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CONTINUED FRACTIONS WITH CIRCULAR TWIN VALUE SETS

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ABSTRACT. We prove that if the continued fraction $K(a_n/1)$ has circular twin value sets $\langle V_0, V_1\rangle$, then $K(a_n/1)$ converges except in some very special cases. The results generalize previous work by Jones and Thron.

1. Introduction and main result

A pair $\langle V_0, V_1\rangle$ of sets from $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called a pair of *twin value sets* for the continued fraction

$$K(a_n/1) := \frac{a_1}{1} \, + \, \frac{a_2}{1} \, + \, \frac{a_3}{1} \, + \dots := \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \dots}}}, \quad a_n \in \mathbb{C} \setminus \{0\} \tag{1.1}$$

if both V_k and its complement V_k^c in $\widehat{\mathbb{C}}$ are non-empty for $k = 0, 1$ and

$$a_{2n-1}/(1 + V_1) \subseteq V_0 \quad \text{and} \quad a_{2n}/(1 + V_0) \subseteq V_1 \quad \text{for } n = 1, 2, 3, \dots. \tag{1.2}$$

Note that we do not require $a_{2n+k} \in V_{k-1}$ for $k = 1, 2$ as was done in the work by Jones and Thron; see for instance their book [7, p. 64]. For given value sets we further define the *corresponding element sets* $\langle E_1, E_2\rangle$ by

$$E_1 := \{a \in \mathbb{C};\ a/(1 + V_1) \subseteq V_0\}, \quad E_2 := \{a \in \mathbb{C};\ a/(1 + V_0) \subseteq V_1\}. \tag{1.3}$$

Here, by definition, $0 \notin E_1$ if $-1 \in \overline{V_1}$ (the closure of V_1 in $\widehat{\mathbb{C}}$) and $0 \notin E_2$ if $-1 \in \overline{V_0}$. The twin element sets $\langle E_1, E_2\rangle$ are *true* if $E_k \setminus \{0\} \neq \emptyset$ for $k = 1$ and 2. We also say that $\langle V_0, V_1\rangle$ are twin value sets for $\langle E_1, E_2\rangle$. For convenience we shall always let $V_2 := V_0$, so that $E_k = \{a \in \mathbb{C};\ a/(1 + V_k) \subseteq V_{k-1}\}$ for $k = 1, 2$.

In this paper we restrict the value sets to be *closed circular domains*; that is, they are closures of simply connected, open, non-empty domains on the Riemann sphere $\widehat{\mathbb{C}}$, bounded by a generalized circle. The points $0, -1, \infty$ are special in the classical continued fraction theory. (See (1.6).) We shall therefore distinguish between closed domains V where

- $\infty \notin V$ (disks),
- ∞ on the boundary ∂V of V (half planes),
- ∞ in the interior V° of V (complements of disks).

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We address the problem: when does $K(a_n/1)$ from $\langle E_1, E_2\rangle$ (i.e. all $a_{2n-1} \in E_1$ and all $a_{2n} \in E_2$) converge? By convergence we mean that the sequence of *approximants* $\{c_n\}$ of $K(a_n/1)$ converges to a $c \in \widehat{\mathbb{C}}$, where

$$\begin{aligned} &c_n := S_n(0) \quad \text{and} \quad S_n(z) := \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1+z}, \\ &\text{i.e.,} \quad S_n := s_0 \circ s_1 \circ s_2 \circ \cdots \circ s_n; \quad s_0(z) := z, \ s_k(z) := \frac{a_k}{1+z}. \end{aligned} \tag{1.4}$$

We say that the *even (odd) part* of $K(a_n/1)$ converges if $\{c_{2n}\}$ ($\{c_{2n+1}\}$) converges in $\widehat{\mathbb{C}}$. A number of papers has been written on this topic. See for instance [7, chapter 4] and the references therein. In particular, the paper [6] by Jones and Thron, published in this journal, gives a very nice and useful presentation of sufficient conditions for convergence. However, these results can be improved, as we shall show in this paper. The very special case where $0 \in \partial V_0$ and $-1 \in \partial V_1$ or vice versa still needs some extra attention, though (see Example 2.9). We shall prove:

Theorem 1.1. *Let $\langle V_0, V_1\rangle$ be closed circular twin value sets with corresponding element sets $\langle E_1, E_2\rangle$ for the continued fraction $K(a_n/1)$. Then the following statements are true:*

A. *Let V_0 and V_1 be disks and $E_2^\circ \neq \emptyset$. Then $K(a_n/1)$ converges to a $c \in V_0$.*
B. *Let V_0 be a disk, V_1 be a half plane and $E_2^\circ \neq \emptyset$. Then $K(a_n/1)$ converges to a $c \in V_0$.*
C. *Let V_0 be a disk and V_1 be the complement of a disk with respective centers C_k and radii R_k such that $|C_k|R_{k+1} \neq R_k|1+C_{k+1}|$ for $k = 0$ or $k = 1$ and $0 \notin \partial^\dagger V_1 := \partial V_1 \cap (-1-\partial V_0)$. Then the even part of $K(a_n/1)$ converges to a $c \in V_0$. If moreover $-1 \notin V_0 \setminus (-1 - V_1^\circ)$, then $K(a_n/1)$ itself converges to c.*
D. *Let V_0 and V_1 be half planes with $0,\ -1 \notin \partial^\dagger V_1$. Then the even and odd parts of $K(a_n/1)$ converge to finite values $\in V_0$. Moreover, $K(a_n/1)$ itself converges if and only if*

$$\sum_{n=1}^{\infty} |b_n| = \infty \quad \text{where } b_{2n} := \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}, \quad b_{2n+1} := \frac{a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n+1}}. \tag{1.5}$$

Remarks 1.2.

1. Since $K_{n=1}^\infty(a_n/1)$ converges in $\widehat{\mathbb{C}}$ if and only if $K_{n=2}^\infty(a_n/1)$ converges in $\widehat{\mathbb{C}}$, we may interchange V_0 and V_1.
2. Theorem 1.1 also covers cases such as, for instance, V_0 a half plane and V_1 a complement of a disk, since $\langle V_0, V_1\rangle$ are twin value sets for the continued fraction $K(a_n/1)$ if and only if $\langle -1 - V_1^c, -1 - V_0^c\rangle$ are twin value sets for $K(a_n/1)$ (see Lemma 4.1). This was also pointed out by Jones and Thron in [6]. Indeed, if V_0 or V_1 contains more than one element, then $Y_0 := V_0 \setminus (-1 - V_1)^\circ \neq \emptyset$ and $Y_1 := V_1 \setminus (-1 - V_0)^\circ \neq \emptyset$, so also $\langle Y_0, Y_1\rangle$ are twin value sets for $K(a_n/1)$, [9, prop. 5.4].
3. It is a well established fact [7, thm. 4.53, p. 128] that (1.5) holds if $\{a_n\}$ has a bounded subsequence.

The classical convergence concept requires that $S_n(0) \to c$, where by (1.4),

$$c_n = S_{n-1}(a_n) = S_n(0) = S_{n+1}(\infty) = S_{n+2}(-1) = S_{n+3}(-1 - a_{n+3}). \tag{1.6}$$

In [2] a more general concept of convergence was introduced: we require that there exist two sequences $\{u_n\}$ and $\{v_n\}$ from $\widehat{\mathbb{C}}$ such that

$$\lim S_n(u_n) = \lim S_n(v_n) = c \quad \text{and} \quad \liminf d(u_n, v_n) > 0, \tag{1.7}$$

where $d(*,*)$ denotes the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$; i.e.,

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}} \quad \text{if } z, w \in \mathbb{C} \tag{1.8}$$

with the natural limit forms if z and/or w is $= \infty$. If (1.7) holds, we say that $K(a_n/1)$ *converges generally* to c. Then, by [2], there exists an *exceptional sequence* $\{z_n^\dagger\} \subseteq \widehat{\mathbb{C}}$ such that

$$\lim S_n(z_n) = c \quad \text{whenever} \quad \liminf d(z_n, z_n^\dagger) > 0. \tag{1.9}$$

If $c \neq \infty$, we can for instance use $z_n^\dagger := \zeta_n := S_n^{-1}(\infty)$ for all n. Or more generally, $\{S_n^{-1}(q)\}$ is an exceptional sequence for every $q \neq c$, also if $c = \infty$. All the exceptional sequences have the same asymptotic behavior.

Classical convergence implies general convergence whereas the converse does not hold. Indeed, there are generally convergent continued fractions $K(a_n/1)$ where $\{z_n^\dagger\}$ has limit points at 0, -1 and ∞ which destroy the classical convergence of $K(a_n/1)$. However, if $K(a_n/1)$ also converges in the classical sense, then it converges to the same value. It is also clear that if the even and odd parts of $K(a_n/1)$ converge to distinct values in the classical sense, then they also converge generally to the same two distinct values.

One might expect to get a nicer theorem with general convergence. However, Theorem 1.1 is already good, except for the disk – complement of disk case. For this case it really pays off to change over to general convergence (here $B(C,R)$ denotes a closed circular disk with center at $C \in \mathbb{C}$ and radius $R > 0$):

Theorem 1.3. *Let $V_0 := B(C_0, R_0)$ and $V_1 := \overline{B(C_1, R_1)^c}$ be twin value sets for the continued fraction $K(a_n/1)$, where $0 \notin \partial^\dagger V_1 := \partial V_1 \cap (-1 - \partial V_0)$ and $|C_k| R_{k+1} \neq R_k |1 + C_{k+1}|$ for $k = 0$ or $k = 1$. Then $K(a_n/1)$ converges generally to a $c \in V_0$.*

The final result in this section describes cases where classical convergence follows from general convergence. We still use the notation $\zeta_n := S_n^{-1}(\infty)$.

Theorem 1.4. *Let $\langle V_0, V_1 \rangle$ be closed twin value sets for the continued fraction $K(a_n/1)$ with $(V_0 \cup V_1)^\circ \neq \emptyset$. Let $K(a_n/1)$ converge generally to c, let $q \neq c$ and let $\tilde{Z}_k$ be the set of limit points for $\{S_{2n+k}^{-1}(q)\}$. Then the following statements hold for fixed $k \in \{1, 2\}$.*

A. *$c \in V_0 \setminus (-1 - V_1^\circ)$ and $\tilde{Z}_k \subseteq (-1 - V_{k-1}) \setminus V_k^\circ$.*
B. *If $-1 \notin \tilde{Z}_k$ or $0 \notin \tilde{Z}_k$, then $S_{2n+k}(0) \to c$. If $\infty \notin \tilde{Z}_k$, then $S_{2n+k-1}(0) \to c$.*
C. *Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. If for each $n \geq n_0$, either $d(a_{2n+k-1}, \tilde{Z}_k) \geq \varepsilon$ or $d(-1 - a_{2n+k+2}, \tilde{Z}_k) \geq \varepsilon$, then $S_{2n+k-1}(0) \to c$.*
D. *If V_0 is bounded, then $\{\zeta_n\}$ is an exceptional sequence for $\{S_n\}$ and $S_{2n}(0) \to c$.*
E. *If $-1 \notin V_0 \setminus (-1 - V_1^\circ)$, then $S_{2n+1}(0) \to c$.*

In section 2 we shall give some explicit expressions for the corresponding element sets $\langle E_1, E_2\rangle$ and some stronger convergence results. Section 3 contains some intermediate results, and in section 4 we prove the results in sections 1 and 2.

