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BOUNDED H_∞-CALCULUS FOR PSEUDODIFFERENTIAL OPERATORS AND APPLICATIONS TO THE DIRICHLET-NEUMANN OPERATOR

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Abstract. Operators of the form $A = a(x, D) + K$ with a pseudodifferential symbol $a(x, \xi)$ belonging to the Hörmander class $S^m_{1,\delta}$, $m > 0$, $0 \leq \delta < 1$, and certain perturbations K are shown to possess a bounded H_∞-calculus in Besov-Triebel-Lizorkin and certain subspaces of Hölder spaces, provided a is suitably elliptic. Applications concern pseudodifferential operators with mildly regular symbols and operators on manifolds of low regularity. An example is the Dirichlet-Neumann operator for a compact domain with $\mathcal{C}^{1+r}$-boundary.

1. Introduction

After its introduction in [15] by McIntosh, the last two decades have seen a continuous increase of interest in the H_∞-calculus of linear operators in general Banach spaces. To a large extent this is due to the strong impact of this calculus on the theory of abstract evolution equations; see [12] for a comprehensive treatise. Let us briefly sketch two typical applications: Assume that $A : \mathcal{D}(A) \subset X \to X$ is a linear, closed operator in a Banach space X that has a bounded H_∞-calculus. Then the fractional powers A^θ of A are well defined and the imaginary powers $A^{i\theta}$ of A are bounded operators, with norm locally uniform in θ. It is well known that the latter property implies that for given $0 < \theta < 1$ the complex interpolation space $[X, \mathcal{D}(A)]_\theta$ coincides (up to an equivalent norm) with the domain of A^θ. Moreover, by the Theorem of Dore and Venni [5], linear evolution equations of the type $u' + Au = f$ involving operators A with bounded imaginary powers possess (under suitable assumptions) the property of the maximal L_p-regularity. It is well known that both of these results are very useful in the study of nonlinear evolution equations.

In concrete situations, often A is a differential operator. For this case there are rather general criteria ensuring the existence of a bounded H_∞-calculus. In [2] and [6] general elliptic systems in L_p-spaces over $\mathbb{R}^n$ or over a compact closed manifold are treated, provided rather mild regularity conditions on the coefficients are satisfied. The paper [4] extends the results of [2] to domains with boundary. In [3] differential operators on conic manifolds with boundary are investigated.

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However, for a lot of interesting applications the limitation to differential operators is much too restrictive. For example, various moving boundary problems of elliptic and/or parabolic type can be reduced to evolution equations for a parametrization of the unknown interface over the initial geometry. Analyzing these evolution equations, it turns out that first of all they are quasilinear or even fully nonlinear, and secondly that these evolution equations involve pseudodifferential operators (which are not differential operators) with mildy regular symbols; cf. [8], [9], [10], and the references therein. Let us discuss a simple but paradigmatic example.

Consider a fluid blob $\Omega(t)$ which moves under the influence of surface tension. We assume that the fluid is initially located in Ω_0 with the boundary Γ_0, and, given $t > 0$, we write $\Gamma(t)$ for the boundary of $\Omega(t)$. If we assume that the flow is governed by Darcy's law, we encounter the following Hele-Shaw problem: Find a family of domains $\{\Omega(t) \mid t > 0\}$ with the corresponding family of boundaries $\{\Gamma(t) \mid t > 0\}$ and a function $u : \bigcup_{t\geq 0} (\Omega(t) \times \{t\}) \to \mathbb{R}$, representing the pressure in $\Omega(t)$ at time t, such that

$$\begin{cases} \Delta u(\cdot,t) = 0 & \text{in } \Omega(t), \\ u(\cdot,t) = \gamma H_{\Gamma(t)} & \text{on } \Gamma(t), \\ V(t) = -\dfrac{\partial u(\cdot,t)}{\partial \nu_{\Gamma(t)}} & \text{on } \Gamma(t), \\ \Gamma(\cdot,0) = \Gamma_0 & \text{at } t = 0. \end{cases} \tag{1.1}$$

Here, γ denotes the surface tension coefficient, a positive constant, and H_Γ stands for the mean curvature of Γ. Furthermore, V is the normal velocity of the family $\{\Gamma(t) \mid t > 0\}$ and ν_Γ stands for the outer normal of Γ. We use the orientation convention that H_Γ is positive if Ω is convex and that V is positive if $\Omega(t)$ is (locally) expanding. Observe that $H_{\Gamma(t)}$ is uniquely determined by $\Gamma(t)$. Hence, given $\Gamma(t)$, there exists a unique solution $u_{\Gamma(t)}$ of the elliptic boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega(t), \qquad u = \gamma H_{\Gamma(t)} \quad \text{on } \Gamma(t).$$

In the last problem the time $t \geq 0$ has to be considered as an independent parameter. Using this notation, the evolution equation for the boundary $\Gamma(t)$, which is contained in (1.1), is given by

$$V = -\frac{\partial u_{\Gamma(t)}}{\partial \nu_{\Gamma(t)}}, \quad t > 0, \quad \Gamma(0) = \Gamma_0. \tag{1.2}$$

Observe that (1.2) is a nonlinear evolution equation of third order for the family $\{\Gamma(t) \mid t > 0\}$; cf. [10]. Introducing a suitable parametrization $v(t)$ of $\Gamma(t)$ and linearizing the first equation of (1.2) at Γ_0, we get the linear initial value problem

$$\partial_t v + \frac{\partial}{\partial \nu_{\Gamma_0}} \mathcal{T}_{\Gamma_0} \mathcal{A} v = F, \quad t > 0, \quad v(0) = 0, \tag{1.3}$$

where $g \mapsto \mathcal{T}_{\Gamma_0} g$ is the solution operator to the elliptic boundary value problem

$$\Delta w = 0 \quad \text{in } \Omega_0, \qquad w = g \quad \text{on } \Gamma_0,$$

and where $\mathcal{A}$ is a linear elliptic differential operator of second order acting on functions over Γ_0. All lower order terms appearing in (1.2) are fused to F. The operator

$$\mathcal{N} = \frac{\partial}{\partial \nu_{\Gamma_0}} \mathcal{T}_{\Gamma_0} \tag{1.4}$$

is the so-called Dirichlet-Neumann operator of Ω_0, a first order pseudodifferential operator acting on functions over Γ_0.

In this paper, we investigate the existence of a bounded H_∞-calculus for pseudodifferential operators and certain perturbations. In particular, the before-mentioned Dirichlet-Neumann operator is an example that fits in this framework.

As a first step, in Section 3, we consider operators $A = a(x, D) + K$ with smooth symbol in the standard Hörmander class $S^m_{1,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m > 0$, $0 \leq \delta < 1$, and K being smoothing compared with $a(x, D)$. Under the assumption of suitable ellipticity conditions for a we prove the existence of a bounded H_∞-calculus for $\lambda_0 + A$ with a sufficiently large $\lambda_0 \geq 0$ in Besov-Triebel-Lizorkin spaces. This extends the above-mentioned results of [2] for differential operators to pseudodifferential operators. The proof relies on a parametrix construction that allows us to express the resolvent $(\lambda - a(x, D))^{-1}$ as a parameter-dependent pseudodifferential operator. Our construction follows that of [11], but also gives precise estimates for the remainders. The idea of using parameter-dependent pseudodifferential calculi for the description of resolvents originated in Seeley's work [17], [18], [19] on complex powers of elliptic operators. Since then, this approach and modifications of it have turned out to be very fruitful and have found a variety of applications.

The next step, performed in Section 4, is to consider pseudodifferential symbols that are not smooth in the variable but only of some Hölder regularity. The key here is the so-called method of symbol smoothing. It allows us to approximate such non-smooth symbols by the usual, smooth symbols considered in Section 3, modulo a remainder that can be treated as a perturbation. The method of symbol smoothing has been introduced in [16] to obtain continuity properties of mildly regular pseudodifferential operators in L_p-spaces. It has been refined later on, yielding a wealth of applications to partial differential equations; see the monographs [22] and [23].

In Section 5 we consider pseudodifferential operators on manifolds of low regularity. As an application of this, we consider in Section 6 operators $A = \mathcal{N}\mathcal{A}$ with the Dirichlet-Neumann operator $\mathcal{N}$ from (1.4) and a differential operator $\mathcal{A}$ on Γ_0. We show that $\lambda_0 + A$ for sufficiently large $\lambda_0 \geq 0$ possesses a bounded H_∞-calculus in various function spaces (Sobolev, Besov, and little Hölder spaces), provided Γ_0 is of Hölder regularity $\mathcal{C}^{k+r}$ with a positive integer k and $0 < r < 1$, and $\mathcal{A}$ is suitably elliptic of order $k-1$ and has $\mathcal{C}^{1+r}$-coefficients. To this end we use and extend the careful analysis of the pseudodifferential structure of $\mathcal{N}$ given in [23].

It is well known that operators with a bounded H_∞-calculus are sectorial and therefore generate analytic semigroups. In fact, in the case of the Dirichlet-Neumann operator there are several generation results available. Rather precise statements for the operator $\mathcal{N}$ on Besov spaces B^s_{pp} with $1 < p < \infty$ are contained in [7]. However, the results in that paper are derived within the smooth category. In [1] it is shown, in particular, that in the case of a $\mathcal{C}^2$-domain the operator $\mathcal{N}$ generates an analytic semigroup on L_p, where $1 \leq p < \infty$. Our results complete also these investigations, since we are now able to verify that the domain of this generator is given by the Sobolev space H^1_p, provided $1 < p < \infty$.

2. The H_∞-calculus for operators in a Banach space

Let us recall some basic facts about the H_∞-calculus for a closed, densely defined operator

$$A : \mathcal{D}(A) \subset X \longrightarrow X$$

in a Banach space X. This calculus was originally introduced by McIntosh [15]. We refer to [12] for a detailed presentation. Given $0 < \theta < \pi$, let

$$\Lambda = \Lambda(\theta) = \{\lambda = re^{i\varphi} \mid r \geq 0,\ \theta \leq \varphi \leq 2\pi - \theta\} \tag{2.1}$$

denote a closed sector in the complex plane, and $\partial\Lambda = \partial\Lambda(\theta)$ its (parameterized) boundary. Assume that $\Lambda \setminus \{0\}$ is contained in the resolvent set of A, that $\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)}$ is uniformly bounded in $0 \neq \lambda \in \Lambda$, and that A is injective with dense range.

We let $H_\infty = H_\infty(\theta)$ denote the space of all functions $f : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ which are holomorphic and bounded and equipped with the supremum norm. The subspace $H = H(\theta)$ consists of all functions which additionally satisfy, for some $s > 0$,

$$\sup_{\lambda \in \mathbb{C}\setminus\Lambda} (|\lambda|^{-s} + |\lambda|^{s})|f(\lambda)| < \infty.$$

Remark 2.1. Any $f \in H_\infty$ has (non-tangential) boundary values $f|_{\partial\Lambda} \in L_\infty(\partial\Lambda)$.

Remark 2.2. Given $f \in H_\infty$, there exists a sequence $(f_j)_{j\in\mathbb{N}} \subset H$ such that $f_j \to f$ locally uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$, and $\|f_j\|_\infty \leq c\,\|f\|_\infty$ for some constant c which is independent of $j \in \mathbb{N}$.

Because of the decay property, for every $f \in H$ the integral

$$f(A) := \frac{1}{2\pi i} \int_{\partial\Lambda} f(\lambda)(\lambda - A)^{-1}\, d\lambda \tag{2.2}$$

converges absolutely in the $\mathcal{L}(X)$-norm and thus defines an operator $f(A) \in \mathcal{L}(X)$. By approximation, the definition of $f(A)$ can be extended to all $f \in H_\infty$: If $(f_j)_{j\in\mathbb{N}} \subset H$ is a sequence as in Remark 2.2, the limit

$$f(A)x = \lim_{j\to\infty} f_j(A)x$$

exists for all $x \in \mathcal{D}(A)$ and does not depend on the specific choice of the sequence. The resulting operator $f(A) : \mathcal{D}(A) \subset X \to X$ is closable. Its closure will be denoted again by $f(A)$.

Definition 2.3. The operator A admits a *bounded H_∞-calculus* for the sector $\mathbb{C}\setminus\Lambda$ if $f(A) \in \mathcal{L}(X)$ for any $f \in H_\infty$ and, with some constant $M \geq 0$,

$$\|f(A)\|_{\mathcal{L}(X)} \leq M\,\|f\|_\infty \qquad \forall\, f \in H.$$

The existence of a bounded H_∞-calculus implies, in particular, the existence of bounded imaginary powers:

Remark 2.4. If A admits a bounded H_∞-calculus for the sector $\mathbb{C} \setminus \Lambda(\theta)$, then A has bounded imaginary powers satisfying, with some constant $M \geq 0$,

$$\|A^{iy}\|_{\mathcal{L}(X)} \leq M\, e^{\theta|y|} \qquad \forall\, y \in \mathbb{R}.$$

In fact, $f(\lambda) = \lambda^{iy} = |\lambda|^{iy}e^{-y\arg\lambda}$ (with $-\pi \leq \arg\lambda < \pi$) belongs to H_∞, satisfies $\|f\|_\infty = e^{\theta|y|}$, and $A^{iy} = f(A)$. Using the Banach-Steinhaus Theorem, we furthermore obtain the following remark.

Remark 2.5. A possesses a bounded H_∞-calculus for $\mathbb{C}\setminus\Lambda$, if there exists an $M\geq 0$ such that

$$\|f(A)\|_{\mathcal{L}(X)}\leq M\,\|f\|_\infty \qquad \forall\, f\in H.$$

Recall that in this case $f(A)$ is defined by the Dunford integral (2.2).

3. H_∞-calculus in $L_p(\mathbb{R}^n)$ for Λ-elliptic pseudodifferential operators

Throughout this section we shall work with pseudodifferential symbols taking values in the space of $(l\times l)$-matrices with complex entries, for some integer $l\geq 1$. For notational convenience we shall not indicate this explicitly in the notation. Also we prefer to denote the standard Sobolev spaces $H_p^s(\mathbb{R}^n,\mathbb{C}^l)$ by $H_p^s(\mathbb{R}^n)$ and the rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n,\mathbb{C}^l)$ by $\mathcal{S}(\mathbb{R}^n)$.

Definition 3.1. The symbol class $S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ with $m\in\mathbb{R}$ and $0\leq\delta<\rho\leq 1$ consists of all smooth functions $a=a(x,\xi)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying

$$\|a\|^k_{\rho,\delta}:=\sup_{\substack{x,\xi\in\mathbb{R}^n\\ |\alpha|+|\beta|\leq k}}|D^\alpha_\xi D^\beta_x a(x,\xi)|\,\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}<\infty$$

for any $k\in\mathbb{N}_0$. As usual, we use the notation $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$ and $D:=-i\partial$. Frequently, we shall simply write $S^m_{\rho,\delta}$ if the dimension of the underlying space is clear.

The system of norms $\|\cdot\|^k_{\rho,\delta}$, $k\in\mathbb{N}_0$, defines a Fréchet topology on $S^m_{\rho,\delta}$.

Let us recall some basic facts about symbols and pseudodifferential operators. Given $a\in S^m_{\rho,\delta}$, we associate a continuous operator $a(x,D):\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ by

$$[a(x,D)u](x)=\int_{\mathbb{R}^n}e^{-ix\xi}a(x,\xi)\widehat{u}(\xi)\,đ\xi, \tag{3.1}$$

where $\widehat{u}$ is the Fourier transform of u and $đ\xi=(2\pi)^{-n}d\xi$. By duality, we extend this operator to $a(x,D):\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$.

