots a_{2n+1}}.$$  

(1.5)

**Remarks 1.2.**

1. Since $K_{n=1}^\infty (a_n/1)$ converges in $\mathbb{C}$ if and only if $K_{n=2}^\infty (a_n/1)$ converges in $\mathbb{C}$, we may interchange $V_0$ and $V_1$.

2. Theorem 1.1 also covers cases such as, for instance, $V_0$ a half plane and $V_1$ a complement of a disk, since $(V_0, V_1)$ are twin value sets for the continued fraction $K(a_n/1)$ if and only if $(-1 - V_1^c, -1 - V_0^c)$ are twin value sets for $K(a_n/1)$ (see Lemma 4.1). This was also pointed out by Jones and Thron in [6]. Indeed, if $V_0$ or $V_1$ contains more than one element, then $V_0 := V_0 \setminus (-1 - V_0^0) \neq \emptyset$ and $Y_1 := V_1 \setminus (-1 - V_0)^c \neq \emptyset$, so also $(Y_0, Y_1)$ are twin value sets for $K(a_n/1)$, [9] prop. 5.4.

3. It is a well established fact [7] thm. 4.53, p. 128 that (1.5) holds if $\{a_n\}$ has a bounded subsequence.

The classical convergence concept requires that $S_n(0) \to c$, where by (1.4),

$$c_n = S_{n-1}(a_n) = S_n(0) = S_{n+1}(\infty) = S_{n+2}(-1) = S_{n+3}(-1 - a_{n+3}).$$

(1.6)
In [2] a more general concept of convergence was introduced: we require that there exist two sequences \( \{u_n\} \) and \( \{v_n\} \) from \( \hat{\mathbb{C}} \) such that

\[
(1.7) \quad \lim_{n \to \infty} S_n(u_n) = \lim_{n \to \infty} S_n(v_n) = c \quad \text{and} \quad \lim \inf d(u_n, v_n) > 0,
\]

where \( d(\ast, \ast) \) denotes the chordal metric on the Riemann sphere \( \hat{\mathbb{C}} \); i.e.,

\[
d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2 \sqrt{1 + |w|^2}}} \quad \text{if} \quad z, w \in \mathbb{C}
\]

with the natural limit forms if \( z \) and/or \( w \) is \( \pm \infty \). If (1.7) holds, we say that \( K(a_n/1) \) converges generally to \( c \). Then, by [2], there exists an exceptional sequence \( \{z_n^+\} \subseteq \hat{\mathbb{C}} \) such that

\[
(1.9) \quad \lim_{n \to \infty} S_n(z_n) = c \quad \text{whenever} \quad \lim \inf d(z_n, z_n^+) > 0.
\]

If \( c \neq \infty \), we can for instance use \( z_n^+ := \zeta_n := S_n^{-1}(\infty) \) for all \( n \). Or more generally, \( \{S_n^{-1}(q)\} \) is an exceptional sequence for every \( q \neq c \), also if \( c = \infty \). All the exceptional sequences have the same asymptotic behavior.

Classical convergence implies general convergence whereas the converse does not hold. Indeed, there are generally convergent continued fractions \( K(a_n/1) \) where \( \{z_n^+\} \) has limit points at \( 0, -1 \) and \( \infty \) which destroy the classical convergence of \( K(a_n/1) \). However, if \( K(a_n/1) \) also converges in the classical sense, then it converges to the same value. It is also clear that if the even and odd parts of \( K(a_n/1) \) converge to distinct values in the classical sense, then they also converge generally to the same two distinct values.

One might expect to get a nicer theorem with general convergence. However, Theorem 1.1 is already good, except for the disk – complement of disk case. For this case it really pays off to change over to general convergence (here \( B(C, R) \) denotes a closed circular disk with center at \( C \in \mathbb{C} \) and radius \( R > 0 \)).

**Theorem 1.3.** Let \( V_0 := B(C_0, R_0) \) and \( V_1 := \overline{B(0, R_1)} \) be twin value sets for the continued fraction \( K(a_n/1) \), where \( 0 \notin \partial V_1 := \partial V_\infty \) and \( |C_k|R_{k+1} \neq R_k|1 + C_{k+1}| \) for \( k = 0 \) or \( k = 1 \). Then \( K(a_n/1) \) converges generally to \( c \in V_0 \).

The final result in this section describes cases where classical convergence follows from general convergence. We still use the notation \( \zeta_n := S_n^{-1}(\infty) \).

**Theorem 1.4.** Let \( (V_0, V_1) \) be closed twin value sets for the continued fraction \( K(a_n/1) \) with \( (V_0 \cup V_1)^c \neq \emptyset \). Let \( K(a_n/1) \) converge generally to \( c \), let \( q \neq c \) and let \( \tilde{Z}_k \) be the set of limit points for \( \{S_n^{-1}(q)\} \). Then the following statements hold for fixed \( k \in \{1, 2\} \):

A. \( c \in V_0 \setminus (-1 - V_1^c) \) and \( \tilde{Z}_k \subseteq (-1 - V_{k-1}) \setminus V_k^c \).

B. If \(-1 \notin \tilde{Z}_k \) or \( 0 \notin \tilde{Z}_k \), then \( S_{2n+k}(0) \to c \). If \( \infty \notin \tilde{Z}_k \), then \( S_{2n+k-1}(0) \to c \).

C. Let \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \). If for each \( n \geq n_0 \), either \( d(a_{2n+k-1}, \tilde{Z}_k) \geq \varepsilon \) or \( d(-1 - a_{2n+k+2}, \tilde{Z}_k) \geq \varepsilon \), then \( S_{2n+k-1}(0) \to c \).

D. If \( V_0 \) is bounded, then \( \{\zeta_n\} \) is an exceptional sequence for \( \{S_n\} \) and \( S_{2n}(0) \to c \).

E. If \(-1 \notin V_0 \setminus (-1 - V_1^c) \), then \( S_{2n+1}(0) \to c \).
In section 2 we shall give some explicit expressions for the corresponding element sets \( \langle E_1, E_2 \rangle \) and some stronger convergence results. Section 3 contains some intermediate results, and in section 4 we prove the results in sections 1 and 2.

**Notation.** We shall use the notation introduced so far, plus some extra. For convenience we list a few of them here:

- \( \overline{A} \), \( A^o \) and \( A^c \) are the closure, the interior, the boundary and the complement of a set \( A \) in \( \mathbb{C} \).
- \( \mathbb{D} \) is the open unit disk \( \{ z \in \mathbb{C}; |z| < 1 \} \).
- \( [z_1, z_2] \) is the closed line segment between the two points \( z_1 \) and \( z_2 \) in \( \mathbb{C} \). Moreover, \( a[r, \infty) := \{ z = ua; \ u \geq r \} \) for \( a \in \mathbb{C} \setminus \{0\} \) and \( r \in \mathbb{R} \).
- \( B(a, r) := \{ z \in \mathbb{C}; |z - a| \leq r \} \) and \( B_d(a, r) := \{ z \in \mathbb{C}; d(z, a) \leq r \} \) for \( a \in \mathbb{C} \) and \( r > 0 \).
- \( H(r, \alpha) \), where \( r, \alpha \in \mathbb{R} \), denotes the closed half plane with \( L := e^{i\alpha}[r, \infty) \subseteq H(r, \alpha) \), whose boundary \( \partial H(r, \alpha) \) is the line through \( r e^{i\alpha} \) orthogonal to \( L \).
- \( \text{rad}(A) \) is the euclidean radius of a circular set \( A \subseteq \mathbb{C} \). \( \text{rad}(A) := \infty \) if \( \infty \notin \overline{A} \).
- \( \text{diam}(A) \) is the euclidean diameter of a set \( A \subseteq \mathbb{C} \).
- \( \text{dist}(z, A) (d(z, A)) \) denotes the euclidean (chordal) distance between a point \( z \in \mathbb{C} \) and a set \( A \subseteq \mathbb{C} \), and \( \text{dist}(A, B) (d(A, B)) \) denotes the euclidean (chordal) distance between two sets \( A, B \subseteq \mathbb{C} \).
- For convenience, \( V_2 := V_0, W_2 := W_0, E_3 := E_1, E_0 = E_2 \), etc. for twin quantities; that is, they are counted modulo 2.
- \( s_m \) denotes the linear fractional transformation \( a_m/(1 + z) \), \( s_m(z) := a_m/(1 + z) \) and so on, and \( S_n := s_1 \circ s_2 \circ \cdots \circ s_n \).
- \( \partial^k V_k := \partial V_k \cap (-1 - \partial V_{k+1}) \) and \( \partial^k V'_k := \partial V_k \cap (-1 - V_{k+1}) \) for \( k = 0, 1 \). Clearly, \( \partial^0 V_0 = -1 - \partial V_1 \), and the condition \( 0 \notin \partial V_k \), \( -1 \notin \partial V_{k+1} \) can be written \( 0 \notin \partial^k V_k \), or equivalently, \( -1 \notin \partial^k V_{k+1} \).
- \( \zeta_n := s_n^{-1}(\infty) \), \( c_n := S_n(0) \) and \( Z_k \) is the (closed) set of limit points for \( \{ \zeta_{n+k} \} \).
- \( W_0 := -1 - W_1 \), \( W_1 := -1 - W_0 \), \( Y_0 := V_0 \setminus (-1 - V_1)^o \) and \( Y_1 := V_1 \setminus (-1 - V_0)^o \) so that \( \langle W_0, W_1 \rangle \) and \( \langle Y_0, Y_1 \rangle \) are alternative closed twin value sets (Remark 1.2.2).
- \( \sum P_n < \infty \) shall mean that there exists an \( n_0 \in \mathbb{N} \) such that \( \sum_{n=n_0}^{\infty} P_n < \infty \) for the non-negative numbers \( P_n \). Hence \( P_n = \infty \) is possible for finitely many \( n \).

