

GENERALIZED DIFFERENTIAL GALOIS THEORY

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ABSTRACT. A Galois theory of differential fields with parameters is developed in a manner that generalizes Kolchin's theory. It is shown that all connected differential algebraic groups are Galois groups of some appropriate differential field extension.

INTRODUCTION

This paper may be viewed as the next step in E. R. Kolchin's work on the foundations of differential Galois theory. In [7, 1948], Kolchin was the first to formulate the Galois theory of differential fields in the current standard of mathematical rigor. In [9, 1953], he defined strongly normal differential field extensions, generalizing Picard-Vessiot extensions, so as to include the nonlinear algebraic groups as Galois groups. In his first book [12, 1973], the properties of these Galois groups are axiomatized as the category of \mathcal{C} -groups for a field of constants \mathcal{C} and are shown to be the Galois groups of strongly normal differential field extensions. In his second book [13, 1985], Kolchin develops more general axioms to define the category of differential algebraic groups. This paper defines a generalization of strongly normal differential field extensions and shows that these extensions have a good Galois theory for which the Galois groups are differential algebraic groups.

A *differential algebraic group* or *E- \mathcal{C} -group* (Definition 1.23), where $E = \{\epsilon_1, \dots, \epsilon_m\}$ is a commuting set of derivations acting on a field \mathcal{C} , is a group that may be thought of as a set of zeros of a system of differential equations in E -derivatives over \mathcal{C} . It is endowed with the E - \mathcal{C} -Zariski topology for which the closed sets are the zeros of a system of differential equations and the operations of the group structure are defined by differential rational functions. In Cassidy's treatment of affine differential algebraic groups [1], this is how they are defined. However, in Kolchin's exposition [13], this definition is a consequence of the extensive development of Kolchin's differential algebraic group axioms.

To see how Kolchin's theory of strongly normal extensions can be enriched, consider two sets of mutually commuting derivations E and Δ acting on a field \mathcal{F} . Let \mathcal{U} be a universal differential extension field of \mathcal{F} with respect to both E and Δ , and let \mathcal{F} contain the Δ -constants \mathcal{C} of \mathcal{U} . Let \mathcal{G} be a subfield of \mathcal{U} containing \mathcal{F} which is closed under the operation of Δ . If \mathcal{G} over \mathcal{F} is a strongly normal extension of Δ -fields, in the sense of Kolchin, the set of Δ -isomorphisms of \mathcal{G} into \mathcal{U} over \mathcal{F} , when \mathcal{U} is viewed as a universal differential extension of \mathcal{F} with respect to Δ , has the structure of an algebraic group defined over \mathcal{C} . All the Δ -isomorphisms are obtained

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by sending the Δ -generators of \mathcal{G} to rational expressions in the Δ -generators of \mathcal{G} and their Δ -derivatives, with Δ -constants in \mathcal{U} as coefficients. These constants are not necessarily constants with respect to E . The generators of \mathcal{G} may satisfy differential equations in E as well as Δ . A Δ -isomorphism of \mathcal{G} into \mathcal{U} will extend to an E and Δ isomorphism of the E and Δ field \mathcal{H} generated by E -derivatives of \mathcal{G} if and only if it maps solutions of the system of differential equations in E and Δ to other solutions of the same system. The Δ -isomorphisms of \mathcal{G} into \mathcal{U} which extend to E and Δ isomorphisms of \mathcal{H} form a differential subgroup H of G defined by differential equations with respect to E . If the field \mathcal{C} of Δ -constants of \mathcal{F} is equal to the field of Δ -constants of \mathcal{H} , then it will be shown that H is a Galois group for \mathcal{H} over \mathcal{F} (Corollary 3.71). That is: subfields of \mathcal{H} closed under both E and Δ are in bijection with subgroups of H defined over \mathcal{C} by E -equations.

In addition to proving the fundamental theorems of a Galois theory, this paper will show that each differential algebraic group is the Galois group of some generalized strongly normal differential field extension (Theorem 3.66). Then there is a short section on the generalized strongly normal extensions that are induced from strongly normal extensions (Section 3.5). At the end of the paper, examples of generalized strongly normal field extensions are constructed for each differential algebraic subgroup of G_a and G_m . This section is dependent on ideas of Johnson, Reinhart and Rubel [4], which are developed in an appendix. Examples with non-linear Galois groups will appear in another paper, and the geometric consequences of this Galois theory is a work in progress.

Several other people have developed Galois theories of differential fields: Drach [3], Vessiot [24], Pommaret [19], Umemura ([22], [23]), Pillay ([16], [17], [18]) and Kovacic [6]. Although the Galois groups of Pillay's theory are differential groups, they are only algebraically finite dimensional. Since a differential algebraic group may have infinite algebraic dimension (even though its differential dimension is finite), the Galois theory developed here includes infinite dimensional groups.

One may speculate as to why this generalized Galois theory was not previously realized. One reason may be that since the simplest new examples are of infinite algebraic dimension the symmetries are difficult to intuit. Also, because the finite dimensional examples all necessitate two commuting derivations, those working in Picard-Vessiot theory do not usually work with two derivations since Kolchin showed that the Picard-Vessiot theory with several derivations is subsumed in that with one derivation [8].

I wish I could thank Professor Kolchin for teaching me his special field of expertise. I also wish to acknowledge the assistance and encouragement of Professors Phyllis Cassidy, Richard C. Churchill, Jerold Kovacic and William Sit, who sat through a series of lectures in 2001–2004 during which I explained the theory presented in this paper. Cassidy and Singer used these ideas to write an exposition of the linear case [2] where they cite this work as my forthcoming thesis.

A SIMPLE EXAMPLE

The following finite dimensional example will serve to further elucidate the nature of the generalization herein and to exhibit its relationship with the standard Picard-Vessiot. Let \mathbb{C} be the complex numbers, and let $\mathbb{C}[t, x]$ be a polynomial ring in two variables with standard derivations D_t and D_x . Consider $\mathcal{F} = \mathbb{C}(t, x, \cos t, \sin t)$ and $\mathcal{G} = \mathcal{F}(\log x \sin t) = \mathcal{F}(\log x)$ as differential fields with

respect to D_t and D_x . Let $\mathcal{C} = \mathbb{C}(t, \cos t, \sin t)$ be the field of D_x -constants of \mathcal{F} , and let \mathcal{U}^{D_x} be the same of \mathcal{U} . Note that in this example the D_t -field generated by \mathcal{G} is \mathcal{G} , and the field of D_x -constants of \mathcal{G} equals that of \mathcal{F} .

Let $\eta = \log x \sin t$, and $\zeta = \sin t/x \in \mathcal{F}$. Then η satisfies the equation $D_x\eta = \zeta$, and \mathcal{G} as a D_x -extension over \mathcal{F} is a strongly normal extension in the sense of Kolchin. The Galois group $\text{Isom}^{D_x}(\mathcal{G}/\mathcal{F}) = \text{Aut}^{D_x}(\mathcal{G}\mathcal{U}^{D_x}/\mathcal{F}\mathcal{U}^{D_x})$ is isomorphic to the additive group \mathcal{U}^{D_x} , is defined over \mathcal{C} , and will be denoted by G_a via this identification. More explicitly, consider $\sigma \in \text{Aut}^{D_x}(\mathcal{G}/\mathcal{F})$. Then, in order for σ to commute with D_x , $\sigma\eta$ must again be a solution to this differential equation, and therefore $\sigma\eta$ must equal $\eta + \rho(\sigma)$, where $\rho(\sigma) \in \mathcal{U}^{D_x}$. There being no other algebraic conditions on $\rho(\sigma)$, the map $\rho : \sigma \mapsto \rho(\sigma)$ defines a group isomorphism between $\text{Isom}^{D_x}(\mathcal{G}/\mathcal{F})$ and the full algebraic group G_a .

Let $\gamma = \cos t/\sin t \in \mathcal{F}$. Then η also satisfies the differential equation $D_t\eta - \gamma\eta = 0$. Indeed, in the (D_x, D_t) -differential polynomial ring $\mathcal{F}\{y\}$, it is easy to verify that the two differential polynomials $A(y) = D_x y - \zeta$ and $B(y) = D_t y - \gamma y$ form a characteristic set of a linear differential ideal $\mathfrak{P} = [A(y), B(y)]$ (relative to any ranking) with η as a generic zero over \mathcal{F} . Consider σ in the subgroup $H = \text{Isom}^{D_t, D_x}(\mathcal{G}/\mathcal{F}) = \text{Aut}^{D_t, D_x}(\mathcal{G}\mathcal{U}^{D_x}/\mathcal{F}\mathcal{U}^{D_x})$ of $\text{Isom}^{D_x}(\mathcal{G}/\mathcal{F}) = G_a$. Then σ must map η to a generic solution of \mathfrak{P} . Thus $0 = B(\sigma(\eta)) = B(\eta + \rho(\sigma)) = B(\rho(\sigma))$, which implies that $\rho(\sigma) = c(\sigma) \sin t$, where $c(\sigma)$ is a constant with respect to D_t . But $\rho(\sigma)$ is a D_x -constant, and so $c(\sigma)$ must be one, too. Conversely, it is clear that given any D_t -constant $k \in \mathcal{U}^{D_x}$, the map σ where $\sigma(\eta) = \eta + k \sin t$ is the unique isomorphism of \mathcal{G} over \mathcal{F} with $c(\sigma) = k$. Therefore, $\rho(H)$ is a differential algebraic subgroup of G_a defined over the D_x -constant field $\mathcal{D} = \mathbb{C}(t, \cos t, \sin t)$ of \mathcal{F} by the prime differential ideal $[B(y)]$ in the D_t -differential polynomial ring $\mathcal{D}\{y\}$, or equivalently, the prime differential ideal $[D_x y, B(y)]$ in the (D_x, D_t) -differential polynomial ring $\mathcal{F}\{y\}$. See a proof just after Proposition 4.80.

1. GROUP OF ISOMORPHISMS

1.1. Notation. To define the category of differential rings, as developed by Ritt and Kolchin, fix a set $\Delta = \{\delta_1, \dots, \delta_m\}$. The objects, called Δ -rings or *differential rings*, are rings on which the set Δ acts as commuting derivations. The morphisms, called Δ -homomorphisms or *differential homomorphisms*, are ring homomorphisms that commute with the action of Δ . Many terms of algebra, such as ideal, field and extension, have straightforward interpretations in the category of Δ -rings and are indicated by the modifier “ Δ ” or “differential”. However, “ Δ -embeddings” are referred to as “ Δ -isomorphisms”, and the now standard term “radical ideal” is used in place of Kolchin’s “perfect ideal”.

Henceforth, all rings are assumed to have characteristic zero. Throughout this chapter, the set of commuting derivations $\Delta = \{\delta_1, \dots, \delta_m\}$ is fixed, and \mathcal{F} is a Δ -field.

Standard notation will now be reviewed from [12]. The Δ -polynomial algebra $\mathcal{F}\{y_1, \dots, y_n\}_\Delta$ over \mathcal{F} in Δ -indeterminates y_1, \dots, y_n is the polynomial ring over \mathcal{F} having one indeterminate for each derivative of y_1, \dots, y_n on which Δ operates in the expected manner. (For details see Kolchin [12, pages 69-71].) If S is a subset of $\mathcal{F}\{y_1, \dots, y_n\}_\Delta$, the Δ -ideal generated by S is denoted by $[S]_\Delta$ (or $[s_1, \dots, s_n]_\Delta$ if $S = \{s_1, \dots, s_n\}$), and the radical Δ -ideal generated by S will be denoted by $\{S\}_\Delta$. Let \mathcal{G} be a Δ -field that is a Δ -extension of \mathcal{F} , and let T be a subset of \mathcal{G} . The Δ -ring

generated by T over \mathcal{F} is denoted by $\mathcal{F}\{T\}_\Delta$ (or $\mathcal{F}\{t_1, \dots, t_n\}_\Delta$ if $T = \{t_1, \dots, t_n\}$), and the Δ -field generated by T over \mathcal{F} is denoted by $\mathcal{F}\langle T \rangle_\Delta$ (or $\mathcal{F}\langle t_1, \dots, t_n \rangle_\Delta$ if $T = \{t_1, \dots, t_n\}$). If T is a finite set, the Δ -ring $\mathcal{F}\{t_1, \dots, t_n\}_\Delta$ and the Δ -field $\mathcal{F}\langle t_1, \dots, t_n \rangle_\Delta$ are said to be *finitely Δ -generated by T over \mathcal{F}* , or, for simplicity, *Δ - \mathcal{F} -finitely generated*. If R is any Δ -ring, the symbol R^Δ denotes the constants of R with respect to Δ , i.e. the elements α of R such that $\delta\alpha = 0$ for every $\delta \in \Delta$.

A Δ -field \mathcal{U} containing a Δ -subfield \mathcal{F} is called *Δ -universal over \mathcal{F}* if the following conditions hold: for each Δ -field \mathcal{G} of \mathcal{U} finitely Δ -generated over \mathcal{F} and for each Δ -field \mathcal{H} (not necessarily contained in \mathcal{U}) finitely Δ -generated over \mathcal{G} , there exists a Δ -isomorphism of \mathcal{H} into \mathcal{U} over \mathcal{G} . The existence of a Δ -universal Δ -extension of any Δ -field is established by Kolchin in [12, Theorem 2, page 134]. Such an extension contains all the solutions to differential equations over \mathcal{F} necessary in Kolchin’s work.

1.2. Specializations. In this section, let \mathcal{F} be a Δ -field, and let \mathcal{U} be a Δ -extension of \mathcal{F} that is Δ -universal over \mathcal{F} . Let \mathcal{G} be a Δ -extension of \mathcal{F} in \mathcal{U} over which \mathcal{U} is universal.

Definition 1.1 (Pre-orders on \mathcal{U}^r). For $\eta = (\eta_1, \dots, \eta_r)$ and $\xi = (\xi_1, \dots, \xi_r)$ in \mathcal{U}^r , define the pre-order by $\eta \xrightarrow{\mathcal{G}} \xi$, called *Δ -specialization over \mathcal{G}* or *Δ - \mathcal{G} -specialization*, if there exists a Δ - \mathcal{G} -homomorphism of $\mathcal{G}\{\eta_1, \dots, \eta_r\}_\Delta$ to $\mathcal{G}\{\xi_1, \dots, \xi_r\}_\Delta$ over \mathcal{G} taking η_i to ξ_i for $i = 1, \dots, r$.

Definition 1.2 (Δ - \mathcal{G} -Specialization of Δ - \mathcal{F} -Isomorphisms). Let $X = \text{Isom}_{\mathcal{F}}^\Delta(\mathcal{G}, \mathcal{U})$. On the set $X^r = X \times \dots \times X$ define a pre-order $\xrightarrow{\mathcal{G}}$ (or, for simplicity, \rightarrow) called *Δ - \mathcal{G} -specialization* (of elements of X^r) as follows: for $\sigma = (\sigma_1, \dots, \sigma_r)$ and $\tau = (\tau_1, \dots, \tau_r) \in X^r$, $\sigma \xrightarrow{\mathcal{G}} \tau$ if there exists a Δ - \mathcal{G} -homomorphism $\phi : \mathcal{G}\{\sigma_1\mathcal{G} \cup \dots \cup \sigma_r\mathcal{G}\}_\Delta \rightarrow \mathcal{G}\{\tau_1\mathcal{G} \cup \dots \cup \tau_r\mathcal{G}\}_\Delta$ such that $\phi(\alpha) = \alpha$ and $\phi(\sigma_i\alpha) = \tau_i\alpha$ for all α in \mathcal{G} and $i = 1, \dots, r$.

In the above definition, note that the Δ -rings $\mathcal{G}\{\sigma_1\mathcal{G} \cup \dots \cup \sigma_r\mathcal{G}\}_\Delta \subset \mathcal{U}$ and $\mathcal{G}\{\tau_1\mathcal{G} \cup \dots \cup \tau_r\mathcal{G}\}_\Delta \subset \mathcal{U}$ are the same as the rings $\mathcal{G}[\sigma_1\mathcal{G} \cup \dots \cup \sigma_r\mathcal{G}]$ and $\mathcal{G}[\tau_1\mathcal{G} \cup \dots \cup \tau_r\mathcal{G}]$, so that ϕ is in fact a Δ - \mathcal{G} -homomorphism from $\mathcal{G}[\sigma_1\mathcal{G} \cup \dots \cup \sigma_r\mathcal{G}]$ to $\mathcal{G}[\tau_1\mathcal{G} \cup \dots \cup \tau_r\mathcal{G}]$. Also, since $\tau_i \circ \sigma_i^{-1}$ is a Δ - \mathcal{F} -isomorphism for all i , ϕ is a Δ - \mathcal{G} -homomorphism if and only if it is a \mathcal{G} -homomorphism (see [12, Lemma 1, page 385]).

Lemma 1.3. Let $\eta = (\eta_1, \dots, \eta_r)$ be a set of Δ -generators of \mathcal{G} over \mathcal{F} . For $\sigma, \tau \in X^r$, $\sigma \xrightarrow{\mathcal{G}} \tau$ as in Definition 1.2 if and only if $(\dots, \sigma_i\eta_j, \dots) \xrightarrow{\mathcal{G}} (\dots, \tau_i\eta_j, \dots)$ as in Definition 1.1.

Proof. (See [12, Lemma 2, page 386] for a statement of the same lemma without a proof.) If $\sigma \xrightarrow{\mathcal{G}} \tau$, the Δ - \mathcal{G} -homomorphism $\phi : \mathcal{G}\{\sigma_1\mathcal{G} \cup \dots \cup \sigma_r\mathcal{G}\}_\Delta \rightarrow \mathcal{G}\{\tau_1\mathcal{G} \cup \dots \cup \tau_r\mathcal{G}\}_\Delta$ of Definition 1.2 restricts to a Δ - \mathcal{G} -homomorphism $\rho : \mathcal{G}\{\dots, \sigma_i\eta_j, \dots\}_\Delta \rightarrow \mathcal{G}\{\dots, \tau_i\eta_j, \dots\}_\Delta$, and $(\dots, \sigma_i\eta_j, \dots) \xrightarrow{\mathcal{G}} (\dots, \tau_i\eta_j, \dots)$.

On the other hand, if $(\dots, \sigma_i\eta_j, \dots) \xrightarrow{\mathcal{G}} (\dots, \tau_i\eta_j, \dots)$, then there is a Δ - \mathcal{G} -homomorphism $\rho : \mathcal{G}\{\dots, \sigma_i\eta_j, \dots\}_\Delta \rightarrow \mathcal{G}\{\dots, \tau_i\eta_j, \dots\}_\Delta$. Let \mathcal{I} be the kernel of ρ . Since the image of ρ is in \mathcal{U} and, therefore, an integral domain, \mathcal{I} is a prime Δ -ideal. Let $\mathcal{G}\{\dots, \sigma_i\eta_j, \dots\}_{\Delta, \mathcal{I}}$ be the localization of $\mathcal{G}\{\dots, \sigma_i\eta_j, \dots\}_\Delta$ at \mathcal{I} , and

let the induced Δ - \mathcal{G} -homomorphism of $\mathcal{G}\{\dots, \sigma_i \eta_j, \dots\}_{\Delta, \mathcal{J}}$ into the quotient field of $\mathcal{G}\{\dots, \tau_i \eta_j, \dots\}_{\Delta}$ be

$$\bar{\rho}: \mathcal{G}\{\dots, \sigma_i \eta_j, \dots\}_{\Delta, \mathcal{J}} \rightarrow QF(\mathcal{G}\{\dots, \tau_i \eta_j, \dots\}_{\Delta}).$$

The Δ - \mathcal{G} -homomorphism $\bar{\rho}$ restricted to $\mathcal{F}\{\sigma_i \eta_1, \dots, \sigma_i \eta_n\}_{\Delta}$ is the Δ - \mathcal{F} -isomorphism $\tau_i \circ \sigma_i^{-1}: \sigma_i \mathcal{G} \rightarrow \tau_i \mathcal{G}$ restricted to $\mathcal{F}\{\sigma_i \eta_1, \dots, \sigma_i \eta_n\}_{\Delta}$. Therefore, $\bar{\rho}$ restricted to $\mathcal{F}\{\sigma_i \eta_1, \dots, \sigma_i \eta_n\}_{\Delta}$ is a Δ - \mathcal{F} -isomorphism. Consequently, $\mathcal{F}\{\sigma_i \eta_1, \dots, \sigma_i \eta_n\}_{\Delta} \cap \mathcal{J} = \{0\}_{\Delta}$, and the nonzero elements of $\mathcal{F}\{\sigma_i \eta_1, \dots, \sigma_i \eta_n\}_{\Delta}$ are invertible in $\mathcal{G}\{\dots, \sigma_i \eta_j, \dots\}_{\Delta, \mathcal{J}}$, i.e., $\sigma_i \mathcal{G} \subseteq \mathcal{G}\{\dots, \sigma_i \eta_j, \dots\}_{\Delta, \mathcal{J}}$ for all i . Since η Δ -generates \mathcal{G} over \mathcal{F} , both $\bar{\rho}$ and $\tau_i \circ \sigma_i^{-1}$ coincide on $\sigma_i \mathcal{G} \subseteq \mathcal{G}\{\dots, \sigma_i \eta_j, \dots\}_{\Delta, \mathcal{J}}$. Therefore $\bar{\rho}$ restricted to $\mathcal{G}\{\sigma_1 \mathcal{G} \cup \dots \cup \sigma_r \mathcal{G}\}_{\Delta}$ is a Δ - \mathcal{G} -homomorphism $\phi: \mathcal{G}\{\sigma_1 \mathcal{G} \cup \dots \cup \sigma_r \mathcal{G}\}_{\Delta} \rightarrow \mathcal{G}\{\tau_1 \mathcal{G} \cup \dots \cup \tau_r \mathcal{G}\}_{\Delta}$ such that $\phi(\alpha) = \alpha$ and $\phi(\sigma_i \alpha) = \tau_i \alpha$ for all α in \mathcal{G} and $i = 1, \dots, r$. Thus, ρ may be extended to the Δ - \mathcal{G} -homomorphism ϕ . By the definition of Δ - \mathcal{G} -specialization of elements of X^r (Definition 1.2), $\sigma \xrightarrow{\mathcal{G}} \tau$. \square

1.3. E-strong isomorphisms. Denote by “ (E, Δ) ” the union of two disjoint sets $E = \{\epsilon_1, \dots, \epsilon_r\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$. However, when this symbol is used as a subscript or superscript the parentheses are removed, e.g., $\mathcal{F}\{y\}_{E, \Delta}$ or $\mathcal{F}^{E, \Delta}$. In this section, \mathcal{F} will denote an (E, Δ) -field, \mathcal{U} an (E, Δ) -field that is (E, Δ) -universal over \mathcal{F} , and \mathcal{C} the Δ -constants of \mathcal{F} . Then $\mathcal{K} = \mathcal{U}^{\Delta}$ may be considered as an E-field. As such, it is E-universal over \mathcal{C} , considered as an E-field. The (E, Δ) -field $\mathcal{G} \subset \mathcal{U}$ will contain \mathcal{F} . If \mathcal{F} and \mathcal{G} are fields contained in a larger field, then $\mathcal{F} \cdot \mathcal{G}$, or more simply $\mathcal{F}\mathcal{G}$, will denote their compositum.

Definition 1.4. Let \mathcal{G} be an (E, Δ) -subfield of \mathcal{U} . An (E, Δ) -isomorphism σ of \mathcal{G} into \mathcal{U} is E-strong if it satisfies the following two conditions:

- (St1) σ leaves invariant every element of \mathcal{G}^{Δ} ;
- (St2) $\sigma \mathcal{G} \subset \mathcal{G} \cdot \mathcal{U}^{\Delta}$ and $\mathcal{G} \subset \sigma \mathcal{G} \cdot \mathcal{U}^{\Delta}$.

An E-strong (E, Δ) -isomorphism is the same as an E-homomorphism which is also a strong Δ -isomorphism in the sense defined by Kolchin in [12, p. 388]. Because of this, some of the proofs in this chapter can often simply quote the results of Kolchin, and, if E is empty, many results of this paper are those of Kolchin.

Note that (St2) is equivalent to $\mathcal{G} \cdot \mathcal{U}^{\Delta} = \sigma \mathcal{G} \cdot \mathcal{U}^{\Delta}$. Also it is clear that any (E, Δ) -automorphism of \mathcal{G} over \mathcal{G}^{Δ} is an E-strong (E, Δ) -isomorphism. For any (E, Δ) -isomorphism σ of \mathcal{G} , let $\mathcal{G}^{\Delta} \langle \sigma \rangle = (\mathcal{G}\sigma \mathcal{G})^{\Delta}$. The first inclusion of (St2) is equivalent to $\mathcal{G}\sigma \mathcal{G} \subset \mathcal{G} \cdot \mathcal{U}^{\Delta}$, which by [12, Corollary 2, p. 88] is equivalent to $\mathcal{G}\sigma \mathcal{G} = \mathcal{G} \cdot \mathcal{G}^{\Delta} \langle \sigma \rangle$. Similarly the second inclusion is equivalent to $\mathcal{G}\sigma \mathcal{G} = \sigma \mathcal{G} \cdot \mathcal{G}^{\Delta} \langle \sigma \rangle$.

If \mathcal{G} is an arbitrary (E, Δ) -extension of \mathcal{F} , it may happen that not all elements σ of $\text{Isom}_{\mathcal{F}}^{E, \Delta}(\mathcal{G}, \mathcal{U})$ are E-strong (E, Δ) -isomorphisms [14, Example 3.147]. However, if there is one E-strong (E, Δ) -isomorphism, the next proposition shows that all its (E, Δ) - \mathcal{G} -specializations are E-strong (E, Δ) -isomorphisms.

Proposition 1.5. *Every (E, Δ) - \mathcal{G} -specialization of an E-strong (E, Δ) -isomorphism of \mathcal{G} is E-strong.*

Proof. Let σ' be an (E, Δ) - \mathcal{G} -specialization of the E-strong (E, Δ) -isomorphism σ of \mathcal{G} . By the definition of (E, Δ) - \mathcal{G} -specialization (Example 1.2), σ' is an (E, Δ) -isomorphism. Now σ is a strong Δ -isomorphism, and hence σ' is also a strong Δ -isomorphism by [12, Proposition 6, p. 390]. Since σ' is an E-homomorphism, σ' is an E-strong (E, Δ) -isomorphism. \square

The following propositions will be used to verify under certain conditions in Theorem 1.24 that the set of E-strong (E, Δ) -isomorphisms of \mathcal{G} over \mathcal{F} verify the axioms of an E-group.

Proposition 1.6. *Let \mathcal{G} be a finitely (E, Δ) -generated (E, Δ) -extension of \mathcal{F} . Then for every E-strong (E, Δ) -isomorphism σ of \mathcal{G} over \mathcal{F} , $\mathcal{G}^\Delta\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^\Delta$ is a finitely E-generated field extension of the E-field \mathcal{G}^Δ .*

Proof. Let $\eta = (\eta_1, \dots, \eta_n)$ be a finite family of (E, Δ) -generators of \mathcal{G} over \mathcal{F} . Let σ be an E-strong (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} . The extension $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\sigma\eta\rangle_{E,\Delta}$ of \mathcal{G} is a finitely (E, Δ) -generated extension by $\sigma\eta$. Let $\xi = (\xi_i)_{i \in I}$ be a family of E-generators of $\mathcal{G}^\Delta\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^\Delta$ over \mathcal{G}^Δ . Since $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{G}^\Delta\langle\sigma\rangle = \mathcal{G}\langle\xi\rangle$, the family ξ also E-generates $\mathcal{G}\sigma\mathcal{G}$ over \mathcal{G} :

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\xi\rangle \\ \uparrow & & \uparrow \\ \mathcal{G}^\Delta & \longrightarrow & \mathcal{G}^\Delta\langle\sigma\rangle = \mathcal{G}^\Delta\langle\xi\rangle. \end{array}$$

Because this extension is (E, Δ) -finitely generated over \mathcal{G} by $\sigma\eta$ and each element of $\sigma\eta$ is in an E-field generated by finitely many of the elements of the family ξ , there is a finite subfamily (ξ_1, \dots, ξ_m) of the family ξ that (E, Δ) -generates $\mathcal{G}\sigma\mathcal{G}$ over \mathcal{G} : that is, $\mathcal{G}\mathcal{G}^\Delta\langle\sigma\rangle = \mathcal{G}\langle\xi_1, \dots, \xi_m\rangle_{E,\Delta} = \mathcal{G} \cdot \mathcal{G}^\Delta\langle\xi_1, \dots, \xi_m\rangle_{E,\Delta}$. Since the elements of ξ are Δ -constants, $\mathcal{G}^\Delta\langle\sigma\rangle = \mathcal{G}^E\langle\xi_1, \dots, \xi_m\rangle$ by [12, Corollary 2, p. 88]. \square

Proposition 1.7. *Let σ and τ be two E-strong (E, Δ) -isomorphisms of \mathcal{G} . Then $\mathcal{G}^\Delta\langle\sigma\rangle\mathcal{G}^\Delta\langle\sigma\tau\rangle = \mathcal{G}^\Delta\langle\sigma\rangle\mathcal{G}^\Delta\langle\tau\rangle = \mathcal{G}^\Delta\langle\sigma\tau\rangle\mathcal{G}^\Delta\langle\tau\rangle$, and $\mathcal{G}^\Delta\langle\sigma^{-1}\rangle = \mathcal{G}^\Delta\langle\sigma\rangle$ as E-fields.*

Proof. By considering the fields in the statement of the proposition as just Δ -fields, and σ and τ as just strong Δ -isomorphisms, Kolchin’s result [12, Proposition 5, p. 390] may be applied to obtain these equalities as fields in \mathcal{U} . Because they are also E-fields, they are equal as E-fields. \square

Proposition 1.8. *Let $\sigma, \sigma', \tau, \tau'$ be E-strong (E, Δ) -isomorphisms of \mathcal{G} .*

- (1) *If (σ', τ') is a specialization of (σ, τ) , then $(\sigma'^{-1}, \sigma'^{-1}\tau')$ is a specialization of $(\sigma^{-1}, \sigma^{-1}\tau)$.*
- (2) *Suppose that σ' and τ' are generic specializations of σ and τ , respectively. If (σ', τ') is a specialization of (σ, τ) , then the induced E-isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ are compatible, and conversely.*
- (3) *Suppose that σ' and τ' are generic specializations of σ and τ , respectively. Let $h : \mathcal{D} \rightarrow \mathcal{D}'$ be an E-homomorphism between subrings of \mathcal{U}^Δ . If h and the induced E-isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ are compatible, then σ'^{-1} is a generic specialization of σ^{-1} and $\sigma'^{-1}\tau'$ is a specialization of $\sigma^{-1}\tau$; when the latter specialization is generic, then h and the induced E-isomorphisms $\mathcal{G}^\Delta\langle\sigma^{-1}\rangle \approx \mathcal{G}^\Delta\langle\sigma'^{-1}\rangle$ and $\mathcal{G}^\Delta\langle\sigma^{-1}\tau\rangle \approx \mathcal{G}^\Delta\langle\sigma'^{-1}\tau'\rangle$ are compatible.*

Proof. Since σ, σ', τ and τ' are E-strong (E, Δ) -isomorphisms, it follows from Kolchin’s corresponding result [12, Proposition 8(a), page 391] for strong Δ -isomorphisms that $(\sigma'^{-1}, \sigma'^{-1}\tau')$ is a specialization of $(\sigma^{-1}, \sigma^{-1}\tau)$ over \mathcal{G} . This remains a specialization over \mathcal{G} when σ, σ', τ and τ' are considered as (E, Δ) -isomorphisms by [12, Lemma 1, page 385].

Part 2 is proved in a manner similar to that of part 1: (σ', τ') is a specialization of (σ, τ) , when σ, σ', τ and τ' are considered as strong Δ -isomorphisms if and only if the induced isomorphisms considered as nondifferential isomorphisms are compatible. The result then follows when σ, σ', τ and τ' are again considered as (E, Δ) -isomorphisms.

Part 3 follows from the same considerations as in the previous part. □

Corollary 1.9. (1) *If σ' is a specialization of σ , then σ'^{-1} is a specialization of σ^{-1} . When the former specialization is generic, then so is the latter, and the induced isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\sigma^{-1}\rangle \approx \mathcal{G}^\Delta\langle\sigma'^{-1}\rangle$ coincide.*

(2) *Suppose that σ' and τ' are generic specializations of σ and τ , respectively, such that the induced E-isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ are compatible. Then $\sigma'\tau'$ is a specialization of $\sigma\tau$. When the last specialization is generic, and $h : \mathcal{D} \rightarrow \mathcal{D}'$ is an E-homomorphism between subrings of \mathcal{U}^Δ such that h and the induced E-isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\tau\rangle \approx \mathcal{G}^\Delta\langle\tau'\rangle$ are compatible, then h and the induced E-isomorphism $\mathcal{G}^\Delta\langle\sigma\tau\rangle \approx \mathcal{G}^\Delta\langle\sigma'\tau'\rangle$ are compatible.*

Proof. The first assertion follows from part 1 of the proposition, in the special case in which $\tau = \sigma, \tau' = \sigma'$. Since $\mathcal{G}^\Delta\langle\sigma^{-1}\rangle = \mathcal{G}^\Delta\langle\sigma\rangle$ (Proposition 1.7), the second assertion follows from part 3 of the proposition, in the special case in which $\tau = \sigma, \tau' = \sigma'$, and h is the induced E-isomorphism $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$.

Because of part 1, one may replace σ, σ' by $\sigma^{-1}, \sigma'^{-1}$. Part 2 then follows from part 3 of the proposition. □

1.4. E-strongly normal extensions.

Definition 1.10. An E-strongly normal extension \mathcal{G} of the (E, Δ) -field \mathcal{F} is a finitely (E, Δ) -generated extension \mathcal{G} of \mathcal{F} such that every (E, Δ) - \mathcal{F} -isomorphism of \mathcal{G} is E-strong (Definition 1.4).

Remark 1.11. If \mathcal{G} over \mathcal{F} is E-strongly normal, it is not necessarily a strongly normal extension for Δ because \mathcal{G} over \mathcal{F} might not be finitely Δ -generated. A strongly normal extension for (E, Δ) is an E-strongly normal extension if each (E, Δ) -isomorphism leaves invariant not only every element of $\mathcal{G}^{E, \Delta}$ but also those of \mathcal{G}^Δ .

Proposition 1.12. *If \mathcal{G} is an E-strongly normal extension of \mathcal{F} , then \mathcal{F} and \mathcal{G} have the same field of Δ -constants.*

Proof. By Definition 1.4 (St1), the Δ -constants in \mathcal{G} are invariant under every isomorphism of \mathcal{G} over \mathcal{F} . Since any element of \mathcal{G} fixed by all E- \mathcal{F} -isomorphisms of \mathcal{G} is in \mathcal{F} [12, Corollary, page 388], the Δ -constants of \mathcal{G} are contained in \mathcal{F} . □

Proposition 1.13. *Let \mathcal{G} be a finitely (E, Δ) -generated extension of \mathcal{F} having the same field of Δ -constants as \mathcal{F} . Let $\sigma_1, \dots, \sigma_r$ be (E, Δ) - \mathcal{F} -isomorphisms of \mathcal{G} such that every (E, Δ) - \mathcal{F} -isomorphism of \mathcal{G} is an (E, Δ) - \mathcal{G} -specialization of one of these. If $\sigma_k \mathcal{G} \subset \mathcal{G}\mathcal{U}^\Delta$ for all k ($1 \leq k \leq r$), then \mathcal{G} is E-strongly normal over \mathcal{F} .*

Proof. Let σ be any (E, Δ) - \mathcal{F} -isomorphism of \mathcal{G} . Since $\mathcal{G}^\Delta = \mathcal{F}^\Delta$, σ fixes \mathcal{G}^Δ . By considering σ as a Δ -homomorphism and the remark after [12, Proposition 6, page 390], $\sigma \mathcal{G} \subset \mathcal{G}\mathcal{U}^\Delta$ since $\sigma_i \mathcal{G} \subset \mathcal{G}\mathcal{U}^\Delta$.

To prove that σ is E-strong, it remains to show $\mathcal{G} \subset \sigma\mathcal{G}\mathcal{U}^\Delta$. Following the technique of the proof in [12, Proposition 10, page 393], one may show that the (E, Δ) - \mathcal{F} -isomorphism $\sigma^{-1} : \sigma\mathcal{G} \approx \mathcal{G}$ can be extended to an (E, Δ) - \mathcal{F} -isomorphism φ of $\mathcal{G}\sigma\mathcal{G}$ because \mathcal{U} is (E, Δ) -universal over \mathcal{F} . The restriction of φ to \mathcal{G} is an (E, Δ) - \mathcal{F} -isomorphism τ of \mathcal{G} over \mathcal{F} . Thus, $\varphi : \mathcal{G}\sigma\mathcal{G} \approx \tau\mathcal{G} \cdot \mathcal{G}$ is an (E, Δ) - \mathcal{F} -isomorphism, $\varphi\mathcal{G} = \tau\mathcal{G}$, $\varphi(\sigma\mathcal{G}) = \mathcal{G}$, and $\varphi(\mathcal{G}^\Delta\langle\sigma\rangle) = \mathcal{G}^\Delta\langle\tau\rangle$. By the final result of the last paragraph, $\tau\mathcal{G} \subseteq \mathcal{G}\mathcal{G}^\Delta\langle\tau\rangle$. Therefore $\mathcal{G} = \varphi^{-1}(\tau\mathcal{G}) \subseteq \varphi^{-1}(\mathcal{G}\mathcal{G}^\Delta\langle\tau\rangle) = \varphi^{-1}\mathcal{G} \cdot \varphi^{-1}(\mathcal{G}^\Delta\langle\tau\rangle) = \sigma\mathcal{G} \cdot \mathcal{G}^\Delta\langle\sigma\rangle \subset \sigma\mathcal{G}\mathcal{U}^\Delta$. \square

Corollary 1.14. *Let \mathcal{G}_1 and \mathcal{G}_2 be extensions of \mathcal{F} such that $\mathcal{G}_1\mathcal{G}_2$ has the same field of Δ -constants as \mathcal{F} . If \mathcal{G}_1 and \mathcal{G}_2 are E-strongly normal over \mathcal{F} , then so is $\mathcal{G}_1\mathcal{G}_2$.*

Proof. Obviously $\mathcal{G}_1\mathcal{G}_2$ is a finitely (E, Δ) -generated extension of \mathcal{F} . If σ is any isomorphism of $\mathcal{G}_1\mathcal{G}_2$ over \mathcal{F} , then the restriction σ_i of σ to \mathcal{G}_i is an E-strong (E, Δ) -isomorphism of \mathcal{G}_i so that $\sigma(\mathcal{G}_1\mathcal{G}_2) = \sigma_1\mathcal{G}_1 \cdot \sigma_2\mathcal{G}_2 \subset \mathcal{G}_1\mathcal{U}^\Delta \cdot \mathcal{G}_2\mathcal{U}^\Delta = (\mathcal{G}_1\mathcal{G}_2) \cdot \mathcal{U}^\Delta$. It follows by Proposition 1.13 that $\mathcal{G}_1\mathcal{G}_2$ is an E-strongly normal extension of \mathcal{F} . \square

Proposition 1.15. *Let \mathcal{G} be any Δ -field in \mathcal{U} . Each E-strong (E, Δ) -isomorphism of \mathcal{G} can be extended to a unique (E, Δ) -automorphism of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ . Conversely, the restriction to \mathcal{G} of each (E, Δ) -automorphism of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ is an E-strong (E, Δ) -isomorphism of \mathcal{G} .*

Proof. By [12, Corollary 1, page 87], \mathcal{G} and \mathcal{U}^Δ are linearly disjoint over \mathcal{G}^Δ . Also, if σ is any E-strong (E, Δ) -isomorphism of \mathcal{G} (Definition 1.4), then $\sigma\mathcal{G}$ and \mathcal{U}^Δ are also linearly disjoint over \mathcal{G}^Δ . Therefore σ can be extended to a unique (E, Δ) -isomorphism $s : \mathcal{G}\mathcal{U}^\Delta \approx \sigma\mathcal{G} \cdot \mathcal{U}^\Delta$ over \mathcal{U}^Δ . Because σ is an E-strong (E, Δ) -isomorphism, $\sigma\mathcal{G}\mathcal{U}^\Delta = \mathcal{G}\mathcal{U}^\Delta$, and s is an (E, Δ) -automorphism of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ . The converse is clear. \square

This proposition canonically identifies the set of all E-strong (E, Δ) -isomorphisms of \mathcal{G} with the set of all (E, Δ) -automorphisms of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ . Because the set of all (E, Δ) -automorphisms of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ has a natural group structure, this identification induces a group structure on the set of all E-strong (E, Δ) -isomorphisms of \mathcal{G} . If \mathcal{F} is an (E, Δ) -subfield of \mathcal{G} , the set of all E-strong (E, Δ) -isomorphisms of \mathcal{G} over \mathcal{F} can be canonically identified with the group G of all (E, Δ) -automorphisms of $\mathcal{G}\mathcal{U}^\Delta$ over $\mathcal{F}\mathcal{U}^\Delta$, which is a subgroup of the group of all (E, Δ) -automorphisms of $\mathcal{G}\mathcal{U}^\Delta$ over \mathcal{U}^Δ .