Pseudodifferential operators behave well under composition: There exists a continuous map

$$(a_1,a_2)\mapsto a_1\#a_2: S^{m_1}_{\rho,\delta}\times S^{m_2}_{\rho,\delta}\longrightarrow S^{m_1+m_2}_{\rho,\delta}$$

such that $a_1(x,D)a_2(x,D)=(a_1\#a_2)(x,D)$. For an explicit formula of the so-called Leibniz product $a_1\#a_2$ see, for example, [11]. In the sense of an asymptotic expansion we have $a_1\#a_2\sim\sum\limits_{\alpha\in\mathbb{N}_0^n}\frac{1}{\alpha!}\partial^\alpha_\xi a_1 D^\alpha_x a_2$, i.e., for any positive integer N,

$$a_1\#a_2-\sum_{|\alpha|=0}^{N-1}\frac{1}{\alpha!}\partial^\alpha_\xi a_1 D^\alpha_x a_2\ \in\ S^{m-(\rho-\delta)N}_{\rho,\delta}.$$

By a well-known theorem on the L_p-continuity of pseudodifferential operators, the closed graph theorem, and the above-mentioned continuity of the Leibniz product one can easily show the following result:

Theorem 3.2. *Let $a\in S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, $1<p<\infty$, and let $\nu=\nu(n,\rho,p):=n(1-\rho)|\frac{1}{2}-\frac{1}{p}|$. Then $a(x,D)$ restricts to a continuous map*

$$a(x,D): H^s_p(\mathbb{R}^n)\longrightarrow H^{s-m-\nu}_p(\mathbb{R}^n)$$

for any real s. Moreover, we have continuity of the mappings

$$a\mapsto a(x,D): S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)\longrightarrow\mathcal{L}(H^s_p(\mathbb{R}^n),H^{s-m-\nu}_p(\mathbb{R}^n)).$$

Note that the mapping property $a(x,D) : H^s_p(\mathbb{R}^n) \to H^{s-m}_p(\mathbb{R}^n)$ for $a \in S^m_{\rho,\delta}$ holds true in general only if either $\rho = 1$ or $p = 2$.

Occasionally, we shall also consider so-called *double symbols* $a \in S^m_{\rho,\delta}(\mathbb{R}^{2n} \times \mathbb{R}^n)$. The definition of this symbol class is analogous to the above one by replacing the variable $x \in \mathbb{R}^n$ by $(x,y) \in \mathbb{R}^{2n}$. The associated operators are of the form

$$[a(x,y,D)u](x) = \iint e^{i(x-y)\xi} a(x,y,\xi) u(y)\, dy đ\xi.$$

However, it can be shown that each operator induced by a double symbol coincides with one induced by a standard symbol.

3.1. **Λ-hypoellipticity.** Although our main applications concern elliptic operators, we shall introduce (respectively, recall) here the notion of Λ-hypoellipticity. From now on, we let $\Lambda = \Lambda(\theta)$ be a fixed sector as in (2.1).

Definition 3.3. A symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m > 0$ is called Λ-*hypoelliptic* if there exist constants $0 < m_0 \le m$, $C_0 \ge 1$, and $R \ge 0$ such that:

(H1) For all $x \in \mathbb{R}^n$ and all $|\xi| \ge R$ we have

$$\sigma(a(x,\xi)) \subset \Omega_\xi := \{z \in \mathbb{C} \mid \tfrac{1}{C_0} \langle\xi\rangle^{m_0} < |z| < C_0 \langle\xi\rangle^m,\ z \notin \Lambda\}.$$

(H2) Given $\alpha, \beta \in \mathbb{N}^n_0$, there exists a $C \ge 0$ such that for all $x \in \mathbb{R}^n$, $|\xi| \ge R$, and $\lambda \in \Lambda$,

$$|(\partial^\alpha_\xi \partial^\beta_x a(x,\xi))(\lambda - a(x,\xi))^{-1}| \le C\ \langle\xi\rangle^{\delta|\beta| - \rho|\alpha|}.$$

In (H1), $\sigma(\cdot)$ denotes the spectrum. In case $m_0 = m$, we call a Λ-*elliptic*.

As a simple consequence of (H2) and the identity $|(\lambda - a)^{-1}| = |\lambda|^{-1}|1 + a(\lambda - a)^{-1}|$ we obtain

$$\sup_{x \in \mathbb{R}^n,\, |\xi| \ge R,\, \lambda \in \Lambda} |\lambda|\, |(\lambda - a(x,\xi))^{-1}| < \infty. \tag{3.2}$$

Lemma 3.4. *There exists a constant $C \ge 0$ such that for all $x \in \mathbb{R}^n$, $|\xi| \ge R$ and all $\lambda \in \Lambda$ with $|\lambda| \le \frac{1}{2C_0} \langle\xi\rangle^{m_0}$ we have*

$$|(\lambda - a(x,\xi))^{-1}| \le C\ \langle\xi\rangle^{-m_0}.$$

Proof. Let x, ξ, and λ be as stated in the lemma. By Cauchy's integral formula,

$$(\lambda - a(x,\xi))^{-1} = \frac{1}{2\pi i} \int_{\mathcal{C}_\xi} \frac{(z - a(x,\xi))^{-1}}{\lambda - z}\, dz,$$

where $\mathcal{C}_\xi$ is the counterclockwise oriented circle centered at zero with radius $\frac{1}{C_0} \langle\xi\rangle^{m_0}$. Since $|z| \le 2|\lambda - z|$ for $z \in \mathcal{C}_\xi$ and by (3.2), the integrand can be estimated in norm from above by $C_1|z|^{-2} = C_0^2 C_1 \langle\xi\rangle^{-2m_0}$ with a suitable constant C_1 independent of x, ξ, and λ. Hence the desired estimate follows with $C = C_0 C_1$. $\square$

Corollary 3.5. *If $a(x,\xi)$ satisfies conditions* (H1) *and* (H2)*, then*

$$\sup_{x \in \mathbb{R}^n,\, |\xi| \ge R,\, \lambda \in \Lambda} (\langle\xi\rangle^{m_0} + |\lambda|)\, |(\lambda - a(x,\xi))^{-1}| < \infty.$$

Proof. Let x and $|\xi| \geq R$ be given. First assume $|\lambda| \leq \frac{1}{2C_0} \langle \xi \rangle^{m_0}$. Then $1 + \frac{1}{2C_0} \geq \langle \xi \rangle^{-m_0} (\langle \xi \rangle^{m_0} + |\lambda|)$. Thus, by Lemma 3.4,

$$|(\lambda - a(x,\xi))^{-1}| \leq C \langle \xi \rangle^{-m_0} \leq C\big(1 + \tfrac{1}{2C_0}\big)(\langle \xi \rangle^{m_0} + |\lambda|)^{-1}.$$

If $|\lambda| \geq \frac{1}{2C_0} \langle \xi \rangle^{m_0}$, then $1 + 2C_0 \geq |\lambda|^{-1}(\langle \xi \rangle^{m_0} + |\lambda|)$. Then (3.2) yields

$$|(\lambda - a(x,\xi))^{-1}| \leq C|\lambda|^{-1} \leq C(1 + 2C_0)(\langle \xi \rangle^{m_0} + |\lambda|)^{-1}$$

for a suitable constant $C \geq 0$. □

Corollary 3.6. *$a \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ is Λ-elliptic if and only if a satisfies condition* (H1) *and*

$$\sup_{x \in \mathbb{R}^n,\, |\xi| \geq R,\, \lambda \in \Lambda} \langle \xi \rangle^m \, |(\lambda - a(x,\xi))^{-1}| < \infty.$$

Proposition 3.7. *For real λ_0 define the symbol $a_0(x,\xi) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ by*

$$a_0(x,\xi) = \lambda_0 + a(x,\xi).$$

If λ_0 is sufficiently large, then a_0 satisfies conditions (H1) *and* (H2) *with $R = 0$* (*and C_0 possibly enlarged*). *In particular,*

$$\sigma(a_0(x,\xi)) \cap \Lambda = \emptyset \qquad \forall\, x, \xi \in \mathbb{R}^n \tag{3.3}$$

and Corollary 3.5 *holds true with $R = 0$.*

Proof. If $\Lambda = \Lambda(\theta)$ we choose

$$\lambda_0 > \frac{1}{\sin(\min\{\theta, \frac{\pi}{2}\})} \sup_{x \in \mathbb{R}^n,\, |\xi| \leq R} |a(x,\xi)|.$$

If $d = \sup_{x, |\xi| \leq R} |a(x,\xi)|$, the spectrum of $a(x,\xi)$ for $|\xi| \leq R$ is contained in the closed ball of radius d. Thus (3.3) for $|\xi| \leq R$ immediately follows, since $|\lambda - \lambda_0| \geq d + \varepsilon$ for all $\lambda \in \Lambda$ for some $\varepsilon > 0$, by the choice of λ_0.

Now let $|\xi| \geq R_0 \geq R$ with R_0 to be chosen below. Clearly, $\sigma(a_0(x,\xi)) \subset \lambda_0 + \Omega_\xi$ and for $z \in \Omega_\xi$, we have

$$\begin{aligned} |\lambda_0 + z| &< \lambda_0 + C_0 \langle \xi \rangle^m \leq (\lambda_0 + C_0) \langle \xi \rangle^m, \\ |\lambda_0 + z| &> \langle \xi \rangle^{m_0} (\tfrac{1}{C_0} - \langle \xi \rangle^{-m_0} \lambda_0) \geq \tfrac{1}{\lambda_0 + C_0} \langle \xi \rangle^{m_0}, \end{aligned} \tag{3.4}$$

where the last estimate holds true for R_0 sufficiently large. Moreover, by the choice of λ_0 and (H1), there exists a compact set K not intersecting Λ with $\sigma(a_0(x,\xi)) \subset K$ for all x and $|\xi| \leq R_0$. However, for a sufficiently large constant C_1,

$$\tfrac{1}{C_1} \langle \xi \rangle^{m_0} < |z| < C_1 \langle \xi \rangle^m \qquad \forall\, z \in K \quad \forall\, |\xi| \leq R_0.$$

Together with (3.4) we conclude that (H1) holds true for a_0 with $R = 0$ and C_0 replaced by $\max(\lambda_0 + C_0, C_1)$.

For multi-indices α, β with $|\alpha| + |\beta| \geq 1$ we have

$$(\partial_\xi^\alpha \partial_x^\beta a_0(x,\xi))(\lambda - a_0(x,\xi))^{-1} = (\partial_\xi^\alpha \partial_x^\beta a(x,\xi))(\lambda - \lambda_0 - a(x,\xi))^{-1}.$$

Since with $\lambda \in \Lambda$ also $\lambda - \lambda_0 \in \Lambda$, we obtain from (H2) that

$$|(\partial_\xi^\alpha \partial_x^\beta a_0(x,\xi))(\lambda - a_0(x,\xi))^{-1}| \leq C_{\alpha\beta} \langle \xi \rangle^{\delta|\beta| - \rho|\alpha|} \tag{3.5}$$

for all $x \in \mathbb{R}^n$, $|\xi| \geq R$, and all $\lambda \in \Lambda$. On the other hand,

$$|(\lambda - \lambda_0 - a(x,\xi))^{-1}| = |\lambda - \lambda_0|^{-1} \Big| \Big(1 - \frac{a(x,\xi)}{\lambda - \lambda_0}\Big)^{-1} \Big|$$

is uniformly bounded in $x \in \mathbb{R}^n$, $|\xi| \le R$ and $\lambda \in \Lambda$, since $|\frac{a(x,\xi)}{\lambda-\lambda_0}| \le \frac{d}{d+\varepsilon} < 1$. Hence (3.5) holds uniformly in $x, \xi \in \mathbb{R}^n$ and $\lambda \in \Lambda$. The case $|\alpha| = |\beta| = 0$ is treated analogously, using (3.2). Therefore a_0 satisfies (H2) with $R = 0$. $\square$

3.2. **A parametrix construction.** Let $a \in S^m_{\rho,\delta}$ satisfy the hypoellipticity conditions (H1) and (H2) with $R = 0$. Then the following result holds true:

Theorem 3.8. *There exists a* $p = p(x,\xi;\lambda) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda)$ *such that*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi;\lambda)| \le C_{\alpha\beta}(\langle\xi\rangle^{m_0} + |\lambda|)^{-1} \langle\xi\rangle^{\delta|\beta|-\rho|\alpha|} \tag{3.6}$$

uniformly in $(x,\xi,\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \Lambda$ *for all* $\alpha, \beta \in \mathbb{N}_0^n$, *as well as*

$$\begin{aligned} &|\partial_\xi^\alpha \partial_x^\beta \{p(x,\xi;\lambda) - (\lambda - a(x,\xi))^{-1}\}| \\ &\qquad \le C_{\alpha\beta}(\langle\xi\rangle^{m_0} + |\lambda|)^{-3} \langle\xi\rangle^{2m-(\rho-\delta)+\delta|\beta|-\rho|\alpha|}. \end{aligned} \tag{3.7}$$

Moreover,

$$\begin{aligned} p(x,D;\lambda)(\lambda - a(x,D)) &= 1 + r_0(x,D;\lambda), \\ (\lambda - a(x,D))p(x,D;\lambda) &= 1 + r_1(x,D;\lambda) \end{aligned} \tag{3.8}$$

with remainders satisfying

$$\sup_{(x,\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^n\times\Lambda} |\partial_\xi^\alpha \partial_x^\beta r_j(x,\xi;\lambda)| \langle\lambda\rangle \langle\xi\rangle^N < \infty \qquad \forall\, \alpha,\beta \in \mathbb{N}_0^n \quad \forall\, N \in \mathbb{N}. \tag{3.9}$$

Apart from the description of the remainders r_0 and r_1, this theorem is proved in Chapter 8 of [11]; the missing part we shall prove below. But first let us describe how the symbol p is constructed. We define $p_0(x,\xi;\lambda) = (\lambda - a(x,\xi))^{-1}$ and then, iteratively,

$$p_j(x,\xi;\lambda) = -\sum_{\substack{k+|\alpha|=j \\ k<j}} \frac{1}{\alpha!}(\partial_\xi^\alpha p_k)(x,\xi;\lambda)(D_x^\alpha a)(x,\xi)p_0(x,\xi;\lambda), \qquad j \ge 1.$$

By induction, each $\partial_\xi^\alpha \partial_x^\beta p_j$, $j \ge 1$, is a finite linear combination of the terms $(\lambda-a)^{-1}(\partial_\xi^{\alpha_1}\partial_x^{\beta_1} a)\cdot\ldots\cdot(\lambda-a)^{-1}(\partial_\xi^{\alpha_k}\partial_x^{\beta_k} a)(\lambda-a)^{-1}$ with $|\alpha_1|+\ldots+|\alpha_k| = |\alpha|+j$, $|\beta_1|+\ldots+|\beta_k| = |\alpha|+j$, and $k \ge 2$. Hence the p_j, $j \ge 1$, fulfill the estimates

$$|\partial_\xi^\alpha \partial_x^\beta p_j(x,\xi;\lambda)| \le C_{\alpha\beta}(\langle\xi\rangle^{m_0} + |\lambda|)^{-3} \langle\xi\rangle^{2m-(\rho-\delta)j+\delta|\beta|-\rho|\alpha|}$$

uniformly in x, ξ, and λ. Now p is defined by an asymptotic summation of the p_j, i.e.

$$p(x,\xi;\lambda) = (\lambda - a(x,\xi))^{-1} + \sum_{j=1}^\infty \chi(c_j\xi)p_j(x,\xi;\lambda),$$

where χ is a zero excision function (i.e. $\chi \equiv 0$ near 0 and $1-\chi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$), and $c_0 > c_1 > \ldots > c_j \xrightarrow{j\to\infty} 0$ sufficiently fast. Then (3.6) and (3.7) are valid.