2. EXPPLICIT ELEMENT SETS AND MORE DETAILED CONVERGENCE CRITERIA

In applications it is useful to know the corresponding element sets \( \langle E_1, E_2 \rangle \) explicitly. We have therefore listed these sets below, along with some more specific convergence criteria for continued fractions \( K(a_n/1) \) with circular twin value sets. Of course we want as few extra conditions as possible, but some situations have to be treated separately:

- \( a_n \to \infty \). The if and only if part of Theorem 1.1D shows that extra conditions are needed in this case. This is true whether we want classical or general convergence.
**The disk – disk case.**

Let $V_k := B(C_k, R_k)$ for some $C_k ∈ \mathbb{C}$ and $R_k > 0$ for $k = 0, 1$. Evidently $E_k = \emptyset$ if $-1 ∈ V_k$, so

\[
|1 + C_k| > R_k \quad \text{for} \quad k = 0, 1
\]

is a necessary condition for $⟨E_1, E_2⟩$ to be true element sets corresponding to $⟨V_0, V_1⟩$. Then we get the following generalization of [6, thm. 5.1]:

**Theorem 2.1.** Let $V_k := B(C_k, R_k)$ for $k = 0, 1$ where $C_k ∈ \mathbb{C}$ and $R_k > 0$ satisfy (2.1) and

\[
|C_k - 1| R_k ≤ |1 + C_k| R_k - 1
\]

for $k = 1, 2$. If (2.2) holds with equality for both $k = 1$ and $k = 2$, we further assume that $σ := \hat{s}_1 \circ \hat{s}_2$ is non-elliptic, where

\[
\hat{a}_k := C_k - 1 (1 + C_k)(1 - R_k^2)/(1 + C_k)^2
\]

Then every continued fraction $K(a_n/1)$ from $⟨E_1, E_2⟩$ converges, where

\[
E_k := \{a ∈ \mathbb{C}; |a - \hat{a}_k| + \frac{R_k}{|1 + C_k|}|a| ≤ \frac{R_k - 1}{|1 + C_k|} |1 + C_k|^2 - R_k^2\}
\]

**Remarks 2.2.**

1. $⟨E_1, E_2⟩$ are the element sets corresponding to $⟨V_0, V_1⟩$. They are true element sets if and only if (2.1) and (2.2) hold. Condition (2.2) is therefore only present to make $⟨E_1, E_2⟩$ true when (2.1) holds. $E_k$ is a one-point set if and only if $E_k = \{\hat{a}_k\}$ as given by (2.3). This happens if and only if $|C_k - 1| R_k = |1 + C_k| R_k - 1$, which happens if and only if $a/(1 + V_k) = V_{k-1}$ for an $a ∈ E_k$, in which case $a = \hat{a}_k ≠ 0$. (See Lemma 4.3.)

2. If $E_k$ contains more than one point, then $E_k$ is a closed convex domain bounded by a cartesian oval with foci at 0 and $\hat{a}_k$ ([12, remark 5, p. 142]), and $E_k^φ ≠ \emptyset$. If $C_k - 1 = 0$, this oval reduces to a circle centered at the origin.

3. Divergence only occurs if and only if $E_k = \{\hat{a}_k\}$ for $k = 1, 2$ and $σ := \hat{s}_1 \circ \hat{s}_2$ is elliptic. This means that $K(a_n/1)$ converges in the classical sense if and only if it converges in the general sense in the disk-disk case.

**The disk – half plane case.**

Let $V_0 := B(C_0, R_0)$ and $V_1 := \{z ∈ \mathbb{C}; \Re(z e^{-iα}) ≥ h \cos α\} ∪ \{∞\} = H(h \cos α, α)$ for some $C_0 ∈ \mathbb{C}$, $R_0 > 0$, $h, α ∈ \mathbb{R}$. It is clear that $a/(1 + V_1) ⊆ V_0$ for $a ≠ 0$ only if $-1 ∉ V_1$ and $0 ∈ V_0$, and that $a/(1 + V_0) ⊆ V_1$ for $a ≠ 0$ only if $-1 ∉ V_0^φ$. Hence we require that

\[
|C_0| ≤ R_0 ≤ |1 + C_0|, \quad |α| < π/2 \quad \text{and} \quad h > -1.
\]
But this leaves the possibility of $0 \in \partial V_1$ and $-1 \in \partial V_0$, a situation that requires caution. We therefore need extra conditions if $0 \in \partial V_1 := \partial V_1 \cap (-1 - \partial V_0)$. Still, we get the following generalization of [6 thm. 5.2]:

**Theorem 2.3.** Let $V_0 := B(C_0, R_0)$ and $V_1 := H(h \cos \alpha, \alpha)$ where $C_0 \in \mathbb{C}$, $R_0 > 0$ and $\alpha, h, R \in \mathbb{R}$ satisfy (2.5), and let

$$a_1^* := 2C_0 e^{ia}(1 + h) \cos \alpha, \quad a_2^* := 2(1 + C_0) he^{ia} \cos \alpha$$

and

$$E_1 := \{a \in \mathbb{C}; |a - a_1^*| + |a| \leq 2R_0(1 + h) \cos \alpha\},$$

$$E_2 := \begin{cases}
E_1 \setminus B(a_1^*, \delta) & \text{if } R_0 = |C_0|, \\
E_1 & \text{otherwise},
\end{cases}$$

$$\bar{E}_{1, \delta} := \begin{cases}
E_2 \setminus B(a_2^*, \delta) & \text{if } R_0 = |1 + C_0| \text{ and } h \geq 0, \\
E_2 & \text{otherwise}
\end{cases}$$

Furthermore, let

$$\bar{E}_{1, \delta} := \begin{cases}
E_1 \setminus B(a_1^*, \delta) & \text{if } R_0 = |C_0|, \\
E_1 & \text{otherwise},
\end{cases}$$

$$\bar{E}_{2, \delta} := \begin{cases}
E_2 \setminus B(a_2^*, \delta) & \text{if } R_0 = |1 + C_0| \text{ and } h \geq 0, \\
E_2 & \text{otherwise}
\end{cases}$$

where $0 < \delta < |a_1^*|$ if $C_0 \neq 0$. Then the following statements are true:

A. Every continued fraction $K(a_n/1)$ from $\langle E_1, \bar{E}_{2, \delta} \rangle$ converges generally.

B. If $0 \notin \partial V_1$, then every continued fraction $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ converges generally.

C. Let $\varepsilon > 0$. If $K(a_n/1)$ is a continued fraction from $\langle E_1, E_2 \rangle$ such that for each $n$ from some $n_0$ on, either $\text{dist}(1 - a_{2n}, \partial V_0) \geq \varepsilon$ or $\text{dist}(a_{2n-1}, \partial V_0) \geq \varepsilon$, then $K(a_n/1)$ converges generally.

D. Let $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ converge generally to $c$. Then $c_{2n} \to c$. If moreover $0 \in V_1^\circ$ or $0 \notin \partial V_0$ or $\lim \inf d(a_{2n-1}, V_0\cap(-1-V_1)) > 0$, then $c_n \to c$.

**Remarks 2.4.**

1. $\langle E_1, E_2 \rangle$ are the element sets corresponding to $\langle V_0, V_1 \rangle$. If $R_0 = |C_0|$, then $E_1$ is the closed line segment $[0, a_1^*]$. Otherwise, $\partial E_1$ is an ellipse with foci at $a_1^*$ and the origin. $\partial E_1$ reduces to a circle if $C_0 = 0$.

2. If $|1 + C_0| = R_0$, then $E_2$ is a ray. Otherwise, $E_2 \neq \emptyset$ and $\partial E_2$ is a hyperbola.

3. If $-1 \notin \partial V_0$ or $0 \in V_1^\circ$, then $\bar{E}_{2, \delta} = E_2$, so every continued fraction $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ converges generally by part A in this case. Let $-1 \in \partial V_0$ and $0 \notin V_1^\circ$. If $0 \notin \partial V_1$ and $0 \in V_0^\circ$, then $\bar{E}_{1, \delta} = E_1$, and every continued fraction from $\langle E_1, E_2 \rangle$ still converges generally by part B.

**The disk - complement of disk case.**

Let $V_0 = B(C_0, R_0)$ and $V_1 = B(C_1, R_1)$. This time $\infty \notin V_0$ and $\infty \in V_1^\circ$, so we evidently need that $0 \in V_0^\circ$ and $-1 \notin V_1$ to get true element sets; that is,

$$|C_0| < R_0 \quad \text{and} \quad |1 + C_1| < R_1.$$
Theorem 2.5. Let $V_0 := B(C_0, R_0)$ and $V_1 := B(C_1, R_1)$ where $C_k \in \mathbb{C}$ and $R_k > 0$ satisfy (2.9), and let

$$E_1 := \begin{cases} \{a; |a - \tilde{a}_1| + |a| \frac{R_0}{|1 + C_0|} \leq \frac{R_0}{|1 + C_1|} (R_0^2 - |1 + C_1|^2) \} & \text{if } C_1 \neq -1, \\ B(0, (R_0 - |C_0|) R_1) & \text{if } C_1 = -1, \end{cases}$$

(2.10)

$$E_2 := \begin{cases} \{a; |a - \tilde{a}_2| - |a| \frac{R_0}{|1 + C_0|} \geq \frac{R_0}{|1 + C_0|} (|1 + C_0|^2 - R_0^2) \} \setminus \{0\} & \text{if } R_0 < |1 + C_0|, \\ \{a; |a| \frac{R_0}{|1 + C_0|} - |a - \tilde{a}_2| \geq \frac{R_0}{|1 + C_0|} (R_0^2 - |1 + C_0|^2) \} & \text{if } R_0 > |1 + C_0| > 0, \\ \{a = r e^{i\theta}; \frac{2}{3} \geq \text{Re}(C_1(1 + C_0) e^{-i\theta}) + R_0 R_1 \} \setminus \{0\} & \text{if } R_0 = |1 + C_0|, \\ \{a; |a| \geq R_0 (R_1 + |C_1|) \} & \text{if } C_0 = -1, \end{cases}$$

where $\tilde{a}_k$ is given by (2.3). Further let $\hat{E}_{1,\delta}$ be given by (2.11), and let $\hat{E}_{2,\delta} := E_2$ if $-1 \notin V_0^c$ and $\hat{E}_{2,\delta}$ be given by (2.11) otherwise, where

$$\hat{E}_{k,\delta} := \begin{cases} E_k \setminus B(\tilde{a}_k, \delta) & \text{if } |C_{k-1}| R_k = R_{k-1} |1 + C_k| > 0, \\ E_k \setminus \{a \in \mathbb{C}; |a| - R_0 R_1 < \delta\} & \text{if } C_{k-1} = 1 + C_k = 0, \\ E_k & \text{otherwise} \end{cases}$$

(2.11)

for given $\delta > 0$ so small that $\hat{E}_{1,\delta} \neq \emptyset$. Then the following statements are true.