Recall the definitions of the E-type, E-dimension and typical E-dimension of a pre E-set in [13, page 31]. If \mathcal{H} over \mathcal{F} (considered as an E-field) is an E-extension that is finitely E-generated by $\rho = (\rho_1, \dots, \rho_n)$, $\omega_{\rho/\mathcal{F}}$ will denote the E-transcendence polynomial of ρ over \mathcal{F} [12, page 117].

Proposition 1.16. *Let \mathcal{G} be an E-strongly normal extension of \mathcal{F} , and let \mathcal{C} denote the field of Δ -constants of \mathcal{F} . For every isomorphism σ of \mathcal{G} over \mathcal{F} , define $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^\Delta$. Then $\mathcal{C}\langle\sigma\rangle$, as an E-field extension of \mathcal{C} , is finitely E-generated over \mathcal{C} . Moreover, \mathcal{G} is finitely E-generated over \mathcal{F} , and, for every isolated isomorphism σ of \mathcal{G} over \mathcal{F} , the E-type (resp. E-dimension, typical E-dimension) of $\mathcal{C}\langle\sigma\rangle$ over \mathcal{C} is equal to the E-type (resp. E-dimension, typical E-dimension) of \mathcal{G} over \mathcal{F} .*

Proof. That $\mathcal{C}\langle\sigma\rangle$ is a finitely E -generated field extension of \mathcal{C} for every isomorphism σ of \mathcal{G} over \mathcal{F} is Proposition 1.6. To show \mathcal{G} is finitely E -generated over \mathcal{F} , let σ be an isolated (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} that specializes to the identity isomorphism. By part b of [13, Corollary, page 388], σ leaves fixed the algebraic closure \mathcal{F}° of \mathcal{F} in \mathcal{G} . Let $\eta = (\eta_1, \dots, \eta_n)$ be a family of (E, Δ) -generators of \mathcal{G} over \mathcal{F} , i.e. $\mathcal{G} = \mathcal{F}\langle\eta\rangle_{E, \Delta}$, and let $\xi = (\xi_1, \dots, \xi_r)$ be a family of E -generators of $\mathcal{C}\langle\sigma\rangle$ over \mathcal{C} , i.e. $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle\xi\rangle_E$. Since $\mathcal{G}\langle\sigma\eta\rangle_{E, \Delta} = \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\langle\xi\rangle_{E, \Delta}$, each coordinate of ξ is in the E -field generated over \mathcal{G} by a finite number of Δ -derivatives of $\sigma\eta$. Denote this set of Δ -derivatives of $\sigma\eta$ by $\vartheta = (\vartheta_1, \dots, \vartheta_s)$. Then ϑ E -generates $\mathcal{G}\sigma\mathcal{G}$ over \mathcal{G} . \square

Claim 1.17. $\mathcal{F}^\circ\langle\vartheta\rangle_E = \sigma\mathcal{G}$.

Proof. By the definition of ϑ , $\mathcal{F}^\circ\langle\vartheta\rangle_E \subset \sigma\mathcal{G}$. Let $\alpha \in \sigma\mathcal{G}$. Then $\alpha \in \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\langle\sigma\eta\rangle_{E, \Delta} = \mathcal{G} \cdot \mathcal{F}^\circ\langle\vartheta\rangle_E$. If $(\gamma_i)_{i \in I}$ is a basis for $\mathcal{F}^\circ\langle\vartheta\rangle_E$ over \mathcal{F}° , then $\alpha = (\sum g_i \gamma_i) / (\sum g'_j \gamma_j)$, with g_i and g'_j in \mathcal{G} and not all the g'_j are 0. Therefore, $\sum g'_j (\gamma_j \alpha) - \sum g_i \gamma_i = 0$, and the family $(\gamma_j \alpha, \gamma_i)$ of elements of $\sigma\mathcal{G}$ is linearly dependent over \mathcal{G} . Since σ is isolated, $\sigma\mathcal{G}$ and \mathcal{G} are algebraically disjoint over \mathcal{F} [12, Comment on page 387]. *A fortiori*, they are also algebraically disjoint over \mathcal{F}° . Since \mathcal{G} is regular over \mathcal{F}° , $\sigma\mathcal{G}$ and \mathcal{G} are linearly disjoint over \mathcal{F}° [15, Theorem 3, page 57]. By this disjointness, the family $(\gamma_j \alpha, \gamma_i)$ is linearly dependent over \mathcal{F}° . So there exists f_i and f'_i elements of \mathcal{F}° , not all 0, such that $\sum f'_j (\gamma_j \alpha) - \sum f_i \gamma_i = 0$. Because the γ_j are linearly independent over \mathcal{F}° , $\sum f'_j \gamma_j \neq 0$. Therefore $\alpha = (\sum f_i \gamma_i) / (\sum f'_j \gamma_j) \in \mathcal{F}^\circ\langle\vartheta\rangle$. \square

Since the E -field $\sigma\mathcal{G} = \mathcal{F}^\circ\langle\vartheta\rangle_E$ is finitely E -generated over \mathcal{F}° , $\mathcal{G} = \mathcal{F}^\circ\langle\sigma^{-1}\vartheta\rangle_E$ (as E -fields) is also finitely E -generated over \mathcal{F}° . Because any intermediate extension of a finitely E -generated extension is finitely E -generated [12, Chapter 2, Proposition 14, p. 112], it follows that \mathcal{F}° is finitely (E, Δ) -generated over \mathcal{F} , and, hence, also finitely E -generated over \mathcal{F} (because \mathcal{F}° is algebraic). Thus \mathcal{G} is finitely E -generated over \mathcal{F} .

Then

$$\omega_{\sigma^{-1}\vartheta/\mathcal{F}} = \omega_{\vartheta/\mathcal{F}} = \omega_{\vartheta/\mathcal{G}}$$

since the first equality would be true for any E - \mathcal{F} -isomorphism σ [12, page 387] and the second equality holds by [12, Comment on page 117] because $\sigma\mathcal{G}$ and \mathcal{G} are algebraically disjoint over \mathcal{F} [12, Comment on page 387]. Also,

$$\omega_{\xi/\mathcal{G}} = \omega_{\xi/e}$$

because \mathcal{G} and $\mathcal{C}\langle\sigma\rangle$ are linearly disjoint over \mathcal{C} [12, Corollary 1, page 87] and [12, Comment on page 117]. Because ϑ and ξ both E -generate $\mathcal{G}\sigma\mathcal{G}$ over \mathcal{G} , the E -birational invariants (E -type, E -dimension, typical E -dimension) of $\omega_{\vartheta/\mathcal{G}}$ and $\omega_{\xi/\mathcal{G}}$ are equal ([12, page 118] or [13, page 7]). By utilizing the above equalities, the E -birational invariants of $\omega_{\sigma^{-1}\vartheta/\mathcal{G}}$ and $\omega_{\xi/e}$ are also the same. Thus the E -type (*resp.* E -dimension, typical E -dimension) of $\mathcal{C}\langle\sigma\rangle$ over \mathcal{C} is equal to the E -type (*resp.* E -dimension, typical E -dimension) of \mathcal{G} over \mathcal{F} .

1.5. Pre E -sets and E -groups. The objects in the category of pre E - \mathcal{C} -sets [13, Chapter 1] are defined as follows.

Definition 1.18. Let \mathcal{C} be an E-field and let \mathcal{V} be a universal E-field extension of \mathcal{C} . A pre E- \mathcal{C} -set (relative to \mathcal{V}) is a set A for which there are given

- (1) for each element $x \in A$, an E-finitely generated field extension $\mathcal{C}\langle x \rangle$ over \mathcal{C} ,
- (2) a preorder on A called E-specialization over \mathcal{C} or, more simply, E- \mathcal{C} -specialization (which shall be indicated by the notation $x \rightarrow x'$),
- (3) for each pair (x, x') in A^2 with $x \leftrightarrow x'$, an E-isomorphism $S_{x',x} : \mathcal{C}\langle x \rangle \approx \mathcal{C}\langle x' \rangle$ over \mathcal{C} , all subject to the following axioms.

DAS1 A has a finite subset Φ such that, for each $x' \in A$, there exists an $x \in \Phi$ with $x \rightarrow x'$.

DAS2a If $x, x', x'' \in A, x \leftrightarrow x',$ and $x' \leftrightarrow x'',$ then $S_{x'',x'} \circ S_{x',x} = S_{x'',x}.$

DAS2b If $x \in A$ and $S : \mathcal{F}\langle x \rangle \approx \mathcal{C}'$ is an E-field isomorphism over \mathcal{C} , then there exists a unique $x' \in A$ with $x \leftrightarrow x'$ such that $\mathcal{F}\langle x' \rangle = \mathcal{C}'$ and $S_{x',x} = S.$

It can be shown [14] that the \mathcal{V} -valued points of any E-scheme in the sense of Kovacic [5] over \mathcal{C} is a pre E- \mathcal{F} -set.

Definition 1.19. A subset B of the pre E- \mathcal{C} -set A is called \mathcal{C} -irreducible (in A) if there exists an $x \in A$ such that B is the set of all elements of A that are E- \mathcal{C} -specializations of x . Such an x will be called a \mathcal{C} -generic element of B . A maximal \mathcal{C} -irreducible subset of A is called a \mathcal{C} -component of A .

Kolchin defines pre E- \mathcal{C} -maps, as below, in a manner such that the composition of two is not necessarily a third.

Definition 1.20. Let A and B be pre E- \mathcal{C} -sets. A pre E- \mathcal{C} -mapping of A to B is a mapping f of a subset A_f of A into B with the following four properties:

- (1) the \mathcal{C} -generic elements of the components of A are contained in A_f ;
- (2) if $x \in A_f,$ then $\mathcal{C}\langle f(x) \rangle \subset \mathcal{F}\langle x \rangle;$
- (3) if $x \in A, x' \in A_f,$ and $x \rightarrow x',$ then $x \in A_f$ and $f(x) \rightarrow f(x');$
- (4) if $x, x' \in A_f$ and $x \leftrightarrow x',$ then $S_{x',x}$ extends $S_{f(x'),f(x)}.$ See Diagram 1.21 below.

Diagram 1.21.

$$\begin{array}{ccc}
 \mathcal{C}\langle x \rangle & \xrightarrow{S_{x',x}} & \mathcal{C}\langle x' \rangle \\
 \uparrow \text{inclusion} & & \uparrow \text{inclusion} \\
 \mathcal{C}\langle f(x) \rangle & \xrightarrow{S_{f(x'),f(x)}} & \mathcal{C}\langle f(x') \rangle
 \end{array}$$

To have morphisms that are composable, pre E- \mathcal{C} -mappings from A to B that are everywhere defined (that is, $A_f = A$) are taken to be the morphisms in the category of pre E- \mathcal{C} -sets.

Definition 1.22. The category of pre E- \mathcal{C} -sets (relative to the universal E-field \mathcal{V}) is the category with pre E- \mathcal{C} -sets as objects and with everywhere defined E- \mathcal{C} -mappings as morphisms.

It can be shown [14] that the functor of \mathcal{V} -valued points is a functor from the category of E- \mathcal{C} -schemes to the category of pre E- \mathcal{C} -sets (relative to the universal E-field \mathcal{V}).

Definition 1.23 ([13, page 33]). An $E\mathcal{C}$ -group (relative to the universal E -field \mathcal{V}) is a set G which has both a group structure (usually written multiplicatively) and a pre $E\mathcal{C}$ -set structure relative to the universal E -field \mathcal{V} , subject to the following axioms.

- DAG1a If $x_1, x_2 \in G$, then $\mathcal{C}\langle x_1x_2 \rangle \subset \mathcal{C}\langle x_1 \rangle\mathcal{C}\langle x_2 \rangle$.
- DAG1b If $x_1, x_2 \in G$, then $\mathcal{C}\langle x_1^{-1}x_2 \rangle \subset \mathcal{C}\langle x_1 \rangle\mathcal{C}\langle x_2 \rangle$.
- DAG2a If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \leftrightarrow x'_1, x_2 \leftrightarrow x'_2$, and $S_{x'_1, x_1}, S_{x'_2, x_2}$ are compatible,¹ then $x_1x_2 \rightarrow x'_1x'_2$. If moreover $x_1x_2 \leftrightarrow x'_1x'_2$, and h is an $E\mathcal{C}$ -homomorphism of finitely E -generated E -overrings of \mathcal{C} in \mathcal{U} such that $h, S_{x'_1, x_1}, S_{x'_2, x_2}$ are compatible, then h and $S_{x'_1x'_2, x_1x_2}$ are compatible.
- DAG2b If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \rightarrow x'_1, x_2 \rightarrow x'_2$, then there exist elements $x_1^*, x_2^* \in G$ with $x_1 \leftrightarrow x_1^*, x_2 \leftrightarrow x_2^*$ such that x_1^*, x_2^* are algebraically disjoint over \mathcal{C} and $x_1^*x_2^* \rightarrow x'_1x'_2$ (i.e., $\mathcal{C}\langle x_1^* \rangle$ and $\mathcal{C}\langle x_2^* \rangle$ are algebraically disjoint over \mathcal{C}), and such that, if $x_1^*x_2^* \leftrightarrow x'_1x'_2$ and $x_2^* \leftrightarrow x'_2$, then $S_{x'_1x'_2, x_1^*x_2^*}, S_{x_2^*, x_2^*}$ are compatible.
- DAG2c If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \leftrightarrow x'_1, x_2 \leftrightarrow x'_2$, and $S_{x'_1, x_1}, S_{x'_2, x_2}$ are compatible, then $x_1^{-1}x_2 \rightarrow x_1'^{-1}x_2'$. If moreover $x_1^{-1}x_2 \leftrightarrow x_1'^{-1}x_2'$, and h is an $E\mathcal{C}$ -homomorphism of finitely E -generated E -overrings of \mathcal{C} in \mathcal{U} such that $h, S_{x'_1, x_1}, S_{x'_2, x_2}$ are compatible, then $h, S_{x_1'^{-1}x_2', x_1^{-1}x_2}$ are compatible.
- DAG2d If $x_1, x_2, x'_1, x'_2 \in G$ and $x_1 \rightarrow x'_1, x_2 \rightarrow x'_2$, then there exist elements $x_1^*, x_2^* \in G$ with $x_1 \leftrightarrow x_2^*, x_2 \leftrightarrow x_1^*$ such that x_1^* and x_2^* are algebraically disjoint over \mathcal{C} and $x_1^{*-1}x_2^* \rightarrow x_1'^{-1}x_2'$.
- DAG3 The unity element 1 of G is contained in a \mathcal{C} -component (Definition 1.19) of G having a \mathcal{C} -generic element x that is regular over \mathcal{C} ; i.e., \mathcal{C} is algebraically closed in $\mathcal{C}\langle x \rangle$.

It can be shown [14] that the functor of \mathcal{V} -valued points applied to an $E\mathcal{C}$ -group scheme of $E\mathcal{C}$ -finite type is an $E\mathcal{C}$ -group (relative to the universal E -field \mathcal{V}).

1.6. E -groups of isomorphisms.

Theorem 1.24. *Let \mathcal{G} be an E -strongly normal extension of the Δ -field \mathcal{F} with field of Δ -constants \mathcal{C} , and let $G = \text{Isom}_{\mathcal{F}}^{E, \Delta}(\mathcal{G}, \mathcal{V})$. With the pre $E\mathcal{C}$ -set structure defined above, G is an $E\mathcal{C}$ -group. Furthermore, the field \mathcal{G} is finitely E -generated, and, as such, the E -type (resp. E -dimension, typical E -dimension) of \mathcal{G} over \mathcal{F} equals the E -type (resp. E -dimension, typical E -dimension) of the $E\mathcal{C}$ -group G .*

Proof. It will be verified that G satisfies the properties of an $E\mathcal{C}$ -group; see Definition 1.23. By Proposition 1.15 and the subsequent remark, the set G has the structure of a group.

Endow G with the following pre $E\mathcal{C}$ -set structure.

- (1) To each $\sigma \in G$ associate $\mathcal{C}\langle \sigma \rangle = (\mathcal{G}\sigma\mathcal{G})^\Delta$, considered as an E -field extension of $\mathcal{C} = \mathcal{G}^\Delta$. This is finitely E -generated by Proposition 1.6.
- (2) For each $(\sigma, \sigma') \in G^2$, let $\sigma \rightarrow \sigma'$ mean that σ' is an $(E, \Delta)\mathcal{F}$ -specialization of σ (Definition 1.2).

¹Let R_1 and R_2 be rings over the ring R , all in a common field, and let ϕ_1 and ϕ_2 be ring homomorphisms of R_1 and R_2 , respectively, into another field over R . Then ϕ_1 and ϕ_2 are compatible if there exists an extension of ϕ_1 and ϕ_2 to the ring $R[R_1, R_2]$.

- (3) For each $(\sigma, \sigma') \in G^2$ with $\sigma \leftrightarrow \sigma'$ (that is, with σ' a generic E - \mathcal{F} -specialization of σ), then there exists a unique \mathcal{G} - (E, Δ) -isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ that, for each $\alpha \in \mathcal{G}$, maps α to α and $\sigma\alpha$ onto $\sigma'\alpha$. The restriction of this (E, Δ) - \mathcal{G} -isomorphism to the Δ -constants yields an E - \mathcal{C} -isomorphism $S_{\sigma',\sigma} : \mathcal{C}\langle\sigma\rangle \approx \mathcal{C}\langle\sigma'\rangle$ over \mathcal{C} which is called the *E- \mathcal{C} -isomorphism induced by the generic (E, Δ) - \mathcal{G} -specialization*.

By [12, Proposition 1(c), page 387], this pre E - \mathcal{C} -set structure satisfies *DAS1*. Axioms *DAS2a* and *DAS2b* follow from the next proposition.

Proposition 1.25. *Let σ be an E -strong (E, Δ) -isomorphism of \mathcal{G} .*

- (1) *If σ' is a generic (E, Δ) - \mathcal{G} -specialization of σ , and σ'' is a generic (E, Δ) - \mathcal{G} -specialization of σ' (and therefore of σ), then the composite of the induced E - \mathcal{G}^Δ -isomorphisms $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$ and $\mathcal{G}^\Delta\langle\sigma'\rangle \approx \mathcal{G}^\Delta\langle\sigma''\rangle$ is the induced E - \mathcal{G}^Δ -isomorphism $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma''\rangle$.*
- (2) *If $S : \mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{C}'$ is any E -isomorphism over \mathcal{G}^Δ , then there exists a unique generic (E, Δ) - \mathcal{G} -specialization σ' of σ such that $\mathcal{G}^\Delta\langle\sigma'\rangle = \mathcal{C}'$, and S is the induced E - \mathcal{G}^Δ -isomorphism $\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}^\Delta\langle\sigma'\rangle$.*

Proof. (1) This follows from the corresponding facts about the (E, Δ) - \mathcal{G} -isomorphisms $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$, $\mathcal{G}\sigma'\mathcal{G} \approx \mathcal{G}\sigma''\mathcal{G}$ and $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma''\mathcal{G}$.
 (2) $\mathcal{G}^\Delta\langle\sigma\rangle$ and \mathcal{G} are linearly disjoint over \mathcal{G}^Δ , as are \mathcal{C}' and \mathcal{G} : therefore S can be extended to an (E, Δ) - \mathcal{G} -isomorphism $T : \mathcal{G}\mathcal{G}^\Delta\langle\sigma\rangle \approx \mathcal{G}\mathcal{C}'$. The composite mapping $\mathcal{G} \approx^\sigma \sigma\mathcal{G} \subset \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{G}^\Delta\langle\sigma\rangle \approx^T \mathcal{G}\mathcal{C}'$ yields an (E, Δ) - \mathcal{G}^Δ -isomorphism $\sigma' : \mathcal{G} \approx T(\sigma\mathcal{G})$. Since $T\sigma\mathcal{G} = \sigma'\mathcal{G}$, $T : \mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$. Therefore σ' is a generic (E, Δ) - \mathcal{G} -specialization of σ , $\mathcal{C}' = \mathcal{G}^\Delta\langle\sigma'\rangle$ [12, Corollary 2, p. 88], and S is the induced E - \mathcal{G}^Δ -isomorphism $S_{\sigma',\sigma}$. The uniqueness is clear. □

Axiom *DAG 1* follows from Proposition 1.7. Axiom *DAG 2a* follows from part 2 of Corollary 1.9. Part 3 of Proposition 1.8 implies Axiom *DAG 2c*.

To prove parts *DAG2b* and *DAG2d*, let $\sigma, \sigma', \tau, \tau'$ be E -strong (E, Δ) -isomorphisms of \mathcal{G} over \mathcal{F} with $\sigma \rightarrow \sigma'$ and $\tau \rightarrow \tau'$. Fix a family $\eta = (\eta_1, \dots, \eta_n)$ of (E, Δ) -generators of \mathcal{G} over \mathcal{F} , and let p (resp. q) denote the defining (E, Δ) -ideal of $\sigma^{-1}\eta$ (resp. $\tau\eta$) in the (E, Δ) -polynomial algebra $\mathcal{G}\{y_1, \dots, y_n\}_{E,\Delta}$ (resp. $\mathcal{G}\{z_1, \dots, z_n\}_{E,\Delta}$). Let \mathcal{G}_a denote the algebraic closure of \mathcal{G} in \mathcal{U} . Then \mathcal{G}_ap and \mathcal{G}_aq have components p_1, \dots, p_r and q_1, \dots, q_s such that the quotient fields $QF(\mathcal{G}_a\{y_1, \dots, y_n\}_{E,\Delta}/p_i)$ for $i = 1, \dots, r$ and $QF(\mathcal{G}_a\{z_1, \dots, z_n\}_{E,\Delta}/q_j)$ for $j = 1, \dots, s$ are regular over \mathcal{G}_a [12, Proposition 3, page 131]. By [13, Corollary, page 132], each (E, Δ) -ideal $r_{kl} = \{p_k \cup q_l\}_{(E,\Delta)}$ of $\mathcal{G}_a\{y_1, \dots, y_n, z_1, \dots, z_n\}_{E,\Delta}$ is prime. Therefore, r_{kl} has a \mathcal{G}_a -generic (E, Δ) -zero $(\eta^{(k,l)}, \xi^{(k,l)})$ where $\eta^{(k,l)}$ is a generic zero of $r_{kl} \cap \mathcal{G}_a\{y_1, \dots, y_n\}_{E,\Delta} = p_k$ and therefore of $p_k \cap \mathcal{G}\{y_1, \dots, y_n\}_{E,\Delta} = p$, so that $\eta^{(k,l)}$ is a \mathcal{G} -generic (E, Δ) -specialization of $\sigma^{-1}\eta$ over \mathcal{G} and hence over \mathcal{F} . Therefore $\eta^{(k,l)}$ is the image of η by an E -strong (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} , which is denoted by σ_{kl}^{-1} and is defined by $\eta^{(k,l)} = \sigma_{kl}^{-1}\eta$. By [12, Lemma 2, page 386], $\sigma^{-1} \leftrightarrow \sigma_{kl}^{-1}$. Similarly $\xi^{(k,l)} = \tau_{kl}\eta$ for some E -strong (E, Δ) -isomorphism τ_{kl} of \mathcal{G} over \mathcal{F} with $\tau \leftrightarrow \tau_{kl}$. By hypothesis $\sigma \rightarrow \sigma'$, whence $\sigma^{-1} \rightarrow \sigma'^{-1}$ (part 1 of the Corollary 1.9) so that $\sigma'^{-1}\eta$ is an (E, Δ) -zero of p and hence of some p_k . Similarly, $\tau'\eta$ is an (E, Δ) -zero of some q_l . Thus,

$(\sigma'^{-1}\eta, \tau'\eta)$ is an (E, Δ) -zero of r_{kl} ; thus $(\sigma_{kl}^{-1}\eta, \tau_{kl}\eta) \rightarrow_{\mathcal{G}_a} (\sigma'^{-1}\eta, \tau'\eta)$ and hence over \mathcal{G} . It follows [12, Lemma 2, page 386], that $(\tau_{kl}, \sigma_{kl}^{-1}) \rightarrow_{\mathcal{G}} (\tau', \sigma'^{-1})$, and hence by Proposition 1.8 (1) and (2), that $(\tau_{kl}^{-1}, \tau_{kl}^{-1}\sigma_{kl}^{-1}) \rightarrow_{\mathcal{G}} (\tau'^{-1}, \tau'^{-1}\sigma'^{-1})$ and that if $\tau_{kl}^{-1}\sigma_{kl}^{-1} \leftrightarrow \tau'^{-1}\sigma'^{-1}$ and $\tau_{kl}^{-1} \leftrightarrow \tau'^{-1}$, then the induced E- \mathcal{C} -isomorphisms $\mathcal{C}\langle\tau_{kl}^{-1}\sigma_{kl}^{-1}\rangle \approx \mathcal{C}\langle\tau'^{-1}\sigma'^{-1}\rangle$ and $\mathcal{C}\langle\tau_{kl}^{-1}\rangle \approx \mathcal{C}\langle\tau'^{-1}\rangle$ are compatible. By part 1 of Corollary 1.9, then $\sigma_{kl}\tau_{kl} \rightarrow \sigma'\tau'$ and if $\sigma_{kl}\tau_{kl} \leftrightarrow \sigma'\tau'$ and $\tau_{kl} \leftrightarrow \tau'$, then the induced E- \mathcal{C} -isomorphisms $\mathcal{C}\langle\sigma_{kl}\tau_{kl}\rangle \approx \mathcal{C}\langle\sigma'\tau'\rangle$ and $\mathcal{C}\langle\tau_{kl}\rangle \approx \mathcal{C}\langle\tau'\rangle$ are compatible. This proves *DAG2b* and (because $\sigma^{-1} \rightarrow \sigma'^{-1}$ whenever $\sigma \rightarrow \sigma'$) also part *DAG2d*.

To prove axiom *DAG3*, one must show that if σ is an isolated isomorphism of \mathcal{G} over \mathcal{F} with $\sigma \rightarrow id_{\mathcal{G}}$, then $\mathcal{C}\langle\sigma\rangle$ is regular over \mathcal{C} . Since $\sigma \rightarrow id_{\mathcal{G}}$, $\sigma\mathcal{F}^\circ = \mathcal{F}^\circ$ [12, Proposition 2(b), page 388]. Since \mathcal{G} is regular over \mathcal{F}° , clearly $\sigma\mathcal{G}$ is regular over $\sigma\mathcal{F}^\circ = \mathcal{F}^\circ$. By [12, Remark, page 387], $\sigma\mathcal{G}$ is algebraically disjoint from \mathcal{G} over \mathcal{F} , and, *a fortiori*, they are algebraically disjoint over \mathcal{F}° . Because \mathcal{G} is regular over \mathcal{F}° , $\sigma\mathcal{G}$ is linearly disjoint from \mathcal{G} over \mathcal{F}° [15, Theorem 3, page 57],

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}\sigma\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{F}^\circ & \longrightarrow & \sigma\mathcal{G}. \end{array}$$

Recall [15, Corollary 6, page 58] that if K and L are field extensions of field k in a larger field and if they linearly disjoint over k , then K is regular over k if and only if KL is regular over L . Therefore, $\mathcal{G}\sigma\mathcal{G}$ is regular over \mathcal{G} . Since \mathcal{G} and $\mathcal{C}\langle\sigma\rangle$ are linearly disjoint over \mathcal{C} [12, Corollary 2, page 88],

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\sigma\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}\langle\sigma\rangle \end{array}$$

and $\mathcal{G}\mathcal{C}\langle\sigma\rangle = \mathcal{G}\sigma\mathcal{G}$; that $\mathcal{G}\sigma\mathcal{G}$ is regular over \mathcal{G} implies $\mathcal{C}\langle\sigma\rangle$ is regular over \mathcal{C} , which is *DAG3*. This establishes G as an E- \mathcal{C} -group.

Definition 1.26. By virtue of Theorem 1.24, the set of E-strong (E, Δ) -isomorphisms of the E-strongly normal extension \mathcal{G} over \mathcal{F} has a natural structure of an E- \mathcal{C} -group relative to the E-universal field \mathcal{U}^Δ . This E- \mathcal{C} -group is called the Galois group of \mathcal{G} over \mathcal{F} , and it is denoted by $G_E(\mathcal{G}/\mathcal{F})$ or $G(\mathcal{G}/\mathcal{F})$. The \mathcal{C} -component of the identity of $G_E(\mathcal{G}/\mathcal{F})$ is denoted by $G_E^\circ(\mathcal{G}/\mathcal{F})$ or $G^\circ(\mathcal{G}/\mathcal{F})$.

Definition 1.27. If G is any E- \mathcal{C} -group, a G -extension of \mathcal{F} is any E-strongly normal extension \mathcal{G} of \mathcal{F} such that $G(\mathcal{G}/\mathcal{F})$ is E- \mathcal{C} -isomorphic to an E- \mathcal{C} -subgroup of $G_{\mathcal{X}}$. When $G(\mathcal{G}/\mathcal{F})$ is E- \mathcal{C} -isomorphic to $G_{\mathcal{X}}$ itself, the (E, Δ) -extension \mathcal{G} over \mathcal{F} is called full. A linear extension of \mathcal{F} is an E- $GL(n)$ -extension of \mathcal{F} for some natural number n .

1.7. Extending the constants.

Definition 1.28 ([13, page 48]). Let \mathcal{C} be an E-field, let \mathcal{V} be another E-field that is E-universal over \mathcal{F} , and let $\mathcal{D} \subset \mathcal{V}$ be an E-field containing \mathcal{F} over which \mathcal{V} is E-universal. Let G be an E- \mathcal{C} -group relative to \mathcal{V} , and let H be an E- \mathcal{D} -group

relative to \mathcal{V} . An E - $(\mathcal{D}, \mathcal{C})$ -homomorphism of H into G is a group homomorphism $f : H \rightarrow G$ that satisfies the following three conditions:

- (1) if $y \in H$, then $\mathcal{D}\langle y \rangle \supset \mathcal{C}\langle f(y) \rangle$,
- (2) if $y, y' \in H$ and $y \rightarrow y'$ over \mathcal{D} , then $f(y) \rightarrow f(y')$ over \mathcal{C} ,
- (3) if $y, y' \in H$ and $y \leftrightarrow y'$ over \mathcal{D} , then $S_{\mathcal{D}, y', y}$ extends $S_{\mathcal{C}, f(y'), f(y)}$.

Definition 1.29 ([13, page 49]). An E - \mathcal{D} -group structure on G is said to be induced (by the given E - \mathcal{C} -group structure on G) if the following two conditions are satisfied:

- (1) the identity map id_G on the set G is an E - $(\mathcal{D}, \mathcal{C})$ -homomorphism from G with the structure of an E - \mathcal{D} -group to G with the structure of the E - \mathcal{C} -group G ;
- (2) every E - $(\mathcal{D}, \mathcal{C})$ -homomorphism of an E - \mathcal{D} -group into G is an E - \mathcal{D} -homomorphism.

The following generalization of [12, Theorem 2, page 396] interprets, for an E -extension \mathcal{C}' of \mathcal{C} in \mathcal{K} ($= \mathcal{U}^\Delta$), the induced E - \mathcal{C}' -group of the E - \mathcal{C} -group $G(\mathcal{G}/\mathcal{F})$.

Theorem 1.30. *Let $\mathcal{G} \subset \mathcal{U}$ be an E -strongly normal extension of \mathcal{F} . Denote the field of Δ -constants of \mathcal{F} by \mathcal{C} , and let $\mathcal{C}' \subset \mathcal{K}$ be an (E, Δ) -extension of \mathcal{C} such that \mathcal{U} is (E, Δ) -universal over $\mathcal{F}\mathcal{C}'$. Then \mathcal{U} is (E, Δ) -universal over $\mathcal{G}\mathcal{C}'$, and $\mathcal{G}\mathcal{C}'$ is an E -strongly normal extension of $\mathcal{F}\mathcal{C}'$ with field of Δ -constants \mathcal{C}' . Furthermore, the E - \mathcal{C}' -group $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is the induced E - \mathcal{C}' -group of the E - \mathcal{C} -group $G(\mathcal{G}/\mathcal{F})$, both of these groups being identified with each other by means of their canonical identifications with the group of (E, Δ) -automorphisms of $\mathcal{G}\mathcal{K}$ over $\mathcal{F}\mathcal{K}$ (see Proposition 1.15)*

$$\begin{array}{ccccc}
 \mathcal{G} & \longrightarrow & \mathcal{G}\mathcal{C}' & \longrightarrow & \mathcal{G}\mathcal{K} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathcal{F} & \longrightarrow & \mathcal{F}\mathcal{C}' & \longrightarrow & \mathcal{F}\mathcal{K}.
 \end{array}$$

Proof. Since $\mathcal{G}\mathcal{C}'$ is finitely (E, Δ) -generated over $\mathcal{F}\mathcal{C}'$, [12, Proposition 4(b), page 133] shows that \mathcal{U} is (E, Δ) -universal over $\mathcal{G}\mathcal{C}'$. That \mathcal{C}' is the field of Δ -constants of $\mathcal{F}\mathcal{C}'$ and $\mathcal{G}\mathcal{C}'$ follows from [12, Corollary 2, page 88]. If σ is any (E, Δ) -isomorphism of $\mathcal{G}\mathcal{C}'$ over $\mathcal{F}\mathcal{C}'$, then the restriction of σ to \mathcal{G} is an (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} and as such is E -strong. Hence, $\sigma(\mathcal{G}\mathcal{C}') = \sigma\mathcal{G}\mathcal{C}' \subset \mathcal{G} \cdot \mathcal{K} \cdot \mathcal{C}' = \mathcal{G}\mathcal{C}' \cdot \mathcal{K}$, and similarly $\mathcal{G}\mathcal{C}' \subset \sigma(\mathcal{G}\mathcal{C}') \cdot \mathcal{K}$: that is, σ is E -strong. Therefore $\mathcal{G}\mathcal{C}'$ is E -strongly normal over $\mathcal{F}\mathcal{C}'$, and $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is a E - \mathcal{C}' -group (Theorem 1.24). Denote by $\mathcal{C}'\langle\sigma\rangle$ the E - \mathcal{C}' -field associated to any $\sigma \in G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$.

Define $\text{id}_G : G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}') \rightarrow G(\mathcal{G}/\mathcal{F})$ by identifying $\sigma \in G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ with the (E, Δ) -automorphism of $\mathcal{G}\mathcal{C}' \cdot \mathcal{K} = \mathcal{G}\mathcal{K}$ over $\mathcal{F}\mathcal{C}' \cdot \mathcal{K} = \mathcal{F}\mathcal{K}$ that extends σ (Proposition 1.15), and then with the E -strong (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} to which σ restricts. Then

$$\mathcal{G}\mathcal{C}'\langle\sigma\rangle = \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle\sigma\rangle = \mathcal{G}\mathcal{C}'\sigma(\mathcal{G}\mathcal{C}') = \mathcal{G}\text{id}_G\sigma\mathcal{G} \cdot \mathcal{C}' = \mathcal{G}\mathcal{C}'\langle\text{id}_G\sigma\rangle\mathcal{C}'$$

and, by [12, Corollary 2, p. 88],

$$(1) \quad \mathcal{C}'\langle\sigma\rangle = \mathcal{C}'\langle\text{id}_G\sigma\rangle\mathcal{C}'.$$

If σ' is an E - \mathcal{C}' -specialization of σ in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$, then $(\sigma'\alpha)_{\alpha \in \mathcal{G}}$ is an (E, Δ) - $\mathcal{G}\mathcal{C}'$ -specialization of $(\sigma\alpha)_{\alpha \in \mathcal{G}}$, and hence over \mathcal{G} , so that $\text{id}_G\sigma'$ is an E - \mathcal{C}' -specialization

of $\text{id}_G\sigma$ in $G(\mathcal{G}/\mathcal{F})$. When the E - \mathcal{C}' -specialization in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is \mathcal{C}' -generic, then there exists an (E, Δ) -isomorphism

$$(2) \quad \mathcal{G}\mathcal{C}'\sigma(\mathcal{G}\mathcal{C}') \approx \mathcal{G}\mathcal{C}'\sigma'(\mathcal{G}\mathcal{C}')$$

over $\mathcal{G}\mathcal{C}'$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ for every $\alpha \in \mathcal{G}$, and this restricts to an (E, Δ) -isomorphism

$$(3) \quad \mathcal{G} \cdot \text{id}_G\sigma\mathcal{G} \approx \mathcal{G} \cdot \text{id}_G\sigma'\mathcal{G}$$

over \mathcal{G} , so that the E - \mathcal{C} -specialization in $G(\mathcal{G}/\mathcal{F})$ is \mathcal{C} -generic. This restricts to the induced E - \mathcal{C} -isomorphism

$$(4) \quad S_{\text{id}_G\sigma', \text{id}_G\sigma}^{\mathcal{C}} : \mathcal{C}\langle \text{id}_G\sigma \rangle \approx \mathcal{C}\langle \text{id}_G\sigma' \rangle,$$

which is also a restriction of the (E, Δ) -isomorphism (2). Moreover, the (E, Δ) -isomorphism (2) also restricts to the induced E - \mathcal{C} -isomorphism

$$(5) \quad S_{\sigma', \sigma}^{\mathcal{C}'} : \mathcal{C}'\langle \sigma \rangle \approx \mathcal{C}'\langle \sigma' \rangle.$$

Therefore, the restriction of $S_{\sigma', \sigma}^{\mathcal{C}'}$ to $\mathcal{C}\langle \text{id}_G\sigma \rangle$ is the induced E - \mathcal{C} -isomorphism $S_{\text{id}_G\sigma', \text{id}_G\sigma}^{\mathcal{C}} : \mathcal{C}\langle \text{id}_G\sigma \rangle \approx \mathcal{C}\langle \text{id}_G\sigma' \rangle$. Therefore, $S_{\sigma', \sigma}^{\mathcal{C}'}$ is an extension of $S_{\text{id}_G\sigma', \text{id}_G\sigma}^{\mathcal{C}}$. This shows that id_G is an E - $(\mathcal{C}', \mathcal{C})$ -homomorphism.