Proof of (3.8) *and* (3.9). We use results of Lemma 2.3 in Chapter 8 of [11] (what we call here a and p is denoted there by p and q, respectively). For $N \in \mathbb{N}$ set

$$\begin{aligned} Q_N(x,\xi;\lambda) &= \sum_{j=0}^{N-1} p_j(x,\xi;\lambda), \\ J_N(x,\xi;\lambda) &= \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha Q_N(x,\xi;\lambda)\, D_x^\alpha(\lambda - a(x,\xi)). \end{aligned}$$

Let us now suppress x and ξ from the notation. Then we can write, for any N,

$$\begin{aligned} r_0(\lambda) &= p(\lambda)\#(\lambda-a)-1 \\ &= [(p(\lambda)-Q_N(\lambda))\#(\lambda-a)]+[Q_N(\lambda)\#(\lambda-a)-J_N(\lambda)]+[J_N(\lambda)-1] \\ &=: s_1(\lambda)+s_2(\lambda)+s_3(\lambda). \end{aligned}$$

By construction of p and [11] we have $\langle\lambda\rangle^3(p(\lambda)-Q_N(\lambda)) \in S^{2m-(\rho-\delta)N}_{\rho,\delta}$ and $\langle\lambda\rangle(J_N(\lambda)-1) \in S^{m-(\rho-\delta)N}_{\rho,\delta}$ uniformly in $\lambda\in\Lambda$. Therefore, $\langle\lambda\rangle s_1(\lambda)$ and $\langle\lambda\rangle s_3(\lambda)$ belong to $S^{3m-(\rho-\delta)N}_{\rho,\delta}$ uniformly in $\lambda\in\Lambda$. Using standard rules for the Leibniz product (see, e.g., Corollary 2 and Lemma 2.4 in Chapter 2 of [11]), we obtain

$$s_2(\lambda) = N\sum_{|\gamma|=N}\int_0^1 \frac{(1-\theta)^{N-1}}{\gamma!} r_{\gamma,\theta}(\lambda)\,d\theta$$

with

$$r_{\gamma,\theta}(x,\xi;\lambda) = \iint e^{-iy\eta}\partial_\xi^\gamma Q_N(x,\xi+\theta\eta;\lambda)D_x^\gamma a(x+y,\xi)\,dy đ\xi,$$

where the integral has to be understood as an oscillatory integral. Now $\langle\lambda\rangle Q_N(\lambda)\in S^0_{\rho,\delta}$ uniformly in $\lambda\in\Lambda$. This yields that $\langle\lambda\rangle r_{\gamma,\theta}(\lambda)\in S^{m-(\rho-\delta)|\gamma|}_{\rho,\delta}$ uniformly in $\lambda\in\Lambda$ and $0\le\theta\le 1$. Thus $\langle\lambda\rangle s_2(\lambda)\in S^{m-(\rho-\delta)N}_{\rho,\delta}$ uniformly in $\lambda\in\Lambda$. Since we may choose N arbitrarily large, (3.9) for $j=0$ follows. The case $j=1$ is verified analogously. □

3.3. H_∞-calculus for Λ-elliptic operators.

Proposition 3.9. *Let $1<p<\infty$ and $a\in S^m_{\rho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$ be Λ-hypoelliptic and either $\rho=1$ or $p=2$. Let $\widetilde{A}$ be the closure of*

$$a(x,D):\mathcal{S}(\mathbb{R}^n)\subset L_p(\mathbb{R}^n)\longrightarrow L_p(\mathbb{R}^n)$$

and, with some $\varepsilon>0$,

$$K\in\mathcal{L}(H_p^{m_0-\varepsilon}(\mathbb{R}^n),L_p(\mathbb{R}^n)).$$

If the unbounded operator

$$A:\mathcal{D}(A):=\mathcal{D}(\widetilde{A})\subset L_p(\mathbb{R}^n)\longrightarrow L_p(\mathbb{R}^n)$$

is defined by the action $a(x,D)+K$, then A is closed and there exists a constant $c\ge 0$ such that $\Lambda_c:=\{\lambda\in\Lambda \mid |\lambda|\ge c\}\subset\varrho(A)$[1] *and*

$$\sup_{\lambda\in\Lambda_c}\langle\lambda\rangle\,\|(\lambda-A)^{-1}\|_{\mathcal{L}(L_p(\mathbb{R}^n))}<\infty.$$

Note that we may assume that $a(x,\xi)$ satisfies (H1) and (H2) with $R=0$, since we can write $A=a_0(x,D)+K_0$ with $a_0(x,\xi)=\lambda_0+a(x,\xi)$ and $K_0=K-\lambda_0$ and then apply Proposition 3.7.

Let us also remark that the action of A is indeed well defined: Theorem 3.8 with $\lambda=0$ (respectively the usual parametrix construction for hypoelliptic operators) yields the existence of a symbol $p\in S^{-m_0}_{\rho,\delta}$ such that $r_0:=p\#a-1$ and $r_1:=a\#p-1$ are symbols in $S^{-\infty}$. This implies that

$$H_p^m(\mathbb{R}^n)\hookrightarrow\mathcal{D}(\widetilde{A})\hookrightarrow H_p^{m_0}(\mathbb{R}^n),^2 \tag{3.10}$$

[1] We write $\varrho(A)$ for the resolvent set of A.

[2] The space $\mathcal{D}(\widetilde{A})$ is endowed with the graph norm.

and we can apply K to elements of $\mathcal{D}(\widetilde{A})$. For the second inclusion note that if u and $a(x,D)u$ belong to $L_p(\mathbb{R}^n)$, then

$$u = p(x,D)(a(x,D)u) - r_0(x,D)u \in H_p^{m_0}(\mathbb{R}^n),$$

since $r_0(x,D)$ is smoothing and $p(x,D) : L_p(\mathbb{R}^n) \to H_p^{m_0}(\mathbb{R}^n)$. It is even true that

$$p(x,D;\lambda) \in \mathcal{L}(L_p(\mathbb{R}^n), \mathcal{D}(\widetilde{A})).$$

To this end let $v \in L_p(\mathbb{R}^n)$ be given. Choose a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ converging to v in $L_p(\mathbb{R}^n)$. Then $u_n := p(x,D;\lambda)v_n \in \mathcal{S}(\mathbb{R}^n)$ and $u_n \to p(x,D;\lambda)v$ in $L_p(\mathbb{R}^n)$. Furthermore, $\widetilde{A}u_n = a(x,D)u_n = \lambda u_n - (1 + r_1(x,D;\lambda))v_n$ converges in $L_p(\mathbb{R}^n)$. Thus $p(x,D;\lambda)$ maps in the domain of $\widetilde{A}$. The continuity follows from the closed graph theorem.

Proof of Proposition 3.9. Let $p(x,\xi;\lambda)$ be the parametrix to $\lambda - a(x,\xi)$ of Theorem 3.8. Since, by (3.9), $\|r_j(x,D;\lambda)\|_{\mathcal{L}(H_p^\mu(\mathbb{R}^n), H_p^\nu(\mathbb{R}^n))} = O(\langle\lambda\rangle^{-1})$ for any $\mu, \nu \in \mathbb{R}$, it immediately follows from (3.8) that $\lambda - \widetilde{A} : \mathcal{D}(\widetilde{A}) \longrightarrow L_p(\mathbb{R}^n)$ is bijective for $\lambda \in \Lambda_c$ with c sufficiently large, and that

$$(\lambda - \widetilde{A})^{-1} = p(x,D;\lambda)(1 + r_1(x,D;\lambda))^{-1}.$$

(3.6) implies

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi;\lambda)| \le C_{\alpha\beta} \langle\lambda\rangle^{-\tau/m_0} \langle\xi\rangle^{-m_0+\tau+\delta|\beta|-\rho|\alpha|}$$

for any $0 \le \tau \le m_0$. This together with Theorem 3.2 (for $\rho = 1$ or $p = 2$) yields

$$\|p(x,D;\lambda)\|_{\mathcal{L}(H_p^s(\mathbb{R}^n), H_p^{s+m_0-\tau}(\mathbb{R}^n))} \le C_{s,\tau} \langle\lambda\rangle^{-\tau/m_0} \qquad \forall\, 0 \le \tau \le m_0 \tag{3.11}$$

and arbitrary $s \in \mathbb{R}$. By (3.11) with $s = 0$, $\tau = m_0$, and by the uniform boundedness in $\lambda \in \Lambda_c$ of $\|(1 + r_1(x,D,\lambda))^{-1}\|_{\mathcal{L}(L_p(\mathbb{R}^n))}$, we obtain

$$\|(\lambda - \widetilde{A})^{-1}\|_{\mathcal{L}(L_p(\mathbb{R}^n))} \le C \langle\lambda\rangle^{-1} \qquad \forall\, \lambda \in \Lambda_c.$$

Using the above representation of $(\lambda - \widetilde{A})^{-1}$, we can write

$$\lambda - A = \lambda - \widetilde{A} - K = (1 - S_1(\lambda))(\lambda - \widetilde{A}), \qquad \lambda \in \Lambda_c,$$

with

$$S_1(\lambda) = K\, p(x,D;\lambda)(1 + r_1(x,D;\lambda))^{-1}.$$

The assumed mapping property of K together with (3.11) for $\tau = \varepsilon$ and $s = 0$ yields

$$\|S_1(\lambda)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} \le C \langle\lambda\rangle^{-\varepsilon/m_0}.$$

This shows the closedness of A,[3] as well as the desired norm estimate for the resolvent of A (with c possibly enlarged). □

Using repeatedly the identity $(1+T)^{-1} = 1 - T + T(1+T)^{-1}T$ we obtain from the previous proof the following corollary.

[3] $(\lambda - A) \in \mathcal{L}(\mathcal{D}(\widetilde{A}), L_p(\mathbb{R}^n))$ since $(\lambda - \widetilde{A}) \in \mathcal{L}(\mathcal{D}(\widetilde{A}), L_p(\mathbb{R}^n))$ and $1 - S_1(\lambda) \in \mathcal{L}(L_p(\mathbb{R}^n))$.

Corollary 3.10. *With the notation and the assumptions of Proposition* 3.9,

$$(\lambda - A)^{-1} = p(x, D; \lambda) + R(\lambda)$$

for all $\lambda \in \Lambda_c$ *with a remainder satisfying*

$$\sup_{\lambda \in \Lambda_c} \langle\lambda\rangle^{1+\varepsilon/m_0} \|R(\lambda)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} < \infty.$$

We now establish the main result of this section, the H_∞-calculus for Λ-elliptic operators.

Theorem 3.11. *Let the notation and the assumptions be as in Proposition* 3.9, *but with* $m_0 = m$ *(i.e. a is* Λ*-elliptic). Then there exists a constant* $\lambda_0 \geq 0$ *such that*

$$\lambda_0 + A : \mathcal{D}(A) = H_p^m(\mathbb{R}^n) \subset L_p(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n)$$

has a bounded H_∞*-calculus with respect to the sector* $\mathbb{C} \setminus \Lambda$.

Proof. Replacing in the definition of A the operator K by $K+\lambda_0$ and using Corollary 3.10, we may assume the resolvent $(\lambda-A)^{-1} = p(x, D; \lambda)+R(\lambda)$ exists for all $\lambda \in \Lambda$, and that $\langle\lambda\rangle^{1+\varepsilon/m} \|R(\lambda)\|_{\mathcal{L}(L_p(\mathbb{R}^n))}$ is uniformly bounded in Λ. Let us now show that after these modifications A has a bounded H_∞-calculus.

By definition, cf. Section 2,

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Lambda} f(\lambda)(\lambda - A)^{-1}\, d\lambda, \qquad f \in H.$$

In view of Remark 2.5 we have to show the existence of an $M \geq 0$ such that

$$\|f(A)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} \leq M\, \|f\|_\infty \qquad \forall\, f \in H. \tag{3.12}$$

Clearly, $\displaystyle\int_{\partial\Lambda} f(\lambda)R(\lambda)\, d\lambda$ can be estimated as in (3.12), since $R(\lambda)$ is absolutely integrable. According to (3.7), $p(x, \xi; \lambda) = (\lambda - a(x, \xi))^{-1} + r(x, \xi; \lambda)$ with

$$|\partial_\xi^\alpha \partial_x^\beta r(x, \xi; \lambda)| \leq C_{\alpha\beta}(\langle\xi\rangle^m + |\lambda|)^{-3} \langle\xi\rangle^{2m-(\rho-\delta)+\delta|\beta|-\rho|\alpha|}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$. Setting

$$r_f(x, \xi) := \int_{\partial\Lambda} f(\lambda) r(x, \xi; \lambda)\, d\lambda,$$

we then obtain

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta r_f(x, \xi)| &\leq C_{\alpha\beta} \|f\|_\infty \int_{\partial\Lambda} (\langle\xi\rangle^m + |\lambda|)^{-3} \langle\xi\rangle^{2m-(\rho-\delta)+\delta|\beta|-\rho|\alpha|}\ d\lambda \\
&= C_{\alpha\beta} \|f\|_\infty \langle\xi\rangle^{-(\rho-\delta)+\delta|\beta|-\rho|\alpha|} \int_{\partial\Lambda} (1 + |\varrho|)^{-3}\, d\varrho \\
&= C'_{\alpha\beta} \|f\|_\infty \langle\xi\rangle^{-(\rho-\delta)+\delta|\beta|-\rho|\alpha|}\,;
\end{aligned}$$

here we used the substitution $\varrho = \langle\xi\rangle^{-m} \lambda$ and the fact that $(\langle\xi\rangle^m + \langle\xi\rangle^m |\varrho|)^{-3} = \langle\xi\rangle^{-3m} (1 + |\varrho|)^{-3}$. Thus Theorem 3.2 yields

$$\Big\| \int_{\partial\Lambda} f(\lambda) r(x, D; \lambda)\, d\lambda \Big\|_{\mathcal{L}(L_p(\mathbb{R}^n))} = \|r_f(x, D)\|_{\mathcal{L}(L_p(\mathbb{R}^n))} \leq M\, \|f\|_\infty \qquad \forall\, f \in H$$

with a suitable constant M. Next consider

$$p_f(x, \xi) := \int_{\partial\Lambda} f(\lambda)(\lambda - a(x, \xi))^{-1}\, d\lambda.$$