A. $(E_1, E_2)$ are the element sets corresponding to $(V_0, V_1)$, and $E_k^c \neq \emptyset$ for $k = 1, 2$.

B. Let $0 \notin \partial^1 V_1$. Then every continued fraction $K(a_n/1)$ from $(\hat{E}_{1,\delta}, E_2)$ or from $(E_1, \hat{E}_{2,\delta})$ converges generally.

C. Let $\varepsilon > 0$. If $K(a_n/1)$ is a continued fraction from $(\hat{E}_{1,\delta}, E_2)$, such that for each $n$ from some $n_0 \in \mathbb{N}$ on, either $\text{dist}(a_{2n-1}, \partial^1 V_0) \geq \varepsilon$ or $\text{dist}(-1 - a_{2n}, \partial^1 V_0) \geq \varepsilon$, then $K(a_n/1)$ converges generally. If $K(a_n/1)$ is a continued fraction from $(E_1, \hat{E}_{2,\delta})$, such that for each $n$ from some $n_0 \in \mathbb{N}$ on, either $\text{dist}(-1 - a_{2n+1}, \partial^1 V_1) \geq \varepsilon$ or $\text{dist}(a_{2n}, \partial^1 V_1) \geq \varepsilon$, then $K(a_n/1)$ converges generally.

D. Let $K(a_n/1)$ from $(E_1, E_2)$ converge generally to $c$. Then $c_{2n} \to c$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. If $-1 \notin V_0^c \setminus (-1 - V_1^c)$ or for each $n \geq n_0$ either $\text{dist}(a_{2n-1}, V_0 \setminus (-1 - V_1)) \geq \varepsilon$ or $\text{dist}(-1 - a_{2n+2}, V_0^c) \geq \varepsilon$, then $K(a_n/1)$ converges to $c$ in the classical sense.

Remarks 2.6.

1. $E_1$ is bounded by a cartesian oval with foci at 0 and $\tilde{a}_1$. If $C_1 = -1$, this oval reduces to a circle. $E_2$ is an unbounded set.

2. Jones and Thron [6] thm. 5.4, [7] thm. 4.11, p.72], proved the expressions for $E_1$ and $E_2$ for the case $|C_0| < R_0 \neq |1 + C_0|$ and $|1 + C_1| < R_1 \leq |C_1|$. Theorem 2.5 generalizes their result.

3. This disk - complement of disk case is quite special in the following sense: the case $a/(1 + V_k) = V_k - 1$ does not necessarily occur only for $a \in \partial E_k$. Therefore $\hat{E}_{k,\delta}$ is not necessarily simply connected or even connected. This means that we do not necessarily have that

$$\overline{G_k} \subseteq E_k^c$$

for $k = 1, 2$ \quad \Rightarrow \quad (G_1, G_2)$ are twin convergence sets

as otherwise this is a normal feature for element sets $(E, E)$ corresponding to simple value sets $(V, V)$. 

The half plane - half plane case.
Let $V_0$ and $V_1$ be closed half planes,

\[(2.12) \quad V_k = \{z \in \mathbb{C}; \text{Re}(z e^{-i\alpha_k}) \geq -g_k \cos \alpha_k\} \cup \{\infty\} = H(-g_k \cos \alpha_k, \alpha_k)\]

for some $\alpha_k, g_k \in \mathbb{R}$. Then $E_k \neq \emptyset$ only if $0 \in V_{k-1}$, and $-1 \notin V_k^0$. Therefore we require

\[(2.13) \quad |\alpha_k| \leq \pi/2 \quad \text{and} \quad 0 \leq g_k \leq 1 \quad \text{for} \quad k = 1, 2.\]

**Theorem 2.7.** Let $\alpha_k, g_k \in \mathbb{R}$ satisfy (2.13) and

\[(2.14) \quad |\alpha_0 + \alpha_1| < \pi \quad \text{and} \quad g_{k-1}(1 - g_k) \neq 1 \quad \text{for} \quad k = 1, 2,\]

and let $K(a_n/1)$ be a continued fraction from $(E_1, E_2)$ given by

\[(2.15) \quad E_k := \{a \in \mathbb{C}; |a| - \text{Re}(ae^{-i(\alpha_0 + \alpha_1)}) \leq 2g_{k-1}(1 - g_k) \cos \alpha_0 \cos \alpha_1\}.\]

Then the even and odd parts of $K(a_n/1)$ converge to finite values in $V_0$, and $K(a_n/1)$ itself converges if and only if (1.5) holds.

**Remarks 2.8.**
1. $(E_1, E_2)$ are the element sets corresponding to $(V_0, V_1)$ in (2.12). If $g_{k-1} = 0$ or if $-1 \in \partial V_k$, then $E_k$ reduces to the ray $e^{i(\alpha_0 + \alpha_1)}(0, \infty)$, possibly including the end point $a = 0$. (Remember, $0 \notin E_k$ if $-1 \notin V_k$ by definition.)
2. If $E_k^2 \neq \emptyset$, then $\partial E_k$ is a parabola with axis along the ray $e^{i(\alpha_0 + \alpha_1)}[-g_{k-1}(1 - g_k) \cos \alpha_0 \cos \alpha_1, \infty)$

and focus at the origin.
3. Theorem 2.7 does not contain any essential news compared to the twin version of Jones’ and Thron’s multiple parabola theorem in [5], [7, thm. 4.43, p. 106] which says that Theorem 2.7 holds under the additional conditions that $0 < g_k < 1$ and $|\alpha_k| < \pi/2$ for $k = 0$ and $k = 1$.

**Example 2.9.** Let $\alpha_0 = \alpha_1 = 0$, $g_0 = 0$ and $g_1 = 1$ in (2.12) and (2.15). Then $0 \in \partial V_0$ and $-1 \in \partial V_1$; i.e., $-1 \in \partial^1 V_1$. For given positive sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ converging to 0, let

\[t_{2n-1} := \varepsilon_n - 1, \quad t_{2n} := \delta_n \quad \text{and} \quad a_n := t_{n-1}(1 + t_n)\]

for all $n$. Then $K(a_n/1)$ is a continued fraction from $(E_1, E_2)$ given by (2.15). By [12 formula (3.3.3), p.216] it follows that

\[S_n(0) - t_0 = -\frac{t_0}{R_n} \quad \text{where} \quad R_n := \sum_{k=0}^{n} \prod_{j=1}^{k} \frac{1 + t_j}{-t_j}.\]

In our situation,

\[\frac{1 + t_{2n-1}}{-t_{2n-1}} \cdot \frac{1 + t_{2n}}{-t_{2n}} = -\frac{\varepsilon_n}{1 - \varepsilon_n} \cdot \frac{1 + \delta_n}{\delta_n} \sim -\frac{\varepsilon_n}{\delta_n} (1 + \varepsilon_n + \delta_n),\]

so $S_{2n}(0)$ may converge or diverge, depending on the asymptotic behavior of $\{\varepsilon_n(1 + \varepsilon_n + \delta_n)/\delta_n\}$. A similar argument also shows that $K(a_n/1)$ may also diverge generally in this case.
3. SOME INTERMEDIATE RESULTS

Let \((V_0, V_1)\) be closed twin value sets for the continued fraction \(K(a_n/1)\). Then it follows from (1.2) and (1.4) that

\[
(3.1) \quad \Delta_n := S_n(V_n) = S_{n-1} \circ s_n(V_n) \subseteq S_{n-1}(V_{n-1}) = \Delta_{n-1} \subseteq \cdots \subseteq \Delta_0 = V_0,
\]

where \(V_{2n} := V_0\) and \(V_{2n+1} := V_1\) for all \(n\). Since all \(s_n\) are (non-singular) linear fractional transformations, so are also \(S_n\) (see (1.4)). Therefore, since \(V_n\) is circular, also \(\Delta_n\) is a circular domain. The nestedness (3.1) implies that \(\Delta_n\) converges to a limit set \(\Delta\). If \(\Delta\) just contains one point, the limit point case, then \(\{S_{2n}\}\) and \(\{S_{2n+1}\}\) converge uniformly in \(V_0\) and \(V_1\) respectively to the limit point \(c\). Since both \(V_0\) and \(V_1\) contain more than one point in our cases, \(K(a_n/1)\) converges generally to \(c\) in this case. If the limit set \(\Delta\) has positive or infinite radius, the limit circle case, we need to investigate further. That \(\Delta\) is a circular set in this case was proved by Thron [7, thm. 4.2B, p. 66].

In special cases classical convergence to \(c\) may be wanted. This may be possible to prove by means of Theorem 1.4. This theorem is partly based on Theorem 3.1 below, which concerns restrained sequences introduced in [4]: we say that a sequence \(\{F_n\}\) of linear fractional transformations is restrained if there exist two sequences \(\{u_n\}\) and \(\{v_n\}\) from \(\overline{\mathbb{C}}\) such that

\[
(3.2) \quad \lim d(F_n(u_n), F_n(v_n)) = 0 \quad \text{and} \quad \liminf d(u_n, v_n) > 0.
\]

If in addition \(\lim F_n(u_n) = c\), then we say that \(\{F_n\}\) converges generally to \(c\). As in (1.9) there exists an exceptional sequence \(\{z_n^\dagger\}\) for \(\{F_n\}\) such that if (3.2) holds, then (see [4])

\[
(3.3) \quad \lim d(F_n(z_n), F_n(u_n)) = 0 \quad \text{whenever} \quad \liminf d(z_n, z_n^\dagger) > 0.
\]

**Theorem 3.1.** Let \((V_0, V_1)\) be closed twin value sets for the continued fraction \(K(a_n/1)\) where \(V_0\) or \(V_1\) contains more than one element. Let \(k \in \{0, 1\}\) be fixed, and let \(\{S_{2n+k}\}\) be restrained with exceptional sequence \(\{z_n^\dagger\}\). Then the limit points for \(\{z_n^\dagger\}\) are contained in \((-1 - V_{k+1}) \setminus V_k^c\), and whenever \(\liminf d(u_n, z_n^\dagger) > 0\), the set \(L\) of the limit points for \(S_{2n+k}(u_n)\) is independent of \(\{u_n\}\) and \(L \subseteq V_0 \setminus (-1 - V_1^c)\).

