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CLASSES OF TIME-DEPENDENT MEASURES, NON-HOMOGENEOUS MARKOV PROCESSES, AND FEYNMAN-KAC PROPAGATORS

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Abstract. We study the inheritance of properties of free backward propagators associated with transition probability functions by backward Feynman-Kac propagators corresponding to functions and time-dependent measures from non-autonomous Kato classes. The inheritance of the following properties is discussed: the strong continuity of backward propagators on the space L^r, the $(L^r - L^q)$-smoothing property of backward propagators, and various generalizations of the Feller property. We also prove that a propagator on a Banach space is strongly continuous if and only if it is separately strongly continuous and locally uniformly bounded.

1. Introduction

In this paper we study the behavior of evolution families (propagators). Examples of such families are solution operators corresponding to initial and final value problems for second order parabolic partial differential equations and also integral operators associated with non-homogeneous transition probability functions. The main objects of our interest in the present paper are forward and backward Feynman-Kac propagators associated with forward and backward free propagators. We formulate and prove our results for backward propagators. However, all the results obtained in the present paper have their counterparts in the case of forward propagators. This is explained in Section 10. We choose backward free and backward Feynman-Kac propagators in this paper because of their links with transition probability functions. Note that in the forward case one uses backward transition probability functions (see Section 10).

The free backward propagator $Y = Y(\tau, t)$ is given by

$$(1)\qquad Y(\tau,t)f(x) = \int_E f(y)P(x,\tau;t,dy) = E_{\tau,x}\left[f(X_t)\right], \quad 0 \le \tau < t \le T,$$

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and the backward Feynman-Kac propagators Y_V and Y_μ are defined by

$$Y_V(\tau,t)f(x) = E_{\tau,x}\left[f(X_t)\exp\{-\int_\tau^t V(s,X_s)ds\}\right], \ \ 0 \le \tau < t \le T, \tag{2}$$

and

$$Y_\mu(\tau,t)f(x) = E_{\tau,x}\left[f(X_t)\exp\{-A_\mu(t,\tau)\}\right], \ \ 0 \le \tau < t \le T. \tag{3}$$

In formula (1), P is a transition probability function on a locally compact second countable Hausdorff topological space E equipped with the Borel σ-algebra $\mathcal{E}$ and a reference measure m. It is assumed that $0 < m(A) < \infty$ for any compact subset A of E with non-empty interior. We will write dx instead of $m(dx)$. It is known that the space E is σ-compact and metrizable (see [26]), and a metric $\rho : E \times E \to [0,\infty)$ generating the topology of the space E will be fixed. In formulae (2) and (3), V is a Borel function on $[0,T]\times E$ and $\mu = \{\mu(t) : 0 \le t \le T\}$ is a family of signed Borel measures of locally finite variation. We call such families time-dependent measures. In formulae (1)–(3), $X = (X_t, \mathcal{F}_t^\tau, P_{\tau,x})$ is an $\mathcal{F}_t^\tau$-progressively measurable non-homogeneous Markov process with state space $(E,\mathcal{E})$ and with P as its transition function. By $\mathcal{M}$ will be denoted the class of all transition probability functions P for which such a process exists, and we will always choose a progressively measurable process X_t to represent P. The restrictions on f, V, and μ in formulae (1)–(3) will be specified below. The symbol A_μ in formula (3) stands for the additive functional of the process X_t that extends the functional $A_V(\tau,t) = \int_\tau^t V(s,X_s)ds$ from functions on the space $[0,T]\times E$ to time-dependent measures (see Section 4). We refer the reader to [6, 8, 9, 10, 15, 28, 31, 35, 41] for more information on transition probability functions, non-homogeneous Markov processes, and progressive measurability. Kolmogorov's paper [27] was an important early work on non-homogeneous Markov processes.

Free backward propagators often arise as solution operators associated with final value problems of the following form:

$$\begin{cases} \frac{\partial u}{\partial \tau} + L(\tau)u = 0, \ \ 0 < \tau < t \le T, \\ u(t) = f, \end{cases}$$

where the generators $L(\tau)$ depend on time. The backward Feynman-Kac propagator Y_V corresponds to the perturbed final value problem,

$$\begin{cases} \frac{\partial u}{\partial \tau} + L(\tau)u - V(\tau)u = 0, \ \ 0 < \tau < t \le T, \\ u(t) = f, \end{cases}$$

while the backward Feynman-Kac propagator Y_μ is associated with the problem

$$\begin{cases} \frac{\partial u}{\partial \tau} + L(\tau)u - \mu(\tau)u = 0, \ \ 0 < \tau < t \le T, \\ u(t) = f. \end{cases} \tag{4}$$

Similarly, free forward propagators often arise as solution operators associated with initial value problems of the following form:

$$\begin{cases} \frac{\partial u}{\partial t} - L(t)u = 0, \ \ 0 \le \tau < t < T, \\ u(\tau) = f. \end{cases}$$

The forward Feynman-Kac propagator U_V corresponds to the perturbed initial value problem,

$$\begin{cases} \frac{\partial u}{\partial t} - L(t)u + V(t)u = 0, \ \ 0 \le \tau < t < T, \\ u(\tau) = f, \end{cases}$$

while the forward Feynman-Kac propagator U_μ is associated with the problem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - L(t)u + \mu(t)u = 0, \ \ 0 \le \tau < t < T, \\ u(\tau) = f. \end{array} \right. \tag{5}$$

The equations in (4) and (5) are understood in the distributional sense (see Section 10 for the definition of free propagators and Feynman-Kac propagators).

Throughout the present paper, we assume that V and μ belong to the non-autonomous Kato classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ of functions and measures, respectively (see Definition 3.1). In the case of the Gaussian transition probability density, these classes were studied in [16, 17, 18, 21]. The classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ are generalizations of a celebrated Kato class K_n introduced by Aizenman and Simon in [1, 37]. The definition of K_n is based on a condition used by Kato in [25]. Aizenman and Simon developed the theory of Schrödinger semigroups with Kato class potentials (see [1, 37]). The Feynman-Kac propagators Y_V and Y_μ with V and μ in the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ are, in a sense, two parameter generalizations of Schrödinger semigroups with Kato class potentials. However, even in the case where P is the Gaussian transition probability function, the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ are not completely analogous to the Kato class. For instance, it was shown in [21] that there exist $V \in \mathcal{P}_f^*$ and $\mu \in \mathcal{P}_m^*$ such that the backward Feynman-Kac propagators Y_V and Y_μ are not bounded on the space L^1. On the other hand, a Schrödinger semigroup with a Kato class potential is always bounded on L^1. More information on the Kato classes of functions and measures and Schrödinger semigroups can be found in [1, 2, 3, 4, 5, 6, 7, 24, 37, 39, 40, 45]. The non-autonomous Kato classes were studied in [16, 17, 18, 19, 20, 21, 22, 30, 31, 34, 36, 38, 42, 43, 44]. The classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ considered in [16, 17, 18, 19, 20, 21, 22] and in the present paper are wider than most of the non-autonomous Kato classes studied before.

We now give an overview of the results obtained in the present paper. In Section 2 we prove that the strong continuity of a propagator is equivalent to its separate strong continuity and uniform local boundedness. Section 3 is devoted to the non-autonomous Kato classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$. In Section 4 we construct two parameter additive functionals of Markov processes corresponding to time-dependent measures. The construction utilizes a special approximation method (see Definition 4.3). Section 5 contains the exponential estimates for the additive functionals defined in Section 4. In Sections 6 and 7 we study the inheritance of properties of free propagators by their Feynman-Kac perturbations. We discuss the strong continuity of propagators on the space L^r, the $(L^r - L^q)$-smoothing property, and various versions of the Feller property. More results concerning the similarities in the behaviour of semigroups (propagators) and their perturbations by potentials can be found in [1, 2, 3, 4, 5, 6, 7, 14, 16, 17, 18, 19, 20, 21, 22, 32, 34, 37, 38, 39, 40, 42, 43, 44, 45]. In Section 8 we establish that the Feller property, the Feller-Dynkin property, and the BUC-property are inherited by Feynman-Kac propagators from free propagators under additional restrictions on functions and time-dependent measures generating Feynman-Kac propagators. In Section 9 we discuss the inheritance problem in the case where transition densities are fundamental solutions to second order conservative parabolic partial differential equations. The last section of the paper (Section 10) concerns the relations between the forward and backward cases.

2. Backward propagators on Banach spaces

We show in this section that a backward propagator is strongly continuous if and only if it is separately strongly continuous and locally uniformly bounded. This result (Theorem 2.2 below) was first formulated without proof in [19]. The proof appeared in the CRM preprint [20].[1] A similar result holds for propagators.

Let $T > 0$ and put $D_T = \{(\tau, t) : 0 \leq \tau \leq t \leq T\}$. If $T = \infty$, then we assume that $D_\infty = \{(\tau, t) : 0 \leq \tau \leq t < \infty\}$. For a Banach space B, the symbol $L(B, B)$ stands for the space of all bounded linear operators on B.

Definition 2.1. A two-parameter family of operators $\{W(t, \tau) \in L(B, B) : (\tau, t) \in D_\infty\}$ is called a propagator on B provided that the following conditions hold:

(i) $W(t, \tau) = W(t, \lambda)W(\lambda, \tau)$ for $0 \leq \tau \leq \lambda \leq t < \infty$;
(ii) $W(\tau, \tau) = I$ for $0 \leq \tau < \infty$.

If the family W is defined on D_T with $T < \infty$, then it will be assumed in (i) and (ii) that $(\tau, t) \in D_T$.

A two parameter family of operators $\{Q(\tau, t) \in L(B, B) : (\tau, t) \in D_\infty\}$ is called a backward propagator on B provided that the following conditions hold:

(1) $Q(\tau, t) = Q(\tau, \lambda)Q(\lambda, t)$ for $0 \leq \tau \leq \lambda \leq t < \infty$;
(2) $Q(t, t) = I$ for $0 \leq t < \infty$.

If the family Q is defined on D_T with $T < \infty$, then it will be assumed in (1) and (2) that $(\tau, t) \in D_T$.

It is easy to see that if $T < \infty$ and Q is a backward propagator on B, then the family of operators defined by $W(t, \tau) = Q(T - t, T - \tau)$ is a propagator on B. Similarly, if W is a propagator on B, then the family Q defined by $Q(\tau, t) = W(T - \tau, T - t)$ is a backward propagator on B. The previous statements show that practically any result about propagators has a counterpart for backward propagators and vice versa.

It is said that a backward propagator Q is strongly continuous if for every $x \in B$, the B-valued function $(\tau, t) \to Q(\tau, t)x$ is continuous. A backward propagator Q is called uniformly bounded if $||Q(\tau, t)||_{B \to B} \leq M$ for all $(\tau, t) \in D_T$. If $T = \infty$, and for every compact subset K of D_∞, $||Q(\tau, t)||_{B \to B} \leq M_K$ for all $(\tau, t) \in K$, then Q is called locally uniformly bounded. A backward propagator Q is called separately strongly continuous if for every fixed t and $x \in B$, the function $\tau \to Q(\tau, t)x$ is continuous on $[0, t]$, and for every fixed τ and $x \in B$, the function $t \to Q(\tau, t)x$ is continuous on $[\tau, T]$ (if $T = \infty$, then we consider the interval $[t, \infty)$ instead of the interval $[t, T]$).

Theorem 2.2. *For a backward propagator Q on B, the following are equivalent:*
(i) the strong continuity;
(ii) the strong separate continuity and the uniform local boundedness.

Proof. Using the uniform boundedness principle, we see that (i) implies (ii). Next, let Q be a strongly separately continuous and locally uniformly bounded propagator. Let $(\tau, t) \in D_T$, and suppose t' and τ' are close to t and τ, respectively. We will first assume that $t > \tau$. Then for τ' close to τ, we have $t > \tau'$. Using the local

[1] After finishing our work on the present paper we found that Theorem 2.2 was also obtained independently but later in [30].

uniform boundedness condition and assuming that $t' \geq t$, we get that for every $x \in B$,

$$\begin{aligned} I &= \|Q(\tau',t')x - Q(\tau,t)x\|_B \\ &\leq \|Q(\tau',t')x - Q(\tau',t)x\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B \\ &\leq \|Q(\tau',t)(Q(t,t')x - x)\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B \\ &\leq M\|Q(t,t')x - x\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B. \end{aligned}$$

By the separate continuity of Q,

$$\lim_{t'\to t,\tau'\to\tau} I = 0. \tag{6}$$

If $t' < t$, then

$$\begin{aligned} I &\leq \|Q(\tau',t')x - Q(\tau',t)x\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B \\ &\leq \|Q(\tau',t')(x - Q(t',t)x)\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B \\ &\leq M\|Q(t',t)x - x\|_B + \|Q(\tau',t)x - Q(\tau,t)x\|_B, \end{aligned}$$

and we again get formula (6).

Finally, let $\tau = t < \tau' \leq t'$. Then the separate continuity of Q implies that for every $\epsilon > 0$ there exists $\lambda > 0$ such that $\lambda > \tau$ and $\|Q(\tau,\lambda)x - x\|_B \leq \epsilon$. It follows from the local uniform boundedness condition that

$$\begin{aligned} I &= \|Q(\tau',t')x - x\|_B \\ &\leq \|Q(\tau',t')x - Q(\tau',\lambda)x\|_B + \|Q(\tau',\lambda)x - Q(\tau,\lambda)x\|_B \\ &\quad + \|Q(\tau,\lambda)x - x\|_B \leq \|Q(\tau',t')(x - Q(t',\lambda)x\|_B \\ &\quad + \|Q(\tau',\lambda)x - Q(\tau,\lambda)x\|_B + \epsilon \leq M\|Q(t',\lambda)x - x\|_B \\ &\quad + \|Q(\tau',\lambda)x - Q(\tau,\lambda)x\|_B + \epsilon \leq M\|Q(t',\lambda)x - Q(\tau,\lambda)x\|_B \\ &\quad + M\|Q(\tau,\lambda)x - x\|_B + \|Q(\tau',\lambda)x - Q(\tau,\lambda)x\|_B + \epsilon \\ &\leq M\|Q(t',\lambda)x - Q(\tau,\lambda)x\|_B \\ &\quad + \|Q(\tau',\lambda)x - Q(\tau,\lambda)x\|_B + (M+1)\epsilon. \end{aligned} \tag{7}$$

The constant M in (7) depends on t. It follows from (7) and from the separate continuity of Q that there exists $\delta > 0$ such that for $\tau \leq \tau' \leq t' < \tau + \delta$, we have $I \leq (2M+2)\epsilon$. Therefore, (6) holds for $\tau = t < \tau' \leq t'$.