Notation. We shall use the notation introduced so far, plus some extra. For convenience we list a few of them here:

- $\overline{A}$, A°, ∂A and A^c are the closure, the interior, the boundary and the complement of a set A in $\widehat{\mathbb{C}}$.
- $\mathbb{D}$ is the open unit disk $\{z\in\mathbb{C};\ |z|<1\}$.
- $[z_1,z_2]$ is the closed line segment between the two points z_1 and z_2 in $\mathbb{C}$. Moreover, $a[r,\infty):=\{z=ua;\ u\ge r\}$ for $a\in\mathbb{C}\setminus\{0\}$ and $r\in\mathbb{R}$.
- $B(a,r):=\{z\in\mathbb{C};\ |z-a|\le r\}$ and $B_d(a,r):=\{z\in\widehat{\mathbb{C}};\ d(z,a)\le r\}$ for $a\in\mathbb{C}$ and $r>0$.
- $H(r,\alpha)$, where $r,\ \alpha\in\mathbb{R}$, denotes the closed half plane with $L:=e^{i\alpha}[r,\infty)\subseteq H(r,\alpha)$, whose boundary $\partial H(r,\alpha)$ is the line through $r\,e^{i\alpha}$ orthogonal to L.
- $\mathrm{rad}(A)$ is the euclidean radius of a circular set $A\subseteq\widehat{\mathbb{C}}$. $\mathrm{rad}(A):=\infty$ if $\infty\in\overline{A}$.
- $\mathrm{diam}(A)$ is the euclidean diameter of a set $A\subset\widehat{\mathbb{C}}$.
- $\mathrm{dist}(z,A)$ $(d(z,A))$ denotes the euclidean (chordal) distance between a point $z\in\widehat{\mathbb{C}}$ and a set $A\subseteq\widehat{\mathbb{C}}$, and $\mathrm{dist}(A,B)$ $(d(A,B))$ denotes the euclidean (chordal) distance between two sets $A,B\subseteq\widehat{\mathbb{C}}$.
- For convenience, $V_2:=V_0$, $W_2:=W_0$, $E_3:=E_1$, $E_0=E_2$, etc. for twin quantities; that is, they are counted modulo 2.
- s_m denotes the linear fractional transformation $a_m/(1+z)$, $s_m^*(z):=a_m^*/(1+z)$ and so on, and $S_n:=s_1\circ s_2\circ\cdots\circ s_n$.
- $\partial^\dagger V_k:=\partial V_k\cap(-1-\partial V_{k+1})$ and $\partial^* V_k:=\partial V_k\cap(-1-V_{k+1})$ for $k=0,1$. Clearly, $\partial^\dagger V_0=-1-\partial^\dagger V_1$, and the condition $0\notin\partial V_k$, $-1\notin\partial V_{k+1}$ can be written $0\notin\partial^\dagger V_k$, or equivalently, $-1\notin\partial^\dagger V_{k+1}$.
- $\zeta_n:=S_n^{-1}(\infty)$, $c_n:=S_n(0)$ and Z_k is the (closed) set of limit points for $\{\zeta_{2n+k}\}$.
- $W_0:=-1-\overline{V_1^c}$, $W_1:=-1-\overline{V_0^c}$, $Y_0:=V_0\setminus(-1-V_1)^\circ$ and $Y_1:=V_1\setminus(-1-V_0)^\circ$ so that $\langle W_0,W_1\rangle$ and $\langle Y_0,Y_1\rangle$ are alternative closed twin value sets (Remark 1.2.2).
- $\sum' P_n<\infty$ shall mean that there exists an $n_0\in\mathbb{N}$ such that $\sum_{n=n_0}^\infty P_n<\infty$ for the non-negative numbers P_n. Hence $P_n=\infty$ is possible for finitely many n.

2. Explicit element sets and more detailed convergence criteria

In applications it is useful to know the corresponding element sets $\langle E_1,E_2\rangle$ explicitly. We have therefore listed these sets below, along with some more specific convergence criteria for continued fractions $K(a_n/1)$ with circular twin value sets. Of course we want as few extra conditions as possible, but some situations have to be treated separately:

- $a_n\to\infty$. The if and only if part of Theorem 1.1D shows that extra conditions are needed in this case. This is true whether we want classical or general convergence.

- $a_{2n-1} \to \tilde{a}_1 \in E_1 \setminus \{0\}$ and $a_{2n} \to \tilde{a}_2 \in E_2 \setminus \{0\}$ where $\tilde{s}_1 \circ \tilde{s}_2$ is an elliptic transformation. If $|a_{2n+k} - \tilde{a}_k| \to 0$ fast enough for $k = 1$ and $k = 2$, then $K(a_n/1)$ diverges generally. $\tilde{s}_1 \circ \tilde{s}_2$ is elliptic if $\tilde{a}_1 = -w_0 w_1$ and $\tilde{a}_2 = -(1+w_0)(1+w_1)$ for some $w_0, w_1 \in \mathbb{C}$ with $w_0(1+w_1) = e^{i\theta} w_1(1+w_0)$ where $e^{i\theta} \neq 1$, [1]. This can happen only if both $\tilde{s}_1(V_1) = V_0$ and $\tilde{s}_2(V_0) = V_1$. (See also Lemma 4.2.)
- $\tilde{a}_k := 0 \in \overline{E_k}$ and $\tilde{a}_{k+1} := -1 \in E_{k+1}$ for $k = 1$ or 2. Also now $K(a_n/1)$ with $a_{2n+k} \to \tilde{a}_k$ for $k = 1, 2$ may converge or diverge depending on *how* $\{a_{2n+k}\}$ approaches $\tilde{a}_k$ (see Example 2.9).

The disk – disk case.

Let $V_k := B(C_k, R_k)$ for some $C_k \in \mathbb{C}$ and $R_k > 0$ for $k = 0, 1$. Evidently $E_k = \emptyset$ if $-1 \in V_k$, so

$$|1 + C_k| > R_k \quad \text{for } k = 0, 1 \tag{2.1}$$

is a necessary condition for $\langle E_1, E_2 \rangle$ to be *true* element sets corresponding to $\langle V_0, V_1 \rangle$. Then we get the following generalization of [6, thm. 5.1]:

Theorem 2.1. *Let $V_k := B(C_k, R_k)$ for $k = 0, 1$ where $C_k \in \mathbb{C}$ and $R_k > 0$ satisfy* (2.1) *and*

$$|C_{k-1}| R_k \leq |1 + C_k| R_{k-1} \tag{2.2}$$

for $k = 1, 2$. If (2.2) *holds with equality for both $k = 1$ and $k = 2$, we further assume that $\sigma := \tilde{s}_1 \circ \tilde{s}_2$ is non-elliptic, where*

$$\tilde{a}_k := C_{k-1}(1 + C_k)(1 - R_k^2/|1 + C_k|^2). \tag{2.3}$$

Then every continued fraction $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ converges, where

$$E_k := \left\{ a \in \mathbb{C};\ |a - \tilde{a}_k| + \frac{R_k}{|1 + C_k|}|a| \leq \frac{R_{k-1}}{|1 + C_k|}(|1 + C_k|^2 - R_k^2) \right\}. \tag{2.4}$$

Remarks 2.2.

1. $\langle E_1, E_2 \rangle$ are the element sets corresponding to $\langle V_0, V_1 \rangle$. They are true element sets if and only if (2.1) and (2.2) hold. Condition (2.2) is therefore only present to make $\langle E_1, E_2 \rangle$ true when (2.1) holds. E_k is a one-point set if and only if $E_k = \{\tilde{a}_k\}$ as given by (2.3). This happens if and only if $|C_{k-1}| R_k = |1 + C_k| R_{k-1}$, which happens if and only if $a/(1 + V_k) = V_{k-1}$ for an $a \in E_k$, in which case $a = \tilde{a}_k \neq 0$. (See Lemma 4.3.)
2. If E_k contains more than one point, then E_k is a closed convex domain bounded by a cartesian oval with foci at 0 and $\tilde{a}_k$ [3, 12, remark 5, p. 142], and $E_k^\circ \neq \emptyset$. If $C_{k-1} = 0$, this oval reduces to a circle centered at the origin.
3. Divergence only occurs if and only if $E_k = \{\tilde{a}_k\}$ for $k = 1, 2$ and $\sigma := \tilde{s}_1 \circ \tilde{s}_2$ is elliptic. This means that $K(a_n/1)$ converges in the classical sense if and only if it converges in the general sense in the disk-disk case.

The disk – half plane case.

Let $V_0 := B(C_0, R_0)$ and $V_1 := \{z \in \mathbb{C};\ \mathrm{Re}(z\, e^{-i\alpha}) \geq h \cos\alpha\} \cup \{\infty\} = H(h\cos\alpha, \alpha)$ for some $C_0 \in \mathbb{C}$, $R_0 > 0$, $h, \alpha \in \mathbb{R}$. It is clear that $a/(1 + V_1) \subseteq V_0$ for $a \neq 0$ only if $-1 \notin V_1$ and $0 \in V_0$, and that $a/(1 + V_0) \subseteq V_1$ for $a \neq 0$ only if $-1 \notin V_0^\circ$. Hence we require that

$$|C_0| \leq R_0 \leq |1 + C_0|, \quad |\alpha| < \pi/2 \quad \text{and} \quad h > -1. \tag{2.5}$$

But this leaves the possibility of $0 \in \partial V_1$ and $-1 \in \partial V_0$, a situation that requires caution. We therefore need extra conditions if $0 \in \partial^\dagger V_1 := \partial V_1 \cap (-1 - \partial V_0)$. Still, we get the following generalization of [6, thm. 5.2]:

Theorem 2.3. *Let $V_0 := B(C_0, R_0)$ and $V_1 := H(h\cos\alpha,\, \alpha)$ where $C_0 \in \mathbb{C}$, $R_0 > 0$ and $\alpha,\, h \in \mathbb{R}$ satisfy (2.5), and let*

$$a_1^* := 2C_0 e^{i\alpha}(1+h)\cos\alpha, \qquad a_2^* := 2(1+C_0)h\, e^{i\alpha}\cos\alpha \tag{2.6}$$

and

$$E_1 := \{a \in \mathbb{C};\ |a - a_1^*| + |a| \le 2R_0(1+h)\cos\alpha\},$$
$$E_2 := \begin{cases} \{a \in \mathbb{C};\ |a|R_0 - \operatorname{Re}(a(1+\overline{C_0})e^{-i\alpha}) \le -h(|1+C_0|^2 - R_0^2)\cos\alpha\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } |1+C_0| > R_0, \\ (1+C_0)e^{i\alpha}[\max\{0, 2h\cos\alpha\},\, \infty) \setminus \{0\} \qquad \text{if } |1+C_0| = R_0. \end{cases} \tag{2.7}$$

Furthermore, let

$$\begin{aligned} \tilde{E}_{1,\delta} &:= \begin{cases} E_1 \setminus B(a_1^*, \delta) & \text{if } R_0 = |C_0|, \\ E_1 & \text{otherwise}, \end{cases} \\ \tilde{E}_{2,\delta} &:= \begin{cases} E_2 \setminus B(a_2^*, \delta) & \text{if } R_0 = |1+C_0| \text{ and } h \ge 0, \\ E_2 & \text{otherwise} \end{cases} \end{aligned} \tag{2.8}$$

where $0 < \delta < |a_1^*|$ if $C_0 \neq 0$. Then the following statements are true:

A. *Every continued fraction $K(a_n/1)$ from $\langle E_1, \tilde{E}_{2,\delta}\rangle$ converges generally.*
B. *If $0 \notin \partial^\dagger V_1$, then every continued fraction $K(a_n/1)$ from $\langle \tilde{E}_{1,\delta}, E_2\rangle$ converges generally.*
C. *Let $\varepsilon > 0$. If $K(a_n/1)$ is a continued fraction from $\langle \tilde{E}_{1,\delta}, E_2\rangle$ such that for each n from some n_0 on, either $\operatorname{dist}(-1-a_{2n}, \partial^* V_0) \ge \varepsilon$ or $\operatorname{dist}(a_{2n-1}, \partial^* V_0) \ge \varepsilon$, then $K(a_n/1)$ converges generally.*
D. *Let $K(a_n/1)$ from $\langle E_1, E_2\rangle$ converge generally to c. Then $c_{2n} \to c$. If moreover $0 \in V_1^\circ$ or $-1 \notin \partial V_0$ or $\liminf d(a_{2n-1}, \overline{V_0^c} \cap (-1 - V_1)) > 0$, then $c_n \to c$.*

Remarks 2.4.