**Proof.** Since either \(V_0\) or \(V_1\) contains at least two points, they both do since \(a_{2n}/(1 + V_0) \subseteq V_1\) and \(a_{2n+1}/(1 + V_1) \subseteq V_0\). Since \(V_k\) contains more than one point, there exists a sequence \(\{v_n\}\) from \(V_k\) with \(\liminf d(v_n, z_n^\dagger) > 0\). By (3.1) it follows that \(S_{2n+k}(V_k) \subseteq V_0\) for all \(n\). It follows from (3.3) that \(L\) is independent of \(\{u_n\}\) when \(\liminf d(u_n, z_n^\dagger) > 0\), and thus \(L \subseteq V_0\). Similarly, by Remark 1.2.2, \(L \subseteq W_0 = -1 - V_1^c\), so \(L \subseteq V_0 \cap W_0 = Y_0 = V_0 \setminus (-1 - V_1^c)\).

Evidently, \(\{z_n^\dagger\}\) can be chosen as \(z_n^\dagger := S_{2n+k}^{-1}(p)\) for any \(p \notin L\). By (3.3) every exceptional sequence has the same asymptotic behavior. Let \(p \notin V_0\). Then \(z_n := S_{2n+k}^{-1}(p) \in V_k^c\) for all \(n\). Similarly, for \(q \in W_k^c\) given by \(W_k := (-1 - V_k^{-V_{k+1}})\), we can choose \(z_n^\dagger := S_{2n+k}^{-1}(q)\) for all \(n\), and then \(z_n^\dagger \in W_k^c\) for all \(n\). (See Remark 1.2.2.) Hence all the limit points of \(\{z_n^\dagger\}\) are \(\subseteq W_k^c \cap V_k^c = (-1 - V_{k+1}) \setminus V_k^c\). \(\square\)

Since \(V_0\) is a circular domain, there exists a linear fractional transformation \(\varphi_0\) such that \(\varphi_0(V_0) = \mathbb{D}\). Hence the following result from [10] is useful to establish convergence in the limit circle case.
Theorem 3.2 ([11] thm. 3.8, 3.10]). Let \( \{t_n\} \) be linear fractional transformations with \( t_n(\mathbb{D}) \subseteq \mathbb{D} \), and let \( T_n := t_1 \circ t_2 \circ \cdots \circ t_n \) for all \( n \in \mathbb{N} \). If \( R := \lim \text{rad}(T_n(\partial \mathbb{D})) > 0 \), and there exists a set \( I \subseteq \mathbb{N} \) such that

\[
\limsup_{n \to \infty} \text{rad}(t_n(\partial \mathbb{D})) < 1 \quad \text{and} \quad \liminf_{n \to \infty} \text{rad}(t_{n-1}^{-1}(\partial \mathbb{D})) > 1,
\]

then \( |T_n^{-1}(\infty)| \to 1 \) and \( \sum_{n=1}^{\infty} |T'_n(0)| < \infty \).

Remarks 3.3.

1. Of course, if \( I \) is bounded, then the first condition in (3.4) is void, and if \( \mathbb{N} \setminus I \) is bounded, then the second one is void.
2. The conclusion \( \sum |T'_n(0)| < \infty \) for the derivatives \( T'_n \) implies that \( \{T_n\} \) is restrained. (Proof: \( T_n \) can be written \( T_n(z) = C_n + R_ne^{i\omega_n} \frac{z - Q_n}{1 - Q_nz} \) for some \( |Q_n| < 1 \) and \( \omega_n \in \mathbb{R} \) when \( T_n(\mathbb{D}) = B(C_n, R_n) \), and thus \( T'_n(z) = R_ne^{i\omega_n}(1 - |Q_n|^2)/(1 - Q_nz)^2 \).

Hence \( T'_n(z) \to 0 \) for all \( z \in \mathbb{D} \). Indeed, \( \sum |T'_n(z)| < \infty \) for every \( z \in \mathbb{D} \).

Let \( \mathcal{M} \) be the family of (non-singular) linear fractional transformations. For given \( V \subseteq \mathbb{C} \) and \( \varepsilon > 0 \) we introduced the subfamily

\[
(3.5) \quad \mathcal{M}_\varepsilon(V) := \{t \in \mathcal{M}: t(V) \subseteq V \setminus B_d(z, \varepsilon) \text{ for some } z \in \partial V \}
\]

in [11]. This notation is useful to convert Theorem 3.2 to our situation:

Corollary 3.4. Let \( k \in \{0, 1\} \) be fixed, and let \( (V_0, V_1) \) be closed circular twin value sets for the continued fraction \( \mathcal{K}(a_n/1) \) where the limit circle case occurs. Furthermore, let \( \sigma_n := s_{2n-1+k} \circ s_{2n+k}, \sigma_0 := \sigma_1 \) and assume that

\[
(3.6) \quad \sigma_n \in \mathcal{M}_\varepsilon(V_k) \text{ for all } n \in I \quad \text{and} \quad \sigma_n^{-1} \in \mathcal{M}_\varepsilon(V_k) \text{ for all } n \in \mathbb{N} \setminus I
\]

for some \( I \subseteq \mathbb{N} \) and \( \varepsilon > 0 \). Then \( \{S_{2n+k}\} \) is restrained and its exceptional sequences \( \{z^1_n\} \) have all their limit points in \( \partial V_k \). If also \( V_0 \) is bounded, then \( \{\zeta_{2n+k}\} \) is an exceptional sequence for \( \{S_{2n+k}\} \) and \( \sum_{n=1}^{\infty} |S_{2n+k}^0(z)| < \infty \) for every finite \( z \in V_k \).

Proof. Let \( \varphi \in \mathcal{M} \) satisfy \( \varphi(V_k) = \mathbb{D} \). Then \( t_n := \varphi \circ \sigma_n \circ \varphi^{-1} \) maps \( \mathbb{D} \) into \( \mathbb{D} \), and \( T_n := t_1 \circ t_2 \circ \cdots \circ t_n = \varphi \circ S_{2n}^{(k)}(z) \circ \varphi^{-1} \) where \( S_{2n}^{(k)} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n \). Condition (3.6) implies (3.4). Hence \( \{T_n\} \) is restrained with exceptional sequence \( \{T_n^{-1}(\infty)\} \) where \( |T_n^{-1}(\infty)| \to 1 \). Therefore \( \{S_{2n}^{(k)}\} \) is restrained with exceptional sequence \( z^1_n := \varphi^{-1} \circ T_n^{-1}(\infty) = (S_{2n}^{(k)})^{-1}(\varphi^{-1}(\infty)) \). That \( \{S_{2n+k}\} \) is restrained with exceptional sequence \( \{z^1_n\} \) follows therefore since \( S_{2n} = S_{2n}^{(0)} \) and \( S_{2n+1} = \sigma_1 \circ S_{2n}^{(1)} \) for the fixed \( \sigma_1 \in \mathcal{M} \). Since \( |T_n^{-1}(\infty)| \to 1 \), i.e., \( \text{dist}(T_n^{-1}(\infty), \partial \mathbb{D}) \to 0 \), it follows that \( d(\varphi^{-1} \circ T_n^{-1}(\infty), \varphi^{-1}(\partial \mathbb{D})) \to 0 \) where \( \varphi^{-1}(\partial \mathbb{D}) = \partial V_k \) and \( \varphi^{-1} \circ T_n^{-1}(\infty) = z^1_n \). That is, all the limit points of \( \{z^1_n\} \) are in \( \partial V_k \).

Let \( V_0 \) be bounded. Then \( \varphi \notin V_0 \), so \( \{\zeta_{2n+k}\} \) is an exceptional sequence for \( \{S_{2n+k}\} \) since \( S_{2n+k}(\zeta_{2n+k}) = \infty \) whereas all the limit points for \( \{S_{2n+k}(u_n)\} \) are in \( V_0 \) when \( \text{lim inf} d(u_n, z^1_n) > 0 \) (Theorem 3.1). It remains to prove that \( \sum |S_{2n+k}^0(z)| < \infty \) for finite \( z \in V_k \).

By Theorem 3.2 and Remark 3.3.2 we know that \( \sum |T'_n(w)| < \infty \) for every \( w \in \mathbb{D} \). First let \( k = 0 \) and choose \( \varphi(z) := (z - C_0)/R_0 \) where \( C_0 \) and \( R_0 \) are the center and radius of \( V_0 \). Let \( z \in V_0 \) be arbitrarily chosen, and let \( w := \varphi(z) \). Then \( w \in \mathbb{D} \) and \( S_{2n}^0(z) = (\varphi^{-1})'(T_n(\varphi(z))) \).
$T_n'(\varphi(z)) \cdot \varphi'(z) = (\varphi^{-1})'(T_n(w)) \cdot T_n'(w) \cdot \frac{1}{R_0} = R_0 \cdot T_n'(w) \cdot \frac{1}{R_0} = T_n'(w)$. Hence
\[ \sum |S_n'(z)| < \infty. \]

Next let $k = 1$ and set $\hat{V}_0 := s_1(V_1)$. Then $\hat{V}_0 = B(\hat{C}_0, \hat{R}_0) \subseteq V_0$ for some fixed $\hat{C}_0 \in \mathbb{C}$ and $\hat{R}_0 > 0$. Furthermore, let $\varphi_1(z) := (z - \hat{C}_0)/\hat{R}_0$ so that $\varphi_1(\hat{V}_0) = \mathbb{D}$ and $t_n := \varphi_1 \circ s_1 \circ s_2n \circ s_{2n+1} \circ s_1^{-1} \circ \varphi_1^{-1}$ maps $\mathbb{D}$ into $\mathbb{D}$. Let a finite $z \in V^*_1$ be arbitrarily chosen, and let $w := \varphi_1 \circ s_1(z)$. Then $w \in \mathbb{D}$ and
\[ S_{2n+1}'(z) = (\varphi_1^{-1})'(T_n \circ \varphi_1 \circ s_1(z)) \cdot T_n'(\varphi_1 \circ s_1(z)) \cdot \varphi_1'(s_1(z)) \cdot s_1'(z) \]
\[ = \hat{R}_0 \cdot T_n'(w) \cdot \frac{1}{\hat{R}_0} \cdot \frac{-a_1}{(1 + z)^2} = \frac{-a_1}{(1 + z)^2} T_n'(w) \]
where $z \neq -1$ since $-1 \notin V_1$ when $V_0$ is bounded. Hence $\sum |S_{2n+1}'(z)| < \infty$. \(\square\)

It follows from (1.6) that $S_n$ can be written
\[ S_n(z) = \begin{cases} c_n - \frac{\zeta_n}{z - \zeta_n} & \text{if } \zeta_n \neq \infty, \\ c_n - c_n - c_{n-2} & \text{if } \zeta_n = \infty. \end{cases} \]

Therefore
\[ S_n'(z) = \begin{cases} \frac{\zeta_n(c_n - c_{n-1})}{(z - \zeta_n)^2} & \text{if } \zeta_n \neq \infty, \\ \frac{c_n - c_{n-2}}{z - \zeta_n} & \text{if } \zeta_n = \infty. \end{cases} \]

Under the conditions of Corollary 3.4 it follows therefore that for arbitrary $\varepsilon > 0$,
\[ \sum |S_{2n+k}(z_n) - c_{2n+k-1}| < \infty \]
whenever $\varepsilon \leq \text{dist}(z_n, Z_k) \leq \frac{1}{\varepsilon}$ for all $n$ and $\infty \notin Z_k$.