Now let H be any E - \mathcal{C}' -group relative to the universal E -field \mathcal{K} , and let $f : H \rightarrow G(\mathcal{G}/\mathcal{F})$ be any E - $(\mathcal{C}', \mathcal{C})$ -homomorphism. To complete the proof of the theorem, it must be shown that $f' = \text{id}_G^{-1}f$ from H to $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$ is an

$$\begin{array}{ccc} H & \xrightarrow{f'} & G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}') \\ & \searrow f & \downarrow \text{id}_G \\ & & G(\mathcal{G}/\mathcal{F}) \end{array}$$

E - \mathcal{C}' -homomorphism [13, Chapter 1, Section 2, p. 37]; that is f' is a homomorphism of groups and an everywhere-defined pre E - \mathcal{C}' -mapping (Definition 1.20). Clearly f' is a homomorphism of groups. By [13, Corollary 1, p. 90], it suffices to show that the restriction, also denoted by f' , of f' to the \mathcal{C}' -generic elements of H is a pre E - \mathcal{C}' -mapping.

Property 1 of the definition of a pre E - \mathcal{C}' -mapping is clear; i.e., the domain of definition of f contains the \mathcal{C}' -generic elements of H . For any $y \in H$, $\mathcal{C}'\langle y \rangle \supset \mathcal{C}\langle f(y) \rangle$ because f is an E - $(\mathcal{C}', \mathcal{C})$ -homomorphism. From this and the equation (1), the following containment may be deduced:

$$\mathcal{C}'\langle y \rangle \supset \mathcal{C}\langle f(y) \rangle \mathcal{C}' = \mathcal{C}\langle \text{id}_G f'(y) \rangle \cdot \mathcal{C}' = \mathcal{C}'\langle f'(y) \rangle.$$

This is property (2) of the definition of a pre E - \mathcal{C}' -mapping.

For properties (3) and (4), if $y \leftrightarrow y'$ in H , then $f(y) \leftrightarrow f(y')$ in $G(\mathcal{G}/\mathcal{F})$ because f is an E - $(\mathcal{C}', \mathcal{C})$ -homomorphism. For the same reason, $S_{y', y}^{\mathcal{C}'}$ extends the induced

E- \mathcal{C} -isomorphism $S_{f(y'),f(y)}^{\mathcal{C}}$,

$$\begin{array}{ccc} \mathcal{C}'\langle y \rangle & \xrightarrow{S_{y',y}^{\mathcal{C}'}} & \mathcal{C}'\langle y' \rangle \\ \cup \uparrow & & \cup \uparrow \\ \mathcal{C}\langle f(y) \rangle & \xrightarrow{S_{f(y'),f(y)}^{\mathcal{C}}} & \mathcal{C}\langle f(y') \rangle, \end{array}$$

and hence $S_{f(y'),f(y)}^{\mathcal{C}}$ and $\text{id}_{\mathcal{C}'}$ are bicompatible. Therefore, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}'\langle y \rangle & \xrightarrow{S_{y',y}^{\mathcal{C}'}} & \mathcal{C}'\langle y' \rangle \\ \cup \uparrow & & \cup \uparrow \\ \mathcal{C}\langle f(y) \rangle \cdot \mathcal{C}' & \xrightarrow{\varphi} & \mathcal{C}\langle f(y') \rangle \cdot \mathcal{C}', \end{array}$$

where the E-isomorphism φ extends $\text{id}_{\mathcal{C}'}$ and $S_{f(y'),f(y)}^{\mathcal{C}}$.

Since \mathcal{G} and $\mathcal{C}[\mathcal{C}'\langle y \rangle]$ are linearly disjoint over \mathcal{C} , as are \mathcal{G} and $\mathcal{C}[\mathcal{C}'\langle y' \rangle]$, it follows that $\text{id}_{\mathcal{G}}$ and $S_{y',y}^{\mathcal{C}'}$ are bicompatible. Therefore, the top square of the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}\mathcal{C}'\langle y \rangle & \xrightarrow{\alpha} & \mathcal{G}\mathcal{C}'\langle y' \rangle \\ \cup \uparrow & & \cup \uparrow \\ \mathcal{G}\mathcal{C}\langle f(y) \rangle \cdot \mathcal{C}' & \xrightarrow{\beta} & \mathcal{G}\mathcal{C}\langle f(y') \rangle \cdot \mathcal{C}' \\ = \uparrow & & = \uparrow \\ \mathcal{G}\mathcal{C}\langle \text{id}_G f'(y) \rangle \mathcal{C}' & \xrightarrow{\beta} & \mathcal{G}\mathcal{C}\langle \text{id}_G f'(y') \rangle \mathcal{C}' \\ = \uparrow & & = \uparrow \\ \mathcal{G}\mathcal{C}'\langle f'(y) \rangle & \xrightarrow{\gamma} & \mathcal{G}\mathcal{C}'\langle f'(y') \rangle \\ = \uparrow & & = \uparrow \\ \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle f'(y) \rangle & \xrightarrow{\lambda} & \mathcal{G}\mathcal{C}' \cdot \mathcal{C}'\langle f'(y') \rangle \\ = \uparrow & & = \uparrow \\ \mathcal{G}\mathcal{C}' \cdot f'(y)(\mathcal{G}\mathcal{C}') & \xrightarrow{\mu} & \mathcal{G}\mathcal{C}' \cdot f'(y')(\mathcal{G}\mathcal{C}') \end{array}$$

where the (E, Δ) -isomorphism α extends $\text{id}_{\mathcal{G}}$ and $S_{y',y}^{\mathcal{C}'}$, and the (E, Δ) -isomorphism β extends φ , $\text{id}_{\mathcal{G}}$, $S_{f(y'),f(y)}^{\mathcal{C}}$ and $\text{id}_{\mathcal{C}'}$. Since $f = \text{id}_G f'$, the third line of this diagram is also

$$\beta : \mathcal{G}\mathcal{C}\langle \text{id}_G f'(y) \rangle \mathcal{C}' \longrightarrow \mathcal{G}\mathcal{C}\langle \text{id}_G f'(y') \rangle \mathcal{C}'$$

extending $\text{id}_{\mathcal{G}}$, $S_{\text{id}_G f'(y'), \text{id}_G f'(y)}$ and $\text{id}_{\mathcal{C}'}$. By equation (1), $\mathcal{C}\langle \text{id}_G f'(y) \rangle \mathcal{C}' = \mathcal{C}'\langle f'(y) \rangle$, and $\mathcal{C}\langle \text{id}_G f'(y') \rangle \mathcal{C}' = \mathcal{C}'\langle f'(y') \rangle$. Because id_G is an E- $(\mathcal{C}', \mathcal{C})$ -homomorphism, as was shown in the first part of this proof, the fourth line

$$\gamma : \mathcal{G} \cdot \mathcal{C}'\langle f'(y) \rangle \longrightarrow \mathcal{G} \cdot \mathcal{C}'\langle f'(y') \rangle$$

is an (E, Δ) -isomorphism extending $\text{id}_{\mathcal{G}}$ and $S_{f'(y'),f'(y)}^{\mathcal{C}'}$. The fifth line of the diagram is the (E, Δ) -isomorphism

$$\lambda : \mathcal{G}\mathcal{C}' \cdot \mathcal{C}' \langle f'(y) \rangle \longrightarrow \mathcal{G}\mathcal{C}' \cdot \mathcal{C}' \langle f'(y') \rangle,$$

which is obtained by writing ‘ $\mathcal{G}\mathcal{C}'$ ’ instead of ‘ \mathcal{G} ’. Clearly, the (E, Δ) -isomorphism λ extends $\text{id}_{\mathcal{G}\mathcal{C}'}$ and $S_{f'(y'),f'(y)}^{\mathcal{C}'}$. By the E-strong normality of $\mathcal{G}\mathcal{C}'$ over $\mathcal{F}\mathcal{C}'$, λ is the same as the (E, Δ) -isomorphism, in the sixth line,

$$\mu : \mathcal{G}\mathcal{C}' \cdot f'(y)(\mathcal{G}\mathcal{C}') \longrightarrow \mathcal{G}\mathcal{C}' \cdot f'(y')(\mathcal{G}\mathcal{C}')$$

that extends $\text{id}_{\mathcal{G}\mathcal{C}'}$ and that maps $f'(y)\alpha$ onto $f'(y')\alpha$ for every $\alpha \in \mathcal{G}\mathcal{C}'$. Therefore $f'(y) \leftrightarrow f'(y')$ in $G(\mathcal{G}\mathcal{C}'/\mathcal{F}\mathcal{C}')$, which is property (3). Property (4) is obtained by restricting the top and bottom lines in the above diagram to the Δ -constants, i.e. $S_{y',y}^{\mathcal{C}'}$ extends $S_{f'(y'),f'(y)}^{\mathcal{C}'}$. \square

Proposition 1.31. *Let \mathcal{G} be an E-strongly normal extension of \mathcal{F} , with \mathcal{C} the subfield of Δ -constants, and let φ be an (E, Δ) -isomorphism of \mathcal{G} over \mathcal{C} such that \mathcal{U} is universal over $\varphi\mathcal{G}$. Then $\varphi\mathcal{G}$ is an E-strongly normal extension of $\varphi\mathcal{F}$. There is a unique (E, Δ) -isomorphism $\mathcal{G} \cdot \mathcal{K} \approx \varphi\mathcal{G} \cdot \mathcal{K}$ over \mathcal{K} that extends φ (that also shall be denoted by φ). When $G(\mathcal{G}/\mathcal{F})$, respectively $G(\varphi\mathcal{G}/\varphi\mathcal{F})$, is canonically identified with the group of (E, Δ) -automorphisms of $\mathcal{G} \cdot \mathcal{K}$ over $\mathcal{F} \cdot \mathcal{K}$, respectively $\varphi\mathcal{G} \cdot \mathcal{K}$ over $\varphi\mathcal{F} \cdot \mathcal{K}$, the formula $T_\varphi(\sigma) = \varphi \cdot \sigma \cdot \varphi^{-1}$ defines an E- \mathcal{C} -isomorphism $T_\varphi : G(\mathcal{G}/\mathcal{F}) \approx G(\varphi\mathcal{G}/\varphi\mathcal{F})$.*

Remark 1.32. When φ is an (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F} , then $\varphi \in G(\mathcal{G}/\mathcal{F})$. After $G(\mathcal{G}/\mathcal{F})$ and $G(\varphi\mathcal{G}/\varphi\mathcal{F})$ are canonically identified with the group of automorphisms of the differential field $\mathcal{G}\mathcal{K} = \varphi\mathcal{G} \cdot \mathcal{K}$ over $\mathcal{F}\mathcal{K}$, then they coincide as groups (but not necessarily as \mathcal{C} -groups), and T_φ is the inner E-automorphism determined by φ .

Proof. Let τ be any (E, Δ) -isomorphism of $\varphi\mathcal{G}$ over $\varphi\mathcal{F}$. The (E, Δ) -isomorphism $\varphi^{-1} : \varphi\mathcal{G} \approx \mathcal{G}$ can be extended to some (E, Δ) -isomorphism $\psi : \varphi\mathcal{G} \cdot \tau\varphi\mathcal{G} \approx \mathcal{G} \cdot \psi\tau\varphi\mathcal{G}$, and evidently the formula $\alpha \mapsto \psi\tau\varphi\alpha$ defines an isomorphism of \mathcal{G} over \mathcal{F} . Therefore, since $\psi\tau\varphi$ is E-strong, the field of constants \mathcal{C}' of $\mathcal{G} \cdot \psi\tau\varphi\mathcal{G}$ has the property that

$$(6) \quad \mathcal{G}\mathcal{C}' = \mathcal{G} \cdot \psi\tau\varphi\mathcal{G} = \psi\tau\varphi\mathcal{G} \cdot \mathcal{C}'.$$

By applying ψ^{-1} to equation (6),

$$\varphi\mathcal{G} \cdot \mathcal{C} \langle \tau \rangle = \varphi\mathcal{G} \cdot \tau\varphi\mathcal{G} = \tau\varphi\mathcal{G} \cdot \mathcal{C} \langle \tau \rangle$$

since ψ^{-1} maps \mathcal{G} onto $\varphi\mathcal{G}$ and \mathcal{C}' onto the field of Δ -constants $\mathcal{C} \langle \tau \rangle$ of $\varphi\mathcal{G} \cdot \tau\varphi\mathcal{G}$. Therefore, τ is E-strong, and, hence, $\varphi\mathcal{G}$ is an E-strongly normal extension of $\varphi\mathcal{F}$.

Since \mathcal{G} and \mathcal{K} are linearly disjoint over \mathcal{C} , as are $\varphi\mathcal{G}$ and \mathcal{K} , φ can be extended to a unique (E, Δ) -isomorphism $\mathcal{G}\mathcal{K} \approx \varphi\mathcal{G} \cdot \mathcal{K}$ over \mathcal{K} , and denote it, too, by φ . Making the canonical identifications, one can see that for each $\sigma \in G(\mathcal{G}/\mathcal{F})$, $\varphi \cdot \sigma \cdot \varphi^{-1} \in G(\varphi\mathcal{G}/\varphi\mathcal{F})$. Therefore one can define a mapping $T_\varphi : G(\mathcal{G}/\mathcal{F}) \approx G(\varphi\mathcal{G}/\varphi\mathcal{F})$ by the formula $T_\varphi(\sigma) = \varphi \cdot \sigma \cdot \varphi^{-1}$, and it is clear that T_φ is a group isomorphism. Since $\varphi\mathcal{G} \cdot \mathcal{C} \langle T_\varphi(\sigma) \rangle = \varphi\mathcal{G} \cdot (\varphi \cdot \sigma \cdot \varphi^{-1})\varphi\mathcal{G} = \varphi(\mathcal{G}\sigma\mathcal{G}) = \varphi(\mathcal{G}\mathcal{C} \langle \sigma \rangle) = \varphi\mathcal{G} \cdot \mathcal{C} \langle \sigma \rangle$, one may infer that $\mathcal{C} \langle T_\varphi(\sigma) \rangle = \mathcal{C} \langle \sigma \rangle$. Furthermore, if $\sigma \leftrightarrow \sigma'$, then there exists an (E, Δ) -isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ over \mathcal{G} mapping $\sigma\alpha$ onto $\sigma'\alpha$ ($\alpha \in \mathcal{G}$) and inducing the E- \mathcal{C} -isomorphism $S_{\sigma',\sigma} : \mathcal{C} \langle \sigma \rangle \approx \mathcal{C} \langle \sigma' \rangle$. Since φ maps $\mathcal{G}\sigma\mathcal{G}$, respectively $\mathcal{G}\sigma'\mathcal{G}$, onto $\varphi\mathcal{G} \cdot T_\varphi(\sigma)\varphi\mathcal{G}$, respectively $\varphi\mathcal{G} \cdot T_\varphi(\sigma')\varphi\mathcal{G}$, and leaves Δ -constants fixed, one obtains

an (E, Δ) -isomorphism $\varphi\mathcal{G} \cdot T_\varphi(\sigma)\varphi\mathcal{G} \approx \varphi\mathcal{G} \cdot T_\varphi(\sigma')\varphi\mathcal{G}$ over $\varphi\mathcal{G}$ mapping $T_\varphi(\sigma)\varphi\alpha$ onto $T_\varphi(\sigma')\varphi\alpha$ ($\alpha \in G$), so that $T_\varphi(\sigma) \leftrightarrow T_\varphi(\sigma')$ and $S_{T_\varphi(\sigma'), T_\varphi(\sigma)} = S_{\sigma', \sigma}$. Thus, T_φ restricted to the \mathcal{C} -generic elements of G is a pre E - \mathcal{C} -map, and T_φ is an E - \mathcal{C} -isomorphism by [13, Corollary 1, p. 90]. \square

2. THE FUNDAMENTAL THEOREMS

2.1. The topology on E -sets. In this section, let \mathcal{F} be an E -field, and let \mathcal{V} be an E -extension of \mathcal{F} that is E -universal over \mathcal{F} . Consider \mathcal{H} an E -extension of \mathcal{F} over which \mathcal{V} need not be E -universal. Also, let A be a pre E - \mathcal{F} -set relative to \mathcal{V} (Section 1.5 or [12, page 29]). Then $x \in A$ is defined to be *rational* over \mathcal{H} if $\mathcal{F}\langle x \rangle \subset \mathcal{H}$ [13, page 29]. In a similar manner, define x to be *algebraic* (*resp.*, *E -algebraic* or *regular*) if $\mathcal{H}\mathcal{F}\langle x \rangle$ is an algebraic (*resp.*, *E -algebraic* or *regular*) extension of \mathcal{H} . Denote by $A_{\mathcal{H}}$ the set of elements of A rational over \mathcal{H} . In particular, $A_{\mathcal{V}}$ is the set A .

Let G be an E - \mathcal{F} -group (Section 1.23 or [13, page 33]). A *homogeneous E - \mathcal{F} -space for G* is a set M on which is given a structure of a homogeneous space for the group G and a structure of a pre E - \mathcal{F} -set subject to axioms, which are similar to those for a Δ - \mathcal{F} -group [13, page 34]. The homogeneous E - \mathcal{F} -space M for G is *principal* if it is principal as a homogeneous space for G and satisfies additional axioms [13, page 35].

A subset V of the pre E - \mathcal{F} -subset A is *\mathcal{F} -irreducible* (*in A*) if there exists $x \in V$ such that V is the set of all E -specializations of x over \mathcal{F} [13, page 30]. Such an x is called an E - \mathcal{F} -generic element of V . If the set B of A is the union of finitely many \mathcal{F} -irreducible subsets of A , then B has the structure of a pre E - \mathcal{F} -set that is induced by the restriction of the pre E - \mathcal{F} -set structure on A . Such a B is called a *pre E - \mathcal{F} -subset* (*of A*). A maximal \mathcal{F} -irreducible subset of A is called an *\mathcal{F} -component* (*of A*).

An E - \mathcal{F} -set is a pre E - \mathcal{F} -subset of a homogeneous E - \mathcal{F} -space for an E - \mathcal{F} -group [13, page 37]. Then the E - \mathcal{H} -subsets of M are the closed subsets of a Noetherian topology on M [13, Theorem 1, page 72], which is called the *E -Zariski topology relative to \mathcal{H}* or more simply the *E - \mathcal{H} -topology*. If $\mathcal{H} = \mathcal{V}$, the reference to \mathcal{V} is usually omitted, and it is called the *E -Zariski topology* or more simply the *E -topology*. Each E - \mathcal{F} -set will be considered to have the topology induced from the E - \mathcal{H} -topology on its ambient homogeneous E - \mathcal{F} -space for an E - \mathcal{F} -group. For an E - \mathcal{F} -set A , the subset $A_{\mathcal{H}} = \{v \in A \mid \mathcal{F}\langle v \rangle \subseteq \mathcal{H}\}$ will be called *E -dense in A* if, for each closed E -closed subset C of A with $A \neq C$, $A_{\mathcal{H}}$ is not contained in C . Kolchin shows that, if \mathcal{H} is constrainedly closed [13, page 79], then $A_{\mathcal{H}}$ is E -dense in A [13, Proposition 3, page 84].

Any E - \mathcal{F} -group G has a natural structure of a principal homogeneous E - \mathcal{F} -space for G , which is called the *regular E - \mathcal{F} -space for G* . Consequently, any pre E - \mathcal{F} -set contained in the E - \mathcal{F} -group G is an E - \mathcal{F} -subset. An *E - \mathcal{F} -subgroup* is a subgroup of G that is an E - \mathcal{F} -subset and satisfies all the E - \mathcal{F} -group axioms [13, page 37]. By [13, Proposition 1, page 87], a subgroup that is also an E - \mathcal{F} -subset is an E - \mathcal{F} -subgroup.

Definition 2.33. The \mathcal{F} -component of the identity of an E -group G is denoted by G° .

2.2. Fundamental theorems. In the rest of this chapter, let \mathcal{F} be an (E, Δ) -field, and let \mathcal{U} be an (E, Δ) -extension of \mathcal{F} which is (E, Δ) -universal over \mathcal{F} . Then

$\mathcal{K} = \mathcal{U}^\Delta$ considered as an E-field is clearly E-universal over $\mathcal{C} = \mathcal{F}^\Delta$ considered as an E-field and, thus, constrainedly closed. Also, \mathcal{G} will denote an E-strongly normal extension of \mathcal{F} .

The following theorem establishes a Galois correspondence between the set of intermediate differential fields of an E-strongly normal extension and the set of E- \mathcal{C} -subgroups of its Galois group when the field of Δ -constants is constrainedly closed. The proofs are very similar to [12, Chapter 6, Section 4].

Theorem 2.34 (First Fundamental Theorem). *Let \mathcal{G} be an E-strongly normal extension of \mathcal{F} with field of Δ -constants \mathcal{C} .*

- (1) *If \mathcal{F}_1 is an (E, Δ) -field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, then \mathcal{G} is E-strongly normal over \mathcal{F}_1 , $G(\mathcal{G}/\mathcal{F}_1)$ is an E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$, and the set of invariants of $G(\mathcal{G}/\mathcal{F}_1)$ in \mathcal{G} is \mathcal{F}_1 .*
- (2) *If G_1 is an E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ and \mathcal{F}_1 denotes the set of invariants of G_1 in \mathcal{G} , then \mathcal{F}_1 is an (E, Δ) -field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, and, if the elements of G_1 rational over \mathcal{C} are E-dense in G_1 , then $G(\mathcal{G}/\mathcal{F}_1) = G_1$.*
- (3) *If \mathcal{C} is constrainedly closed [13, page 79] as an E-field, parts (1) and (2) establish a bijective correspondence between (E, Δ) -subfields \mathcal{F}_1 with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$ and E-subgroups $G_1 \subseteq G(\mathcal{G}/\mathcal{F})$.*

Remark 2.35. It would be preferable to remove the hypothesis of constrainedly closed from part (3). For a certain type of small E- \mathcal{C} -subgroup, this is accomplished in Corollary 2.39, and, for subgroups of G_a and G_m , in Proposition 4.75 and Proposition 4.85. Also, if \mathcal{C} is E-universal over some E-field, then \mathcal{C} is constrainedly closed.

Proof. To prove part (1), let \mathcal{F}_1 be an (E, Δ) -field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$. Every (E, Δ) -isomorphism of \mathcal{G} over \mathcal{F}_1 is over \mathcal{F} , too, and hence is E-strong. Therefore \mathcal{G} is E-strongly normal over \mathcal{F}_1 , and the Galois group $G(\mathcal{G}/\mathcal{F}_1)$ is an E- \mathcal{C} -group by Theorem 1.24. It is obviously a subgroup and an E- \mathcal{C} -subset [13, pages 30 and 37] of $G(\mathcal{G}/\mathcal{F})$. Thus, $G(\mathcal{G}/\mathcal{F}_1)$ is an E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$. By definition, every element of \mathcal{F}_1 is an invariant of $G(\mathcal{G}/\mathcal{F}_1)$ in \mathcal{G} , and, by Proposition [12, Corollary, page 388], every such invariant is in \mathcal{F}_1 .

For part (2), let \mathcal{F}_1 be the set of invariants of G_1 in G . It is obvious that \mathcal{F}_1 is an (E, Δ) -field with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{G}$, and therefore, by part (1), $G(\mathcal{G}/\mathcal{F}_1)$ is E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$. Of course $G_1 \subset G(\mathcal{G}/\mathcal{F}_1)$. It must be shown that $G_1 = G(\mathcal{G}/\mathcal{F}_1)$ under the hypothesis that the elements of G_1 rational over \mathcal{C} are E-dense in G_1 .

Assume that $G_1 \neq G(\mathcal{G}/\mathcal{F}_1)$. Fix E- \mathcal{C} -generic elements $\sigma_1, \dots, \sigma_r$ of the E- \mathcal{C} -components of G_1 . By assumption, there exists an element $\tau \in G(\mathcal{G}/\mathcal{F}_1)$ that is not an E-specialization of any σ_k . Fixing elements $\eta_1, \dots, \eta_n \in \mathcal{G}$ with $\mathcal{F}\langle \eta_1, \dots, \eta_n \rangle_{E, \Delta} = \mathcal{G}$, by Lemma 1.3, for each index k there exists a differential polynomial $F_k \in \mathcal{G}\{y_1, \dots, y_n\}_{(E, \Delta)}$ that vanishes at $(\sigma_k \eta_1, \dots, \sigma_k \eta_n)$ but not at $(\tau \eta_1, \dots, \tau \eta_n)$. Then F_k vanishes at $(\sigma \eta_1, \dots, \sigma \eta_n)$ for all σ in the component of σ_k . The product $\prod_i F_i$ is a differential polynomial in $\mathcal{G}\{y_1, \dots, y_n\}_{(E, \Delta)}$ that vanishes at $(\sigma \eta_1, \dots, \sigma \eta_n)$ for every $\sigma \in G_1$ but not for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Let F be such a differential polynomial with as few nonzero terms as possible. Also suppose that one of the coefficients in F is 1. Consider any $\sigma' \in (G_1)_{\mathcal{C}}$. Then σ' is an (E, Δ) -automorphism of \mathcal{G} over \mathcal{F} . Since $F^{\sigma'}(\sigma \eta_1, \dots, \sigma \eta_n) = \sigma'(F(\sigma'^{-1} \sigma \eta_1, \dots, \sigma'^{-1} \sigma \eta_n))$, $F^{\sigma'}$ vanishes at $(\sigma \eta_1, \dots, \sigma \eta_n)$ for every $\sigma \in G_1$, because $\sigma'^{-1} \sigma \in G_1$. Therefore $F - F^{\sigma'}$ does

too. Since $F - F^{\sigma'}$ has fewer terms than F , $F - F^{\sigma'}$ must vanish at $(\sigma\eta_1, \dots, \sigma\eta_n)$ for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Hence for any $\alpha \in \mathcal{G}$, $F - \alpha(F - F^{\sigma'})$ vanishes at $(\sigma\eta_1, \dots, \sigma\eta_n)$ for every $\sigma \in \mathcal{G}_1$ but not for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. If $F - F^{\sigma'}$ were not zero, one could choose α so that $F - \alpha(F - F^{\sigma'})$ has fewer terms than F and is nonzero. Therefore $F - F^{\sigma'} = 0$ for $\sigma' \in (G_1)_e$.

By part (1), the set $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^\sigma\}$ is the E- \mathcal{C} -group leaving invariant the (E, Δ) - \mathcal{F} -field generated by the coefficients of F . In particular, it is an E- \mathcal{C} -subset of $G(\mathcal{G}/\mathcal{F})$ and a closed subset of the E- \mathcal{C} -topology on $G(\mathcal{G}/\mathcal{F})$. If the closed set $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^\sigma\} \cap G_1$ were not all of G_1 , there would be an element of $(G_1)_e$ not in $\{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^\sigma\} \cap G_1$ since (by hypothesis) $(G_1)_e$ is E-dense in G_1 . Therefore, $G_1 \subseteq \{\sigma \in G(\mathcal{G}/\mathcal{F}) \mid F = F^\sigma\}$, or $F = F^\sigma$ for all $\sigma \in G_1$. Since \mathcal{F}_1 is the (E, Δ) -field invariant under the action of G_1 , $F \in \mathcal{F}_1\{y_1, \dots, y_n\}$. However, then $F^\sigma = F$ for every $\sigma \in G(\mathcal{G}/\mathcal{F}_1)$, so that $F(\sigma\eta_1, \dots, \sigma\eta_n) = \sigma F(e\eta_1, \dots, e\eta_n) = 0$, since the identity e of G_1 is contained in G_1 . This contradiction shows that $G_1 = G(\mathcal{G}/\mathcal{F}_1)$ under the hypothesis that the elements of G_1 rational over \mathcal{C} are E-dense in G_1 .

For part (3), the hypothesis that \mathcal{C} is constrainedly closed implies that the elements of G_1 rational over \mathcal{C} are E-dense in G_1 ([13, Proposition 3, page 84]). \square

After two preliminary lemmas, the next corollary characterizes the E- \mathcal{C} -subgroups of $G(\mathcal{G}/\mathcal{F})$ with fixed field \mathcal{H} having the property that \mathcal{G} over \mathcal{H} is strongly normal (in the sense of Kolchin).

Lemma 2.36. *Let G and K be field extensions of F . Let H' be a subfield of GK containing K . Put $H = G \cap H'$. If H' and G are linearly disjoint over H and if K and H are linearly disjoint over F , then $H' = HK$:*

$$\begin{array}{ccc}
 G & \longrightarrow & GK \\
 \uparrow & & \uparrow \\
 H = G \cap H' & \longrightarrow & H' \\
 \uparrow & & \uparrow \\
 F & \longrightarrow & K.
 \end{array}$$

Proof. Evidently $HK \subset H'$. Consider any element $\varphi \in H'$. Fix a basis (c_i) of K over F . By considering φ as an element of GK , one may write $\varphi = (\sum \beta_i c_i) / (\sum \gamma_j c_j)$, where the β_i and γ_j are elements of G , and therefore $\sum \gamma_j (c_j \varphi) - \sum \beta_i c_i = 0$. Thus the elements $c_j \varphi$ and c_i of H' are linearly dependent over G . By the first hypothesis, they must be linearly dependent over H ; that is, there exist elements β'_i and γ'_j of H , not all 0, such that $\sum \gamma'_j (c_j \varphi) - \sum \beta'_i c_i = 0$. By the second hypothesis, the elements c_j of K are linearly independent over H , and therefore $\sum \gamma'_j c_j \neq 0$, so that $\varphi = (\sum \beta'_i c_i) / (\sum \gamma'_j c_j) \in HK$. This shows that $HK = H'$. \square

Lemma 2.37. *Let \mathcal{G} over \mathcal{F} be an E-strongly normal extension of (E, Δ) -fields, and let $G = G(\mathcal{G}/\mathcal{F})$, the associated E- \mathcal{C} -group of (E, Δ) -isomorphisms. Let H be an E- \mathcal{C} -subgroup of G and \mathcal{H} be the (E, Δ) -field of invariants of H in \mathcal{G} . If $\mathcal{C}\langle\sigma\rangle \subset \mathcal{C}U^{E, \Delta}$ for all $\sigma \in H$, then \mathcal{G} over \mathcal{H} as an (E, Δ) -extension is strongly normal in the sense of Kolchin.*

Proof. For all $\sigma \in H$, $\mathcal{C}\langle\sigma\rangle \subset \mathcal{C}\mathcal{U}^{\mathbb{E},\Delta}$ implies $\sigma\mathcal{G} \subset \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle \subset \mathcal{G}(\mathcal{C}\mathcal{U}^{\mathbb{E},\Delta}) = \mathcal{G}\mathcal{U}^{\mathbb{E},\Delta}$. Since σ leaves invariant Δ -constants, it also leaves invariant (\mathbb{E}, Δ) -constants. By [12, Proposition 10, page 393], \mathcal{G} over \mathcal{H} as an (\mathbb{E}, Δ) -extension is strongly normal as an (\mathbb{E}, Δ) -extension in the sense of Kolchin. \square

Lemma 2.38. *Let F be an \mathbb{E} -field, and let $\mathcal{V} \supset F$ be an \mathbb{E} -field that is \mathbb{E} -universal over F . Let B be an \mathbb{E} - \mathcal{F} -set. Let $C = F^{\mathbb{E}}$, and let C_a be the algebraic closure of C in $\mathcal{V}^{\mathbb{E}}$. If $B_{\mathcal{V}} \subset B_{F\mathcal{V}^{\mathbb{E}}}$, then B_{FC_a} is \mathbb{E} -dense in B .*

Proof. Since \mathcal{V} is a constrainedly closed extension of F ([13, Proposition 3, page 84]), $B_{\mathcal{V}}$ is dense in B [13, Proposition 3, page 84]. However, each point of B rational over \mathcal{V} is rational over $F\mathcal{V}^{\mathbb{E}}$ by assumption. But an element constrained over F rational over $F\mathcal{V}^{\mathbb{E}}$ is, in fact, rational over FC_a because an \mathbb{E} -extension constrained over \mathcal{C} has \mathbb{E} -constants algebraic over \mathcal{C} [12, Proposition 7(d), page 142]. Therefore, the set B_{FC_a} is \mathbb{E} -dense in B . \square

The formulation of the following corollary was influenced by Chapter 3 of Sit’s thesis [20], in which he considers Δ -subfields of $\mathcal{F}(t)_{\Delta}$ over which $\mathcal{F}(t)_{\Delta}$ is strongly normal in the sense of Kolchin, where t is a Δ -indeterminate over the Δ -field \mathcal{F} . For instance, the previous lemma is a generalization of [20, Lemma 2.1, page 652] from an affine \mathbb{E} -Zariski closed subset of \mathcal{V}^n to an \mathbb{E} - \mathcal{F} -subset that is not necessarily affine. In this corollary, these ideas have been combined with those of Kolchin in the second part of his proof of the fundamental theorem for strongly normal extensions ([12, Theorem 3, page 398]).

Corollary 2.39. *Let $\mathcal{L} = \mathcal{U}^{\mathbb{E},\Delta}$ and let $G = G(\mathcal{G}/\mathcal{F})$. Let \mathcal{J} be the set of (\mathbb{E}, Δ) -subfields \mathcal{H} of \mathcal{G} containing \mathcal{F} such that \mathcal{G} over \mathcal{H} is strongly normal as an (\mathbb{E}, Δ) -field extension (in the sense of Kolchin), and let \mathcal{S} be the set of \mathbb{E} - \mathcal{C} -subgroups H of G such that $H_{\mathcal{U}^{\Delta}} \subset H_{\mathcal{C}\mathcal{L}}$. Then there is a Galois correspondence between \mathcal{J} and \mathcal{S} .*

Proof. Let $\mathcal{H} \in \mathcal{J}$. Then by part (1) of the First Fundamental Theorem 2.34, there exists an \mathbb{E} - \mathcal{C} -subgroup $H = G(\mathcal{G}/\mathcal{H})$ of G such that the (\mathbb{E}, Δ) -field of invariants of H is \mathcal{H} . Let $\sigma \in H_{\mathcal{U}^{\Delta}}$. Because \mathcal{G} over \mathcal{H} is strongly normal as an (\mathbb{E}, Δ) -extension (in the sense of Kolchin), σ is a strong (in the sense of Kolchin) (\mathbb{E}, Δ) -isomorphism of \mathcal{G} over \mathcal{H} , and $\mathcal{G}\sigma\mathcal{G} \subset \mathcal{G} \cdot \mathcal{L}$:

$$\begin{array}{ccc}
 \mathcal{G} & \longrightarrow & \mathcal{G}\mathcal{L} \\
 \uparrow & & \uparrow \\
 (\mathcal{G}\mathcal{L})^{\Delta} \cap \mathcal{G} = \mathcal{G}^{\Delta} & \longrightarrow & (\mathcal{G}\mathcal{L})^{\Delta} \\
 \uparrow & & \uparrow \\
 \mathcal{G}^{\mathbb{E},\Delta} & \longrightarrow & \mathcal{L}.
 \end{array}$$

Apply Lemma 2.36 to the case $G = \mathcal{G}$, $K = \mathcal{L}$, $F = \mathcal{G}^{\mathbb{E},\Delta}$, $H' = (\mathcal{G}\mathcal{L})^{\Delta}$ and $H = \mathcal{G}^{\Delta}$. By the linear disjointness of \mathbb{E} -constants [12, Corollary 1, page 87], \mathcal{G}^{Δ} and \mathcal{L} are linearly disjoint over $\mathcal{G}^{\mathbb{E},\Delta}$, and \mathcal{G} and $(\mathcal{G}\mathcal{L})^{\Delta}$ are linearly disjoint over \mathcal{G}^{Δ} by the linear disjointness of Δ -constants. This lemma then implies $(\mathcal{G}\mathcal{L})^{\Delta} = \mathcal{G}^{\Delta}\mathcal{L}$. Then, $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G})^{\Delta} \subset (\mathcal{G} \cdot \mathcal{L})^{\Delta} = \mathcal{G}^{\Delta} \cdot \mathcal{L} = \mathcal{C}\mathcal{L}$. Therefore, $\sigma \in H_{\mathcal{C}\mathcal{L}}$, and $H \in \mathcal{S}$.

Let $H \in \mathcal{S}$, and let \mathcal{H} be the corresponding subfield of invariants of H in \mathcal{G} . If the elements of H rational over \mathcal{C} are \mathbb{E} -dense in H , by part (1) of the First Fundamental

Theorem 2.34, \mathcal{H} is a differential field with $\mathcal{F} \subset \mathcal{H} \subset \mathcal{G}$ and $H = G(\mathcal{G}/\mathcal{H})$. By Lemma 2.37, \mathcal{G} over \mathcal{H} is strongly normal as an (E, Δ) -extension (in the sense of Kolchin), and $\mathcal{H} \in \mathcal{J}$.

Let $\mathcal{D} = \mathcal{C}^E = \mathcal{G}^{(E, \Delta)}$, and let \mathcal{D}_a be the algebraic closure of \mathcal{D} in \mathcal{L} . For all $H \in \mathcal{S}$, the set of elements of H rational over $\mathcal{C}\mathcal{D}_a$ is E -dense in H by Lemma 2.38. (For the affine case, see a lemma of Sit [20, Chapter 2, Section 2].) By results in the two paragraphs above, if $\mathcal{D}_a \subseteq \mathcal{C}$ or equivalently if $\mathcal{D}_a\mathcal{C} = \mathcal{C}$, then there is a Galois correspondence between \mathcal{J} and \mathcal{S} .

To prove the corollary without assuming $\mathcal{D}_a \subseteq \mathcal{C}$, let $H \in \mathcal{S}$, and let \mathcal{H} be the set of invariants of H in \mathcal{G} . It will be shown that $H = G(\mathcal{G}/\mathcal{H})$. Let \mathcal{H}' denote the set of invariants of H in $\mathcal{G}\mathcal{D}_a$. Then \mathcal{H}' is an (E, Δ) -field with $\mathcal{F}\mathcal{D}_a \subset \mathcal{H}' \subset \mathcal{G}\mathcal{D}_a$ and $\mathcal{G} \cap \mathcal{H}' = \mathcal{H}$:

$$\begin{array}{ccc}
 \mathcal{G} & \longrightarrow & \mathcal{G}\mathcal{D}_a \\
 \uparrow & & \uparrow \\
 \mathcal{G} \cap \mathcal{H}' = \mathcal{H} & \longrightarrow & \mathcal{H}' \\
 \uparrow & & \uparrow \\
 \mathcal{F} & \longrightarrow & \mathcal{F}\mathcal{D}_a \\
 \uparrow & & \uparrow \\
 \mathcal{D} & \longrightarrow & \mathcal{D}_a.
 \end{array}$$

Claim 2.40. The fields \mathcal{G} and \mathcal{H}' are linearly disjoint over \mathcal{H} .