1. $\langle E_1, E_2\rangle$ are the element sets corresponding to $\langle V_0, V_1\rangle$. If $R_0 = |C_0|$, then E_1 is the closed line segment $[0, a_1^*]$. Otherwise, ∂E_1 is an ellipse with foci at a_1^* and the origin. ∂E_1 reduces to a circle if $C_0 = 0$.
2. If $|1+C_0| = R_0$, then E_2 is a ray. Otherwise, $E_2^\circ \neq \emptyset$ and ∂E_2 is a hyperbola.
3. If $-1 \notin \partial V_0$ or $0 \in V_1^\circ$, then $\tilde{E}_{2,\delta} = E_2$, so every continued fraction $K(a_n/1)$ from $\langle E_1, E_2\rangle$ converges generally by part A in this case. Let $-1 \in \partial V_0$ and $0 \notin V_1^\circ$. If $0 \notin \partial V_1$ and $0 \in V_0^\circ$, then $\tilde{E}_{1,\delta} = E_1$, and every continued fraction from $\langle E_1, E_2\rangle$ still converges generally by part B.

The disk – complement of disk case.
Let $V_0 = B(C_0, R_0)$ and $V_1 = \overline{B(C_1, R_1)^c}$. This time $\infty \notin V_0$ and $\infty \in V_1^\circ$, so we evidently need that $0 \in V_0^\circ$ and $-1 \notin V_1$ to get true element sets; that is,

$$|C_0| < R_0 \quad \text{and} \quad |1 + C_1| < R_1. \tag{2.9}$$

Theorem 2.5. *Let $V_0 := B(C_0, R_0)$ and $V_1 := \overline{B(C_1, R_1)^c}$ where $C_k \in \mathbb{C}$ and $R_k > 0$ satisfy* (2.9), *and let*

$$E_1 := \begin{cases} \{a;\ |a-\tilde{a}_1| + |a|\frac{R_1}{|1+C_1|} \leq \frac{R_0}{|1+C_1|}(R_1^2 - |1+C_1|^2)\} & \textit{if } C_1 \neq -1, \\ B(0,\, (R_0 - |C_0|)R_1) & \textit{if } C_1 = -1, \end{cases}$$

$$(2.10)\quad E_2 := \begin{cases} \{a;\ |a-\tilde{a}_2| - |a|\frac{R_0}{|1+C_0|} \geq \frac{R_1}{|1+C_0|}(|1+C_0|^2 - R_0^2)\} \setminus \{0\} & \textit{if } R_0 < |1+C_0|, \\ \{a;\ |a|\frac{R_0}{|1+C_0|} - |a-\tilde{a}_2| \geq \frac{R_1}{|1+C_0|}(R_0^2 - |1+C_0|^2)\} & \textit{if } R_0 > |1+C_0| > 0, \\ \{a = r\, e^{i\theta};\ \frac{r}{2} \geq \mathrm{Re}(C_1(1+C_0)e^{-i\theta}) + R_0R_1\} \setminus \{0\} & \textit{if } R_0 = |1+C_0|, \\ \{a;\ |a| \geq R_0(R_1 + |C_1|)\} & \textit{if } C_0 = -1, \end{cases}$$

where $\tilde{a}_k$ is given by (2.3). *Further let $\widehat{E}_{1,\delta}$ be given by* (2.11), *and let $\widehat{E}_{2,\delta} := E_2$ if $-1 \notin V_0^\circ$ and $\widehat{E}_{2,\delta}$ be given by* (2.11) *otherwise, where*

$$(2.11)\qquad \widehat{E}_{k,\delta} := \begin{cases} E_k \setminus B(\tilde{a}_k, \delta)^\circ & \textit{if } |C_{k-1}|R_k = R_{k-1}|1+C_k| > 0, \\ E_k \setminus \{a \in \mathbb{C};\ ||a| - R_0R_1| < \delta\} & \textit{if } C_{k-1} = 1 + C_k = 0, \\ E_k & \textit{otherwise} \end{cases}$$

for given $\delta > 0$ so small that $\widehat{E}_{1,\delta} \neq \emptyset$. Then the following statements are true.

A. *$\langle E_1, E_2 \rangle$ are the element sets corresponding to $\langle V_0, V_1 \rangle$, and $E_k^\circ \neq \emptyset$ for $k = 1, 2$.*
B. *Let $0 \notin \partial^\dagger V_1$. Then every continued fraction $K(a_n/1)$ from $\langle \widehat{E}_{1\delta}, E_2 \rangle$ or from $\langle E_1, \widehat{E}_{2,\delta} \rangle$ converges generally.*
C. *Let $\varepsilon > 0$. If $K(a_n/1)$ is a continued fraction from $\langle \widehat{E}_{1,\delta}, E_2, \rangle$ such that for each n from some $n_0 \in \mathbb{N}$ on, either* $\mathrm{dist}(a_{2n-1}, \partial^\dagger V_0) \geq \varepsilon$ *or* $\mathrm{dist}(-1-a_{2n}, \partial^\dagger V_0) \geq \varepsilon$, *then $K(a_n/1)$ converges generally. If $K(a_n/1)$ is a continued fraction from $\langle E_1, \widehat{E}_{2,\delta}, \rangle$ such that for each n from some $n_0 \in \mathbb{N}$ on, either* $\mathrm{dist}(-1-a_{2n+1}, \partial^\dagger V_1) \geq \varepsilon$ *or* $\mathrm{dist}(a_{2n}, \partial^\dagger V_1) \geq \varepsilon$, *then $K(a_n/1)$ converges generally.*
D. *Let $K(a_n/1)$ from $\langle E_1, E_2 \rangle$ converge generally to c. Then $c_{2n} \to c$. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. If $-1 \notin V_0 \setminus (-1 - V_1^\circ)$ or for each $n \geq n_0$ either* $\mathrm{dist}(a_{2n-1}, \overline{V_0^c} \cap (-1 - V_1)) \geq \varepsilon$ *or $d(-1 - a_{2n+2}, \overline{V_0^c}) \geq \varepsilon$, then $K(a_n/1)$ converges to c in the classical sense.*

Remarks 2.6.

1. E_1 is bounded by a cartesian oval with foci at 0 and $\tilde{a}_1$. If $C_1 = -1$, this oval reduces to a circle. E_2 is an unbounded set.
2. Jones and Thron [6, thm. 5.4], [7, thm. 4.11, p.72], proved the expressions for E_1 and E_2 for the case $|C_0| < R_0 \neq |1 + C_0|$ and $|1 + C_1| < R_1 \leq |C_1|$. Theorem 2.5 generalizes their result.
3. This disk - complement of disk case is quite special in the following sense: the case $a/(1 + V_k) = V_{k-1}$ does not necessarily occur only for $a \in \partial E_k$. Therefore $\widehat{E}_{k,\delta}$ is not necessarily simply connected or even connected. This means that we do not necessarily have that

$$\overline{G_k} \subseteq E_k^\circ \ \text{ for } k = 1, 2 \qquad \Rightarrow \qquad \langle G_1, G_2 \rangle \text{ are twin convergence sets}$$

as otherwise this is a normal feature for element sets $\langle E, E \rangle$ corresponding to simple value sets $\langle V, V \rangle$.

The half plane – half plane case.

Let V_0 and V_1 be closed half planes,

$$V_k = \{z \in \mathbb{C};\ \mathrm{Re}(z\, e^{-i\alpha_k}) \geq -g_k \cos\alpha_k\} \cup \{\infty\} = H(-g_k \cos\alpha_k, \alpha_k) \tag{2.12}$$

for some $\alpha_k,\, g_k \in \mathbb{R}$. Then $E_k \neq \emptyset$ only if $0 \in V_{k-1}$, and $-1 \notin V_k^\circ$. Therefore we require

$$|\alpha_k| \leq \pi/2 \quad \text{and} \quad 0 \leq g_k \leq 1 \quad \text{for } k = 1, 2. \tag{2.13}$$

Theorem 2.7. *Let $\alpha_k,\, g_k \in \mathbb{R}$ satisfy* (2.13) *and*

$$|\alpha_0 + \alpha_1| < \pi \quad \textit{and } g_{k-1}(1 - g_k) \neq 1 \textit{ for } k = 1, 2, \tag{2.14}$$

and let $K(a_n/1)$ be a continued fraction from $\langle E_1, E_2\rangle$ given by

$$E_k := \{a \in \mathbb{C};\ |a| - \mathrm{Re}(a\, e^{-i(\alpha_0+\alpha_1)}) \leq 2g_{k-1}(1 - g_k)\cos\alpha_0 \cos\alpha_1\}. \tag{2.15}$$

Then the even and odd parts of $K(a_n/1)$ converge to finite values in V_0, and $K(a_n/1)$ itself converges if and only if (1.5) *holds.*

Remarks 2.8.

1. $\langle E_1, E_2\rangle$ are the element sets corresponding to $\langle V_0, V_1\rangle$ in (2.12). If $g_{k-1} = 0$ or if $-1 \in \partial V_k$, then E_k reduces to the ray $e^{i(\alpha_0+\alpha_1)}(0, \infty)$, possibly including the end point $a = 0$. (Remember, $0 \notin E_k$ if $-1 \in V_k$ by definition.)
2. If $E_k^\circ \neq \emptyset$, then ∂E_k is a parabola with axis along the ray
$$e^{i(\alpha_0+\alpha_1)}[-g_{k-1}(1 - g_k)\cos\alpha_0 \cos\alpha_1, \infty)$$
and focus at the origin.
3. Theorem 2.7 does not contain any essential news compared to the twin version of Jones' and Thron's multiple parabola theorem in [5], [7, thm. 4.43, p. 106] which says that Theorem 2.7 holds under the additional conditions that $0 < g_k < 1$ and $|\alpha_k| < \pi/2$ for $k = 0$ and $k = 1$.