For fixed (x,ξ), the integrand is holomorphic outside $\{|\lambda|\le 2\|a\|^0_{\rho,\delta}\,\langle\xi\rangle^m\}\cup\Lambda$ (see Definition 3.1 for the norm $\|\cdot\|^0_{\rho,\delta}$) and decays there like $|\lambda|^{-1-\varepsilon}$ for some $\varepsilon>0$ (recall the decay property of functions $f\in H$). Thus we may use Cauchy's Theorem to replace in the definition of $p_f(x,\xi)$ the path $\partial\Lambda$ by a path $\mathcal{C}_\xi$, which looks as follows: It has two radial parts defined by $\partial\Lambda\cap\{|\lambda|\le 2\|a\|^0_{\rho,\delta}\,\langle\xi\rangle^m\}$ and a circular part defined by $(\mathbb{C}\setminus\Lambda)\cap\{|\lambda|=2\|a\|^0_{\rho,\delta}\,\langle\xi\rangle^m\}$. On the circular part, $|(\lambda-a(x,\xi))^{-1}|\le\langle\xi\rangle^{-m}/\|a\|^0_{\rho,\delta}$. This together with Corollary 3.5 (with $R=0$) yields

$$|\partial^\alpha_\xi\partial^\beta_x p_f(x,\xi)|\le C_{\alpha\beta}\|f\|_\infty\,\langle\xi\rangle^{\delta|\beta|-\rho|\alpha|}\,\mathrm{length}(\mathcal{C}_\xi)\,\langle\xi\rangle^{-m}\le C'_{\alpha\beta}\|f\|_\infty\,\langle\xi\rangle^{\delta|\beta|-\rho|\alpha|},$$

since the length of $\mathcal{C}_\xi$ can be estimated by $4(1+\pi)\|a\|^0_{\rho,\delta}\,\langle\xi\rangle^m$. As before, using Theorem 3.2, this yields

$$\Big\|\int_{\partial\Lambda} f(\lambda)(\lambda-p)^{-1}(x,D)\,d\lambda\Big\|_{\mathcal{L}(L_p(\mathbb{R}^n))}=\|p_f(x,D)\|_{\mathcal{L}(L_p(\mathbb{R}^n))}\le M\,\|f\|_\infty$$

for all $f\in H$ for a suitable M. This finishes the proof of Theorem 3.11. □

4. Extension to other spaces and mildly regular symbols

4.1. **Extension to other function and distribution spaces.** The results of Section 3.3 are not restricted to the frame of Sobolev spaces, but also extend to other (half-)scales. Let us denote by $\mathcal{H}^s$, $s\ge 0$, a scale of Banach spaces having the following properties:

- (S1) $\mathcal{H}^s\hookrightarrow\mathcal{H}^{s'}\hookrightarrow\mathcal{S}'(\mathbb{R}^n)$ whenever $s\ge s'\ge 0$.
- (S2) $\mathcal{H}^\infty:=\bigcap_{t\ge 0}\mathcal{H}^t$ is dense in each $\mathcal{H}^s$.
- (S3) If $a\in S^m_{\varrho,\delta}$ with $m\in\mathbb{R}$, then $a(x,D)\in\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})$ provided $s\ge 0$ and $s-m\ge 0$.

In (S3), $0\le\delta<\varrho\le 1$ are fixed. By the closed graph theorem it follows that the map $a\mapsto a(x,D):S^m_{\varrho,\delta}\to\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})$ is continuous whenever $s\ge 0$ and $s-m\ge 0$.

Theorem 4.1. *Let $a\in S^m_{\varrho,\delta}$ be Λ-elliptic and $K\in\mathcal{L}(\mathcal{H}^{m-\varepsilon},\mathcal{H}^0)$ for some $\varepsilon>0$. If*

$$A:\mathcal{H}^m\subset\mathcal{H}^0\longrightarrow\mathcal{H}^0$$

is defined by the action of $a(x,D)+K$, then A is closed and there exists a $\lambda_0\ge 0$ such that λ_0+A has a bounded H_∞-calculus with respect to the sector $\mathbb{C}\setminus\Lambda$.

The proof is the same as in Section 3.3 after replacing $H^s_p(\mathbb{R}^n)$ by $\mathcal{H}^s$ and $L_p(\mathbb{R}^n)$ by $\mathcal{H}^0$, using a corresponding version of Proposition 3.9, with $a(x,D)$ initially defined on $\mathcal{H}^\infty$.

Let us give two examples. For the proofs see Theorem 6.2.2 of [24].

Example 4.2. With (fixed) $1\le p,q<\infty$ and $t\in\mathbb{R}$, the scales

$$\mathcal{H}^s=F^{s+t}_{p,q}(\mathbb{R}^n)\quad\text{or}\quad\mathcal{H}^s=B^{s+t}_{p,q}(\mathbb{R}^n),\qquad s\ge 0,$$

of Triebel-Lizorkin spaces and Besov spaces, respectively, satisfy (S1)–(S3) for $\varrho=1$ and arbitrary $0\le\delta<1$.

Also the limiting case $p = q = \infty$ is interesting. Let us set

$$\mathcal{C}^s_*(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n), \qquad s \in \mathbb{R}. \tag{4.1}$$

These spaces shall be referred to as *Hölder-Zygmund spaces*. As they will appear frequently in this paper (for $s > 0$), let us recall an explicit description of these spaces (cf. [24], Section 2.6.5): If $s = k + r$ with $k \in \mathbb{N}_0$ and $0 < r < 1$, then u belongs to $\mathcal{C}^s_*(\mathbb{R}^n)$ if and only if $u \in BUC^k(\mathbb{R}^n)$[4] and $\partial^\alpha u$ is uniformly Hölder continuous of exponent r for all $|\alpha| = k$, i.e.

$$\sup_{x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^r} < \infty \qquad \forall\, |\alpha| = k. \tag{4.2}$$

If $s = k + 1$, we require $u \in BUC^k(\mathbb{R}^n)$ and

$$\sup_{x \neq y} \frac{|\partial^\alpha u(x) - 2\partial^\alpha u((x+y)/2) + \partial^\alpha u(y)|}{|x-y|} < \infty \qquad \forall\, |\alpha| = k. \tag{4.3}$$

Summing up the supremum norms of all derivatives up to order k and either the terms in (4.2) or (4.3) defines the norm of u in $\mathcal{C}^s_*(\mathbb{R}^n)$[5]. Since $BUC^\infty(\mathbb{R}^n)$ is not dense in $\mathcal{C}^s_*(\mathbb{R}^n)$, we define

$$c^s_*(\mathbb{R}^n) = \text{closure of } BUC^\infty(\mathbb{R}^n) \text{ in } \mathcal{C}^s_*(\mathbb{R}^n), \qquad s \in \mathbb{R}, \tag{4.4}$$

the so-called *little* Hölder-Zygmund spaces.

Example 4.3. Let $t \in \mathbb{R}$. Then the scale $\mathcal{H}^s = c^{s+t}_*(\mathbb{R}^n)$, $s \geq 0$, satisfies (S1)–(S3) for $\varrho = 1$ and arbitrary $0 \leq \delta < 1$.

4.2. **Mildly regular pseudodifferential operators.** If E is a Banach space, let us define the symbol class $S^m(\mathbb{R}^n; E)$, $m \in \mathbb{R}$, as the space of all smooth functions $\widetilde{a} : \mathbb{R}^n \to E$ with $\sup_{\xi \in \mathbb{R}^n} \|\partial^\alpha_\xi \widetilde{a}(\xi)\|_E \,\langle \xi \rangle^{-m+|\alpha|} < \infty$ for any multi-index $\alpha \in \mathbb{N}^n_0$.

Definition 4.4. $\mathcal{C}^r_* S^m_{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n)$ with $r > 0$, $m \in \mathbb{R}$, and $0 \leq \gamma < 1$ denotes the space of all functions a defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\widetilde{a}(\xi) := a(\cdot, \xi) \in S^m(\mathbb{R}^n_\xi; BC(\mathbb{R}^n)) \cap S^{m+r\gamma}(\mathbb{R}^n_\xi; \mathcal{C}^r_*(\mathbb{R}^n));$$

here, $BC(\mathbb{R}^n)$ denotes the space of all bounded continuous functions and $\mathcal{C}^r_*(\mathbb{R}^n)$ are the Hölder-Zygmund spaces. For the sake of simplicity, we often simply write $\mathcal{C}^r_* S^m_{1,\gamma}$.

If $a \in \mathcal{C}^r_* S^m_{1,\gamma}$ and we define $a(x, D)$ by (3.1), then

$$a(x, D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow BC(\mathbb{R}^n)$$

continuously. Thus, in general, $a(x, D)$ does *not* extend to an operator on $\mathcal{S}'(\mathbb{R}^n)$. Hence it might not be clear how to apply $a(x, D)$ to spaces $\mathcal{H}^s$ not having $\mathcal{S}(\mathbb{R}^n)$ as a dense subspace. However, sometimes there is a canonical way to do so. We shall not go into details here, but only state such a result (see, e.g., [14], Proposition 2.4, Lemma 2.9) for Besov and Hölder-Zygmund spaces. It relies on decomposing a symbol a as

$$a = \sum_{k=4}^{\infty} \sum_{j=0}^{k-4} a_{jk} + \sum_{k=0}^{\infty} \sum_{j=k-3}^{k+3} a_{jk} + \sum_{j=4}^{\infty} \sum_{k=0}^{j-4} a_{jk}$$

[4]We write $BUC^k(\mathbb{R}^n)$ for the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^n$, having bounded and uniformly continuous derivatives up to order k, endowed with its natural norm.

[5]It is worth noting that $\mathcal{C}^k_*(\mathbb{R}^n) \neq BUC^k(\mathbb{R}^n)$ if $k \in \mathbb{N}$.

where $a_{jk}(\cdot,\xi) = \mathcal{F}^{-1}\varphi_j * a(\cdot,\xi)\varphi_k(\xi)$ with a Littlewood-Paley decomposition of unity φ_j, $j \in \mathbb{N}_0$. Since $a_{jk} \in S^{-\infty}_{1,0}$, the operators $a_{jk}(x,D)$ are defined on $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 4.5. *If $a \in \mathcal{C}^r_* S^m_{1,\gamma}$ with $0 \le \gamma < 1$, then, continuously,*

$$a(x,D) : B^{s+m}_{pq}(\mathbb{R}^n) \longrightarrow B^s_{pq}(\mathbb{R}^n) \qquad \forall\ -(1-\gamma)r < s < r \quad \forall\, 1 \le p,q \le \infty.$$

Note that there is no problem giving meaning to $a(x,D)u$ for a $u \in \mathcal{C}^s_*$, $s > 0$, with compact support, since in this case also $u \in L_2(\mathbb{R}^n)$. This is sufficient for studying operators on compact manifolds.

Corollary 4.6. *Each $a \in \mathcal{C}^r_* S^m_{1,\gamma}$ with $0 \le \gamma < 1$ induces continuous maps*

$$a(x,D) : c^{s+m}_*(\mathbb{R}^n) \longrightarrow c^s_*(\mathbb{R}^n) \qquad \forall\ -(1-\gamma)r < s < r.$$

Proof. It is known (cf. Proposition 0.2.1 in [13]) that

$$c^s_*(\mathbb{R}^n) = \text{closure of } \mathcal{C}^t_*(\mathbb{R}^n) \text{ in } \mathcal{C}^s_*(\mathbb{R}^n) \qquad \forall\, t > s. \tag{4.5}$$

Thus the result follows by choosing $s < t < r$ and using that $a(x,D) \in \mathcal{L}(\mathcal{C}^{t+m}_*(\mathbb{R}^n), \mathcal{C}^t_*(\mathbb{R}^n))$ by Proposition 4.5. $\square$

There is a nice way to approximate symbols with limited regularity in the x-variable by the usual smooth pseudodifferential symbols. This method is usually referred to as *symbol smoothing*. For the following result see (3.27) in [23, Section 1.3].

Theorem 4.7. *If $a \in \mathcal{C}^r_* S^m_{1,\gamma}$ and $\gamma < \delta < 1$, there exists an $a_\delta \in S^m_{1,\delta}$ with $a - a_\delta \in \mathcal{C}^r_* S^{m-r(\delta-\gamma)}_{1,\delta}$.*

In the following theorem let $\mathcal{H}^s$ denote either $c^s_*(\mathbb{R}^n)$, $H^s_p(\mathbb{R}^n)$ with $1 < p < \infty$, or $B^s_{pq}(\mathbb{R}^n)$ with $1 \le p,q < \infty$.

Theorem 4.8. *Let $a \in \mathcal{C}^r_* S^m_{1,\gamma}$ with $m > 0$ and $0 \le \gamma < 1$ be Λ-elliptic, i.e.*

$$\sigma(a(x,\xi)) \subset \Omega_\xi \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, |\xi| \ge R,$$

for some $R \ge 0$ and Ω_ξ as in (H1), *and*

$$|(\lambda - a(x,\xi))^{-1}| \le C\,\langle\xi\rangle^m \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, |\xi| \ge R \quad \forall\, \lambda \in \Lambda.$$

Let $-(1-\gamma)r < s < r$, and let

$$A : \mathcal{H}^{s+m} \subset \mathcal{H}^s \longrightarrow \mathcal{H}^s$$

act as $a(x,D) + K$, where $K \in \mathcal{L}(\mathcal{H}^{s+m-\varepsilon}, \mathcal{H}^s)$ for some $\varepsilon > 0$. Then A is closed and there exists a constant $\lambda_0 \ge 0$ such that $\lambda_0 + A$ has a bounded H_∞-calculus with respect to $\mathbb{C} \setminus \Lambda$.

Proof. Choose a δ with $\gamma < \delta < 1$ such that $-(1-\delta)r < s$ and $s + r(\delta-\gamma) < r$. Associate a_δ with a as in Theorem 4.7. Then we write $A = a_\delta(x,D) + \widetilde{K}$ with $\widetilde{K} = K + (a-a_\delta)(x,D)$. Due to Proposition 4.5 and/or Corollary 4.6, $(a-a_\delta)(x,D)$ belongs to $\mathcal{L}(\mathcal{H}^{s+m-\varepsilon}, \mathcal{H}^s)$ for $\varepsilon = r(\delta-\gamma)$. Hence $\widetilde{K}$ has the same mapping property as K. Since $a - a_\delta$ has order strictly less than m, it is easy to see that also a_δ has the assumed properties of a, i.e. a_δ is Λ-elliptic in the sense of Definition 3.3 by Corollary 3.6. Now the claim follows from Theorem 4.1, applied to the scale $\widetilde{\mathcal{H}}^t = \mathcal{H}^{s+t}$, $t \ge 0$. $\square$

Example 4.9. A particular case of Theorem 4.8 is that of a differential operator $a(x,D)$, i.e.

$$a(x,\xi)=\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \qquad a_\alpha\in\mathcal{C}^r_*(\mathbb{R}^n)$$

(the lower order terms are, under suitable conditions on the coefficients, subsumed in the remainder K). If there exists a compact set $D\subset\mathbb{C}\setminus\Lambda$ with

$$\sigma(a(x,\xi))\subset D \qquad \forall\, x\in\mathbb{R}^n \quad \forall\,|\xi|=1,$$

then a satisfies the assumptions of Theorem 4.8.

4.3. **Operators on smooth manifolds.** If M is a smooth compact Riemannian manifold and E a smooth vector bundle over M, it is meaningful to speak of the spaces $L^m_{\varrho,\delta}(M,E)$ of pseudodifferential operators on M, whenever $1-\varrho\le\delta<\varrho$. In fact, an operator

$$\widetilde{A}:\mathcal{C}^\infty(M,E)\longrightarrow\mathcal{C}^\infty(M,E)$$

acting on smooth sections into E belongs to this class if, in any local trivialization of E, the operator $\widetilde{A}$ is a pseudodifferential operator with symbol in $S^m_{\varrho,\delta}(\mathbb{R}^n\times\mathbb{R}^n)$, $n=\dim M$ (with values in the square matrices of size corresponding to the dimension of E). Also it makes sense to speak of Λ-(hypo)ellipticity, and the parametrix construction described in Section 3.2 extends to the case of smooth manifolds (for details see [11] and Section 5). Therefore, Theorem 4.1 straightforwardly extends to operators on M.