This completes the proof of Theorem 2.2. □

3. Non-autonomous Kato classes of functions and measures

In this section we introduce and study the non-autonomous Kato classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$. The subscripts f and m in $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ are the first letter of the words "function" and "measure". Let V be a Borel function on the set $[0,T] \times E$, where $T > 0$ is a fixed number, and let μ be a family $\mu = \{\mu(t) : 0 \leq t \leq T\}$ of signed Borel measures on $(E, \mathcal{E})$. Recall that we called such families time-dependent measures. By $|\mu| = \{|\mu(t)| : 0 \leq t \leq T\}$ is denoted the family consisting of variations of the measures $\mu(t)$ with $0 \leq t \leq T$. Let P be a transition probability function. For a function V as above, put

$$N(V)(\tau,t,x) = \int_\tau^t Y(\tau,s)V(s)(x)ds, \ \ (\tau,t) \in D_T, \ \ x \in E. \tag{8}$$

The function $N(V)$ is, in a sense, a potential of V. In the case of a time-dependent measure μ we assume that P has a density p. Then we define the potential N_μ by

$$N(\mu)(\tau,t,x)=\int_\tau^t Y(\tau,s)\mu(s)(x)ds,\ \ (\tau,t)\in D_T,\ \ x\in E. \tag{9}$$

It is assumed in (8) and (9) that the integrals on the right-hand side make sense. The next definition concerns backward non-autonomous Kato classes.

Definition 3.1. Let $P\in\mathcal{M}$. Then it is said that V belongs to the class $\hat{\mathcal{P}}_f^*$ provided that

$$\sup_{(\tau,t)\in D_T}\sup_{x\in E}N(|V|)(\tau,t,x)<\infty.$$

Let $V\in\hat{\mathcal{P}}_f^*$. Then it is said that V belongs to the class $\mathcal{P}_f^*$ provided that

$$\lim_{t-\tau\to 0+}\sup_{x\in E}N(|V|)(\tau,t,x)=0.$$

Suppose that $P\in\mathcal{M}$ has a density p. Then it is said that μ belongs to the class $\hat{\mathcal{P}}_m^*$ provided that

$$\sup_{\tau:(\tau,t)\in D_T}\sup_{x\in E}N(|\mu|)(\tau,t,x)<\infty.$$

If $\mu\in\hat{\mathcal{P}}_m^*$, then it is said that μ belongs to the class $\mathcal{P}_m^*$ provided that

$$\lim_{t-\tau\to 0+}\sup_{x\in E}N(|\mu|)(\tau,t,x)=0.$$

In the case of the heat semigroup, the classes in Definition 3.1 were introduced in [17, 21].

Let $V\in\hat{\mathcal{P}}_f^*$, $\mu\in\hat{\mathcal{P}}_m^*$, and denote

$$||V||_f=\sup_{\tau:0\le\tau\le T}\sup_{x\in E}N(|\mu|)(\tau,T,x) \tag{10}$$

and

$$||\mu||_m=\sup_{\tau:0\le\tau\le T}\sup_{x\in E}N(|\mu|)(\tau,T,x). \tag{11}$$

It will be shown below that under certain restrictions the non-autonomous Kato classes in Definition 3.1 equipped with the norms defined by (10) and (11) are Banach spaces.

Remark 3.2. Let l denote the Lebesgue measure on the σ-algebra $\mathcal{B}_{[0,T]}$ of all Borel subsets of $[0,T]$. By $l_{[\tau,T]}$ is denoted the restriction of l to $[\tau,T]$. For every $\tau\in[0,T]$ and $x\in E$, define a measure $\xi_{\tau,x}$ on the σ-algebra $\sigma(\mathcal{B}_{[\tau,T]}\times\mathcal{E})$ as follows: For $U\in\sigma(\mathcal{B}_{[\tau,T]}\times\mathcal{E})$,

$$\xi_{\tau,x}(U)=\int_U P(\tau,x;u,dy)du.$$

Then, for $V\in\hat{\mathcal{P}}_f^*$, the condition $||V||_f=0$ means that for all $\tau\in[0,T)$ and $x\in E$, $V(u,y)=0$ holds $\xi_{\tau,x}$-a.e. on $[\tau,T]\times E$. If P has a density p such that $p(\tau,x;u,y)>0$ for all τ, x, u, and y, then the condition $||V||_f=0$ is equivalent to the condition $V(u,y)=0$ $l\times m$-a.e. on $[0,T]\times E$. If there exists a density p, and if $\mu\in\hat{\mathcal{P}}_m^*$, then the condition $||\mu||_m=0$ means that

$$\int_\tau^T\int_E p(\tau,x;u,y)d|\mu(u)|(y)du=0$$

for all τ and x. If p is strictly positive, then we get the following equivalent condition: $\mu(u) = 0$ for l-a.a. $u \in [0, T]$.

Taking into account the identifications described in Remark 4.3, we see that $(\hat{\mathcal{P}}_f^*, ||\cdot||_f)$ and $(\hat{\mathcal{P}}_m^*, ||\cdot||_m)$ are normed spaces. In addition, they are Banach spaces.

Lemma 3.3. *Let $P \in \mathcal{M}$. Then $(\hat{\mathcal{P}}_f^*, ||\cdot||_f)$ is a Banach space, and $(\mathcal{P}_f^*, ||\cdot||_f)$ is its closed subspace. Moreover, if P has a strictly positive density p, then $(\hat{\mathcal{P}}_m^*, ||\cdot||_m)$ is a Banach space, and $\mathcal{P}_m^*$ is its closed subspace.*

Proof. We will prove that if p is strictly positive, then the space $\hat{\mathcal{P}}_m^*$ is complete, and $\mathcal{P}_m^*$ is a closed subspace of $\hat{\mathcal{P}}_m^*$. The rest of the proof of Lemma 3.3 is similar.

Let $\mu_k \in \hat{\mathcal{P}}_m^*$, $k \geq 1$, be a sequence such that

$$\sum_{k=1}^{\infty} ||\mu_k||_m = \sum_k \sup_{\tau: 0 \leq \tau \leq T} \sup_{x \in E} \int_\tau^T du \int_E p(\tau, x; u, y) d|\mu_k(u, y)| < \infty. \tag{12}$$

Then for every $x \in E$, we have

$$\int_0^T du \int_E p(0, x; u, y) d \sum_{k=1}^{\infty} |\mu_k(u, y)| < \infty.$$

Therefore, there exists a Borel set $U_x \in [0, T]$ such that $l(U_x) = T$ and

$$\int_E p(0, x; u, y) d \sum_{k=1}^{\infty} |\mu_k(u, y)| < \infty \tag{13}$$

for all $u \in U_x$. Fix $x \in E$. Then (13) implies that for every $j \geq 1$ and $u \in U_x$,

$$\sum_{k=1}^{\infty} |\mu_k(u)|(A_{j,u}) < \infty,$$

where $A_{j,u} = \{y \in E : p(0, x; u, y) > j^{-1}\}$. Hence, $\sum \mu_k(u)$ is a finite signed Borel measure on every set $A_{j,u}$ for all $u \in U_x$. Since the strict positivity of p implies the equality $\bigcup_{j=1}^{\infty} A_{j,u} = E$ for all $u \in U_x$, the measure $\mu(u) = \sum \mu_k(u)$ is a signed Borel measure on E for all $u \in U_x$, and hence l-a.e. on $[0, T]$. It follows from (12) that $\mu \in \hat{\mathcal{P}}_m^*$, and it is not difficult to show that the series $\sum_{k=1}^{\infty} \mu_k$ converges to μ in the space $\hat{\mathcal{P}}_m^*$. This proves the completeness of $\hat{\mathcal{P}}_m^*$.

Now let $\nu_k \in \mathcal{P}_m^*$, $k \geq 1$, be such that $\nu_k \to \nu$ in $\hat{\mathcal{P}}_m^*$. We have

$$\begin{aligned} \int_\tau^t Y(\tau, u)|\mu(u)|(x) du &\leq \int_\tau^t Y(\tau, u)|\mu(u) - \mu_k(u)|(x) du \\ &\quad + \int_\tau^t Y(\tau, u)|\mu_k(u)|(x) du. \end{aligned} \tag{14}$$

It follows from (14) that $\mu \in \mathcal{P}_m^*$. Hence, the class $\mathcal{P}_m^*$ is a closed subspace of the space $\hat{\mathcal{P}}_m^*$.

This completes the proof of Lemma 3.3. □

The next result provides a description of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ in terms of the potential operator N.

Lemma 3.4. *(a) Let $P \in \mathcal{M}$ and $V \in \hat{\mathcal{P}}_f^*$. Then $V \in \mathcal{P}_f^*$ if and only if*

$$\lim_{t'-t\to 0+} \sup_{\tau:0\le\tau\le t} \sup_{x\in E}[N(|V|)(\tau,t',x) - N(|V|)(\tau,t,x)] = 0. \tag{15}$$

(b) Suppose that $P \in \mathcal{M}$ has a density p, and let $\mu \in \hat{\mathcal{P}}_m^*$. Then $\mu \in \mathcal{P}_m^*$ if and only if (15) holds with μ instead of V.

Proof. Let $V \in \mathcal{P}_f^*$. Then we have

$$N(|V|)(\tau,t',x) - N(|V|)(\tau,t,x) = \int_t^{t'} Y(\tau,u)|V(u)|(x)du = Y(\tau,t)N(|V|)(t,t')(x).$$

It follows that

$$\sup_{x\in E}[N(|V|)(\tau,t',x) - N(|V|)(\tau,t,x)] \le \sup_{x\in E} N(|V|)(t,t',x).$$

It is clear that the previous estimate implies (15).

Now assume that (15) holds. Then we have

$$\lim_{t-\tau\to 0} \sup_{x\in E} N(|V|)(\tau,t,x) = \lim_{t-\tau\to 0} \sup_{x\in E}[N(|V|)(\tau,t,x) - N(|V|)(\tau,\tau,x)] = 0.$$

Therefore, $V \in \mathcal{P}_f^*$. This establishes part (a) of Lemma 3.4. The proof of part (b) is similar. □

Remark 3.5. It is easy to see from the proof of Lemma 3.4 that

$$\lim_{t'-t\to 0+} \sup_{\tau:0\le\tau\le t} \sup_{x\in E}|N(V)(\tau,t',x) - N(V)(\tau,t,x)| = 0.$$

4. Power type estimates and the functional A_μ

In this section we explain how to construct the additive functional A_μ, $\mu \in \mathcal{P}_m^*$, appearing in the definition of the backward Feynman-Kac propagator Y_μ. Such constuctions are standard in the time-homogeneous case (see, e.g., Section 5.1 in [13]). Moreover, under certain restrictions, there is a one-to-one correspondence between additive functionals and measures (the Revuz correspondence; see, e.g., [35], Chapter 10). Although the structure of the proof of the existence result for the functional A_μ below is similar to that in the time-homogeneous case, we decided to include the proof of Theorem 4.6 since the non-homogeneous case has its own peculiarities. Note that the class of time-dependent measures $\mathcal{P}_m^*$ is not completely analogous to the classical Kato class of measures, and that the estimates for the additive functionals A_V and A_μ, obtained below for V and μ, which are not necessarily positive, have independent interest.

The following functions will be used in Sections 4 and 5:

$$\begin{aligned} M(V)(\tau,t) &= \sup_{r:\tau\le r\le t} \sup_{x\in E}|N(V)(r,t,x)| \\ &= \sup_{r:\tau\le r\le t} \sup_{x\in E}\left|\int_r^t Y(\tau,s)V(s)(x)ds\right| \end{aligned} \tag{16}$$

and

$$\begin{aligned} M(\mu)(\tau,t) &= \sup_{r:\tau\le r\le t} \sup_{x\in E}|N(\mu)(r,t,x)| \\ &= \sup_{r:\tau\le r\le t} \sup_{x\in E}\left|\int_r^t Y(\tau,s)\mu(s)(x)ds\right|. \end{aligned} \tag{17}$$

The next result is an approximation lemma for functions and time-dependent measures. Note that one cannot always approximate a time-dependent measure by functions in the norm topology of the space $\hat{\mathcal{P}}_m^*$ (see Lemma 3.3). Therefore, one has to look for weaker approximations.

Lemma 4.1. (*a*) *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. For $k \geq 1$, $0 \leq \tau \leq T$, and $x \in E$, put*

$$g_k(\tau, x) = kN(V)(\tau, \min(\tau + \frac{1}{k}, T), x). \tag{18}$$

Then the following conditions hold: $g_k \in \mathcal{P}_f^*$ for all $k \geq 1$,

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(V - g_k)(\tau, t, x)| = 0, \tag{19}$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\geq 1} \sup_{x\in E} N(|g_k|)(\tau, t, x) = 0. \tag{20}$$

(*b*) *Suppose that $P \in \mathcal{M}$ has a density p, and let $\mu \in \mathcal{P}_m^*$. For $k \geq 1$, $0 \leq \tau \leq T$, and $x \in E$, put*

$$g_k(\tau, x) = kN(\mu)(\tau, \min(\tau + \frac{1}{k}, T), x). \tag{21}$$

Then the conditions in part (a) of Lemma 4.1 hold with μ instead of V.

Proof. (*a*) We have

$$\begin{aligned} N(g_k)(\tau, t, x) &= k\int_\tau^t Y(\tau, s)ds \int_s^{\min(s+\frac{1}{k},T)} Y(s, u)V(u)(x)du \\ &= k\int_\tau^t ds \int_s^{\min(s+\frac{1}{k},T)} Y(\tau, u)V(u)(x)du \\ &= k\int_\tau^{\min(t+\frac{1}{k},T)} Y(\tau, u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds, \end{aligned} \tag{22}$$

where $\chi_{C_k(u)}$ is the characteristic function of the set $C_k(u) = \{s : s \leq u \leq \min(s + \frac{1}{k}, T)\}$. It follows from (22) that

$$\begin{aligned} N(g_k)(\tau, t, x) = k&\int_\tau^{\min(\tau+\frac{1}{k},T)} Y(\tau, u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds \\ &+ \int_{\min(\tau+\frac{1}{k},t)}^t Y(\tau, u)V(u)(x)du \\ &+ k\int_t^{\min(t+\frac{1}{k},T)} Y(\tau, u)V(u)(x)du \int_\tau^t \chi_{C_k(u)}(s)ds. \end{aligned} \tag{23}$$

Using (23), we obtain

$$\begin{aligned} |N(V - g_k)(\tau, t, x)| \leq &\int_\tau^{\min(\tau+\frac{1}{k},t)} Y(\tau, u)|V(u)|(x)du \\ &+ k\int_\tau^{\min(\tau+\frac{1}{k},t)} Y(\tau, u)|V(u)|(x)du \int_\tau^t \chi_{C_k(u)}(s)ds \\ &+ k\int_t^{\min(t+\frac{1}{k},T)} Y(\tau, u)|V(u)|(x)du \int_\tau^t \chi_{C_k(u)}(s)ds. \end{aligned}$$

Since the Lebesgue measure of the set $C_k(u)$ does not exceed $\frac{1}{k}$, we get

$$\begin{aligned}|N(V-g_k)(\tau,t,x)| &\le 2N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x)\\ &\quad + Y(\tau,t)\int_t^{\min(t+\frac{1}{k},T)} Y(t,u)|V(u)|(x)du\\ &= 2N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x)\\ &\quad + Y(\tau,t)N(|V|)(t,\min(t+\frac{1}{k},T)(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{x\in E}|N(V-g_k)(\tau,t,x)| &\le 2\sup_{x\in E}N(|V|)(\tau,\min(\tau+\frac{1}{k},t),x)\\ &\quad + \sup_{x\in E}N(|V|)(t,\min(t+\frac{1}{k},T),x).\end{aligned} \tag{24}$$

Now it is clear that (24) and the definition of the class $\mathcal{P}_f^*$ imply condition (19) in Lemma 4.1.