Example 2.9. Let $\alpha_0 = \alpha_1 = 0$, $g_0 = 0$ and $g_1 = 1$ in (2.12) and (2.15). Then $0 \in \partial V_0$ and $-1 \in \partial V_1$; i.e., $-1 \in \partial^\dagger V_1$. For given positive sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ converging to 0, let

$$t_{2n-1} := \varepsilon_n - 1, \quad t_{2n} := \delta_n \quad \text{and} \quad a_n := t_{n-1}(1 + t_n)$$

for all n. Then $K(a_n/1)$ is a continued fraction from $\langle E_1, E_2\rangle$ given by (2.15). By [12, formula (3.3.3), p.216] it follows that

$$S_n(0) - t_0 = -\frac{t_0}{R_n} \quad \text{where } R_n := \sum_{k=0}^{n} P_k \text{ and } P_k := \prod_{j=1}^{k} \frac{1 + t_j}{-t_j}.$$

In our situation,

$$\frac{1 + t_{2n-1}}{-t_{2n-1}} \cdot \frac{1 + t_{2n}}{-t_{2n}} = -\frac{\varepsilon_n}{1 - \varepsilon_n} \cdot \frac{1 + \delta_n}{\delta_n} \sim -\frac{\varepsilon_n}{\delta_n}(1 + \varepsilon_n + \delta_n),$$

so $S_{2n}(0)$ may converge or diverge, depending on the asymptotic behavior of $\{\varepsilon_n(1 + \varepsilon_n + \delta_n)/\delta_n\}$. A similar argument also shows that $K(a_n/1)$ may also diverge generally in this case.

3. Some intermediate results

Let $\langle V_0, V_1\rangle$ be closed twin value sets for the continued fraction $K(a_n/1)$. Then it follows from (1.2) and (1.4) that

$$\Delta_n := S_n(V_n) = S_{n-1}\circ s_n(V_n) \subseteq S_{n-1}(V_{n-1}) = \Delta_{n-1} \subseteq \cdots \subseteq \Delta_0 = V_0, \tag{3.1}$$

where $V_{2n} := V_0$ and $V_{2n+1} := V_1$ for all n. Since all s_n are (non-singular) linear fractional transformations, so are also S_n (see (1.4)). Therefore, since V_n is circular, also Δ_n is a circular domain. The nestedness (3.1) implies that Δ_n converges to a limit set Δ. If Δ just contains one point, *the limit point case*, then $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge uniformly in V_0 and V_1 respectively to the limit point c. Since both V_0 and V_1 contain more than one point in our cases, $K(a_n/1)$ converges generally to c in this case. If the limit set Δ has positive or infinite radius, *the limit circle case*, we need to investigate further. That Δ is a circular set in this case was proved by Thron [7, thm. 4.2B, p. 66].

In special cases classical convergence to c may be wanted. This may be possible to prove by means of Theorem 1.4. This theorem is partly based on Theorem 3.1 below, which concerns restrained sequences introduced in [4]: we say that a sequence $\{F_n\}$ of linear fractional transformations is *restrained* if there exist two sequences $\{u_n\}$ and $\{v_n\}$ from $\widehat{\mathbb{C}}$ such that

$$\lim d(F_n(u_n),\, F_n(v_n)) = 0 \quad\text{and}\quad \liminf d(u_n, v_n) > 0. \tag{3.2}$$

If in addition $\lim F_n(u_n) = c$, then we say that $\{F_n\}$ *converges generally* to c. As in (1.9) there exists an exceptional sequence $\{z_n^\dagger\}$ for $\{F_n\}$ such that if (3.2) holds, then (see [4])

$$\lim d(F_n(z_n),\, F_n(u_n)) = 0 \quad\text{whenever}\quad \liminf d(z_n, z_n^\dagger) > 0. \tag{3.3}$$

Theorem 3.1. *Let $\langle V_0, V_1\rangle$ be closed twin value sets for the continued fraction $K(a_n/1)$ where V_0 or V_1 contains more than one element. Let $k\in\{0,1\}$ be fixed, and let $\{S_{2n+k}\}$ be restrained with exceptional sequence $\{z_n^\dagger\}$. Then the limit points for $\{z_n^\dagger\}$ are contained in $(-1-V_{k+1})\setminus V_k^\circ$, and whenever* $\liminf d(u_n, z_n^\dagger) > 0$, *the set L of the limit points for $S_{2n+k}(u_n)$ is independent of $\{u_n\}$ and $L\subseteq V_0\setminus(-1-V_1^\circ)$.*

Proof. Since either V_0 or V_1 contains at least two points, they both do since $a_{2n}/(1+V_0)\subseteq V_1$ and $a_{2n+1}/(1+V_1)\subseteq V_0$. Since V_k contains more than one point, there exists a sequence $\{v_n\}$ from V_k with $\liminf d(v_n, z_n^\dagger) > 0$. By (3.1) it follows that $S_{2n+k}(V_k)\subseteq V_0$ for all n. It follows from (3.3) that L is independent of $\{u_n\}$ when $\liminf d(u_n, z_n^\dagger) > 0$, and thus $L\subseteq V_0$. Similarly, by Remark 1.2.2, $L\subseteq W_0 = -1-\overline{V_1^c}$, so $L\subseteq V_0\cap W_0 = Y_0 = V_0\setminus(-1-V_1^\circ)$.

Evidently, $\{z_n^\dagger\}$ can be chosen as $z_n^\dagger := S_{2n+k}^{-1}(p)$ for any $p\notin L$. By (3.3) every exceptional sequence has the same asymptotic behavior. Let $p\notin V_0$. Then $z_n^\dagger := S_{2n+k}^{-1}(p)\in V_k^c$ for all n. Similarly, for $q\in W_k^c$ given by $W_k := (-1-\overline{V_{k+1}^c})$ we can choose $z_n^\dagger := S_{2n+k}^{-1}(q)$ for all n, and then $z_n^\dagger\in W_k^c$ for all n. (See Remark 1.2.2.) Hence all the limit points of $\{z_n^\dagger\}$ are $\subseteq \overline{W_k^c}\cap\overline{V_k^c} = (-1-V_{k+1})\setminus V_k^\circ$. □

Since V_0 is a circular domain, there exists a linear fractional transformation φ_0 such that $\varphi_0(V_0) = \overline{\mathbb{D}}$. Hence the following result from [10] is useful to establish convergence in the limit circle case.

Theorem 3.2 ([10, thm. 3.8, 3.10])**.** *Let $\{t_n\}$ be linear fractional transformations with $t_n(\mathbb{D}) \subseteq \mathbb{D}$, and let $T_n := t_1 \circ t_2 \circ \cdots \circ t_n$ for all $n \in \mathbb{N}$. If $R := \lim \operatorname{rad}(T_n(\overline{\mathbb{D}})) > 0$, and there exists a set $I \subseteq \mathbb{N}$ such that*

$$\limsup_{n\in I, n\to\infty} \operatorname{rad}(t_n(\partial\mathbb{D})) < 1 \quad \text{and} \quad \liminf_{n\in\mathbb{N}\setminus I, n\to\infty} \operatorname{rad}(t_{n-1}^{-1}(\partial\mathbb{D})) > 1, \tag{3.4}$$

then $|T_n^{-1}(\infty)| \to 1$ and $\sum_{n=1}^{\infty} |T_n'(0)| < \infty$.

Remarks 3.3.

1. Of course, if I is bounded, then the first condition in (3.4) is void, and if $\mathbb{N} \setminus I$ is bounded, then the second one is void.
2. The conclusion $\sum |T_n'(0)| < \infty$ for the derivatives T_n' implies that $\{T_n\}$ is restrained. (Proof: T_n can be written

$$T_n(z) = C_n + R_n e^{i\omega_n} \frac{z - Q_n}{1 - \overline{Q}_n z} \quad \text{for some } |Q_n| < 1 \text{ and } \omega_n \in \mathbb{R}$$

when $T_n(\overline{\mathbb{D}}) = B(C_n, R_n)$, and thus $T_n'(z) = R_n e^{i\omega_n}(1 - |Q_n|^2)/(1 - \overline{Q}_n z)^2$. Hence $T_n'(z) \to 0$ for all $z \in \mathbb{D}$.) Indeed, $\sum |T_n'(z)| < \infty$ for every $z \in \mathbb{D}$.

Let $\mathcal{M}$ be the family of (non-singular) linear fractional transformations. For given $V \subseteq \widehat{\mathbb{C}}$ and $\varepsilon > 0$ we introduced the subfamily

$$\mathcal{M}_\varepsilon(V) := \{t \in \mathcal{M};\ t(V) \subseteq V \setminus B_d(z, \varepsilon) \text{ for some } z \in \partial V\} \tag{3.5}$$

in [11]. This notation is useful to convert Theorem 3.2 to our situation:

Corollary 3.4. *Let $k \in \{0, 1\}$ be fixed, and let $\langle V_0, V_1 \rangle$ be closed circular twin value sets for the continued fraction $K(a_n/1)$ where the limit circle case occurs. Furthermore, let $\sigma_n := s_{2n-1+k} \circ s_{2n+k}$, $\sigma_0 := \sigma_1$ and assume that*

$$\sigma_n \in \mathcal{M}_\varepsilon(V_k) \text{ for all } n \in I \quad \text{and} \quad \sigma_{n-1}^{-1} \in \mathcal{M}_\varepsilon(V_k^c) \text{ for all } n \in \mathbb{N} \setminus I \tag{3.6}$$

for some $I \subseteq \mathbb{N}$ and $\varepsilon > 0$. Then $\{S_{2n+k}\}$ is restrained and its exceptional sequences $\{z_n^\dagger\}$ have all their limit points $\in \partial V_k$. If also V_0 is bounded, then $\{\zeta_{2n+k}\}$ is an exceptional sequence for $\{S_{2n+k}\}$ and $\sum_{n=1}^{\infty} |S_{2n+k}'(z)| < \infty$ for every finite $z \in V_k^\circ$.

Proof. Let $\varphi \in \mathcal{M}$ satisfy $\varphi(V_k) = \overline{\mathbb{D}}$. Then $t_n := \varphi \circ \sigma_n \circ \varphi^{-1}$ maps $\mathbb{D}$ into $\mathbb{D}$, and $T_n := t_1 \circ t_2 \circ \cdots \circ t_n = \varphi \circ S_{2n}^{(k)} \circ \varphi^{-1}$ where $S_{2n}^{(k)} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$. Condition (3.6) implies (3.4). Hence $\{T_n\}$ is restrained with exceptional sequence $\{T_n^{-1}(\infty)\}$ where $|T_n^{-1}(\infty)| \to 1$. Therefore $\{S_{2n}^{(k)}\}$ is restrained with exceptional sequence $z_n^\dagger := \varphi^{-1} \circ T_n^{-1}(\infty) = (S_{2n}^{(k)})^{-1}(\varphi^{-1}(\infty))$. That $\{S_{2n+k}\}$ is restrained with exceptional sequence $\{z_n^\dagger\}$ follows therefore since $S_{2n} = S_{2n}^{(0)}$ and $S_{2n+1} = s_1 \circ S_{2n}^{(1)}$ for the fixed $s_1 \in \mathcal{M}$. Since $|T_n^{-1}(\infty)| \to 1$, i.e., $\operatorname{dist}(T_n^{-1}(\infty), \partial\mathbb{D}) \to 0$, it follows that $d(\varphi^{-1} \circ T_n^{-1}(\infty), \varphi^{-1}(\partial\mathbb{D})) \to 0$ where $\varphi^{-1}(\partial\mathbb{D}) = \partial V_k$ and $\varphi^{-1} \circ T_n^{-1}(\infty) = z_n^\dagger$. That is, all the limit points of $\{z_n^\dagger\}$ are $\in \partial V_k$.