To prove this, consider elements $\varphi_1, \dots, \varphi_s \in \mathcal{H}'$ that are linearly dependent over \mathcal{G} . It must be shown that they are linearly dependent over \mathcal{H} . It may be assumed that $s > 1$ and no $s - 1$ of them are linearly dependent over \mathcal{G} . Then there exist nonzero elements $\alpha_1, \dots, \alpha_s \in \mathcal{G}$ with $\sum_{1 \leq j \leq s} \alpha_j \varphi_j = 0$. Dividing by α_s one may suppose that $\alpha_s = 1$. For any $\sigma \in H$, since \mathcal{H}' is invariant under H , $\sum_{1 \leq j \leq s} (\sigma \alpha_j) \varphi_j = 0$, and therefore $\sum_{1 \leq j \leq s-1} (\sigma \alpha_j - \alpha_j) \varphi_j = 0$.

Take $\sigma \in H_{\mathcal{C}\mathcal{D}_a}$ so that, by definition, $\mathcal{C}\langle \sigma \rangle$ is algebraic over \mathcal{C} . By part (1) of the definition of a pre E -set, $\mathcal{C}\langle \sigma \rangle$ is finitely E -generated over \mathcal{C} . Therefore, the degree of $\mathcal{G}\mathcal{C}\langle \sigma \rangle$ over \mathcal{G} is finite. Let $f_i : \mathcal{G}\mathcal{C}\langle \sigma \rangle \rightarrow \mathcal{G}\mathcal{D}_a$ for $i = 1, \dots, t$ denote the finite set of isomorphisms (not necessarily differential) over \mathcal{G} . Suppose $f_1 = \text{id}$. It is simple to show that each f_i is, in fact, an (E, Δ) -isomorphism because $\mathcal{G}\mathcal{C}\langle \sigma \rangle$ over \mathcal{G} is algebraic. By Lemma 2.37, σ is a strong isomorphism of \mathcal{G} over \mathcal{F} (in the sense of Kolchin) such that $(\mathcal{G}\sigma\mathcal{G})^\Delta = \mathcal{C}\langle \sigma \rangle \subset \mathcal{C}\mathcal{D}_a$ and $\mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle \sigma \rangle \subset \mathcal{G}\mathcal{D}_a$. Consider the (E, Δ) -isomorphisms of \mathcal{G} defined as $\sigma_i = f_i \sigma$ for $i = 1, \dots, t$. So $\sigma = \sigma_1$. For each i , by the definitions of f_i and σ_i , there exist (E, Δ) - \mathcal{G} -isomorphisms $\psi_i : \mathcal{G}\sigma\mathcal{G} \rightarrow \mathcal{G}\sigma_i\mathcal{G}$ such that $\psi_i(\alpha) = \alpha$ and $\psi_i(\sigma\alpha) = \sigma_i\alpha$ for all $\alpha \in \mathcal{G}$. That is, $\sigma \leftrightarrow \sigma_i$ in G . Since $\sigma \rightarrow \sigma_i$ and since H is E -closed in $G(\mathcal{G}/\mathcal{F})$, $\sigma_i \in H$.

Thus $\sum_{1 \leq j \leq s-1} (\sigma_k \alpha_j - \alpha_j) \varphi_j = 0$ (for $1 \leq k \leq t$). If $\sigma \alpha_1 - \alpha_1 \neq 0$, then, because f_k is an isomorphism over \mathcal{G} , $0 \neq f_k(\sigma \alpha_1 - \alpha_1) = f_k \sigma \alpha_1 - f_k \alpha_1 = \sigma_k \alpha_1 - \alpha_1$. So, one may divide by $\sigma_k \alpha_1 - \alpha_1$ for each k to obtain

$$(7) \quad \sum_{1 \leq j \leq s-1} (\sigma_k \alpha_1 - \alpha_1)^{-1} (\sigma_k \alpha_j - \alpha_j) \varphi_j = 0 \quad (\text{for } 1 \leq k \leq t).$$

Set $\alpha'_j = \sum_{1 \leq k \leq t} (\sigma_k \alpha_1 - \alpha_1)^{-1} (\sigma_k \alpha_j - \alpha_j) = \text{Tr} (\sigma \alpha_1 - \alpha_1)^{-1} (\sigma \alpha_j - \alpha_j)$ (Tr is the trace of $\mathcal{G}\mathcal{D}_a$ over \mathcal{G}). By summing the equations (7), one would have $\sum_{1 \leq j \leq s-1} \alpha'_j \varphi_j = 0$, $\alpha'_j \in \mathcal{G}$ ($1 \leq j \leq s-1$), $\alpha'_1 = \text{Tr} 1 \neq 0$. This contradicts the linear independence of $\varphi_1, \dots, \varphi_{s-1}$ over \mathcal{G} . Therefore, $\sigma \alpha_1 = \alpha_1$ for every $\sigma \in H_{e\mathcal{D}_a}$. Since $H_{e\mathcal{D}_a}$ is E-dense in H , $\sigma \alpha_1 = \alpha_1$ for every $\sigma \in H_{\mathcal{U}\Delta}$. Therefore, $\alpha_1 \in \mathcal{H}$. Similarly, $\alpha_k \in \mathcal{H}$ for every index k , so that $\varphi_1, \dots, \varphi_s$ are linearly dependent over \mathcal{H} . This establishes the claim.

By the claim and Lemma 2.36, $\mathcal{H}' = \mathcal{H}\mathcal{D}_a$. It follows from Theorem 1.30 that $G(\mathcal{G}/\mathcal{H}') = G(\mathcal{G}\mathcal{D}_a/\mathcal{H}\mathcal{D}_a) = G(\mathcal{G}\mathcal{D}_a/\mathcal{H}')$. Because it has been shown that $H_{e\mathcal{D}_a}$ is E-dense in H and \mathcal{H}' is the (E, Δ) -subfield of invariants of H in $\mathcal{G}\mathcal{D}_a$, the Galois correspondence (part (2) of the First Fundamental Theorem 2.34) implies $G(\mathcal{G}\mathcal{D}_a/\mathcal{H}') = H$ and, thus, $G(\mathcal{G}/\mathcal{H}') = H$. Since $\mathcal{H} \in \mathcal{J}$, this establishes the Galois correspondence of the theorem. \square

Corollary 2.41. *Assume that \mathcal{C} is constrainedly closed over \mathcal{F} as an E-field. Let \mathcal{F}_1 and \mathcal{F}_2 be (E, Δ) -differential fields contained in \mathcal{G} and containing \mathcal{F} . Then $G(\mathcal{G}/\mathcal{F}_1\mathcal{F}_2) = G(\mathcal{G}/\mathcal{F}_1) \cap G(\mathcal{G}/\mathcal{F}_2)$, and $G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$ is the smallest E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{G}/\mathcal{F}_1)G(\mathcal{G}/\mathcal{F}_2)$.*

Proof. Observe that an (E, Δ) -isomorphism of \mathcal{G} leaves invariant every element of $\mathcal{F}_1\mathcal{F}_2$ if and only if it leaves invariant every element of \mathcal{F}_1 and every element of \mathcal{F}_2 . Thus the first assertion is true, because, under the assumptions of this corollary, the Galois correspondences of the First Fundamental Theorem and Corollary 2.39 imply the subgroups of the Galois groups are uniquely determined by the invariant subfields.

For the second assertion, the smallest E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{G}/\mathcal{F}_1)G(\mathcal{G}/\mathcal{F}_2)$ is of the form $G(\mathcal{G}/\mathcal{F}')$, where $\mathcal{F}' \subset \mathcal{F}_1 \cap \mathcal{F}_2$, so that $G(\mathcal{G}/\mathcal{F}') \supset G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$. On the other hand, $G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$ is an E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{G}/\mathcal{F}_1)$ and $G(\mathcal{G}/\mathcal{F}_2)$, so that $G(\mathcal{G}/\mathcal{F}') \subset G(\mathcal{G}/\mathcal{F}_1 \cap \mathcal{F}_2)$. \square

Theorem 2.42. *Assume that \mathcal{C} is constrainedly closed over \mathcal{F} as an E-field. Let $\mathcal{F}_1 \subseteq \mathcal{G}$ be an (E, Δ) -field containing \mathcal{F} . Then the following four conditions are equivalent.*

- (1) \mathcal{F}_1 is an E-strongly normal extension of \mathcal{F} .
- (2) For each element $\alpha \in \mathcal{F}_1$ with $\alpha \notin \mathcal{F}$, there exists an E-strong isomorphism σ_1 of \mathcal{F}_1 over \mathcal{F} such that $\sigma_1 \alpha \neq \alpha$.
- (3) $G(\mathcal{G}/\mathcal{F}_1)$ is a normal E-subgroup of $G(\mathcal{G}/\mathcal{F})$.
- (4) $\sigma \mathcal{F}_1 \subset \mathcal{F}_1 \mathcal{U}^\Delta$ for every $\sigma \in G(\mathcal{G}/\mathcal{F})$.

When these conditions are satisfied, then, for each $\sigma \in G(\mathcal{G}/\mathcal{F})$, the restriction σ_1 of σ to \mathcal{F}_1 is an element of $G(\mathcal{F}_1/\mathcal{F})$, and the formula $\sigma \mapsto \sigma_1$ defines a surjective E- \mathcal{C} -homomorphism $G(\mathcal{G}/\mathcal{F}) \rightarrow G(\mathcal{F}_1/\mathcal{F})$ with kernel $G(\mathcal{G}/\mathcal{F}_1)$.

Remark 2.43. In the proof below, only the implication condition (4) implies condition (3) and uses part (3) of the First Fundamental Theorem.

Proof. If condition (1) is satisfied, then, by part (1) of the First Fundamental Theorem 2.34, the set of invariants of $G(\mathcal{F}_1/\mathcal{F})$ in \mathcal{F}_1 is \mathcal{F} , so that part (2) is satisfied. Let condition (2) be satisfied. The normalizer N of $G(\mathcal{F}_1/\mathcal{F})$ in $G(\mathcal{G}/\mathcal{F})$ is an E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ containing $G(\mathcal{F}_1/\mathcal{F})$ [13, Corollary 2, page 103]. By the First Fundamental Theorem and Corollary 2.39, there exists a differential field

\mathcal{F}_2 with $\mathcal{F} \subset \mathcal{F}_2 \subset \mathcal{F}_1$ such that $G(\mathcal{G}/\mathcal{F}_2) = N$. By the universality of \mathcal{U} , if σ_1 is any E-strong isomorphism of \mathcal{F}_1 over \mathcal{F} , σ_1 can be extended to an E- \mathcal{F} -isomorphism of \mathcal{G} , that is, to an element $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then for any $\tau \in G(\mathcal{G}/\mathcal{F}_1)$ and any $\beta \in \mathcal{F}_1$, $\sigma\beta = \sigma_1\beta \in \mathcal{F}_1\mathcal{U}^\Delta$; hence $\tau\sigma\beta = \sigma\beta$ and $\sigma^{-1}\tau\sigma\beta = \beta$, so that $\sigma^{-1}\tau\sigma \in G(\mathcal{G}/\mathcal{F}_1)$. Thus, $\sigma \in N$, so that σ_1 leaves invariant every element of \mathcal{F}_2 . Since σ is an extension of an arbitrary element of $G(\mathcal{F}_1/\mathcal{F})$, it follows by condition (2) that $\mathcal{F}_2 = \mathcal{F}$, that is, $N = G(\mathcal{G}/\mathcal{F})$. Therefore, condition (3) is proved from condition (2). Next, let condition (3) be satisfied. Consider any $\sigma \in G(\mathcal{G}/\mathcal{F})$ and any $\beta \in \mathcal{F}_1$. For every $\tau \in G(\mathcal{G}/\mathcal{F}_1)$, $\sigma^{-1}\tau\sigma \in G(\mathcal{G}/\mathcal{F}_1)$, so that $\sigma^{-1}\tau\sigma\beta = \beta$ and $\tau\sigma\beta = \sigma\beta$. Since by Theorem 4.30, $G(\mathcal{G}/\mathcal{F}_1) = G(\mathcal{G}\mathcal{C}\langle\sigma\rangle/\mathcal{F}_1\mathcal{C}\langle\sigma\rangle)$, and since $\sigma\beta \in \mathcal{G}\sigma\mathcal{G} = \mathcal{G}\mathcal{C}\langle\sigma\rangle$, $\sigma\beta$ is an invariant of $G(\mathcal{G}\mathcal{C}\langle\sigma\rangle/\mathcal{F}_1\mathcal{C}\langle\sigma\rangle)$ in $\mathcal{G}\mathcal{C}\langle\sigma\rangle$, and hence, by the first part of the First Fundamental Theorem and Corollary 2.39, $\sigma\beta \in \mathcal{F}_1\mathcal{C}\langle\sigma\rangle$. Therefore condition (4) is satisfied. Let condition (4) be satisfied. If σ' is any isomorphism of \mathcal{F}_1 over \mathcal{F} , then σ' can be extended to an element $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then because of condition (4) $\sigma'\mathcal{F}_1 = \sigma\mathcal{F}_1 \subset \mathcal{F}_1\mathcal{K}$. It follows by [12, Proposition 10, page 393], that condition (1) is satisfied. Therefore all four conditions are equivalent.

Let the conditions be satisfied. It is obvious that the restriction mapping defined by the formula $\sigma \mapsto \sigma_1$ is a group homomorphism $G(\mathcal{G}/\mathcal{F}) \rightarrow G(\mathcal{G}/\mathcal{F}_1)$ with kernel $G(\mathcal{G}/\mathcal{F}_1)$. It has already been observed that every isomorphism of \mathcal{F}_1 over \mathcal{F} can be extended to an isomorphism of \mathcal{G} , and this shows that the homomorphism is surjective. It remains to prove that it is an E- \mathcal{C} -homomorphism. First of all, $\mathcal{C}\langle\sigma\rangle = (\mathcal{G}\sigma\mathcal{G}) \cap \mathcal{U}^\Delta \supset (\mathcal{F}_1\sigma_1\mathcal{F}_1) \cap \mathcal{U}^\Delta = \mathcal{C}\langle\sigma_1\rangle$. Next, if σ' is an E-specialization of σ in $G(\mathcal{G}/\mathcal{F})$, then, by definition, $\sigma \xrightarrow{\mathcal{G}} \sigma'$ (Definition 1.2). By Lemma 1.3, this is equivalent to $(\sigma'\alpha)_{\alpha \in \mathcal{G}}$ being an (E, Δ) - \mathcal{G} -specialization of $(\sigma\alpha)_{\alpha \in \mathcal{G}}$ over \mathcal{G} , so that a fortiori $(\sigma'_1\alpha)_{\alpha \in \mathcal{G}}$ is an (E, Δ) - \mathcal{G} -specialization of $(\sigma_1\alpha)_{\alpha \in \mathcal{G}}$ over \mathcal{F}_1 ; that is, σ'_1 is a differential specialization of σ_1 by Lemma 1.3. Finally, if σ' is a generic specialization of σ , then by the above, σ'_1 is a generic specialization of σ_1 . Since the induced E- \mathcal{C} -isomorphism $S_{\sigma',\sigma}: \mathcal{C}\langle\sigma\rangle \approx \mathcal{C}\langle\sigma'\rangle$ is a restriction of the (E, Δ) - \mathcal{G} -isomorphism $\mathcal{G}\sigma\mathcal{G} \approx \mathcal{G}\sigma'\mathcal{G}$ mapping $\sigma\alpha$ onto $\sigma'\alpha$ ($\alpha \in \mathcal{G}$), and the induced E- \mathcal{C} -isomorphism $S_{\sigma'_1,\sigma_1}: \mathcal{C}\langle\sigma_1\rangle \approx \mathcal{C}\langle\sigma'_1\rangle$ is a restriction of the (E, Δ) - \mathcal{G} -isomorphism $\mathcal{F}_1\sigma_1\mathcal{F}_1 \approx \mathcal{F}_1\sigma'_1\mathcal{F}_1$ over \mathcal{F}_1 mapping $\sigma\alpha$ onto $\sigma'\alpha$ ($\alpha \in \mathcal{F}_1$), it is evident that $S_{\sigma',\sigma}$ is an extension of $S_{\sigma'_1,\sigma_1}$. This shows that the restriction mapping is an E- \mathcal{C} -homomorphism and completes the proof of the theorem. \square

Corollary 2.44. *Assume that \mathcal{C} is constrainedly closed over \mathcal{F} as an E-field. Let \mathcal{F}° denote the algebraic closure of \mathcal{F} in \mathcal{G} . Then $G(\mathcal{G}/\mathcal{F}^\circ) = G^\circ(\mathcal{G}/\mathcal{F})$ (Definition 2.33), \mathcal{F}° is an E-strongly normal extension of \mathcal{F} , and $G(\mathcal{F}^\circ/\mathcal{F}) \approx G(\mathcal{G}/\mathcal{F})/G^\circ(\mathcal{G}/\mathcal{F})$. In particular, the degree of \mathcal{F}° over \mathcal{F} equals the index of $G^\circ(\mathcal{G}/\mathcal{F})$ in $G(\mathcal{G}/\mathcal{F})$, so that \mathcal{F} is algebraically closed in \mathcal{G} if and only if $G(\mathcal{G}/\mathcal{F})$ is connected, and \mathcal{G} is algebraic over \mathcal{F} if and only if $G(\mathcal{G}/\mathcal{F})$ is finite.*

Proof. By Definition 1.18, there exists an isolated (E, Δ) - \mathcal{F} -isomorphism $\sigma_0 \in G^\circ(\mathcal{G}/\mathcal{F})$ such that $\sigma_0 \xrightarrow{\mathcal{G}} \text{id}$. By part b of [13, Corollary to Proposition 2, page 388], the set of invariants of $G^\circ(\mathcal{G}/\mathcal{F})$ is \mathcal{F}° . Therefore, by the First Fundamental Theorem, $G^\circ(\mathcal{G}/\mathcal{F}) = G(\mathcal{G}/\mathcal{F}^\circ)$. As $G^\circ(\mathcal{G}/\mathcal{F})$ is a normal E- \mathcal{C} -subgroup of $G(\mathcal{G}/\mathcal{F})$ [13, Theorem 1, page 39], the previous theorem shows that \mathcal{F}° is E-strongly normal over \mathcal{F} and $G(\mathcal{F}^\circ/\mathcal{F}) \approx G(\mathcal{G}/\mathcal{F})/G^\circ(\mathcal{G}/\mathcal{F})$. \square

Corollary 2.45. *Assume that \mathcal{C} is constrainedly closed over \mathcal{F} as an E -field. Assume that \mathcal{GH} and \mathcal{F} have the same field of Δ -constants. Then $\mathcal{G} \cap \mathcal{H}$ is an E -strongly normal extension of \mathcal{F} .*

Proof. By Corollary 1.14, \mathcal{GH} is E -strongly normal over \mathcal{F} . By Theorem 2.42, $G(\mathcal{GH}/\mathcal{G})$ and $G(\mathcal{GH}/\mathcal{H})$ are normal E - \mathcal{C} -subgroups of $G(\mathcal{GH}/\mathcal{F})$, so that their product is also [13, Corollary 2, page 109]. By Corollary 2.41, the product is $G(\mathcal{GH}/\mathcal{G} \cap \mathcal{H})$. Since it is normal in $G(\mathcal{GH}/\mathcal{F})$, it follows by Theorem 2.42 that $\mathcal{G} \cap \mathcal{H}$ is E -strongly normal over \mathcal{F} . \square

Theorem 2.46. *Assume that \mathcal{C} is constrainedly closed over \mathcal{F} as an E -field. Let \mathcal{E} be an extension of \mathcal{F} such that \mathcal{U} is universal over \mathcal{E} as an E -strongly normal extension and the field of Δ -constants of \mathcal{GE} is \mathcal{C} . Then \mathcal{GE} is an E -strongly normal extension of \mathcal{E} , for each element $\tau \in G(\mathcal{GE}/\mathcal{E})$ the restriction τ_1 of τ to \mathcal{G} is an element of $G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$, and the formula $\tau \mapsto \tau_1$ defines an E - \mathcal{C} -isomorphism $G(\mathcal{GE}/\mathcal{E}) \approx G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$.*

Proof. For any (E, Δ) - \mathcal{E} -isomorphism τ of \mathcal{GE} , τ_1 is obviously an (E, Δ) -isomorphism of \mathcal{G} over $\mathcal{G} \cap \mathcal{E}$ and hence is E -strong. Therefore,

$$\tau(\mathcal{GE}) \subseteq \mathcal{GE} \cdot \tau(\mathcal{GE}) = \mathcal{GE}\tau\mathcal{G} \cdot \mathcal{E} = \mathcal{G}\tau_1\mathcal{G} \cdot \mathcal{E} = \mathcal{G}\mathcal{C}\langle\tau_1\rangle \cdot \mathcal{E} = \mathcal{G}\mathcal{E}\mathcal{C}\langle\tau_1\rangle \subset \mathcal{G}\mathcal{E}\mathcal{U}^\Delta.$$

It follows from Proposition 1.13 that \mathcal{GE} is E -strongly normal over \mathcal{E} .

Clearly the formula $\tau \mapsto \tau_1$ defines an injective group homomorphism $G(\mathcal{GE}/\mathcal{E}) \rightarrow G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$. It also follows from the above sequence of equalities that $\mathcal{GE}\mathcal{C}\langle\tau\rangle = \mathcal{G}\mathcal{E}\mathcal{C}\langle\tau_1\rangle$ and by [12, Corollary 2, page 88], $\mathcal{C}\langle\tau\rangle = \mathcal{C}\langle\tau_1\rangle$. If τ and τ' are elements of $G(\mathcal{GE}/\mathcal{E})$ and $\tau \rightarrow \tau'$, then $(\tau'\beta)_{\beta \in \mathcal{GE}}$ is an (E, Δ) -specialization of $(\tau\beta)_{\beta \in \mathcal{GE}}$ over \mathcal{GE} , so that $(\tau'_1\beta)_{\beta \in \mathcal{G}}$ is an (E, Δ) -specialization of $(\tau_1\beta)_{\beta \in \mathcal{G}}$ over \mathcal{G} , whence $\tau_1 \rightarrow \tau'_1$. If moreover $\tau \leftrightarrow \tau'$, then $\tau_1 \leftrightarrow \tau'_1$, and the (E, Δ) -isomorphism $\mathcal{GE} \cdot \tau(\mathcal{GE}) \approx \mathcal{GE} \cdot \tau'(\mathcal{GE})$ over \mathcal{GE} mapping $\tau\beta$ onto $\tau\beta'$ ($\beta \in \mathcal{GE}$) is an extension of the (E, Δ) -isomorphism $\mathcal{G}\tau_1\mathcal{G} \approx \mathcal{G}\tau'_1\mathcal{G}$ over \mathcal{G} mapping $\tau_1\beta$ onto $\tau'_1\beta$ ($\beta \in \mathcal{G}$). Since these two (E, Δ) -isomorphisms are extensions of the induced E -isomorphisms $S_{\tau',\tau} : \mathcal{C}\langle\tau\rangle \approx \mathcal{C}\langle\tau'\rangle$ and $S_{\tau'_1,\tau_1} : \mathcal{C}\langle\tau_1\rangle \approx \mathcal{C}\langle\tau'_1\rangle$, and since $\mathcal{C}\langle\tau\rangle = \mathcal{C}\langle\tau_1\rangle$ and $\mathcal{C}\langle\tau'\rangle = \mathcal{C}\langle\tau'_1\rangle$, $S_{\tau',\tau} = S_{\tau'_1,\tau_1}$. It follows that the injective group homomorphism is an E - \mathcal{C} -homomorphism. Its image is an E - \mathcal{C} -subgroup G_1 of $G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$. If an element $\alpha \in \mathcal{G}$ is an invariant of G_1 , then it is an invariant of $G(\mathcal{GE}/\mathcal{E})$, whence $\alpha \in \mathcal{E}$. Thus, the set of invariants of G_1 in \mathcal{G} is $\mathcal{G} \cap \mathcal{E}$, so that $G_1 = G(\mathcal{G}/\mathcal{G} \cap \mathcal{E})$ by the First Fundamental Theorem and Corollary 2.39. \square

3. DISJOINTNESS FROM DERIVATIVES

In this chapter, Kolchin's concept of disjointness is defined and used in two ways to construct E -strongly normal extensions. First, an E -strongly normal extension will be constructed with Galois group E -isomorphic to any given connected E -group. The method of proof of this result is new even for algebraic groups in Kolchin's setting [10, Theorem 2, page 880] and does not require the field of constants to be algebraically closed as does the result of Kolchin. A second use of these extensions will be to define a functor from pre Δ' -sets to pre Δ -sets. This takes a Δ' -group to a Δ -group and is compatible with the Galois theory (Section 3.5). By combining this result with the First Fundamental Theorem, for any Δ' -subgroup of an algebraic group, a Δ' -strongly normal extension is obtained with that Δ' -subgroup as its Galois group.

In this section, Δ is the union of two disjoint subsets Δ' and Δ'' and \mathcal{F} is a Δ -field.

3.1. Definition of Δ'' -free.

Definition 3.47. Let \mathcal{A} be a Δ - \mathcal{F} -algebra. Let Δ' be a finite commuting subset of the vector space of derivations of \mathcal{U} spanned by Δ over \mathcal{F} . Let \mathcal{A}' be a Δ' - \mathcal{F} -subalgebra of \mathcal{A} such that \mathcal{A}' generates \mathcal{A} as a Δ - \mathcal{F} -algebra. Define \mathcal{A} to be Δ/Δ' - \mathcal{F} -free over \mathcal{A}' if any Δ' - \mathcal{F} -homomorphism of \mathcal{A}' into a Δ -field extension of \mathcal{F} can be extended to a Δ - \mathcal{F} -homomorphism of \mathcal{A} . If Δ is the disjoint union of two subsets Δ' and Δ'' , define \mathcal{A} to be Δ'' - \mathcal{F} -free over \mathcal{A}' if \mathcal{A} is Δ/Δ' - \mathcal{F} -free over \mathcal{A}' .

Kolchin [13, Section 7, page 19] uses the terminology “ \mathcal{A}' and Δ are Δ' -disjoint over \mathcal{F} ” instead of \mathcal{A} is Δ/Δ' - \mathcal{F} -free over \mathcal{A}' . Although Kolchin’s terminology does not refer to the ring \mathcal{A} that \mathcal{A}' Δ -generates. But \mathcal{A} is implicit in Kolchin’s definition because the Δ' -algebra \mathcal{A}' is assumed to be contained in some larger unspecified Δ -algebra, so that \mathcal{A} is uniquely determined by \mathcal{A}' and the Δ -algebra containing it.

The following proposition shows that if \mathcal{A} is Δ/Δ' - \mathcal{F} -free over \mathcal{A}' the Δ' - \mathcal{F} -isomorphism class of \mathcal{A}' determines the Δ - \mathcal{F} -isomorphism class of \mathcal{A}'_{Δ} .

Proposition 3.48. *Let \mathcal{A} and \mathcal{B} be Δ - \mathcal{F} -algebras that are integral domains. Let \mathcal{A}' and \mathcal{B}' be Δ' - \mathcal{F} -subalgebras of \mathcal{A} and \mathcal{B} such that \mathcal{A} is Δ/Δ' - \mathcal{F} -free over \mathcal{A}' and \mathcal{B} is Δ/Δ' - \mathcal{F} -free over \mathcal{B}' . If \mathcal{A}' and \mathcal{B}' are Δ' - \mathcal{F} -isomorphic, then $\mathcal{A} = \mathcal{A}'_{\Delta}$ and $\mathcal{B} = \mathcal{B}'_{\Delta}$ are Δ - \mathcal{F} -isomorphic.*

Proof. In the definition of Δ/Δ'' -free, the extension Δ -homomorphism is clearly unique because it is determined by the action of the Δ' -homomorphism on Δ' -ring generators. Let $\varphi' : \mathcal{A}' \rightarrow \mathcal{B}'$ be a given Δ' - \mathcal{F} -isomorphism, and let $\chi' : \mathcal{B}' \rightarrow \mathcal{A}'$ be the inverse Δ' - \mathcal{F} -isomorphism. Then φ' and χ' extend to unique Δ - \mathcal{F} -homomorphisms $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ and $\chi : \mathcal{B} \rightarrow \mathcal{A}$. The composite Δ - \mathcal{F} -homomorphism $\chi\varphi : \mathcal{A} \rightarrow \mathcal{A}$ is the unique Δ - \mathcal{F} -homomorphism extending the identity Δ' - \mathcal{F} -isomorphism of \mathcal{A}' and, therefore, is the identity Δ - \mathcal{F} -isomorphism of \mathcal{A} . Similarly, $\varphi\chi$ is the identity, and, therefore, φ is a Δ - \mathcal{F} -isomorphism. \square

Corollary 3.49. *Let \mathcal{A} be an integral domain and Δ/Δ' - \mathcal{F} -free over \mathcal{A}' . Then each Δ' -automorphism of \mathcal{A}' extends uniquely to a Δ -automorphism of $\mathcal{A} = \mathcal{A}'_{\Delta}$.*

The following is the first basic proposition of Kolchin about this concept of Δ/Δ' - \mathcal{F} -free extensions [13, Proposition 9, page 20].

Proposition 3.50. *Let $\eta = (\eta_j)_{j \in J}$ be a family of elements of a Δ -field extension \mathcal{U} that is Δ -universal over \mathcal{F} , let Δ' be a commutative linearly independent subset of $\mathcal{F}\Delta$, and let \mathfrak{P}' and \mathfrak{P} denote, respectively, the defining Δ' -ideal of η in $\mathcal{F}\{(y_j)_{j \in J}\}_{\Delta'}$ and the defining Δ -ideal of η in $\mathcal{F}\{(y_j)_{j \in J}\}_{\Delta}$. Then the following three conditions are equivalent:*

- (1) $\mathcal{F}\{\eta\}_{\Delta}$ is Δ'' -free over $\mathcal{F}\{\eta\}_{\Delta'}$;
- (2) $\mathcal{F}\{\eta\}_{\Delta}$ is Δ'' -free over $\mathcal{F}\langle \eta \rangle_{\Delta'}$;
- (3) $\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$.

The equivalence of condition (1) and condition (3) in this proposition shows that $\mathcal{F}\{\eta\}_\Delta$ is Δ'' -free over $\mathcal{F}\{\eta\}_{\Delta'}$ if and only if $\{\mathfrak{P}'\}_\Delta$ is the defining Δ -ideal of η in $\mathcal{F}\{(y_j)_{j \in J}\}_\Delta$. This observation enables one to construct a Δ' - \mathcal{F} -algebra $\mathcal{B}' \subset \mathcal{U}$ which is Δ' -isomorphic over \mathcal{F} to \mathcal{A}' and such that \mathcal{B}'_Δ is Δ/Δ' - \mathcal{F} -free over \mathcal{B}' [13, Proposition 11, page 22]. Just take $\xi = (\xi_j)_{j \in J}$ to be an \mathcal{F} -generic Δ -zero of $\{\mathfrak{P}'\}_\Delta$. Then, by the equivalence stated, $\mathcal{F}\{\xi\}_\Delta$ is Δ/Δ' - \mathcal{F} -free over \mathcal{B}' , and \mathcal{B}' is Δ' -isomorphic over \mathcal{F} to \mathcal{A}' because ξ , as is η , is an \mathcal{F} -generic Δ' -zero of \mathfrak{P}' .

The proof that condition (3) implies condition (1) is a straightforward application of the definition of Δ'' -freeness. For simplicity, assume that the indexing set J is finite, i.e. $\eta = (\eta_1, \dots, \eta_n)$. Let $\varphi' : \mathcal{A}' \rightarrow \mathcal{U}$ be a Δ' - \mathcal{F} -homomorphism. Then, $\varphi'(\eta)$ is a Δ' -zero of \mathfrak{P}' and, thus, a Δ -zero of $\mathfrak{P} = \{\mathfrak{P}'\}_\Delta$. Since η is an \mathcal{F} -generic Δ -zero of its defining ideal \mathfrak{P} , $\eta \rightarrow \varphi'(\eta)$, and, thus, φ' extends to a Δ - \mathcal{F} -homomorphism $\varphi : \mathcal{A}_\Delta = \mathcal{F}\{\eta\}_\Delta \rightarrow \mathcal{F}\{\varphi'\eta\}_\Delta$.

The following proof that condition (1) implies condition (3) is slightly different than that of Kolchin and will serve to motivate the next proposition. Clearly, $\mathfrak{P} \supset \{\mathfrak{P}'\}_\Delta$. Assume that there exists $F \in \mathfrak{P} \subset \mathcal{F}\{y_1, \dots, y_n\}_\Delta$ with $F \notin \{\mathfrak{P}'\}_\Delta$. Since $\mathfrak{P} \cap \mathcal{F}\{y_1, \dots, y_n\}_{\Delta'} = \mathfrak{P}'$ [13, Proposition 8, page 16], the Δ -polynomial $F \notin \mathcal{F}\{y_1, \dots, y_n\}_{\Delta'}$ and, thus, must involve some Δ'' -derivatives of some y_i . Since \mathcal{U} is Δ -universal over \mathcal{F} , one may choose a Δ -zero $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{U}^n$ of $\{\mathfrak{P}'\}_\Delta \subset \mathcal{F}\{y_1, \dots, y_n\}_\Delta$ such that $F(\xi) \neq 0$. Because ξ is a zero of \mathfrak{P}' , there is a Δ' -homomorphism of $\mathcal{F}\{\eta\}_{\Delta'}$ onto $\mathcal{F}\{\xi\}_{\Delta'}$ sending η to ξ . This Δ' -homomorphism cannot extend to a Δ -homomorphism from $\mathcal{F}\{\eta\}_\Delta$ to $\mathcal{F}\{\xi\}_\Delta$ because $F(\eta) = 0$ and $F(\xi) \neq 0$. Therefore, \mathcal{A} is not Δ/Δ' - \mathcal{F} -free over \mathcal{A}' .

The existence of the Δ -polynomial F ‘prevents’ \mathcal{A} from being Δ/Δ' - \mathcal{F} -free over \mathcal{A}' . Since $F \notin \mathcal{F}\{y_1, \dots, y_n\}_{\Delta'}$, proper Δ'' -derivatives of Δ' -derivatives of (y_1, \dots, y_n) are present in F . Since η is a Δ -zero of F , some Δ'' -derivatives of Δ' -derivatives of η are algebraically dependent over \mathcal{A}' . Thus, the algebraic independence of certain of the ring generators of \mathcal{A} over \mathcal{A}' is a necessary condition for freeness. This is made precise in the following proposition, which is a generalization of the results of Sit (with Δ' empty) [21, Corollaries 1 and 2, page 25].

Proposition 3.51. *Let $\xi = (\xi_1, \dots, \xi_n)$ be Δ -generators of a Δ -field over \mathcal{F} . Assume that Δ is the union of two disjoint subsets Δ' and Δ'' . Then the following statements are equivalent.*

- (1) *The Δ - \mathcal{F} -algebra $\mathcal{F}\{\mathcal{F}\langle \xi \rangle_{\Delta'}\}_\Delta$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle \xi \rangle_{\Delta'}$.*
- (2) *Every transcendence basis for the field $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} is Δ'' -algebraically independent over \mathcal{F} .*
- (3) *There exists one transcendence basis for the field $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} that is Δ'' -algebraically independent over \mathcal{F} .*

Proof. Because there always exists a transcendence basis for the field $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} , condition (2) implies condition (3). Assuming condition (3), let the transcendence basis $(t_i)_{i \in I}$ for the field $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} be Δ'' -algebraically independent over \mathcal{F} . \square

Claim 3.52. $\mathcal{F}\{\mathcal{F}\langle \xi \rangle_{\Delta'}\}_\Delta = \mathcal{F}\langle \xi \rangle_{\Delta'}[(\theta'' t_i)_{i \in I, \theta'' \in \Theta_{\Delta''}}]$.

Proof. The right-hand side is clearly contained in the left. To prove the claim, it must be shown that all the Δ'' -derivatives of $\alpha \in \mathcal{F}\langle \xi \rangle_{\Delta'}$ are in the right-hand side. If $\alpha \in \mathcal{F}((t_i)_{i \in I})$, this is clear by the formula for the derivative of a quotient. If α is algebraic over $\mathcal{F}((t_i)_{i \in I})$ and not in $\mathcal{F}((t_i)_{i \in I})$, let $f(x) \in \mathcal{F}((t_i)_{i \in I})[x]$ be

the minimal polynomial for α . For $\delta'' \in \Delta''$, let $S_f(x) = df/dx$, and let $f^{\delta''}(x)$ be the polynomial obtained from $f(x)$ by differentiating the coefficients of $f(x)$ with respect to δ'' . Then, $S_f(\alpha)\delta''\alpha + f^{\delta''}(\alpha) = 0$. Since the degree of $S_f(x)$ in x is less than the degree of the minimal polynomial, $S_f(\alpha) \neq 0$. Then, $\delta''\alpha = -f^{\delta''}(\alpha)/S_f(\alpha)$ is an element of the right-hand side because the coefficients of $f^{\delta''}(x)$, $f^{\delta''}(\alpha)$ and $1/S_f(\alpha)$ are in the right-hand side. \square

To show condition (1), let $\rho : \mathcal{F}\langle\xi\rangle_{\Delta'} \rightarrow \mathcal{H}$ be a Δ' - \mathcal{F} -homomorphism to a Δ' - \mathcal{F} -field \mathcal{H} . The $\theta''t_i$ for all $i \in I$ and all $\theta'' \in \Theta_{\Delta''}$ of positive order, in addition to being algebraically independent over \mathcal{F} , are algebraically independent over $\mathcal{F}\langle\xi\rangle_{\Delta'}$ because an algebraic relation over $\mathcal{F}\langle\xi\rangle_{\Delta'}$ would contradict the algebraic independence of the family $(\theta''t_i)_{i \in I, \theta'' \in \Theta_{\Delta''}}$ over \mathcal{F} . Therefore, one may extend ρ to an \mathcal{F} -homomorphism of $\mathcal{F}\langle\xi\rangle_{\Delta'}[(\theta''t_i)_{i \in I, \theta'' \in \Theta_{\Delta''}}]$ by defining $\rho(\theta''t_i) = \theta''\rho(t_i)$ for all $i \in I$ and all $\theta'' \in \Theta_{\Delta''}$. To complete the proof of condition (1), it will be shown that ρ is a Δ' - \mathcal{F} -isomorphism.