(For the notation $\sum'$ and $Z_k$, see the list of notation in section 1.) This leads to the following result, where $W_k := -1 - V_{k+1}^*$ and $\partial^*V_k := \partial V_k \cap (-1 - V_{k+1})$ as usual.

**Theorem 3.5.** Let $k \in \{0, 1\}$ be fixed. Let $\langle V_0, V_1 \rangle$ be closed circular twin value sets for the continued fraction $K(a_n/1)$ where $V_0$ is bounded, the limit circle case occurs and (3.6) holds for our $k$ for some $I \subseteq \mathbb{N}$ and $\varepsilon > 0$. Then $Z_k \subseteq \partial^*V_k$, $-k \notin Z_k$, $0 \notin Z_0$ and $Z_1$ and $Z_k$ are bounded, $\sum' |c_{2n} - c_{2n-1}| < \infty$, and the following statements are true.

A. Let $\varepsilon > 0$. If $(k - 1) \notin Z_k$ or if for each $n$ from some $n_0$ on, either
\[ \text{dist}(a_{2n+k-1}, Z_k) \geq \varepsilon \text{ or dist}(-1 - a_{2n+k}, Z_k) \geq \varepsilon, \]
then $\sum' |c_n - c_{n-1}| < \infty$.

B. If also the limit circle case occurs for $S_{2n}(W_0)$ and
\[ \sigma_n \in \mathcal{M}_c(W_k) \text{ for } n \in I \quad \text{and} \quad \sigma_n^{-1} \in \mathcal{M}_c(W_k) \text{ for } n \in \mathbb{N} \setminus I \]
for some $I \subseteq \mathbb{N}$ and $\varepsilon > 0$ for $\sigma_n$ as in Corollary 3.4, then $Z_k \subseteq \partial^*V_k$.

**Proof.** Under our conditions, $\{S_{2n+k}\}$ is restrained with exceptional sequence $\{\zeta_{2n+k}\}$, $Z_k \subseteq (-1 - V_{k+1}) \cap \partial^*V_k = \partial^*V_k$ and $Z_{k+1} \subseteq (-1 - V_{k+1}) \setminus V_{k+1}$ (Theorem 3.1 and Corollary 3.4). Now, $V_0$ is bounded, so $-1 \notin V_1$, and thus $0 \notin Z_0$ and $-k \notin Z_k$, and $Z_0$ and $Z_1$ are bounded. Since $S_{2n}(0) = c_{2n}$ and $S_{2n+1}(-1) = c_{2n-1}$, it follows therefore from (3.9) that $\sum' |c_{2n} - c_{2n-1}| < \infty$.

A. It suffices to prove that either $\sum' |c_{2n-2} - c_{2n-1}| < \infty$ or $\sum' |c_{2n+m} - c_{2n+m-2}| < \infty$ for an $m \in \{0, 1\}$. First let $(k - 1) \notin Z_k$. If $k = 0$, this means that
\( \sum' |S_{2n}(-1) - c_{2n-1}| < \infty \) by (3.9) where \( S_{2n}(-1) = c_{2n-2} \). If \( k = 1 \), then \( 0 \not\in \mathbb{Z}_1 \) and \( \sum' |S_{2n+1}(0) - c_{2n}| < \infty \). Next let \( I := \{ n \in \mathbb{N}; \text{dist}(a_{2n+k-1}, \mathbb{Z}_k) \geq \varepsilon \} \). Then
\( \sum_{n \in I} |S_{2n+k-2}(a_{2n+k-1}) - c_{2n+k-3}| < \infty \) where \( S_{2n+k-2}(a_{2n+k-1}) = c_{2n+k-1} \) and \( \sum_{n \not\in I} |S_{2n+k}(-1 - a_{2n+k}) - c_{2n+k-1}| < \infty \) where \( S_{2n+k}(-1 - a_{2n+k}) = c_{2n+k-3} \), which means that \( \sum' |c_{2n+k-1} - c_{2n+k-1}| < \infty \).

B. \( \langle W_0, W_1 \rangle \) are twin value sets for \( K(a_n/1) \) (Remark 1.2.2). They satisfy the conditions in Corollary 3.4, so the exceptional sequences for \( \{S_{2n+k}\} \) all have their limit points in \( \partial W_k \). Hence \( Z_k \subseteq \partial V_k \cap \partial W_k = \partial V_k \cap (-1 - \partial V_{k+1}) = \partial V_k \). \( \square \)

4. Proofs

Inspired by (3.5) we define

\[
(4.1) \quad \mathcal{M}(V, W) := \{ t \in \mathcal{M}; t(V) \subseteq W \setminus B_d(z, \varepsilon) \text{ for some } z \in \partial W \},
\]

\[
(4.2) \quad \mathcal{E}(V) := \{ (A, B) \subseteq \mathbb{C}^2; \exists \varepsilon > 0 \text{ s.t. } a_1 \circ a_2 \in \mathcal{M}(V) \text{ for all } \langle a_1, a_2 \rangle \in \langle A, B \rangle \},
\]

\[
(4.3) \quad \mathcal{E}(V, W) := \{ A \subseteq \mathbb{C}; \exists \varepsilon > 0 \text{ s.t. } s \in \mathcal{M}(V, W) \text{ for all } a \in A \}.
\]

**Proof of Theorem 1.4.** Since \( K(a_n/1) \) converges generally to \( c \) whereas \( q \neq c \), the sequence \( \{S_n\} \) is restrained with exceptional sequence \( z_n^c := S_n^{-1}(q) \). Part A follows from Theorem 3.1. The result in B follows from (1.9) since \( S_{2n+k}(-1) = c_{2n+k-2} \) and \( S_{2n+k}(\infty) = c_{2n+k-1} \). Similarly, part C follows from (1.9) since \( S_{2n+k-2}(a_{2n+k-1}) = c_{2n+k-1} \) and \( S_{2n+k+2}(-1 - a_{2n+k+2}) = c_{2n+k-1} \).

To prove part D we observe that if \( V_0 \) is bounded, then \( c \neq \infty \) and \( \infty \not\in \partial \mathcal{Z}_1 \) by part A. Hence \( \{z_n\} \) is exceptional and \( S_{2n+1}(\infty) = c_{2n} \to c \) by part B. Finally, if \( -1 \notin V_0 \setminus (-1 - V_0) \), i.e., \( 0 \notin (-1 - V_0) \setminus V_0 \), then \( 0 \notin \partial \mathcal{Z}_1 \), and part E follows from part B. (The same holds true if \( 0 \notin V_0 \), but \( 0 \notin V_0 \Rightarrow \infty \notin V_1 \Rightarrow -1 \notin V_0 \). ) \( \square \)

**Lemma 4.1.** For given closed twin value sets \( \langle V_0, V_1 \rangle \), let \( U_k := -1 - V_{k+1}^c \) for \( k = 0, 1 \), and let \( k \in \{0, 1\} \) be a fixed number. Then \( s(U_k) \subseteq U_{k+1} \) if and only if \( s(V_k) \subseteq V_{k+1} \) and \( s(U_k) = U_{k+1} \) if and only if \( s(V_k) = V_{k+1} \). Similarly, if \( A \subseteq \mathbb{C} \) is a closed set with \( 0, \infty \notin A \), then \( A \in \mathcal{E}(U_k, U_{k+1}) \) if and only if \( A \in \mathcal{E}(V_k, V_{k+1}) \).

**Proof.** Let \( a/(1 + V_k) \subseteq V_{k+1} \). Since \( V_k \) is closed, the set \( V_k^c \) is open and non-empty, and both \( V_k, V_{k+1}, U_k \) and \( U_{k+1} \) contain finite elements. Therefore

\[
as \left(\frac{a}{1 + V_k}\right) = -\frac{a}{V_{k+1}} = -\left(\frac{a}{V_{k+1}}\right)^c \subseteq (-1 - V_k)^c = U_{k+1}.
\]

This actually proves the first two equivalences since \( U \) and \( V \) can be interchanged in this inclusion. Let \( a/(1 + V_k) \subseteq V_{k+1} \setminus B_d(z, \varepsilon) \) for some finite \( z \in \partial V_{k+1} = -1 - \partial U_k \) and \( \varepsilon > 0 \). That is, \( a/U_{k+1} \supseteq -V_{k+1} \setminus B_d(z, \varepsilon) \) and \( U_k \cup B_d(z, \varepsilon) = 1 + U_k \cup B_d(z, \varepsilon) \) where \( z^* := -1 - z \in \partial U_k \). That is, \( s(U_{k+1}) \supseteq U_k \cup B_d(z^*, \varepsilon) \), and \( s(U_k) \subseteq B_d(z^*, \varepsilon) \). Let \( D := B_d(z^*, \varepsilon) \setminus U_k \) so that \( U_k \cap D = \emptyset \) and \( U_k \cup D = U_k \cup B_d(z^*, \varepsilon) \). Then

\[
as \left(\frac{a}{1 + U_k}\right) = \frac{a}{1 + U_k} \cup \frac{a}{1 + D} \subseteq U_{k+1}; \quad \text{i.e.,} \quad \frac{a}{1 + U_k} \subseteq U_{k+1} \setminus \frac{a}{1 + D}
\]

where \( a/(1 + z^*) \subseteq V_{k+1} \). Therefore \( a/(1 + U_k) \subseteq U_{k+1} \setminus B_d(\frac{a}{1 + z^*}, \varepsilon) \) where \( \varepsilon' := \text{dist}(\frac{a}{1 + z^*}, \frac{a}{1 + U_k}) \). Since \( 0, \infty \notin A \), the quantity \( \varepsilon' \) has a positive lower bound for \( a \in A \). Therefore \( A \in \mathcal{E}(U_k, U_{k+1}) \). This proves the last equivalence. \( \square \)
Lemma 4.2. Let $V_0$, $V_1$ be closed circular domains, and let $a_1, a_2 \in \mathbb{C}\setminus\{0\}$ satisfy $a_k/(1 + V_k) \subseteq V_{k-1}$ for $k = 1, 2$. Then $\sigma := s_1 \circ s_2$ is an elliptic transformation if and only if $s_k(V_k) = V_{k-1}$ for $k = 1, 2$ and $\sigma$ has exactly two distinct fixed points $w_0, w_1 \notin \partial V_0$.