Theorem 4.10. *Let $\mathcal{H}^s(M,E)$, $s\ge 0$, denote a half-scale of spaces, satisfying the obvious analogs of conditions* (S1)–(S3) *in Section* 4.1. *Let $\widetilde{A}\in L^m_{\varrho,\delta}(M,E)$ be Λ-elliptic and $K\in\mathcal{L}(\mathcal{H}^{m-\varepsilon}(M,E),\,\mathcal{H}^0(M,E))$ for some $\varepsilon>0$. If*

$$A:\mathcal{H}^m(M,E)\subset\mathcal{H}^0(M,E)\longrightarrow\mathcal{H}^0(M,E)$$

is defined by the action of $\widetilde{A}+K$, then A is closed and there exists a $\lambda_0\ge 0$ such that λ_0+A has a bounded H_∞-calculus with respect to the sector $\mathbb{C}\setminus\Lambda$.

Examples for such scales are the Besov-Triebel-Lizorkin spaces, i.e. $\mathcal{H}^s(M,E)=F^{s+t}_{p,q}(M,E)$ or $\mathcal{H}^s(M,E)=B^{s+t}_{p,q}(M,E)$ with $1\le p,q<\infty$ and $t\in\mathbb{R}$, as well as the little Hölder-Zygmund spaces $\mathcal{H}^s(M,E)=c^{s+t}_*(M,E)$.

5. Operators on manifolds of low regularity

The aim of this section is to extend the results from the previous section for operators living on $\mathbb{R}^n$ to operators on a compact n-dimensional Riemannian $\mathcal{C}^{k+r}$-manifold, where $k\in\mathbb{N}$ is a positive integer and $0<r<1$.

Let M be such a manifold. By using a partition of unity and local coordinates, we may define the Sobolev spaces $H^s_p(M)$, $1<p<\infty$, $|s|<k+r$, and the Hölder-Zygmund spaces $\mathcal{C}^s_*(M)$, $0<s\le k+r$, as well as the little Hölder-Zygmund spaces $c^s_*(M)$, $0<s<k+r$. Note that

$$c^s_*(M)=\text{closure of } \mathcal{C}^t_*(M) \text{ in } \mathcal{C}^s_*(M) \text{ for any } s<t\le k+r; \tag{5.1}$$

cf. (4.5). By interpolation, one then also obtains Besov spaces $B^s_{pq}(M)$ for $1<p,q<\infty$ and $|s|<k+r$. One may also define such spaces of sections into a vector bundle E. In the remaining part of this section we shall denote by $\mathcal{H}^s$ either $H^s_p(M)$, $B^s_{pq}(M)$ or $c^s_*(M)$. We frequently shall write $\mathcal{C}^s(M)$ or $c^s(M)$ if it is clear from the context that $s>0$ is not an integer.

Let us introduce some further notation: We fix a real m with $0 < m < k - r$ and a real s with $0 < s < k + r - m$ in the case of Hölder-Zygmund spaces and with $-k - r < s < k + r - m$ in the case of Sobolev or Besov spaces. Moreover, we let $U_1, \ldots, U_N$ denote a fixed covering of M with coordinate maps $\kappa_j : U_j \to V_j \subset \mathbb{R}^n$. Finally, let

$$A : \mathcal{H}^{s+m} \subset \mathcal{H}^s \longrightarrow \mathcal{H}^s. \tag{5.2}$$

We shall prove the following theorem:

Theorem 5.1. *Let A from* (5.2) *satisfy the following three conditions:*

(A1) *There exists an $\varepsilon > 0$ such that for all $\varphi, \psi \in \mathcal{C}^{k+r}(M)$ with disjoint support,*

$$\varphi\, A\, \psi \;\in\; \mathcal{L}(\mathcal{H}^{s+m-\varepsilon}, \mathcal{H}^{s+\varepsilon}).$$

(A2) *There exist symbols $a_j(x,\xi) \in S^m_{1,\delta}$, $1 \le j \le N$, and an $\varepsilon > 0$ such that*

$$\varphi \left(A - \kappa_j^* a_j(x,D)\right) \psi \;\in\; \mathcal{L}(\mathcal{H}^{s+m-\varepsilon}, \mathcal{H}^{s+\varepsilon})$$

for any choice of $\varphi, \psi \in \mathcal{C}^{k+r}(M)$ with support in U_j.

(A3) *All a_j from* (A2) *are Λ-elliptic* (*cf. Definition* 3.3 *and Corollary* 3.6).[6]

In (A2), $0 \le \delta < 1$ *and κ_j^* denotes the pull-back of operators from V_j to U_j via κ_j. Then A is closed and there exists a $\lambda_0 \ge 0$ such that $\lambda_0 + A$ has a bounded H_∞-calculus with repect to $\mathbb{C} \setminus \Lambda$.*

Let us add two remarks before we prove the above theorem: If $\mathcal{H}^s = H^s_2(M)$, we can allow in (A2) that $a_j \in S^m_{\varrho,\delta}$ for $0 \le \delta < \varrho \le 1$. Secondly, using Proposition 3.7, we may assume that the a_j satisfy the Λ-ellipticity conditions with $R = 0$. Finally, by (A1) and (A2), A acts as a continuous operator $\mathcal{H}^{s+m} \to \mathcal{H}^s$.

Now let $p_j(x,\xi;\lambda)$ be a parametrix to $(\lambda - a_j(x,\xi))$ as constructed in Theorem 3.8. The key to Theorem 5.1 is to show that patching together the $\kappa_j^* p_j(x,D;\lambda)$ on M via a partition of unity yields a suitable parametrix for $\lambda - A$.

Lemma 5.2. *If $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ have disjoint support, then, for all $\mu, \nu \in \mathbb{R}$,*

$$\|\varphi\, p_j(x,D;\lambda)\, \psi\|_{\mathcal{L}(H^\mu_p(\mathbb{R}^n), H^\nu_p(\mathbb{R}^n))} \le C_{\mu\nu} \langle\lambda\rangle^{-2} \qquad \forall\, \lambda \in \Lambda.$$

Proof. Let $r(x,\xi;\lambda) = p_j(x,\xi;\lambda) - (\lambda - a_j(x,\xi))^{-1}$. Due to (3.7), $\langle\lambda\rangle^3 r(x,\xi;\lambda) \in S^{2m}_{1,\delta}$ uniformly in λ. Thus the desired estimate holds for $\varphi\, r(x,D;\lambda)\psi$. Now choose $\widetilde{\varphi}, \widetilde{\psi} \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with disjoint supports that are $\equiv 1$ on the supports of φ and ψ, respectively. Then, by integration by parts, $\varphi\, p_j(x,D;\lambda)\psi - \varphi\, r(x,D;\lambda)\psi = \varphi\, q_j(x,D;\lambda)\psi$ with

$$q_j(x,y,\xi,\lambda) = \frac{\widetilde{\varphi}(x)\widetilde{\psi}(y)}{|x-y|^2} \Delta_\xi (\lambda - a_j(x,\xi))^{-1}.$$

By the chain rule, $\langle\lambda\rangle^2 q_j(x,y,\xi;\lambda) \in S^{m-2}_{1,\delta}$ (double symbol) uniformly in λ. This gives the result. □

Lemma 5.3. *Let $U_{jk} := U_j \cap U_k \ne \emptyset$ and $\varphi, \psi \in \mathcal{C}^{k+r}(M)$ be supported in U_{jk}. Then*

$$\|\varphi\{\kappa_j^* p_j(x,D;\lambda) - \kappa_k^* p_k(x,D;\lambda)\}\psi\|_{\mathcal{L}(\mathcal{H}^{s-\varepsilon/2}, \mathcal{H}^{s+m-\tau+\varepsilon/2})} \le C_\tau \langle\lambda\rangle^{-(\varepsilon+\tau)/m}$$

for all $0 \le \tau \le m$, uniformly in $\lambda \in \Lambda$.

[6] In fact, it is sufficient that the a_j satisfy conditions (H1) and (H2) uniformly in $x \in U_j$.

Proof. Setting $V_{jk} = \kappa_j(U_{jk})$ and $\kappa = \kappa_j \circ \kappa_k^{-1} : V_{kj} \to V_{jk}$, the statement follows if we can show that

$$\|\varphi\{p_k(x,D;\lambda) - \kappa^* p_j(x,D;\lambda)\}\psi\| \le C_\tau \langle\lambda\rangle^{-(\varepsilon+\tau)/m} \tag{5.3}$$

for any $\varphi, \psi \in \mathcal{C}^\infty_{\text{comp}}(V_{kj})$; here the norm means the operator-norm as a map in the spaces on $\mathbb{R}^n$ corresponding to the scale $\mathcal{H}^s$. To this end choose $\psi_1, \psi_2, \psi_3 \in \mathcal{C}^\infty_{\text{comp}}(V_{kj})$ with $\psi_j \equiv 1$ on the support of φ and ψ and $\psi_{j+1} \equiv 1$ on the support of ψ_j. To have a neat notation, let us set $P_j(\lambda) = p_j(x,D;\lambda)$, $A_j(\lambda) = \lambda - a_j(x,D)$, and $A_j = -a_j(x,D)$. By (3.6) and (3.8), we have

$$\varphi\kappa^* P_j(\lambda)\psi \equiv \varphi\kappa^* P_j(\lambda)\psi_2 A_k(\lambda) P_k(\lambda)\psi$$

modulo a remainder $R_1(\lambda) \in \mathcal{L}(H^\mu_p, H^\nu_p)$ for all $\mu, \nu \in \mathbb{R}$ with operator norm being $O(\langle\lambda\rangle^{-2})$. The term of the right-hand side we write as

$$\begin{aligned} &\varphi\kappa^*\{(\kappa_*\psi_1)\, P_j(\lambda)\, (\kappa_*\psi_2)\, \kappa_* A_k(\lambda)\, (\kappa_*\psi_3)\} P_k(\lambda)\psi \\ &\quad + \varphi\, \kappa^* P_j(\lambda)\, \psi_2\, A_k\, (1-\psi_3)\, P_k(\lambda)\, \psi. \end{aligned} \tag{5.4}$$

By (3.6), the second term is a remainder of the same kind as $R_1(\lambda)$. By (A2),

$$\varphi\, (\kappa_j^* A_j - \kappa_k^* A_k)\, \psi : \mathcal{H}^{s+m-\varepsilon} \longrightarrow \mathcal{H}^{s+\varepsilon} \tag{5.5}$$

(for any choice of φ and ψ). Moreover, by (3.6),

$$\langle\lambda\rangle^{-\tau/m}\, p_j(x,\xi;\lambda) \ \in\ S^{-m+\tau}_{1,\delta} \qquad \forall\, 0 \le \tau \le m \tag{5.6}$$

uniformly in λ. Combining this with (5.5), we can replace in the first term of (5.4) the factor $\kappa_* A_k(\lambda)$ by $A_j(\lambda)$, modulo a remainder $R_2(\lambda)$ satisfying estimates as in (5.3). After this, using Lemma 5.2, we can cancel the factor $(\kappa_*\psi_2)$ modulo a remainder of the sort $R_1(\lambda)$. We arrive at

$$\varphi\kappa^* P_j(\lambda)\psi \equiv \varphi\kappa^*\{(\kappa_*\psi_1)\, P_j(\lambda) A_j(\lambda)\, (\kappa_*\psi_3)\} P_k(\lambda)\psi.$$

This term equals $\varphi\, P_k(\lambda)\, \psi$ modulo a remainder as $R_1(\lambda)$, again by (3.8). $\square$

Let us now define on M the operator

$$P(\lambda) = \sum_{j=1}^N \widetilde{\varphi}_j\, \kappa_j^* p_j(x,D;\lambda)\, \widetilde{\psi}_j \tag{5.7}$$

for a fixed choice of functions $\widetilde{\varphi}_j, \widetilde{\psi} \in \mathcal{C}^{k+r}(M)$ with support in U_j, with $\widetilde{\psi}_j \equiv 1$ near the support of φ_j, and with $\sum \widetilde{\varphi}_j \equiv 1$ on M.

Lemma 5.4. *Let $\varphi, \psi \in \mathcal{C}^{k+r}(M)$ have support in U_j. Then*

$$\|\varphi\{P(\lambda) - \kappa_j^* p_j(x,D;\lambda)\}\psi\|_{\mathcal{L}(\mathcal{H}^{s-\varepsilon/2}, \mathcal{H}^{s+m-\tau+\varepsilon/2})} \le C_\tau \langle\lambda\rangle^{-(\varepsilon+\tau)/m}$$

for all $0 \le \tau \le m$, uniformly in $\lambda \in \Lambda$.

Proof. Again, let us write $P_j(\lambda) = p_j(x,D;\lambda)$. Then

$$\varphi\, P(\lambda)\, \psi = \sum_{k=1}^N \widetilde{\varphi}_k \varphi\, \kappa_j^* P_j(\lambda)\, \psi\widetilde{\psi}_k + \sum_{k=1}^N \widetilde{\varphi}_k \varphi\{\kappa_k^* P_k(\lambda) - \kappa_j^* P_j(\lambda)\}\psi\widetilde{\psi}_k.$$

The second sum on the right-hand side is the desired behaviour due to Lemma 5.3. The first sum equals

$$\varphi\, \kappa_j^* P_j(\lambda)\, \psi - \sum_{k=1}^N \widetilde{\varphi}_k \varphi\, \kappa_j^* P_j(\lambda)\, \psi(1-\widetilde{\psi}_k).$$

Since $\widetilde{\varphi}_k\varphi$ and $\psi(1-\widetilde{\psi}_k)$ have disjoint support, the second term is as requested due to Lemma 5.2. $\square$

Proposition 5.5. *There exists a constant $C \geq 0$ such that, for all $\lambda \in \Lambda$,*

$$\|(\lambda - A)P(\lambda) - 1\|_{\mathcal{L}(\mathcal{H}^s)} + \|P(\lambda)(\lambda - A) - 1\|_{\mathcal{L}(\mathcal{H}^{s+m})} \leq C\,\langle\lambda\rangle^{-\varepsilon/m}.$$

In particular, A is closed and $\lambda - A$ is invertible for all $\lambda \in \Lambda$ with sufficiently large absolute value.

Proof. Set $P_j(\lambda) = p_j(x, D; \lambda)$, $A_j(\lambda) = \lambda - a_j(x, D)$, and $A_j = -a_j(x, D)$. Moreover, denote in the following by '$\equiv$' equivalence modulo a remainder $R(\lambda)$ with $\|R(\lambda)\|_{\mathcal{L}(\mathcal{H}^s)} = O(\langle\lambda\rangle^{-\varepsilon/m})$.