To show ρ is a Δ'' - \mathcal{F} -homomorphism, since ρ restricted to $\mathcal{F}[(\theta''t_i)_{i \in I, \theta'' \in \Theta_{\Delta''}}]$ clearly is, it must be shown that $\rho\delta''\alpha = \delta''\rho\alpha$ for all δ'' in $\Theta_{\Delta''}$ and for $\alpha \in \mathcal{F}\langle\xi\rangle_{\Delta'}$ algebraic over $\mathcal{F}((t_i)_{i \in I})$. If α is not in $\mathcal{F}((t_i)_{i \in I})$, as before, let $f(x) \in \mathcal{F}((t_i)_{i \in I})[x]$ be the minimal polynomial for α . Then, for $\delta \in \Delta''$, $S_f(\alpha)\delta''\alpha + f^{\delta''}(\alpha) = 0$, $S_f(\alpha) \neq 0$, and $\delta''\alpha = -f^{\delta''}(\alpha)/S_f(\alpha)$ is an element of $\mathcal{F}\langle\xi\rangle_{\Delta'}[(\theta''t_i)_{i \in I, \theta'' \in \Theta_{\Delta''}}]$, the domain of ρ . Since ρ restricted to $\mathcal{F}\langle\xi\rangle_{\Delta'}$ is an isomorphism, $\rho\alpha$ satisfies $(\rho f)(x)$ and $S_{\rho f}(\rho\alpha) = \rho(S_f(\alpha)) \neq 0$. Apply δ'' to $(\rho f)(\alpha) = 0$ to obtain $S_{\rho f}(\rho\alpha)\delta''\rho\alpha + (\rho f)^{\delta''}(\rho\alpha) = 0$ and $\delta''\rho\alpha = -(\rho f)^{\delta''}(\rho\alpha)/S_{\rho f}(\rho\alpha)$. Since the coefficients of f are in $\mathcal{F}((t_i)_{i \in I})$ where ρ and δ'' commute, $(\rho f)^{\delta''}(x) = \rho(f^{\delta''})(x)$. Therefore,

$$\begin{aligned} \delta''\rho\alpha &= -(\rho f)^{\delta''}(\rho\alpha)/S_{\rho f}(\rho\alpha) = -\rho(f^{\delta''})(\rho\alpha)/S_{\rho f}(\rho\alpha) \\ &= -\rho((f^{\delta''})(\alpha))/\rho(S_f(\alpha)) = -\rho((f^{\delta''})(\alpha)/S_f(\alpha)) = \rho\delta''\alpha. \end{aligned}$$

This Δ'' - \mathcal{F} -homomorphism ρ is also a Δ' - \mathcal{F} -homomorphism because ρ restricted to $\mathcal{F}\langle\xi\rangle_{\Delta'}$ was assumed to be a Δ' - \mathcal{F} -isomorphism and because, for all θ'' in $\Theta_{\Delta''}$ and all δ' in Δ' ,

$$\rho(\delta'\theta''t_i) = \rho(\theta''\delta't_i) = \theta''\rho(\delta't_i) = \theta''\delta'\rho(t_i) = \delta'\theta''\rho(t_i) = \delta'\rho(\theta''t_i).$$

Therefore, ρ is a Δ -homomorphism of $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$. This shows $\mathcal{F}\{\mathcal{F}\langle\xi\rangle_{\Delta'}\}_{\Delta}$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle\xi\rangle_{\Delta'}$.

Assume condition (1). Let $(t_i)_{i \in I}$ be a transcendence basis of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ over \mathcal{F} . Let $(y_i)_{i \in I}$ be a family of Δ'' -indeterminates over $\mathcal{F}\langle\xi\rangle_{\Delta'}$. Define an isomorphism over \mathcal{F} of fields $\varphi : \mathcal{F}((t_i)_{i \in I}) \mapsto \mathcal{F}((y_i)_{i \in I})$ such that $\varphi(t_i) = y_i$ for each $i \in I$. Then because each element of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ is algebraic over $\mathcal{F}((t_i)_{i \in I})$, φ extends to an isomorphism of $\mathcal{F}\langle\xi\rangle_{\Delta'}$ into an algebraically closed field containing $\mathcal{F}\langle(y_i)_{i \in I}\rangle_{\Delta''}$. Endow the image \mathcal{H} of φ with the unique Δ' -structure such that φ is a Δ' - \mathcal{F} -isomorphism mapping each t_i to y_i for i in I . Then $\mathcal{H}\{(y_i)_{i \in I}\}_{\Delta''}$ has a structure of a Δ'' - \mathcal{F} -algebra because the elements in \mathcal{H} not in $\mathcal{F}\langle(y_i)_{i \in I}\rangle_{\Delta''}$ are algebraic over $\mathcal{F}\langle(y_i)_{i \in I}\rangle_{\Delta''}$ and, as shown in the proof of the claim, have uniquely determined Δ'' -derivatives in $\mathcal{H}\{(y_i)_{i \in I}\}_{\Delta''}$. The Δ' -structure on \mathcal{H} may be extended to all of $\mathcal{H}\{(y_i)_{i \in I}\}_{\Delta''}$ by defining $\delta'(\theta''y_i) = \theta''\delta'y_i$ for each θ'' in $\Theta_{\Delta''}$, each $\delta' \in \Delta'$ and $i \in I$. Because $\delta'\delta''y_i = \delta''\delta'y_i$, the derivation $\delta'\delta'' - \delta''\delta'$ on $\mathcal{F}\langle(y_i)_{i \in I}\rangle_{\Delta''}$ is the zero derivation. Since it extends uniquely to the zero derivation on \mathcal{H} , $\delta'\delta''\beta = \delta''\delta'\beta$

for β in \mathcal{H} not in $\mathcal{F}(y_i)_{i \in I}$. This shows that there is a well-defined Δ -structure on $\mathcal{H}_{\Delta''}$.

Because condition (3) implies condition (1), $\mathcal{H}_{\Delta''}$ is Δ'' - \mathcal{F} -free over \mathcal{H} . By Proposition 3.48, since φ from $\mathcal{F}\langle \xi \rangle_{\Delta'}$ to \mathcal{H} is a Δ' -isomorphism, $(\mathcal{F}\langle \xi \rangle_{\Delta'})_{\Delta}$ and $\mathcal{H}\{(y_i)_{i \in I}\}_{\Delta''}$ are Δ -isomorphic over \mathcal{F} by an isomorphism that sends t_i to y_i . Because the $(y_i)_{i \in I}$ are Δ'' -algebraically independent over \mathcal{F} , the $(t_i)_{i \in I}$ are Δ'' -algebraically independent over \mathcal{F} also.

The goal of the rest of this section is to analyze the constants of free extensions.

Proposition 3.53 ([12, Exercise 8, page 159]). *Let \mathcal{U} a Δ -field universal over \mathcal{F} . Let $t_1, \dots, t_n \in \mathcal{U}$ be Δ -algebraically independent over \mathcal{F} . Then each element u of $\mathcal{F}\langle t_1, \dots, t_n \rangle_{\Delta}$ not in \mathcal{F} is Δ -transcendental over \mathcal{F} .*

Proof. Let $u = P(t_1, \dots, t_n)/Q(t_1, \dots, t_n)$ where $P, Q \in \mathcal{F}\{y_1, \dots, y_n\}_{\Delta}$ such that $PQ \notin \mathcal{F}$ and $\gcd(P, Q) = 1$. Choose orderly rankings for $\mathcal{F}\{y_1, \dots, y_n\}_{\Delta}$ and $\mathcal{F}\{z\}_{\Delta}$. Assume $g \in \mathcal{F}\{z\}_{\Delta}$ is of lowest rank among the nonzero Δ -polynomials satisfied by u . Let $g = I_d v_g^d + I_{d-1} v_g^{d-1} + \dots + I_0$ where d is a positive integer, v_g is the leader of g , and the I_k are Δ -polynomials in $\mathcal{F}\{z\}_{\Delta}$ of lower rank than v_g . Because I_0 and I_d are of lower rank than g , $I_0(u) \neq 0$ and $I_d(u) \neq 0$. If $\text{ord } v_g = 0$, substitute P/Q for z , clear denominators and observe P divides Q . But $\gcd(P, Q) = 1$, so it may be assumed that $\text{ord } v > 0$. Let v_g, v_P and v_Q be the leaders of g, P and Q , respectively, and S_g, S_P and S_Q the separants. Write $v_g = \theta z$, where θ is the nonempty product of r derivations from Δ . □

Claim 3.54. $v_g(P/Q) = \theta(P/Q) = [Q^{r-1}(S_P \theta v_P Q - P S_Q \theta v_Q) + W]/Q^{r+1}$ such that W is the sum of terms of rank lower than the maximum rank of θv_P and θv_Q .

Proof. The claim is clearly true for $r = 1$. Assume the claim is true for r . By differentiating $v_g(P/Q) = \theta(P/Q) = (S_P \theta v_P Q - P S_Q \theta v_Q)/Q^2 + W/Q^{r+1}$ with respect to one of the $\delta \in \Delta$, $\delta v_g(P/Q) =$

$$\delta \theta(P/Q) = (S_P \delta \theta v_P Q - P S_Q \delta \theta v_Q)/Q^2 + V/Q^3 + (\delta W Q - (r+1)W \delta Q)/Q^{r+2}$$

such that the rank of V is lower than the maximum rank of $\delta \theta v_P$ and $\delta \theta v_Q$. Since $\delta W Q$ and $(r+1)W \delta Q$ also have lower than the maximum rank of $\delta \theta v_P$ and $\delta \theta v_Q$, after adding the three fractions, the claim is true for $r+1$. □

Let t be a positive integer such that $Q^t \cdot I_j(P/Q)v_g^j(P/Q)$ is a Δ -polynomial, in $\mathcal{F}\{y_1, \dots, y_n\}_{\Delta}$, for each $j = 0, \dots, d$. By substituting u into $Q^t g(z)$, one obtains the zero Δ -polynomial

$$Q^t g(u) = Q^t (I_d(P/Q)v_g^d(P/Q) + I_{d-1}(P/Q)v_g^{d-1}(P/Q) + \dots + I_0(P/Q)).$$

If $\text{rank } P > \text{rank } Q$, then, by the claim, the sum of the highest ranking terms of $Q^t g(P/Q)$ is the Δ -polynomial $Q^t I_n(P/Q)(Q^{r-1} S_P \theta v_P Q)^d$ which is equal to zero because $Q^t g(u) = 0$. Therefore, since $I_n(P/Q) \neq 0$ and $Q \neq 0$, it follows that $S_P = 0$ and $P \in \mathcal{F}$. Thus, $Q \in \mathcal{F}$ because $\text{rank } P > \text{rank } Q$. This is contrary to the assumption $PQ \notin \mathcal{F}$. If $\text{rank } Q > \text{rank } P$, the same type of contradiction results.

If $\text{ord } P = \text{ord } Q$, by the claim,

$$\begin{aligned} & Q^t I_n(P/Q) Q^{(r-1)d} (S_P \theta v_P Q - P S_Q \theta v_Q)^d \\ &= Q^t I_n(P/Q) Q^{(r-1)d} (S_P Q - P S_Q)^d (\theta v_P) \end{aligned}$$

is the sum of the highest ranking terms of $Q^t g$ and is equal to 0. Therefore, $(S_P Q - P S_Q) = 0$. Then, P divides S_P because $\gcd(P, Q) = 1$. But, this is impossible because S_P has lower rank than P .

Corollary 3.55. *Let \mathcal{U} be a Δ -field Δ -universal over \mathcal{F} . Let $t_1, \dots, t_n \in \mathcal{U}$ be Δ -algebraically independent over \mathcal{F} . Then $\mathcal{F}(t_1, \dots, t_n)^\Delta = \mathcal{F}^\Delta$.*

Proof. The condition that an element be a Δ -constant is a Δ -relation on that element. This is impossible by the previous proposition. \square

The next lemma is well known.

Lemma 3.56 (The Algebraic Constant Lemma). *Let \mathcal{G} over \mathcal{F} be an extension of Δ -fields. A Δ -constant of \mathcal{G} algebraic over \mathcal{F} is algebraic over the Δ -constants of \mathcal{F} .*

Proof. Let α be a Δ -constant of \mathcal{G} algebraic over \mathcal{F} . Let $f(x) \in \mathcal{F}[x]$ be the minimal polynomial of α over \mathcal{F} . Write $f(x) = \sum_{i=1, \dots, d} a_i x^i$ for $a_i \in \mathcal{F}$. Then, for each $\delta \in \Delta$, $S_f(\alpha)\delta\alpha + f^\delta(\alpha) = 0$, where $S_f(x)$ is the derivative of f with respect to x and $f^\delta(x)$ is the polynomial obtained by applying δ to the coefficients of $f(x)$. Since $\delta\alpha = 0$, $f^\delta(\alpha) = 0$. Because the leading coefficient of $f(x)$ is 1, the degree of $f^\delta(x)$ is less than that of $f(x)$. Since $f(x)$ is the minimal polynomial of α , $f^\delta(x) = 0$. Consequently, $\delta a_i = 0$ for $i = 1, \dots, d$ and all $\delta \in \Delta$. Therefore, the coefficients of $f(x)$ are δ -constants in \mathcal{F} , and α is algebraic over \mathcal{F}^Δ . \square

Lemma 3.57 (No New Δ'' -Constant Lemma). *Assume that Δ is the union of two disjoint subsets Δ' and Δ'' . Let $\xi = (\xi_1, \dots, \xi_n)$ be a finite family of elements of \mathcal{U} . If the Δ -ring $\mathcal{F}\langle \mathcal{F}(\xi)_{\Delta'} \rangle_\Delta$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle \xi \rangle_{\Delta'}$, then the Δ'' -constants of $\mathcal{F}\langle \xi \rangle_\Delta$ are contained in the algebraic closure of $\mathcal{F}^{\Delta''}$ in $\mathcal{F}\langle \xi \rangle_{\Delta'}$. If $\mathcal{F}\langle \xi \rangle_{\Delta'}$ is a regular extension of \mathcal{F} , $\mathcal{F}\langle \xi \rangle_\Delta$ and \mathcal{F} have the same Δ'' -constants.*

Proof. By Proposition 3.51, there is a transcendence basis $(t_i)_{i \in I}$ for the field $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} that is Δ'' -algebraically independent over \mathcal{F} . By Corollary 3.55, the Δ'' -constants of $\mathcal{F}\langle (t_i)_{i \in I} \rangle_{\Delta''}$ are in \mathcal{F} .

Let $\gamma \in \mathcal{F}\langle \xi \rangle_\Delta$ be a Δ'' -constant and assume $\gamma \notin \mathcal{F}\langle (t_i)_{i \in I} \rangle_{\Delta''}$. Then γ is algebraic over $\mathcal{F}\langle (t_i)_{i \in I} \rangle_{\Delta''}$ because ξ and all its Δ'' -derivatives are algebraic over $\mathcal{F}\langle (t_i)_{i \in I} \rangle_{\Delta''}$. The Algebraic Constant Lemma 3.56 can then be applied to show γ is algebraic over the Δ'' -constants of $\mathcal{F}\langle (t_i)_{i \in I} \rangle_{\Delta''}$, which is equal to $\mathcal{F}^{\Delta''}$ by Corollary 3.55. If $\mathcal{F}\langle \xi \rangle_{\Delta'}$ is regular over \mathcal{F} , then $\mathcal{F}\langle \xi \rangle_\Delta$ is regular over \mathcal{F} ([13, Proposition 10(c), page 21]) and, therefore, $\gamma \in \mathcal{F}$. \square

If the Δ -ring $\mathcal{F}\langle \mathcal{F}(\xi)_{\Delta'} \rangle_\Delta$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle \xi \rangle_{\Delta'}$ and if $\mathcal{F}\langle \xi \rangle_\Delta$ is a not regular extension of \mathcal{F} , there may be some Δ'' -constants in $\mathcal{F}\langle \xi \rangle_{\Delta'}$ algebraic over \mathcal{F} . For example, take $\mathcal{F} = \mathbb{Q}$, $\Delta' = \emptyset$ and $\mathfrak{P}_{\Delta'} = (y^2 + 1) \subset \mathbb{Q}[y]$ a prime ideal. Then $\{\mathfrak{P}\}_{\Delta', \Delta''} = \{y^2 + 1\}_{\Delta''} \subset \mathbb{Q}\{y\}_{\Delta''}$ is a Δ -prime ideal ([13, Proposition 8, page 16]). Let ξ be a \mathbb{Q} -generic zero of $\{y^2 + 1\}_{\Delta''}$ in \mathcal{U} . Then, by Proposition 3.50, $\mathbb{Q}\langle \xi \rangle_\Delta$ is Δ - \mathbb{Q} -free over $\mathbb{Q}\langle \xi \rangle$, and, since $\delta'' y \in \{y^2 + 1\}_\Delta$ for $\delta'' \in \Delta''$, ξ is a Δ'' -constant of $\mathbb{Q}\langle \xi \rangle$. In fact, the same technique shows that, if $\mathfrak{P}_{\Delta'} = (f)$ where $f \in \mathbb{Q}[y]$ is an irreducible polynomial, $\delta'' \xi = 0$ for a \mathbb{Q} -generic zero ξ of $\{f\}_{\Delta''}$ in \mathcal{U} .

The next two propositions analyze Δ' -constants of $\mathcal{F}\langle \xi \rangle_\Delta$ instead of the Δ'' -constants.

Proposition 3.58. *Let $\xi = (\xi_1, \dots, \xi_n)$ be a finite family of elements of \mathcal{U} . If the Δ -ring $\mathcal{F}\{\mathcal{F}\langle \xi \rangle_{\Delta'}\}_{\Delta}$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle \xi \rangle_{\Delta'}$, and if the ξ are Δ' -independent over \mathcal{F} , then $(\mathcal{F}\langle \xi \rangle_{\Delta'})^{\Delta'} = ((\mathcal{F}\langle \xi \rangle_{\Delta})^{\Delta'})^{\Delta'} = \mathcal{F}^{\Delta'}$.*

Proof. The set of all the Δ' -derivatives of ξ is a transcendence basis for $\mathcal{F}\langle \xi \rangle_{\Delta'}$ over \mathcal{F} . By Proposition 3.51, they are Δ'' -algebraically independent over \mathcal{F} , and all the Δ'' -derivatives of ξ are Δ' -independent. By Corollary 3.55, there are no new Δ' -constants, and the conclusion follows. \square

Proposition 3.59. *Let $\text{card } \Delta' = \text{card } \Delta'' = 1$ and $\xi = (\xi_1)$, and let $\xi \in \mathcal{U}$. Let $f(y) \in \mathcal{F}^{\Delta}\{y\}_{\Delta'}$ such that $f(y) = \sum a_{ij}y^i(\delta'y)^j$ with $a_{ij} \in \mathcal{F}^{\Delta}$. Assume $f(\xi) = 0$ and $S(\xi) \neq 0$ where $S(y)$ is the separant of f relative to an orderly ranking of $\mathcal{F}^{\Delta}\{y\}_{\Delta'}$. Also assume the Δ -ring $\mathcal{F}\{\mathcal{F}\langle \xi \rangle_{\Delta'}\}_{\Delta}$ is Δ'' - \mathcal{F} -free over $\mathcal{F}\langle \xi \rangle_{\Delta'}$. If $f(y)$ is of order zero, i.e. $a_{ij} = 0$ for $j > 0$, then $\delta'\xi = 0$ and $\delta''\xi = 0$. If not, then $\delta'\xi_1/\delta''\xi_1$ is a Δ' -constant of $\mathcal{F}\langle \xi \rangle_{\Delta}$ not in $\mathcal{F}\langle \xi \rangle_{\Delta'}$.*

Proof. Let \mathfrak{P}' and \mathfrak{P} denote, respectively, the prime defining Δ' -ideals of ξ in $\mathcal{F}\{y\}_{\Delta'}$ and the defining Δ -ideals of ξ in $\mathcal{F}\{y\}_{\Delta}$. By Proposition 3.50, $\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$.

If $\delta'y$ is not present in f , then ξ is algebraic over \mathcal{F}^{Δ} . Let $g \in \mathcal{F}^{\Delta}[y]$ be the minimal polynomial for ξ . Clearly, $g \in \mathfrak{P}'$, and $\delta'g \in \mathfrak{P}'$. Let $S(y)$ be dg/dy . Because g is the minimal polynomial, $S(y) \notin \mathfrak{P}'$. Since $\delta'g = S(y)\delta'y$ and since \mathfrak{P}' is prime, $\delta'y \in \mathfrak{P}' \subset \mathfrak{P}$, and $\delta'\xi = 0$. Similarly, $\delta''\xi = 0$.

If $\delta'y$ is present in f , $\delta'f = S(y)\delta'^2y + (\partial f/\partial y)\delta'y$ and $\delta''f = S(y)\delta''\delta'y + (\partial f/\partial y)\delta''y$ are elements of $\mathfrak{P} = \{\mathfrak{P}'\}_{\Delta}$, where $S(y) = \partial f/\partial \delta'y$ and $S(y) \notin \mathfrak{P}$. Then,

$$\delta''y \cdot \delta'f - \delta'y \cdot \delta''f = S(y)(\delta''y\delta'^2y - \delta'y\delta''\delta'y)$$

is also an element of \mathfrak{P} . Since $S(y) \notin \mathfrak{P}$, $\delta''y\delta'^2y - \delta'y\delta''\delta'y$ is. Because ξ is a Δ -zero of \mathfrak{P} , $\delta''\xi\delta'^2\xi - \delta'y\delta''\delta'\xi = 0$, and $\delta'(\delta'\xi/\delta''\xi) = 0$. Since $\delta'\xi \in \mathcal{F}\langle \xi \rangle_{\Delta'}$ and $\delta''\xi \notin \mathcal{F}\langle \xi \rangle_{\Delta'}$, clearly $\delta'\xi/\delta''\xi \notin \mathcal{F}\langle \xi \rangle_{\Delta'}$. \square

The last proposition applies to the familiar Weierstrass \wp -function (a Δ -zero of $f(y) = (\delta'y)^2 - y^3 - ay - b$) and the exponential function (a Δ -zero of $f(y) = y - \delta'y$), in which case a new Δ' -constant is $\delta'\xi/\delta''\xi = \xi/\delta''\xi$.

3.2. The E-group induced from an algebraic group. In this section, let \mathcal{F} be a Δ -field and let Δ' be a commutative linearly independent subset of the vector space spanned by Δ over \mathcal{F} . Let \mathcal{U} be a Δ -universal extension of \mathcal{F} . In [13, Chapter 2, Section 3, page 56], Kolchin develops a procedure for associating to each Δ' - \mathcal{F} -group G (relative to the Δ' -field \mathcal{U}) a Δ - \mathcal{F} -group G_{Δ} (relative to the Δ -field \mathcal{U}), which is called the induced Δ - \mathcal{F} -group. The elements of G_{Δ} are defined to be the same as those of G . If the Δ' -subfield of \mathcal{U} associated with x in G is $\mathcal{F}\langle x \rangle_{\Delta'}$, the Δ -subfield of \mathcal{U} associated with x in G_{Δ} is $\mathcal{F}\langle \mathcal{F}\langle x \rangle_{\Delta'} \rangle_{\Delta}$.

Heuristically, to each open affine B of G defined by a Δ' -ideal \mathfrak{P}' of $\mathcal{F}\{y_1, \dots, y_n\}_{\Delta'}$, one may associate the open affine B_{Δ} of G_{Δ} defined by the Δ -ideal $\{\mathfrak{P}'\}_{\Delta}$ of $\mathcal{F}\{y_1, \dots, y_n\}_{\Delta}$. To the element x of G , thought of as a Δ' -zero in \mathcal{U}^n of \mathfrak{P}' , corresponds the element x of G_{Δ} , thought of as a Δ -zero of $\{\mathfrak{P}'\}_{\Delta}$. The Δ' -rational functions giving the group law on G are also Δ -rational functions on G_{Δ} and give the group law on G_{Δ} . An \mathcal{F} -generic element v of G , which is a generic zero of some \mathfrak{P}' as above, will be an \mathcal{F} -generic element of G_{Δ} if and only if it is a generic zero of $\{\mathfrak{P}'\}_{\Delta}$ [13, Theorem 3(2c), page 58]. The discussion in the last section implies v

will be an \mathcal{F} -generic element of G_Δ if v is an \mathcal{F} -generic element of G and $(\mathcal{F}\langle x \rangle_{\Delta'})_\Delta$ is Δ/Δ' - \mathcal{F} -free over $\mathcal{F}\langle x \rangle_{\Delta'}$.

Definition 3.60 ([13, page 56]). Let Δ' be a commutative linearly independent subset of $\mathcal{F}\Delta$. Let G be a Δ' - \mathcal{F} -group (relative to the Δ' -field \mathcal{U}), and let H be a Δ - \mathcal{F} -group (relative to the Δ -field \mathcal{U}). A (Δ, Δ') - \mathcal{F} -homomorphism of H into G is a group homomorphism $f : H \rightarrow G$ that satisfies the following three conditions:

- (1) if $y \in H$, then $\mathcal{F}\langle f(y) \rangle_{\Delta'} \subset \mathcal{F}\langle y \rangle_\Delta$,
- (2) if $y, y' \in H$ and $y \xrightarrow{\Delta} y'$, then $f(y) \xrightarrow{\Delta'} f(y')$,
- (3) if $y, y' \in H$ and $y \xrightarrow{\Delta} y'$, then $S_{\Delta, y', y}$ extends $S_{\Delta', f(y'), f(y)}$.

Definition 3.61 ([13, page 57]). Let G be a Δ' - \mathcal{F} -group relative to the universe \mathcal{U} . A Δ - \mathcal{F} -group structure on G , denoted by G_Δ , is said to be *induced* (by the given Δ' - \mathcal{F} -group structure on G) if the following two conditions are satisfied:

- (1) id_G is a (Δ, Δ') - \mathcal{F} -homomorphism;
- (2) every (Δ, Δ') - \mathcal{F} -homomorphism of a Δ - \mathcal{F} -group into G is a Δ - \mathcal{F} -homomorphism.

3.3. Varying the universal field. For \mathcal{F} a Δ -field, the functor “extending the universal field of \mathcal{F} ” has been developed by Kolchin. (See [13, Chapter 2, Section 1, Varying the universal differential field, page 45] and [13, Chapter 8, Section 10, The Lie-Cassidy-Kovacic method, page 247]). Let \mathcal{V} and \mathcal{U} be Δ -extensions of \mathcal{F} that are Δ -universal over \mathcal{F} and such that $\mathcal{U} \subseteq \mathcal{V}$. The functor “extending the universal field of \mathcal{F} ” takes the category of Δ - \mathcal{F} -groups (relative to \mathcal{U}) and Δ - \mathcal{F} -group homomorphisms to the category of Δ - \mathcal{F} -groups (relative to \mathcal{V}) and Δ - \mathcal{F} -group homomorphisms. Heuristically, a set defined as the Δ -zeros in \mathcal{U} of a system of Δ -equations is associated with the set of Δ -zeros in \mathcal{V} of the same system of Δ -equations.

3.4. The existence theorem. The purpose of this section is to prove every connected E-group is isomorphic to the Galois group of an E-strongly normal extension.

Let \mathcal{F} be an E-field, and let \mathcal{V} be an E-extension of \mathcal{F} that is E-universal over \mathcal{F} . Let G be a connected E- \mathcal{F} -group (relative to the E-field \mathcal{V}). Let $\mathcal{H} \subset \mathcal{V}$ be an E-extension of \mathcal{F} , with \mathcal{V} not necessarily universal over \mathcal{H} . Let χ be an E-derivation (χ commutes with the action of E) of \mathcal{H} into \mathcal{V} over \mathcal{F} . For each element g of G rational over \mathcal{H} , evaluation at g of E- \mathcal{F} -functions on G defined at g composed with χ is a local E-derivation at g . If g is E- \mathcal{H}^χ -affine, this local derivation can be extended to a unique tangent vector to G at g [13, Section 8, Chapter 8]. By right translating this tangent vector to all of G , one obtains an element $l\chi(g)$ of the Lie algebra $\mathcal{L}_\Delta(G)$ of invariant E-derivations of G which is called the logarithmic derivative of g relative to χ [13, page 236]. Thus, for any local derivation χ at $g \in G$, there exists a unique element $l\chi(g)$ of the Lie algebra $\mathcal{L}_\Delta(G)$ with the property that

$$l\chi(g)(f)(g) = \chi(f(g))$$

for every E- \mathcal{F} -function f defined at g .

In the remainder of this section, let \mathcal{F} be an (E, Δ) -field, let $\mathcal{C} = \mathcal{F}^\Delta$, and let \mathcal{U} be an (E, Δ) -extension of \mathcal{F} that is (E, Δ) -universal over \mathcal{F} . Let G be a connected E- \mathcal{C} -group (relative to the E-universal field \mathcal{U}^Δ). By extending the universal E-field from \mathcal{U}^Δ to \mathcal{U} , considered as an E-field (Section 3.3 or [13, Chapter 2, Section 1,

page 44]), G may be considered as an E - \mathcal{C} -group (relative to the E -field \mathcal{U}). For each δ in Δ and any g in $G_{\mathcal{U}}$, the logarithmic derivative is $l\delta(g) \in \mathcal{L}_E(G)$.

The following lemma is one of the well-known properties of the logarithmic derivative [13, Proposition 8, page 236] and will be used a few times.

Lemma 3.62. *Let $x, y \in G_{\mathcal{U}}$. If $l\delta x = l\delta y$ for all $\delta \in \Delta$, there exists an element $c \in G_{\mathcal{U}^\Delta}$ such that $c = x^{-1}y$.*

Proof. Assume $l\delta x = l\delta y$ for all $\delta \in \Delta$. By [13, Remark after Theorem 3, page 237], for $w, z \in G$, $l\delta(wz) = l\delta(w) + \tau_w^\#(l\delta(z))$ where $\tau_w^\#$ is the isomorphism of the Lie algebra induced by conjugation with w . By letting $w = x$ and $z = x^{-1}y$, $l\delta(y) = l\delta(x) + \tau_x^\#(l\delta(x^{-1}y))$. So $0 = \tau_x^\#(l\delta(x^{-1}y))$ and $0 = l\delta(x^{-1}y)$. Then $c = x^{-1}y \in G_{\mathcal{U}^\Delta}$ [13, Proposition 8(c), page 236]. \square

Definition 3.63. The element $\alpha \in G_{\mathcal{U}}$ is a Δ - G -primitive over \mathcal{F} if the logarithmic derivative $l\delta(\alpha) \in \mathcal{L}_{E,\mathcal{F}}(G)$ for each $\delta \in \Delta$. A Δ - G -primitive extension is an extension of \mathcal{F} of the form $\mathcal{F}\langle\alpha\rangle$ where α is a Δ - G -primitive over \mathcal{F} .

Proposition 3.64. *Let α be a Δ - G -primitive over \mathcal{F} such that the field of Δ -constants of $\mathcal{F}\langle\alpha\rangle_{E,\Delta}$ is \mathcal{C} . Then $\mathcal{F}\langle\alpha\rangle_{E,\Delta}$ is an E -strongly normal extension of \mathcal{F} (relative to the (E, Δ) -field \mathcal{U}), and the map $c : G(\mathcal{F}\langle\alpha\rangle_{E,\Delta}/\mathcal{F}) \mapsto G$ defined by $c(\sigma) = \alpha^{-1}\sigma\alpha$ defines an injective E - \mathcal{C} -homomorphism of E - \mathcal{C} -groups (relative to the E -field \mathcal{U}^Δ).*

Proof. Since α is a Δ - G -primitive over \mathcal{F} , $l\delta(\alpha) \in \mathcal{L}_{E,\mathcal{F}}(G)$ for each $\delta \in \Delta$ so that, for any (E, Δ) -isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{E,\Delta}$ over \mathcal{F} , $\sigma(l\delta(\alpha)) = l\delta(\alpha)$ for $\delta \in \Delta$. Also, $l\delta(\sigma\alpha) = \sigma(l\delta(\alpha))$ for all $\delta \in \Delta$ by [13, Proposition 8(b), page 236]. Therefore, $l\delta(\sigma\alpha) = l\delta(\alpha)$, and, by Lemma 3.62, $c(\sigma) = \alpha^{-1}\sigma\alpha$ is an element of $G_{\mathcal{U}^\Delta}$. Since

$$\begin{aligned} \sigma(\mathcal{F}\langle\alpha\rangle_{E,\Delta}) &\subset \mathcal{F}\langle\alpha\rangle_{E,\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{E,\Delta}) = \mathcal{F}\langle\alpha, \sigma\alpha\rangle_{E,\Delta} \\ &= \mathcal{F}\langle\alpha, c(\sigma)\rangle_{E,\Delta} = \mathcal{F}\langle\alpha\rangle_{E,\Delta}\mathcal{C}\langle c(\sigma)\rangle_E, \end{aligned}$$

$\mathcal{F}\langle\alpha\rangle$ is E -strongly normal over \mathcal{F} by Proposition 1.13. By definition, $\mathcal{C}\langle\sigma\rangle = (\mathcal{F}\langle\alpha\rangle_{E,\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{E,\Delta}))^\Delta$. Therefore, $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle c(\sigma)\rangle_E$ by [12, Corollary 2 to Theorem 1, page 88]. For any $\sigma, \tau \in G(\mathcal{F}\langle\alpha\rangle_{E,\Delta}/\mathcal{F})$, c is a group homomorphism since $\alpha c(\sigma\tau) = \sigma\tau\alpha = \sigma(\alpha c(\tau)) = \sigma\alpha \circ c(\tau) = \alpha c(\sigma)c(\tau)$. If σ is in the kernel of c , $\sigma\alpha = \alpha c(\sigma) = \alpha$ and, hence, $\sigma = \text{id}_{\mathcal{F}\langle\alpha\rangle}$ because α (E, Δ) -generates $\mathcal{F}\langle\alpha\rangle_{E,\Delta}$. Therefore c is injective.

To prove that c is an E - \mathcal{C} -homomorphism, it will be shown to be a pre E - \mathcal{C} -mapping (Definition 1.20). Then, since c is a homomorphism, [13, Corollary 1, page 90] implies that c is an E - \mathcal{C} -homomorphism. Parts (1), (2) and (3) of Definition 1.20 follow by taking the domain to consist only of \mathcal{C} -generic elements and from the fact that $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle c(\sigma)\rangle_E$. To show part (4) of the definition, take $\sigma \leftrightarrow \sigma'$ to be two \mathcal{C} -generic elements. By the definition of \mathcal{C} -generic E -specialization in $G(\mathcal{F}\langle\alpha\rangle_{E,\Delta}/\mathcal{F})$, there exists an (E, Δ) - $\mathcal{F}\langle\alpha\rangle_{E,\Delta}$ -isomorphism $\varphi : \mathcal{F}\langle\alpha\rangle_{E,\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{E,\Delta}) \approx \mathcal{F}\langle\alpha\rangle_{E,\Delta}\sigma'(\mathcal{F}\langle\alpha\rangle_{E,\Delta})$ that maps $\sigma\beta$ onto $\sigma'\beta$ for each $\beta \in \mathcal{F}\langle\alpha\rangle$. Therefore, $\varphi(c(\sigma)) = \varphi(\alpha^{-1}\sigma\alpha) = \alpha^{-1}\sigma'\alpha = c(\sigma')$. Thus, the induced E - \mathcal{C} -isomorphism $S_{c(\sigma'),c(\sigma)}$ obtained by restricting φ to $\mathcal{C}\langle\sigma\rangle = \mathcal{C}\langle c(\sigma)\rangle_E$ is exactly the induced E - \mathcal{C} -isomorphism $S_{\sigma',\sigma}$, and $c(\sigma) \leftrightarrow c(\sigma')$. \square

The following lemma has a pivotal role in the next theorem.

Lemma 3.65. *Let G be a connected E - \mathcal{C} -group (relative to the \mathcal{U}). Let η and ξ be elements of $G_{\mathcal{U}}$, i.e. elements of G rational over \mathcal{U} . Assume η is \mathcal{C} -generic in $G_{E,\Delta}$ and $\mathcal{C}\langle\eta\rangle^\Delta = \mathcal{C}$. If $l\delta(\eta) = l\delta(\xi)$ for all $\delta \in \Delta$, then ξ is \mathcal{C} -generic in $G_{E,\Delta}$, and $\eta \leftrightarrow \xi$ in $G_{E,\Delta}$.*

Proof. By Lemma 3.62, there exists $\gamma \in G_{\mathcal{U}\Delta}$ such that $\eta\gamma = \xi$. By the theorem on the linear disjointness of Δ -constants [12, Corollary 1, page 87], $\mathcal{C}\langle\eta\rangle$ and $\mathcal{C}\langle\gamma\rangle$ are linearly disjoint over \mathcal{C} . By [13, Theorem 1(d), page 39], $\eta\gamma$ is a \mathcal{C} -generic element of $G_{E,\Delta}$. Since $\eta\gamma = \xi$, ξ is \mathcal{C} -generic. Because G is connected, so is $G_{E,\Delta}$ (see the proof of the theorem below), and $\eta \leftrightarrow \xi$ in $G_{E,\Delta}$. \square

For the proof the next theorem, one uses the fact that the elements of G (relative to the E -field \mathcal{U}^Δ) are contained in $(G_{E,\Delta})_{\mathcal{U}\Delta}$, as the following discussion indicates. An E - \mathcal{C} -group G (relative to the E -field \mathcal{U}^Δ) is given. Let $G_{\mathcal{U}}$ (relative to \mathcal{U}) be the E - \mathcal{C} -group obtained from G (relative to \mathcal{U}^Δ) by extending the universal differential field from \mathcal{U}^Δ to \mathcal{U} . The elements of G (relative to \mathcal{U}^Δ) are the elements $(G_{\mathcal{U}})_{\mathcal{U}\Delta}$ of the E -group $G_{\mathcal{U}}$ (relative to \mathcal{U}) rational over \mathcal{U}^Δ . Let $G_{E,\Delta}$ (relative to the (E, Δ) -field \mathcal{U}) be the (E, Δ) - \mathcal{C} -group obtained from the E - \mathcal{C} -group $G_{\mathcal{U}}$ (relative to \mathcal{U}) by extending the derivations from E to (E, Δ) . From the discussion in the preceding section on the (E, Δ) - \mathcal{C} -group $G_{E,\Delta}$, the elements of the E - \mathcal{C} -group $G_{\mathcal{U}}$ are included in the elements of the (E, Δ) - \mathcal{C} -group $G_{E,\Delta}$. Therefore the elements of the E - \mathcal{C} -group G (relative to \mathcal{U}^Δ) are elements $(G_{E,\Delta})_{\mathcal{U}\Delta}$ of the (E, Δ) - \mathcal{C} -group $G_{E,\Delta}$ (relative to the (E, Δ) -field \mathcal{U}).

Theorem 3.66. *Let G be a connected E - \mathcal{C} -group (relative to the E -field \mathcal{U}^Δ). Let η be a \mathcal{C} -generic element of $G_{E,\Delta}$. Then, $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ is E -strongly normal over $\mathcal{F} = \mathcal{C}\langle l\delta_1\eta\rangle_{E,\Delta} \cdots \mathcal{C}\langle l\delta_m\eta\rangle_{E,\Delta}$ (relative to the (E, Δ) -field \mathcal{U}) such that the Galois group $G(\mathcal{G}/\mathcal{F})$ (relative to the E -field \mathcal{U}^Δ) is E - \mathcal{C} -isomorphic to G .*

Proof. Since the E - \mathcal{C} -group G (relative to the E -field \mathcal{U}^Δ) is connected, the E - \mathcal{C} -group G (relative to the E -field \mathcal{U}) is connected [13, Section 1, page 44]. This implies that the (E, Δ) - \mathcal{C} -group $G_{E,\Delta}$ (relative to the (E, Δ) -field \mathcal{U}) is connected [13, Theorem 3, page 58].