Let V_0 be bounded. Then $\infty \notin V_0$, so $\{\zeta_{2n+k}\}$ is an exceptional sequence for $\{S_{2n+k}\}$ since $S_{2n+k}(\zeta_{2n+k}) = \infty$ whereas all the limit points for $\{S_{2n+k}(u_n)\}$ are $\in V_0$ when $\liminf d(u_n, z_n^\dagger) > 0$ (Theorem 3.1). It remains to prove that $\sum |S_{2n+k}'(z)| < \infty$ for finite $z \in V_k^\circ$. By Theorem 3.2 and Remark 3.3.2 we know that $\sum |T_n'(w)| < \infty$ for every $w \in \mathbb{D}$. First let $k = 0$ and choose $\varphi(z) := (z - C_0)/R_0$ where C_0 and R_0 are the center and radius of V_0. Let $z \in V_0^\circ$ be arbitrarily chosen, and let $w := \varphi(z)$. Then $w \in \mathbb{D}$ and $S_{2n}'(z) = (\varphi^{-1})'(T_n(\varphi(z))) \cdot$

$T_n'(\varphi(z)) \cdot \varphi'(z) = (\varphi^{-1})'(T_n(w)) \cdot T_n'(w) \cdot \frac{1}{R_0} = R_0 \cdot T_n'(w) \cdot \frac{1}{R_0} = T_n'(w)$. Hence $\sum |S_{2n}'(z)| < \infty$.

Next let $k = 1$ and set $\widehat{V}_0 := s_1(V_1)$. Then $\widehat{V}_0 = B(\widehat{C}_0, \widehat{R}_0) \subseteq V_0$ for some fixed $\widehat{C}_0 \in \mathbb{C}$ and $\widehat{R}_0 > 0$. Furthermore, let $\varphi_1(z) := (z - \widehat{C}_0)/\widehat{R}_0$ so that $\varphi_1(\widehat{V}_0) = \overline{\mathbb{D}}$ and $t_n := \varphi_1 \circ s_1 \circ s_{2n} \circ s_{2n+1} \circ s_1^{-1} \circ \varphi_1^{-1}$ maps $\mathbb{D}$ into $\mathbb{D}$. Let a finite $z \in V_1^\circ$ be arbitrarily chosen, and let $w := \varphi_1 \circ s_1(z)$. Then $w \in \mathbb{D}$ and

$$\begin{aligned} S_{2n+1}'(z) &= (\varphi_1^{-1})'(T_n \circ \varphi_1 \circ s_1(z)) \cdot T_n'(\varphi_1 \circ s_1(z)) \cdot \varphi_1'(s_1(z)) \cdot s_1'(z) \\ &= \widehat{R}_0 \cdot T_n'(w) \cdot \frac{1}{\widehat{R}_0} \cdot \frac{-a_1}{(1+z)^2} = \frac{-a_1}{(1+z)^2} T_n'(w) \end{aligned}$$

where $z \neq -1$ since $-1 \notin V_1$ when V_0 is bounded. Hence $\sum |S_{2n+1}'(z)| < \infty$. $\square$

It follows from (1.6) that S_n can be written

$$S_n(z) = \begin{cases} c_{n-1} - \dfrac{\zeta_n(c_n - c_{n-1})}{z - \zeta_n} & \text{if } \zeta_n \neq \infty, \\ c_n - (c_{n-2} - c_n)z & \text{if } \zeta_n = \infty. \end{cases} \tag{3.7}$$

Therefore

$$S_n'(z) = \begin{cases} \dfrac{\zeta_n(c_n - c_{n-1})}{(z - \zeta_n)^2} = -\dfrac{S_n(z) - c_{n-1}}{z - \zeta_n} & \text{if } \zeta_n \neq \infty, \\ c_n - c_{n-2} & \text{if } \zeta_n = \infty. \end{cases} \tag{3.8}$$

Under the conditions of Corollary 3.4 it follows therefore that for arbitrary $\varepsilon > 0$,

$$\begin{aligned} &\sum{}' |S_{2n+k}(z_n) - c_{2n+k-1}| < \infty \\ &\text{whenever } \ \varepsilon \leq \operatorname{dist}(z_n, Z_k) \leq \tfrac{1}{\varepsilon} \ \text{ for all } \ n \ \text{ and } \ \infty \notin Z_k. \end{aligned} \tag{3.9}$$

(For the notation $\sum'$ and Z_k, see the list of notation in section 1.) This leads to the following result, where $W_k := -1 - \overline{V_{k+1}^c}$ and $\partial^* V_k := \partial V_k \cap (-1 - V_{k+1})$ as usual.

Theorem 3.5. *Let $k \in \{0, 1\}$ be fixed. Let $\langle V_0, V_1 \rangle$ be closed circular twin value sets for the continued fraction $K(a_n/1)$ where V_0 is bounded, the limit circle case occurs and (3.6) holds for our k for some $I \subseteq \mathbb{N}$ and $\varepsilon > 0$. Then $Z_k \subseteq \partial^* V_k$, $-k \notin Z_k$, $0 \notin Z_0$ and Z_1 and Z_k are bounded, $\sum' |c_{2n} - c_{2n-1}| < \infty$, and the following statements are true.*

A. *Let $\varepsilon > 0$. If $(k-1) \notin Z_k$ or if for each n from some n_0 on, either $\operatorname{dist}(a_{2n+k-1}, Z_k) \geq \varepsilon$ or $\operatorname{dist}(-1 - a_{2n+k}, Z_k) \geq \varepsilon$, then $\sum' |c_n - c_{n-1}| < \infty$.*

B. *If also the limit circle case occurs for $S_{2n}(W_0)$ and*

$$\sigma_n \in \mathcal{M}_\varepsilon(W_k) \ \textit{for } n \in I \quad \textit{and} \quad \sigma_{n-1}^{-1} \in \mathcal{M}_\varepsilon(W_k^c) \ \textit{for } n \in \mathbb{N} \setminus I \tag{3.10}$$

for some $I \subseteq \mathbb{N}$ and $\varepsilon > 0$ for σ_n as in Corollary 3.4, then $Z_k \subseteq \partial^\dagger V_k$.

Proof. Under our conditions, $\{S_{2n+k}\}$ is restrained with exceptional sequence $\{\zeta_{2n+k}\}$, $Z_k \subseteq (-1 - V_{k+1}) \cap \partial V_k = \partial^* V_k$ and $Z_{k+1} \subseteq (-1 - V_k) \setminus V_{k+1}^\circ$ (Theorem 3.1 and Corollary 3.4). Now, V_0 is bounded, so $-1 \notin V_1$, and thus $0 \notin Z_0$ and $-k \notin Z_k$, and Z_k and Z_1 are bounded. Since $S_{2n}(0) = c_{2n}$ and $S_{2n+1}(-1) = c_{2n-1}$, it follows therefore from (3.9) that $\sum' |c_{2n} - c_{2n-1}| < \infty$.

A. It suffices to prove that either $\sum' |c_{2n-2} - c_{2n-1}| < \infty$ or $\sum' |c_{2n+m} - c_{2n+m-2}| < \infty$ for an $m \in \{0, 1\}$. First let $(k-1) \notin Z_k$. If $k = 0$, this means that

$\sum' |S_{2n}(-1) - c_{2n-1}| < \infty$ by (3.9) where $S_{2n}(-1) = c_{2n-2}$. If $k = 1$, then $0 \notin Z_1$ and $\sum' |S_{2n+1}(0) - c_{2n}| < \infty$. Next let $I := \{n \in \mathbb{N};\ \mathrm{dist}(a_{2n+k-1}, Z_k) \geq \varepsilon\}$. Then $\sum'_{n\in I} |S_{2n+k-2}(a_{2n+k-1}) - c_{2n+k-3}| < \infty$ where $S_{2n+k-2}(a_{2n+k-1}) = c_{2n+k-1}$ and $\sum'_{n\notin I} |S_{2n+k}(-1 - a_{2n+k}) - c_{2n+k-1}| < \infty$ where $S_{2n+k}(-1 - a_{2n+k}) = c_{2n+k-3}$, which means that $\sum' |c_{2n+k+1} - c_{2n+k-1}| < \infty$.

B. $\langle W_0, W_1\rangle$ are twin value sets for $K(a_n/1)$ (Remark 1.2.2). They satisfy the conditions in Corollary 3.4, so the exceptional sequences for $\{S_{2n+k}\}$ have all their limit points in ∂W_k. Hence $Z_k \subseteq \partial V_k \cap \partial W_k = \partial V_k \cap (-1 - \partial V_{k+1}) = \partial^\dagger V_k$. □

4. Proofs

Inspired by (3.5) we define

$$\mathcal{M}_\varepsilon(V, W) := \{t \in \mathcal{M};\ t(V) \subseteq W \setminus B_d(z, \varepsilon) \text{ for some } z \in \partial W\}, \tag{4.1}$$

$$\mathcal{E}(V) := \{\langle A, B\rangle \subseteq \mathbb{C}^2;$$

$$\exists\, \varepsilon > 0 \text{ s.t. } s_1 \circ s_2 \in \mathcal{M}_\varepsilon(V) \text{ for all } \langle a_1, a_2\rangle \in \langle A, B\rangle\}, \tag{4.2}$$

$$\mathcal{E}(V, W) := \{A \subseteq \mathbb{C};\ \exists\, \varepsilon > 0 \text{ s.t. } s \in \mathcal{M}_\varepsilon(V, W) \text{ for all } a \in A\}. \tag{4.3}$$

Proof of Theorem 1.4. Since $K(a_n/1)$ converges generally to c whereas $q \neq c$, the sequence $\{S_n\}$ is restrained with exceptional sequence $z_n^\dagger := S_n^{-1}(q)$. Part A follows from Theorem 3.1. The result in B follows from (1.9) since $S_{2n+k}(-1) = c_{2n+k-2}$ and $S_{2n+k}(\infty) = c_{2n+k-1}$. Similarly, part C follows from (1.9) since $S_{2n+k-2}(a_{2n+k-1}) = c_{2n+k-1}$ and $S_{2n+k+2}(-1 - a_{2n+k+2}) = c_{2n+k-1}$.