Choose $\psi_j \in \mathcal{C}^{k+r}(M)$ that are supported in U_j such that $\psi_j \equiv 1$ near the support of $\widetilde{\varphi}_j$, and such that $\widetilde{\psi}_j \equiv 1$ near the support of ψ_j. Then

$$\begin{aligned} A(\lambda)P(\lambda) = \sum_{j=1}^{N} \Big\{ &\widetilde{\varphi}_j\, A(\lambda)\, \psi_j\, P(\lambda)\, \widetilde{\psi}_j + \widetilde{\varphi}_j\, A(\lambda)\, \psi_j\, P(\lambda)\, (1 - \widetilde{\psi}_j) \\ &+ \widetilde{\varphi}_j\, A\, (1 - \psi_j)\, P(\lambda)\, (1 - \widetilde{\psi}_j) \Big\} \equiv \sum_{j=1}^{N} \widetilde{\varphi}_j\, A(\lambda)\, \psi_j\, P(\lambda)\, \widetilde{\psi}_j, \end{aligned}$$

by Lemma 5.2 and (5.6) (with $\tau = \varepsilon$) together with (A1). Similarly, using (A1), (5.6) (either with $\tau = 0$ or $\tau = m$), and Lemma 5.4, we obtain

$$A(\lambda)P(\lambda) \equiv \sum_{j=1}^{N} \widetilde{\varphi}_j\, \kappa_j^* A_j(\lambda)\, \psi_j\, \kappa_j^* P_j(\lambda)\, \widetilde{\psi}_j$$

and then

$$A(\lambda)P(\lambda) \equiv \sum_{j=1}^{N} \widetilde{\varphi}_j\, \kappa_j^* \{A_j(\lambda)\, P_j(\lambda)\}\, \widetilde{\psi}_j + \sum_{j=1}^{N} \widetilde{\varphi}_j\, \kappa_j^* A_j\, (1 - \psi_j)\, \kappa_j^* P_j(\lambda)\, \widetilde{\psi}_j \equiv 1.$$

The composition $P(\lambda)(\lambda - A)$ is treated in the same way. $\square$

Theorem 5.6. *There exists a $c \geq 0$ such that $\lambda - A$ is invertible for all $\lambda \in \Lambda_c = \{\lambda \in \Lambda \mid |\lambda| \geq c\}$ and, cf.* (5.7),

$$(\lambda - A)^{-1} = \sum_{j=1}^{N} \widetilde{\varphi}_j\, \kappa_j^* p_j(x, D; \lambda)\, \widetilde{\psi}_j + R(\lambda) \qquad \forall\, \lambda \in \Lambda_c$$

with a remainder satisfying

$$\sup_{\lambda \in \Lambda_c} \langle\lambda\rangle^{1+\varepsilon/m}\, \|R(\lambda)\|_{\mathcal{L}(\mathcal{H}^s)} < \infty.$$

In particular,

$$\sup_{\lambda \in \Lambda_c} \langle\lambda\rangle^{1}\, \|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}^s)} < \infty.$$

Proof. Let $S(\lambda) = (\lambda - A)P(\lambda) - 1$. Then for sufficiently large c,

$$(\lambda - A)^{-1} = P(\lambda)(1 + S(\lambda))^{-1} = P(\lambda)\left\{1 - S(\lambda) + S(\lambda)(1 + S(\lambda))^{-1} S(\lambda)\right\}.$$

Now $R(\lambda) = -P(\lambda)S(\lambda) + P(\lambda)S(\lambda)(1 + S(\lambda))^{-1}S(\lambda)$ has the required behaviour due to Proposition 5.5 and (5.6) with $\tau = m$. $\square$

Using Theorem 5.6, the proof of Theorem 5.1 is analogous to that of Theorem 3.11.

6. The Dirichlet-Neumann operator for $\mathcal{C}^{1+r}$-domains

We shall prove that the Dirichlet-Neumann operator for $\mathcal{C}^{1+r}$-domains admits a bounded H_∞-calculus for any sector Λ not containing the positive reals. This is achieved by showing that this operator fits into the scenario developed in the previous Section 5. In turn, this relies on an analysis of this operator, which is performed in Section 3.4 of [23].

Definition 6.1. A $\mathcal{C}^{1+r}$-domain, $0 < r < 1$, in $\mathbb{R}^{n+1}$ is an open, connected, and bounded set $\Omega \subset \mathbb{R}^{n+1}$ with the following property: To each $x_0 \in \partial\Omega$ there exists a system of coordinates of $\mathbb{R}^{n+1}$ with origin x_0, a $\delta > 0$ and a function $\varrho \in \mathcal{C}^{1+r}(\mathbb{R}^n)$ such that

$$\Omega \cap W_\delta(x_0) = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < \varrho(x_1, \dots, x_n)\} \cap W_\delta(x_0),$$

where $W_\delta(x_0)$ is the cube of side length 2δ, centered in x_0.

As in Section 5 we shall denote by $\mathcal{H}^s$ either the scale of Sobolev spaces, the scale of Besov spaces, or the scale of little Hölder-Zygmund spaces on $\partial\Omega$. Moreover, we shall fix an $0 \le s < r$, which we also assume to be positive if we consider Hölder-Zygmund spaces.

Denoting by $\mathcal{T} f$ the solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega$$

(which exists and is unique for sufficiently regular f),

$$f \mapsto \mathcal{N} f := \frac{\partial}{\partial \nu} \mathcal{T} f \qquad (\nu \text{ being the outer normal})$$

is the so-called *Dirichlet-Neumann* operator with respect to Ω.

Proposition 6.2 ([23], Proposition 4.9)**.** $\mathcal{N} : \mathcal{H}^{1+s} \to \mathcal{H}^s$ *continuously.*

We shall now describe more precisely the structure of $\mathcal{N}$. It can be written as

$$\mathcal{N} = (-\tfrac{1}{2} + K^*) S^{-1} =: -\tfrac{1}{2} S^{-1} + R \tag{6.1}$$

(cf. (4.94) in [23]) with S and R having the following properties:

Proposition 6.3 ([23], Proposition 4.6)**.** $S : \mathcal{H}^s \to \mathcal{H}^{1+s}$ *is an isomorphism.*

Proposition 6.4. *There exists an $\varepsilon > 0$ such that $R : \mathcal{H}^{1+s-\varepsilon} \to \mathcal{H}^{s+\varepsilon}$.*

The last proposition follows from (4.52), (4.53) and Proposition 4.6 in [23]. Actually the cited statements of [23] are formulated only for Sobolev and Hölder-Zygmund spaces. However, the corresponding results follow for little Hölder-Zygmund spaces from (5.1), while for Besov spaces we use interpolation.

6.1. **The single layer potential.** S has a certain pseudodifferential structure, which we shall discuss now. To do so, we first recall the construction of S: If $m = n + 1$, i.e. $\Omega \subset \mathbb{R}^{n+1}$, and $n \ge 2$ (this we assume from now on), then

$$(Sf)(x) = -\frac{1}{\omega_{n+1}(n-1)} \int_{\partial\Omega} \frac{1}{|x-y|^{n-1}} f(y)\, d\sigma(y), \qquad x \in \partial\Omega,$$

where ω_{n+1} is the volume of the unit sphere in $\mathbb{R}^{n+1}$. If we let

$$h(w) = -\frac{1}{\omega_{n+1}(n-1)}|w|^{1-n}$$

denote the standard fundamental solution of Δ in $\mathbb{R}^{n+1}$, then

$$(Sf)(x) = \int_{\partial\Omega} h(x-y)f(y)\,d\sigma(y), \qquad x \in \partial\Omega.$$

Now we cover $\partial\Omega$ with finitely many neighborhoods $U_1, \ldots, U_N$ as described in Definition 6.1. We choose functions $\varphi_j, \psi_j \in \mathcal{C}^{1+r}(\partial\Omega)$ supported in $U_j \cap \partial\Omega$, such that $\psi_j \equiv 1$ near the support of φ_j and $\sum\limits_j \varphi_j \equiv 1$ on $\partial\Omega$. Then

$$S = \sum_{j=1}^N \varphi_j\, S\, \psi_j + \sum_{j=1}^N \varphi_j\, S\,(1-\psi_j) =: \sum_{j=1}^N \varphi_j\, S\, \psi_j + R_1, \tag{6.2}$$

and R_1 is an integral operator on $\partial\Omega$ with kernel in $\mathcal{C}^{1+r}(\partial\Omega\times\partial\Omega)$. Each of the operators $\varphi_j\, S\, \psi_j$ is localized in $U_j \cap \partial\Omega$. To understand their structure in the corresponding local coordinates, we have to analyze operators of the form

$$(Hf)(x) = \varphi(x)\int_{\mathbb{R}^n} h(x-y, \varrho(x)-\varrho(y))f(y)\psi(y)\sqrt{1+|\nabla\varrho(y)|^2}\,dy$$

with $\varrho \in \mathcal{C}^{1+r}(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with $\psi \equiv 1$ near the support of φ. For notational convenience, we define

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}, \qquad \phi(x) = (x, \varrho(x)).$$

By Taylor expansion we then can write (cf. (4.1) in [23])[7]

$$h(\phi(x)-\phi(y)) = h(\phi'(x)(x-y)) + h_1(x,y) \qquad \forall\, x \neq y,$$

where ϕ' denotes the (total) derivative and

$$h_1(x,y) = \int_0^1 \langle \nabla h([\phi'(x)+\tau\Psi(x,y)](x-y)), \Psi(x,y)(x-y)\rangle\, d\tau$$

with $\Psi \in \mathcal{C}^r(\mathbb{R}^n\times\mathbb{R}^n)$ given by $\Psi(x,y) = \displaystyle\int_0^1 [\phi'(tx+(1-t)y)-\phi'(x)]\,dt$. From the interplay of pseudodifferential operators and (singular) integral operators one can deduce that $H = a(x,y,D)$ with a double symbol $a(x,y,\xi) \in \mathcal{C}^r S^{-1}_{1,0}$. In particular, $H : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{s+\varepsilon}$ for some $\varepsilon > 0$.

However, H has a better behaviour than an arbitrary operator with symbol in $\mathcal{C}^r S^{-1}_{1,0}$. Performing a Taylor expansion as above,

$$\begin{aligned}\frac{\partial}{\partial x_j} h(\phi(x)-\phi(y)) &= (\partial_j h)(\phi(x)-\phi(y)) + (\partial_j\varrho)(x)(\partial_{n+1}h)(\phi(x)-\phi(y))\\ &= (\partial_j h)(\phi'(x)(x-y)) + (\partial_j\varrho)(x)(\partial_{n+1}h)(\phi'(x)(x-y)) + \widetilde{h}_1(x,y)\\ &= \frac{\partial}{\partial z_j} h(\phi'(x)z)\Big|_{z=x-y} + \widetilde{h}_1(x,y).\end{aligned}$$

[7]Note that $t\phi'(x)(x-y)+(1-t)(\phi(x)-\phi(y)) = (x-y, t\langle\nabla\varrho(x), x-y\rangle+(1-t)(\varrho(x)-\varrho(y))) \neq 0$ for $x \neq y$.

Therefore one obtains (cf. (4.18) in [23]) that, for any $1 \le j \le n$,

$$\begin{aligned} \partial_j(Hf)(x) = \varphi(x)\,\text{p.v.-}\!\!\int \frac{\partial}{\partial z_j} h(\phi'(x)z)\Big|_{z=x-y} \sqrt{1+|\nabla\varrho(y)|^2}\,\psi(y)f(y)\,dy \\ + B_{0,j} + B_{1,j}. \end{aligned} \tag{6.3}$$

Here, $B_{i,j} = b_{i,j}(x,y,D)$ with a double symbol $b_{0,j}(x,y,\xi) \in \mathcal{C}^r S^{-1}_{1,0}$ and a double symbol $b_{1,j}(x,y,\xi) \in \mathcal{C}^r S^0_{1,0}$ that vanishes for $x = y$.

Let us now calculate the pseudodifferential symbol of the singular integral operator in (6.3): According to Example 7.4 (with $a = \nabla\varrho(x)$), we obtain for the Fourier transform

$$\begin{aligned} \mathcal{F}_{z\to\xi}\Big(\text{p.v.}\frac{\partial}{\partial z_j}(h(\phi'(x)z))\Big) &= i\xi_j \mathcal{F}_{z\to\xi} h(\phi'(x)z) \\ &= -\frac{i}{2}\left\{(1+|\nabla\varrho(x)|^2)|\xi|^2 - \langle\xi,\nabla\varrho(x)\rangle^2\right\}^{-1/2}\xi_j. \end{aligned}$$

For convenience, let us now set

$$\widetilde{p}_\varrho(x,\xi) = \sqrt{1+|\nabla\varrho(x)|^2}\left\{(1+|\nabla\varrho(x)|^2)|\xi|^2 - \langle\xi,\nabla\varrho(x)\rangle^2\right\}^{-1/2}. \tag{6.4}$$

Note that $1/\widetilde{p}_\varrho(x,\xi)$ is just the local expression for the length of a covector ξ in the point $(x,\varrho(x))$ of the graph of ϱ in the canonical local coordinates.

Corollary 6.5. *Let $\chi(\xi)$ be an arbitrary (fixed) zero excision function. Then*

$$\partial_j H = \varphi\,\widetilde{p}_{\varrho,j}(x,D)\,\psi + \widetilde{B}_j$$

with $\widetilde{p}_{\varrho,j}(x,\xi) = -\frac{i}{2}\widetilde{p}_\varrho(x,\xi)\chi(\xi)\xi_j \in \mathcal{C}^r S^0_{1,0}$ and a remainder $\widetilde{B}_j : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{s+\varepsilon}$ for some $\varepsilon > 0$.

Proof. Choose a $\widetilde{\psi} \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^n)$ with $\widetilde{\psi} \equiv 1$ near the support of ψ and set

$$q(x,y,\xi) = -\frac{i}{2}\varphi(x)\widetilde{\psi}(y)\sqrt{1+|\nabla\varrho(y)|^2}\,\frac{\widetilde{p}_\varrho(x,\xi)}{\sqrt{1+|\nabla\varrho(x)|^2}}\,\xi_j.$$

We have just shown that $\partial_j H = q(x,y,D)\psi + B_{0,j} + B_{1,j}$ with $B_{i,j}$ from (6.3). Now we write

$$q(x,y,\xi) = \chi(\xi)q(x,x,\xi) + \chi(\xi)\big(q(x,y,\xi) - q(x,x,\xi)\big) + (1-\chi)(\xi)q(x,y,\xi).$$

The operator associated with the third term is an integral operator with kernel in $\mathcal{C}^r_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$, hence has the required mapping properties for the remainder. The operator associated with the second symbol, call it $\widetilde{B}_{1,j}$, is like $B_{1,j}$, i.e. has a double symbol in $\mathcal{C}^r S^0_{1,0}$ that vanishes for $x = y$. From Propositions 9.5 and 9.18 in Chapter 1 of [23] we then derive that $B_{0,j} + B_{1,j} + \widetilde{B}_{1,j}$ has the desired behaviour of the remainder. It remains to observe that $\chi(\xi)q(x,x,\xi) = \varphi(x)\widetilde{p}_{\varrho,j}(x,\xi)$. □

By symbol smoothing, we can choose a real valued symbol $p_{\varrho,\delta}(x,\xi) \in S^{-1}_{1,\delta}$ such that

$$\chi(\xi)\widetilde{p}_\varrho(x,\xi) - p_{\varrho,\delta}(x,\xi) \in \mathcal{C}^r S^{-1-r\delta}_{1,\delta}.$$

Combining Corollary 6.5 with Proposition 2.1.E of [22] we obtain:

Proposition 6.6. *Let $0 < \delta < 1$ such that $s < (1-\delta)r$. Then*

$$\partial_j H = \varphi\, p_{\varrho,\delta,j}(x,D)\,\psi + B_{\delta,j}$$

with $p_{\varrho,\delta,j}(x,\xi) = -\frac{i}{2} p_{\varrho,\delta}(x,\xi)\xi_j \in S^0_{1,\delta}$ and a remainder $B_{\delta,j} : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{s+\varepsilon}$ for some $\varepsilon > 0$.