By Proposition 3.50, $\mathcal{C}\{\mathcal{C}\langle\eta\rangle_E\}_\Delta$ is Δ -free over $\mathcal{C}\langle\eta\rangle_E$. Because $G_{E,\Delta}$ is connected, $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ is a regular extension of \mathcal{C} by the third axiom for E -groups. The No New Δ'' -Constant Lemma 3.57 then implies that the Δ -constants of $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ are in \mathcal{C} .

Set $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ and $\mathcal{F} = \mathcal{C}\langle l\delta_1\eta\rangle_{E,\Delta} \cdots \mathcal{C}\langle l\delta_m\eta\rangle_{E,\Delta}$. Since for each $\delta \in \Delta$, $l\delta : G_{E,\Delta} \rightarrow (\mathcal{L}_{E,\mathcal{F}}(G))_{E,\Delta}$ is a pre (E, Δ) -mapping [13, Corollary, page 243], $\mathcal{C}\langle l\delta\eta\rangle_{E,\Delta} \subseteq \mathcal{C}\langle\eta\rangle_{E,\Delta}$ for each $\delta \in \Delta$. Therefore, $\mathcal{F} \subset \mathcal{G}$, and $\mathcal{G}^\Delta = \mathcal{F}^\Delta = \mathcal{C}$. By construction, η is a Δ - G -primitive over \mathcal{F} . By Proposition 3.64, \mathcal{G} is strongly E -normal over \mathcal{F} , and the map $c : G(\mathcal{G}/\mathcal{F}) \mapsto G$ defined by $c(\sigma) = \eta^{-1}\sigma\eta$ is an injective E - \mathcal{C} -homomorphism.

To show that c is surjective, let β be any element of the connected E - \mathcal{C} -group G (relative to the universal E -field \mathcal{U}^Δ). Using the identification of the elements of the E - \mathcal{C} -group G (relative to the E -field \mathcal{U}^Δ) with the subset $(G_{E,\Delta})_{\mathcal{U}\Delta}$ of the elements of the (E, Δ) - \mathcal{C} -group $G_{E,\Delta}$ (relative to the (E, Δ) -field \mathcal{U}), consider β as an element of $G_{E,\Delta}$. Because $l\delta(\eta\beta) = l\delta(\eta) + \tau_\eta^*l\delta(\beta) = l\delta(\eta)$, Lemma 3.65 implies $\eta \leftrightarrow \eta\beta$. Then, by part (3) in the definition of a pre set, there is an (E, Δ) -isomorphism $S_{(E,\Delta),\eta\beta,\eta} : \mathcal{C}\langle\eta\rangle_{E,\Delta} \approx \mathcal{C}\langle\eta\beta\rangle_{E,\Delta}$ over \mathcal{C} . Let $\sigma = S_{(E,\Delta),\eta\beta,\eta}$. By DAS 2b in the definition of a pre set, there exists a unique element x of $G_{E,\Delta}$

such that $\eta \leftrightarrow x$, $S_{(E,\Delta),x,\eta} = \sigma$ and $\sigma(\mathcal{C}\langle\eta\rangle_{E,\Delta}) = \mathcal{C}\langle x\rangle_{E,\Delta}$. This element x is the definition of $\sigma\eta$ [13, page 30]. Therefore, $\sigma\eta = \eta\beta$. For all $\delta \in \Delta$, the computation $\sigma l\delta(\eta) = l\delta(\sigma\eta) = l\delta(\eta\beta) = l\delta(\eta) + \tau_\eta^* l\delta(\beta) = l\delta(\eta)$ shows that \mathcal{F} is invariant under σ , and, hence, $\sigma \in G(\mathcal{G}/\mathcal{F})$. Then, c is surjective since $c(\sigma) = \eta^{-1}\sigma\eta = \beta$. Because a bijective E- \mathcal{C} -homomorphism of E- \mathcal{C} -groups is an E- \mathcal{C} -isomorphism [13, Corollary 4, page 97], c is an E- \mathcal{C} -isomorphism. \square

For a given E-group, the procedure in the next corollary constructs an E-strongly normal extension in two stages.

Corollary 3.67. *Assume $\Delta = \{\delta\}$. Let G be a connected E- \mathcal{C} -group (relative to the E-field \mathcal{U}^Δ). Let $G_{E,\Delta}$ be the (E, Δ) - \mathcal{C} -group (relative to the (E, Δ) -field \mathcal{U}) obtained by first extending the universal E-field from \mathcal{U}^Δ to \mathcal{U} and then by extending the derivations from E to (E, Δ) . First choose a \mathcal{C} -generic element a of $\mathcal{L}_{E,\mathcal{C}}(G)_{E,\Delta}$, and then choose an element b of $G_{E,\Delta}$ such that $l\delta(b) = a$. Then b is a \mathcal{C} -generic element of $G_{E,\Delta}$, and $\mathcal{C}\langle b\rangle_{(E,\Delta)}$ over $\mathcal{C}\langle a\rangle_{(E,\Delta)}$ is E-strongly normal (relative to the (E, Δ) -field \mathcal{U}) with Galois group E- \mathcal{C} -isomorphic to G .*

Proof. There exist a \mathcal{C} -generic element a of $\mathcal{L}_{E,\mathcal{C}}(G)_{E,\Delta}$ because of the definition of pre (E, Δ) -sets. That b exists follows from the surjectivity of the logarithmic derivative [13, Proposition 11, page 240].

Let η be a \mathcal{C} -generic element of $G_{E,\Delta}$. Set $\mathcal{G} = \mathcal{C}\langle\eta\rangle_{E,\Delta}$ and $\mathcal{F} = \mathcal{C}\langle l\delta\eta\rangle_{E,\Delta}$. By the previous theorem, \mathcal{G} over \mathcal{F} is an E-strongly normal extension with Galois group $G(\mathcal{G}/\mathcal{F})$ which is E- \mathcal{C} -isomorphic to G (relative to the universal E-field \mathcal{U}^Δ). The proof of this corollary will be accomplished by showing that $\mathcal{C}\langle b\rangle_{E,\Delta}$ is (E, Δ) -isomorphic to $\mathcal{C}\langle\eta\rangle_{E,\Delta}$ over \mathcal{C} .

Because η is a \mathcal{C} -generic element of $G_{E,\Delta}$ and the logarithmic derivative $l\delta$ is a surjective (E, Δ) - \mathcal{C} -mapping, $l\delta\eta$ is a \mathcal{C} -generic element of $\mathcal{L}_{E,\mathcal{C}}(G)_{E,\Delta}$ because, if t is any element of $\mathcal{L}_{E,\mathcal{C}}(G)_{E,\Delta}$ and ξ is an element of $G_{E,\Delta}$ such that $l\delta\xi = t$, then $\eta \rightarrow \xi$ implies $l\delta\eta \rightarrow l\delta\xi = t$ since $l\delta$ is a pre (E, Δ) -mapping [13, Corollary, page 242]. Because a and $l\delta\eta$ are both \mathcal{C} -generic elements of $\mathcal{L}_{E,\mathcal{C}}(G)_{E,\Delta}$, there exists an (E, Δ) -isomorphism φ over \mathcal{C} from $\mathcal{C}\langle a\rangle_{E,\Delta}$ to $\mathcal{C}\langle l\delta\eta\rangle_{E,\Delta}$. Because \mathcal{U} is (E, Δ) -universal over $\mathcal{C}\langle a\rangle_{E,\Delta}$, φ extends to an (E, Δ) - \mathcal{C} -isomorphism, also called φ , from $\mathcal{C}\langle b\rangle_{E,\Delta}$ to \mathcal{U} .

Since b is an element of $G_{E,\Delta}$, by DAS 2b in the definition of pre sets, there exists a unique x in $G_{E,\Delta}$ with $b \leftrightarrow x$ such that $\mathcal{C}\langle x\rangle_{E,\Delta} = \varphi(\mathcal{C}\langle b\rangle_{E,\Delta})$ and $S_{(E,\Delta),c,x,b} = \varphi$. Since (E, Δ) -isomorphisms over \mathcal{C} commute with the logarithmic derivative [13, Proposition 8, page 236], $l\delta(x) = l\delta(\varphi b) = \varphi(l\delta(b)) = \varphi a = l\delta(\eta)$. By Lemma 3.65, x is a \mathcal{C} -generic element of $G_{E,\Delta}$, and $x \leftrightarrow \eta$. Therefore, $b \leftrightarrow \eta$, and b is a \mathcal{C} -generic element of $G_{E,\Delta}$. Because $S_{(E,\Delta),\eta,b} : \mathcal{C}\langle b\rangle_{E,\Delta} \approx \mathcal{C}\langle\eta\rangle_{E,\Delta}$ is an (E, Δ) - \mathcal{C} -isomorphism and $S_{(E,\Delta),\eta,b}(\mathcal{C}\langle a\rangle_{E,\Delta}) = \mathcal{C}\langle l\delta(\eta)\rangle_{E,\Delta}$, by Proposition 1.31, $\mathcal{C}\langle b\rangle_{(E,\Delta)}$ over $\mathcal{C}\langle a\rangle_{(E,\Delta)}$ is E-strongly normal with Galois group E- \mathcal{C} -isomorphic to G . \square

3.5. The E-strongly normal extension corresponding to the E-group induced from an algebraic group. This section is a precise explanation of the heuristics described in the fourth paragraph of the introduction. In particular, given a linear differential operator L in the variable x such that the coefficients are in the (D_t, D_x) -field \mathcal{F} , let \mathcal{G}' be the extension D_x -field of the coefficient field \mathcal{F} generated by a fundamental system of D_x -zeros of L . Furthermore, assume the D_x -constants of \mathcal{G}' equal those of \mathcal{F} so that the extension \mathcal{G}' over \mathcal{F} is strongly

normal with Galois group G . Let \mathcal{G} be the (D_t, D_x) -field generated by \mathcal{G}' such that the D_x -constants of \mathcal{G} equal those of \mathcal{F} , which is true if the function field consists of analytic functions of two variables. Then \mathcal{G} is a D_t -strongly normal extension of \mathcal{F} , and the Galois group H is a D_t -group. Corollary 3.71 shows that H is embedded via a D_t -homomorphism to the D_t -group G_{D_t} induced from G by the extension of derivations (Section 3.2). An open problem is to compute the D_t -Galois groups of classical differential equations depending on parameters, such as the hypergeometric differential equation.

If A is a Δ -ring which is a subset of an (E, Δ) -ring, A_E will denote the (E, Δ) -ring generated by A . If A is a Δ -ring which is a subset of an (E, Δ) -field, $A_{(E)}$ will denote the (E, Δ) -field generated by A . Always $(A^\Delta)_{(E)} \subset (A_{(E)})^\Delta$. Also, please note that, if A and B are two Δ -rings which are subsets of an (E, Δ) -field, $(A[B])_{(E)} = A_{(E)} \cdot B_{(E)}$.

In this section, the following notation will be used. Let \mathcal{U} be an (E, Δ) -field that is (E, Δ) -universal over some (E, Δ) -field. Consider E as the union of two disjoint subsets E' and E'' . Let \mathcal{F}' be an (E', Δ) -subfield of \mathcal{U} such that \mathcal{U} is universal over $\mathcal{F}_{(E'')}$ as (E, Δ) -fields. This implies that \mathcal{U} considered as an (E', Δ) -field is also (E', Δ) -universal over \mathcal{F}' . Let \mathcal{G}' be an (E', Δ) -subfield of \mathcal{U} which is an E' -strongly normal extension of \mathcal{F}' relative to the universal (E', Δ) -field \mathcal{U} . Also, let $\mathcal{G} = (\mathcal{G}')_{(E'')}$, $\mathcal{F} = (\mathcal{F}')_{(E'')}$, $\mathcal{C}' = \mathcal{G}'^\Delta = \mathcal{F}'^\Delta$ and $\mathcal{C} = \mathcal{G}^\Delta$. This definition of \mathcal{C} is a change in notation from the usual $\mathcal{C} = \mathcal{F}^\Delta$ (see Remark 3.69) and we have

$$\begin{array}{ccccc}
 \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{G} = \mathcal{G}\mathcal{C} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}\mathcal{C} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathcal{C}' & \longrightarrow & \mathcal{F}^\Delta & \longrightarrow & \mathcal{C}.
 \end{array}$$

All the results in this section relate the Galois groups of the (E', Δ) -fields \mathcal{G}' over \mathcal{F}' to the Galois groups of the (E, Δ) -fields $\mathcal{G}\mathcal{C} = \mathcal{G}$ over $\mathcal{F}\mathcal{C}$ and constitute a straightforward application of basic definitions. In one's first reading of this material, the reader may assume that E' is empty. The theorems are presented in increased generality, with E' not empty, because no extra work is involved and they might be useful.

Lemma 3.68. *Let \mathcal{G}' be an (E', Δ) -subfield of \mathcal{U} which is an E' -strongly normal extension of the E' -field \mathcal{F}' relative to the (E', Δ) -universal (E', Δ) -field \mathcal{U} . Assume \mathcal{U} is (E, Δ) -universal over $\mathcal{F} = \mathcal{F}'_{(E'')}$. Then any (E, Δ) -isomorphism σ of $\mathcal{G} = \mathcal{G}\mathcal{C}$ into \mathcal{U} over $\mathcal{F}\mathcal{C}$ is E -strong. Furthermore, $(\mathcal{G}\sigma\mathcal{G})^\Delta = ((\mathcal{G}'\sigma\mathcal{G}')_{(E'')})^\Delta = \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')}$, and $\mathcal{C}\langle\sigma\rangle = \mathcal{C} \cdot \mathcal{C}'\langle\sigma\rangle_{(E'')}$.*

Remark 3.69. The field generated by the E'' -derivatives of \mathcal{G}' may contain new Δ -constants not in the field generated by the E'' -derivatives of \mathcal{F}' . An example of a strongly normal extension of Δ -fields \mathcal{G}' over \mathcal{F}' with this property is any \mathcal{G}' generated by a Weierstrassian over a field of Δ -constants \mathcal{F}' . (See [12, Examples, page 405] and Corollary 3.59.) This means that, in the lemma, for σ to be E -strong it must leave fixed a field \mathcal{C} of Δ -constants that might include Δ -constants not in \mathcal{C}' .

Proof. Because σ is an (E, Δ) -isomorphism of \mathcal{G} over $\mathcal{F}\mathcal{C}$ and $\mathcal{C} = \mathcal{G}^\Delta$, σ leaves the Δ -constants \mathcal{C} of \mathcal{G} invariant. Since σ restricted to \mathcal{G}' is E' -strong, $\sigma\mathcal{G}' \subset \mathcal{G}'\mathcal{U}^\Delta$ and $\mathcal{G}' \subset \sigma\mathcal{G}'\mathcal{U}^\Delta$. Then,

$$\sigma\mathcal{G} = \sigma(\mathcal{G}'_{(E'')}) = (\sigma\mathcal{G}')_{(E'')} \subset (\mathcal{G}'\mathcal{U}^\Delta)_{(E'')} = \mathcal{G}'_{(E'')}(\mathcal{U}^\Delta)_{(E'')} = \mathcal{G}\mathcal{U}^\Delta$$

and

$$\mathcal{G} = \mathcal{G}'_{(E'')} \subset (\sigma\mathcal{G}'\mathcal{U}^\Delta)_{(E'')} = (\sigma\mathcal{G}')_{(E'')}(\mathcal{U}^\Delta)_{(E'')} = \sigma(\mathcal{G}'_{(E'')})\mathcal{U}^\Delta = \sigma(\mathcal{G})\mathcal{U}^\Delta.$$

Therefore, σ is E -strong.

For the first equality,

$$(\mathcal{G}\sigma\mathcal{G})^\Delta = (\mathcal{G}'_{(E'')}\sigma(\mathcal{G}'_{(E''))})^\Delta = (\mathcal{G}'_{(E'')}\sigma\mathcal{G}')^\Delta = ((\mathcal{G}'\sigma\mathcal{G}')_{(E'')})^\Delta.$$

Since the E' -strong normality of σ implies $\mathcal{G}'\sigma\mathcal{G}' = \mathcal{G}'(\mathcal{G}'\sigma\mathcal{G}')^\Delta$, the above sequence of equalities is equal to

$$\begin{aligned} ((\mathcal{G}'(\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')})^\Delta &= (\mathcal{G}'_{(E'')}((\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')})^\Delta = (\mathcal{G} \cdot ((\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')})^\Delta \\ &= (\mathcal{G} \cot \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')})^\Delta = \mathcal{C}((\mathcal{G}'\sigma\mathcal{G}')^\Delta)_{(E'')}, \end{aligned}$$

where the last equality follows from [12, Corollary 2, page 88] because \mathcal{G} and the Δ -constants $\mathcal{C}((\mathcal{G}\sigma\mathcal{G})^\Delta)_{(E'')}$ are linearly disjoint over \mathcal{C} . The last equality of the proposition follows from the first two equalities and the definitions of $\mathcal{C}\langle\sigma\rangle$ and $\mathcal{C}'\langle\sigma\rangle$ as $(\mathcal{G}\sigma\mathcal{G})^\Delta$ and $(\mathcal{G}'\sigma\mathcal{G}')^\Delta$. \square

Proposition 3.70. *Let \mathcal{G}' be an (E', Δ) -subfield of \mathcal{U} which is an E' -strongly normal extension of \mathcal{F}' relative to the universal (E', Δ) -field \mathcal{U} . Then \mathcal{G} is an E -strongly normal extension of $\mathcal{F}\mathcal{C}$ relative to the universal (E, Δ) -field \mathcal{U} . Define the map ρ from the E - \mathcal{C} -group $G(\mathcal{G}/\mathcal{F}\mathcal{C})$ to the E' - \mathcal{C} -group $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ that associates to an (E, Δ) - $\mathcal{F}\mathcal{C}$ -isomorphism of \mathcal{G} its restriction to $\mathcal{G}'\mathcal{C}$. Then ρ is an injective (E, E') - \mathcal{C} -homomorphism (Definition 3.60). Furthermore, $\mathcal{C}\langle\sigma\rangle = \mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle_{(E'')}$.*

Proof. Because \mathcal{G}' over \mathcal{F}' is finitely (E', Δ) -generated, \mathcal{G} over \mathcal{F} and, therefore, \mathcal{G} over $\mathcal{F}\mathcal{C}$ are finitely (E, Δ) -generated. By Lemma 3.68, any (E, Δ) -isomorphism of \mathcal{G} over $\mathcal{F}\mathcal{C}$ is E -strong, and, since $\mathcal{G}^\Delta = (\mathcal{F}\mathcal{C})^\Delta$, \mathcal{G} over $\mathcal{F}\mathcal{C}$ is E -strongly normal.

By Theorem 1.30, $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ is the induced E' - \mathcal{C} -group of the E' - \mathcal{C} -group $G(\mathcal{G}'/\mathcal{F}')$, both being identified with each other by means of their canonical identifications with the group of (E', Δ) -automorphisms of $\mathcal{G}'\mathcal{U}^\Delta$ over $\mathcal{F}'\mathcal{U}^\Delta$. That ρ is a group homomorphism is clear by identifying the E -group $G(\mathcal{G}/\mathcal{F}\mathcal{C})$ with (E, Δ) -automorphisms of $\mathcal{G}\mathcal{U}^\Delta$ over $\mathcal{F}\mathcal{C}\mathcal{U}^\Delta = \mathcal{F}\mathcal{U}^\Delta$ and the E' -group $G(\mathcal{G}'\mathcal{C}/\mathcal{F}'\mathcal{C})$ with (E', Δ) -automorphisms of $\mathcal{G}'\mathcal{U}^\Delta$ over $\mathcal{F}'\mathcal{U}^\Delta$ and observing that the restriction ρ preserves composition in these groups. Because any set of (E', Δ) -generators of the (E', Δ) -field $\mathcal{G}'\mathcal{C}$ over $\mathcal{F}'\mathcal{C}$ are (E, Δ) -generators of the (E, Δ) -field $\mathcal{G}\mathcal{C}$ over $\mathcal{F}\mathcal{C}$, ρ is injective.

To show ρ is an (E, E') - \mathcal{C} -homomorphism each part of Definition 3.60 will be verified. For $\sigma \in G(\mathcal{G}/\mathcal{F}\mathcal{C})$, $\mathcal{C}\langle\sigma\rangle = \mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle_{(E'')}$ by Lemma 3.68. Since $\mathcal{C} \cdot \mathcal{C}'\langle\rho(\sigma)\rangle = \mathcal{C}\langle\rho(\sigma)\rangle$ (Theorem 1.30), it follows that $\mathcal{C}\langle\sigma\rangle \supset \mathcal{C}\langle\rho(\sigma)\rangle$. If $\sigma \rightarrow \tau$ for $\sigma, \tau \in G(\mathcal{G}/\mathcal{F}\mathcal{C})$, then, by the definition of specialization, there is an (E, Δ) -homomorphism $\varphi : \mathcal{G}[\sigma\mathcal{G}] \rightarrow \mathcal{G}[\tau\mathcal{G}]$ over \mathcal{G} such that $\varphi(\sigma\alpha) = \tau\alpha$ for all $\alpha \in \mathcal{G}$. Since $\mathcal{G}'\mathcal{C} \subset \mathcal{G}$, the restriction of φ to $\mathcal{G}'\mathcal{C}[\rho(\sigma)(\mathcal{G}'\mathcal{C})]$ is an (E', Δ) -homomorphism $\mathcal{G}'\mathcal{C}[\rho(\sigma)(\mathcal{G}'\mathcal{C})] \rightarrow \mathcal{G}'\mathcal{C}[\rho(\sigma)(\mathcal{G}'\mathcal{C})]$ over $\mathcal{G}'\mathcal{C}$ which takes $\rho(\sigma)\alpha$ to $\rho(\tau)\alpha$ for all $\alpha \in \mathcal{G}'\mathcal{C}$. Therefore, by definition, $\rho(\sigma) \rightarrow \rho(\tau)$. If $\sigma \leftrightarrow \tau$, then the (E, Δ) -homomorphism φ ,

defined above, is an (E, Δ) -isomorphism and, therefore, extends to an (E, Δ) -isomorphism, also denoted by φ , of the E -field $\mathcal{G}\sigma\mathcal{G}$ to the field $\mathcal{G}\tau\mathcal{G}$. The restriction of this (E, Δ) -isomorphism to $(\mathcal{G}\sigma\mathcal{G})^\Delta = \mathcal{C}\langle\sigma\rangle$ is the induced E - \mathcal{C} -isomorphism $S_{E;\tau,\sigma} : \mathcal{C}\langle\sigma\rangle_E \rightarrow \mathcal{C}\langle\tau\rangle_E$. The (E, Δ) -isomorphism φ also restricts to an (E', Δ) - \mathcal{C} -isomorphism from the (E', Δ) -field $\mathcal{G}'\rho(\sigma)\mathcal{G}'$ to the (E', Δ) -field $\mathcal{G}'\rho(\tau)\mathcal{G}'$, which in turn restricts to the induced E' - \mathcal{C} -isomorphism $S_{E';\rho(\tau),\rho(\sigma)} : \mathcal{C}\langle\rho(\sigma)\rangle_{E'} \rightarrow \mathcal{C}\langle\rho(\tau)\rangle_{E'}$. Since $\mathcal{C}\langle\rho(\sigma)\rangle_{E'} \subset \mathcal{C}\langle\sigma\rangle_E$, $S_{E;\tau,\sigma}$ extends $S_{E';\rho(\tau),\rho(\sigma)}$. \square

This proposition, in the case E' is empty, can be used to produce examples of E -strongly normal extensions. Start with a Δ -extension \mathcal{G}' over \mathcal{F}' which is strongly normal (in the sense of Kolchin) such that the coefficients of the differential equations defining \mathcal{G}' over \mathcal{F}' depend on a parameter t . Assume that the Δ -field \mathcal{F} is closed with respect to differentiation by t . Differentiate the elements of \mathcal{G}' with respect to t to generate a $\{d/dt, \Delta\}$ -field extension \mathcal{G} . Then if $(\mathcal{G})^\Delta \subset \mathcal{F}$, \mathcal{G} over \mathcal{F} is $\{d/dt\}$ -strongly normal over \mathcal{F} .

Corollary 3.71. *In the above proposition, assume $\mathcal{C} = \mathcal{G}^\Delta \subset \mathcal{F}^\Delta = \mathcal{C}'$. Then the injective (E, E') - \mathcal{C} -homomorphism $\rho : G(\mathcal{G}/\mathcal{F}) \rightarrow G(\mathcal{G}'/\mathcal{F}')$ identifies the E - \mathcal{C} -group $G(\mathcal{G}/\mathcal{F})$ with an E - \mathcal{C} -subgroup of the E - \mathcal{C} -group $G(\mathcal{G}'/\mathcal{F}')_E$ induced from the E' - \mathcal{C} -group $G(\mathcal{G}'/\mathcal{F}')$ by extending the derivations to E (Definition 3.61).*

Proof. Kolchin proved that the induced E - \mathcal{C} -group $G(\mathcal{G}'/\mathcal{F}')_E$ always exists [13, Theorem 3, page 58]. By Definition 3.61 of the induced E -group, the (E, E') - \mathcal{C} -homomorphism ρ of the last proposition extends to a unique E - \mathcal{C} -homomorphism $\bar{\rho} : G(\mathcal{G}/\mathcal{F}) \rightarrow G(\mathcal{G}'/\mathcal{F}')_E$. It is also injective because ρ and $\bar{\rho}$ are equal on the elements of $G(\mathcal{G}/\mathcal{F})$. The image of an E - \mathcal{C} -group under an E - \mathcal{C} -homomorphism is an E - \mathcal{C} -subgroup [13, Proposition 4, page 92]. Because ρ is a bijective E - \mathcal{C} -homomorphism of $G(\mathcal{G}/\mathcal{F})$ to its image, the E - \mathcal{C} -group $G(\mathcal{G}/\mathcal{F})$ and its image in $G(\mathcal{G}'/\mathcal{F}')_E$ are E - \mathcal{C} -isomorphic [13, Corollary 4, page 97]. \square

4. EXAMPLES

In this chapter, \mathcal{F} will denote an (E, Δ) -field, and \mathcal{U} will denote an (E, Δ) -field universal over \mathcal{F} . The field \mathcal{K} of Δ -constants of \mathcal{U} is, as an E -field, E -universal over the Δ -constants \mathcal{C} of \mathcal{F} .

4.1. G_a^E -extensions. Denote the additive (E, Δ) - \mathbb{Q} -group [13, page 28] (relative to \mathcal{U}) by the symbol $G_a^{E,\Delta}$. The elements of $G_a^{E,\Delta}$ are those of \mathcal{U} , and its group structure is that of the field \mathcal{U} under addition. Similarly, G_a^E will denote the additive E - \mathbb{Q} -group (relative to \mathcal{K}) with elements those of \mathcal{K} . Let $\kappa \in \mathfrak{F}(G_a^{E,\Delta})$ be the canonical coordinate function on $G_a^{E,\Delta}$. Then, $\delta_i \kappa \in \mathfrak{F}(G_a^{E,\Delta})$, and the E - \mathcal{F} -mapping $l\Delta = (\delta_1 \kappa, \dots, \delta_m \kappa) : G_a^{E,\Delta} \rightarrow (G_a^{E,\Delta})^n$ [13, Proposition 6, page 129] is the logarithmic derivation on $G_a^{E,\Delta}$ relative to Δ [12, Example 1, page 352]. By [13, Proposition 3, page 89], it is an (E, Δ) - \mathcal{F} -homomorphism. The kernel of $l\Delta$ is the (E, Δ) - \mathcal{F} -subgroup consisting of (E, Δ) -zeros of the (E, Δ) -ideal $[\delta_1 y, \dots, \delta_m y] \subset \mathcal{F}\{y\}_{E,\Delta}$ and can be identified with G_a^E relative to the E -universal field \mathcal{K} .

Definition 4.72. An element $\alpha \in \mathcal{U}$ is Δ -primitive over \mathcal{F} if $l\Delta\alpha \in \mathcal{F}^m$; that is, for suitable elements $a_1, \dots, a_m \in \mathcal{F}$, α satisfies the system of differential equations

$$\delta_i \alpha = a_i \quad (1 \leq i \leq m).$$

Let α be Δ -primitive over \mathcal{F} , and suppose that the field of Δ -constants of $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ is $\mathcal{C} = \mathcal{F}^\Delta$. For any (\mathbb{E}, Δ) -isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ over \mathcal{F} , $(\delta_1(\sigma\alpha), \dots, \delta_m(\sigma\alpha)) = (\sigma(\delta_1\alpha), \dots, \sigma(\delta_m\alpha)) = (\delta_1\alpha, \dots, \delta_m\alpha)$; hence the difference $c(\sigma) = \sigma\alpha - \alpha$ is in the kernel of the above homomorphism $l\delta$ and a Δ -constant. As

$$\begin{aligned} \mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}) &= \mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}\mathcal{F}\langle\sigma\alpha\rangle_{\mathbb{E},\Delta} \\ &= \mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}\mathcal{F}\langle\alpha + c(\sigma)\rangle_{\mathbb{E},\Delta} = \mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}\mathcal{C}\langle c(\sigma)\rangle_{\mathbb{E},\Delta}, \end{aligned}$$

it follows that $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ is \mathbb{E} -strongly normal over \mathcal{F} , and

$$\mathcal{C}\langle\sigma\rangle_{\mathbb{E}} = (\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}\sigma(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}))^\Delta = \mathcal{C}\langle c(\sigma)\rangle_{\mathbb{E}}.$$

For any two elements $\sigma, \sigma' \in G(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}/\mathcal{F})$ (regarded as elements of $\text{Aut}_{\mathbb{E},\Delta}(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}/\mathcal{F})$ by means of Proposition 1.15),

$$\alpha + c(\sigma\sigma') = \sigma\sigma'\alpha = \sigma(\alpha + c(\sigma')) = \sigma\alpha + c(\sigma') = \alpha + c(\sigma) + c(\sigma')$$

since $c(\sigma') \in \mathcal{K}$ and, thus, $\sigma(c(\sigma')) = c(\sigma')$. Therefore, $c(\sigma\sigma') = c(\sigma) + c(\sigma')$, and, evidently, $c(\sigma) = 0$ only when $\sigma = \text{id}_{\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}}$. This proves the first part of the following proposition, and the remainder is the same as that of Proposition 3.64.

Proposition 4.73. *Let α be a Δ -primitive over \mathcal{F} , and suppose that the field of Δ -constants of $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ is $\mathcal{C} = \mathcal{F}^\Delta$. Then, each (\mathbb{E}, Δ) - \mathcal{F} -isomorphism σ of $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ into \mathcal{U} is of the form $\sigma\alpha = \alpha + c(\sigma)$ for $c(\sigma) \in \mathcal{K}$. In addition, $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ is \mathbb{E} -strongly normal over \mathcal{F} , and the mapping $c : G(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}/\mathcal{F}) \rightarrow G_a^{\mathbb{E}}$ defined by $c(\sigma) = \sigma\alpha - \alpha$ for $\sigma \in G(\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}/\mathcal{F})$ is an injective \mathbb{E} - \mathcal{C} -homomorphism of \mathbb{E} -groups relative to the \mathbb{E} -universal field \mathcal{K} . Consequently, $\mathcal{F}\langle\alpha\rangle_{\mathbb{E},\Delta}$ is a $G_a^{\mathbb{E}}$ -extension of \mathcal{F} .*

Proposition 4.74. *Let G be an \mathbb{E} - \mathcal{C} -subgroup of $G_a^{\mathbb{E}}$. Let $\mathcal{L} \subseteq \mathcal{C}\{y\}_{\mathbb{E}}$ be the linear \mathbb{E} -ideal defining G [13, page 151]. Let $b \in \mathcal{U}$ be a \mathcal{C} -generic (\mathbb{E}, Δ) -zero of $\mathcal{L}_{\Delta,\mathbb{E}} \subseteq \mathcal{C}\{y\}_{\mathbb{E},\Delta}$. Let $a = (a_1, \dots, a_m) = l\Delta b$. Put $\mathcal{F} = \mathcal{C}\langle a \rangle_{\mathbb{E},\Delta}$ and $\mathcal{G} = \mathcal{F}\langle b \rangle_{\mathbb{E},\Delta}$. Then \mathcal{G} over \mathcal{F} is an \mathbb{E} -strongly normal extension with Galois group \mathbb{E} - \mathcal{C} -isomorphic to G .*

Proof. This is a special case of Theorem 3.66. □

Let \mathcal{G} be \mathbb{E} -strongly normal over \mathcal{F} with Galois group $G \subset G_a^{\mathbb{E}}$. Theorem 1.24 shows that G is an \mathbb{E} - \mathcal{C} -group where $\mathcal{C} = \mathcal{F}^\Delta$. By [13, page 151], G is a set of \mathbb{E} -zeros of a linear \mathbb{E} -ideal $\mathcal{L}_G \subseteq \mathcal{C}\{\kappa\}_{\mathbb{E}}$, where κ is the canonical coordinate function on $G_a^{\mathbb{E}}$. Each \mathbb{E} - \mathcal{C} -subgroup $H \subseteq G$ is also the \mathbb{E} -zeros of a linear \mathbb{E} -ideal $\mathcal{L}_H \subseteq \mathcal{C}\{y\}_{\mathbb{E}}$ such that $\mathcal{L}_G \subseteq \mathcal{L}_H$. Recall, by the definition of a linear \mathbb{E} -ideal \mathcal{L} , $\mathcal{L} = [\mathcal{L}_1]_{\mathbb{E}}$ where \mathcal{L}_1 is the subset of elements of \mathcal{L} of degree one. For each $H \subset G$, the following proposition exhibits the subfield of \mathcal{G} invariant under the action of H and, thus, specifies the Galois correspondence, even if $\mathcal{C} = \mathcal{F}^\Delta$ is not constrainedly closed.

Proposition 4.75. *Let \mathcal{G} be an \mathbb{E} -strongly normal extension of \mathcal{F} with Galois group $G(\mathcal{G}/\mathcal{F}) \subseteq G_a^{\mathbb{E}}$. Assume that $\mathcal{G} = \mathcal{F}\langle b \rangle_{\mathbb{E},\Delta}$ where $b \in \mathcal{U}$ is a Δ -primitive over \mathcal{F} . Then, there exists a Galois correspondence which to each \mathbb{E} - \mathcal{C} -subgroup H of $G(\mathcal{G}/\mathcal{F})$ associates the (\mathbb{E}, Δ) -subfield $\mathcal{H} = \mathcal{F}\langle (L(b))_{L(y) \in \mathcal{L}_H} \rangle_{\mathbb{E},\Delta} \subseteq \mathcal{G}$, where $\mathcal{L}_H \subseteq \mathcal{C}\{\kappa\}_{\mathbb{E}}$ is the linear \mathbb{E} -ideal defining H and κ is the canonical coordinate*

function on $\mathcal{K} = G_a^E$:

$$\begin{array}{ccc}
 G(\mathcal{G}/\mathcal{H}) & \xrightarrow{\quad} & \mathcal{F} \\
 \uparrow & & \downarrow \\
 H & \xrightarrow{\quad} & \mathcal{H} = \mathcal{G}^H = \mathcal{F}\langle(L(b))_{L(y) \in \mathcal{L}_H}\rangle_{E,\Delta} \\
 \uparrow & & \downarrow \\
 1 & \xrightarrow{\quad} & \mathcal{G}.
 \end{array}$$

Proof. Since $\mathcal{L}_H = [\mathcal{L}_{H,1}]_E$, it follows that $\mathcal{H} = \mathcal{F}\langle(L(b))_{L(y) \in \mathcal{L}_{H,1}}\rangle_{E,\Delta}$. Let $\sigma \in G(\mathcal{G}/\mathcal{H})$. By Proposition 4.73, $\sigma(b) = b + c(\sigma)$ for $c(\sigma) \in \mathcal{K}$. For all $L(y) \in \mathcal{L}_{H,1}$,

$$L(b) = \sigma(L(b)) = L(\sigma(b)) = L(b + c(\sigma)) = L(b) + L(c(\sigma));$$

thus $L(c(\sigma)) = 0$. Therefore, $c(\sigma)$ is an E-zero of \mathcal{L}_H , $\sigma \in H$ and $H \supset G(\mathcal{G}/\mathcal{H})$. If $\sigma \in H$,

$$(8) \quad \sigma(L(b)) = L(\sigma(b)) = L(b + c(\sigma)) = L(b) + L(c(\sigma)) = L(b)$$

for $L(y) \in \mathcal{L}_1$, and $H \subset G(\mathcal{G}/\mathcal{H})$. □

For simplicity, assume $\Delta = \{\delta\}$ throughout the remainder of this section. In the next proposition, if b is Δ -primitive over \mathcal{F} , the Galois group of $\mathcal{G} = \mathcal{F}\langle b \rangle_{E,\Delta}$ over \mathcal{F} is completely determined by $a = \delta b \in \mathcal{F}$.

Proposition 4.76. *Let b be a Δ -primitive over \mathcal{F} , and let $\mathcal{G} = \mathcal{F}\langle b \rangle_{E,\Delta}$. Assume that $\mathcal{G}^\Delta = \mathcal{F}^\Delta$. Let $a = \delta b$, let $\mathcal{L}_{a,1} = \{L(y) \in \mathcal{C}\{y\}_{E,1} \mid L(a) \in \delta\mathcal{F}\}$, and let $\mathcal{L}_a = [\mathcal{L}_{a,1}]_E$. Let $G = \text{Gal}(\mathcal{G}/\mathcal{F})$, and let $c : G \rightarrow \mathcal{V}$ be the E- \mathcal{F} -homomorphism defined by $c(\sigma) = \sigma(b) - b$. Then, the defining E-ideal $\mathfrak{A}_{c(G)} \subset \mathcal{C}\{y\}_E$ of $c(G)$ is \mathcal{L}_a .*

Proof. By Proposition 4.73, \mathcal{G} over \mathcal{F} is E-strongly normal, and G is an E- \mathcal{C} -group. By [13, page 151], the E- \mathcal{C} -group $c(G) \subseteq G_a^E$ is the set of E-zeros of a linear E-ideal $\mathfrak{A}_{c(G)} \subset \mathcal{C}\{y\}_E$. Also, Proposition 4.73 shows that each $\sigma \in G$ is of the form $\sigma(b) = b + c(\sigma)$ for an E-zero $c(\sigma)$ of $\mathfrak{A}_{c(G)}$.

For each linear $L(y) \in \mathfrak{A}_{c(G)}$, Equation 8 above shows $L(b)$ is invariant under all elements of G . Thus, $L(b) \in \mathcal{F}$, and $L(b) = f$ for some $f \in \mathcal{F}$. Hence,

$$L(a) = L(\delta b) = \delta(L(b)) = \delta f.$$

Therefore, $\mathfrak{A}_{c(G)} \subseteq \mathcal{L}_a$.