Since

$$\begin{aligned}N(|g_k|)(\tau,t,x) &\le k\int_\tau^t Y(\tau,s)ds\int_s^{\min(s+\frac{1}{k},T)} Y(s,u)|V(u)|(x)du\\ &= k\int_\tau^t ds\int_s^{\min(s+\frac{1}{k},T)} Y(\tau,u)|V(u)|(x)du\\ &= k\int_\tau^{\min(t+\frac{1}{k},T)} Y(\tau,u)|V(u)|(x)du\int_{\tau,t}\chi_{C_k(u)}(s)ds,\end{aligned}$$

we get

$$N(|g_k|)(\tau,t,x)\le k\min(t-\tau,\frac{1}{k})N(|V|)(\tau,\min(t+\frac{1}{k},T),x). \tag{25}$$

It is not hard to see that the condition $g_k\in\mathcal{P}_f^*$, $k\ge 1$, follows from (25).

It remains to show that condition (20) holds. Let $\epsilon>0$. Then (25) and Lemma 3.4 imply that there exist $\delta_1>0$ and $k_0>1$ such that

$$\sup_{x\in E}N(|g_k|)(\tau,t,x)<\epsilon \tag{26}$$

for all $t-\tau<\delta_1$ and $k\ge k_0$. Moreover, since $g_k\in\mathcal{P}_f^*$, there exists $\delta_2>0$ such that $\delta_2<\delta_1$ and (26) holds for all $t-\tau<\delta_2$ and $k\le k_0$. Hence, (26) holds for all $k\ge 1$ and $t-\tau<\delta_2$, and we get (20).

This completes the proof of part (a) of Lemma 4.1. The proof of part (b) is similar. □

Remark 4.2. Suppose that the conditions in part (b) of Lemma 4.1 hold. Then it follows from (19) that

$$\lim_{k\to\infty}M(g_k)(\tau,t)=M(\mu)(\tau,t).$$

Moreover, (25) implies

$$\limsup_{k\to\infty}N(|g_k|)(\tau,t,x)\le N(|\mu|)(\tau,t,x)$$

and

$$\limsup_{k\to\infty} M(|g_k|)(\tau,t) \leq M(|\mu|)(\tau,t).$$

In the next definition, we introduce a special approximation method. It is based on the properties of the approximating sequence g_k in Lemma 4.1.

Definition 4.3. Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and $V_k \in \mathcal{P}_f^*$, $k \geq 1$. By definition, the sequence V_k approaches V in the potential sense provided that

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(V-V_k)(\tau,t,x)| = 0$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\geq 1} \sup_{x\in E} N(|V_k|)(\tau,t,x) = 0.$$

Suppose that $P \in \mathcal{M}$ has a density p. By definition, a sequence $\mu_k \in \mathcal{P}_m^*$, $k \geq 1$, approaches $\mu \in \mathcal{P}_f^*$ in the potential sense provided that

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} \sup_{x\in E} |N(\mu-\mu_k)(\tau,t,x)| = 0$$

and

$$\lim_{t-\tau\to 0+} \sup_{k\geq 1} \sup_{x\in E} N(|\mu_k|)(\tau,t,x) = 0.$$

Remark 4.4. It follows from (18), (21), and from the definition of the class $\hat{\mathcal{P}}_f^*$ that the functions g_k in Lemma 4.1 are bounded. Hence, for any $V \in \mathcal{P}_f^*$ ($\mu \in \mathcal{P}_m^*$), there exists a sequence of bounded functions approaching the function V (the time-dependent measure μ) in the potential sense.

Lemma 4.5. *Suppose that*

$$\lim_{t-\tau\to 0+} \sup_{k\geq 1} \sup_{x\in E} N(|V_k|)(\tau,t,x) = 0.$$

Then $\sup_k ||V_k||_f < \infty$. *The same result is true for time-dependent measures.*

Proof. If the condition in Lemma 4.5 holds, then there exists $\delta > 0$ such that for $t-\tau < \delta$ and $k \geq 1$,

$$\sup_{x\in E} \sup_{k\geq 1} N(|V_k|)(\tau,t,x) < 1. \tag{27}$$

For every $(\tau,t) \in D_T$ with $\tau < t$, there exists a partition $\tau = t_0 < \cdots < t_n = t$ such that $\max\{|t_{j+1}-t_j| : 0 \leq j \leq n-1\} < \delta$ and $n < \delta^{-1}T$. Moreover,

$$\begin{aligned} N(|V_k|)(\tau,t,x) &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} Y(\tau,s)|V_k|(x)ds \\ &= \sum_{j=0}^{n-1} Y(\tau,t_j) \int_{t_j}^{t_{j+1}} Y(t_j,s)|V_k|(x)ds \\ &= \sum_{j=0}^{n-1} Y(\tau,t_j) N(|V_k|)(t_j,t_{j+1})(x). \end{aligned}$$

By (27), we have

$$\sup_{k\geq 1} \sup_{x\in E} N(|V_k|)(\tau,t,x) \leq \sum_{k\geq 1}^{n-1} N(|N_k|)(t_j,t_{j+1},x) \leq n < \frac{T}{\delta}.$$

This completes the proof of Lemma 4.5 in the case of functions. The case of measures is similar. □

Let V be a non-negative function from the class $\mathcal{P}_f^*$. Then the functional A_V defined by $A_V(\tau,t)=\int_\tau^t V(s,X_s)ds$ is a non-decreasing continuous additive functional. More precisely, it possesses the following properties:

(1) For all $\tau\le t$, the random variable $A_V(\tau,t)$ is $\mathcal{F}_t^\tau$-measurable.
(2) For all τ and $x\in E$, $A_V(\tau,\tau)=0$ $P_{\tau,x}$-a.s.
(3) For all $\tau\le t$ and $x\in E$, the function $t\to A_V(\tau,t)$, $\tau\le t\le T$, is non-decreasing and continuous $P_{\tau,x}$-a.s.
(4) For all $\tau\le\lambda\le t$, $A_V(\tau,t)=A_V(\tau,\lambda)+A_V(\lambda,t)$ $P_{\tau,x}$-a.s.
(5) For all $\tau\le t$ and $x\in E$, $E_{\tau,x}\left[A_V(\tau,t)\right]=N(V)(\tau,t,x)$.

Theorem 4.6. *Let $P\in\mathcal{M}$ be a transition probability function possessing a density p. Then for every family μ of nonnegative measures from the class $\mathcal{P}_m^*$, there exists a functional $A_\mu(\tau,t)$, $(\tau,t)\in D_T$, for which conditions (1)–(5) above hold.*

Proof of Theorem 4.6. We will need the following lemma:

Lemma 4.7. *Let $P\in\mathcal{M}$ and $V\in\mathcal{P}_f^*$. Then for every $(\tau,t)\in D_T$, $x\in E$, and any integer $n\ge 2$,*

$$|E_{\tau,x}A_V(\tau,t)^n|\le n!N(|V|)(\tau,t,x)M(|V|)(\tau,t)^{n-2}M(V)(\tau,t). \tag{28}$$

□

Proof. Using the Markov property and taking into account that $V\in\mathcal{P}_f^*$, we obtain

$$\begin{aligned}
E_{\tau,x}A_V(\tau,t)^2 &= 2E_{\tau,x}\int_\tau^t V(s,X_s)ds\int_s^t V(u,X_u)du\\
&= 2E_{\tau,x}\int_\tau^t V(s,X_s)ds\int_s^t E_{\tau,x}(V(u,X_u)|\mathcal{F}_s^\tau)du\\
&= 2E_{\tau,x}\int_\tau^t V(s,X_s)ds\int_s^t E_{s,z}V(u,X_u)du|_{z=X_s}\\
&\le 2\int_\tau^t dsY(\tau,s)|V(s)|(x)ds\sup_{s:\tau\le s\le t}\sup_{y\in E}|\int_s^t Y(s,u)V(u)(y)du|.
\end{aligned} \tag{29}$$

Now it is clear that (29) implies (28) with $n=2$.

Next, let $n>2$ be any positive integer. By induction, we get

$$\begin{aligned}
&E_{\tau,x}A_V(\tau,t)^n\\
= \;& n!E_{\tau,x}\int_\tau^t V(t_1,X_{t_1})dt_1\int_{t_1}^t V(t_2,X_{t_2})dt_2\cdots\int_{t_{n-1}}^t V(t_n,X_{t_n})dt_n\\
\le \;& n!\int_\tau^t Y(\tau,s)|V(s)|(x)ds\sup_{r:\tau\le r\le t}\sup_{y\in E}[\int_r^t Y(r,u)|V(u)|(y)du]^{n-2}\\
\times \;& \sup_{r:\tau\le r\le t}\sup_{y\in E}|\int_r^t Y(r,u)V(u)(y)du|.
\end{aligned}$$

It is clear that the previous estimate implies (28).

Corollary 4.8. *Suppose that the conditions in Lemma* 4.7 *are satisfied. Then for any odd integer* $n \geq 3$,

(30)
$$E_{\tau,x}|A_V(\tau,t)|^n \leq \sqrt{(n-1)!(n+1)!}N(|V|)(\tau,t,x)M(|V|)(\tau,t)^{n-2}M(V)(\tau,t).$$

□

Proof. If $n \geq 3$ is odd, then

$$E_{\tau,x}|A_V(\tau,t)|^n \leq \{E_{\tau,x}A_V(\tau,t)^{n-1}\}^{\frac{1}{2}}\{E_{\tau,x}A_V(\tau,t)^{n+1}\}^{\frac{1}{2}}.$$

Now it is clear that (30) follows from Lemma 4.7. □

Lemma 4.7 and Corollary 4.8 provide pointwise estimates for the expression $E_{\tau,x}|A_V(\tau,t)|^n$. The next result shows that stronger estimates hold.

Lemma 4.9. *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{P}_f^*$. *Then for any* τ *with* $0 \leq \tau \leq T$, *any* $\delta > 0$ *such that* $\tau + \delta \leq T$, *and any even integer* $n \geq 2$, *the following estimate holds:*

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\leq t\leq\tau+\delta} A_V(\tau,t)^n \leq c_n M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta), \tag{31}$$

where

$$c_n = 2^n[(\frac{n}{n-1})^n n! + 1]. \tag{32}$$

Moreover, for any odd integer $n \geq 3$, *we have*

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\leq t\leq\tau+\delta} |A_V(\tau,t)|^n \leq c_n M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta), \tag{33}$$

where $c_n = \sqrt{c_{n-1}c_{n+1}}$.

Remark 4.10. For $n = 1$, we have

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\leq t\leq\tau+\delta} |A_V(\tau,t)| \leq \{c_2 M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}}. \tag{34}$$

Estimate (31) easily follows from (28) with $n = 2$.

Proof of Lemma 4.9. We will prove estimate (31). Estimate (33) follows from (31) and Hölder's inequality.

Let $n \geq 2$ be an even integer, and let $V \in \mathcal{P}_f^*$. For given τ, x, and t with $\tau \leq t \leq \tau + \delta$, put $M_t = E_{\tau,x}(A_V(\tau,\tau+\delta)|\mathcal{F}_t^\tau)$. Then M_t is an $\mathcal{F}_t^\tau$-martingale. Moreover, it belongs to the space L^n (see (28)). Using the Markov property, we see that for every t with $\tau \leq t \leq \tau + \delta$,

$$\begin{aligned} M_t &= A_V(\tau,t) + \int_t^{\tau+\delta} E_{\tau,x}(V(s,X_s)|\mathcal{F}_t^\tau)ds \\ &= A_V(\tau,t) + \int_t^{\tau+\delta} Y(t,s)V(s)(X_t)ds \end{aligned}$$

$P_{\tau,x}$-a.s. Hence, M_t is a modification of the functional

$$\tilde{M}_t = A_V(\tau,t) + N(V)(t,\tau+\delta,X_t).$$

Fix a partition $\tau = t_0 < t_1 < \cdots < t_k = \tau + \delta$. By Doob's inequality (see [35]), we get

$$\begin{aligned} & E_{\tau,x} \sup_{j:0\le j\le k} A_V(\tau,t_j)^n \\ \le\ & 2^n E_{\tau,x} \sup_{j:0\le j\le k} M_{t_j}^n + 2^n E_{\tau,x} \sup_{j:0\le j\le k} |N(V)(t_j,\tau+\delta,X_{t_j})|^n \\ \le\ & 2^n(\frac{n}{n-1})^n E_{\tau,x} A_V(\tau,\tau+\delta)^n + 2^n[\sup_{z\in E} \sup_{s:\tau\le s\le\tau+\delta} |N(V)(s,\tau+\delta,z)|]^n \\ \le\ & 2^n(\frac{n}{n-1})^n E_{\tau,x} A_V(\tau,\tau+\delta)^n + 2^n M(V)(\tau,\tau+\delta)^n. \end{aligned} \tag{35}$$

It follows from (28) and (35) that

$$E_{\tau,x} \sup_{j:0\le j\le k} A_V(\tau,t_j)^n \le c_n M(|V|)(\tau,\tau+\delta)^{n-1} M(V)(\tau,\tau+\delta) \tag{36}$$

for all $x \in E$ where c_n is defined by (32). Next we choose a sequence of refinements of the partition $\tau = t_0 < t_1 < \cdots < t_k = \tau + \delta$ on the left-hand side of (36) such that the maximum length of the partition intervals tends to 0, and pass to the limit, using the monotone convergence theorem and the continuity of the functional $A_V(\tau,t)$ with respect to the variable t. This establishes estimate (31) and completes the proof of Lemma 4.9. □

Let us return to the proof of Theorem 4.6. For a non-negative family $\mu \in \mathcal{P}_m^*$, define the sequence g_k by (21). Using (31) and (33), we obtain

$$\sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_{g_k-g_j}(\tau,t)|^n \le c_n M(|g_k-g_j|)(\tau,\tau+\delta)^{n-1} M(g_k-g_j)(\tau,\tau+\delta).$$

It follows from Lemma 4.1 that

$$\lim_{j,k\to\infty} \sup_{\tau:0\le\tau\le T} \sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_{g_k}(\tau,t) - A_{g_j}(\tau,t)|^n = 0.$$

Hence, there exists a functional A_μ such that

$$\lim_{k\to\infty} \sup_{\tau:0\le\tau\le T} \sup_{x\in E} E_{\tau,x} \sup_{t:\tau\le t\le\tau+\delta} |A_\mu(\tau,t) - A_{g_k}(\tau,t)|^n = 0. \tag{37}$$

Using the fact that every functional A_{g_k} satisfies conditions (1)-(5) in Theorem 4.6, we can prove that the functional A_μ defined by (37) also satisfies these conditions.