Proof. Let $\sigma$ be elliptic. Since $\sigma(V_0) \subseteq V_0$, it follows from \[\text{thm. 1.4} \] that $\sigma(V_0) = V_0$. Since $V_0 = s_1 \circ s_2(V_0) \subseteq s_1(V_1) \subseteq V_0$, this means that $s_1(V_1) = V_0$ and $s_2(V_0) = V_1$. It is clear that $\sigma$ has two distinct fixed points $w_0, w_1$ and that $\partial V_0$ is a fixed circle (or fixed line) for $\sigma$. Hence $\partial V_0$ separates the two fixed points.

Conversely, assume that $s_k(V_k) = V_{k-1}$ for $k = 1, 2$ and that $\sigma$ has two distinct fixed points $w_0 \notin \partial V_0$. Then $\sigma(\partial V_0) = \partial V_0$, which means that $\sigma$ is either hyperbolic, parabolic, elliptic or the identity transformation. Since $\sigma$ has exactly two distinct fixed points, the parabolic case and the identity case are ruled out. Since none of the fixed points lie on $\partial V_0$, the hyperbolic case is ruled out, so $\sigma$ is elliptic. \[\square\]

Lemma 4.3 (The disk – disk case). Let $V_k := B(C_k, R_k)$ for $k = 0, 1$, where $C_k \in \mathbb{C}$ and $R_k > 0$ satisfy (2.1). Then $\langle E_1, E_2 \rangle$ given by (2.4) are the corresponding element sets. Let $k \in \{1, 2\}$ be fixed. Then $E_k \neq \emptyset$ if and only if (2.2) holds. If (2.2) holds with strict inequality, then $E_0^\circ \neq \emptyset$. If (2.2) holds with equality, then $E_k = \{\tilde{a}_k\}$ is given by (2.3) and $\tilde{a}_k \neq 0$. If $E_k^\circ \neq \emptyset$, then $\langle E_k, E_{k+1} \rangle \in \mathcal{E}(V_{k-1})$.

Proof. For fixed $k \in \{1, 2\}$ and $a \neq 0$ we have

\[a/(1 + V_k) \subseteq V_{k-1}\]

\[\text{if and only if } (\tilde{C}_{k-1} - C_{k-1}) + \tilde{R}_{k-1} \leq R_{k-1}, \text{ that is, if and only if } \tilde{a}_k \in E_k, \text{ where } E_k \text{ is given by (2.4).} \]

Since $R_k < |1 + C_k|$, we see from (2.4) that $E_k \neq \emptyset$ if and only if $\tilde{a}_k \in E_k$, which proves that (2.2) is necessary and sufficient. It also proves that $\tilde{a}_k$ is the only point in $E_k$ if and only if (2.2) holds with equality, and that $E_k^\circ \neq \emptyset$ otherwise. This means that if $E_k^\circ \neq \emptyset$, then $s \in \mathcal{M}_{\varepsilon_a}(V_k, V_{k-1})$ for some $\varepsilon_a > 0$ for every $a \in E_k$. Since $E_k$ is compact in $\mathbb{C}$ ($-1 \notin V_k$ when $V_{k-1}$ is bounded), this means that $E_k \in \mathcal{E}(V_k, V_{k-1})$. Finally, since $s_k \circ s_{k+1}(V_{k+1}) \subseteq s_k(V_k)$ for all $(a_k, a_{k+1}) \in \langle E_k, E_{k+1} \rangle$, it follows that $\langle E_k, E_{k+1} \rangle \in \mathcal{E}(V_{k-1})$. \[\square\]

Proof of Theorem 2.1. If $|C_{k-1}| R_k = |1 + C_k| R_{k-1}$ for $k = 1$ and $k = 2$, then $K(a_n/1)$ with all $a_{2n-1} = \hat{a}_1$ and $a_{2n} = \hat{a}_2$ is the only continued fraction from $\langle E_1, E_2 \rangle$. It converges if and only if $s_1 \circ s_2$ is non-elliptic. Let (2.2) hold with strict inequality for at least one $k \in \{1, 2\}$. Without loss of generality we assume that $E_1^\circ \neq \emptyset$. (See Remark 1.2.1.)

Assume first that the limit point case occurs. Then $K(a_n/1)$ converges generally to a value $c \in V_0$. It follows by Lemma 1.4D that $e_{2n} \to c$. Since also $V_1$ is bounded, we have $-1 \notin V_0$, so also $e_{2n+1} \to c$ by Lemma 1.4E.

Assume next that the limit circle case occurs. By Lemma 4.3 we know that $\limsup \mathrm{rad}(s_{2n-1} \circ s_{2n}(V_0)) < \mathrm{rad}(V_0)$, and so $Z_0 \subseteq \partial V_0$ by Theorem 3.5. Now, $-1 \notin V_0$ implies that $-1 \notin \partial V_0$. Hence $\sum \left| c_n - c_{n-1} \right| < \infty$ by Theorem 3.5A, and thus $K(a_n/1)$ converges. \[\square\]

Lemma 4.4 (The disk – half plane case). Let $V_0 := B(C_0, R_0)$ and $V_1 := H(h \cos \alpha, \alpha)$ where $C_0 \in \mathbb{C}$ and $R_0, h, \alpha \in \mathbb{R}$ satisfy (2.5). Then $\langle E_1, E_2 \rangle$ given by
(2.7) are the corresponding element sets, and \( \hat{E}_{k, \delta} \) given by (2.8) satisfies \( \hat{E}_{k, \delta} \in E(V_k, V_{k-1}) \) for \( k = 1, 2 \) and \( 0 < \delta < |a^*_1| \).

**Proof.** For \( a \neq 0 \) we have

\[
(4.5) \quad \frac{a}{1 + V_1} = B \left( \frac{a e^{-i\alpha}}{2(1+h) \cos \alpha}, \frac{|a|}{2(1+h) \cos \alpha} \right)
\]

which is \( \subseteq V_0 \) if and only if \( \frac{|a| e^{-i\alpha}}{2(1+h) \cos \alpha} - C_0 |a| + \frac{|a|}{2(1+h) \cos \alpha} \leq R_0 \), i.e., if and only if \( a \in E_1 \). Since \( 0 \notin V_0 \) and \( 0/(1 + V_1) = \{0\} \), we also have \( 0 \in E_1 \). Similarly, for \( a \neq 0 \),

\[
(4.6) \quad \frac{a}{1 + V_0} = \begin{cases} 
B \left( \frac{a(1 + C_0) e^{-i\alpha}}{|1 + C_0|^2 - R_0^2}, \frac{|a| R_0}{|1 + C_0|^2 - R_0^2} \right) =: B(\bar{C}_1, \bar{R}_1) & \text{if } |1 + C_0| > R_0, \\
H \left( |a|/(2R_0), \arg(a(1 + C_0)) \right) & \text{if } |1 + C_0| = R_0,
\end{cases}
\]

and thus \( a/(1 + V_0) \subseteq V_1 \) if and only if

\[
(4.7) \quad \begin{cases} 
\text{Re} \left( \frac{a(1 + C_0)}{|1 + C_0|^2 - R_0^2} e^{-i\alpha} \right) - \frac{|a| R_0}{|1 + C_0|^2 - R_0^2} \geq h \cos \alpha & \text{if } |1 + C_0| > R_0, \\
\arg(a(1 + C_0)) = \alpha & \text{and } \frac{|a|}{2R_0} \geq h \cos \alpha & \text{if } |1 + C_0| = R_0,
\end{cases}
\]

i.e., if and only if \( a \in E_2 \). If \( -1 \in V_0 \), i.e., \( |1 + C_0| = R_0 \), then \( 0 \notin E_2 \) by definition. Hence \( \langle E_1, E_2 \rangle \) are the element sets corresponding to \( (V_0, V_1) \).

By (4.5) it follows that \( a/(1 + V_1) = V_0 \) if and only if \( R_0 = |C_0| \) and \( C_0 = a e^{-i\alpha}/(2(1 + h) \cos \alpha) \), i.e., \( a = a^*_1 \). Since \( -1 \notin V_1 \), the set \( E_1 \) is compact, so this shows that \( E_1 \in E(V_1, V_0) \) if \( R_0 > |C_0| \). Let \( R_0 = |C_0| \). Since \( \bar{E}_{1, \delta} \subseteq E_1 \) is a compact set not containing \( a^*_1 \), \( \bar{E}_{1, \delta} \in E(V_1, V_0) \).

Next we study \( \hat{E}_{2, \delta} \). First let \( |1 + C_0| = R_0 \). By (4.6) it follows that \( a/(1 + V_0) = V_1 \) for \( a \neq 0 \) if and only if \( h > 0 \) and \( q := \frac{a(1 + C_0)}{|1 + C_0|^2 - R_0^2} = h e^{i\alpha} \cos \alpha \), i.e., \( a = a^*_2 \). In this case \( a^*_2 \neq 0 \) and \( E_2 \) is the ray \( \hat{E}_2 = a^*_2 [1, \infty) \) and \( \hat{E}_{2, \delta} = a^*_2 [1 + \delta/|a^*_2|, \infty) \). Hence \( \hat{E}_{2, \delta} \) is a closed set in \( \mathbb{C} \) with \( 0 \notin \hat{E}_{2, \delta} \), and even if \( a_{2m} \to \infty \) as \( m \to \infty \), the set \( \frac{a_{2m}}{1 + V_0} \) will not approach \( V_1 \). (Indeed, it approaches the point set \( \{\infty\} \) since \( V_0 \) is bounded.) Therefore \( \hat{E}_{2, \delta} \in E(V_0, V_1) \) if \( h > 0 \). If \( h < 0 \), then \( \text{dist}(q - h e^{i\alpha} \cos \alpha) > |h| \cos \alpha > 0 \), and \( E_2 \in E(V_0, V_1) \). If \( h = 0 \), then \( \hat{E}_{2, \delta} = [\delta, \infty) e^{i\gamma} \) with \( \gamma := \alpha + \arg(1 + C_0) \), and \( a/(1 + V_0) \) is the half plane \( H(|a|/2R_0, \arg(a(1 + C_0))) = H(|a|/2R_0, \alpha) \) for \( a \in E_2 \). Hence also now \( \hat{E}_{2, \delta} \in E(V_0, V_1) \).