Let us for a moment set $P_j = \varphi\, p_{\varrho,\delta,j}(x,D)\,\psi$ and $B_j = B_{\delta,j}$. Then $\partial_j H = P_j + B_j$ for $1 \le j \le n$ according to Proposition 6.6. Thus

$$(1-\partial_j^2)H = (1-\partial_j)(1+\partial_j)H = -\partial_j P_j + (P_j + (1-\partial_j)(H+B_j)).$$

Summing up these identities for $1 \le j \le n$ and applying $(n-\Delta)^{-1}$ from the left to the resulting identity, we obtain

$$H = -(n-\Delta)^{-1}(\partial_1 P_1 + \ldots + \partial_n P_n) + R_1 \tag{6.5}$$

with $R_1 = (n-\Delta)^{-1} \sum_{j=1}^n P_j + (1-\partial_j)(H+B_j)$. The first term on the right-hand side of (6.5) equals $\varphi\, p(x,D)\,\psi + r(x,D)$ with

$$p(x,\xi) = -\frac{i}{n+|\xi|^2} \sum_{j=1}^n p_{\varrho,\delta,j}(x,\xi)\xi_j = -\frac{1}{2}\frac{|\xi|^2}{n+|\xi|^2} p_{\varrho,\delta}(x,\xi)$$

and a symbol $r \in S^{-2+\delta}_{1,\delta}$. Since $\frac{|\xi|^2}{n+|\xi|^2}$ differs from 1 by a symbol from $S^{-2}_{1,0}$, we arrive at the following result:

Proposition 6.7. *Let $0 < \delta < 1$ such that $s < (1-\delta)r$. Then there exists an $\varepsilon > 0$ such that*

$$H = -\frac{1}{2}\varphi\, p_{\varrho,\delta}(x,D)\,\psi + R_\delta$$

with a remainder $R_\delta : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{1+s+\varepsilon}$.

Combining this result with (6.2) we derive that S has a nice pseudodifferential structure. For its formulation, let $\kappa_j : U_j \cap \partial\Omega \to V_j \subset \mathbb{R}^n$ be the canonical local coordinates of $U_j \cap \partial\Omega$ (recall that after possibly translating and rotating Ω, $U_j \cap \partial\Omega$ is the graph of $\varrho_j \in \mathcal{C}^{1+r}$).

Theorem 6.8. *Let $0 < \delta < 1$ such that $s < (1-\delta)r$. Then there exist real-valued symbols*

$$p_{j,\delta}(x,\xi) \in S^{-1}_{1,\delta}, \qquad 1 \le j \le N,$$

having the following property: For any choice of functions $\widetilde{\varphi}_j, \widetilde{\psi}_j \in \mathcal{C}^{1+r}(\partial\Omega)$ with support in $U_j \cap \partial\Omega$ and such that the $\widetilde{\varphi}_j$ form a partition of unity on $\partial\Omega$ and $\widetilde{\psi}_j \equiv 1$ near the support of $\widetilde{\varphi}_j$,

$$S = -\frac{1}{2}\sum_{j=1}^N \widetilde{\varphi}_j\, \kappa_j^* p_{j,\delta}(x,D)\,\widetilde{\psi}_j \;+\; R_\delta \tag{6.6}$$

(κ_j^* denotes the pull-back of operators from V_j to U_j by κ_j) with a remainder $R_\delta : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{1+s+\varepsilon}$ for some $\varepsilon > 0$. If $|\xi|_{x,j}$ denotes the local expression with respect

to κ_j for the length of the covector ξ in the point $\kappa_j^{-1}(x)$, then

$$p_{j,\delta}(x,\xi) - \chi(\xi)\frac{1}{|\xi|_{x,j}} \in \mathcal{C}^r S_{1,\delta}^{-1-r\delta} \tag{6.7}$$

for any zero excision function $\chi(\xi)$.

6.2. **The Dirichlet-Neumann operator.** With the notation from (6.1) and Proposition 6.4, $\mathcal{N} = -\frac{1}{2}S^{-1} + R$. Using Theorem 6.8, we shall now derive that $\mathcal{N}$ also has the pseudodifferential structure needed to apply the results of Section 5. By (6.7) and (6.4) it is obvious that

$$p_{j,\delta}(x,\xi) \geq \frac{1}{2}|\xi|_{x,j}^{-1} \geq C|\xi|^{-1} \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, |\xi| \geq R$$

for suitable constants $C > 0$ and $R \geq 0$. In particular, $p_{j,\delta}(x,\xi)$ is an elliptic symbol. Therefore, there exists a parametrix $a_{j,\delta}(x,\xi) \in S^1_{1,\delta}$, i.e.

$$l_{j,\delta} := 1 - a_{j,\delta}\#p_{j,\delta} \in S_{1,0}^{-\infty}, \qquad r_{j,\delta} := 1 - p_{j,\delta}\#a_{j,\delta} \in S_{1,0}^{-\infty}.$$

If χ is a suitable zero excision function, then, by parametrix construction, $a_{j,\delta}(x,\xi)$ differs from $\chi(\xi)/p_{j,\delta}(x,\xi)$ by a symbol belonging to $S^{\delta}_{1,\delta}$. From (6.7) it follows that $\chi(\xi)/p_{j,\delta}(x,\xi)$ differs from $\chi(\xi)|\xi|_{x,j}$ by a symbol[8] in $\mathcal{C}^r S_{1,\delta}^{1-r\delta}$. Thus we derive that

$$a_{j,\delta}(x,\xi) - \chi(\xi)|\xi|_{x,j} \in \mathcal{C}^r S_{1,\delta}^{1-r\delta} \tag{6.8}$$

provided, in addition, $0 < \delta < \frac{1}{1+r}$. Consequently, for suitable constants $C > 0$ and $R \geq 0$,

$$a_{j,\delta}(x,\xi) \geq \frac{1}{2}|\xi|_{x,j} \geq C|\xi| \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, |\xi| \geq R. \tag{6.9}$$

Theorem 6.9. *Let $0 < \delta < \frac{1}{1+r}$ such that $s < (1-\delta)r$. Let $a_{j,\delta}(x,\xi) \in S^1_{1,\delta}$ be the symbols constructed above. Then, for any choice of functions $\varphi_j, \psi_j \in \mathcal{C}^{1+r}(\partial\Omega)$ supported in $U_j \cap \partial\Omega$, such that the φ_j form a partition of unity on $\partial\Omega$ and $\psi_j \equiv 1$ near the support of φ_j,*

$$\mathcal{N} = \sum_{j=1}^{N} \varphi_j\, \kappa_j^* a_{j,\delta}(x,D)\, \psi_j \;+\; K_\delta \tag{6.10}$$

with a remainder $K_\delta : \mathcal{H}^{1+s-\varepsilon} \to \mathcal{H}^{s+\varepsilon}$ for some $\varepsilon > 0$.

Proof. For convenience, let us suppress the index δ from the notation, and set $\widetilde{S} = -2S$. For $\phi, \psi \in \mathcal{C}^{1+r}(\partial\Omega)$ supported in $U_j \cap \partial\Omega$ we deduce from (6.6), by choosing $\varphi_j \equiv 1$ near the supports of ϕ and ψ, that $\phi\{\widetilde{S} - \kappa_j^* p_j(x,D)\}\psi : \mathcal{H}^{s-\varepsilon} \to \mathcal{H}^{1+s+\varepsilon}$. From this it follows that

$$\phi\{\kappa_i^* p_i(x,D) - \kappa_j^* p_j(x,D)\}\psi : \mathcal{H}^{s-\varepsilon} \longrightarrow \mathcal{H}^{1+s+\varepsilon} \tag{6.11}$$

for all $\phi, \psi \in \mathcal{C}^{1+r}(\partial\Omega)$ supported in $U_i \cap U_j \cap \partial\Omega \neq \emptyset$. Since a_j is a parametrix to p_j, we derive (cf. Lemma 5.3) for such ϕ, ψ that

$$\phi\{\kappa_i^* a_i(x,D) - \kappa_j^* a_j(x,D)\}\psi : \mathcal{H}^{1+s-\varepsilon} \longrightarrow \mathcal{H}^{s+\varepsilon}. \tag{6.12}$$

[8] If r denotes the symbol in (6.7), then $p^{-1} = |\xi|(1+|\xi|r)^{-1} = |\xi| - r|\xi|^2(1+r|\xi|)^{-1}$.

Now let us set $\widetilde{\mathcal{N}} := \sum\limits_{j=1}^{N} \varphi_j^0\, \kappa_j^* a_j(x,D)\, \psi_j^0$ with a fixed choice of the φ_j^0 and ψ_j^0. Moreover, we choose $\varphi_j, \psi_j, \psi_j' \in \mathcal{C}^{1+r}(\partial\Omega)$ supported in $U_j \cap \partial\Omega$ in such a way that $\psi_j' \equiv 1$ near the support of φ_j and $\psi_j \equiv 1$ near the support of ψ_j'. Then

$$\widetilde{S}\widetilde{\mathcal{N}} = \sum_{j=1}^{N} \varphi_j\, \widetilde{S}\, \psi_j'\, \widetilde{\mathcal{N}}\, \psi_j + \sum_{j=1}^{N} \varphi_j\, \widetilde{S}\, \psi_j'\, \widetilde{\mathcal{N}}\,(1-\psi_j) + \sum_{j=1}^{N} \varphi_j\, \widetilde{S}\,(1-\psi_j')\, \widetilde{\mathcal{N}}.$$

Due to the disjoint supports of ψ_j' and $1-\psi_j$ as well as of φ_j and $1-\psi_j'$, the second and third terms on the right-hand side map $\mathcal{H}^{1+s-\varepsilon} \to H^{1+s+\varepsilon}$. Due to (6.11) and (6.12), a remainder of the same kind arises if we replace in the first term $\widetilde{S}$ by $\kappa_j^* p_j(x,D)$ and $\widetilde{\mathcal{N}}$ by $\kappa_j^* a_j(x,D)$, respectively. Thus

$$\widetilde{S}\widetilde{\mathcal{N}} \equiv \sum_{j=1}^{N} \varphi_j\, \kappa_j^*(p_j(x,D)a_j(x,D))\, \psi_j + \sum_{j=1}^{N} \varphi_j\, \kappa_j^* p_j(x,D)\,(1-\psi_j')\, \kappa_j^* a_j(x,D)\, \psi_j,$$

and therefore $\widetilde{S}\widetilde{\mathcal{N}} = 1 - T$ with $T : \mathcal{H}^{1+s-\varepsilon} \to \mathcal{H}^{1+s+\varepsilon}$. Then, by Proposition 6.4,

$$\mathcal{N} = -\frac{1}{2} S^{-1} + R = \widetilde{S}^{-1} + R = \widetilde{\mathcal{N}} + \widetilde{S}^{-1} T + R,$$

and the result follows by setting $K = \widetilde{S}^{-1} T + R$. □

Combining the previous Theorem 6.9 and (6.9) yields that the Dirichlet-Neumann operator $\mathcal{N}$ satisfies the assumptions of Theorem 5.1. Thus we obtain:

Theorem 6.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{1+r}$-domain. Consider the Dirichlet-Neumann operator $\mathcal{N}$ either as the unbounded operator*

$$\begin{aligned} &\mathcal{N} : H_p^{1+s}(\partial\Omega) \subset H_p^s(\partial\Omega) \longrightarrow H_p^s(\partial\Omega), \qquad 1 < p < \infty, \quad 0 \le s < r, \\ &\mathcal{N} : B_{pq}^{1+s}(\partial\Omega) \subset B_{pq}^s(\partial\Omega) \longrightarrow B_{pq}^s(\partial\Omega), \qquad 1 < p,q < \infty, \quad 0 \le s < r, \end{aligned}$$

or

$$\mathcal{N} : c^{1+s}(\partial\Omega) \subset c^s(\partial\Omega) \longrightarrow c^s(\partial\Omega), \qquad 0 < s < r.$$

Then $\mathcal{N}$ is closed and to any sector $\Lambda \subset \mathbb{C}$ not containing the positive reals, there exists a constant $\lambda_0 \ge 0$ such that $\lambda_0 + \mathcal{N}$ has a bounded H_∞-calculus with respect to $\mathbb{C} \setminus \Lambda$.

6.3. **Composition with differential operators.** Let $k \in \mathbb{N}$ and $0 < r < 1$. Given a Riemannian $\mathcal{C}^{k+r}$-manifold M, it makes sense to speak about differential operators $\mathcal{A}$ of order $k-1$ with $\mathcal{C}^{1+r}$-coefficients. With $\mathcal{A}$ we can associate its principal symbol $\sigma_\psi^{k-1}(A)$, which is a function on the cotangent bundle of M. By homogeneity, it is uniquely determined by its restriction to the cosphere bundle of M.

Definition 6.11. With the above notation, the differential operator $\mathcal{A}$ on M is called Λ-elliptic if

$$\sigma\left(\sigma_\psi^{k-1}(\mathcal{A})(x,\xi)\right) \cap \Lambda = \emptyset \qquad \forall\, \xi \ne 0.$$

Alternatively to the requirement in Definition 6.11 we could ask the existence of a compact subset D of $\mathbb{C} \setminus \Lambda$ such that $\sigma(\sigma_\psi^{k-1}(\mathcal{A})(x,\xi)) \subset D$ whenever $|\xi| = 1$.

We have the following generalization of Theorem 6.10:

Theorem 6.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathcal{C}^{k+r}$-domain, $\mathcal{N}$ the associated Dirichlet-Neumann operator, and $\mathcal{A}$ a Λ-elliptic differential operator of order $k-1$ with scalar $\mathcal{C}^{1+r}$-coefficients. Consider $A := \mathcal{N}\mathcal{A}$ either as the unbounded operator*

$$A : H_p^{k+s}(\partial\Omega) \subset H_p^s(\partial\Omega) \longrightarrow H_p^s(\partial\Omega), \qquad 1 < p < \infty, \quad 0 \le s < r,$$
$$A : B_{pq}^{k+s}(\partial\Omega) \subset B_{pq}^s(\partial\Omega) \longrightarrow B_{pq}^s(\partial\Omega), \qquad 1 < p, q < \infty, \quad 0 \le s < r,$$

or

$$A : c^{k+s}(\partial\Omega) \subset c^s(\partial\Omega) \longrightarrow c^s(\partial\Omega), \qquad 0 < s < r.$$

Then A is closed and there exists a constant $\lambda_0 \ge 0$ such that $\lambda_0 + A$ has a bounded H_∞-calculus with respect to $\mathbb{C} \setminus \Lambda$.

Proof. We shall show that A satisfies conditions (A1)–(A3) of Theorem 5.1.

Let $\varphi, \psi \in \mathcal{C}^{k+r}(\partial\Omega)$ have disjoint support. Choose a $\psi' \in \mathcal{C}^{k+r}(\partial\Omega)$ whose support is disjoint from that of φ and with $\psi' \equiv 1$ near the support of ψ. Then

$$\varphi\, A\, \psi = \varphi\, \mathcal{N}\, \psi'\, \mathcal{A}\, \psi + \varphi\, \mathcal{N}\, (1-\psi')\, \mathcal{A}\, \psi = \varphi\, \mathcal{N}\, \psi'\, \mathcal{A}\, \psi,$$

since $\mathcal{A}$ is a local operator. From $\varphi \mathcal{N} \psi' \in \mathcal{L}(\mathcal{H}^{1+s-\varepsilon}, \mathcal{H}^{s+\varepsilon})$ for some $\varepsilon > 0$, it follows that $\varphi A \psi' \in \mathcal{L}(\mathcal{H}^{k+s-\varepsilon}, \mathcal{H}^{s+\varepsilon})$; i.e. A satisfies condition (A1).