To prove part D we observe that if V_0 is bounded, then $c \neq \infty$ and $\infty \notin \tilde{Z}_1$ by part A. Hence $\{\zeta_n\}$ is exceptional and $S_{2n+1}(\infty) = c_{2n} \to c$ by part B. Finally, if $-1 \notin V_0 \setminus (-1 - V_1^\circ)$, i.e., $0 \notin (-1 - V_0) \setminus V_1^\circ$, then $0 \notin \tilde{Z}_1$, and part E follows from part B. (The same holds true if $0 \notin V_0$, but $0 \notin V_0 \Rightarrow \infty \notin V_1 \Rightarrow -1 \notin V_0$.) □

Lemma 4.1. *For given closed twin value sets $\langle V_0, V_1\rangle$, let $U_k := -1 - V_{k+1}^c$ for $k = 0, 1$, and let $k \in \{0, 1\}$ be a fixed number. Then $s(U_k) \subseteq U_{k+1}$ if and only if $s(V_k) \subseteq V_{k+1}$ and $s(U_k) = U_{k+1}$ if and only if $s(V_k) = V_{k+1}$. Similarly, if $A \subseteq \mathbb{C}$ is a closed set with $0, \infty \notin A$, then $A \in \mathcal{E}(U_k, U_{k+1})$ if and only if $A \in \mathcal{E}(V_k, V_{k+1})$.*

Proof. Let $a/(1+V_k) \subseteq V_{k+1}$. Since V_k is closed, the set V_k^c is open and non-empty, and both V_k, V_{k+1}, U_k and U_{k+1} contain finite elements. Therefore

$$\frac{a}{1+U_k} = -\frac{a}{V_{k+1}^c} = -\left(\frac{a}{V_{k+1}}\right)^c \subseteq (-1 - V_k)^c = U_{k+1}.$$

This actually proves the first two equivalences since U and V can be interchanged in this inclusion. Let $a/(1+V_k) \subseteq V_{k+1} \setminus B_d(z, \varepsilon)$ for some finite $z \in \partial V_{k+1} = -1 - \partial U_k$ and $\varepsilon > 0$. That is, $a/U_{k+1} \supseteq -(V_{k+1} \setminus B_d(z, \varepsilon))^c = -(-1 - U_k) \cup B_d(z, \varepsilon) = 1 + U_k \cup B_d(z^*, \varepsilon)$ where $z^* := -1 - z \in \partial U_k$. That is, $s^{-1}(U_{k+1}) \supseteq U_k \cup B_d(z^*, \varepsilon)$, so $s(U_k \cup B_d(z^*, \varepsilon)) \subseteq U_{k+1}$. Let $D := B_d(z^*, \varepsilon) \setminus U_k$ so that $U_k \cap D = \emptyset$ and $U_k \cup D = U_k \cup B_d(z^*, \varepsilon)$. Then

$$\frac{a}{1 + U_k \cup B_d(z^*, \varepsilon)} = \frac{a}{1+U_k} \cup \frac{a}{1+D} \subseteq U_{k+1}; \quad \text{i.e.,} \quad \frac{a}{1+U_k} \subseteq U_{k+1} \setminus \frac{a}{1+D}$$

where $a/(1 + z^*) \in U_{k+1}$. Therefore $a/(1 + U_k) \subseteq U_{k+1} \setminus B_d(\frac{a}{1+z^*}, \varepsilon^*)$ where $\varepsilon^* := \mathrm{dist}(\frac{a}{1+z^*}, \frac{a}{1+\partial B_d(z,\varepsilon)})$. Since $0, \infty \notin A$, the quantity ε^* has a positive lower bound for $a \in A$. Therefore $A \in \mathcal{E}(U_k, U_{k+1})$. This proves the last equivalence. □

Lemma 4.2. *Let V_0, V_1 be closed circular domains, and let $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ satisfy $a_k/(1+V_k) \subseteq V_{k-1}$ for $k = 1, 2$. Then $\sigma := s_1 \circ s_2$ is an elliptic transformation if and only if $s_k(V_k) = V_{k-1}$ for $k = 1, 2$ and σ has exactly two distinct fixed points $w_0, w_1 \notin \partial V_0$.*

Proof. Let σ be elliptic. Since $\sigma(V_0) \subseteq V_0$, it follows from [11, thm. 1.4] that $\sigma(V_0) = V_0$. Since $V_0 = s_1 \circ s_2(V_0) \subseteq s_1(V_1) \subseteq V_0$, this means that $s_1(V_1) = V_0$ and $s_2(V_0) = V_1$. It is clear that σ has two distinct fixed points w_0, w_1 and that ∂V_0 is a fixed circle (or fixed line) for σ. Hence ∂V_0 separates the two fixed points.

Conversely, assume that $s_k(V_k) = V_{k-1}$ for $k = 1, 2$ and that σ has two distinct fixed points $\notin \partial V_0$. Then $\sigma(\partial V_0) = \partial V_0$, which means that σ is either hyperbolic, parabolic, elliptic or the identity transformation. Since σ has exactly two distinct fixed points, the parabolic case and the identity case are ruled out. Since none of the fixed points lie on ∂V_0, the hyperbolic case is ruled out, so σ is elliptic. □

Lemma 4.3 (The disk – disk case)**.** *Let $V_k := B(C_k, R_k)$ for $k = 0, 1$, where $C_k \in \mathbb{C}$ and $R_k > 0$ satisfy* (2.1)*. Then $\langle E_1, E_2 \rangle$ given by* (2.4) *are the corresponding element sets. Let $k \in \{1, 2\}$ be fixed. Then $E_k \neq \emptyset$ if and only if* (2.2) *holds. If* (2.2) *holds with strict inequality, then $E_k^\circ \neq \emptyset$. If* (2.2) *holds with equality, then $E_k = \{\tilde{a}_k\}$ is given by* (2.3) *and $\tilde{a}_k \neq 0$. If $E_k^\circ \neq \emptyset$, then $\langle E_k, E_{k+1} \rangle \in \mathcal{E}(V_{k-1})$.*

Proof. For fixed $k \in \{1, 2\}$ and $a \neq 0$ we have

$$\frac{a}{1+V_k} = B\left(\frac{a(1+\overline{C}_k)}{|1+C_k|^2 - R_k^2}, \frac{|a|R_k}{|1+C_k|^2 - R_k^2}\right) =: B(\widehat{C}_{k-1}, \widehat{R}_{k-1}) \tag{4.4}$$

and $a/(1+V_k) \subseteq V_{k-1}$ if and only if $|\widehat{C}_{k-1} - C_{k-1}| + \widehat{R}_{k-1} \leq R_{k-1}$, that is, if and only if $a \in E_k$, where E_k is given by (2.4). Since $R_k < |1 + C_k|$, we see from (2.4) that $E_k \neq \emptyset$ if and only if $\tilde{a}_k \in E_k$, which proves that (2.2) is necessary and sufficient. It also proves that $\tilde{a}_k$ is the only point in E_k if and only if (2.2) holds with equality, and that $E_k^\circ \neq \emptyset$ otherwise. This means that if $E_k^\circ \neq \emptyset$, then $s \in \mathcal{M}_{\varepsilon_a}(V_k, V_{k-1})$ for some $\varepsilon_a > 0$ for every $a \in E_k$. Since E_k is compact in $\mathbb{C}$ ($-1 \notin V_k$ when V_{k-1} is bounded), this means that $E_k \in \mathcal{E}(V_k, V_{k-1})$. Finally, since $s_k \circ s_{k+1}(V_{k+1}) \subseteq s_k(V_k)$ for all $\langle a_k, a_{k+1} \rangle \in \langle E_k, E_{k+1} \rangle$, it follows that $\langle E_k, E_{k+1} \rangle \in \mathcal{E}(V_{k-1})$. □

Proof of Theorem 2.1. If $|C_{k-1}|R_k = |1 + C_k|R_{k-1}$ for $k = 1$ and $k = 2$, then $K(a_n/1)$ with all $a_{2n-1} = \tilde{a}_1$ and $a_{2n} = \tilde{a}_2$ is the only continued fraction from $\langle E_1, E_2 \rangle$. It converges if and only if $\tilde{s}_1 \circ \tilde{s}_2$ is non–elliptic. Let (2.2) hold with strict inequality for at least one $k \in \{1, 2\}$. Without loss of generality we assume that $E_1^\circ \neq \emptyset$. (See Remark 1.2.1.)

Assume first that the limit point case occurs. Then $K(a_n/1)$ converges generally to a value $c \in V_0$. It follows by Lemma 1.4D that $c_{2n} \to c$. Since also V_1 is bounded, we have $-1 \notin V_0$, so also $c_{2n+1} \to c$ by Lemma 1.4E.

Assume next that the limit circle case occurs. By Lemma 4.3 we know that $\limsup \operatorname{rad}(s_{2n-1} \circ s_{2n}(V_0)) < \operatorname{rad}(V_0)$, and so $Z_0 \subseteq \partial^* V_0$ by Theorem 3.5. Now, $-1 \notin V_0$ implies that $-1 \notin \partial^* V_0$. Hence $\sum' |c_n - c_{n-1}| < \infty$ by Theorem 3.5A, and thus $K(a_n/1)$ converges. □

Lemma 4.4 (The disk – half plane case)**.** *Let $V_0 := B(C_0, R_0)$ and $V_1 := H(h \cos\alpha, \alpha)$ where $C_0 \in \mathbb{C}$ and R_0, h, $\alpha \in \mathbb{R}$ satisfy* (2.5)*. Then $\langle E_1, E_2 \rangle$ given by*

(2.7) *are the corresponding element sets, and* $\tilde{E}_{k,\delta}$ *given by* (2.8) *satisfies* $\tilde{E}_{k,\delta} \in \mathcal{E}(V_k, V_{k-1})$ *for* $k = 1, 2$ *and* $0 < \delta < |a_1^*|$.

Proof. For $a \neq 0$ we have

$$\frac{a}{1+V_1} = B\left(\frac{a\,e^{-i\alpha}}{2(1+h)\cos\alpha}, \frac{|a|}{2(1+h)\cos\alpha}\right) \tag{4.5}$$

which is $\subseteq V_0$ if and only if $|\frac{a\,e^{-i\alpha}}{2(1+h)\cos\alpha} - C_0| + \frac{|a|}{2(1+h)\cos\alpha} \le R_0$, i.e., if and only if $a \in E_1$. Since $0 \in V_0$ and $0/(1+V_1) = \{0\}$, we also have $0 \in E_1$. Similarly, for $a \neq 0$,

$$\frac{a}{1+V_0} = \begin{cases} B\left(\frac{a(1+\overline{C_0})}{|1+C_0|^2 - R_0^2}, \frac{|a|R_0}{|1+C_0|^2-R_0^2}\right) =: B(\widehat{C}_1, \widehat{R}_1) & \text{if } |1+C_0| > R_0, \\ H\left(|a|/(2R_0), \arg(a(1+\overline{C_0}))\right) & \text{if } |1+C_0| = R_0, \end{cases} \tag{4.6}$$

and thus $a/(1+V_0) \subseteq V_1$ if and only if

$$\begin{aligned} \operatorname{Re}\left(\frac{a(1+\overline{C_0})}{|1+C_0|^2-R_0^2}\,e^{-i\alpha}\right) - \frac{|a|R_0}{|1+C_0|^2-R_0^2} \ge h\cos\alpha \quad & \text{if } |1+C_0| > R_0, \\ \arg\left(a(1+\overline{C_0})\right) = \alpha \quad \text{and} \quad \frac{|a|}{2R_0} \ge h\cos\alpha \quad & \text{if } |1+C_0| = R_0, \end{aligned} \tag{4.7}$$

i.e., if and only if $a \in E_2$. If $-1 \in V_0$, i.e., $|1+C_0| = R_0$, then $0 \notin E_2$ by definition. Hence $\langle E_1, E_2\rangle$ are the element sets corresponding to $\langle V_0, V_1\rangle$.