This completes the proof of Theorem 4.6.

It is not difficult to see that the functional A_μ does not depend on the choice of a sequence g_k such that g_k approaches μ in the potential sense. Actually, more is true.

Lemma 4.11. *Let μ be a non-negative family from the class $\mathcal{P}_m^*$, and let A_1 and A_2 be two functionals satisfying conditions (1)-(5) in Theorem 4.6. Then for every $0 \le \tau \le T$ and $x \in R^n$, the processes $A_1(\tau,t)$ and $A_2(\tau,t)$ are indistinguishable.*

The proof of Lemma 4.11 is standard (see, e.g., the proof of Theorem 5.1.2 in [13]).

In the next definition, we extend the functional A_μ to the case of signed time-dependent measures.

Definition 4.12. Let $\mu \in \mathcal{P}_m^*$. Denote by μ^+ and μ^- the positive and the negative variation of the family μ, respectively. Then the functional A_μ is defined as follows:

$$A_\mu(\tau,t) = A_{\mu^+}(\tau,t) - A_{\mu^-}(\tau,t).$$

Lemma 4.13. *Let $P \in \mathcal{M}$ be a transition probability function possessing a density p, and let $\mu \in \mathcal{P}_m^*$. Then estimates* (31) *and* (33) *hold with μ instead of V.*

Proof. We will prove estimate (31) for a time-dependent measure μ. It is clear that estimate (33) for μ follows from (31).

Let $\mu \in \mathcal{P}_m^*$, and let g_k be the sequence constructed for μ in Lemma 4.1. Then $g_k \in \mathcal{P}_f^*$. Applying estimate (31) to the sequence g_k, we see that

$$\sup_{x \in E} E_{\tau,x} \sup_{t:\tau \le t \le \tau+\delta} |A_{g_k}(\tau,t)|^n \le c_n M(|g_k|)(\tau,t)^{n-1} M(g_k)(\tau,t). \tag{38}$$

It follows from Remark 4.2 that

$$\limsup_{k \to \infty} M(|g_k|)(\tau,t) \le M(|\mu|)(\tau,t) \tag{39}$$

and

$$M(g_k)(\tau,t) \to M(\mu)(\tau,t) \tag{40}$$

as $k \to \infty$. Now using (37), (38), (39), and (40) we see that (31) holds for μ.

This completes the proof of Lemma 4.13. □

5. Exponential estimates for non-autonomous functionals

The non-autonomous multiplicative functionals $(\tau,t) \mapsto \exp\{-\int_\tau^t V(s,X_s)\}ds$ and $(\tau,t) \mapsto \exp\{-A_\mu(t,\tau)\}$ play an important role in the theory of Feynman-Kac propagators. These functionals are called the Kac functionals. We begin this section with a non-autonomous version of Khas′minski's Lemma. This lemma and similar results imply that the Feynman-Kac propagators Y_V and Y_μ with $V \in \mathcal{P}_f^*$ and $\mu \in \mathcal{P}_f^*$ are uniformly bounded on the space $L_{\mathcal{E}}^\infty$.

Lemma 5.1. (*a*) *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and let $(\tau,t) \in D_T$ be such that $M(|V|)(\tau,t) < 1$. Then*

$$\sup_{x \in E} E_{\tau,x} \exp\{\int_\tau^t |V(s,X_s)|ds\} \le \frac{1}{1 - M(|V|)(\tau,t)}. \tag{41}$$

(*b*) *If $P \in \mathcal{M}$ has a density p, $\mu \in \mathcal{P}_m^*$, and $(\tau,t) \in D_T$ is such that $M(|\mu|)(\tau,t) < 1$, then*

$$\sup_{x \in E} E_{\tau,x} \exp\{A_{|\mu|}(\tau,t)\} \le \frac{1}{1 - M(|\mu|)(\tau,t)}. \tag{42}$$

Lemma 5.1 can be obtained as follows. Estimate (28) gives

$$E_{\tau,x} \frac{A_{|V|}(\tau,t)^n}{n!} \le M(|V|)(\tau,t)^n. \tag{43}$$

It is clear that (43) implies (41). Now arguing as in the proof of Lemma 4.13, we see that estimate (42) follows from (41).

The next assertions contain more exponential estimates.

Lemma 5.2. (*a*) *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and let $(\tau,t) \in D_T$ be such that $M(|V|)(t,\tau) < 1$. Then*

$$\begin{aligned} E_{\tau,x} \exp\{|A_V(\tau,t)|\} &\le 1 + \{2N(|V|)(\tau,t,x)M(V)(\tau,t)\}^{\frac{1}{2}} \\ &\quad + \frac{2\sqrt{3}}{3} \frac{N(|V|)(\tau,t,x)M(V)(\tau,t)}{1 - M(|V|)(\tau,t)}. \end{aligned} \tag{44}$$

(*b*) *If* $P \in \mathcal{M}$ *has a density* p, $\mu \in \mathcal{P}_m^*$, *and* $(\tau, t) \in D_T$ *is such that* $M(|\mu|)(\tau, t) < 1$, *then estimate* (44) *holds with* μ *instead of* V.

Proof. Estimate (44) follows from Lemma 4.7, Corollary 4.8, and the fact that $\sqrt{(m-1)!(m+1)!} \leq \frac{2\sqrt{3}}{3}(m!)$ for all $m \geq 3$. The proof of part (*b*) is similar. Here we reason as in the proof of Lemma 4.13. □

Lemma 5.3. (*a*) *Let* $P \in \mathcal{M}$, $q \geq 1$, $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, *and let* V *and* W *be functions from the class* $\mathcal{P}_f^*$. *Suppose that* $(\tau, t) \in D_T$ *is such that* $M(rq|W|)(\tau, t) < 1$ *and* $M(r'q|V - W|)(\tau, t) < 1$. *Then*

$$
\begin{aligned}
&E_{\tau,x}|\exp\{A_V(\tau,t)\} - \exp\{A_W(\tau,t)\}|^q \\
&\leq \frac{1}{(1 - M(rq|W|)(\tau,t))^{\frac{1}{r}}}\{[2N(r'q|V-W|)(\tau,t,x)M(r'q(V-W))(\tau,t)]^{\frac{1}{2}} \\
&\quad + \frac{2\sqrt{3}}{3}\frac{N(r'q|V-W|)(\tau,t,x)M(r'q(V-W))(\tau,t)}{1 - M(r'q|V-W|)(\tau,t)}\}^{\frac{1}{r'}}.
\end{aligned}
\tag{45}
$$

(*b*) *Suppose that* $P \in \mathcal{M}$ *has a density* p, *and let* $q \geq 1$, $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, *and* $\mu, \nu \in \mathcal{P}_m^*$. *Let* $(\tau, t) \in D_T$ *be such that*

$$M(rq|\nu|)(\tau,t) < 1 \quad \textit{and} \quad M(r'q|\mu - \nu|)(\tau,t) < 1.$$

Then

$$
\begin{aligned}
&E_{\tau,x}|\exp\{A_\mu(\tau,t)\} - \exp\{A_\nu(\tau,t)\}|^q \\
&\leq \frac{1}{(1 - M(rq|\nu|)(\tau,t))^{\frac{1}{r}}}\{[2N(r'q|\mu-\nu|)(\tau,t,x)M(r'q(\mu-\nu))(\tau,t)]^{\frac{1}{2}} \\
&\quad + \frac{2\sqrt{3}}{3}\frac{N(r'q|\mu-\nu|)(\tau,t,x)M(r'q(\mu-\nu))(\tau,t)}{1 - M(r'q|\mu-\nu|)(\tau,t)}\}^{\frac{1}{r'}}.
\end{aligned}
\tag{46}
$$

Proof. Part (*a*). It is not difficult to prove that

$$|e^a - 1|^b \leq e^{|a|b} - 1, \tag{47}$$

for $a \in R$ and $b \geq 1$. Using (47) and Hölder's inequality, we get

$$
\begin{aligned}
&E_{\tau,x}|\exp\{A_V(\tau,t)\} - \exp\{A_W(\tau,t)\}|^q \\
\leq\ &\{E_{\tau,x}\exp\{A_{rqW}(\tau,t)\}\}^{\frac{1}{r}}\{E_{\tau,x}|\exp\{A_{V-W}(\tau,t)\} - 1|^{r'q}\}^{\frac{1}{r'}} \\
\leq\ &\{E_{\tau,x}\exp\{A_{rq|W|}(\tau,t)\}^{\frac{1}{r}}\{E_{\tau,x}\exp\{|A_{r'q(V-W)}(\tau,t)|\} - 1\}^{\frac{1}{r'}}.
\end{aligned}
\tag{48}
$$

Now it is clear that (45) follows from (41), (44) and (48).

Part (*b*). We will first use Lemma 4.1 to find the approximating sequences g_k (for μ) and h_k (for ν). By the assumptions, $M(rq|\nu|)(\tau,t) < 1$ and $M(r'q|\mu-\nu|)(\tau,t) < 1$. Using Remark 4.2, we see that there exists a sequence k' of positive integers such that

$$
\begin{aligned}
&M(rq|h_{k'}|)(\tau,t) < 1, \quad M(r'q|g_{k'} - h_{k'}|)(\tau,t) < 1, \\
&\lim_{k'\to\infty} A_{g_{k'}}(\tau,t) = A_\mu(\tau,t), \quad \lim_{k'\to\infty} A_{h_{k'}}(\tau,t) = A_\nu(\tau,t), \\
&\limsup_{k'\to\infty} M(|h_{k'}|)(\tau,t) \leq M(|\mu|)(\tau,t), \\
&\limsup_{k'\to\infty} M(|g_{k'} - h_{k'}|)(\tau,t) \leq M(|\mu-\nu|)(\tau,t), \\
&\limsup_{k'\to\infty} N(|g_{k'} - h_{k'}|)(\tau,t,x) \leq N(|\mu-\nu|)(\tau,t,x),
\end{aligned}
$$

and

$$\lim_{k'\to\infty} M(g_{k'}-h_{k'})(\tau,t)=M(\mu-\nu)(\tau,t).$$

It is not hard to get from (45) that

$$\begin{aligned}
&E_{\tau,x}|\exp\{A_{g_{k'}}(\tau,t)\}-\exp\{A_{h_{k'}}(\tau,t)\}|^q \le \frac{1}{(1-M(rq|h_{k'}|)(\tau,t))^{\frac{1}{r}}}\\
&\quad\times\{[2N(r'q|g_{k'}-h_{k'}|)(\tau,t,x)M(r'q(g_{k'}-h_{k'}))(\tau,t)]^{\frac{1}{2}}\\
&\quad+\frac{2\sqrt{3}}{3}\frac{N(r'q|g_{k'}-h_{k'}|)(\tau,t,x)M(r'q(g_{k'}-h_{k'}))(\tau,t)}{1-M(r'q|g_{k'}-h_{k'}|)(\tau,t)}\}^{\frac{1}{r'}}.
\end{aligned}\tag{49}$$

Using Fatou's Lemma in (49), we see that estimate (46) holds.

This completes the proof of Lemma 5.3. □

It follows from formula (32) that $c_n \le c2^n n!$. However, this estimate is not strong enough to derive an exponential estimate for the functional A_V by using (31) and (33). We will obtain such an estimate by modifying the proof of Lemma 4.9.

Theorem 5.4. *(a) Let $P\in\mathcal{M}$ and $V\in\mathcal{P}_f^*$. Then for every τ with $0\le\tau\le T$ and every $\delta>0$ such that $\tau+\delta\le T$ and $M(|V|)(\tau,\tau+\delta)<1$, the following estimate holds:*

$$\begin{aligned}
&\sup_{x\in E} E_{\tau,x}\exp\{\sup_{t:\tau\le t\le\tau+\delta}|A_V(\tau,t)|\}\le\exp\{M(V)(\tau,\tau+\delta)\}\\
&\times\quad(1+c\{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}}\\
&+\quad c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}).
\end{aligned}\tag{50}$$

(b) Suppose that $P\in\mathcal{M}$ has a density p, and let $\mu\in\mathcal{P}_m^*$. Then for every τ with $0\le\tau\le T$ and every $\delta>0$ such that $\tau+\delta\le T$ and $M(|\mu|)(\tau,\tau+\delta)<1$, the following estimate holds:

$$\begin{aligned}
&\sup_{x\in E} E_{\tau,x}\exp\{\sup_{t:\tau\le t\le\tau+\delta}|A_\mu(\tau,t)|\}\le\exp\{M(\mu)(\tau,\tau+\delta)\}\\
&\times\quad(1+c\{M(|\mu|)(\tau,\tau+\delta)M(\mu)(\tau,\tau+\delta)\}^{\frac{1}{2}}\\
&+\quad c\frac{M(|\mu|)(\tau,\tau+\delta)M(\mu)(\tau,\tau+\delta)}{1-M(|\mu|)(\tau,\tau+\delta)}).
\end{aligned}$$

Proof. Using the same notation as in the proof of Lemma 4.9 and applying Doob's inequality, we see that for every $n\ge 2$,

$$\begin{aligned}
E_{\tau,x}\sup_{j:0\le j\le k}|M_{t_j}|^n &\le (\frac{n}{n-1})^n E_{\tau,x}|A_V(\tau,\tau+\delta)|^n\\
&\le (\frac{n}{n-1})^n n! M(|V|)(\tau,\tau+\delta)^{n-1}M(V)(\tau,\tau+\delta).
\end{aligned}$$

Dividing the previous inequality by $n!$, adding the resulting inequalities, and finally using (31) and the equality $M_t = A_V(\tau,t) + N(V)(t,t+\delta,X_t)$, we get

$$\begin{aligned}
E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|M_{t_j}|\} &\le 1 + E_{\tau,x}\sup_{j:0\le j\le k}|M_{t_j}| \\
&\quad + c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)} \\
&\le 1 + E_{\tau,x}\sup_{j:0\le j\le k}|A_V(\tau,t_j)| \\
&\quad + E_{\tau,x}\sup_{j:0\le j\le k}|N(V)(t_j,\tau+\delta,X_{t_j})| \\
&\quad + c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)} \\
&\le 1 + \{E_{\tau,x}\sup_{j:0\le j\le k}|A_V(\tau,t_j)|^2\}^{\frac{1}{2}} \\
&\quad + M(V)(\tau,\tau+\delta) + c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)} \\
&\le 1 + \{c_2M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} \\
&\quad + M(V)(\tau,\tau+\delta) + c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}. \qquad (51)
\end{aligned}$$

We also have

$$\begin{aligned}
& E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|M_{t_j}|\} \\
\ge\ & E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)| - \sup_{j:0\le j\le k}|N(V)(t_j,\tau+\delta,X_{t_j})|\} \\
\ge\ & \exp\{-M(V)(\tau,\tau+\delta)\}E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\}. \qquad (52)
\end{aligned}$$

It follows from (51) and (52) that

$$\begin{aligned}
& E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\} \le \exp\{M(V)(\tau,\tau+\delta)\} \\
\times\ & [1 + \{c_2M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} \\
+\ & M(V)(\tau,\tau+\delta) + c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& E_{\tau,x}\exp\{\sup_{j:0\le j\le k}|A_V(\tau,t_j)|\} \le \exp\{M(V)(\tau,\tau+\delta)\} \\
\times\ & (1 + c\{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)\}^{\frac{1}{2}} \\
+\ & c\frac{M(|V|)(\tau,\tau+\delta)M(V)(\tau,\tau+\delta)}{1-M(|V|)(\tau,\tau+\delta)}). \qquad (53)
\end{aligned}$$

Now we see that for a sequence of refinements of the partition $\tau = t_0 < t_1 < \cdots < t_k = \tau+\delta$ on the left-hand side of (53) such that the maximum length of the partition intervals tends to 0, we can pass to the limit using the monotone convergence theorem and the continuity of the functional $A_V(\tau,t)$ with respect to t. It follows that estimate (50) holds. The proof of part (b) of Theorem 5.4 is similar. Here we use the ideas in the proof of part (b) of Lemma 5.3. $\square$

6. The L^r-boundedness and the $(L^r - L^q)$-smoothing property of backward Feynman-Kac propagators

In this section we discuss the inheritance of properties of free propagators by the corresponding Feynman-Kac propagators. This discussion will be continued in Section 7. Various inheritance theorems are known for the Kato class perturbations of semigroups generated by homogeneous Markov processes (see, e.g., [6]). Thus, it may seem natural to use these results to solve the inheritance problem for backward Feynman-Kac propagators by employing the Howland semigroup,

$$S_t f(\tau, x) = Y(\tau, \min(\tau + t, T)) f(\min(\tau + t, T))(x),$$

associated with the free backward propagator Y and similar semigroups associated with the backward Feynman-Kac propagators Y_V and Y_μ. In probabilistic terms, this amounts to replacing the process X_t by the space-time process $\widehat{X}_t = (t, X_t)$ associated with X_t and considering functions $f(t,x)$ where $(t,x) \in [0,T] \times E$. However, this approach to the inheritance problem for propagators often fails. The reason for this failure is that additional restrictions on the time-behavior of the free backward propagator Y are needed if we would like to apply the results obtained in [6] to Howland semigroups. These restrictions are not imposed in the inheritance theorems for backward Feynman-Kac propagators in the present section and in Section 7.