Next let \( |1 + C_0| > R_0 \). Then it follows from (4.6) that \( a/(1 + V_0) = B(\bar{C}_1, \bar{R}_1) \) is a disk not containing the origin for \( a \neq 0 \). If \( a = 0 \), then \( a/(1 + V_0) = \{0\} \) since \( -1 \notin V_0 \). Hence, there is no possibility of \( B(\bar{C}_1, \bar{R}_1) \to V_1 \) unless \( \bar{R}_1 \to \infty \); i.e., \( |a| \to \infty \), but then \( a/(1 + V_0) \to \{\infty\} \) since \( V_0 \) is bounded. Hence \( E_2 \in E(V_0, V_1) \) in this case.

**Proof of Theorem 2.3.** Let \( K(a_n/1) \) be a continued fraction from \( \langle E_1, E_2 \rangle \). If \( \text{rad}(S_{2n}(V_0)) \to 0 \) or \( \text{rad}(S_{2n}(W_0)) \to 0 \) or \( \text{diam}(S_{2n}(V_0)) \to 0 \), then \( K(a_n/1) \) clearly converges generally. Assume in the proof of parts A–C below that \( \text{diam}(S_{2n}(V_0)) \to \delta > 0 \), and thus \( \text{rad}(S_{2n}(V_0)) \to R > 0 \) and \( \text{rad}(S_{2n}(W_0)) \to R^* > 0 \).
A. Let \( K(a_n/1) \) be from \( \langle E_1, \hat{E}_{2,\delta} \rangle \). Then \( s_{2n} \circ s_{2n+1}(V_1) \subseteq s_{2n}(V_0) \) where \( a_{2n} \in \hat{E}_{2,\delta} \in \mathcal{E}(V_0, V_1) \) by Lemma 4.4, so \( \langle \hat{E}_{2,\delta}, E_1 \rangle \in \mathcal{E}(V_1) \). Therefore \( K(a_n/1) \) converges in the classical sense if \( 0 \not\in Z_1 \) (Theorem 3.5A with \( k = 1 \)).

Let \( 0 \in Z_1 \). Since by Theorem 3.5, \( Z_1 \subseteq \partial^* V_1 \), this means that \( 0 \in \partial V_1 \), and \(-1 \in V_0 \), which means that \(-1 \in \partial V_0 \) by (2.5), so indeed, \( 0 \not\in \partial V_1 \). Then \( h = 0 \), and thus \( \hat{a}_2 = 0 \), and \( R_0 = |1 + C_0| \) and \( \hat{E}_{2,\delta} = e^{\gamma}|\delta, \infty| \) where \( \gamma := \alpha + \arg(1 + C_0) \).

This means that \( \text{dist}(\hat{E}_{2,\delta}, \partial^* V_1) > 0 \) unless \( \hat{E}_{2,\delta} \subseteq \partial V_1 \). Now, \( \Re(C_0) \geq -\frac{1}{2} \) when \(-1 \in \partial V_0 \) since \( 0 \not\in V_0 \) by (2.5) and \( V_0 \) is a disk. Therefore \( \gamma \not= \alpha \pm \frac{\pi}{2} \), and \( \hat{E}_{2,\delta} \not\subseteq \partial V_1 \). Hence \( K(a_n/1) \) still converges by Theorem 3.5A.

B. Let \( K(a_n/1) \) be from \( \langle \hat{E}_{1,\delta}, E_2 \rangle \) and let \( 0 \not\in \partial^* V_1 \). If \( \hat{E}_{2,\delta} = E_2 \), then the situation is covered by part B, so let \( R_0 = |1 + C_0| \) and \( h \geq 0 \). That is, \(-1 \in \partial V_0 \) and \( 0 \not\in V_0 \) and so, \( 0 \not\in V_1 \) under our conditions. Now, \( a_{2n-1} \in \hat{E}_{1,\delta} \in \mathcal{E}(V_1, V_0) \) by Lemma 4.4, so \( \langle \hat{E}_{1,\delta}, E_2 \rangle \in \mathcal{E}(V_0) \). The result follows therefore from Theorem 3.5A since \( 0 \not\in V_1 \) implies \( -1 \not\in \partial V_0 \), and thus \( -1 \not\in Z_0 \).

C. Let \( K(a_n/1) \) be from \( \langle \hat{E}_{1,\delta}, E_2 \rangle \). By Lemma 4.4, \( \langle \hat{E}_{1,\delta}, E_2 \rangle \in \mathcal{E}(V_0) \). Hence \( Z_0 \subseteq \partial^* V_0 \) by Theorem 3.5. The convergence follows therefore from Theorem 3.5A.

D. That \( c_{2n} \to c \) follows from Theorem 1.4D. We know that \( Z_k \subseteq (-1 - V_{k+1}) \setminus V_k \) by Theorem 1.4A. Therefore \( 0 \not\in Z_1 \) if \( 0 \not\in V_1 \) or \( -1 \not\in V_0 \), which in our situation holds if \(-1 \not\in \partial V_0 \), and \( c_n \to c \) by Theorem 1.4E.

The conditions on \( \{a_n\} \) imply that \( \text{dist}(a_{2n-1}, Z_0) \geq \varepsilon \) from some \( n \) on (Theorem 3.5), and thus \( c_{2n-1} \to c \) by Theorem 1.4C. \( \square \)

**Lemma 4.5** (The disk - complement of disk case). Let \( V_0 := B(C_0, R_0) \) and \( V_1 := B(C_1, R_1)^c \) where \( C_0, C_1 \in \mathbb{C} \) and \( R_0, R_1 > 0 \) satisfy (2.9). Let \( E_k \) and \( \hat{E}_{k,\delta} \) be given as in Theorem 2.5. Then \( (E_1, E_2) \) are the element sets corresponding to \( \langle V_0, V_1 \rangle \) and \( \langle \hat{E}_{1,\delta}, E_2 \rangle \in \mathcal{E}(V_0) \) and \( \langle \hat{E}_{2,\delta}, E_1 \rangle \in \mathcal{E}(V_1) \).

**Proof.** For \( a \neq 0 \) the set \( a/(1 + V_1) \) is a circular disk \( B(\hat{C}_0, \hat{R}_0) \) where

\[
\hat{C}_0 = \frac{a(1 + C_1)}{|1 + C_1|^2 - R_1^2}, \quad \hat{R}_0 = \frac{|a| R_1}{R_1^2 - |1 + C_1|^2}.
\]

It is \( \subseteq V_0 \) if and only if \( |\hat{C}_0 - C_0| + \hat{R}_0 \leq R_0 \), i.e., if and only if \( a \in E_1 \). It is equal to \( V_0 \) if and only if \( \hat{C}_0 = C_0 \) and \( \hat{R}_0 = R_0 \), i.e., if and only if either

\[
C_1 \neq -1, \quad a = \hat{a}_1 \quad \text{and} \quad |C_0| R_1 = R_0 |1 + C_1|
\]

or \( C_1 = -1, \ C_0 = 0 \) and \( |a| = R_0 R_1 \).

Since \( \hat{E}_{1,\delta} \) is a closed, bounded set in \( \mathbb{C} \) with \( a/(1 + V_1) \neq V_0 \) for all \( a \in \hat{E}_{1,\delta} \), we have \( \hat{E}_{1,\delta} \in \mathcal{E}(V_1, V_0) \). Since \( s_1 \circ s_2(V_0) \subseteq s_1(V_1) \) this proves that \( \langle \hat{E}_{1,\delta}, E_2 \rangle \in \mathcal{E}(V_0) \).

Let \( |1 + C_0| < R_0 \). Then \( a/(1 + V_0) \) is the exterior of a disk. Indeed, \( a/(1 + V_0^c) = B(\hat{C}_1, \hat{R}_1)^c \) where

\[
\hat{C}_1 = \frac{a(1 + C_0)}{|1 + C_0|^2 - R_0^2}, \quad \hat{R}_1 = \frac{|a| R_0}{R_0^2 - |1 + C_0|^2}.
\]

It is \( \subseteq V_1 \) if and only if \( |\hat{C}_1 - C_1| + \hat{R}_1 \leq \hat{R}_1 \), i.e., if and only if \( a \in E_2 \). It is equal to \( V_1^c \) if and only if \( \hat{C}_1 = C_1 \) and \( \hat{R}_1 = R_1 \), i.e., if and only if either

\[
-1 \in V_0, \ C_0 \neq -1, \ a = \hat{a}_2 \quad \text{and} \quad |C_1| R_0 = R_1 |1 + C_0|
\]

or \( C_0 = -1, \ C_1 = 0 \) and \( |a| = R_0 R_1 \).
These cases are excluded for \( a \in \hat{E}_{2,\delta} \). From (2.10) we see that \( 0 \notin \mathcal{E}_{2} \) when \( |1 + C_{0}| < R_{0} \). We need to check whether \( \hat{a}_{2n_{k}}/(1 + V_{0}) \to V_{1} \) is possible for \( a_{2n_{k}} \in E_{2} \) if \( a_{2n_{k}} \to \infty \). But this is no problem since \( V_{0} \) is bounded, and thus \( \lim_{n \to \infty} a/(1 + V_{0}) = \{ \infty \} \). Therefore \( \hat{E}_{2,\delta} \in \mathcal{E}(V_{0}, V_{1}) \), and thus \( (\hat{E}_{2,\delta}, E_{1}) \in \mathcal{E}(V_{1}) \).

Next, let \( |1 + C_{0}| = R_{0} \). Then for \( a \neq 0, a/(1 + V_{0}) \) is the half plane given by (4.6). Hence \((E_{2}, E_{1}) \in \mathcal{E}(V_{1})\) and \(a/(1 + V_{0}) \subseteq V_{1}\) if and only if

\[
\Re \left( C_{1} \frac{1 + C_{0}}{|1 + C_{0}|} e^{-i\theta} \right) + R_{1} \leq \frac{|a|}{2R_{0}} \quad \text{where } \theta := \arg a,
\]

which gives the expression for \( E_{2} \) in this case. \((0 \notin E_{2}\) since \(-1 \in V_{0}\).