By (4.5) it follows that $a/(1+V_1) = V_0$ if and only if $R_0 = |C_0|$ and $C_0 = a\,e^{-i\alpha}/[2(1+h)\cos\alpha]$, i.e., $a = a_1^*$. Since $-1 \notin V_1$, the set E_1 is compact, so this shows that $E_1 \in \mathcal{E}(V_1, V_0)$ if $R_0 > |C_0|$. Let $R_0 = |C_0|$. Since $\tilde{E}_{1,\delta} \subseteq E_1$ is a compact set not containing a_1^*, $\tilde{E}_{1,\delta} \in \mathcal{E}(V_1, V_0)$.

Next we study $\tilde{E}_{2,\delta}$. First let $|1+C_0| = R_0$. By (4.6) it follows that $a/(1+V_0) = V_1$ for $a \neq 0$ if and only if $h > 0$ and $q := \frac{a}{2R_0}\frac{1+\overline{C_0}}{|1+C_0|} = h\,e^{i\alpha}\cos\alpha$, i.e., $a = a_2^*$. In this case $a_2^* \neq 0$ and E_2 is the ray $E_2 = a_2^*[1,\infty)$ and $\tilde{E}_{2,\delta} = a_2^*[1+\delta/|a_2^*|, \infty)$. Hence $\tilde{E}_{2,\delta}$ is a closed set in $\mathbb{C}$ with $0 \notin \tilde{E}_{2,\delta}$, and even if $a_{2n_m} \to \infty$ as $m \to \infty$, the set $a_{2n_m}/(1+V_0)$ will not approach V_1. (Indeed, it approaches the point set $\{\infty\}$ since V_0 is bounded.) Therefore $\tilde{E}_{2,\delta} \in \mathcal{E}(V_0, V_1)$ if $h > 0$. If $h < 0$, then $\operatorname{dist}(q - h\,e^{i\alpha}\cos\alpha) > |h|\cos\alpha > 0$, and $E_2 \in \mathcal{E}(V_0, V_1)$. If $h = 0$, then $\tilde{E}_{2,\delta} = [\delta, \infty)e^{i\gamma}$ with $\gamma := \alpha + \arg(1+C_0)$, and $a/(1+V_0)$ is the half plane $H(|a|/2R_0, \arg(a(1+\overline{C_0}))) = H(|a|/2R_0, \alpha)$ for $a \in E_2$. Hence also now $\tilde{E}_{2,\delta} \in \mathcal{E}(V_0, V_1)$.

Next let $|1+C_0| > R_0$. Then it follows from (4.6) that $a/(1+V_0) = B(\widehat{C}_1, \widehat{R}_1)$ is a disk not containing the origin for $a \neq 0$. If $a = 0$, then $a/(1+V_0) = \{0\}$ since $-1 \notin V_0$. Hence, there is no possibility of $B(\widehat{C}_1, \widehat{R}_1) \to V_1$ unless $\widehat{R}_1 \to \infty$; i.e., $|a| \to \infty$, but then $a/(1+V_0) \to \{\infty\}$ since V_0 is bounded. Hence $E_2 \in \mathcal{E}(V_0, V_1)$ in this case. □

Proof of Theorem 2.3. Let $K(a_n/1)$ be a continued fraction from $\langle E_1, E_2\rangle$. If $\operatorname{rad}(S_{2n}(V_0)) \to 0$ or $\operatorname{rad}(S_{2n}(W_0)) \to 0$ or $\operatorname{diam}(S_{2n}(Y_0)) \to 0$, then $K(a_n/1)$ clearly converges generally. Assume in the proof of parts A–C below that $\operatorname{diam}(S_{2n}(Y_0)) \to \tilde{d} > 0$, and thus $\operatorname{rad}(S_{2n}(V_0)) \to R > 0$ and $\operatorname{rad}(S_{2n}(W_0)) \to R^* > 0$.

A. Let $K(a_n/1)$ be from $\langle E_1, \tilde{E}_{2,\delta}\rangle$. Then $s_{2n} \circ s_{2n+1}(V_1) \subseteq s_{2n}(V_0)$ where $a_{2n} \in \tilde{E}_{2,\delta} \in \mathcal{E}(V_0, V_1)$ by Lemma 4.4, so $\langle \tilde{E}_{2,\delta}, E_1\rangle \in \mathcal{E}(V_1)$. Therefore $K(a_n/1)$ converges in the classical sense if $0 \notin Z_1$ (Theorem 3.5A with $k = 1$).

Let $0 \in Z_1$. Since by Theorem 3.5, $Z_1 \subseteq \partial^* V_1$, this means that $0 \in \partial V_1$, and $-1 \in V_0$, which means that $-1 \in \partial V_0$ by (2.5), so indeed, $0 \in \partial^\dagger V_1$. Then $h = 0$, and thus $a_2^* = 0$, and $R_0 = |1+C_0|$ and $\tilde{E}_{2,\delta} = e^{i\gamma}[\delta, \infty)$ where $\gamma := \alpha + \arg(1+C_0)$. This means that $\operatorname{dist}(\tilde{E}_{2,\delta},\, \partial^* V_1) > 0$ unless $\tilde{E}_{2,\delta} \subseteq \partial V_1$. Now, $\operatorname{Re}(C_0) \geq -\frac{1}{2}$ when $-1 \in \partial V_0$ since $0 \in V_0$ by (2.5) and V_0 is a disk. Therefore $\gamma \neq \alpha \pm \frac{\pi}{2}$, and $\tilde{E}_{2,\delta} \not\subseteq \partial V_1$. Hence $K(a_n/1)$ still converges by Theorem 3.5A.

B. Let $K(a_n/1)$ be from $\langle \tilde{E}_{1,\delta}, E_2\rangle$ and let $0 \notin \partial^\dagger V_1$. If $\tilde{E}_{2,\delta} = E_2$, then the situation is covered by part B, so let $R_0 = |1 + C_0|$ and $h \geq 0$. That is, $-1 \in \partial V_0$ and $0 \notin V_1^\circ$, and so, $0 \notin V_1$ under our conditions. Now, $a_{2n-1} \in \tilde{E}_{1,\delta} \in \mathcal{E}(V_1, V_0)$ by Lemma 4.4, so $\langle \tilde{E}_{1,\delta}, E_2\rangle \in \mathcal{E}(V_0)$. The result follows therefore from Theorem 3.5A since $0 \notin V_1$ implies that $-1 \notin \partial^* V_0$, and thus $-1 \notin Z_0$.

C. Let $K(a_n/1)$ be from $\langle \tilde{E}_{1,\delta}, E_2\rangle$. By Lemma 4.4, $\langle \tilde{E}_{1,\delta}, E_2\rangle \in \mathcal{E}(V_0)$. Hence $Z_0 \subseteq \partial^* V_0$ by Theorem 3.5. The convergence follows therefore from Theorem 3.5A.

D. That $c_{2n} \to c$ follows from Theorem 1.4D. We know that $Z_k \subseteq (-1-V_{k+1}) \setminus V_k^\circ$ by Theorem 1.4A. Therefore $0 \notin Z_1$ if $0 \in V_1^\circ$ or $-1 \notin V_0$, which in our situation holds if $-1 \notin \partial V_0$, and $c_n \to c$ by Theorem 1.4E.

The conditions on $\{a_n\}$ imply that $\operatorname{dist}(a_{2n-1}, Z_0) \geq \varepsilon$ from some n on (Theorem 3.5), and thus $c_{2n-1} \to c$ by Theorem 1.4C. □

Lemma 4.5 (The disk – complement of disk case)**.** *Let* $V_0 := B(C_0, R_0)$ *and* $V_1 := \overline{B(C_1, R_1)^c}$ *where* $C_0, C_1 \in \mathbb{C}$ *and* $R_0, R_1 > 0$ *satisfy* (2.9). *Let* E_k *and* $\widehat{E}_{k,\delta}$ *be given as in Theorem 2.5. Then* $\langle E_1, E_2\rangle$ *are the element sets corresponding to* $\langle V_0, V_1\rangle$, *and* $\langle \widehat{E}_{1,\delta}, E_2\rangle \in \mathcal{E}(V_0)$ *and* $\langle \widehat{E}_{2,\delta}, E_1\rangle \in \mathcal{E}(V_1)$.

Proof. For $a \neq 0$ the set $a/(1 + V_1)$ is a circular disk $B(\widehat{C}_0, \widehat{R}_0)$ where

$$\widehat{C}_0 = \frac{a(1+\overline{C_1})}{|1+C_1|^2 - R_1^2}, \quad \widehat{R}_0 = \frac{|a|R_1}{R_1^2 - |1+C_1|^2}. \tag{4.8}$$

It is $\subseteq V_0$ if and only if $|\widehat{C}_0 - C_0| + \widehat{R}_0 \leq R_0$, i.e., if and only if $a \in E_1$. It is equal to V_0 if and only if $\widehat{C}_0 = C_0$ and $\widehat{R}_0 = R_0$, i.e., if and only if either

$$\begin{aligned} &C_1 \neq -1,\ \ a = \tilde{a}_1 \ \text{ and } \ |C_0|R_1 = R_0|1+C_1| \\ &\qquad\qquad \text{or} \quad C_1 = -1,\ \ C_0 = 0 \ \text{ and } \ |a| = R_0R_1. \end{aligned} \tag{4.9}$$

Since $\widehat{E}_{1,\delta}$ is a closed, bounded set in $\mathbb{C}$ with $a/(1 + V_1) \neq V_0$ for all $a \in \widehat{E}_{1,\delta}$, we have $\widehat{E}_{1,\delta} \in \mathcal{E}(V_1, V_0)$. Since $s_1 \circ s_2(V_0) \subseteq s_1(V_1)$ this proves that $\langle \widehat{E}_{1,\delta}, E_2\rangle \in \mathcal{E}(V_0)$.

Let $|1+C_0| < R_0$. Then $a/(1+V_0)$ is the exterior of a disk. Indeed, $a/(1+V_0^\circ) = B(\widehat{C}_1, \widehat{R}_1)^c$ where

$$\widehat{C}_1 = \frac{a(1+\overline{C_0})}{|1+C_0|^2 - R_0^2}, \quad \widehat{R}_1 = \frac{|a|R_0}{R_0^2 - |1+C_0|^2}. \tag{4.10}$$

It is $\subseteq V_1$ if and only if $|\widehat{C}_1 - C_1| + R_1 \leq \widehat{R}_1$, i.e., if and only if $a \in E_2$. It is equal to V_1° if and only if $\widehat{C}_1 = C_1$ and $\widehat{R}_1 = R_1$, i.e., if and only if either

$$\begin{aligned} &-1 \in V_0^\circ,\ \ C_0 \neq -1,\ \ a = \tilde{a}_2 \ \text{ and } \ |C_1|R_0 = R_1|1+C_0| \\ &\qquad\qquad\qquad \text{or} \quad C_0 = -1,\ \ C_1 = 0 \ \text{ and } \ |a| = R_0R_1. \end{aligned} \tag{4.11}$$

These cases are excluded for $a \in \widehat{E}_{2,\delta}$. From (2.10) we see that $0 \notin \overline{E_2}$ when $|1+C_0| < R_0$. We need to check whether $a_{2n_k}/(1+V_0) \to V_1$ is possible for $a_{2n_k} \in E_2$ if $a_{2n_k} \to \infty$. But this is no problem since V_0 is bounded, and thus $\lim_{a\to\infty} a/(1+V_0) = \{\infty\}$. Therefore $\tilde{E}_{2,\delta} \in \mathcal{E}(V_0, V_1)$, and thus $\langle \widehat{E}_{2,\delta}, E_1 \rangle \in \mathcal{E}(V_1)$.