Our first results in this section concern the behavior of the backward Feynman-Kac propagators Y_V and Y_μ on the scale of Lebesgue spaces L^r with respect to the reference measure m. By $L^r_{\mathcal{E}}$ is denoted the space of all Borel functions from L^r.

Theorem 6.1. (a) *Let $P \in \mathcal{M}$. Then for any $V \in \mathcal{P}^*_f$, Y_V is a backward propagator on $L^\infty_{\mathcal{E}}$.*
(b) *Suppose that $P \in \mathcal{M}$ has a density p, and let $V \in \mathcal{P}^*_f$. Then Y_V is a backward propagator on L^∞.*
(c) *Suppose that $P \in \mathcal{M}$ has a density p, and let $\mu \in \mathcal{P}^*_m$. Then Y_μ is a backward propagator on L^∞.*

The proof of Theorem 6.1 is a standard application of Khas′minski's Lemma and the propagator properties.

Remark 6.2. The following estimate holds in part (b) of Theorem 6.1:

$$||Y_V(\tau, t)||_{\infty \to \infty} \le \exp\{A([\frac{t-\tau}{\delta}] + 1)\},$$

where $\delta > 0$ is any number such that $\rho(\delta) = \sup\{M(|V|)(\eta, \lambda) : \lambda - \eta < \delta\} < 1$ and $A = \ln \frac{1}{1-\rho(\delta)}$. Similar estimates hold in parts (a) and (c).

Our next result explains why the approximation in the potential sense is useful in the theory of Feynman-Kac propagators.

Theorem 6.3. *Let $P \in \mathcal{M}$, and let $V \in \mathcal{P}^*_f$ and $V_k \in \mathcal{P}^*_f$ be such that V_k approaches V in the potential sense. Then*

$$\lim_{k \to \infty} \sup_{(\tau,t) \in D_T} ||Y_V(\tau, t) - Y_{V_k}(\tau, t)||_{L^\infty_{\mathcal{E}} \to L^\infty_{\mathcal{E}}} = 0.$$

Suppose that $P \in \mathcal{M}$ has a density p, and let $\mu \in \mathcal{P}^*_m$ and $\mu_k \in \mathcal{P}^*_m$ be such that μ_k approaches μ in the potential sense. Then

$$\lim_{k \to \infty} \sup_{(\tau,t) \in D_T} ||Y_\mu(\tau, t) - Y_{\mu_k}(\tau, t)||_{\infty \to \infty} = 0.$$

Proof. We will only prove the second part of Theorem 6.3. The proof of the first part is similar. Let μ and μ_k be such as in the formulation of Theorem 6.3, and let $f \in L^\infty$. Then by part (b) of Lemma 5.3 with $q = 1$ and $r = 2$, there exists $\delta_0 > 0$ such that

$$\begin{aligned} &|Y_\mu(\tau,t)f(x) - Y_{\mu_k}(\tau,t)f(x)| \\ &\le \alpha ||f||_\infty (\{M(\mu-\mu_k)(\tau,t)\}^{\frac{1}{2}} + M(\mu-\mu_k)(\tau,t)) \end{aligned} \tag{54}$$

for all τ and t with $t - \tau < \delta_0$ and all $x \in E$. In (54), the constant α does not depend on x, t, τ, and k. It follows from (54) that

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T : t-\tau<\delta_0} ||Y_\mu(\tau,t) - Y_{\mu_k}(\tau,t)||_{L^\infty \to L^\infty_{\mathcal{E}}} = 0. \tag{55}$$

Next, we will get rid of the restriction $t-\tau < \delta_0$ in formula (55). Consider a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ of the interval $[0,T]$ such that $t_{j+1} - t_j < \delta_0$ for all j with $0 \le j \le n-1$. Then estimate (54) holds provided that t and τ belong to the same interval $[t_j, t_{j+1}]$. Using this fact and the properties of backward propagators, we can finish the proof of the second part of Theorem 6.3. We will illustrate how to do it by considering a special case where $t_j \le \tau \le t_{j+1} \le t \le t_{j+2}$ with $0 \le j \le n-2$. The general case is similar. By the uniform boundedness of the propagators Y_{μ_k} on the space L^∞ (this follows from Remark 6.2),

$$\begin{aligned} &||Y_\mu(\tau,t) - Y_{\mu_k}(\tau,t)||_{\infty\to\infty} \\ &\le ||Y_\mu(\tau,t_{j+1})Y_\mu(t_{j+1},t) - Y_{\mu_k}(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\ &\le ||Y_\mu(\tau,t_{j+1})Y_\mu(t_{j+1},t) - Y_\mu(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\ &\quad + ||Y_\mu(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t) - Y_{\mu_k}(\tau,t_{j+1})Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\ &\le \alpha ||Y_\mu(t_{j+1},t) - Y_{\mu_k}(t_{j+1},t)||_{\infty\to\infty} \\ &\quad + \alpha ||Y_\mu(\tau,t_{j+1}) - Y_{\mu_k}(\tau,t_{j+1})||_{\infty\to\infty}. \end{aligned} \tag{56}$$

Let us recall that $t - t_{j+1} < \delta_0$ and $t_{j+1} - \tau < \delta_0$. Taking into account (55) and (56), we see that an equality similar to (55) holds in the case where $t_j \le \tau \le t_{j+1} \le t \le t_{j+2}$ with $0 \le j \le n-2$.

This completes the proof of Theorem 6.3. □

Corollary 6.4. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and define g_k by* (18). *Then*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_V(\tau,t) - Y_{g_k}(\tau,t)||_{L^\infty_{\mathcal{E}} \to L^\infty_{\mathcal{E}}} = 0.$$

Suppose that $P \in \mathcal{M}$ has a density p, and let $\mu \in \mathcal{P}_m^*$. Define a sequence of functions g_k by (21). *Then*

$$\lim_{k\to\infty} \sup_{(\tau,t)\in D_T} ||Y_\mu(\tau,t) - Y_{g_k}(\tau,t)||_{\infty\to\infty} = 0.$$

Corollary 6.4 follows from Theorem 6.3 and from the fact that the sequence g_k defined by (18) approaches V in the potential sense, and the sequence g_k defined by (21) approaches μ in the potential sense (see Lemma 4.1).

The next lemma will be important in the sequel.

Lemma 6.5. (a) *Let $P \in \mathcal{M}$. Then for any $V \in \mathcal{P}_f^*$, we have*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{L^\infty_{\mathcal{E}} \to L^\infty_{\mathcal{E}}} = 0. \tag{57}$$

(*b*) *Suppose that $P \in \mathcal{M}$ has a density p. Then for any $V \in \mathcal{P}_f^*$,*

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{\infty\to\infty} = 0. \tag{58}$$

(*c*) *Suppose that $P \in \mathcal{M}$ has a density p. Then for any $\mu \in \mathcal{P}_m^*$,*

$$\lim_{t-\tau\to 0} ||Y_\mu(\tau,t) - Y(\tau,t)||_{\infty\to\infty} = 0. \tag{59}$$

Proof. It follows from part (a) of Lemma 5.1 and from the definition of the class $\mathcal{P}_f^*$ that

$$\begin{aligned}
\limsup_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{\infty\to\infty} &\leq \limsup_{t-\tau\to 0+} \sup_{x\in E} (E_{\tau,x}\exp\{A_{|V|}(\tau,t)\} - 1) \\
&\leq \limsup_{t-\tau\to 0+} \sup_{x\in E} \frac{M(|V|)(\tau,t)}{1 - M(|V|)(\tau,t)} = 0.
\end{aligned}$$

This gives equality (58). The proof of (57) and (59) is similar. We use part (b) of Lemma 5.1 in the proof of (59). □

The next result provides sufficient conditions for the existence of backward Feynman-Kac propagators on the space L^s.

Theorem 6.6. *Let $1 < s < \infty$ and $1 \leq r < s$. Then the following are true:*
(a) *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Suppose that the free backward propagator Y satisfies $Y(\tau,t) \in L(L^r_{\mathcal{E}}, L^r_{\mathcal{E}})$ for all $(\tau,t) \in D_T$. Then Y_V is a backward propagator on $L^s_{\mathcal{E}}$. If, in addition, Y is uniformly bounded on $L^r_{\mathcal{E}}$ and strongly continuous on $L^s_{\mathcal{E}}$, then Y_V is a strongly continuous backward propagator on $L^s_{\mathcal{E}}$.*
(b) *If $P \in \mathcal{M}$ has a density p, and if $Y(\tau,t) \in L(L^r, L^r)$ for all $(\tau,t) \in D_T$, then Y_V is a backward propagator on L^s. If, in addition, Y is uniformly bounded on L^r and strongly continuous on L^s, then Y_V is a strongly continuous backward propagator on L^s.*
(c) *Suppose that $P \in \mathcal{M}$ has a density p and let $\mu \in \mathcal{P}_m^*$. If $Y(\tau,t) \in L(L^r, L^r)$ for all $0 \leq \tau < t \leq T$, then Y_μ is a backward propagator on L^s. If, in addition, Y is uniformly bounded on L^r and strongly continuous on L^s, then Y_μ is a strongly continuous backward propagator on L^s.*

Remark 6.7. We do not know whether Theorem 6.6 holds for $r = s$. In the case of the heat semigroup, Theorem 6.6 may fail for $s = 1$. This was established in [21].

Proof of Theorem 6.6. Part (b). Assume that the conditions in part (b) of Theorem 6.6 hold, and let $g \in L^s$. It follows from Hölder's inequality and Remark 6.2 that

$$|Y_V(\tau,t)g(x)| \leq c\{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}}, \tag{60}$$

where $c \geq 1$ depends on s, r, and V. Now we see that (60) implies

$$||Y_V(\tau,t)||_{s\to s} \leq c||Y(\tau,t)||^{\frac{r}{s}}_{r\to r}. \tag{61}$$

Therefore, Y_V is a propagator on L^s. The proofs of the corresponding assertions in parts (a) and (c) are similar. □

Remark 6.8. Inequality (61) provides a norm estimate for the backward propagator Y_V.

Let us return to the proof of part (b) of Theorem 6.6. We will need the following lemma:

Lemma 6.9. *Let $1 < s < \infty$ and $1 \le r < s$. Then the following are true:*
(a) Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Suppose that the free backward propagator Y is uniformly bounded on $L^r_{\mathcal{E}}$. Then we have

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{L^s_{\mathcal{E}}\to L^s_{\mathcal{E}}} = 0. \tag{62}$$

(b) If $P \in \mathcal{M}$ has a density p, $V \in \mathcal{P}_f^*$, and Y is uniformly bounded on L^r, then

$$\lim_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{s\to s} = 0. \tag{63}$$

(c) If $P \in \mathcal{M}$ has a density p, $\mu \in \mathcal{P}_m^*$, and Y is uniformly bounded on L^r, then

$$\lim_{t-\tau\to 0+} ||Y_\mu(\tau,t) - Y(\tau,t)||_{s\to s} = 0. \tag{64}$$

Remark 6.10. Lemmas 6.5 and 6.9 were obtained in [17, 21] in the case of the heat semigroup. A similar result concerning time-independent perturbations of semigroups on L^1 was obtained earlier in [32, Lemma 4.2].

Proof of Lemma 6.9. We begin with the proof of part (b) of Lemma 6.9. It follows from part (b) of Theorem 6.6 that Y and Y_V are backward propagators on L^s. Let $g \in L^s$. Then, using Hölder's inequality and inequality (47), we obtain

$$\begin{aligned}
& |Y_V(\tau,t)g(x) - Y(\tau,t)g(x)| \\
\le\ & \{E_{\tau,x}|g(X_t)|^{\frac{s}{r}}\}^{\frac{r}{s}}\{E_{\tau,x}|\exp\{A_V(\tau,t)\} - 1|^{\frac{s}{s-r}}\}^{\frac{s-r}{s}} \\
\le\ & \{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{r}{s}}\{E_{\tau,x}\exp\{\frac{s}{s-r}A_{|V|}(\tau,t)\} - 1\}^{\frac{s-r}{s}}.
\end{aligned} \tag{65}$$

It follows from part (a) of Lemma 5.1, the definition of the class $\mathcal{P}_f^*$, the uniform boundedness of Y on L^r, and estimate (65) that

$$\begin{aligned}
& \limsup_{t-\tau\to 0+} ||Y_V(\tau,t) - Y(\tau,t)||_{s\to s} \\
\le\ & a(s,r,V)\limsup_{t-\tau\to 0+}\sup_{x\in R^n}\{E_{\tau,x}\exp\{\frac{s}{s-r}A_{|V|}(\tau,t)\} - 1\}^{\frac{s-r}{s}} \\
\le\ & c(s,r,V)\limsup_{t-\tau\to 0+}[\frac{\frac{s}{s-r}M(|V|)(\tau,t)}{1 - \frac{s}{s-r}M(|V|)(\tau,t)}]^{\frac{s-r}{s}} = 0.
\end{aligned}$$

This gives equality (63). The proof of equality (62) is similar. In the proof of equality (64), we use part (b) of Lemma 5.1, instead of part (a) of Lemma 5.1.