Finally, let \(|1 + C_{0}| > R_{0} \). Then \(a/(1 + V_{0}) = B(\hat{C}_{1}, -\hat{R}_{1})\) for \( a \neq 0\), where \( \hat{C}_{1} \) and \( \hat{R}_{1} \) are given by (4.10). Therefore \(a/(1 + V_{0}) \subseteq V_{1}\) if and only if \(|\hat{C}_{1} - C_{1}| \geq R_{1} + |\hat{R}_{1}|\), i.e., if and only if \(a \in E_{2}\). Moreover, \((E_{2}, E_{1}) \in \mathcal{E}(V_{1})\). \(\square\)

**Proof of Theorem 2.5.** A. The expressions for \( E_{1} \) and \( E_{2} \) follow from Lemma 4.5. We need to check that \( E_{k}^{0} \neq \emptyset \) for \( k = 1, 2 \). This clearly holds for \( E_{1} \) since \( |C_{0}| < R_{0} \) and \( |1 + C_{1}| < R_{1} \), and thus \( 0 \in E_{1}^{0} \). It is also clear that \( E_{2}^{0} \neq \emptyset \) if \( C_{0} = -1 \) or if \( R_{0} = |1 + C_{0}| \). Let \( R_{0} < |1 + C_{0}| \) and \( C_{1} \neq 0 \). Then \( \hat{a}_{2} \neq 0 \) and \(-t\hat{a}_{2} \in E_{2}^{0} \) for all \( t > 0 \) sufficiently large. If \( R_{0} < |1 + C_{0}| \) and \( C_{1} = 0 \), then \( \hat{a}_{2} = 0 \) and \( E_{2} = \{ a : |a| \geq R_{1}(|1 + C_{0}| + R_{0}) \} \), so again \( E_{2}^{0} \neq \emptyset \). If \( R_{0} > |1 + C_{0}| > 0 \) and \( C_{1} \neq 0 \), then \( \hat{a}_{2} \in E_{2}^{0} \) for all \( t > 0 \) sufficiently large, and thus \( E_{2}^{0} \neq \emptyset \). Finally, if \( R_{0} > |1 + C_{0}| \) and \( C_{1} = 0 \), then \( \hat{a}_{2} = 0 \) and all \( a \) with \( |a| \geq R_{0}^{2} - |1 + C_{0}|^{2} \) are in \( E_{2} \).

Let \( K(a_{n}/1) \) be a continued fraction from \((E_{1}, E_{2})\). If \( \operatorname{rad}(S_{2n}(V_{0})) \to 0 \) or \( \operatorname{rad}(S_{2n}(W_{0})) \to 0 \) or \( \operatorname{diam}(S_{2n}(Y_{0})) \to 0 \), then \( K(a_{n}/1) \) clearly converges generally. Assume in the proof of parts B and C below that \( \operatorname{diam}(S_{2n}(Y_{0})) \to 0 \) and \( \operatorname{rad}(S_{2n}(V_{0})) \to 0 \) and \( \operatorname{rad}(S_{2n}(W_{0})) \to 0 \).

B. We first observe that \( W_{0} = B(-1 - C_{1}, R_{1}) \) and \( W_{1} = B(-1 - C_{0}, R_{0})^{c} \) in this case. By Lemma 4.1 the element sets \( E_{1} \) and \( E_{2} \) do not change if we replace \( (V_{0}, V_{1}) \) by \( (W_{0}, W_{1}) \) (although their representation (2.10) changes), and neither do the conditions in (2.11). Indeed, \( \hat{E}_{1,\delta} \) and \( \hat{E}_{2,\delta} \) do not change either, since

\[
\hat{a}_{k} = C_{k-1}(1 + C_{k})(1 - R_{k}^{2}/|1 + C_{k}|^{2}) = (-1 - C_{k})(-C_{k-1})(1 - R_{k}^{2}/|C_{k-1}|^{2})
\]

when \( |C_{k-1}|R_{k} = R_{k-1}|1 + C_{k}| > 0 \). Therefore \( \hat{E}_{1,\delta} \in \mathcal{E}(W_{1}, W_{0}) \cap \mathcal{E}(V_{1}, V_{0}) \) by Lemma 4.5.

There is one condition that is changed, though, and that is the condition \(-1 \notin V_{0}^{0} \), which is equivalent to \( 0 \in W_{1} \). This means that if \(-1 \notin V_{0}^{0} \), then \( E_{2} \in \mathcal{E}(W_{0}, W_{1}) \), whereas, by (4.11), \( E_{2} \notin \mathcal{E}(W_{0}, W_{1}) \) if also \(-1 \in W_{0}^{0} \) and \( |C_{1}|R_{0} = R_{1}|1 + C_{0}| \geq 0 \). However, this case cannot occur since

\[
-1 \notin V_{0}^{0} \iff |1 + C_{0}| \geq R_{0} \quad \text{and} \quad -1 \in W_{0}^{0} \iff |C_{1}| < R_{1},
\]

which give \( |C_{1}|R_{0} < R_{1}|1 + C_{0}| \). Therefore, also now \( \hat{E}_{2,\delta} \in \mathcal{E}(W_{0}, W_{1}) \cap \mathcal{E}(V_{0}, V_{1}) \) by Lemma 4.5. This means that \( Z_{0} \in \partial^{1}V_{0} \) by Theorem 3.5B. Since \( \partial^{1}V_{0} = -1 - \partial^{1}V_{1} \), the convergence follows from Theorem 3.5A, both if \( K(a_{n}/1) \) is from \((\hat{E}_{1,\delta}, E_{2})\) or from \((E_{1}, \hat{E}_{2,\delta})\).

C. By the proof of part B, \( Z_{0} \subseteq \partial^{1}V_{0} \) when \( K(a_{n}/1) \) is from \((\hat{E}_{1,\delta}, E_{2})\), and \( Z_{1} \subseteq \partial^{1}V_{1} \) when \( K(a_{n}/1) \) is from \((E_{1}, \hat{E}_{2,\delta})\). The result follows therefore from Theorem 3.5A.
D: Let $K(a_n/1)$ from $(E_1, E_2)$ converge generally to $c$. Then $c_{2n} \to c$ and $\tilde{Z}_k = Z_k$ for $k = 0, 1$ by Theorem 1.4D. Therefore $Z_0 \subseteq \bigcup_{1} \cap (1 - V_1)$ (Theorem 1.4A). It follows therefore from Theorem 1.4E and C with $k = 0$ that also $c_{2n-1} \to c$. □

Proof of Theorem 2.7. Let $k \in \{1, 2\}$ be fixed. First let $-1 \not\in \partial V_k$. Then $g_k < 1, |\alpha_k| < \pi/2$ and

$$
a/(1 + V_k) = B(\tilde{C}_k, \tilde{R}_k), \quad \tilde{C}_k := \frac{a e^{-i \alpha_k}}{2(1 - g_k) \cos \alpha_k}, \quad \tilde{R}_k := \frac{|a|}{2(1 - g_k) \cos \alpha_k}
$$

for $a \neq 0$. This set is contained in $V_{k-1}$ if and only if $\text{Re}(\tilde{C}_k e^{-i \alpha_k}) - \tilde{R}_k \geq -g_{k-1} \cos \alpha_{k-1}$, which proves the expression for $E_k$ in this case. Next let $-1 \in \partial V_k$. Then $1/(1 + V_k) = H(0, -\alpha_k)$. Hence $a/(1 + V_k) \subseteq V_{k-1}$ for $a \neq 0$ if and only if $\arg(a) = \alpha_{k-1} + \alpha_k$. Since either $g_k = 1$ or $|\alpha_k| = \pi/2$ when $-1 \not\in \partial V_k$, the expression (2.15) for $E_k$ is still valid. Therefore $(E_1, E_2)$ given by (2.15) are the element sets corresponding to $(\tilde{V}_0, \tilde{V}_1)$.

If $0, -1 \not\in V_k$ for both $k = 0$ and $k = 1$, then the convergence follows from the twin version of the multiple parabola theorem proved in [5]. (See Remark 2.8.3.) Otherwise, by (2.14), there exist $\tilde{g}_0, \tilde{g}_1, \tilde{\alpha}_0$ and $\tilde{\alpha}_1$ such that

$$
|\alpha_0| < \frac{\pi}{2}, \quad \tilde{\alpha}_1 < \frac{\pi}{2} \quad \text{and} \quad \tilde{\alpha}_0 + \tilde{\alpha}_1 = \alpha_0 + \alpha_1,
$$

$$
0 < \tilde{g}_0, \quad 0 < \tilde{g}_1 \quad \text{and} \quad \tilde{g}_k(1 - \tilde{g}_{k-1}) \geq g_k(1 - g_{k-1}) \text{ for } k = 1, 2.
$$

Let $\tilde{E}_1$ and $\tilde{E}_2$ be given by (2.15) with $g_0, g_1, \alpha_0$ and $\alpha_1$ replaced by $\tilde{g}_0, \tilde{g}_1, \tilde{\alpha}_0$ and $\tilde{\alpha}_1$. Then $E_1 \subseteq \tilde{E}_1$ and $E_2 \subseteq \tilde{E}_2$, and the convergence follows again from the twin version of the multiple parabola theorem. □

Proof of Theorem 1.3. Since $|C_k| R_{k+1} \neq R_k|1 + C_{k+1}|$ for $k = 0$ or $k = 1$, we have $\tilde{E}_{k+1, \delta} = E_{k+1}$ in (2.11) for this $k$, and $K(a_n/1)$ converges generally by Theorem 2.5B. □

Proof of Theorem 1.1. A. Since $E_2^0 = \emptyset$ if and only if $E_2 = \{\alpha_2\}$ in this case, which happens if and only if $|C_1|R_0 = |1 + C_0|R_1$, it follows from Theorem 2.1 that $K(a_n/1)$ converges.

B. By (2.7) we always have $-1 \not\in V_0$ when $E_2^0 \neq \emptyset$. Hence $\tilde{E}_{2, \delta} = E_2$, and $K(a_n/1)$ converges generally by Theorem 2.3B. Theorem 1.4D shows therefore that its even part converges, and Theorem 1.4E shows that its odd part converges.

C. It follows from Theorem 1.3 that $K(a_n/1)$ converges generally in this case. Therefore its even part converges by Theorem 1.4D. The convergence of $K(a_n/1)$ follows from Theorem 1.4E.

D. (2.14) holds under our conditions, and the result follows from Theorem 2.7. □

References

11. Lorentzen, Lisa and Ruscheweyh, St., Simple convergence sets for continued fractions $K(a_n/1)$, Math. Anal. and Appl. 179(2) (1993), 349–370. MR1249825 (95b:40001)

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