Next, let $|1+C_0| = R_0$. Then for $a \neq 0$, $a/(1+V_0)$ is the half plane given by (4.6). Hence $\langle E_2, E_1 \rangle \in \mathcal{E}(V_1)$ and $a/(1+V_0) \subseteq V_1$ if and only if

$$\mathrm{Re}\left(C_1 \frac{1+C_0}{|1+C_0|}\, e^{-i\theta} \right) + R_1 \leq \frac{|a|}{2R_0} \quad \text{where } \theta := \arg a,$$

which gives the expression for E_2 in this case. ($0 \notin E_2$ since $-1 \in V_0$.)

Finally, let $|1+C_0| > R_0$. Then $a/(1+V_0) = B(\widehat{C}_1, -\widehat{R}_1)$ for $a \neq 0$, where $\widehat{C}_1$ and $\widehat{R}_1$ are given by (4.10). Therefore $a/(1+V_0) \subseteq V_1$ if and only if $|\widehat{C}_1 - C_1| \geq R_1 + |\widehat{R}_1|$, i.e., if and only if $a \in E_2$. Moreover, $\langle E_2, E_1 \rangle \in \mathcal{E}(V_1)$. □

Proof of Theorem 2.5. A. The expressions for E_1 and E_2 follow from Lemma 4.5. We need to check that $E_k^\circ \neq \emptyset$ for $k = 1, 2$. This clearly holds for E_1 since $|C_0| < R_0$ and $|1+C_1| < R_1$, and thus $0 \in E_1^\circ$. It is also clear that $E_2^\circ \neq \emptyset$ if $C_0 = -1$ or if $R_0 = |1+C_0|$. Let $R_0 < |1+C_0|$ and $C_1 \neq 0$. Then $\tilde{a}_2 \neq 0$ and $-t\tilde{a}_2 \in E_2^\circ$ for all $t > 0$ sufficiently large. If $R_0 < |1+C_0|$ and $C_1 = 0$, then $\tilde{a}_2 = 0$ and $E_2 = \{a;\, |a| \geq R_1(|1+C_0| + R_0)\}$, so again $E_2^\circ \neq \emptyset$. If $R_0 > |1+C_0| > 0$ and $C_1 \neq 0$, then $t\tilde{a}_2 \in E_2^\circ$ for all $t > 0$ sufficiently large, and thus $E_2^\circ \neq \emptyset$. Finally, if $R_0 > |1+C_0|$ and $C_1 = 0$, then $\tilde{a}_2 = 0$ and all a with $|a| \geq R_0^2 - |1+C_0|^2$ are $\in E_2$.

Let $K(a_n/1)$ be a continued fraction from $\langle E_1, E_2 \rangle$. If $\mathrm{rad}(S_{2n}(V_0)) \to 0$ or $\mathrm{rad}(S_{2n}(W_0)) \to 0$ or $\mathrm{diam}(S_{2n}(Y_0)) \to 0$, then $K(a_n/1)$ clearly converges generally. Assume in the proof of parts B and C below that $\mathrm{diam}(S_{2n}(Y_0)) \to \tilde{d} > 0$, and thus $\mathrm{rad}(S_{2n}(V_0)) \to R > 0$ and $\mathrm{rad}(S_{2n}(W_0)) \to R^* > 0$.

B. We first observe that $W_0 = B(-1-C_1, R_1)$ and $W_1 = \overline{B(-1-C_0, R_0)^c}$ in this case. By Lemma 4.1 the element sets E_1 and E_2 do not change if we replace $\langle V_0, V_1 \rangle$ by $\langle W_0, W_1 \rangle$ (although their representation (2.10) changes), and neither do the conditions in (2.11). Indeed, $\widehat{E}_{1,\delta}$ and $\widehat{E}_{2,\delta}$ do not change either, since

$$\tilde{a}_k = C_{k-1}(1+C_k)(1 - R_k^2/|1+C_k|^2) = (-1-C_k)(-C_{k-1})(1 - R_{k-1}^2/|C_{k-1}|^2)$$

when $|C_{k-1}|R_k = R_{k-1}|1+C_k| > 0$. Therefore $\widehat{E}_{1,\delta} \in \mathcal{E}(W_1, W_0) \cap \mathcal{E}(V_1, V_0)$ by Lemma 4.5.

There is one condition that is changed, though, and that is the condition $-1 \notin V_0^\circ$, which is equivalent to $0 \in W_1$. This means that if $-1 \notin V_0^\circ$, then $E_2 \in \mathcal{E}(V_0, V_1)$, whereas, by (4.11), $E_2 \notin \mathcal{E}(W_0, W_1)$ if also $-1 \in W_0^\circ$ and $|C_1|R_0 = R_1|1+C_0| \geq 0$. However, this case cannot occur since

$$-1 \notin V_0^\circ \;\Leftrightarrow\; |1+C_0| \geq R_0 \quad \text{and} \quad -1 \in W_0^\circ \;\Leftrightarrow\; |C_1| < R_1,$$

which give $|C_1|R_0 < R_1|1+C_0|$. Therefore, also now $\widehat{E}_{2,\delta} \in \mathcal{E}(W_0, W_1) \cap \mathcal{E}(V_0, V_1)$ by Lemma 4.5. This means that $Z_k \in \partial^\dagger V_k$ by Theorem 3.5B. Since $\partial^\dagger V_0 = -1 - \partial^\dagger V_1$, the convergence follows from Theorem 3.5A, both if $K(a_n/1)$ is from $\langle \widehat{E}_{1,\delta}, E_2 \rangle$ or from $\langle E_1, \widehat{E}_{2,\delta} \rangle$.

C. By the proof of part B, $Z_0 \subseteq \partial^\dagger V_0$ when $K(a_n/1)$ is from $\langle \widehat{E}_{1,\delta}, E_2 \rangle$, and $Z_1 \subseteq \partial^\dagger V_1$ when $K(a_n/1)$ is from $\langle E_1, \widehat{E}_{2,\delta} \rangle$. The result follows therefore from Theorem 3.5A.

D: Let $K(a_n/1)$ from $\langle E_1, E_2\rangle$ converge generally to c. Then $c_{2n} \to c$ and $\tilde{Z}_k = Z_k$ for $k = 0, 1$ by Theorem 1.4D. Therefore $Z_0 \subseteq \overline{V_0^c} \cap (-1 - V_1)$ (Theorem 1.4A). It follows therefore from Theorem 1.4E and C with $k = 0$ that also $c_{2n-1} \to c$. □

Proof of Theorem 2.7. Let $k \in \{1, 2\}$ be fixed. First let $-1 \notin \partial V_k$. Then $g_k < 1$, $|\alpha_k| < \pi/2$ and

$$a/(1 + V_k) = B(\tilde{C}_k, \tilde{R}_k), \quad \tilde{C}_k := \frac{a\, e^{-i\alpha_k}}{2(1 - g_k)\cos\alpha_k}, \;\; \tilde{R}_k := \frac{|a|}{2(1 - g_k)\cos\alpha_k}$$

for $a \neq 0$. This set is contained in V_{k-1} if and only if $\mathrm{Re}(\tilde{C}_k e^{-i\alpha_{k-1}}) - \tilde{R}_k \geq -g_{k-1}\cos\alpha_{k-1}$, which proves the expression for E_k in this case. Next let $-1 \in \partial V_k$. Then $1/(1 + V_k) = H(0, -\alpha_k)$. Hence $a/(1 + V_k) \subseteq V_{k-1}$ for $a \neq 0$ if and only if $\arg(a) = \alpha_{k-1} + \alpha_k$. Since either $g_k = 1$ or $|\alpha_k| = \pi/2$ when $-1 \in \partial V_k$, the expression (2.15) for E_k is still valid. Therefore $\langle E_1, E_2\rangle$ given by (2.15) are the element sets corresponding to $\langle V_0, V_1\rangle$.

If $0, -1 \notin V_k$ for both $k = 0$ and $k = 1$, then the convergence follows from the twin version of the multiple parabola theorem proved in [5]. (See Remark 2.8.3.) Otherwise, by (2.14), there exist $\tilde{g}_0$, $\tilde{g}_1$, $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ such that

$$\begin{aligned}
&|\tilde{\alpha}_0| < \tfrac{\pi}{2}, \quad |\tilde{\alpha}_1| < \tfrac{\pi}{2} \quad \text{and } \tilde{\alpha}_0 + \tilde{\alpha}_1 = \alpha_0 + \alpha_1,\\
&0 < \tilde{g}_0 < 1, \quad 0 < \tilde{g}_1 < 1 \quad \text{and } \tilde{g}_k(1 - \tilde{g}_{k-1}) \geq g_k(1 - g_{k-1}) \text{ for } k = 1, 2.
\end{aligned}$$

Let $\tilde{E}_1$ and $\tilde{E}_2$ be given by (2.15) with g_0, g_1, α_0 and α_1 replaced by $\tilde{g}_0$, $\tilde{g}_1$, $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$. Then $E_1 \subseteq \tilde{E}_1$ and $E_2 \subseteq \tilde{E}_2$, and the convergence follows again from the twin version of the multiple parabola theorem. □

Proof of Theorem 1.3. Since $|C_k|R_{k+1} \neq R_k|1 + C_{k+1}|$ for $k = 0$ or $k = 1$, we have $\widehat{E}_{k+1,\delta} = E_{k+1}$ in (2.11) for this k, and $K(a_n/1)$ converges generally by Theorem 2.5B. □

Proof of Theorem 1.1. A. Since $E_2^\circ = \emptyset$ if and only if $E_2 = \{\tilde{a}_2\}$ in this case, which happens if and only if $|C_1|R_0 = |1 + C_0|R_1$, it follows from Theorem 2.1 that $K(a_n/1)$ converges.

B. By (2.7) we always have $-1 \notin V_0$ when $E_2^\circ \neq \emptyset$. Hence $\tilde{E}_{2,\delta} = E_2$, and $K(a_n/1)$ converges generally by Theorem 2.3B. Theorem 1.4D shows therefore that its even part converges, and Theorem 1.4E shows that its odd part converges.

C. It follows from Theorem 1.3 that $K(a_n/1)$ converges generally in this case. Therefore its even part converges by Theorem 1.4D. The convergence of $K(a_n/1)$ follows from Theorem 1.4E.

D. (2.14) holds under our conditions, and the result follows from Theorem 2.7. □

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