Let us continue the proof of part (b) of Theorem 6.6. Suppose that Y is locally uniformly bounded on L^r and strongly continuous on L^s. We have already shown that Y_V is a backward propagator on L^s. Moreover, Y_V is uniformly bounded on L^s (see estimate (61)). Therefore, in order to prove the strong continuity of Y_V it suffices to show that Y_V is separately strongly continuous (see Theorem 2.2).

Let $(\tau,t) \in D_T$, and suppose that $t' \ge t$ and $g \in L^s$. Then

$$\begin{aligned}
& ||Y_V(\tau,t')g - Y_V(\tau,t)g||_s = ||Y_V(\tau,t)(Y_V(t,t')g - g)||_s \\
\le\ & M||Y_V(t,t')g - g||_s \\
\le\ & M||g||_s||Y_V(t,t') - Y(t,t')||_{s\to s} + M||Y(t,t')g - g||_s.
\end{aligned}$$

It follows from Lemma 6.9 and from the strong continuity of Y that

$$\lim_{t' \to t+} ||Y_V(\tau, t')g - Y_V(\tau, t)g||_s = 0. \tag{66}$$

Similarly, we get

$$\lim_{t' \to t-} ||Y_V(\tau, t')g - Y_V(\tau, t)g||_s = 0. \tag{67}$$

Now assume that $\tau' \leq \tau$. Then

$$\begin{aligned} ||Y_V(\tau', t)g - Y_V(\tau, t)g||_s &= ||(Y_V(\tau, \tau') - I)Y_V(\tau, t)g||_s \\ &\leq ||Y_V(\tau', \tau) - Y(\tau', \tau)||_{s\to s}||Y_V(\tau, t)g||_s \\ &\quad +||Y(\tau', \tau)Y_V(\tau, t)g - Y_V(\tau, t)g||_s. \end{aligned}$$

Using (61), Lemma 6.9, and the strong continuity of Y, we see that

$$\lim_{\tau' \to \tau-} ||Y_V(\tau', t)g - Y_V(\tau, t)g||_s = 0. \tag{68}$$

Finally, let $\tau < \tau' < t$, and let λ be such that $\tau' < \lambda < t$. Then

$$\begin{aligned} ||Y_V(\tau', t)g - Y_V(\tau, t)g||_s &= ||(Y_V(\tau', \lambda) - Y_V(\tau, \lambda))Y_V(\lambda, t)g||_s \\ &\leq ||(Y(\tau', \lambda) - Y(\tau, \lambda))Y_V(\lambda, t)g||_s \\ &\quad + ||Y_V(\tau', \lambda) - Y(\tau', \lambda)||_{s\to s}||Y_V(\lambda, t)g||_s \\ &\quad + ||Y_V(\tau, \lambda) - Y(\tau, \lambda)||_{s\to s}||Y_V(\lambda, t)g||_s \\ &\leq ||(Y(\tau', \lambda) - Y(\tau, \lambda))Y_V(\lambda, t)g||_s \\ &\quad + M||Y_V(\tau', \lambda) - Y(\tau', \lambda)||_{s\to s}||g||_s \\ &\quad + M||Y_V(\tau, \lambda) - Y(\tau, \lambda)||_{s\to s}||g||_s \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{69}$$

For every $\epsilon > 0$, fix λ such that $\tau < \lambda < t$ and $I_2 + I_3 \leq \frac{\epsilon}{2}$ for all τ' with $\tau < \tau' < \lambda$. This can be done using Lemma 6.9. It is not hard to see that the strong continuity of Y implies the existence of $\delta > 0$ such that $I_1 \leq \frac{\epsilon}{2}$ for all τ' with $\tau < \tau' \leq \tau + \delta < \lambda$. Therefore, (69) gives

$$\lim_{\tau' \to \tau+} ||Y_V(\tau', t)g - Y_V(\tau, t)g||_s = 0, \tag{70}$$

and it follows from (66), (67), (68), and (70) that Y_V is separately strongly continuous.

This completes the proof of Theorem 6.6. □

The next result concerns the smoothing properties of backward Feynman-Kac propagators.

Theorem 6.11. *Let $1 < s < q \leq \infty$ and $1 \leq r < s$. Then the following are true:*
(a) Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Suppose that $Y(\tau, t) \in L(L^r_{\mathcal{E}}, L^{\frac{rq}{s}}_{\mathcal{E}})$ for all $0 \leq \tau < t \leq T$. Then $Y_V(\tau, t) \in L(L^s_{\mathcal{E}}, L^q_{\mathcal{E}})$ for all $0 \leq \tau < t \leq T$.
(b) If $P \in \mathcal{M}$ has a density p, $V \in \mathcal{P}_f^*$, and $Y(\tau, t) \in L(L^r, L^{\frac{rq}{s}})$ for all $0 \leq \tau < t \leq T$, then $Y_V(\tau, t) \in L(L^s, L^q)$.
(c) If $P \in \mathcal{M}$ has a density p, $\mu \in \mathcal{P}_m^*$, and $Y(\tau, t) \in L(L^r, L^{\frac{rq}{s}})$ for all $0 \leq \tau < t \leq T$, then $Y_\mu(\tau, t) \in L(L^s, L^q)$.

Proof. We will prove part (b) in the case $q \neq \infty$. The proof in the case $q = \infty$ is similar.

Let $g \in L^s$. Using estimate (60), we get

$$||Y_V(\tau,t)g||_q \leq c\{\int_E \{Y(\tau,t)|g|^{\frac{s}{r}}(x)\}^{\frac{rq}{s}} dx\}^{\frac{1}{q}}.$$

It follows from the assumptions in Theorem 6.11 that

$$||Y_V(\tau,t)g||_q \leq c||Y(\tau,t)||^{\frac{r}{s}}_{r\rightarrow\frac{rq}{s}}||g||_s. \tag{71}$$

Now it is clear that part (b) of Theorem 6.11 follows from (71). The proofs of parts (a) and (c) are similar. □

7. Feller, Feller-Dynkin, and BUC-property of backward Feynman-Kac propagators

In this section we turn our attention to the behavior of the free backward propagator Y and the backward Feynman-Kac propagators Y_V and Y_μ on spaces of continuous functions on E. By BC is denoted the space of all bounded continuous functions on E equipped with the norm $||f||_C = \sup_{x\in E}|f(x)|$. The symbol C_0 stands for the space of all continuous functions on E vanishing at infinity, and by BUC is denoted the space of all bounded uniformly continuous functions on E. It is known that C_0 is a closed subspace of BUC, and BUC is a closed subspace of BC.

Definition 7.1. A backward BC-propagator is called a backward Feller propagator. A backward C_0-propagator is called a backward Feller-Dynkin propagator. If a backward $L^\infty_{\mathcal{E}}$-propagator Q is such that $Q(\tau,t) \in L(L^\infty_{\mathcal{E}}, BC)$ for all $0 \leq \tau < t \leq T$, then it is said that Q satisfies the strong Feller condition. If a backward $L^\infty_{\mathcal{E}}$-propagator Q is such that $Q(\tau,t) \in L(L^\infty_{\mathcal{E}}, BUC)$ for all $0 \leq \tau < t \leq T$, then it is said that Q satisfies the strong BUC-condition.

Remark 7.2. If Q is a backward L^∞-propagator, then we may replace the space $L^\infty_{\mathcal{E}}$ by the space L^∞ in the definition of the strong Feller and the strong BUC-condition.

Theorem 7.3. *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}^*_f$. Then the following assertions hold:*
(a) If Y satisfies the strong Feller condition, then Y_V also satisfies the same condition.
(b) If Y satisfies the strong BUC-condition, then Y_V also satisfies the same condition.

We do not know whether Theorem 7.3 holds for backward Feller, Feller-Dynkin, or BUC propagators. However, this is true under additional restrictions.

Theorem 7.4. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}^*_f$, and suppose that Y satisfies the strong Feller condition. Then the following assertions hold:*
(a) If Y is a backward Feller-Dynkin propagator, then Y_V also has the same property.
(b) If Y is a strongly continuous backward Feller-Dynkin propagator, then Y_V also has the same property.

If Y satisfies the strong BUC-condition, then part (b) of Theorem 7.3 implies that Y_V is a backward BUC-propagator. Moreover, the following theorem holds:

Theorem 7.5. *Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and suppose that Y satisfies the strong BUC-condition. If Y is a strongly continuous backward BUC-propagator, then Y_V possesses the same property.*

Proof of Theorem 7.3. We will prove part (a) of Theorem 7.3. The proof of Part (b) is similar. □

Lemma 7.6. (a) *Let $P \in \mathcal{M}$ and $V \in \mathcal{P}_f^*$. Then for all $x, x' \in E$, $0 \le \tau < t \le T$, $g \in L^\infty_{\mathcal{E}}$, and $\lambda > 0$ with $\tau + \lambda < t$,*

$$\begin{aligned}(72)\quad &|Y_V(\tau,t)g(x') - Y_V(\tau,t)g(x)| \\ &\quad \le 2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_\infty \\ &\quad\quad + |Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)|.\end{aligned}$$

(b) *Suppose that $P \in \mathcal{M}$ has a density p. Then* (72) *holds for all $g \in L^\infty$.*
(c) *Suppose that $P \in \mathcal{M}$ has a density p. If $\mu \in \mathcal{P}_m^*$, then for every $x, x' \in E$, $0 \le \tau < t \le T$, $g \in L^\infty$, and $\lambda > 0$ with $\tau + \lambda < t$,*

$$\begin{aligned}&|Y_\mu(\tau,t)g(x') - Y_\mu(\tau,t)g(x)| \\ &\quad \le 2||Y_\mu(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_\mu(\tau+\lambda,t)g||_\infty \\ &\quad\quad + |Y(\tau,\tau+\lambda)Y_\mu(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_\mu(\tau+\lambda,t)g(x)|.\end{aligned}$$

Proof of Lemma 7.6. We will prove part (a) of Lemma 7.6. The proofs of parts (b) and (c) are similar. We have

$$\begin{aligned}&|Y_V(\tau,t)g(x') - Y_V(\tau,t)g(x)| \\ &= |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ &\le |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x')| \\ &\quad + |Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ &\quad + |Y_V(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x) - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| \\ &\le 2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_\infty \\ &\quad + |Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)|.\end{aligned}$$

This completes the proof of Lemma 7.6. □

Let us go back to the proof of part (a) of Theorem 7.3. Suppose that the conditions in part (a) of Theorem 7.3 hold, and let $g \in L^\infty_{\mathcal{E}}$. Since Y_V is a uniformly bounded backward $L^\infty_{\mathcal{E}}$-propagator (see Remark 6.2), we have

$$||Y_V(\tau,t)||_{\infty\to\infty} < M \tag{73}$$

for all $(\tau,t) \in D_T$. It follows from (73) and Lemma 6.9 that for every $\epsilon > 0$ there exists $\lambda > 0$ such that $\tau + \lambda < t$ and

$$2||Y_V(\tau,\tau+\lambda) - Y(\tau,\tau+\lambda)||_{\infty\to\infty}||Y_V(\tau+\lambda,t)g||_\infty < \frac{\epsilon}{2}. \tag{74}$$

Moreover, for λ such as above and any fixed $x \in E$ there exists $\delta > 0$ such that

$$|Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x') - Y(\tau,\tau+\lambda)Y_V(\tau+\lambda,t)g(x)| < \frac{\epsilon}{2} \tag{75}$$

for all x' such that $\rho(x', x) < \delta$. This follows from (73) and the assumption that Y is a backward strong Feller propagator. Now it is easy to see that part (a) of Theorem 7.3 can be obtained from (74), (75), and Lemma 7.6.

This completes the proof of Theorem 7.3.

Proof of Theorem 7.4. (a) Let $g \in C_0$. Then $Y_V(\tau, t)g \in BC$ for all $(\tau, t) \in D_T$, by part (a) of Theorem 7.3.

For every $\epsilon > 0$ there exists a compact set K_ϵ such that $|g(x)| < \epsilon$ for all $x \in E \backslash K_\epsilon$. Moreover, Urysohn's Lemma implies that there exists a continuous function g_ϵ on E with compact support such that $g_\epsilon(x) = 1$ for $x \in K_\epsilon$. It follows that

$$\begin{aligned} |Y_V(\tau, t)g(x)| &\leq |Y_V(\tau, t)gg_\epsilon(x)| + |Y_V(\tau, t)g(1 - g_\epsilon)(x)| \\ &\leq c_{\tau,t}[\{Y(\tau, t)|gg_\epsilon|^2(x)\}^{\frac{1}{2}} + \epsilon]. \end{aligned}$$

It is clear that this implies part (a) of Theorem 7.4.

(b) Let Y be a strongly continuous Feller-Dynkin propagator. By part (a) of Theorem 7.4, Y_V is a Feller-Dynkin propagator. Arguing as in the proof of the strong continuity of Y_V in the space L^s in Theorem 6.6 and using the C-norm instead of the L^s-norm, we see that Y_V is strongly continuous on C_0. □

Proof of Theorem 7.5. It is clear that Y_V is a backward BUC-propagator (see Theorem 7.3). Now we can obtain the strong continuity of Y_V on the space BUC, reasoning as in the proof of the strong continuity of Y_V on the space L^s in Theorem 6.6 and using the C-norm instead of the L^s-norm. □

The next theorem provides sufficient conditions for the continuity of the function $(\tau, x) \to Y_V(\tau, t)g(x)$ on the set $[0, t) \times E$. By ξ is denoted the topology on the space BC generated by the uniform convergence of functions on compact subsets of the space E.

Theorem 7.7. *Let $P \in \mathcal{M}$, and suppose that Y satisfies the following conditions:*

(i) *Y is a backward strong Feller propagator.*

(ii) *For every function $h \in BC$ such that $h = Y_V(r, s)g$ with $0 \leq r < s \leq T$ and $g \in BC$, the mapping $(u, v) \mapsto Y(u, v)h$ of the space $\{(u, v) : 0 \leq u \leq v \leq T\}$ into the space (BC, ξ) is continuous.*

Then for any $V \in \mathcal{P}_f^*$, $t \in (0, T]$, and $g \in L_{\mathcal{E}}^\infty$, the function $(\tau, x) \mapsto Y_V(\tau, t)g(x)$ is continuous on the space $[0, t) \times E$.