On the other hand, let $L(y) \in \mathcal{L}_{a,1}$. Then $L(a) = \delta f$ for $f \in \mathcal{F}$, and $L(b) - f$ is a Δ -constant because $\delta(L(b) - f) = L(\delta b) - \delta f = L(a) - \delta f = 0$. Therefore, $L(b) - f \in \mathcal{C} \subseteq \mathcal{F}$, and $L(b) \in \mathcal{F}$. Hence, for all $\sigma \in G$, $\sigma(L(b)) = L(b)$, and the computation

$$L(c(\sigma)) = L(\sigma(b) - b) = L(\sigma(b)) - L(b) = \sigma(L(b)) - L(b) = 0$$

shows that $\mathfrak{A}_{c(G)} \supseteq \mathcal{L}_a$. □

The following is a simple example of an E-strongly normal extension \mathcal{G} over \mathcal{F} such that the transcendence degree of \mathcal{G} over \mathcal{F} is infinite in the usual algebraic sense. Let $\mathcal{F} \subset \mathcal{U}$ be an (E, Δ) -field containing an element a that is linearly E- \mathcal{F}^Δ -independent modulo $\delta\mathcal{F}$ (Definition 5.101). For instance, any $a \in \mathcal{C}(t)_{E,\Delta}$, $a \notin \mathcal{F}$, where t is (E, Δ) -independent over \mathcal{F} satisfies this condition by Proposition 3.53.

Let $b \in \mathcal{U}$ be an (E, Δ) -zero of the (E, Δ) -ideal $\{\delta y - a\}_{E, \Delta} \subset \mathcal{F}\{y\}_{E, \Delta}$. Let $\mathcal{G} = \mathcal{F}\langle b \rangle_{E, \Delta}$. By Corollary 5.102, $(\mathcal{F}\langle b \rangle_{E, \Delta})^\Delta = \mathcal{F}^\Delta$. Therefore, \mathcal{G} is E -strongly normal over \mathcal{F} by Proposition 4.73. Since b is E -independent over \mathcal{F} , $\mathcal{G} = \mathcal{F}\langle b \rangle_{E, \Delta}$ has infinite transcendence degree over \mathcal{F} . In fact $c(G(\mathcal{G}/\mathcal{F})) = G_a^E$, because if a nonzero $L(y) \in \mathcal{C}\{y\}_{E, 1}$ is in the defining E -ideal of $c(G(\mathcal{G}/\mathcal{F}))$ by the previous proposition, there exists an $f \in \mathcal{F}$ such that $L(a) = \delta f$. This contradicts the fact that $1, b, \epsilon b, \epsilon^2 b, \dots$ are linearly independent over \mathcal{F} (Proposition 5.96).

Corollary 4.77. *Assume $E = \{\epsilon\}$ and $\Delta = \{\delta\}$. Let \mathcal{H} be an algebraically closed (E, Δ) -field such that $\mathcal{H}^\Delta = \mathcal{H}$, let $\mathcal{F} = \mathcal{H}\langle x \rangle_{E, \Delta}$, where $x \in \mathcal{U}$, $\epsilon x = 0$ and $\delta x = 1$, and as usual let $\mathcal{C} = \mathcal{F}^\Delta = \mathcal{H}^\Delta$. Then, there is no Δ -primitive E -strongly normal extension of \mathcal{F} with Galois group E - \mathcal{C} -isomorphic to G_a^E .*

Remark 4.78. This remains true if the hypothesis that \mathcal{H} be algebraically closed is omitted; the following proof must be modified to take the structure of irreducibles into account in the partial fraction decomposition.

Proof. Assume that there exists a Δ -primitive E -strongly normal extension \mathcal{G} of \mathcal{F} with Galois group G that is E - \mathcal{C} -isomorphic to G_a^E . Let $b \in \mathcal{U}$ be a Δ -primitive over \mathcal{F} such that $\delta b = a \in \mathcal{F}$ and $\mathcal{G} = \mathcal{F}\langle b \rangle_{E, \Delta}$. Let $a = p(x) + \sum_{i,j} \frac{h_{i,j}}{(x-h_i)^j}$, for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of a . If $h_{i,1} = 0$ for all i , then $a = \delta f$ for $f \in \mathcal{F}$, and $b - f \in \mathcal{G}$ is a Δ -constant not in \mathcal{F} , which contradicts the assumption that \mathcal{G} over \mathcal{F} is E -strongly normal (Proposition 1.12). Therefore, $h_{i,1} \neq 0$ for at least one i , and there exists a nonzero $L(y) =$

$$\begin{vmatrix} h_{1,1} & h_{2,1} & \dots & h_{r,1} & y \\ \epsilon h_{1,1} & \epsilon h_{2,1} & \dots & \epsilon h_{r,1} & \epsilon y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \epsilon^r h_{1,1} & \epsilon^r h_{2,1} & \dots & \epsilon^r h_{r,1} & \epsilon^r y \end{vmatrix}$$

$\in \mathcal{H}\{y\}_{E, 1}$ such that the finitely many $h_{i,1}$ span over $\mathcal{H}^{E, \Delta}$ the linear space of E -zeros of $L(y)$. By Lemma 4.79 below, since $L(h_{i,1}) = 0$ for all i , $L(a) \in \delta\mathcal{F}$. By Proposition 4.76, $L(y)$ is contained in the defining E -ideal of $c(G)$, which contradicts the assumption that G is E - \mathcal{C} -isomorphic to G_a^E . \square

Lemma 4.79. *Assume $E = \{\epsilon\}$ and $\Delta = \{\delta\}$. Let \mathcal{H} be an algebraically closed (E, Δ) -field such that $\mathcal{H}^\Delta = \mathcal{H}$, and let $\mathcal{F} = \mathcal{H}\langle x \rangle_{E, \Delta}$, where $x \in \mathcal{U}$, $\epsilon x = 0$ and $\delta x = 1$. Let $M(y) \in \mathcal{H}^\Delta\{y\}_{E, 1}$. For $\alpha \in \mathcal{F}$, let $\alpha = p(x) + \sum_{i,j} \frac{h_{i,j}}{(x-h_i)^j}$ for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of α . Then, $M(\alpha) \in \delta\mathcal{F}$ if and only if $M(h_{i,1}) = 0$ for all i .*

Proof. The only terms in the above representation of α not in $\delta\mathcal{F}$ are those with $j = 1$. Since $\delta M(y) = M(\delta y)$, if $j > 1$, $M(\frac{h_{i,j}}{(x-h_i)^j}) \in \delta\mathcal{F}$ because $\frac{h_{i,j}}{(x-h_i)^j} \in \delta\mathcal{F}$. Therefore, the condition $M(\alpha) \in \delta\mathcal{F}$ is equivalent to $M(\sum_i \frac{h_{i,1}}{(x-h_i)}) \in \delta\mathcal{F}$. Since

$$\epsilon(\frac{h_{i,1}}{(x-h_i)}) = \frac{\epsilon h_{i,1}}{(x-h_i)} - \frac{h_{i,1}\epsilon h_i}{(x-h_i)^2},$$

by induction $\frac{\epsilon^k h_{i,1}}{(x-h_i)} = \frac{\epsilon^k h_{i,1}}{(x-h_i)} +$ an element of $\delta\mathcal{F}$. By the linearity of M , $M(\alpha) \in \delta\mathcal{F}$ is equivalent to $\sum_i \frac{M(h_{i,1})}{(x-h_i)} \in \delta\mathcal{F}$. This is true if and only if $M(h_{i,1}) = 0$ for all i since all the $\frac{1}{x-h_i}$ are linearly independent over \mathcal{H} modulo $\delta\mathcal{F}$ (Corollary 5.93). \square

The next two results establish procedures for the construction of all G_a^E -extensions under the condition $E = \{\epsilon\}$.

Proposition 4.80. *Assume $E = \{\epsilon\}$ and $\Delta = \{\delta\}$. Let \mathcal{H} be an (E, Δ) -field, and let $h \in \mathcal{H}$. Let \mathcal{U} be (E, Δ) -universal over \mathcal{H} . Let $L(y) \in \mathcal{H}^\Delta\{y\}_{E,1}$ of positive order n with the coefficient of the highest-order term equal to 1.*

- (1) *There exists $a \in \mathcal{U}$ an (E, Δ) -zero of $[L(y) - \delta h]_{E,\Delta} \subset \mathcal{H}\{y\}_{E,\Delta}$ such that $a, \epsilon a, \dots, \epsilon^{n-1}a$ are linearly independent over $(\mathcal{H}\langle a \rangle_{E,\Delta})^\Delta$ modulo $\delta(\mathcal{H}\langle a \rangle_{E,\Delta})$.*
- (2) *There exists $b \in \mathcal{U}$ an (E, Δ) -zero of $M = [\delta y - a, L(y) - h]_{E,\Delta} \subset \mathcal{H}\langle a \rangle_{E,\Delta}\{y\}_{E,\Delta}$.*

Put $\mathcal{F} = \mathcal{H}\langle a \rangle_{E,\Delta}$ and $\mathcal{G} = \mathcal{F}\langle b \rangle_{E,\Delta}$. Then, \mathcal{G} is an E -strongly normal extension of \mathcal{F} , and $\mathcal{L} = [L]_E$ is the defining E -ideal of $c(G(\mathcal{G}/\mathcal{F})) \subseteq G_a^E$.

Proof. Let $a \in \mathcal{U}$ be an \mathcal{H} -generic (E, Δ) -zero of $[L(y) - \delta h]_{E,\Delta}$. Clearly, a satisfies (1). To show there exists $b \in \mathcal{U}$ that satisfies (2), [12, Lemmas 5 and 6, page 137] will be applied to show that M is a proper prime (E, Δ) -ideal. Since \mathcal{U} is (E, Δ) -universal over \mathcal{H} , there exists an (E, Δ) -zero $b \in \mathcal{U}$ as required.

To apply [12, Lemma 5, page 137], $\{\delta y - a, L(y) - h\}$ must be a coherent autoreduced set of M relative to some fixed ranking. It is clearly autoreduced. The coherence follows by letting $L'(y) = L(y) - \epsilon^n y$ and computing

$$\begin{aligned} \delta(L(y) - h) - \epsilon^n(\delta y - a) &= \delta L'(y) + \epsilon^n a - \delta h \\ &= \delta L'(y) + \epsilon^n a - \delta h - (L(a) - \delta h) = \delta L'(y) - L'(a) = L'(\delta y - a). \end{aligned}$$

To show M is a proper (E, Δ) -ideal, assume that it is not. Then $1 \in M$. Since 1 is partially reduced with respect to $\{\delta y - a, L(y) - h\}$, [12, Lemma 5, page 137] implies that $1 \in (\delta y - a, L(y) - h) \subset \mathcal{H}\langle a \rangle_{E,\Delta}\{y\}_{E,\Delta}$, which is impossible because 1 is reduced with respect to $\{\delta y - a, L(y) - h\}$ and is not zero. To show M is a prime (E, Δ) -ideal, since the separants and initials of a coherent autoreduced set are 1, it is sufficient to observe that the ideal $(\delta y - a, L(y) - h)$ is prime by [12, Lemma 6, page 137]. It is well known that an inhomogeneous linear ideal is prime.

Next, $\mathcal{G}^\Delta = \mathcal{F}^\Delta$ follows from (1) and Proposition 5.96 since $\delta(\epsilon^k b) = \epsilon^k a$ for $k = 0, \dots, n - 1$ and the set $b, \epsilon b, \dots, \epsilon^{n-1}b$ generate \mathcal{G} as a field extension of \mathcal{F} . Because \mathcal{G} is (E, Δ) -generated by a Δ -primitive element over \mathcal{F} , Proposition 4.73 implies that \mathcal{G} is E -strongly normal over \mathcal{F} . Since $L(a) = \delta h$, Proposition 4.76 implies $L(y) \in \mathcal{L}$. Suppose there were a linear $N(y) \in \mathcal{L}$ of lower order than L . Then, again by Proposition 4.76, there exist $f \in \mathcal{F}$ such that $N(a) = \delta f$. Then, $N(b) - f$ is a Δ -constant in \mathcal{F} . Therefore, $N(b) = f_1 \in \mathcal{F}$. However, by Proposition 5.96, $1, b, \epsilon b, \dots, \epsilon^{n-1}b$ are linearly independent over \mathcal{F} . This contradiction shows that L is linear of minimal order in \mathcal{L} . Therefore, $\mathcal{L} = [L(y)]_E$. □

One may apply this proposition to the example of the introduction. Let $\mathcal{H} = \mathbb{C}\langle t, x, \cos t, \sin t \rangle$ ($\epsilon t = 1, \epsilon x = 0, \delta t = 0, \delta x = 0$), $h = 0$, $\gamma = \cos t / \sin t$, $a = \sin t / x \in \mathcal{H}$, and let $L(y) = \epsilon y - \gamma y \in \mathcal{H}^E\{y\}_{E,1}$. Then a is an (E, Δ) -zero of $[L(y)]_{E,\Delta}$, and a is linearly independent over $(\mathcal{H}\langle a \rangle_{E,\Delta})^\Delta = (\mathcal{H})^\Delta = \mathbb{C}\langle t, \cos t, \sin t \rangle$ modulo $\delta(\mathcal{H}\langle a \rangle_{E,\Delta}) = \delta(\mathcal{H})$ by Corollary 5.93. Let $b = \log x \sin t$. Then b is an (E, Δ) -zero of $[\delta y - a, L(y)]_{E,\Delta}$. By Proposition 4.80, $\mathcal{G} = \mathcal{H}\langle b \rangle_{E,\Delta}$ is an E -strongly normal extension of $\mathcal{F} = \mathcal{H}\langle a \rangle_{E,\Delta}$, and $[L(y)]_E$ is the defining ideal of the Galois group in G_a^E .

The following corollary reformulates the previous proposition so that other examples may be constructed easily.

Corollary 4.81. *Let F be an (E, Δ) -field, let $h \in F$, and let $d_1, \dots, d_n \in F^E \subset \mathcal{U}$. Let $L(y) \in F^\Delta\{y\}_{E,1}$ of positive order n with the coefficient of the highest-order term equal to 1, and, for $i = 1, \dots, n$, let $e_i \in \mathcal{U}^E$ be a Δ -zero of $\delta y - d_i \in F\{y\}_\Delta$. Assume*

- (1) d_1, \dots, d_n are linearly independent over F^Δ modulo δF ,
- (2) there exist $\eta_1, \dots, \eta_n \in F^\Delta$ such that η_1, \dots, η_n are E -zeros of $L(y)$ linearly independent over $F^{E,\Delta}$, and
- (3) there exists an (E, Δ) -zero $\eta \in F$ of $L(y) - h \in F\{y\}_{E,\Delta}$.

Let $b = \eta + \sum_i \eta_i e_i$. Then $F\langle b \rangle_{E,\Delta}$ is E -strongly normal over F , and we have that $c(G(F\langle b \rangle_{E,\Delta}/F)) \subseteq G_a^E$ is defined by the E -ideal $\mathcal{L} = [L(y)]_E$.

Proof. The assumptions of the proposition are satisfied by taking $\mathcal{H} = F$, $\mathcal{F} = F$ and $a = \delta\eta + \sum_i \eta_i d_i \in F$. Clearly

$$\begin{aligned} L(a) &= L(\delta b) = \delta(L(b)) = \delta(L(\eta + \sum_i \eta_i e_i)) = \delta(L(\eta) + \sum_i L(\eta_i e_i)) \\ &= \delta(L(\eta) + \sum_i L(\eta_i) e_i) = \delta(L(\eta)) = \delta h \end{aligned}$$

shows that a is an (E, Δ) -zero of $[L(y) - \delta h]_{E,\Delta}$. The computations

$$\delta b = \delta(\eta + \sum_i \eta_i e_i) = \delta\eta + \sum_i \eta_i \delta e_i = \delta\eta + \sum_i \eta_i d_i = a$$

and $L(b) = L(\eta + \sum_i \eta_i e_i) = L(\eta) + \sum_i e_i L(\eta_i) = L(\eta) = h$ demonstrate that b is an (E, Δ) -zero of $[\delta y - a, L(y) - h]_{E,\Delta} \subset F\{y\}_{E,\Delta}$.

It remains to be shown that $a, \epsilon a, \dots, \epsilon^{n-1} a$ are linearly independent over F^Δ modulo δF . Since d_1, \dots, d_n are linearly independent over F^Δ modulo δF , Proposition 5.96 implies that $(F\langle e_1, \dots, e_n \rangle_{E,\Delta})^\Delta = F^\Delta$ and e_1, \dots, e_n are algebraically independent over F . Because η_1, \dots, η_n are linearly independent over $F^{E,\Delta}$, the matrix $(\epsilon^{i-1} \eta_j)_{i=1, \dots, n; j=1, \dots, n}$ is invertible [12, Theorem 1, page 86], and, therefore, the map φ of $F\langle e_1, \dots, e_n \rangle_{E,\Delta}$ defined by $\varphi(e_i) = \sum_j \epsilon^{i-1} \eta_j e_j$ is an automorphism of $F\langle e_1, \dots, e_n \rangle_{E,\Delta}$ over F . The composition ρ of this automorphism with the translation that sends $\varphi(e_i)$ to $\epsilon^{i-1} \eta + \varphi(e_i)$ is an automorphism of $F\langle e_1, \dots, e_n \rangle_{E,\Delta}$ such that $\rho(e_i) = \epsilon^{i-1} \eta + \sum_j \epsilon^{i-1} \eta_j e_j$. Therefore, $\rho(e_1), \dots, \rho(e_n)$ are also algebraically independent over F , and $(F\langle \rho(e_1), \dots, \rho(e_n) \rangle_{E,\Delta})^\Delta = F^\Delta$. Proposition 5.96 implies that $\delta(\rho(e_1)), \dots, \delta(\rho(e_n))$ are linearly independent over F^Δ modulo δF . The observation that for each i ,

$$\begin{aligned} \delta(\rho(e_i)) &= \delta(\epsilon^{i-1} \eta + \sum_j \epsilon^{i-1} \eta_j e_j) = \delta\epsilon^{i-1} \eta + \sum_j \epsilon^{i-1} \eta_j \delta(e_j) \\ &= \delta\epsilon^{i-1} \eta + \sum_j \epsilon^{i-1} \eta_j d_j = \epsilon^{i-1} (\delta\eta + \sum_j \eta_j d_j) = \epsilon^{i-1} a \end{aligned}$$

completes the proof. □

A particularly simple example may be obtained by taking, in this last corollary, $F = \mathbb{C}(t, x)$ ($\epsilon t = 1, \epsilon x = 0, \delta t = 0, \delta x = 1$), $d_i = 1/(x - i)$ for $i = 0, \dots, n - 1$, $e_i = \ln(x - i)$, $L = \epsilon^n y$, $h = 0$ and $\eta = 0$. By Corollary 5.93, $1/(x - i)$ for $i = 1, \dots, n$ are linearly independent over $\mathbb{C}(t)$ modulo δF . Let $\eta_1 = 1, \dots, \eta_n = t^{n-1}$ be a fundamental system for $\epsilon^n y$, and let

$$c = \ln x + t \ln(x - 1) + \dots + t^{n-1} \ln(x - (n - 1)).$$

Then, $F\langle c \rangle_{E,\Delta} = F(c, \ln x, \dots, \ln(x - (n - 1)))$, and $F\langle c \rangle_{E,\Delta}$ is E -strongly normal over $\mathbb{C}(t, x)$. The operation of an element $g = \alpha_0 + t\alpha_1 + \dots + t^{n-1}\alpha_{n-1}$ of the

Galois group $\mathfrak{Z}_{[\epsilon^n y]_{\mathbb{E}}} = \{v \in \mathcal{V} \mid \epsilon^n v = 0\} = \{\alpha_0 + t\alpha_1 + \dots + t^{n-1}\alpha_{n-1} \mid \alpha_i \in \mathbb{C}\}$ is defined by $gc = (\alpha_0 + \ln x) + t(\alpha_1 + \ln(x - 1)) + \dots + t^{n-1}(\alpha_{n-1} + \ln(x - (n - 1)))$. If $f = x$, η may be taken to be $t^n x / (n)!$. Then

$$c = t^n x / (n)! + \ln x + t \ln(x - 1) + \dots + t^{n-1} \ln(x - (n - 1)),$$

and the Galois group is the same.

4.2. $G_m^{\mathbb{E}}$ -extensions. Denote the multiplicative (\mathbb{E}, Δ) - \mathbb{Q} -group [13, page 28] (relative to \mathcal{U}) by the symbol $G_m^{\mathbb{E}, \Delta}$. The elements of $G_m^{\mathbb{E}, \Delta}$ are those of \mathcal{U}^* , and its group structure is that of the field \mathcal{U} under multiplication. Similarly, $G_m^{\mathbb{E}}$ will denote the multiplicative \mathbb{E} - \mathbb{Q} -group (relative to \mathcal{K}) with elements those of \mathcal{K}^* . Let $\kappa \in \mathfrak{F}(\mathcal{U}^*)$ be the canonical coordinate function on $G_m^{\mathbb{E}, \Delta}$. Then, $\delta_i \kappa / \kappa \in \mathfrak{F}(\mathcal{U}^*)$, and the \mathbb{E} - \mathcal{F} -mapping $l\Delta = (\delta_1 \kappa / \kappa, \dots, \delta_m \kappa / \kappa) : G_m^{\mathbb{E}, \Delta} \rightarrow (G_m^{\mathbb{E}, \Delta})^m$ [13, Proposition 6, page 129] is the logarithmic derivative on $G_m^{\mathbb{E}, \Delta}$ relative to Δ [12, Example 2, page 352]. By [13, Proposition 3, page 89], it is an (\mathbb{E}, Δ) - \mathcal{F} -homomorphism. The kernel of $l\Delta$ is the (\mathbb{E}, Δ) - \mathcal{F} -subgroup consisting of (\mathbb{E}, Δ) -zeros of the (\mathbb{E}, Δ) -ideal $[\delta_1 y, \dots, \delta_m y] \subset \mathcal{F}\{y\}_{\mathbb{E}, \Delta}$ and can be identified with $G_m^{\mathbb{E}}$ relative to the \mathbb{E} -universal field \mathcal{K} .

Definition 4.82. An element $\alpha \in \mathcal{U}^*$ is Δ -exponential over \mathcal{F} if $(\alpha^{-1} \delta_1 \alpha, \dots, \alpha^{-1} \delta_m \alpha) \in \mathcal{F}^m$; that is, if for suitable elements $a_1, \dots, a_m \in \mathcal{F}$, α satisfies the system of differential equations

$$\delta_i \alpha = a_i \alpha \quad (1 \leq i \leq m).$$

Let α be Δ -exponential over \mathcal{F} , and suppose that the field of Δ -constants of $\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}$ is \mathbb{C} ($= \mathcal{F}^\Delta$). For any isomorphism σ of $\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}$ over \mathcal{F} and $\delta \in \Delta$,

$$\begin{aligned} (\alpha^{-1} \sigma \alpha)^{-1} \delta (\alpha^{-1} \sigma \alpha) &= (\alpha^{-1} \sigma \alpha)^{-1} [\delta (\alpha^{-1} \sigma \alpha) + \alpha^{-1} \delta (\sigma \alpha)] \\ &= (\sigma \alpha)^{-1} \alpha [-\alpha^{-1} \delta \alpha \alpha^{-1} \sigma \alpha + \alpha^{-1} \delta (\sigma \alpha)] = -\alpha^{-1} \delta \alpha + (\sigma \alpha)^{-1} \delta (\sigma \alpha) \\ &= -\alpha^{-1} \delta \alpha + \sigma (\alpha^{-1} \delta \alpha) = -\alpha^{-1} \delta \alpha + \alpha^{-1} \delta \alpha = 0. \end{aligned}$$

Therefore,

$$l\Delta(\alpha^{-1} \sigma \alpha) = ((\alpha^{-1} \sigma \alpha)^{-1} \delta_1 (\alpha^{-1} \sigma \alpha), \dots, (\alpha^{-1} \sigma \alpha)^{-1} \delta_m (\alpha^{-1} \sigma \alpha)) = 0.$$

Hence the element $c(\sigma) = \alpha^{-1} \sigma \alpha$ is in the kernel of $l\Delta$ and is a Δ -constant. Just as in the case of an element Δ -primitive over \mathcal{F} , $\mathcal{F}\langle \alpha \rangle$ is \mathbb{E} -strongly normal over \mathcal{F} because

$$\begin{aligned} \mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta} \sigma(\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}) &= \mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta} \mathcal{F}\langle \sigma \alpha \rangle_{\mathbb{E}, \Delta} \\ &= \mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta} \mathcal{F}\langle \alpha \cdot c(\sigma) \rangle_{\mathbb{E}, \Delta} = \mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta} \mathbb{C}\langle c(\sigma) \rangle_{\mathbb{E}, \Delta}. \end{aligned}$$

The mapping $c : G(\mathcal{F}\langle \alpha \rangle / \mathcal{F}) \rightarrow G_m^{\mathbb{E}}$ is clearly a group homomorphism. It is injective because $1 = c(\sigma) = \alpha^{-1} \sigma \alpha$ implies $\alpha = \sigma \alpha$ and $\sigma = \text{id}_{\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}}$. This proves the first part of the following proposition, and the remainder is a special case of Proposition 3.64.

Proposition 4.83. *Let α be a Δ -exponential over \mathcal{F} , and suppose that $\mathbb{C} = (\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta})^\Delta$. Then, each (\mathbb{E}, Δ) - \mathcal{F} -isomorphism σ of $\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}$ into \mathcal{U} is of the form $\sigma \alpha = \alpha \cdot c(\sigma)$ for $c(\sigma) \in \mathcal{K}^*$. In addition, $\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}$ is \mathbb{E} -strongly normal over \mathcal{F} , and the mapping $c : G(\mathcal{F}\langle \alpha \rangle / \mathcal{F}) \rightarrow G_m^{\mathbb{E}}$ defined by $c(\sigma) = \alpha^{-1} \sigma \alpha$ for $\sigma \in G(\mathcal{F}\langle \alpha \rangle / \mathcal{F})$ is an injective \mathbb{E} - \mathbb{C} -homomorphism of \mathbb{E} -groups relative to the \mathbb{E} -field \mathcal{K} . Consequently, $\mathcal{F}\langle \alpha \rangle_{\mathbb{E}, \Delta}$ is a $G_m^{\mathbb{E}}$ -extension of \mathcal{F} .*

Proposition 4.84. *Let G be a connected E- \mathcal{C} -subgroup of G_m^E . Let \mathfrak{P} be the prime E-ideal in $\mathcal{C}\{y\}_E$ defining G . Let b be a generic zero in \mathcal{U} of $\mathfrak{P}_{\Delta,E} \subset \mathcal{C}\{y\}_{E,\Delta}$. Let $a = l\Delta b/b$. Put $\mathcal{F} = \mathcal{C}\langle a \rangle_{E,\Delta}$, and $\mathcal{G} = \mathcal{F}\langle b \rangle_{E,\Delta}$. Then \mathcal{G} over \mathcal{F} is an E-strongly normal extension with Galois group G .*

Proof. This is a special case of Theorem 3.66.

The E-subgroups of G_m^E are the algebraic subgroups $\mu_r = \{v \in G_m^E \mid v^r = 1\}$ for every positive integer r and $G_{\mathcal{L}} = \{v \in \mathcal{V}^* \mid L(lE(v)) = 0 \text{ for } L(y) \in \mathcal{L}\}$ where $\mathcal{L} \subset \mathcal{F}\{y\}_E$ is a linear E-ideal [1, Chapter 4]. For $\mu_s \subseteq \mu_r$, it is necessary and sufficient for s to be a divisor of r , and, for $G_{\mathcal{L}} \subseteq G_{\mathcal{L}'}$, it is necessary and sufficient for $\mathcal{L} \supseteq \mathcal{L}'$. Additionally, each subgroup of the form $G_{\mathcal{L}}$ is connected and contains μ_r for each r [1, Chapter 4]. \square

The following proposition exhibits the Galois correspondence even if $\mathcal{C} = \mathcal{F}^\Delta$ is not constrainedly closed.

Proposition 4.85. *Assume that \mathcal{G} is an E-strongly normal extension of \mathcal{F} that is (E, Δ) -generated over \mathcal{F} by a Δ -exponential b over \mathcal{F} . Let $G = \text{Gal}(\mathcal{G}/\mathcal{F}) \subseteq \mathcal{V}^*$ be the Galois group.*

- (1) *If $G = \mu_r$, then each E- \mathcal{C} -subgroup H is E- \mathcal{C} -isomorphic to μ_s for some divisor s of r , and $\mathcal{G}^H = \mathcal{F}\langle b^s \rangle_{E,\Delta}$.*
- (2) *If $G = G_{\mathcal{L}}$, then each E- \mathcal{C} -subgroup H is E- \mathcal{C} -isomorphic to either μ_s for some positive integer s or $G_{\mathcal{L}'}$ such that $\mathcal{L} \subseteq \mathcal{L}'$. If $H = \mu_s$, $\mathcal{G}^H = \mathcal{F}\langle b^s \rangle_{E,\Delta}$. If $H = G_{\mathcal{L}'}$, $\mathcal{G}^H = \mathcal{F}\langle (L(\epsilon b/b))_{L(y) \in \mathcal{L}'} \rangle_{E,\Delta}$.*

Proof. Let $\sigma \in G_m^E$. Proposition 4.83 implies $\sigma_\zeta(b) = \zeta b$ for some $\zeta \in \mathcal{K}^*$. If σ leaves $\mathcal{F}\langle b^s \rangle_{E,\Delta}$ invariant for some positive integer s , then $\sigma(b^s) = (\sigma(b))^s = (\zeta b)^s = \zeta^s b^s$ implies $\zeta^s = 1$ and $\sigma^s = \text{id}$. Therefore, $\sigma \in \mu_s$. If σ leaves $\mathcal{F}\langle (L(\epsilon b/b))_{L(y) \in \mathcal{L}'} \rangle_{E,\Delta}$ invariant, since $\sigma(b) = \zeta b$, for each $L \in \mathcal{L}'$,

$$\begin{aligned} L(\epsilon\zeta/\zeta) &= L(\epsilon(\zeta b)/(\zeta b) - \epsilon b/b) = L(\epsilon(\zeta b)/(\zeta b)) - L(\epsilon b/b) \\ &= L(\epsilon(\sigma(b))/\sigma(b)) - L(\epsilon b/b) = \sigma(L(\epsilon b/b)) - L(\epsilon b/b) = 0, \end{aligned}$$

and $\sigma \in G_{\mathcal{L}'}$.

If $G = \mu_r$, each E-subgroup $H = \mu_s$, for s a divisor of r , clearly leaves invariant $\mathcal{F}\langle b^s \rangle_{E,\Delta}$. Conversely, by the above result, if an element of G leaves $\mathcal{F}\langle b^s \rangle_{E,\Delta}$ invariant, it is in H .

If $G = G_{\mathcal{L}}$ and $H = \mu_s$, then the last paragraph shows that $\mathcal{G}^H = \mathcal{F}\langle b^s \rangle_{E,\Delta}$. If $G = G_{\mathcal{L}}$ and $H = G_{\mathcal{L}'}$, then H leaves invariant $\mathcal{F}\langle (L(\epsilon b/b))_{L(y) \in \mathcal{L}'} \rangle_{E,\Delta}$ because, for $\sigma \in H$,

$$\begin{aligned} \sigma(L'(\epsilon b/b)) &= L'(\epsilon(\sigma(b))/(\sigma b)) = L'(\epsilon(\zeta b)/(\zeta b)) \\ &= L'(\epsilon b/b + \epsilon\zeta/\zeta) = L'(\epsilon b/b) + L'(\epsilon\zeta/\zeta) = L'(\epsilon b/b), \end{aligned}$$

the (E, Δ) -field $\mathcal{F}\langle (L'(\epsilon b/b))_{L'(y) \in \mathcal{L}'} \rangle_{E,\Delta}$ is invariant under $G_{\mathcal{L}'}$. From the result in the first paragraph of this proof, $\mathcal{G}^H = \mathcal{F}\langle (L(\epsilon b/b))_{L(y) \in \mathcal{L}'} \rangle_{E,\Delta}$. \square

The following proposition characterizes certain E-exponential G_m^E -extensions by the structure of \mathcal{F} .

Proposition 4.86. *Let $\Delta = \{\delta\}$, and let \mathcal{G} be an E-strongly normal extension of \mathcal{F} , that is, (E, Δ) - \mathcal{F} -generated by a transcendental Δ -exponential c over \mathcal{F} . Let*

$a = \delta c/c$, let $\mathcal{L}_{a,1} = \{L(y) \in \mathcal{C}\{y\}_{E,1} \mid L(\epsilon a) \in \delta\mathcal{F}\}$, and let $\mathcal{L}_a = [\mathcal{L}_{a,1}]_E$. Then $\text{Gal}(\mathcal{G}/\mathcal{F}) = G_{\mathcal{L}_a}$.

Proof. By Proposition 4.84, $\text{Gal}(\mathcal{G}/\mathcal{F}) \subset G_m^E$, and, since c is transcendental over \mathcal{F} , $\text{Gal}(\mathcal{G}/\mathcal{F}) = G_{\mathcal{L}}$ for some E-ideal $\mathcal{L} \subset \mathcal{C}\{y\}_E$. Let $\sigma \in G_m^E$. Proposition 4.83 implies $\sigma(b) = \zeta b$ for some $\zeta \in G_{\mathcal{L}}$ so that $L(\epsilon\zeta/\zeta) = 0$ for every $L(y) \in \mathcal{L}$.

Let $b = \epsilon c/c$. Clearly, $\delta b = \epsilon a$. Let $L(y) \in \mathcal{L}$ of degree one. Then $L(\epsilon c/c)$ is invariant under all elements of G because

$$\begin{aligned} \sigma(L(\epsilon c/c)) &= L((\epsilon\sigma(c))/\sigma(c)) = L(\epsilon(\zeta c)/(\zeta c)) \\ &= L(\epsilon c/c + \epsilon\zeta/\zeta) = L(\epsilon c/c) + L(\epsilon\zeta/\zeta) = L(\epsilon c/c). \end{aligned}$$

Thus $L(\epsilon c/c) \in \mathcal{F}$, and $L(\epsilon c/c) = f$ for some $f \in \mathcal{G}$. The computation

$$L(\epsilon a) = L(\delta b) = \delta(L(b)) = \delta(L(\epsilon c/c)) = \delta f$$

shows $L(y) \in \mathcal{L}_a$, and $\mathcal{L} \subseteq \mathcal{L}_a$ since \mathcal{L} is generated by elements of degree 1.

On the other hand, let $L(y) \in \mathcal{L}_{a,1}$. Then $L(\epsilon a) = \delta f$ for $f \in \mathcal{F}$, and $L(b) - f$ is a Δ -constant because $\delta(L(b) - f) = L(\delta b) - \delta f = L(\epsilon a) - \delta f = 0$. Therefore, $L(b) - f \in \mathcal{C} \subseteq \mathcal{F}$, and $L(b) \in \mathcal{F}$. Hence, for all $\sigma \in G_{\mathcal{L}}$, $\sigma(L(b)) = L(b)$, and the computation

$$\begin{aligned} L(\epsilon v/v) &= L(\sigma(\epsilon c/c) - \epsilon c/c) = L(\sigma(\epsilon c/c)) - L(\epsilon c/c) \\ &= \sigma(L(\epsilon c/c)) - L(\epsilon c/c) = \sigma(L(b)) - L(b) = 0 \end{aligned}$$

shows $L(y) \in \mathcal{L}$ and $\mathcal{L} \supseteq \mathcal{L}_a$ since \mathcal{L}_a is generated by elements of degree 1. □

Corollary 4.87. *Let \mathcal{H} be an algebraically closed (E, Δ) -field such that $\mathcal{H}^\Delta = \mathcal{H}$, and let $\mathcal{F} = \mathcal{H}\langle x \rangle_{E, \Delta}$, where $x \in \mathcal{U}$, $\epsilon x = 0$ and $\delta x = 1$. Then, there is no E-strongly normal extension of \mathcal{F} that is (E, Δ) -generated by a Δ -exponential over \mathcal{F} and has Galois group E- \mathcal{H} -isomorphic to G_m^E .*

Remark 4.88. This remains true if the hypothesis that \mathcal{H} be algebraically closed is omitted; the following proof must be modified to take the structure of irreducibles into account in the partial fraction decomposition.

Proof. Assume that such an E-strongly normal extension \mathcal{G} of \mathcal{F} exists. Let $b \in \mathcal{U}$ be a Δ -exponential over \mathcal{F} such that $\delta b = ab$ for $a \in \mathcal{F}$, and $\mathcal{G} = \mathcal{F}\langle b \rangle_{E, \Delta}$. Let $\epsilon a = p(x) + \sum_{i,j} \frac{h_{i,j}}{(x-h_i)^j}$ for $p(x) \in \mathcal{H}[x]$ and $h_i, h_{i,j} \in \mathcal{H}$, be the partial fraction decomposition of ϵa .

By Proposition 4.86, since the Galois group is G_m^E , there does not exist a nonzero $L(y) \in \mathcal{H}\{y\}_{E,1}$ such that $L(\epsilon a) \in \delta\mathcal{F}$. If all of the $h_{i,1} = 0$, then $\epsilon a \in \delta\mathcal{F}$, and $L(\epsilon a) \in \delta\mathcal{F}$ for $L(y) = y$. If there exists a nonzero $h_{i,1}$, then there exists a nonzero $L(y) =$

$$\begin{vmatrix} h_{1,1} & h_{2,1} & \dots & h_{r,1} & y \\ \epsilon h_{1,1} & \epsilon h_{2,1} & \dots & \epsilon h_{r,1} & \epsilon y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \epsilon^r h_{1,1} & \epsilon^r h_{2,1} & \dots & \epsilon^r h_{r,1} & \epsilon^r y \end{vmatrix}$$

$\in \mathcal{H}\{y\}_{E,1}$ such that the finitely many $h_{i,1}$ span over $\mathcal{H}^{E, \Delta}$ the linear space of E-zeros of $L(y)$. By Lemma 4.79, since $L(h_{i,1}) = 0$ for all i , $L(\epsilon a) \in \delta\mathcal{F}$. □

The following proposition shows how to construct an E-strongly normal extension for a given connected E-subgroup of G_m^E .