Let us now use the notation of Theorem 6.9, but assume $\varphi_j, \psi_j \in \mathcal{C}^{k+r}(\partial\Omega)$. By symbol smoothing, it is obvious that $\mathcal{A} \equiv \sum_{j=1}^{N} \varphi_j\, \kappa_j^* \widetilde{a}_j(x,D)\, \psi_j$ modulo $\mathcal{L}(\mathcal{H}^{k+s-\varepsilon},\ \mathcal{H}^{1+s+\varepsilon})$ for some $\varepsilon > 0$, and with Λ-elliptic symbols $\widetilde{a}_j \in S^{k-1}_{1,\delta}$. Then it is straightforward to derive that $A \equiv \sum_{j=1}^{N} \varphi_j\, \kappa_j^* (a_{j,\delta} \widetilde{a}_j)(x,D)\, \psi_j$ modulo $\mathcal{L}(\mathcal{H}^{k+s-\varepsilon}, \mathcal{H}^{s+\varepsilon})$ (cf. the calculation of $\widetilde{S}\widetilde{N}$ in the proof of Theorem 6.9; note also that the Leibniz product $a_{j,\delta} \# \widetilde{a}_j$ coincides with $a_{j,\delta}\widetilde{a}_j$ modulo $S^{m-1+\delta}_{1,\delta}$). From this (A2) follows. Finally, the identity

$$(\lambda - a_{j,\delta}(x,\xi)\widetilde{a}_j(x,\xi))^{-1} = a_{j,\delta}(x,\xi)^{-1} \left(\lambda a_{j,\delta}(x,\xi)^{-1} - \widetilde{a}_j(x,\xi)\right)^{-1}$$

together with (6.9) and the Λ-ellipticity of $\widetilde{a}_j$ shows that also $a_{j,\delta}\widetilde{a}_j$ is Λ-elliptic; i.e. (A3) holds. □

7. Appendix: Calculating a Fourier transform

In the following assume $n \ge 2$ and let $A : \mathbb{R}^n \to \mathbb{R}^{n+1}$ denote a linear injective mapping. We write A^{-1} for the inverse of A on $\operatorname{im} A$. Further we let $v^\perp \in \mathbb{R}^{n+1}$ be a unit vector orthogonal to $\operatorname{im} A$ and let π denote the orthogonal projection onto $\operatorname{im} A$; i.e. $\pi z = z - \langle z, v^\perp \rangle v^\perp$.

Lemma 7.1. *Let $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$ and $\psi := \varphi \circ A$. Then $\psi \in \mathcal{S}(\mathbb{R}^n)$ and*

$$\widehat{\psi}(\xi) = \frac{1}{|\det B|} \int \widehat{\varphi}\big((A^{-1}\pi)^* \xi + \tau v^\perp\big)\, đ\tau, \tag{7.1}$$

where $B \in \mathcal{L}(\mathbb{R}^{n+1})$ is defined by $Bw = B(w', w_{n+1}) = Aw' + w_{n+1} v^\perp$.

Proof. As $\varphi \circ B$ is rapidly decreasing we can write, by the Fourier inversion formula,

$$\psi(x) = \varphi(B(x,0)) = \lim_{\varepsilon \to 0} \iint e^{-ir\tau} \kappa(\varepsilon\tau) \varphi(B(x,r))\, dr\, đ\tau,$$

where $\kappa \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R})$ with $\kappa(0) = 1$. After integration by parts, the double integral on the right-hand side can be estimated from above by

$$\iint |\kappa(\varepsilon\tau)\,\langle\tau\rangle^{-2}\,(1 + D_r^2)\varphi(B(x,r))|\,dr\,đ\tau \le c\,\langle x\rangle^{-n-1}$$

uniformly in ε. By dominated convergence, we thus obtain

$$\widehat{\psi}(\xi) = \lim_{\varepsilon\to 0} \iiint e^{-ix\xi} e^{-ir\tau}\kappa(\varepsilon\tau)\varphi(B(x,r))\,dr\,dx\,đ\tau.$$

A change of variables yields

$$\begin{aligned}\widehat{\psi}(\xi) &= \lim_{\varepsilon\to 0} \frac{1}{|\det B|} \iiint e^{-i\langle B^{-1}(x,r),(\xi,\tau)\rangle}\kappa(\varepsilon\tau)\varphi(x,r)\,dr\,dx\,đ\tau \\ &= \lim_{\varepsilon\to 0} \frac{1}{|\det B|} \int \kappa(\varepsilon\tau)\varphi((B^{-1})^*(\xi,\tau))\,đ\tau.\end{aligned}$$

It remains to observe that $B^{-1}z = ((A^{-1}\pi)z, \langle z, v^\perp\rangle)$ and therefore

$$\langle B^{-1}z,(\xi,\tau)\rangle = \langle (A^{-1}\pi)z,\xi\rangle + \langle z, v^\perp\rangle\tau = \langle z,(A^{-1}\pi)^*\xi + \tau v^\perp\rangle;$$

i.e. $(B^{-1})^*(\xi,\tau) = (A^{-1}\pi)^*\xi + \tau v^\perp$. □

We shall now extend formula (7.1) to other classes of distributions. Let $k = k(w) \in \mathcal{C}^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ be positive homogeneous of degree $-n+1$. Then k is a regular distribution; its Fourier transform $\widehat{k}$ again is smooth outside 0 and positive homogeneous of degree -2.

Proposition 7.2. *Let k be as described above. The Fourier transform of $h := k\circ A \in \mathcal{C}^\infty(\mathbb{R}^n\setminus\{0\})$ is given by*

$$\widehat{h}(\xi) = \frac{1}{|\det B|}\int \widehat{k}\big((A^{-1}\pi)^*\xi + \tau v^\perp\big)\,đ\tau, \qquad \xi \ne 0; \tag{7.2}$$

it belongs to $\mathcal{C}^\infty(\mathbb{R}^n\setminus\{0\})$ and is positive homogeneous of degree -1.

Proof. Since h is smooth outside 0 and homogeneous of degree $-n+1$, its Fourier transform is also smooth on $\mathbb{R}^n\setminus\{0\}$ and homogeneous of degree -1. If $\xi\ne 0$, then $0 \ne (A^{-1}\pi)^*\xi \in \operatorname{im} A$. Hence the integral in (7.2) defines a smooth function outside 0, since the integrand is dominated by $c\,\langle\tau\rangle^{-2}$. Also it is homogeneous of degree -1. Recall that distributions of homogeneity -1 are uniquely determined by their restrictions to $\mathbb{R}^n\setminus\{0\}$.

Now let $\chi = \chi(|w|)$ be a zero excision function. We define $\chi_\varepsilon(w) = \chi(w/\varepsilon)$ and $h_\varepsilon = (\chi_\varepsilon k)\circ A$.

Claim 1: On $\mathbb{R}^n\setminus\{0\}$ both $\widehat{h_\varepsilon}$ and $\dfrac{1}{|\det B|}\displaystyle\int \widehat{\chi_\varepsilon k}\big((A^{-1}\pi)^*\xi + \tau v^\perp\big)\,đ\tau$ coincide:

By the homogeneity, we have $\chi_\varepsilon k \in S^{-n+1}(\mathbb{R}^{n+1})$ and $h_\varepsilon \in S^{-n+1}(\mathbb{R}^n)$. Now let $d > -n+1$ be fixed and choose a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^{n+1})$ that converges to $\chi_\varepsilon k$ in $S^d(\mathbb{R}^{n+1})$. Set $\psi_j = \varphi_j\circ A$.

The Fourier transform induces a continuous map from $S^d(\mathbb{R}^{n+1})$ to $\mathcal{S}(\mathbb{R}^{n+1}\setminus\{0\})$. If $|\xi| \ge t$ with an arbitrary $t > 0$ we thus can estimate, using Lemma 7.1,

$$\begin{aligned}&\Big|\widehat{\psi_j}(\xi) - \frac{1}{|\det B|}\int \widehat{\chi_\varepsilon k}\big((A^{-1}\pi)^*\xi + \tau v^\perp\big)\,đ\tau\Big| \\ &\qquad \le \frac{1}{|\det B|}\int \big|\mathcal{F}(\varphi_j - \chi_\varepsilon k)\big((B^{-1})^*(\xi,\tau)\big)\big|\,đ\tau \le C_t|||\varphi_j - \chi_\varepsilon k|||\int \langle\tau\rangle^{-2}\,d\tau\end{aligned}$$

with a constant C_t only depending on t and some semi-norm $|||\cdot|||$ of $S^d(\mathbb{R}^{n+1})$. Clearly, the right-hand side tends to 0 for $j\to\infty$. But also $\psi_j\to h_\varepsilon$ uniformly in $\mathbb{R}^n$, in particular, in $\mathcal{S}'(\mathbb{R}^n)$. Therefore $\widehat{\psi_j}\to\widehat{h_\varepsilon}$ in $\mathcal{S}'(\mathbb{R}^n)$. This proves Claim 1.

Claim 2: $\displaystyle\frac{1}{|\det B|}\int\mathcal{F}((1-\chi_\varepsilon)k)\big((A^{-1}\pi)^*\xi+\tau v^\perp\big)\,đ\tau\in\mathcal{C}_b^\infty(\mathbb{R}^n_\xi)$ tends uniformly to 0 as $\varepsilon\to 0$:

First observe that, by the homogeneity of k,

$$\mathcal{F}((1-\chi_\varepsilon)k)(z)=\int e^{-iwz}(1-\chi)(w/\varepsilon)k(w)\,dw=\varepsilon^2\mathcal{F}((1-\chi)k)(\varepsilon z).$$

Since $\mathcal{F}((1-\chi_\varepsilon)k)$ is a symbol of order -2, the claim then follows from

$$\int\mathcal{F}((1-\chi_\varepsilon)k)\big((B^{-1})^*(\xi,\tau)\big)\,đ\tau=\varepsilon\int\mathcal{F}((1-\chi)k)\big((B^{-1})^*(\varepsilon\xi,\tau)\big)\,đ\tau$$

and the fact that $\big\langle(B^{-1})^*(\varepsilon\xi,\tau)\big\rangle^{-2}\le c\,\langle\tau\rangle^{-2}$ with a constant independent of τ,ξ, and ε.

We can now conclude the desired result: In (7.2) we write $k=\chi_\varepsilon k+(1-\chi_\varepsilon)k$ and obtain, by Claim 1,

$$\frac{1}{|\det B|}\int\widehat{k}\big((A^{-1}\pi)^*\xi+\tau v^\perp\big)\,đ\tau=\widehat{h_\varepsilon}+\frac{1}{|\det B|}\int\mathcal{F}((1-\chi_\varepsilon)k)\big((A^{-1}\pi)^*\xi+\tau v^\perp\big)\,đ\tau$$

on $\mathbb{R}^n\setminus\{0\}$. The second term tends to 0 by Claim 2, while $h_\varepsilon\xrightarrow{\varepsilon\to 0}h$ in $\mathcal{S}'(\mathbb{R}^n)$, by dominated convergence. Thus $\widehat{h_\varepsilon}\xrightarrow{\varepsilon\to0}\widehat{h}$ in $\mathcal{S}'(\mathbb{R}^n)$. $\square$

We now specialize Proposition 7.2 to a particular choice of the embedding A.

Corollary 7.3. *Let $k=k(w)\in\mathcal{C}^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ be positive homogeneous of degree $-n+1$ and $h:=k\circ A$, where*

$$Ax=(x,\langle x,a\rangle),\qquad x\in\mathbb{R}^n,$$

with a fixed vector $a\in\mathbb{R}^n$. Then, with the above notation,

$$v^\perp=\frac{1}{\sqrt{1+|a|^2}}(a,-1),\qquad (A^{-1}\pi)^*\xi=(\xi,0)-\frac{\langle\xi,a\rangle}{\sqrt{1+|a|^2}}v^\perp,$$

and the Fourier transform of h is

$$\widehat{h}(\xi)=\frac{1}{\sqrt{1+|a|^2}}\int\widehat{k}\big((A^{-1}\pi)^*\xi+\tau v^\perp\big)\,đ\tau=\int\widehat{k}(\xi-\tau a,\tau)\,đ\tau,\qquad\xi\neq0.\tag{7.3}$$

Proof. $(A^{-1}\pi)z$ is given by the first n components of πz, i.e.

$$(A^{-1}\pi)z=z'-\frac{\langle z,v^\perp\rangle}{\sqrt{1+|a|^2}}\,a.$$

This yields the formula for $(A^{-1}\pi)^*$. The matrix representation of B with respect to the standard basis of $\mathbb{R}^{n+1}$ is

$$B=\begin{pmatrix}E & \frac{a}{\sqrt{1+|a|^2}}\\ a^t & -\frac{1}{\sqrt{1+|a|^2}}\end{pmatrix},$$

where E is the $n\times n$ identity matrix. Thus, by an elementary calculation,

$$\det B=\frac{1}{\sqrt{1+|a|^2}}\det\begin{pmatrix}E & a\\ a^t & -1\end{pmatrix}=-\frac{1}{\sqrt{1+|a|^2}}(1+|a|^2)=-\sqrt{1+|a|^2}.$$

This shows the first part of (7.3). Inserting the explicit expressions for $(A^{-1}\pi)^*$ and $v^\perp$,

$$\begin{aligned}\int \widehat{k}\big((A^{-1}\pi)^*\xi + \tau v^\perp\big)\, đ\tau &= \int \widehat{k}\Big((\xi,0) + (\tau - \tfrac{\langle \xi, a\rangle}{\sqrt{1+|a|^2}})v^\perp\Big)\, đ\tau \\ &= \int \widehat{k}\big((\xi,0) + \tau v^\perp\big)\, đ\tau \\ &= \int \widehat{k}\Big(\xi + \tau\tfrac{a}{\sqrt{1+|a|^2}}, -\tau\tfrac{1}{\sqrt{1+|a|^2}}\Big)\, đ\tau \\ &= \sqrt{1+|a|^2}\int \widehat{k}(\xi - \tau a, \tau)\, đ\tau;\end{aligned}$$

i.e. the second part of (7.3) is valid. □

Example 7.4. Let the notation be as in Corollary 7.3. If now k is the standard fundamental solution of the Laplacian in $\mathbb{R}^{n+1}$, i.e. $\widehat{k}(\eta) = -\frac{1}{|\eta|^2}$, then

$$\begin{aligned}\widehat{h}(\xi) &= -\frac{1}{\sqrt{1+|a|^2}}\int \frac{1}{|(A^{-1}\pi)^*\xi|^2 + \tau^2}\, đ\tau \\ &= -\frac{1}{\sqrt{1+|a|^2}}\frac{1}{|(A^{-1}\pi)^*\xi|}\int \frac{1}{1+\tau^2}\, đ\tau = -\frac{1}{2}\,\frac{1}{\sqrt{|\xi|^2(1+|a|^2) - \langle \xi, a\rangle^2}}.\end{aligned}$$

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