Proof. Suppose that the conditions in Theorem 7.7 hold. Using part (a) of Theorem 7.3, we see that Y_V is a backward strong Feller propagator. Given $t \in (0, T]$ and $g \in L_{\mathcal{E}}^\infty$, fix $x \in E$ and τ with $0 \leq \tau < t$. Suppose that τ' is close to τ and $x' \in U(x)$, where $U(x)$ is a relatively compact neighborhood of x. Then we have

$$\begin{aligned} & |Y_V(\tau', t)\, g(x') - Y_V(\tau, t)g(x)| \\ \leq\; & |Y_V(\tau', t)\, g(x') - Y_V(\tau, t)g(x')| + |Y_V(\tau, t)g(x') - Y_V(\tau, t)g(x)| \\ =\; & J_1 + J_2. \end{aligned} \tag{76}$$

Since Y_V is a backward strong Feller propagator,

$$\lim_{x' \to x} J_2 = 0. \tag{77}$$

Next, we will estimate the quantity $\sup_{x' \in U(x)} J_1$. Let us first suppose that $\tau' < \tau$. Then we have

$$\begin{aligned}
\sup_{x' \in U(x)} J_1 &\leq \sup_{x' \in U(x)} \left| \left(Y_V\left(\tau', \tau\right) - I \right) Y_V(\tau, t) g\left(x'\right) \right| \\
&\leq \sup_{x' \in U(x)} \left| \left(Y_V\left(\tau', \tau\right) - Y\left(\tau', \tau\right) \right) Y_V(\tau, t) g\left(x'\right) \right| \\
&\quad + \sup_{x' \in U(x)} \left| \left(Y\left(\tau', \tau\right) - I \right) Y_V(\tau, t) g\left(x'\right) \right| \\
&\leq M \left| Y_V\left(\tau', \tau\right) - Y\left(\tau', \tau\right) \right|_{\infty \to \infty} |g|_\infty \\
&\quad + \sup_{x' \in U(x)} \left| \left(Y\left(\tau', \tau\right) - I \right) Y_V(\tau, t) g\left(x'\right) \right|. \qquad (78)
\end{aligned}$$

Put $h = Y_V(\tau, t)g$. Then for any small $\varepsilon > 0$ we have

$$h = Y_V(\tau, t - \varepsilon) Y_V(t - \varepsilon, t) g = Y_V(\tau, t - \varepsilon) h_1.$$

Since condition (ii) in Theorem 7.7 holds, the function h_1 belongs to the space BC. It follows from (78), condition (ii) in Theorem 7.7, and Lemma 6.5 that

$$\lim_{\tau' \uparrow \tau} \sup_{x' \in U(x)} J_1 = 0. \qquad (79)$$

Next, suppose that $\tau < \tau'$. Then for every λ with $\tau' < \lambda < t$,

$$\begin{aligned}
\sup_{x' \in U(x)} J_1 &\leq \sup_{x' \in U(x)} \left| \left(Y_V\left(\tau', \lambda\right) - Y_V\left(\tau, \lambda\right) \right) Y_V(\lambda, t) g\left(x'\right) \right| \\
&\leq \sup_{x' \in U(x)} \left| \left(Y_V\left(\tau', \lambda\right) - Y\left(\tau', \lambda\right) \right) Y_V(\lambda, t) g\left(x'\right) \right| \\
&\quad + \sup_{x' \in U(x)} \left| \left(Y_V\left(\tau, \lambda\right) - Y\left(\tau, \lambda\right) \right) Y_V(\lambda, t) g\left(x'\right) \right| \\
&\quad + \sup_{x' \in U(x)} \left| \left(Y\left(\tau', \lambda\right) - Y\left(\tau, \lambda\right) \right) Y_V(\lambda, t) g\left(x'\right) \right| \\
&\leq M |g|_\infty \left| Y_V\left(\tau', \lambda\right) - Y\left(\tau', \lambda\right) \right|_{\infty \to \infty} \\
&\quad + M |g|_\infty \left| Y_V\left(\tau, \lambda\right) - Y\left(\tau, \lambda\right) \right|_{\infty \to \infty} \\
&\quad + \sup_{x' \in U(x)} \left| \left(Y\left(\tau', \lambda\right) - Y\left(\tau, \lambda\right) \right) Y_V(\lambda, t) g\left(x'\right) \right| \\
&= C_1 + C_2 + C_3. \qquad (80)
\end{aligned}$$

Using Lemma 6.5, we see that the following statement holds: for every $\varepsilon > 0$ there exists $\lambda \in (\tau, t)$ such that if $\tau < \tau' < \lambda$, then $C_1 + C_2 < \frac{1}{2}\varepsilon$. Moreover,

$$Y_V(\lambda, t) g = Y(\lambda, t - \delta) Y_V(t - \delta, t) g = Y(\lambda, t - \delta) h, \qquad (81)$$

where $h \in BC$. Now condition (ii) in Theorem 7.7 and (81) imply that there exists $\eta > 0$ such that $\tau < \tau' < \tau + \eta < \lambda$ and $C_3 \leq \frac{1}{2}\varepsilon$. Hence, (80) gives

$$\lim_{\tau' \downarrow \tau} \sup_{x' \in U(x)} J_1 = 0. \qquad (82)$$

Now it is clear that Theorem 7.7 follows from (76), (77), (79), and (82). ☐

Corollary 7.8. *Let $P \in \mathcal{M}$, and suppose that Y is a backward strong Feller propagator. Suppose also that for every $g \in BC$, the mapping $(u, v) \mapsto Y(u, v)g$ of the space $\{(u, v) : 0 \leq u \leq v \leq T\}$ into the space (BC, ξ) is continuous. Then for any*

$V \in \mathcal{P}_f^*$, $t \in (0,T]$, and $g \in L^\infty_{\mathcal{E}}$, the function $(\tau, x) \to Y_V(\tau,t)g(x)$ is continuous on the set $[0,t) \times E$.

Corollary 7.9. *Let $P \in \mathcal{M}$, and suppose that Y is a strongly continuous backward BUC-propagator. Suppose also that Y possesses the strong BUC-property. Then for any $V \in \mathcal{P}_f^*$, $t \in (0,T]$, and $g \in L^\infty_{\mathcal{E}}$, the function $(\tau, x) \to Y_V(\tau,t)g(x)$ is continuous on the set $[0,t) \times E$.*

It is not hard to see that Corollaries 7.8 and 7.9 follow from Theorem 7.7. The proof is left as an exercise for the reader.

Remark 7.10. If a transition probability function $P \in \mathcal{M}$ has a density p, then Theorems 7.3–7.7 and Corollaries 7.8 and 7.9 hold for any time-dependent measure μ from the class $\mathcal{P}_m^*$. The proofs of these results for $\mu \in \mathcal{P}_m^*$ are similar to the proofs in the case $V \in \mathcal{P}_f^*$.

8. Subclasses of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$

We do not know whether the Feller-Dynkin property or the BUC-property are inherited by the backward Feynman-Kac propagators Y_V and Y_μ with $V \in \mathcal{P}_f^*$ and $\mu \in \mathcal{P}_m^*$ from the backward free propagator Y. Note that Theorems 7.4 and 8.7 contain additional assumptions. It will be shown in this section that if V and μ belong to appropriate subclasses of the classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$, then the Feller-Dynkin property and the BUC-property are inherited.

The next lemma concerns the non-autonomous Dyson series. Such assertions are standard, and we do not include the proof.

Lemma 8.1. *(a) Let $P \in \mathcal{M}$, $V \in \mathcal{P}_f^*$, and $g \in L^\infty_{\mathcal{E}}$. If t and τ are such that $M(|V|)(\tau,t) < 1$, then*

$$Y_V(\tau,t)g(x) - Y(\tau,t)g(x) = \sum_{k\geq 1}(-1)^k \int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots$$

$$\int_{t_{k-1}}^t Y(\tau,t_1)V(t_1)Y(t_1,t_2)V(t_2)\cdots Y(t_{k-1},t_k)V(t_k)Y(t_k,t)g(x)dt_k. \tag{83}$$

(b) If $P \in \mathcal{M}$ has a density p, $V \in \mathcal{P}_f^*$, $g \in L^\infty$, and $(\tau,t) \in D_T$ is such that $M(|V|)(\tau,t) < 1$, then equality (83) holds.
(c) If $P \in \mathcal{M}$ has a density p, $\mu \in \mathcal{P}_m^*$, $g \in L^\infty$, and $(\tau,t) \in D_T$ is such that $M(|\mu|)(\tau,t) < 1$, then

$$Y_\mu(\tau,t)g(x) - Y(\tau,t)g(x) = \sum_{k\geq 1}(-1)^k \int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots$$

$$\int_{t_{k-1}}^t Y(\tau,t_1)\mu(t_1)Y(t_1,t_2)\mu(t_2)\cdots Y(t_{k-1},t_k)\mu(t_k)Y(t_k,t)g(x)dt_k.$$

Definition 8.2. The function classes $\mathcal{P}_{f,c}^*$ and $\mathcal{P}_{f,u}^*$ are defined as follows:

$$V \in \mathcal{P}_{f,c}^* \Longleftrightarrow V \in \mathcal{P}_f^* \text{ and } N(V)(\tau,t,\cdot) \in BC \text{ for all } (\tau,t) \in D_T,$$

$$V \in \mathcal{P}_{f,u}^* \Longleftrightarrow V \in \mathcal{P}_f^* \text{ and } N(V)(\tau,t,\cdot) \in BUC \text{ for all } (\tau,t) \in D_T.$$

Definition 8.3. The class $\mathcal{D}^*_{f,c}$ is defined as follows: A function $V \in \mathcal{P}^*_f$ belongs to this class if there exists a sequence $V_k \in \mathcal{P}^*_f$ such that $V_k(t,\cdot) \in BC$ for all $k \geq 1$ and $0 \leq t \leq T$, the sequence V_k approaches V in the potential sense and $\int_0^T ||V_k(t)||_C dt < \infty$ for all $k \geq 1$. The class $\mathcal{D}^*_{f,u}$ is defined similarly. Here the following restrictions are imposed: $V_k(t,\cdot) \in BUC$ for all $k \geq 1$ and $0 \leq t \leq T$, the sequence V_k approaches V in the potential sense and $\int_0^T ||V_k(t)||_C dt < \infty$ for all $k \geq 1$.

Remark 8.4. If P has a density p, then the classes of time-dependent measures $\mathcal{P}^*_{m,c}$, $\mathcal{P}^*_{m,u}$, $\mathcal{D}^*_{m,c}$, and $\mathcal{D}^*_{m,u}$ can be defined similarly.

Lemma 8.5. *The following assertions hold:*
1. $\mathcal{P}^*_{f,c} \subset \mathcal{D}^*_{f,c}$ *and* $\mathcal{P}^*_{m,c} \subset \mathcal{D}^*_{m,c}$.
2. $\mathcal{P}^*_{f,u} \subset \mathcal{D}^*_{f,u}$ *and* $\mathcal{P}^*_{m,u} \subset \mathcal{D}^*_{m,u}$.
3. *If* $V \in \mathcal{P}^*_f$*, and there exists a sequence* $V_k \in \mathcal{P}^*_{f,c}$ *such that* V_k *approaches* V *in the potential sense, then* $V \in \mathcal{P}^*_{f,c}$*. Similarly, if* $\mu \in \mathcal{P}^*_m$*, and there exists a sequence* $V_k \in \mathcal{P}^*_{f,c}$ *such that* V_k *approaches* V *in the potential sense, then* $\mu \in \mathcal{P}^*_{m,c}$.
4. *If* $V \in \mathcal{P}^*_f$*, and there exists a sequence* $V_k \in \mathcal{P}^*_{f,u}$ *such that* V_k *approaches* V *in the potential sense, then* $V \in \mathcal{P}^*_{f,u}$*. Similarly, if* $\mu \in \mathcal{P}^*_m$*, and there exists a sequence* $V_k \in \mathcal{P}^*_{f,u}$ *such that* V_k *approaches* V *in the potential sense, then* $\mu \in \mathcal{P}^*_{m,u}$.

Proof. Part 1. Let $V \in \mathcal{P}^*_{f,c}$, and let g_k be the sequence defined by (18). Then we have $g_k(t,\cdot) \in BC$ for all $k \geq 1$ and $0 \leq t \leq T$. Moreover, g_k approaches V in the potential sense (see Lemma 4.1). It remains to prove that

$$\int_0^T ||g_k(t)||_C dt < \infty \tag{84}$$

for all $k \geq 1$. We have

$$\int_0^T ||g_k(t)||_C dt \leq k \int_0^T \sup_{x \in E} N(|V|)(\tau, \min(\tau + \frac{1}{k}, T), x) dt < \infty,$$

since $V \in \mathcal{P}^*_f$. This establishes (84).

The proof of part 1 of Lemma 8.5 is thus completed. The proof for the measures and also that of part 2 of Lemma 8.5 is similar.

Part 3. Let $V \in \mathcal{P}^*_f$, and assume that there exists a sequence $V_k \in \mathcal{P}^*_{f,c}$ such that V_k approaches V in the potential sense. Using Definition 8.2 and Lemma 4.1, we see that $V \in \mathcal{P}^*_{f,c}$. The proof for the measures and that of part 4 of Lemma 8.5 is similar.

This completes the proof of Lemma 8.5. □

The next assertions concern the inheritance of the Feller, the Feller-Dynkin, and the BUC-properties.

Theorem 8.6. *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{D}^*_{f,c}$*. Then the following assertions hold:*
(*a*) *If* Y *is a backward Feller propagator, then* Y_V *has the same property.*
(*b*) *If* Y *is a backward Feller-Dynkin propagator, then* Y_V *has the same property. If, in addition,* Y *is strongly continuous on* C_0*, then* Y_V *is also strongly continuous on* C_0.

Theorem 8.7. *Let* $P \in \mathcal{M}$ *and* $V \in \mathcal{D}^*_{f,u}$*. If* Y *is a backward* BUC*-propagator, then* Y_V *has the same property. If, in addition,* Y *is strongly continuous on* BUC*, then* Y_V *is also strongly continuous on* BUC.

Remark 8.8. Theorem 8.6 (Theorem 8.7) holds for a time-dependent measure $\mu \in \mathcal{D}^*_{m,c}$ ($\mu \in \mathcal{D}^*_{m,u}$), provided that $P \in \mathcal{M}$ has a density p.

Proof of Theorems 8.6 *and* 8.7. We start with the proof of the inheritance of the Feller-Dynkin property in part (b) of Theorem 8.6. Let $V \in \mathcal{D}^*_{f,c}$, $g \in C_0$, and let $V_k \in \mathcal{P}^*_f$ be a sequence of functions such that $V_k(t,\cdot) \in BC$ for all $k \geq 1$ and $0 \leq t \leq T$, and, moreover, V_k approaches V in the potential sense. Then using part (a) of Lemma 4.1 and estimate (45) with $q = 1$ and $r = 2$, we get

$$\begin{aligned}
&||Y_V(\tau,t)g - Y_{V_k}(\tau,t)g||_C \\
&\leq ||g||_C \frac{1}{(1 - M(2|V|)(\tau,t))^{\frac{1}{2}}} \{[2M(2|V-V_k|)(\tau,t)M(2(V-V_k))(\tau,t)]^{\frac{1}{2}} \\
&\quad + \frac{2\sqrt{3}}{3} \frac{M(2|V-V_k|)(\tau,t)M(2(V-V_k))(\tau,t)}{1 - M(2|V-V_k|)(\tau,t)}\}^{\frac{1}{2}}
\end{aligned} \tag{85}$$

for all $k \geq k_0$ and $t - \tau < \delta$, where $\delta > 0$ is small and does not depend on k. It follows from (85) and Definition 4.3 that for $t - \tau < \delta$, we have

$$\lim_{k\to\infty} ||Y_V(\tau,t)g - Y_{V_k}(\tau,t)g||_C = 0. \tag{86}$$

Hence, it suffices to prove that the Feller-Dynkin property is inherited if a function $W \in \mathcal{P}^*_f$ is such that $W(t,\cdot) \in BC$ for all $0 \leq t \leq T$ and $\int_0^T ||W(t)||_C dt < \infty$. Indeed, suppose that the Feller-Dynkin property is inherited for such functions. Let $V \in \mathcal{D}^*_{f,c}$. Then using Definition 8.3 and our assumption, we see that for every $k \geq 1$, Y_{V_k} is a backward Feller-Dynkin propagator. It follows from (86) and from the fact that C_0 is a closed subspace of BC that $Y_V(\tau,t)g \in C_0$ for all $g \in C_0$ and $t - \tau < \delta$. Now the properties of backward propagators show that Y_V is a backward Feller-Dynkin propagator. This establishes the inheritance of the Feller-Dynkin property in part (b) of Theorem 8.6 for all $V \in \mathcal{D}^*_{f,c}$.