Proposition 4.89. *Assume $E = \{\epsilon\}$ and $\Delta = \{\delta\}$. Let the (E, Δ) -field \mathcal{U} be (E, Δ) -universal over the (E, Δ) -field \mathcal{D} of Δ -constants.*

- (1) *Let $G_{\mathcal{L}} = \{v \in G_m^E \mid M(\epsilon v/v) = 0, M(y) \in \mathcal{L}\}$ be a connected E -subgroup of G_m^E defined over an (E, Δ) -subfield $\mathcal{D} \subset \mathcal{U}^\Delta$ where $L(y) \in \mathcal{D}\{y\}_{E,1}$ of positive order n with the coefficient of the highest-order term equal to 1 and $\mathcal{L} = [L(y)]_E$.*
- (2) *Let the (E, Δ) -field $\mathcal{C} \subset \mathcal{U}^\Delta$ be a strongly normal extension of \mathcal{D} , considered as an E -field, that is, E -generated over \mathcal{D} by a fundamental system $1, \eta_1, \dots, \eta_n$ of E -zeros of $L(\epsilon y)$.*
- (3) *Let the (E, Δ) -field $\mathcal{B} \subset \mathcal{U}^E$ be finitely Δ -generated over \mathcal{C}^E , satisfy the condition $\mathcal{B}^\Delta = \mathcal{C}^E$, and contain the elements f_1, \dots, f_n that are assumed to be linearly independent over \mathcal{B}^Δ modulo $\delta\mathcal{B}$.*
- (4) *Let $\mathcal{F} = \mathcal{C} \cdot \mathcal{B}$, and let $f \in \mathcal{F}$. Let $\eta \in \mathcal{F}$ be an (E, Δ) -zero η of $L(\epsilon y) - f \in \mathcal{F}\{y\}_{E,\Delta}$.*
- (5) *For each $i = 1, \dots, n$, let $g_i \in \mathcal{U}^E$ be a Δ -primitive of f_i , i.e. $\delta g_i = f_i$.*
- (6) *Let $\mathcal{H} = \mathcal{F}\langle g_1, \dots, g_n \rangle_{E,\Delta}$, and let c be an \mathcal{H} -generic (E, Δ) -zero of*

$$N = [\delta y - (\delta\eta + \sum_i \eta_i f_i)y, \epsilon y - (\epsilon\eta + \sum_i \epsilon\eta_i g_i)y]_{E,\Delta} \subset \mathcal{H}\{y\}_{E,\Delta},$$

$$\begin{array}{ccccc} \eta_i \in \mathcal{C} & \longrightarrow & \mathcal{F} = \mathcal{C} \cdot \mathcal{B} & \longrightarrow & \mathcal{H} = \mathcal{F}\langle g_1, \dots, g_n \rangle_{E,\Delta} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D} & \longrightarrow & \mathcal{D} \cdot \mathcal{B} & \longrightarrow & \mathcal{D}\mathcal{B}\langle g_1, \dots, g_n \rangle_{E,\Delta} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{C}^E = \mathcal{B}^\Delta & \longrightarrow & f_i \in \mathcal{B} & \longrightarrow & \mathcal{B}\langle g_1, \dots, g_n \rangle_{E,\Delta} \end{array}$$

Then $\mathcal{F}\langle c \rangle_{E,\Delta}$ is E -strongly normal over \mathcal{F} with Galois group $G_{\mathcal{L}}$.

Remark 4.90. If the elements of the (E, Δ) -fields in the proposition are of analytic functions of two variables, c may be taken to be $\exp(\eta + \sum_i \eta_i g_i)$.

Proof. Since $\mathcal{B}\langle g_1, \dots, g_n \rangle_{E,\Delta}$ and \mathcal{C} are linearly disjoint over $\mathcal{C}^E = \mathcal{B}^\Delta$ [12, Corollary 1, page 87], $\mathcal{B}\langle g_1, \dots, g_n \rangle_{E,\Delta}$ and \mathcal{F} are linearly disjoint over \mathcal{B} [15, Proposition 1, page 50]. Since f_1, \dots, f_n are assumed to be linearly independent over \mathcal{B}^Δ modulo $\delta\mathcal{B}$, Proposition 5.96 implies $1, g_1, \dots, g_n$ are linearly independent over \mathcal{B} , which, by the linear disjointness, are also linearly independent over \mathcal{F} . Proposition 5.96 implies f_1, \dots, f_n are linearly independent over \mathcal{F}^Δ modulo $\delta\mathcal{F}$, g_1, \dots, g_n are algebraically independent over \mathcal{F} and $\mathcal{H}^\Delta = \mathcal{F}^\Delta$.

Let $a = \delta\eta + \sum_i \eta_i f_i \in \mathcal{F}$ and $b = \epsilon\eta + \sum_i \epsilon\eta_i g_i \in \mathcal{H}$. Clearly, $\epsilon a = \delta b$. For any orderly ranking, the set $\{\delta y - ay, \epsilon y - by\}$ is coherent and autoreduced because $\epsilon(\delta y - ay) - \delta(\epsilon y - by) = 0$. By [12, Lemma 5, page 137], N is prime. No nonzero polynomial $p(y) \in \mathcal{H}[y] \subset \mathcal{H}\{y\}_{E,\Delta}$ is contained in N because if $p(y) \in N$, then because $p(y)$ is partially reduced with respect to $\{\delta y - ay, \epsilon y - by\}$ [12, Lemma 5, page 137] implies $p(y) \in (\delta y - ay, \epsilon y - by)$. This is impossible since $p(y)$ is reduced and nonzero. By taking $p(y) = 1$, the argument above shows N is proper. Therefore, there exists a nonzero (E, Δ) -zero $c \in \mathcal{U}$ that is not algebraic over \mathcal{H} . This and the fact that $\mathcal{H}[c] = \mathcal{H}\{c\}_{E,\Delta}$ imply that c is transcendental over \mathcal{H} .

The Wronskian matrix $(\epsilon^j \eta_i)_{i=1, \dots, n; j=1, \dots, n}$ is invertible because $\epsilon\eta_1, \dots, \epsilon\eta_n$ is a fundamental system of zeros for $L(y)$. Therefore, the following system of linear

equations obtained by repeatedly differentiating $b = \epsilon\eta + \sum_i \epsilon\eta_i g_i$ by ϵ may be solved for g_1, \dots, g_n :

$$\begin{aligned} b &= \epsilon\eta + \sum_i \epsilon\eta_i g_i, \\ \epsilon b &= \epsilon^2\eta + \sum_i \epsilon^2\eta_i g_i, \\ &\dots \\ \epsilon^{n-1}b &= \epsilon^n\eta + \sum_i \epsilon^n\eta_i g_i. \end{aligned}$$

Because $\eta \in \mathcal{F}$, $\mathcal{F}(b, \epsilon b, \dots, \epsilon^{n-1}b) = \mathcal{F}(g_1, \dots, g_n)$. From this and the fact that $b = \epsilon c/c \in \mathcal{F}\langle c \rangle_{E,\Delta}$, all $g_i \in \mathcal{F}\langle c \rangle_{E,\Delta}$. Since g_1, \dots, g_n are algebraically independent over \mathcal{F} , so are $b, \epsilon b, \dots, \epsilon^{n-1}b$. Because $\delta(\epsilon^i b) = \epsilon^i(\delta b) = \epsilon^i(\epsilon a) = \epsilon^{i+1}a$, Proposition 5.96 implies $\epsilon a, \dots, \epsilon^n a$ are linearly independent over \mathcal{F}^Δ modulo $\delta\mathcal{F}$.

To show $(\mathcal{F}\langle c \rangle_{E,\Delta})^\Delta = \mathcal{F}^\Delta$, since $\mathcal{F}^\Delta = \mathcal{H}^\Delta$, $(\mathcal{H}\langle c \rangle_{E,\Delta})^\Delta = \mathcal{H}^\Delta$ must be proved. Let $\alpha \in \mathcal{H}\langle c \rangle_{E,\Delta}$ be a nonzero Δ -constant. First assume $\alpha \in \mathcal{H}\{c\}_{\Delta,E} = \mathcal{H}[c]$. Since c is transcendental over \mathcal{H} , one may uniquely write $\alpha = a_r c^r + a_{r-1} c^{r-1} + \dots + a_0$ where $a_r \neq 0$ and $a_i \in \mathcal{H}$ for $i = 0, \dots, r$. Then $\delta\alpha = A_r c^r + A_{r-1} c^{r-1} + \dots + A_0$ where $A_i = \delta a_i + i a a_i$ for $i = 0$ to r . Since $\delta\alpha = 0$ and the powers of c are linearly independent over \mathcal{H} , it follows that $A_i = 0$ for $i = 0, \dots, r$. By Corollary 5.99, $a_i \in \mathcal{F}$ for $i = 1$ to r . Therefore $\epsilon a_r/a_r \in \mathcal{F}$, and $\delta(\epsilon a_r/a_r) = \epsilon(\delta a_r/a_r) = \epsilon(-ra) = -r\epsilon a$, which unless $r = 0$ contradicts the linear independence of the family $\epsilon a, \dots, \epsilon^n a$ over \mathcal{F}^Δ modulo $\delta\mathcal{F}$. Hence, $\alpha = a_0 \in \mathcal{F}^\Delta$. Similarly, if $1/\alpha \in \mathcal{H}[c]$, then $1/\alpha \in \mathcal{F}^\Delta$.

Second, if neither α nor $1/\alpha$ is in $\mathcal{H}[c]$, let $\alpha = A/B$ where A and B are in $\mathcal{H}[c]$ of positive degree such that A has the minimal degree among all such choices of A and B . It may be assumed that $\delta B \neq 0$ because otherwise $\delta A = 0$ and $A \in \mathcal{H}^\Delta$. Since $\delta\alpha = 0$, $A/B = \delta A/\delta B$. Write $A = a_r c^r + \dots + a_0$, $a_i \in \mathcal{H}$ for $i = 0$ to r , and $B = b_s c^s + \dots + b_0$, $b_i \in \mathcal{H}$ for $i = 0$ to s . Both a_0 and b_0 may not be 0 because then the numerator and the denominator of α may be divided by c resulting in a fraction representing α with a lower degree numerator. If $b_0 = 0$ and $a_0 \neq 0$, divide the numerator and the denominator by a_0 . Then the derivatives of both have no constant terms and may be divided by c again to produce an equivalent fraction with lower degree numerator. If $b_0 \neq 0$ and $a_0 = 0$, apply the same reasoning. If $b_0 \neq 0$ and $a_0 \neq 0$, from $\delta gf = g\delta f$, by comparing the zeroth degree terms in c , it follows that $\delta b_0 a_0 = b_0 \delta a_0$. Therefore $\delta(a_0/b_0) = 0$. Divide the numerator and the denominator both by b_0 . The zeroth degree terms in c of both the numerator and the denominator are Δ -constants. Differentiate them and divide both by c to produce an equivalent fraction with lower degree numerator. So, $\alpha \in \mathcal{F}^\Delta$.

Since c is a Δ -exponential over \mathcal{F} and $(\mathcal{F}\langle c \rangle_{E,\Delta})^\Delta = \mathcal{F}^\Delta$, Proposition 4.83 implies $\mathcal{F}\langle c \rangle_{E,\Delta}$ over \mathcal{F} is E-strongly normal. It remains to show that the Galois group G of $\mathcal{F}\langle c \rangle_{E,\Delta}$ over \mathcal{F} is $G_{\mathcal{L}}$. Because c is transcendental over \mathcal{F} , G is not finite, and $G = G_{\mathcal{M}}$ for some linear E-ideal $\mathcal{M} \subset \mathcal{F}^\Delta\{y\}_E$. Since it may be verified that $L(\epsilon a) = \delta f$, Proposition 4.86 implies $L(y) \in \mathcal{M}$. For a linear $M(y) \in \mathcal{M}$, the same proposition implies $M(\epsilon a) = \delta h$ for some $h \in \mathcal{F}$. Then $M(b) - h \in \mathcal{F}^\Delta \subseteq \mathcal{F}$, and $M(b) = \bar{h}$ for $\bar{h} \in \mathcal{F}$. Since $1, b, \epsilon b, \dots, \epsilon^{n-1}b$ are linearly independent over \mathcal{F} , $M(y)$ has order greater than or equal to the order of $L(y)$. Hence $\{L(y)\}_E = \mathcal{M} = \mathcal{L}$, and $G = G_{\mathcal{L}}$. □

A particularly simple example may be obtained by taking, in Proposition 4.89, $\mathcal{D} = \mathbb{C}$, $L = \epsilon^n y$, $\mathcal{C} = \mathcal{D}(t)$ with $\epsilon t = 1$ and $\delta t = 0$, $\mathcal{B} = \mathbb{C}(x)$, $\mathcal{F} = \mathbb{C}(t, x)$,

$\eta_i = t^i$ for $i = 1, \dots, n$, $f_i = 1/(x + i - 1)$ for $i = 1, \dots, n$, $g_i = \ln(x + i - 1)$ for $i = 1, \dots, n$ and $\eta = 0$. A fundamental system of E-zeros of $\epsilon^{n+1}y$ is $1, t, t^2, \dots, t^n$. By Corollary 5.93, $1/(x), 1/(x + 1), \dots, 1/(x + n - 1)$ are linearly independent over $\mathbb{C} = \mathcal{B}^\Delta$ modulo $\delta\mathcal{B}$. Then, $a = t/(x) + t^2/(x + 1) + \dots + t^n/(x + n - 1)$, and $b = \ln(x) + 2t \ln(x + 1) + \dots + nt^{n-1} \ln(x + n - 1)$. One may take

$$c = \exp(t \ln x + t^2 \ln(x + 1) + \dots + t^n \ln(x + n - 1)).$$

Then, $\mathcal{F}\langle c \rangle_{E, \Delta} = \mathcal{F}(c, \ln x, \dots, \ln(x + n - 1))$, and $\mathcal{F}\langle c \rangle_{E, \Delta}$ is E-strongly normal over \mathcal{F} . The operation of the Galois group $G_{[\epsilon^n y]_E} = \{v \in \mathcal{V}^* \mid \epsilon^n(\epsilon v/v) = 0\} = \{\exp(\alpha_0 + t\alpha_1 + \dots + t^n \alpha_n) \mid \alpha_i \in \mathbb{C}\}$ on c is induced by addition in the exponents. If $f = x$, η may be taken to be $t^{n+1}x/(n + 1)!$. Then,

$$c = \exp(t^{n+1}x/(n + 1)! + t \ln x + t^2 \ln(x + 1) + \dots + t^n \ln(x + n - 1)),$$

and the Galois group is the same.

5. APPENDIX

Throughout this section, let $\Delta = \{\delta\}$, and write δw as w' for some Δ -ring element w . The following proposition and its corollaries determine the Δ -zeros of $\delta y - \alpha$ from the factorization of α .

Proposition 5.91. *Let R be a Δ -ring that is a factorial domain of characteristic zero. Extend δ to a derivation of the quotient field Q of R . For any $\alpha \in Q$, write the reduced fraction $\alpha = \prod p_i^{n_i} / \prod q_j^{m_j}$ where the p_i and q_j are nonassociate irreducible elements of R and the n_i and m_j are positive integers. If $q'_j \notin (q_j)$, then q_j is in the denominator of the reduced fraction of α' with an exponent of $m_j + 1$.*

Proof. Examine the numerator of α' :

$$\begin{aligned} & (\prod p_i^{n_i})' \prod q_j^{m_j} - \prod p_i^{n_i} (\prod q_j^{m_j})' \\ &= (\prod p_i^{n_i})' \prod q_j^{m_j} - \sum_j (\prod p_i^{n_i}) m_j q'_j q_j^{m_j-1} \prod_{k \neq j} q_k^{m_k}. \end{aligned}$$

The only term not divisible by $q_j^{m_j}$ is $(\prod p_i^{n_i}) m_j q'_j q_j^{m_j-1} \prod_{k \neq j} q_k^{m_k}$. So the power of q_i in the factorization of the numerator is $m_i - 1$. Since $q_i^{2m_i}$ is present in the denominator of the derivative formula for α' , in the reduced fraction of α' the irreducible element q_j is present in the denominator with an exponent of $m_j + 1$. \square

Corollary 5.92. *Let k be a field of characteristic zero, and let $k(x)$ be the rational function field in one indeterminate x such that $x' = 1$ and $a' = 0$ for every $a \in k$. For any $\alpha \in k(x)$, write the reduced fraction $\alpha = \prod p_i^{n_i} / \prod q_j^{m_j}$ where the p_i and q_j are different irreducible elements of $k[x]$ and the n_i and m_j are positive integers. If one $m_j = 1$, α is not a derivative of any element of $k(x)$.*

Corollary 5.93. *Let \mathcal{U} be a Δ -universal extension of the constant field \mathcal{C} . Let $x \in \mathcal{U}$ be a Δ -zero of $y' - 1 \in \mathcal{C}\{y\}_\Delta$. For $i = 1, \dots, n$, let $p_i \in \mathcal{C}[x]$ be nonassociate and irreducible. Then the reciprocals of the p_i are linearly independent over \mathcal{C} modulo $(\mathcal{C}(x))'$.*

Proof. Express any linear combination $\sum_i c_i/p_i$ ($c_i \in \mathbb{C}$ and all $c_i \neq 0$) of the reciprocals of the p_i over \mathbb{C} as a rational fraction α in reduced form. Since the numerator is not divisible by any p_i , the denominator of α has each p_i as a factor with exponent exactly 1. Now apply Corollary 5.92.

In the proof of the next proposition, the following order on polynomials will be utilized. (See [4, Lemma 3, page 58] for a similar argument.) Let z_1, \dots, z_n be algebraic indeterminates over \mathcal{F} . Let $g \in \mathcal{F}[z_1, \dots, z_n]$, and let d be the degree of g in the indeterminates z_1, \dots, z_n , with the convention $\deg 0 = -1$. Write $g = \sum_M \alpha_M M$ where the M are monomials in z_1, \dots, z_n and $\alpha_M \in \mathcal{F}$. Let $c(g)$ denote the number of terms $\alpha_M M$ ($\alpha_M \neq 0$) of degree d in g . Define the level(g) to be $(\deg g, c(g))$ in the lexicographical order on $\mathbb{N} \times \mathbb{N}$.

Let $a_i \in \mathcal{F}$ for $i = 1, \dots, n$, and define a Δ -ring structure $\mathcal{F}[z_1, \dots, z_n]_\Delta$ on $\mathcal{F}[z_1, \dots, z_n]$ by $z'_i = a_i$ for $i = 1, \dots, n$. □

Lemma 5.94. *Assume that a_1, \dots, a_n are linearly independent over \mathbb{C} modulo $\delta\mathcal{F}$. For each $g \in \mathcal{F}[z_1, \dots, z_n]_\Delta$ of degree d , $\deg g' \geq d - 1$. If $g \neq 0$ and at least one of the nonzero coefficients of a term of degree d is in \mathbb{C} , then $\text{level}(g') < \text{level}(g)$.*

Proof. Write g in the form

$$g = \sum_{\deg M=d} \alpha_M M + \sum_{\deg N=d-1} \alpha_N N + P$$

where $\alpha_M, \alpha_N \in \mathcal{F}$, $P \in \mathcal{F}[z_1, \dots, z_n]$ and $\deg P < d - 1$. Then, since

$$\delta(\alpha_M M) = \delta\alpha_M M + \sum_i \sum_{z_i|M} n_i \alpha_M a_i \frac{M}{z_i}$$

for a monomial M of positive degree and integers n_i ,

$$g' = \sum_{\deg M=d} \alpha'_M M + \sum_{\deg N=d-1} (\alpha'_N + \sum_{\deg L=d, L=Nz_i} n_{L,N} \alpha_L a_i) N + Q$$

where $n_{L,N}$ are positive integers, $Q \in \mathcal{F}[z_1, \dots, z_n]$ and $\deg Q < d - 1$.

Assume that $\alpha'_M \neq 0$ for at least one monomial M of degree d in g . Then $\deg g' = \deg g > d - 1$. If also $\alpha'_{M'} = 0$ for at least one monomial M' of degree d in g , then $c(g') < c(g)$. Therefore, $\text{level}(g') < \text{level}(g)$.

Assume the negative of the assumption of the last paragraph: $\alpha'_M = 0$ for all monomials M of degree d in g . If $g \neq 0$, then $\deg g' < \deg g$. Therefore, $\text{level}(g') < \text{level}(g)$. To show $\deg g' \geq d - 1$, first assume $\deg g \leq 0$. Then $g \in \mathbb{C}$, and $\deg g' = -1 \geq d - 1$. On the other hand, if $\deg g > 0$, choose a monomial N of degree $d - 1$ such that, for some i , Nz_i is present in g , i.e., $a_{Nz_i} \neq 0$. In g' , the coefficient of N , $\alpha'_N + \sum_{L=Nz_i} \alpha_L a_i$, is not equal to 0 because a_1, \dots, a_n are assumed to be linearly independent over \mathbb{C} modulo $\delta\mathcal{F}$. This proves $\deg g' = d - 1$. □

Lemma 5.95. *Assume that a_1, \dots, a_n are linearly independent over \mathbb{C} modulo $\delta\mathcal{F}$. The Δ - \mathcal{F} -ring $\mathcal{F}[z_1, \dots, z_n]_\Delta$ is Δ -simple, i.e., has no proper nontrivial Δ -ideal.*

Proof. Let $\mathfrak{P} \subset \mathcal{F}[z_1, \dots, z_n]_\Delta$ be a proper Δ -ideal. Assume there exists a nonzero element of \mathfrak{P} . Let $g \in \mathfrak{P}$ have the lowest level of all nonzero elements of \mathfrak{P} . Since \mathfrak{P} is proper and, therefore, has no nonzero elements of degree 0, $d = \deg g > 0$. Multiply g by a nonzero element of \mathcal{F} to ensure that one of the terms of degree d has 1 for a coefficient. This new nonzero element, which again is denoted by g , is also in \mathfrak{P} and has level less than or equal to all of the nonzero elements of \mathfrak{P} . By

Lemma 5.94, $\text{level}(g') < \text{level}(g)$. Since $g' \in \mathfrak{P}$, $g' = 0$. However, by the first part of the same lemma, $-1 = \deg g' \geq d - 1 \geq 0$ since $d > 0$. This contradiction shows \mathfrak{P} is the zero Δ -ideal. \square

Proposition 5.96. *Let \mathcal{U} be a Δ -universal extension of the Δ -field \mathcal{F} , and let $\mathcal{C} = \mathcal{F}^\Delta$. For $i = 1, \dots, n$, let $a_i \in \mathcal{F}$, and let $b_i \in \mathcal{U}$ be such that $b_i' = a_i$. The following four conditions are equivalent:*

- (1) a_1, \dots, a_n are linearly independent over \mathcal{C} modulo $\delta\mathcal{F}$,
- (2) b_1, \dots, b_n are algebraically independent over \mathcal{F} , and $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ is Δ -simple,
- (3) $1, b_1, \dots, b_n$ are linearly independent over \mathcal{F} , and $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta = \mathcal{C}$,
- (4) $1, b_1, \dots, b_n$ are linearly independent over \mathcal{F} , and $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta = \mathcal{C}$.

Proof. $1 \implies 2$. Define a Δ -ring structure $\mathcal{F}\langle z_1, \dots, z_n \rangle_\Delta$ on $\mathcal{F}\langle z_1, \dots, z_n \rangle$ by $z_i' = a_i$ for $i = 1, \dots, n$. Clearly, $\mathcal{F}\langle z_1, \dots, z_n \rangle_\Delta = \mathcal{F}\langle z_1, \dots, z_n \rangle_\Delta$. To show $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta = \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ is Δ -simple, define a surjective Δ - \mathcal{F} -homomorphism $\rho : \mathcal{F}\langle z_1, \dots, z_n \rangle_\Delta \rightarrow \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ over \mathcal{F} by $\rho(z_i) = b_i$ for $i = 1, \dots, n$. Then ρ is a Δ - \mathcal{F} -isomorphism because the kernel of ρ , which is a Δ -ideal, must be the zero ideal by Lemma 5.95. Therefore, $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$, the codomain of ρ , also has no nontrivial Δ -ideal, and b_1, \dots, b_n are algebraically independent over \mathcal{F} because z_1, \dots, z_n are algebraically independent over \mathcal{F} and $\rho(z_i) = b_i$ for every i .

$2 \implies 3$. Let g be a nonzero element of $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta$. Because g is a Δ -constant, $(g) \subset \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ is a Δ -ideal and must be the unit Δ -ideal by (2). Because, by assumption, $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ is a polynomial ring in the algebraically independent indeterminates b_1, \dots, b_n , $g \in \mathcal{F}$ and $g \in \mathcal{C} = \mathcal{F}^\Delta$. That b_1, \dots, b_n are algebraically independent over \mathcal{F} clearly implies that $1, b_1, \dots, b_n$ are linearly independent over \mathcal{F} .

$3 \implies 1$. Assume a_1, \dots, a_n are linearly dependent over \mathcal{C} modulo $\delta\mathcal{F}$, i.e., $\sum_i \alpha_i a_i = \delta f$ for $\alpha_i \in \mathcal{C}$ and $f \in \mathcal{F}$. Since $1, b_1, \dots, b_n$ are assumed to be linearly independent over \mathcal{F} , the element $\sum_i \alpha_i b_i - f$ ($\in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$) is not in \mathcal{F} and is a Δ -constant of $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$. This contradicts (3). Therefore, the a_1, \dots, a_n are linearly independent over \mathcal{C} modulo $\delta\mathcal{F}$. This proves (1).

$3 \iff 4$. For the nonobvious implication, assume (3). Assume $g \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$. Then

$$\mathfrak{a} = \{a \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta \mid ag \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta\}$$

is a Δ -ideal because g is a Δ -constant. Since it is nonzero and $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ is Δ -simple, $1 \in \mathfrak{a}$, which implies $g \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$. \square

Corollary 5.97 (The Ostrowski Theorem). *If b_1, \dots, b_n are algebraically dependent over \mathcal{F} and $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta = \mathcal{C}$, then $1, b_1, \dots, b_n$ are linearly dependent over \mathcal{F} .*

Proof. (See [12, Exercise 4, page 407] or [11, page 1155].) The contrapositive of $4 \implies 2$ is that, if $\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ has a nontrivial Δ -ideal or b_1, \dots, b_n are algebraically dependent over \mathcal{F} , then $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta \neq \mathcal{C}$ or $1, b_1, \dots, b_n$ are linearly dependent over \mathcal{F} . Therefore, if b_1, \dots, b_n are algebraically dependent over \mathcal{F} and $(\mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta)^\Delta = \mathcal{C}$, then $1, b_1, \dots, b_n$ are linearly dependent over \mathcal{F} . \square

Corollary 5.98. *Let \mathcal{U} be a Δ -universal extension of the constant field \mathcal{C} . Let $x \in \mathcal{U}$ be a Δ -zero of $y' - 1 \in \mathcal{C}\langle y \rangle_\Delta$. For $i = 1, \dots, n$, let $c_i \in \mathcal{C}$ such that $c_i \neq c_j$ if*

$i \neq j$, and let $b_i \in \mathcal{U}$ be a Δ -zero of the Δ -polynomial $y' - \frac{1}{x+c_i} \in \mathcal{C}(x)\{y\}_\Delta$. Then b_1, \dots, b_n are algebraically independent over $\mathcal{C}(x)$, and $(\mathcal{C}(x)\langle b_1, \dots, b_n \rangle_\Delta)^\Delta = \mathcal{C}$.

Proof. By Corollary 5.93 with irreducible $p_i(x) = x + c_i$ for $i = 1, \dots, n, \frac{1}{x+c_1}, \dots, \frac{1}{x+c_n}$ are linearly independent over \mathcal{C} modulo $\delta(\mathcal{C}(x))$. Then apply Proposition 5.96. □

Corollary 5.99. *Assume $\Delta = \{\delta\}$. Let the conditions of the last corollary be satisfied, let $a \in \mathcal{F}$, and let $\xi \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ be a Δ -zero of $\delta y - ay \in \mathcal{F}\{y\}_\Delta$. Then $\xi \in \mathcal{F}$.*

Proof. Let $\xi = A/B$ where $A, B \in \mathcal{F}\{b_1, \dots, b_n\}_\Delta$ where A and B are relatively prime and both A and B are not elements of \mathcal{F} . Then $\delta AB - A\delta B - aAB = 0$. If $A \notin \mathcal{F}$, then A divides δA . This is impossible because then the proper ideal $(A) \subset \mathcal{F}\{b_1, \dots, b_n\}_\Delta$ would be a Δ -ideal, which is contrary to (2) of the proposition. If $B \notin \mathcal{F}$, the argument is similar. □

Corollary 5.100. *Assume $\Delta = \{\delta\}$. Let the conditions of the last corollary be satisfied, let $a \in \mathcal{F}$, and let $\xi \in \mathcal{F}\langle b_1, \dots, b_n \rangle_\Delta$ be a Δ -zero of $\delta y - ay \in \mathcal{F}\{y\}_\Delta$. Then $\xi \in \mathcal{F}$.*

Proof. Let $\xi = A/B$ where $A, B \in \mathcal{F}\{b_1, \dots, b_n\}_\Delta$ where A and B are relatively prime and both A and B are not elements of \mathcal{F} . Then $\delta AB - A\delta B - aAB = 0$. If $A \notin \mathcal{F}$, then A divides δA . This is impossible because then the proper ideal $(A) \subset \mathcal{F}\{b_1, \dots, b_n\}_\Delta$ would be a Δ -ideal, which is contrary to (2) of the proposition. If $B \notin \mathcal{F}$, the argument is similar. □

Definition 5.101. Let \mathcal{W} be a Δ -vector space over a Δ -field \mathcal{F} . Any set $\Sigma \subseteq \mathcal{W}$ is Δ -linearly independent over \mathcal{F} if the family $(\theta\alpha)_{\theta \in \Theta, \alpha \in \Sigma}$ is linearly independent over \mathcal{F} . Let \mathcal{R} be a Δ -ring. A family $(\alpha_i)_{i \in I}$ of elements of a Δ -overring of \mathcal{R} is Δ -algebraically independent over \mathcal{R} or, more simply, Δ - \mathcal{R} -algebraically independent, or Δ - \mathcal{R} -independent, if the family $(\theta\alpha)_{\theta \in \Theta, \alpha \in \Sigma}$ is algebraically independent over \mathcal{R} .

Corollary 5.102. *Let \mathcal{U} be an (E, Δ) -universal extension of the (E, Δ) -field \mathcal{F} , and let $\mathcal{C} = \langle a_i \mid i = 1, \dots, n \rangle$, let $a_i \in \mathcal{F}$, and let $b_i \in \mathcal{U}$ be such that $b'_i = a_i$. The following four conditions are equivalent:*

- (1) a_1, \dots, a_n are E -linearly independent over \mathcal{C} modulo $\delta\mathcal{F}$,
- (2) b_1, \dots, b_n are E -algebraically independent of the family over \mathcal{F} , and $\mathcal{F}\{b_1, \dots, b_n\}_{E, \Delta}$ is Δ -simple,
- (3) $1, b_1, \dots, b_n$ are E -linearly independent over \mathcal{F} , and $(\mathcal{F}\{b_1, \dots, b_n\}_{E, \Delta})^\Delta = \mathcal{C}$,
- (4) $1, b_1, \dots, b_n$ are E -linearly independent over \mathcal{F} , and $(\mathcal{F}\langle b_1, \dots, b_n \rangle_{E, \Delta})^\Delta = \mathcal{C}$.

Proof. For each positive integer ν , let $\Psi(\nu)$ be the set of monomials in E of order less than or equal to ν . Then for each ν, i and $\psi \in \Psi(\nu)$, ψb_i is a Δ -zero of the Δ -polynomial $y' - \psi a_i$. Since \mathcal{U} is clearly also a Δ -universal extension of the Δ -field \mathcal{F} , Proposition 5.96 may be applied to the families $(\psi b_i)_{\psi \in \Psi(\nu), i=1, \dots, n}$ and $(\psi a_i)_{\psi \in \Psi(\nu), i=1, \dots, n}$ for each ν .

The equivalence of the four parts of the proposition may be verified by the following four observations, which are true because each E-algebraic relation only has a finite number of E-derivatives:

- (1) $(\Psi_i a_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}$ are linearly independent over \mathbb{C} modulo $\delta\mathcal{F}$ for all ν if and only if the a_1, \dots, a_n are E-linearly independent over \mathbb{C} modulo $\delta\mathcal{F}$,
- (2) $(\Psi_i a_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}$ are algebraically independent over \mathbb{C} modulo $\delta\mathcal{F}$ for all ν if and only if the b_1, \dots, b_n are E-algebraically independent over \mathbb{C} modulo $\delta\mathcal{F}$,
- (3) $\mathcal{F}\{(\psi b_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}\}_{E, \Delta}$ is Δ -simple for all ν if and only if $\mathcal{F}\{b_i\}_{E, \Delta}$ is Δ -simple,
- (4) $(\mathcal{F}\{(\psi b_i)_{\psi \in \Psi(\nu), 1 \leq i \leq n}\}_{E, \Delta})^\Delta = \mathbb{C}$ for all ν if and only if $(\mathcal{F}\{b_i\}_{E, \Delta})^\Delta = \mathbb{C}$.

□

Corollary 5.103. *Assume $\Delta = \{\delta\}$. Let the conditions of the last corollary be satisfied, let $a \in \mathcal{F}$, and let $\xi \in \mathcal{F}\langle b_1, \dots, b_n \rangle_{E, \Delta}$ be a Δ -zero of $\delta y - ay \in \mathcal{F}\langle y \rangle_\Delta$. Then $\xi \in \mathcal{F}$.*

Proof. The proof is the same as 5.99. □

The main objective of [4] by Johnson, Reinhart and Rubel is to construct a prime (E, Δ) -ideal $\mathfrak{P} \subset \mathcal{F}\langle y \rangle_{E, \Delta}$ such that all (E, Δ) -zeros $\zeta \in \mathcal{U}$ of \mathfrak{P} generate (E, Δ) -field extensions $\mathcal{F}\langle \zeta \rangle_{E, \Delta}$ over \mathcal{F} that have infinite transcendence degree over \mathcal{F} . Using the techniques just developed, the next proposition presents new simpler examples of such prime ideals. Recall $\Delta = \{\delta\}$.

Lemma 5.104. *Let z be an (E, Δ) -indeterminate over the (E, Δ) -field \mathcal{H} . Let $a \in \mathcal{H}\langle z \rangle_E$ and $a \notin \mathcal{H}$. Then*

- (1) 1 and a are E-linearly independent over \mathcal{H} ,
- (2) $a \notin \delta\mathcal{H}\langle z \rangle_{E, \Delta}$, i.e., a has no Δ -primitive in $\mathcal{H}\langle z \rangle_{E, \Delta}$,
- (3) $(\mathcal{H}\langle z \rangle_{E, \Delta})^\Delta = \mathcal{H}^\Delta$ and
- (4) a is E-linearly independent over \mathcal{H}^Δ modulo $\delta\mathcal{H}\langle z \rangle_{E, \Delta}$.

Proof. (1) Apply Proposition 3.53.

- (2) Let $E = \{\epsilon_1, \dots, \epsilon_n\}$ and choose the ranking on the (E, Δ) -indeterminate z such that the rank of $\delta^r \epsilon_1^{r_1}, \dots, \epsilon_n^{r_n} z$ is (r, r_1, \dots, r_n) in the lexicographical order on \mathbb{N}^{n+1} . Extend this to a ranking of $\mathcal{H}\langle z \rangle_{E, \Delta}$. For an element $f \in \mathcal{H}\langle z \rangle_{E, \Delta}$, let S_f denote the separant of f .

Let $b \in \mathcal{H}\langle z \rangle_{E, \Delta}$ be represented as the quotient c/d with $c, d \in \mathcal{H}\langle z \rangle_{E, \Delta}$ and $d \neq 0$ such that the maximum of the rank of c and the rank of d is the least possible among all such representations. Let w be the highest ranking derivative of z present in c or d .

Suppose $a = b'$ where $b = c/d$ as above. If the rank of $w = (0, \dots, 0)$, then $c \in \mathcal{H}$, $d \in \mathcal{H}$, $c/d \in \mathcal{H}$, and $a = (c/d)' \in \mathcal{H}$. Assume the rank of w is greater than $(0, \dots, 0)$ and write $(c/d)' =$

$$\frac{c' \cdot d - c \cdot d'}{d^2} = \frac{S_c d - c S_d}{d^2} w' + \frac{\text{terms of rank } < \text{rank } w'}{d^2}.$$

If $(S_c d - c S_d) \neq 0$, then $(c/d)' \notin \mathcal{H}\langle z \rangle_E$ because $w' \notin \mathcal{H}\langle z \rangle_E$. If $(S_c d - c S_d) = 0$, then, since $c \neq 0$ and $d \neq 0$, $S_d \neq 0$ because otherwise $S_c = 0$. But $S_c/S_d = c/d = b$ is a representation of b such that S_c and S_d have lower rank than c and d , which is contrary to the assumptions on c and d .

- (3) For each positive integer ν , let $\Psi(\nu)$ be the monomials in E of order less than or equal to ν . Since $(\psi z)_{\psi \in \Psi(\nu)}$ is a finite set of Δ -indeterminates over \mathcal{H} and each Δ -constant of $\mathcal{H}\langle z \rangle_{E,\Delta}$ is in $\mathcal{H}\langle (\psi z)_{\psi \in \Psi(\nu)} \rangle_{\Delta}$ for some ν , Corollary 3.55 implies that $(\mathcal{H}\langle z \rangle_{E,\Delta})^{\Delta} = \mathcal{H}^{\Delta}$.
- (4) Every E -linear combination of a over \mathcal{H} is not in \mathcal{H} because a and 1 are E -linearly independent over \mathcal{H} (part 1), is in $\mathcal{H}\langle z \rangle_E$ by assumption, and is not in $\delta\mathcal{H}\langle z \rangle_{E,\Delta}$ by part 2. Therefore a is E -independent over \mathcal{H} modulo $\delta\mathcal{H}\langle z \rangle_{E,\Delta}$. A fortiori, a is E -linearly independent over \mathcal{H}^{Δ} modulo $\delta\mathcal{H}\langle z \rangle_{E,\Delta}$, since $\mathcal{H}^{\Delta} \subseteq \mathcal{H}$.

□

Proposition 5.105. *Let E be nonempty, and $\Delta = \{\delta\}$. Let z be an (E, Δ) -indeterminate over the (E, Δ) -field \mathcal{H} , and, let y be an (E, Δ) -indeterminate over the (E, Δ) -field $\mathcal{F} = \mathcal{H}\langle z \rangle_{E,\Delta}$. Let $a \in \mathcal{H}\langle z \rangle_E$, and $a \notin \mathcal{H}$. Then for all (E, Δ) -zeros b of the prime (E, Δ) -ideal $[\delta y - a]_{E,\Delta} \subset \mathcal{F}\langle y \rangle_{E,\Delta}$, b is E -algebraically independent over \mathcal{F} , and the algebraic transcendence degree of $\mathcal{F}\langle b \rangle_{E,\Delta}$ over \mathcal{F} is infinite.*

Proof. By part 4 of Lemma 5.104, a is E -linearly independent over $\mathcal{F}^{\Delta} = \mathcal{H}^{\Delta}$ modulo $\delta(\mathcal{F})$ (see [4, Theorem 5, page 59]). This is the condition 1 of Corollary 5.102. For any $b \in \mathcal{U}$ that is an (E, Δ) -zero of the prime (E, Δ) -ideal $[\delta y - a]_{E,\Delta} \subset \mathcal{F}\langle y \rangle_{E,\Delta}$, condition 2 of Corollary 5.102 implies b is E -algebraically independent over \mathcal{F} . Since E is nonempty, $\mathcal{G} = \mathcal{F}\langle b \rangle_{E,\Delta}$ has E -transcendence degree one over \mathcal{F} and has infinite algebraic transcendence degree over \mathcal{F} . □

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