Our final goal is to prove the inheritance of the Feller-Dynkin property in part (b) of Theorem 8.6 for a function $V \in \mathcal{P}^*_f$ for which $V(t,\cdot) \in BC$ for all $0 \leq t \leq T$ and $\int_0^t ||V(t)||_C dt < \infty$. Let $g \in C_0$, and assume that Y is a backward Feller-Dynkin propagator. Then using formula (83), we see that there exists $\delta < 0$ such that

$$\begin{aligned}
&Y_V(\tau,t)g(x) - Y(\tau,t)g(x) \\
&= \sum_{j\geq 1} (-1)^j \int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{j-1}}^t Y(\tau,t_1)V(t_1)Y(t_1,t_2)V(t_2) \\
&\qquad \cdots Y(t_{j-1},t_j)V(t_j)Y(t_j,t)g(x)dt_j
\end{aligned} \tag{87}$$

for all τ and t with $t - \tau < \delta$. The family Y consists of contractions on $L^\infty_{\mathcal{E}}$ which map the space C_0 into itself. Moreover, the definition of the class $\mathcal{P}^*_{f,c}$ shows that $V(t_k,\cdot) \in BC$ for every fixed t. The integrands in (87) are Borel functions of the variables $t_1, \dots, t_j$ and belong to the space C_0 in the variable x. Our next goal is to show that the integrals in (87) also belong to the space C_0. This can be seen using the Dominated Convergence Theorem since

$$\int_\tau^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{j-1}}^t \sup_{x\in E} Y(\tau,t_1)|V(t_1)|Y(t_1,t_2)|V(t_2)|$$
$$\cdots Y(t_{j-1},t_j)|V(t_j)|Y(t_j,t)g(x)dt_j$$
$$\leq \int_\tau^t dt_1 \int_\tau^t dt_2 \cdots \int_\tau^t ||V(t_1)||_C||V(t_2)||_C \cdots ||V(t_j)||_C||g||dt_j$$
$$= ||g||_C(\int_\tau^t ||V(t)||_C dt)^j < \infty.$$

It follows from the convergence of the series in (87) in the space BC and from the fact that C_0 is a closed subspace of BC that the function on the right-hand side of (87) belongs to the space C_0. By our assumption, we have $Y(\tau,t)g \in C_0$ for all $k \geq k_0$ and $t-\tau < \delta$. Using the properties of backward propagators, we see that Y_V is a backward Feller-Dynkin propagator. If, in addition, Y is a strongly continuous backward propagator on C_0, then we can prove the strong continuity of Y_V on C_0 using the same methods as in the proof of part (b) of Theorem 6.6 with $s = \infty$.

This completes the proof of part (b) of Theorem 8.6. The proofs of part (a) and that of Theorem 8.7 are similar. □

9. Examples

A rich source of transition probability densities is the theory of second order parabolic partial differential equations on R^n. It is known that under certain restrictions, fundamental solutions of such equations are transition probability densities. Numerous results concerning the existence of fundamental solutions in the case of equations with time-dependent coefficients can be found in [11, 12, 23, 29, 33]. In the present section we will discuss what follows from the results obtained in Sections 6 and 7 for such transition probability densities.

Consider the following final value problem on R^n:

$$\begin{cases} \frac{\partial u}{\partial \tau} + Lu = 0, \ \ 0 \leq \tau < t \leq T, \\ u(t) = f. \end{cases} \tag{88}$$

In (88), L stands for a differential operator given by

$$L = \sum_{i,j=1}^n a_{ij}(\tau,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(\tau,x)\frac{\partial}{\partial x_i} \tag{89}$$

(non-divergence form), or by

$$Lu = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}[a_{ij}(\tau,x)\frac{\partial}{\partial x_j}] + \sum_{i=1}^n b_i(\tau,x)\frac{\partial u}{\partial x_i} \tag{90}$$

(divergence form). Solutions to problem (88) with L in the divergence form are understood in the weak sense.

It is known that if L is as in formula (89), and if the following conditions hold:

(1) the functions a_{ij} and b_i are bounded and measurable on $[0,T]\times R^n$,

(2) there exists a constant $\gamma>0$ such that for all $(\tau,x)\in[0,T]\times R^n$ and any collection of real numbers $\lambda_1,\dots,\lambda_n$,

$$\sum_{i,j=1}^{n} a_{ij}(\tau,x)\lambda_i\lambda_j \geq \gamma\sum_{i=1}^{n}\lambda_i^2,$$

(3) there exists a constant δ with $0<\delta\leq 1$ such that

$$\sum_{i,j=1}^{n}|a_{ij}(\tau_1,x_1)-a_{ij}(\tau_2,x_2)|+\sum_{i=1}^{n}|b_i(\tau_1,x_1)-b_i(\tau_2,x_2)| \leq C(|x_1-x_2|^{\delta}+|\tau_1-\tau_2|^{\delta})$$

for all $(\tau_1,x_1),(\tau_2,x_2)\in[0,T]\times R^n$,

then there exists a unique fundamental solution $p(\tau,x;t,y)$ of the final value problem (88). The function p satisfies the following conditions: it is jointly continuous, strictly positive, and the Gaussian estimates hold for p and its derivatives. For $f\in C_0^{\infty}$ and $t>0$, the function $u(\tau,x)=\int_{R^n}f(y)p(\tau,x;t,y)dy$ belongs to the space $C_b^{1,2}([0,t]\times R^n)$ and satisfies (88) (see, e.g., [11, 12, 23, 33]). The fundamental solution p is simultaneously a transition probability density. It follows from the upper Gaussian estimate for p that there exists a continuous Markov process X_t with p as its transition density. It is not hard to prove that the backward free propagator Y, associated with the density p, is (L^r-L^q)-smoothing for all $1\leq r\leq q\leq\infty$ and possesses the strong Feller and the strong BUC-property. Moreover, Y is a backward BC-propagator, a backward BUC-propagator, and a backward Feller-Dynkin propagator. It is not hard to prove that Y is strongly continuous on the spaces C_0 and BUC. Here we need the following well-known assertion concerning general transition probability densities:

Lemma 9.1. *For every function $f\in BUC$ and $\epsilon>0$, the following estimate holds:*

$$||f-Y(\tau,t)f||_C\leq\sup_{x,y\in E:\rho(x,y)\leq\epsilon}|f(x)-f(y)|+2||f||_C\sup_{x\in E}\int_{y:\rho(x,y)>\epsilon}p(\tau,x;t,y)dy.$$

We also employ the Gaussian estimates for p and Theorem 2.2 in the proof of the strong continuity of Y on the C_0 and BUC.

It follows from the properties of the backward propagator Y listed above that the backward Feynman-Kac propagators Y_V and Y_μ with $V\in\mathcal{P}_f^*$ and $\mu\in\mathcal{P}_m^*$ satisfy the conditions $Y_V(\tau,t)\in L(L^r,L^q)$ and $Y_\mu(\tau,t)\in L(L^r,L^q)$ for all $(\tau,t)\in D_T$ and $1<r\leq q\leq\infty$. Moreover, they possess the strong Feller and the strong BUC-property and are strongly continuous backward Feller-Dynkin propagators and strongly continuous backward BUC-propagators (see Theorems 6.1, 6.6, 6.11, 7.4, and 7.5).

Next, suppose that L is as in (90). A very general existence theorem for fundamental solutions was obtained in this case in [29] (see Theorem 1 in [29]). The fundamental solution $p(\tau,x;t,y)$ of the final value problem (88) in [29] satisfies the Gaussian estimates. Under the conditions in Theorem 1 in [29] the backward free propagator Y satisfies the condition $Y(\tau,t)\in L(L^r,L^q)$ for all $(\tau,t)\in D_T$ and $1\leq r\leq q\leq\infty$ (this can be shown using the upper Gaussian estimate for

p). It follows from the upper Gaussian estimate and the continuity of p that the strong Feller property also holds for Y. Moreover, Y is a strongly continuous Feller-Dynkin propagator (this fact can be obtained from the strong Feller property, the Gaussian estimate, and Lemma 9.1 using the ideas in the proof of part (b) of Theorem 6.6). However, the validity of the strong BUC-property for Y is not clear. Using the results obtained in Sections 6 and 7 of the present paper, we see that the backward Feynman-Kac propagators Y_V and Y_μ with $V \in \mathcal{P}_f^*$ and $\mu \in \mathcal{P}_m^*$ satisfy $Y_V \in L(L^r, L^q)$ and $Y_\mu \in L(L^r, L^q)$ for all $(\tau, t) \in D_T$ and $1 < r \leq q \leq \infty$. They also possess the strong Feller property and are strongly continuous backward Feller-Dynkin propagators.

10. Backward transition functions and forward propagators

In this section we discuss forward Feynman-Kac propagators. There is a simple connection between the forward and backward cases which uses the idea of time reversal. We will consider the case where $T < \infty$.

Suppose that $\tilde{P}(\tau, A; t, y)$ satisfies the following conditions:

(1) For fixed τ, A, and t, $\tilde{P}$ is a non-negative Borel function on E.
(2) For fixed τ, t, and y, $\tilde{P}$ is a Borel measure on $\mathcal{E}$.
(3) The normality condition, $\tilde{P}(\tau, E; t, y) = 1$, holds for all τ, t, and y.
(4) The Chapman-Kolmogorov equation,
$$\tilde{P}(\tau, A; t, y) = \int_E \tilde{P}(\tau, A; \lambda, x)\tilde{P}(\lambda, dx; t, y),$$
holds for all $\tau < \lambda < t$, A, and y.

Then $\tilde{P}$ is called a backward transition probability function.

The free propagator associated with $\tilde{P}$ is defined on the space $L^\infty_{\mathcal{E}}$ by
$$\begin{cases} U(t, \tau)g(y) = \int_E g(x)\tilde{P}(\tau, dx; t, y) \\ U(t, t)f = f, \end{cases}$$
for all τ, t, and $f \in L^\infty_{\mathcal{E}}$. If $\tilde{P}$ possesses density $\tilde{p}$, then
$$\begin{cases} U(t, \tau)g(y) = \int_E g(x)\tilde{p}(\tau, x; t, y)dx \\ U(t, t)f = f, \end{cases}$$
for all $x \in E$, $0 \leq \tau < t < \infty$, and $f \in L^\infty$.

The time reversal η is the function $\eta(t) = T - t$ where $t \in [0, T]$. For a function V on $[0, T] \times E$ and a time-dependent measure μ, we put
$$\eta(V)(t, x) = (\eta(t), x) \quad \text{and} \quad \eta(\mu)(t) = \mu(\eta(t)).$$
One of the links between the forward and backward cases is as follows. If $\tilde{P}$ is a backward transition probability function, then
$$P(\tau, x; t, A) = \tilde{P}(\eta(t), A; \eta(\tau), x) \tag{91}$$
is a transition probability function. If $(X, \mathcal{F}_t^\tau, P_{\tau,x})$ is a non-homogeneous progressively measurable Markov process on $(\Omega, \mathcal{F})$ with transition probability function P, then we can define a progressively measurable backward Markov process $\tilde{X}$ on $(\Omega, \mathcal{F})$ by putting $\tilde{X}_t = X_{\eta(t)}$.

Suppose that a backward transition probability function $\tilde{P}$ is given. If V is a Borel function on $[0, T] \times E$, then we will say that V belongs to the class $\mathcal{P}_f$ provided that $\eta(V) \in \mathcal{P}_f^*$. Similarly, if $\tilde{P}$ possesses density $\tilde{p}$ and μ is a time-dependent measure, we will say that μ belongs to the class $\mathcal{P}_m$ provided that $\eta(\mu) \in \mathcal{P}_m^*$. The potentials of the function V and the measure μ are defined by

$$\tilde{N}(V)(t,\tau,x) = \int_\tau^t U(t,s)V(s)(x)ds$$

and

$$\tilde{N}(\mu)(t,\tau,x) = \int_\tau^t U(t,s)\mu(s)(x)ds,$$

respectively. If $\tilde{P}$ possesses density $\tilde{p}$, then the additive functional C_μ corresponding to a time-dependent measure $\mu \in \mathcal{P}_m$ is given by

$$C_\mu(t,\tau) = A_{\eta(\mu)}(T-t, T-\tau). \tag{92}$$

Here we should take into account the correspondence between $\tilde{P}$ and P, expressed by (91), and use Theorem 4.6. Since $\eta(\mu) \in \mathcal{P}_m^*$, we only need the progressive measurability of the process X (or equivalently, the progressive measurability of the process $\tilde{X}$) in order for the right-hand side of (92) to be defined.

Let $\tilde{P}$ be a backward transition probability function, and suppose that the process $\tilde{X}$ is progressively measurable. Then for any $V \in \mathcal{P}_f$, the Feynman-Kac propagator U_V is defined on the space $L_{\mathcal{E}}^\infty$ by

$$U_V(t,\tau)g(y) = E_{\eta(t),y}\, g(X_{\eta(\tau)}) \exp\{-\int_\tau^t V(s, X_{\eta(s)})ds\}.$$

Similarly, if $\tilde{P}$ possesses density $\tilde{p}$ and if the process $\hat{X}$ is progressively measurable, then the Feynman-Kac propagator U_μ is defined on the space $L_{\mathcal{E}}^\infty$ by

$$U_\mu(t,\tau)g(y) = E_{\eta(t),y}\, g(X_{\eta(\tau)}) \exp\{-C_\mu(t,\tau)\}.$$

It is not hard to see that all the results for backward Feynman-Kac propagators obtained in the present paper can be reformulated for forward propagators using time reversal. Here we assume that a backward transition probability function $\widetilde{P}$ is given, and pass from the Kato classes $\mathcal{P}_f^*$ and $\mathcal{P}_m^*$ and the backward propagators Y, Y_V, and Y_μ to the classes $\mathcal{P}_f$ and $\mathcal{P}_m$ and the forward propagators U, U_V, and U_μ, respectively, taking into account formula (91).

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